ENCYCLOPÆDIA METROPOLITANA;

OR,

UNIVERSAL DICTIONARY OF KNOWLEDGE,

On an Original Plan:

COMPRISING THE TWOFOLD ADVANTAGE OF

A PHILOSOPHICAL AND AN ALPHABETICAL ARRANGEMENT,

WITH APPROPRIATE ENGRAVINGS.

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PREFACE

TO THE

ENCYCLOPÆDIA METROPOLITANA.

GENERAL OBSERVATIONS.

As the Encyclopædia Metropolitana is now placed before the public as a complete work, it appears essential to offer a few remarks on the objects proposed in this great undertaking, and the manner in which its early professions have been realized. The Prospectus, written by the late eminent poet and philosopher, S. T. Coleridge, and Dr. Stoddart, and the Introductory Essay on the Principles of Method, which accompanied the first part of the work, sufficiently explain the plan on which it was intended to conduct the Encyclopædia. The scheme put forth in those two remarkable productions certainly proceeded on a more enlarged and philosophical view, both of the general relations existing between different branches of human knowledge, and of the proper mode of exhibiting those relations and the principles of each science in an Encyclopædia, than had ever formed the basis of any similar work. A very brief historical notice respecting Encyclopædias will confirm this assertion.

"With the Ancients," it was remarked in the Prospectus, "the term ENCYCLOPÆDIA explained itself. It was really *Instruction in a Cycle*, i. e., the cycle of the seven liberal Arts and Sciences that constituted the course of education for the higher class of citizens; grammar being the first, and each of the others having its particular place in the cycle determined by its dependency on the preceding." No work of this nature, however, has descended to us from ancient times, although the name of *Encyclopædia* has sometimes been applied to the Antiquities of Varro and the *Historia Naturalis* of Pliny. *Speusippus*, the Academic, and Aristotle, in his last work on the Sciences (περὶ ἐπιστήμων), are referred to by *Krug** as having been amongst the earliest compilers of similar works. But in the Middle Ages they were not uncommon under the title of *Summæ, Specula*, &c. One of

* In his Philosophical Lexicon.

the most celebrated of these is the *Speculum historiale, naturale et doctrinale*, by *Vincent* of *Beauvais* (Vincentius Bellovacensis), in the XIIIth century, to which a *Speculum morale* was afterwards added. In the XVIth century several works of an Encyclopædic character appeared, such as *Ringelberg's Cyclopædia*, Basle, 1541; *Paulus de Scala Epistemon*, Basle, 1559; *Reisch's Margarita Philosophica, Martini Idea Philosophica*, &c. The work of Ringelberg, a small thick volume, nearly represents the ancient notion of an Encyclopædia, and consists of concise treatises on Grammar, Logic, Rhetoric, &c. The nature of the work may, in some degree, be perceived from the title, which runs thus: *Joachimi Fortii Ringelbergii Andoverpiani Lucubrationes, vel potius absolutissima ἐγκυκλοπαιδεια, nempe Liber de ratione studii utrusque linguæ, Grammaticæ, Dialecticæ, Rhetoricæ, Mathematicæ, et sublimioris Philosophicæ Multa*, &c. It is possible that this work of Ringelberg may have led the way to Alsted's more elaborate Encyclopædia, which is generally referred to as the most celebrated of the early Encyclopædias. Its author, John Henry Alsted, born in Herborn of Nassau, 1588, was one of the Divines who attended the Synod of Dort. His Encyclopædia, after several smaller editions had appeared, was published at Lyons in 1649, in 4 volumes, folio. Its plan is not unlike that of Ringelberg, but the subjects it embraces are more varied, and each is more elaborately treated. It is preceded by an analysis and compendium of the whole work. It contains thirty-five books. The 1st book is entitled Hexilogia (or doctrina de habitu mentis); the 2d, Technologia; 3rd, Archelogia; 4th, Didactica; 5th, Lexicons and Nomenclature of each Science; 6th, Grammar; 7th, Rhetoric, &c. In the early part of the work will be found Short Grammars and Lexicons of Latin, Greek, and Hebrew.

It is sufficient to mention these among the earlier Encyclopædias to show how near they approach to the ancient idea attached to the word Encyclopædia, and how far they differ from more modern works which bear the same name.

In recent times, in fact, the term has almost exclusively been applied to dictionaries of general knowledge, or works in which the arts and sciences, and most branches of human knowledge, are treated of in alphabetical order. In France many dictionaries of this kind appeared towards the end of the XVIIth and during the course of the XVIIIth century, among which the *Dictionnaire Universel* of M. l'Abbé Furetière (Amst. 1690, with a Preface by Bayle), afterwards published by M. de Beauval, and subsequently re-edited by M. Brutel de la Riviere at the Hague, in 1727, bears the highest character. The celebrated *Dictionnaire de Trevoux* (as we learn from the preface by the last editor of the above Dictionary) was only a pirated edition of this work. It is, like most of the general dictionaries of the same age and country, chiefly confined to the definition of scientific terms, with a very brief account of each science, &c. But in England, about the beginning of the last century, the *Lexicon Technicum* of Harris, and the *Cyclopædia* of Chambers, prepared the way for the more elaborate and extensive undertakings which have appeared during the last fifty years in so great numbers. In most of these the alphabetical arrangement has been adopted, although it has been adhered to with greater strictness in some instances than in others. The chief difference has consisted in this circumstance,—that in

some of these works, indeed in most of them, treatises of more or less completeness are given under the general name of each science, such as OPTICS, ASTRONOMY, SURGERY, &c.; and a reference to the treatise is made under each of the technical terms which belong to it; while in others, under these technical terms, a short account is given of the meaning of the word, and the most useful information respecting the portion of the science to which it belongs is inserted there. Thus in this latter case the laws of Refraction, the treatment of Aneurism, and the doctrine of Precession would be given under those terms respectively; while in the former plan nothing but a definition would be given, with a reference to OPTICS, SURGERY, and ASTRONOMY. The former plan is the most common, and, as it is easy to perceive, partakes more of a scientific and systematic character. Still some of the disadvantages of any mere alphabetical arrangement pointed out in the original Prospectus to this Encyclopædia must remain under whatever modifications it may be adopted, and with whatever ability it may be executed. In some of the smaller Encyclopædias an attempt has been made to obviate this inconvenience by a division into various branches of knowledge, and by giving, in separate volumes, the historical and geographical articles in one dictionary, the arts and sciences in another, and so forth. But this arrangement has not formed the basis of any very extensive undertaking in our own language.

The plan of treating each science separately has, however, been adopted in the latest and most elaborate work published in France—the *Encyclopédie Méthodique*, the publication of which commenced in the year 1782, but was not concluded till about ten years ago. This great work consists of 201 volumes, including 47 volumes of plates. It is, however, nothing more or less than a collection of classified dictionaries, with a few dissertations interspersed. For example, the section devoted to Law, and called *Jurisprudence*, consists of a Law Dictionary, in ten volumes, to which a Preliminary Discourse is prefixed: the "*Histoire Naturelle*" is also in ten volumes, of which the First Volume consists of a Preliminary Discourse, followed by a Dictionary of Quadrupeds; a Discourse on Ornithology, followed by a Dictionary of Birds, which is concluded in the Second Volume. The Second Volume contains, besides the conclusion of the Dictionary of Birds, a Discourse on Ophiology, with a Dictionary of Serpents. The Third Volume contains Fishes; and Vols. 4 to 10 contain Insects, on the same plan as the preceding volumes. History consists of an Historical Dictionary in 6 volumes; and the whole Encyclopædia consists of Dictionaries arranged in the same manner. Of the earlier French Encyclopédie, to the name of which so much infamy attaches, it is not necessary here to speak. It was alphabetical in its arrangement, and the *Encyclopédie Méthodique* was probably intended to supersede its use by a more methodical system and better principles.

The last work to which we shall call attention is the celebrated Encyklopädie of Ersch and Grüber. Germany offers great facilities for the execution of any literary work requiring the combination of men of varied acquirements and indefatigable industry; and it would be impossible to deny that articles of first-rate merit are to be found in this work written by German scholars and mathematicians of the highest character. But at present it is difficult

to form any judgment upon the work as a whole. For although the alphabet has been drawn up into three brigades, and an attack on each commenced with the courage and perseverance characteristic of Germans, the enemy's position is not yet stormed,—in other words, the work, after these operations have proceeded for about a quarter of a century, is still incomplete, much of the alphabet is unpublished, and some of the most important sciences remain to be treated.

It is scarcely worth while here to do more than just to notice the class of works which have latterly been common in Germany under the name of Conversations-Lexicon, some of which have found their way by means of translation into other countries. The scientific portions are usually very superficial, hardly advancing beyond the mere definitions and the class of information supplied in the French Dictionnaire Universel, already described (see p. vi.); while on subjects of historical and miscellaneous information a great deal of useful matter, though sometimes not untinged with unsound principles, is brought forward in a popular and attractive manner. No notice is here taken of Oriental Encyclopædias, as they scarcely affect European Literature. There is a list of them, with much information on the subject, by V. Hammer Purgstall, in Ersch and Grüber's Encyklopädie, Art. Encyklopädie [orientalische].

From this brief review of the various classes of works bearing the name of Encyclopædia, it will be seen that no great work has ever yet taken the same ground with the present undertaking, and attempted to make a separation between those subjects which demand an alphabetical arrangement and those which are far more conveniently treated in a systematic manner. A very few words will be sufficient to place this in a clear light. It is presumed that Encyclopædias are required by different classes of readers. By some they will be looked upon as repertories of general information; and to this class of readers the facility of reference afforded by the alphabetical arrangement is, no doubt, a matter of convenience. And yet, if their reference is for the purpose of acquainting themselves with some of the principles of a science, or refreshing their memory on some point connected with its details, it is quite obvious that it can make no difference to them whether that science is found placed in its alphabetical order, or in a separate volume with other sciences to which it bears a close relation. Indeed, if mere facility of reference were the only object, and the reader has neither time nor grasp of mind to take in more than is contained under the single term to which he refers, then the old plan, now almost abandoned in England, of giving sciences piecemeal, must have the preference over every arrangement which gives the technical terms and the details of any science in one comprehensive treatise, whether inserted in its alphabetical order or ranged with its sister sciences. But no work of real value ought to contemplate so limited an utility, nor attempt to meet a demand for such desultory and superficial information. One step, therefore, is clearly gained when the several details and technical terms belonging to one science are gathered together into one treatise, even when the place of that treatise is determined by no regard to system, nor to any other circumstance than the first letter in its name. But the Encyclopædia Metropolitana makes another step in advance, and that advance is of more importance than at first sight it seems to be.

One of the advantages offered by this arrangement is, that it brings the work under the class of publications really deserving the name of an Encyclopædia, i. e., *instruction in a methodical order*. The sciences which are capable of mutual dependency are thus brought into one volume; and those who really desire instruction in them may read them in their natural sequence, and ensure by that means a progressive proficiency in them. It is not a small advantage, particularly in the exact sciences, to find such an arrangement adopted as would enable a student to pursue them even without the assistance of a tutor. It may safely be affirmed, that any person of good mathematical abilities, who followed the course of treatises in the first and second volumes of Pure and Mixed Sciences in this Encyclopædia would become by that means a mathematician of a very high character, and be enabled to master the most difficult and delicate speculations of continental mathematicians.

If, again, in Sciences, where, although there is a mutual dependency, yet each science may be pursued separately by one acquainted with a few mathematical truths, the advantages of this systematic treatment are so great, must it not be tenfold greater in regard to all historical information, where nothing can be isolated, but all is intimately connected. In the alphabetical arrangement a concise history of each country may be given under its name, but then it is isolated from all collateral matter and all contemporary history. But even this system is not always adopted; but the clumsy and unscientific mode of exhibiting the history of each country under the name of its sovereigns is often followed. To obviate the inconvenience arising from this fragmentary kind of history, the Encyclopædia Metropolitana has exhibited the history of the world at first in a series of biographical sketches, and then in a continuous history of each remarkable country, combined with an ecclesiastical history remarkably full and rich in the most interesting epochs of the Christian Church. But of these portions of the work, the Scientific and the Historical, we shall have to speak more in detail hereafter; our concern at present lies only with the mode of arrangement.

These two portions of the work, however, still leave untouched a considerable portion of that Miscellaneous information for which it is usual to refer to an Encyclopædia; and accordingly a very large proportion of the work is devoted to this class of subjects, and combined with the most philosophical dictionary of the English language hitherto published. While, therefore, we deprecate the practice of extravagantly lauding every particular article furnished to this Encyclopædia, we feel justified in observing, that the plan on which it was projected, by a peculiar adjustment of the systematic and alphabetical arrangements, has happily avoided the greater inconveniences of each, while it has at the same time combined their chief advantages. That which is capable of being learned systematically is so exhibited, while any portion of it may be referred to with ease as a separate article; and the alphabetical arrangement has been restricted to that which scarcely admits of any other with convenience to the reader. The plan may have some slight inconveniences of its own, but these advantages far more than counterbalance them. Indeed one of the greatest disadvantages entailed on the work, viz., the fragmentary manner in which each portion was published in the separate parts, is now wholly removed by the completion of the Encyclopædia, and its formation into volumes.

It may be proper here to state exactly the nature of the several divisions of the work, as set forth in the original Prospectus, and as subsequently modified in one or two slight particulars.

FIRST DIVISION.

PURE SCIENCES. — 2 Vols.

- FORMAL.
  - Universal Grammar.
  - Logic:—Rhetoric.
  - Mathematics.
  - Metaphysics.
- REAL.
  - Morals.
  - Law.
  - Theology.

SECOND DIVISION.

MIXED AND APPLIED SCIENCES. — 6 Vols.

- MIXED.
  - Mechanics.
  - Hydrostatics.
  - Pneumatics.
  - Optics.
  - Astronomy.
- APPLIED.
  - I. EXPERIMENTAL PHILOSOPHY.
    - Magnetism:—Electro-Magnetism.
    - Electricity, Galvanism.
    - Heat.
    - Light.
    - Chemistry.
    - Sound.
    - Meteorology.
    - Figure of the Earth.
    - Tides and Waves.
  - II. THE FINE ARTS.
    - Architecture.
    - Sculpture.
    - Painting.
    - Heraldry.
    - Numismatics.
    - Poetry.
    - Music.
    - Engraving.
  - III. THE USEFUL ARTS.
    - Agriculture.
    - Horticulture.
    - Commerce.
    - Political Economy.
    - Carpentry.
    - Fortification.
    - Naval Architecture.
    - Manufactures.
  - IV. NATURAL HISTORY.
    - Inanimate:—Crystallography, Geology, Mineralogy.
    - Insentient:—Phytonomy, Botany.
    - Animate:—Zoology.
  - V. APPLICATION OF NATURAL HISTORY.
    - Anatomy.
    - Materia Medica.
    - Medicine.
    - Surgery.

THIRD DIVISION.

BIOGRAPHICAL AND HISTORICAL. — 5 Vols.

Biography CHRONOLOGICALLY arranged, interspersed with introductory Chapters of National History, Political Geography, and Chronology, and accompanied with correspondent Maps and Charts. The far larger portion of HISTORY being thus conveyed, not only in its most interesting, but in its most philosophical, because most natural and real form; while the remaining and connecting facts are interwoven in the several preliminary chapters.

FOURTH DIVISION.

MISCELLANEOUS AND LEXICOGRAPHICAL. — 12 Vols.

Alphabetical, Miscellaneous, and Supplementary:—containing a GAZETTEER, or complete Vocabulary of Geography; and a Philosophical and Etymological LEXICON of the English Language, or the History of English Words;—the citations arranged according to the Age of the Works from which they are selected, yet with every attention to the independent beauty or value of the sentences chosen, which is consistent with the higher ends of a clear insight into the original and acquired meaning of every word.

INDEX.—Being a digested and complete Body of Reference to the whole Work.

We now proceed to speak of each portion separately.

SCIENTIFIC DIVISION.

PURE AND MIXED SCIENCES.

The principles on which the plan of the Encyclopædia Metropolitana was formed having been already explained, we now proceed to consider the various divisions in detail, beginning with the scientific portion of the work.

It is obvious that, besides the mere separation of scientific from historical and literary matter, another very remarkable division is afforded by the nature of the sciences themselves. On this natural line of distinction the subdivisions of the scientific volumes of the Encyclopædia Metropolitana are founded.

In the first place, those sciences are grouped together, the principles of which belong to the pure Reason, (*e. g.*, Algebra, Geometry, Grammar, Logic, &c.). They are combined together as preliminary to the knowledge of those which depend partly on abstract principles and partly on close observation of the phenomena around us, and thus belong to the truths received by the Understanding.* And here, again, there is also a manifest difference between those sciences in which so great a progress has been made by the human mind that their fundamental principles may be considered permanently fixed, such as Mechanics, Hydrostatics, Astronomy, &c.; and those which depend chiefly on observation of the external world and a large collection of facts and a careful induction from those facts, such as Geology, and perhaps Chemistry. This distinction has not been overlooked in the Encyclopædia Metropolitana.

It would be idle to pretend to give treatises upon the latter which shall permanently embody all the principles which belong to them. This would be to profess to perform impossibilities. All that can be done is to represent their *present condition;* and the names of those who have contributed these portions of the Encyclopædia are a sufficient guarantee that this is effectually provided for.

But with respect to the exact sciences, whether pure or mixed, more is required, and much more has here been performed. The principles of these sciences have long been established; but the efficiency of any treatise depends much on the mode in which they are exhibited, and the value of the whole series in some degree on the manner in which they are combined. In both these respects the Encyclopædia Metropolitana may challenge competition with any existing work. The order in which these sciences are exhibited, and the plan on which the mathematical portion of the Encyclopædia is conceived, resemble considerably that of the series of Elementary Treatises projected many years ago for the University of Cambridge by Dr. Wood the late Dean of Ely, and Professor Vince; but with this difference, that the present volumes are far more comprehensive in the subjects they embrace, and far more elaborate and scientific in their execution. But this very simi-

* The Reason and the Understanding are here distinguished according to the views of German philosophers, and much in the same manner as in the works of the late S. T. Coleridge.

larity shows that the Encyclopædia Metropolitana has attained one of its professed objects,—systematic instruction and scientific information conveyed—not in a confused mass, but in the natural sequence of the sciences.

Indeed this portion of the work has met with a degree of approbation in many quarters, but especially in the University of Cambridge, which no other Encyclopædia has ever yet received. And this preference relates, we may observe, to sciences which have obtained a stated position, and are not liable to be superseded by any new discoveries. Geology and Chemistry indeed, and other sciences founded on observation and experiment, are constantly enlarging their boundaries and changing even some of their elementary principles. But no such change takes place, or indeed, we may confidently assert, ever can take place, in pure Mathematics, or the more exact branches of the Mixed Sciences. The utmost which may be expected in these, is some extension of their present boundaries. The principles already established may *implicitly* contain results not yet developed from them, and some of the known elementary principles may perhaps be thrown into a different *form*, but they are established on too firm a basis ever to be overturned. Again, as physical science employs in its advancement some of the results of those refined speculations in pure mathematics which are at present only truths belonging to the Reason, and have no connection with the world in which we live, there may be discovered another set of results which may give to the mind of man a more ample dominion over the phenomena of the material world. Still these are results which require nothing to be *un*learned; they are a mere advance in the quantity of our knowledge, and in the number of the results we can elicit from them. The student who has really mastered these sciences in the systematic form in which they are arranged here, will never in the course of the longest life find occasion to *un*learn any portion of what he has here acquired, and will find no difficulty whatever in adding to his stores any new results which the mental energy and labour of mankind may hereafter develop from principles now known.

We have been thus particular in stating the advantages of the *arrangement* adopted, because we deem it a matter of considerable importance; but we now proceed to speak more in detail of the execution of each portion.

The distinction between Pure and Mixed Mathematics is of primary importance. In the manner in which mathematical inquiries are now conducted, our progress in mixed mathematical science mainly depends on our command of the principles of pure mathematics. It is indeed almost an acknowledged fact, that, in some respects, we have a superfluity of knowledge of these principles. Our application of Mathematics to Natural Philosophy is so far from having exhausted all our stores of Pure Mathematics, that although there are still many problems too intricate for solution with our present means, yet there is also a large mass of results in pure mathematics which as yet have no specific application, and may be considered as stores reserved for future use.

The mere names of the authors of the Treatises on Pure Mathematics are sufficient to prove that the work is worthy of the present state of science, and that its most important Treatises are contributed by those who have themselves been foremost in the onward march of science. The elaborate Treatise on ARITHMETIC, by the present Dean of Ely (Dr. Peacock), Lowndian Professor of Mathematics in the University of Cambridge, is interesting alike to the scholar, the mathematician, and the speculator in metaphysics. The brief but comprehensive Treatise on TRIGONOMETRY, by Professor Airy, now Astronomer Royal, although on so elementary a subject, is of considerable value from the general elegance of its demonstrations.

The publications of the Rev. H. P. Hamilton on ANALYTICAL GEOMETRY and CONIC SECTIONS, and that of Professor Barlow on the THEORY of NUMBERS, are so well known and so highly esteemed that any eulogium on the essays supplied by these gentlemen on these subjects respectively would be entirely superfluous.

The Treatises of Professor Levy on the DIFFERENTIAL and INTEGRAL CALCULUS are written with a comprehensive brevity which recommends them as an introduction to those important branches of Analytical Mathematics, and are calculated to carry the student to a very high point of proficiency.

The GEOMETRY, ALGEBRA, and GEOMETRICAL ANALYSIS complete the volume in a manner worthy of the treatises with which they are associated.

These sciences are however in some degree elementary; and although by them the student would be so far advanced as to enter upon the works of some of the ablest analysts, it would be unworthy of such a publication as the Encyclopædia Metropolitana to leave either untouched or imperfectly treated the more refined applications of the higher Calculus. It will be found accordingly, that in the second volume of pure sciences the highest branches of mathematical analysis have been treated by writers conversant with all its intricacies, and that the mathematical student is furnished in them with results of far greater variety and of a more subtle nature than can at present be used in the application of analysis to Mixed Mathematics. On this subject it is unnecessary to do more than just to enumerate the names of the treatises and their respective authors, whose eminence in mathematical attainments is universally acknowledged.

The CALCULUS of VARIATIONS, and the CALCULUS of FINITE DIFFERENCES, supplied by the Rev. T. G. Hall, Professor of Mathematics in King's College, London, are treated with the clearness which his long and successful course of mathematical teaching has enabled him to give to these refined and subtle portions of analysis.

The CALCULUS of FUNCTIONS and the THEORY of PROBABILITY are the work of Professor De Morgan. The former of these subjects may at present be considered almost in its infancy; but there can be no doubt that this author has here brought forward much that is calculated to expedite its development. The Treatise on Probabilities (a subject which has

exercised the talents of the greatest mathematicians even down to the times of La Place) is, as might be expected, one of the most complete in any language.

And lastly, the Treatise on DEFINITE INTEGRALS completes the series of these elaborate essays on the higher branches of mathematical analysis. The name of Professor Moseley is a sufficient warrant that this essay is also of the highest character.

Without wishing, therefore, to offer any undue eulogium on the treatises enumerated above, we may confidently ask that portion of the public which is qualified to judge of their merits, to compare the whole system of Pure Mathematics here presented to them with that in any similar work, whether of this country or of the continent, on the grounds of arrangement, clearness, ability, and completeness. From any ordeal of this sort, however severe, this Encyclopædia will not shrink; and it is confidently believed that no parties connected with it would have reason to regret the comparison.

From Pure Mathematics we proceed in natural order to their application to physical phenomena. Of these sciences, some belong to the more elementary branches of physical knowledge, and others to a higher and more advanced stage. Now the treatises on—

HYDRODYNAMICS, MECHANICS, HYDROSTATICS, OPTICS, PLANE ASTRONOMY,

have been written by Professor Barlow with an express view to this distinction. They are elementary enough to enable any student, with a competent knowledge of Pure Mathematics, to overcome their difficulties; and yet they are so based on scientific principles, that they will also prepare him to enter readily on the higher branches of Mixed Mathematics. In Mechanics, more especially, a foundation is laid for the succeeding investigations of Physical Astronomy, which is in fact only one of the higher branches of Analytical Physics.

While, however, for these portions of the work we claim only that high share of approbation due to the presentation of ascertained results and knowledge already acquired, in an elegant and useful form, there are some treatises in the volumes devoted to the Mixed Sciences which demand a separate notice, as enlarging the boundaries of our scientific knowledge. Of this class are the Treatises on LIGHT and SOUND, by Sir J. F. W. Herschel.

The Treatise on LIGHT, by Sir J. F. W. Herschel, from the position it has already obtained in the scientific world, both in England and on the Continent, cannot require any comment or recommendation here. We shall merely cite it as furnishing the best refutation to the words of its author respecting the decline of Science in England.*

The simple mention of Sir J. F. W. Herschel's name is a sufficient recommendation to the Treatise on PHYSICAL ASTRONOMY. It proves at once that it must be an Essay of the highest order of merit, and worthy of the present state of the Science. Indeed the name of Sir J. F. W. HERSCHEL stands so confessedly at the head of Physical Science in England, that the conductors of this Encyclopædia may justly be proud that he has contributed so largely to its pages.

* Herschel—Essay on SOUND. Mixed Sciences, vol. ii. p. 810.

But although Plane and Physical Astronomy had been thus ably treated, it was considered that something more was required; and the late Captain Kater kindly furnished the very useful and able Treatise on NAUTICAL ASTRONOMY, a subject with which his acquaintance was at once profound and practical.

MAGNETISM and ELECTRO-MAGNETISM are treated by Professor Barlow with the same ability and research which he has displayed in the other essays contributed by him. It cannot be needful to recommend the Essay on GALVANISM, as Dr. Roget's scientific character is too firmly established to leave any doubt as to its merit.

The author of the Treatises on ELECTRICITY, HEAT, and CHEMISTRY, the late Rev. F. Lunn, was one whose merits as an experimental philosopher and chemist were not so extensively known as they deserved to be; but in Cambridge, in a considerable circle of persons qualified to judge in these matters, his talents were justly appreciated, and his acquirements acknowledged to be of the highest order. The treatises themselves, it is believed, will amply justify their favourable anticipations.

The third volume of Mixed Sciences is chiefly devoted to the Fine Arts; but there are two or three essays in the early part of the volume which belong to the more exact sciences, viz., the Essay on the FIGURE OF THE EARTH, by the present Astronomer Royal, and his Treatise on the TIDES. With regard to the former, much novelty was hardly to be expected; but the Editor believes he is justified in stating that this Treatise contains the most complete combination and discussion of observations relating to the subject which has yet appeared in England. But the treatise into which this great mathematician has thrown all his power is the Theory of the Tides. It was remarked in 1833, by Mr. Lubbock, on the subject of the tides, that "there is no branch of physical astronomy in which so much remains to be accomplished."* The Astronomer Royal, in this treatise, has made a large step in advance in this science; he has, at all events, demonstrated the unsoundness of the equilibrium theory and the inapplicability of the theory of Laplace. The latter he has explained in such a manner as to bring it within the reach of good mathematicians; whereas, in the manner it was presented by its author in the *Mécanique Céleste*, none but persons of very high mathematical ability and undaunted perseverance would venture to encounter its difficulties. Still the theory was inapplicable; and the Astronomer Royal gave all the leisure he could command, for some years, to the consideration of these questions, and to an endeavour to place this great problem on a firm foundation. The Editor does not pretend to speak on this point from his own knowledge; but the terms in which some of the most distinguished mathematicians of Cambridge have spoken to him of this treatise prove that they consider it to have advanced the knowledge of this difficult subject in no ordinary degree. Indeed, the Editor believes that he may confidently assert that every previous treatise on the subject is entirely superseded by this theory, and that it will prove, for many years to come, the only sound foundation of our knowledge of the Theory of the Tides.

* "Report on the Tides," published in the Report of the First and Second Meetings of the British Association for the Advancement of Science.

A few more treatises in these volumes require separate mention,—the METEOROLOGY of the late Mr. Harvey, and the CRYSTALLOGRAPHY of Mr. Brooke. Although not anxious to quote *opinions* on articles in this Enelyclopædia, the Editor may be permitted to call attention to the following incidental notice of the article in Professor Forbes's Report on Meteorology, addressed to the British Association at its second meeting:—"We shall occasionally avail ourselves of the information contained in this work (the Elémens de Physique of M. Pouillet), as well as of a useful compendium of facts *contained in the article Meteorology, in the Encyclopædia Metropolitana, now in the course of publication.*" —Report, p. 206. The testimony of Professor Forbes is of first-rate authority, and above all suspicion.

Of the CRYSTALLOGRAPHY and MINERALOGY of Mr. Brooke it is not necessary to speak particularly, but we may again quote the same volume of Reports for the testimony of a competent witness to the value of Mr. Brooke's labours in these sciences:—

"Mr. Phillips and Mr. Brooke have contributed to the stock of *crystallography* observations more numerous and exact, probably, than any other two names could rival."—Dr. Whewell's Report on Mineralogy at the Second Meeting of the British Association.

The names of Mr. Phillips and Dr. Daubeny will sufficiently recommend the Treatise on GEOLOGY, as exhibiting an adequate representation of that science at the time of its publication. And, even in this hasty enumeration, the Essays on CARPENTRY, by P. Nicholson, Esq.; on FORTIFICATION, by Major Michell and Captain Procter; and on NAVAL ARCHITECTURE, by the late Mr. Harvey, must not be passed over. We can only say here, as in so many other instances, the names guarantee the value of their contributions.

Before we leave this class of Mixed Sciences, we must call attention to the novel feature exhibited in the sixth volume of the series, viz., a systematic account of the ARTS and MANUFACTURES of Great Britain. There is, probably, no writer who would be able to do such ample justice to so extensive a range of matter, requiring both theoretical and practical knowledge, as Mr. Barlow; but that nothing might be wanting to the completeness of this portion of the work, Professor Babbage was engaged to give a Preliminary Discourse on the *Principles of Manufactures;* and it may confidently be asked, to what other source could the conductors of the work have appealed on so difficult and general a subject where the answer to that appeal would have afforded such entire confidence in the result?

We have now enumerated all the articles in these volumes which appertain to the more exact sciences and to those connected with physical phenomena. The remainder are devoted to another class of subjects,—Natural History, Physiology, Medical Sciences, the Useful Arts, Belles Lettres, and the Fine Arts.

The Treatises on BOTANY and HORTICULTURE are supplied by G. Don, Esq., whose profound acquaintance with every department of knowledge which belongs to the vegetable

kingdom is known to all botanists and florists. The Treatise on POLITICAL ECONOMY was written by N. W. Senior, Esq.

The following enumeration of the remaining Treatises in the volumes devoted to the Mixed and Applied Sciences will show that the range of subjects to which attention is directed is wide and comprehensive, and the intrinsic merit of the Essays themselves will prove that no pains have been spared to do justice to these interesting topics. They embrace a Series of Treatises on ARCHITECTURE, SCULPTURE, PAINTING, ENGRAVING, HERALDRY, NUMISMATICS, POETRY, MUSIC, AGRICULTURE, and COMMERCE.

In the first volume of Pure Sciences Sir J. Stoddart has given a lucid and able summary of the General Principles of GRAMMAR, of which it is unnecessary to speak in detail.

The LOGIC and RHETORIC of the present Archbishop of Dublin require no commendation here, as they have already, for many years, been published in a separate form, and taken their place among the standard works of our language.

The Treatise on LAW is the work of three gentlemen,—Richard Jebb, Esq., Professor Graves, and Archer Polson, Esq. It was originally intended that the whole should have been executed by Mr. Jebb; but ill health having rendered it inconvenient to him to furnish the conclusion, it was intrusted to Professor Graves and Mr. Polson, who were fully acquainted by Mr. Jebb with the plan on which he projected it, and kindly undertook to complete its execution. The portion accomplished by Mr. Jebb embraces one of the most difficult portions of philosophy—the general foundations of law and morals; and the Editor is happy to state that testimony from the very highest quarters has been given to the profoundness of the views entertained by Mr. Jebb, and the ability with which they are developed.

In regard to two of the Treatises in the volumes devoted to Pure Sciences, viz., the MORAL AND METAPHYSICAL PHILOSOPHY, and the OUTLINES OF THEOLOGY, a few words of explanation are required. They appear, it must be acknowledged, under a form different from that which seems to be contemplated in the original scheme of this work.

That scheme apparently was intended to comprise formal and scientific treatises on these important subjects; but every person at all conversant with these matters will acknowledge that such a Treatise could have but little value, if it were confined to the limits which a general work like the present must necessarily prescribe. A course was therefore adopted, by which, it is hoped, the most important principles of these sciences are brought forward in the manner most likely to conduce to the advantage of those who study them. In the present state of metaphysical knowledge, it would be presumptuous to put forth any system of Metaphysics; but a general History of Moral and Metaphysical Philosophy affords the most convenient opportunity for displaying the principles on which the greatest philosophers have hitherto endeavoured to form their systems, for pointing out their difficulties, and for marking how far each has contributed to the progress of the science. Such a sketch, however, required the hand of a master; and the Editor confidently believes that the

Treatise on Moral and Metaphysical Philosophy which is here given is calculated fully to sustain the deservedly high reputation of the Rev. F. D. Maurice.

Of the Outlines of Theology, it does not become the Editor to say more than that to acknowledge with gratitude the very able assistance of Professor Corrie, to whom two chapters are due. Much of the matter which usually falls under the head of Theology had already been anticipated in the Miscellaneous and Historical Departments; and it was the object of the Editor to devote the comparatively small space which he could command only to the most important portions of the subject, and to render this Treatise as practically useful as possible. He has endeavoured to avoid passing controversies, but to bring forward the sound and genuine doctrines of the Church of England; and perhaps he may be allowed to add that, in pursuance of this object, he has spared no pains or labour.

HISTORICAL DIVISION.

From the time that this Encyclopædia was consigned to the management of Archdeacon Lyall, and subsequently to that of the Rev. Edward Smedley, its Historical Division became enriched with contributions from some of the most eminent writers of the day.

It will be impossible, in the rapid sketch of the contents of the several volumes which a Preface admits, to specify every paper; but as every contribution (except in part of the first volume) is assigned to its proper author at the beginning of each volume, such a course is unnecessary, either for the information of the public, or as a tribute of respect to the distinguished authors themselves. It will be observed, on a general survey of their names, that ample care has been taken to enlist among the contributors to this department writers not only of splendid endowments, but also of the highest attainments in different classes of historical knowledge. There will be found contributions from Bishop Blomfield, Dr. Whewell, Serjeant Talfourd, Dr. Arnold, Dr. Hinds, Rev. J. A. Jeremie, Rev. G. C. Renouard, Rev. J. H. Newman, Bishop Russell, Archdeacon Hale, Archdeacon Lyall, Rev. J. B. S. Carwithen, Dr. Hampden, Rev. R. Garnet, Major Mountain, Rev. J. H. B. Mountain, Dr. W. C. Taylor, Captain Procter, Rev. J. E. Riddle, Rev. T. G. Ormerod, T. Roscoe, Esq., W. M'Pherson, Esq., Rev. R. L. Browne, Rev. H. Thompson, Rev. J. G. Dowling, Rev. J. W. Blakesley, Rev. J. B. Ottley, W. Lowndes, Esq., Q.C., &c. &c.

A good work on general history has long been a great desideratum in our literature. The summaries of Tytler and Russell are too brief, and the Universal History, independently of the heavy manner in which it is written, is too long. It is presumed that the historical volumes of the Encyclopædia Metropolitana will be found to meet this want in an efficient manner. The histories are written by men of undoubted ability; and historical dissertations, such as those on the Crusades and the Feudal System, are introduced into the text at the most convenient periods, for the illustration of the subjects involved.

In the original Prospectus it was intimated that the History would be given in the form of Biography, chronologically arranged. Such an arrangement, however conve-

nient in regard to Ancient History, when the History of Greece or Rome was virtually the history of the world, would scarcely admit of any modification by which a modern universal history could be treated biographically. The interests even of Europe alone are too complicated in modern times to be treated in any other way than by a separate history of each country. Accordingly, it will be found that the former plan has been exchanged for national histories from about the middle of the third volume, an exchange which every reader will acknowledge to have been not only advantageous, but imperatively required.

The first volume, beginning from the earliest accounts of mankind, brings down the History to about the year 200, B.C. It contains, besides the usual course of Ancient History, an Essay on Greek Philosophy, connected with the life of Socrates, by the present Bishop of London; and a Life of Archimedes, with a Sketch of Greek Mathematics, by Dr. Whewell; with many other papers, which it is obviously impossible here to specify.

The second volume continues the secular history to the age of the Antonines, and lays a foundation for the future chapters of Ecclesiastical History in an elaborate account of the first appearance of Christianity, and of the apostolic age. The following dissertations, unconnected with the general course of the History, but of great importance in a philosophical point of view, may be particularly specified as giving great value to this volume,—PLATO, ARISTOTLE, SENECA, the STOICS, CICERO, ROMAN PHILOSOPHY, HISTORIANS OF ROME, SEXTUS EMPIRICUS, the PYRRHONISTS, &c. Nor would it be proper to pass over, without a distinct reference, the elaborate History of Latin Poetry, which has been generally acknowledged as a valuable accession to our literature.

The third volume contains an account of the Decline of the Roman Empire, the Rise of the Empire of Charlemagne, and of the Modern System of Europe, as well as an elaborate History of Mohammed, and the origin of Saracenic Power. It brings down the History to about the end of the thirteenth century, and comprises, besides the Secular History, an ample Ecclesiastical History of the same period. The historical dissertations with which it is enriched are Essays on MOHAMMED; on the HERESIES OF THE SECOND AND THIRD CENTURIES; PLOTINUS and the LATER ECLECTICS; the CRUSADES; the FEUDAL SYSTEM; THOMAS AQUINAS, and the SCHOLASTIC SYSTEM.

The fourth and fifth volumes continue the Modern History to the settlement of Europe under the Treaties of 1815. The Ecclesiastical History is also continued to the same period.

The Editor would also desire to call attention to the copious Chronological Tables inserted at convenient intervals in this division of the work.

The historical volumes of the Encyclopædia Metropolitana, it will be seen, have been formed on the principle of giving an accurate and ample general history. As every Encyclopædia is now expected to embody a large amount of history, the only question left for consideration was, *how* to meet this demand in the most efficient manner. The plan

most commonly adopted in works of this nature, of giving the history of each country in the article assigned by alphabetical order to that country, appeared liable to some objections, which might be obviated by removing the history to separate volumes, and giving to it a certain degree of continuity. The convenience of this method is obvious; and the names of the contributors employed upon this important portion of the work bear ample testimony to the exertions which must have been made to obtain the co-operation of so many writers of high endowments.

MISCELLANEOUS PORTION.

Although the Miscellaneous Division of this Encyclopædia occupies a larger number of volumes than any other, it requires a less extended notice. It will be impossible to mention separately every article, or even every contributor of merit; but all that is required in this Preface is to explain in some degree the *principle* on which this portion of the work was executed, and to indicate the authors of some of the most remarkable series of papers.

The most remarkable features in this division of the Encyclopædia are clearly—

1. The English Lexicon.
2. The Geography.
3. The Natural History.
4. The strictly Miscellaneous Articles.

It is unnecessary here to speak in any detail on the subject of the Lexicon. Its plan was duly described in the Prospectus and the special Preface to the Lexicon itself. To that plan a steady adherence has been maintained; and the universal approbation with which this Lexicon has been received, precludes the necessity of enlarging either on the plan itself or on the gigantic labour involved in its execution. The plan of giving the quotations of each word *chronologically* has the advantage of embodying in a philosophical Lexicon a *history of our own language*. They are generally full of interest; but the labour of searching them out and arranging them is one of which those who have never engaged in any similar occupation can form no adequate notion. Once achieved, the work is performed for ever; and Dr. Richardson may be contented to think that he has here left a *κτῆμα ἐς ἀεί* of infinite value to his countrymen.

Before we speak of the Geography and Natural History, and the articles on Law, we may be allowed to insert a few words relating to the highly gifted individual to whom this Encyclopædia owes so many of its advantages and attractions; we mean the late Rev. Edward Smedley. Besides the advantages derived from the confidence reposed in him during his editorship by so many men of distinguished literary merit, he not only threw into the historical volumes of this book very elaborate chapters, containing the results of deep historical research, but gave to the Miscellaneous Division a series of articles which embodied a vast store of curious and recondite information, communicated in a manner at once instructive and agreeable. The copious stores of his own mind, and his vast fund of

acquired knowledge, enabled him to enrich this department of the Encyclopædia with a class of articles which stamp a peculiar character on those volumes of the work which he superintended,—a character which it would have been in vain to seek to supply from any other source. His death was a loss to literature in general; it left a void which it was difficult to supply, and we may be thankful that it was not more severely felt in this Encyclopædia. The arrangements he had already made were so efficient that the succeeding Editors found little difficulty in carrying on what he had begun, and completing what he had either overlooked or left unfinished. The editorship was placed on his decease in the hands of the late Rev. Hugh James Rose, B.D., Principal of King's College, London. It would not become the present Editor to speak of one so closely connected with himself, of the high purpose which he ever set before him in all his undertakings, and the noble endowments with which those high purposes were ever prosecuted. Of these it would be a grateful task to speak, but this is not a fitting place. We confine ourselves here to the simple fact that he made such engagements as materially benefited the work, and facilitated the completion of it on the plan which had been projected and adhered to as far as was practicable.

We proceed now to add a few words on some of the most remarkable sections of the Miscellaneous Division. And first, on the Geography.

It will be observed, that the arrangement in this department, although in the alphabetical portion of the work, is not strictly alphabetical. It has been the practice, through the chief portion of the Encyclopædia, to describe whole regions at once, and give accounts of remarkable places and smaller divisions of territory under the larger geographical division to which they belong. Thus, for example, if the reader wished to turn to the account of SMYRNA, he must look under NATOLIA; for UTRECHT, he would look at NETHERLANDS; and so forth. That this is a sacrifice in some degree of facility of reference, cannot be denied; but at the same time it gives a more philosophical and systematic consistency to the geographical section; and, as the work is now complete, the *Index* will obviate every difficulty of this character. In any case in which it is uncertain where a town or district may be described, a single reference to the Index will be enough.

For the whole of the articles on Geography, the proprietors feel that they may fairly advance the claim of having obtained the co-operation of persons more than competent to bring forward whatever is most valuable for a work like this from all usually accessible sources of information. In this respect the Encyclopædia Metropolitana claims to take a high station among similar works; and the names of those gentlemen who have contributed the articles on European and American Geography are a sufficient pledge of the ability and care with which they are executed. The gentlemen to whose labours this department is chiefly indebted, are the following:—T. Myers, Esq., Captain Bonnycastle, R.E., C. Vignoles, Esq., C.E., H. Lloyd, Esq., G. H. Smith, Esq., A. Jacob, Esq., W. D. Cooley, Esq., and Cyrus Redding, Esq.

But there is one class of geographical articles which demands an especial mention. They are indeed SUI GENERIS, and may be said to be wholly without a rival in any similar work

in our own language. These are the articles on Ancient, Oriental, and African Geography, which, throughout the work, were supplied by the Rev. G. C. Renouard, late Fellow of Sidney-Sussex College, Cambridge, and formerly Chaplain at Smyrna. It is not merely the extensive familiarity with every class of language, ancient and modern, and with all the storehouses of information in them, which give the value to his researches, but it is the extraordinary zeal and industry which he has invariably bestowed in conjunction with these great advantages on his favourite pursuit of geography. No one but the Editor of this Encyclopædia is probably aware of the amount of time and labour bestowed by Mr. Renouard on each of these articles. This circumstance, and his extensive familiarity with the original sources of information in all languages, render his contributions *unique* in the history of similar undertakings; and the Editor believes that if these essays were collected together, and published as a system of Oriental Geography, they would surpass in accuracy and value anything at present existing in our own or any other European language.

We pass on now to the Section of Natural History. This is divided chiefly into Botany and Zoology. In these two sciences the *Genera* will be found described in their alphabetical order, while their scientific arrangement and the principles of the sciences form part of the treatises in the volumes devoted to the Mixed Sciences.

For these two departments, the services of several eminent naturalists were engaged. In Botany, T. Edwards, Esq., and Mr. Don, &c. In Zoology, T. Bell, Esq, F.L.S., &c., J. E. Gray, Esq., F.L.S., &c., of the British Museum, J. F. Stephens, Esq., and Mr. South. To Mr. South the Encyclopædia is much indebted for the very great accuracy with which he has composed his descriptions, and for the varied and interesting information he has interwoven with the subject of most of these articles.* It will also be observed that a very copious Law Dictionary is incorporated with this portion of the work, furnished by a variety of able contributors engaged in the study and the practice of the Law. The articles supplied by each contributor are indicated in the volumes in which they occur.

Besides the miscellaneous articles of the late Editor, the Geographical Gazetteer, and the Law Dictionary, included in this portion of the Encyclopædia, a large number of articles, some of them of very great importance and value, will be found scattered through the volumes of the Miscellaneous Division, which it is obviously impossible here to particularize. Attention may, however, be called, amongst a variety of others, to the Biblical articles, by the Rev. T. H. Horne; to the Philological and Oriental articles, by the Rev. G. C. Renouard; the Scientific articles (as *e. g.*, Dialling, Surveying, Weights and Measures, &c.), by Mr. Barlow; Meteoric Stones, by Professor Miller; Stove and Ventilation, by C. Hood, Esq., F.R.S., &c.; Stucco, by T. L. Donaldson, Professor of Architecture in University College, London; the Theological articles, by Archdeacon Hale; Writing, and other articles, by the Rev. R. Garnet; and to a number of others, which cannot here be enumerated, but for

* These will sometimes be found to supersede other articles on similar subjects. Thus, Balæna includes an account of Whale Fisheries, &c.

which the able and distinguished writers will receive due credit in the volumes to which their labours belong.

MEDICAL VOLUME.

Every portion of the Encyclopædia has now been considered except the Physiological and Medical Volume.

The Zoology combines General Physiology with Comparative Anatomy, and is the work of J. F. South, Esq., Surgeon of St. Thomas's Hospital, (assisted in one portion of Physiology, by F. Le Gros Clark, Esq., and T. Solly, Esq., both of St. Thomas's Hospital). For this treatise one merit, and that not of any ordinary kind, may be claimed. It is usual, in works of this kind, to give the best information derived from the best authorities. But Mr. South, whose acquaintance with these authorities is most extensive, on comparing the descriptions in books of the very highest character with the specimens themselves (particularly those of Osteology), preserved in the Museum of the College of Surgeons, found that he could never entirely rely upon them, and accordingly determined to describe, in every instance in which it was practicable, from the specimens themselves. Of the labour thus entailed upon him, and of the value which this circumstance must give to his details, it is unnecessary to say one single word.

Of the Anatomy, by Mr. South and Mr. Le Gros Clark, and the Materia Medica, by Dr. G. Johnson, it may be said that their names are sufficient pledge that these Treatises are of first-rate character.

The Treatise on Medicine, by Dr. Robert Williams, of St. Thomas's Hospital, is an attempt to give a more philosophical view of the classification of disease than has hitherto been taken in any works of modern date. The work of Dr. Williams on Morbid Poisons, and his essays read before the College of Physicians, have obtained him the highest reputation among the members of his own profession. No person can read his treatise without a deep interest; and the Editor is willing to believe that it will add to the fame of its author, and invest him with the credit of having triumphed over obstacles hitherto thought an insuperable bar to any philosophical arrangement of disease.

To W. Bowman, Esq., the Encyclopædia is indebted for an able outline of Surgical Practice. His qualifications for treating that subject are amply testified by his long experience as Demonstrator at King's College, London, and by his publication on Physiology in conjunction with Dr. Todd. This volume, the contents of which will, it is hoped, prove interesting to all classes of readers, is closed by a comprehensive Treatise on Veterinary Art, by W. C. Spooner, Esq.

Before concluding this Preface, there are two subjects to which some allusion is required,—the Plates which accompany the work, and the general Index.

The Plates are for the most part the work of those two eminent engravers, Messrs. Lowry. They speak for themselves, and require only a simple inspection to prove their

beauty and excellence, and the ample justice which the engraver has done to the subject before him.

With regard to the Index, it is proper to observe that it was begun at an early period in the publication of the Encyclopædia, when it was intrusted to the Rev. J. Hindle, who, after completing his references to the portion then published, added those which were required for the succeeding Parts, as each appeared. The consequence is, that the Index, instead of being a hasty work got up under the disadvantage of an overwhelming mass of references to arrange in a short time, occupied the attention of a very competent person for several years. It is hoped that if it does not fulfil the promise of giving a reference to the English name of every scientific subject, it will be found to contain amply sufficient to facilitate a reference to all that is most important and interesting.

The foregoing enumeration of the principal parts of the Encyclopædia embodies all the observations which the Editor considers it necessary to make in recommending the work to the patronage of the public. The exertions made by the Proprietors to procure the just fulfilment of the high expectations formed of the work, and of the promises they had made, as well as the perseverance with which they have conducted this important publication to its completion, amidst the many obstacles which must necessarily arise in so extensive an undertaking, entitle them to high consideration from that portion of the Public which is interested in works of a sterling and substantial character. From the present position of Literature, and the system now in fashion of publishing small and superficial works which may be cheaply produced, and are really of no intrinsic value, it is probable that a long period must elapse before any similar undertaking will be entered upon, from the enormous outlay of capital it requires, and the uncertainty of remuneration which it offers. It is hoped, therefore, that this great national work, for such it really is, may meet with that patronage which the Proprietors feel confident it fairly and fully deserves. They feel assured that, whether it be viewed as a whole or in its separate divisions, it embodies a mass of information at once extensive, accurate, and scientifically arranged, which must place it in a pre-eminent and triumphant position. Whatever its measure of success may be in a pecuniary point of view, they may justly feel a high gratification in having been instrumental, under Providence, in bringing to a successful termination a work which, whether its literary merit or the soundness of its moral and religious views be regarded, must ever be considered as an inestimable benefit to their country and a permanent ornament to its literature.

H. J. ROSE.

GENERAL INTRODUCTION;

OR,

A PRELIMINARY TREATISE ON METHOD.

" Non simpliciter nil sciri posse ; sed nil nisi certo ordine certâ viâ sciri posse." BACON.

SECTION I.

ON THE PHILOSOPHICAL PRINCIPLES OF METHOD.

Introduction. Section I. Nature of the work.

THE word ENCYCLOPÆDIA is too familiar to modern literature to require, in this place, any detailed explanation. It is current amongst us as the title of various Dictionaries of Science, whose professed object is to furnish a compendium of human knowledge, whatever may be their plan. But to *methodize* such a compendium has either never been attempted, or the attempt has failed, from the total disregard of those general connecting principles, on which Method essentially depends. In presenting, therefore, to the Public an entirely new work, intended to be methodically arranged, we are not insensible to the difficulties of our undertaking; but we trust that we have found a clue to the labyrinth in those considerations which we are now about to submit to the reader.

As METHOD is thus avowed to be the principal aim and distinguishing feature of our publication, it becomes us, at the commencement, clearly to explain what we mean in this Introduction by that word ; to exhibit the principles on which alone a correct philosophical Method can be founded ; to illustrate those principles by their application to distinct studies and to the history of the human mind ; and lastly to apply them to the general concatenation of the several arts and sciences, and to the most perspicuous,

Introduction. Section I.

elegant, and useful manner of developing each particular study. Such are the objects of this Essay, which we conceive must form a necessary Introduction to a work, that is designated in its title from the place whence it originates,—the ENCYCLOPÆDIA METROPOLITANA; but claims from its mode of execution to be also called "*a* METHODICAL *Compendium of Human Knowledge*."

The word Method.

The word METHOD (μέθοδος), being of Grecian origin, first formed and applied by that acute, ingenious, and accurate people, to the purposes of scientific arrangement; it is in the Greek language that we must seek for its primary and fundamental signification. Now, in Greek, it literally means *a way, or path, of transit*. Hence the first idea of Method is *a progressive transition* from one step in any course to another; and where the word Method is applied with reference to many such transitions in continuity, it necessarily implies a principle of UNITY WITH PROGRESSION. But that which unites, and makes many things *one* in the mind of man, must be an act of the mind itself, a manifestation of intellect, and not a spontaneous and uncertain production of circumstances. This act of the mind, then, this leading thought, this "key note" of the harmony, this "subtile, cementing, subterraneous" power, borrowing a phrase from the nomenclature of legislation, we may not inaptly call the INITIATIVE of all Method. It is manifest, that the wider the sphere of transition is, the more comprehensive and commanding must be the initiative: and if we would discover an *universal Method* by which every step in our progress through the whole circle of art and science should be directed, it is absolutely necessary that we should seek it in the very interior and central essence of the human intellect.

The Science of Method.

To this point we are led by mere reflection on the meaning of the word Method. We discover that it cannot, otherwise than by abuse, be applied to a dead and arbitrary arrangement, containing in itself no principle of progression. We discover, that there is a *Science of Method;* and that *that* science, like all others, must necessarily have its *principles;* which it therefore becomes our duty to consider, in so far at least as they may be necessary to the arrangement of a Methodical Encyclopædia.

Its objects, and relations.

All things, in us, and about us, are a chaos, without Method: and so long as the mind is entirely passive, so long as there is an habitual submission of the understanding to mere events and images, as such, without any attempt to classify and arrange them, so long the chaos must continue. There may be transition, but there can never be progress; there may be sensation, but there cannot be thought: for the total absence of Method renders thinking impracticable; as we find that partial defects of Method proportionably render thinking a trouble and a fatigue. But as soon as the mind becomes accustomed to contemplate, not *things* only, but likewise *relations* of things,

Introduction. there is immediate need of some path or way of transit from one to the other of the Section I. things related;—there must be some law of agreement or of contrast between them; there must be some mode of comparison; in short, there must be Method. We may, therefore, assert that the *relations of things* form the prime objects, or so to speak, the *materials of Method:* and that the contemplation of those relations is the indispensable condition of thinking methodically.

Of these relations of things, we distinguish two principal kinds. One of them is the relation by which we understand that a thing *must be:* the other, that by which we merely perceive that it *is*. The one, we call the relation of LAW, using that word in its highest and original sense, namely, that of *laying down* a rule to which the subjects of the law must necessarily conform. The other, we call the relation of THEORY.

Relation of Law. The relation of LAW is in its absolute perfection conceivable only of GOD, that Supreme Light, and Living Law, "in whom we live and move, and have our being;" who is *εν παντι*, and *προ των παντων*. But yet the human mind is capable of viewing some relations of things as necessarily existent; that is to say, as predetermined by a truth in the mind itself, pregnant with the consequence of other truths in an indefinite progression. Of such truths, some continue always to exist in and for the mind alone, forming the *pure sciences*, moral or intellectual; whilst others, though originating in the mind, constitute what are commonly called the great laws of nature, and form the groundwork of the *mixed sciences*, such as those of Mechanics and Astronomy.

Relation of Theory. The second relation is that of THEORY, in which the existing forms and qualities of objects, discovered by observation, suggest a given arrangement of them to the mind, not merely for the purposes of more easy remembrance and communication; but for those of understanding, and sometimes of controlling them. The studies to which this class of relations is subservient, are more properly called *scientific arts* than sciences. Medicine, Chemistry, and Physiology, are examples of a Method founded on this second sort of relation, which, as well as the former, always supposes the necessary connection of cause and effect.

Fine Arts. The relations of law and theory have each their Methods. Between these two, lies the Method of the FINE ARTS, a Method in which certain great truths, composing, what are usually called the *laws of taste*, necessarily predominate; but in which there are also other laws, dependent on the external objects of sight and sound, which these arts embrace. To prove the comparative value and dignity of the first relation, it will be sufficient to observe that what is called "tinkling" verse is disagreeable to the accomplished critic in poetry, and that a fine musical taste is soon dissatisfied with the

Introduction. Section I.

Harmonica, or any similar instrument of glass or steel, because the *body* of the sound (as the Italians phrase it), or that effect which is derived from the materials, encroaches too far on the effect derived from the proportions of the notes, which proportions are in fact laws of the mind, analogous to the laws of Arithmetic and Geometry.

Principle of union.

We have stated, that Method implies both an *uniting* and a *progressive* power. Now the relations of things are not united in human conception at random—*humano capiti—cervicem equinam;* but there is some rule, some mode of union, more or less strictly necessary. Where it is absolutely necessary, we have called it a relation of law; and as by law we mean the laying down the rule, so the rule laid down we call, in the ancient and proper sense of the word, an *Idea:* and consequently the words Idea and Law, are correlative terms, differing only as object and subject, as being and truth. It is extremely necessary to advert to this use of the word Idea; since, in modern philosophy, almost any and every exercise of any and every mental faculty, has been abusively called by this name, to the utter confusion and *unmethodising* of the whole science of the human mind, and indeed of all other knowledge whatsoever.

Ideas.

Definite or instinctive.

The idea may exist in a clear, distinct, definite form, as that of a circle in the mind of an accurate geometrician; or it may be a mere *instinct,* a vague appetency toward something which the mind incessantly hunts for but cannot find, like a name which has escaped our recollection, or the impulse which fills the young poet's eye with tears, he knows not why. In the infancy of the human mind, all our ideas are instincts; and language is happily contrived to lead us from the vague to the distinct, from the imperfect to the full and finished form: the boy knows that his hoop is round, and this, in after years, helps to teach him, that in a circle, all the lines drawn from the centre to the circumference, are equal. It will be seen, in the sequel, that this distinction between the instinctive approach toward an idea, and the idea itself, is of high importance in methodising art and science.

Principle of progression.

From the first, or initiative idea, as from a seed, successive ideas germinate. Thus, from the idea of a triangle, necessarily follows that of equality between the sum of its three angles, and two right angles. This is the *principle* of an indefinite, not to say infinite, *progression;* but this progression, which is truly Method, requires not only the proper choice of an initiative, but also the following it out through all its ramifications. It requires, in short, a constant wakefulness of mind; so that if we wander but in a single instance from our path, we cannot reach the goal, but by retracing our steps to the point of divergency, and thence beginning our progress anew. Thus, a ship beating off and on an unknown coast, often takes, in nautical phrase, "a new departure;" and thus it is necessary often to recur to that regulating process, which the French

Introduction. language so happily expresses by the word *s'orienter*, i. e. to find out the east for ourselves, and so to put to rights our faulty reckoning. Section I.

The habit of Method, should always be present and effective; but in order to render it so, a certain training, or education of the mind, is indispensably necessary. Events and images, the lively and spirit-stirring machinery of the external world, are like light, and air, and moisture, to the seed of the mind, which would else rot and perish. In all processes of mental evolution the objects of the senses must stimulate the mind; and the mind must in turn assimilate and digest the food which it thus receives from without. Method, therefore, must result from the due mean, or balance, between our passive impressions and the mind's re-action on them. So in the healthful state of the human body, waking and sleep, rest and labour, reciprocally succeed each other, and mutually contribute to liveliness, and activity, and strength. There are certain stores proper, and as it were, indigenous to the mind, such as the ideas of number and figure, and the logical forms and combinations of conception or thought. The mind that is rich and exuberant in this intellectual wealth, is apt, like a miser, to dwell upon the vain contemplation of its riches, is disposed to generalize and methodize to excess, ever philosophising, and never descending to action;—spreading its wings high in the air above some beloved spot, but never flying far and wide over earth and sea, to seek food, or to enjoy the endless beauties of nature; the fresh morning, and the warm noon, and the dewy eve. On the other hand, still less is to be expected, toward the methodising of science, from the man who flutters about in blindness, like the bat; or is carried hither and thither, like the turtle sleeping on the wave, and fancying, because he moves, that he is in progress. State of mind adapted to.

The paths in which we may pursue a methodical course are manifold: at the head of each stands its peculiar and guiding idea; and those ideas are as regularly subordinate in dignity, as the paths to which they point are various and eccentric in direction. The world has suffered much, in modern times, from a subversion of the natural and necessary order of science; from elevating the terrestrial, as it has been called, above the celestial; and from summoning reason and faith to the bar of that limited physical experience, to which by the true laws of Method, they owe no obedience. The subordination, of which we here speak, is not that which depends on immediate practical utility: for the utility of human powers, in their practical application, depends on the circumstances of the moment; and at one time strength is essential to our very existence, at another time skill: and even Cæsar in a fever could cry— Proper direction of.

——Give me some drink Titinius,
As a sick girl.——

In truth there is scarcely any one of the powers or faculties with which the Divine Goodness has endowed his creatures, which may not in its turn be a source of paramount benefit and usefulness; for every thing around us is full of blessings: nor is there any line of honest occupation in which we would dare to affirm, that by a proper exercise of the talent committed to his charge, an individual might not justly advance himself to highest praise. But we now allude to the subordination which necessarily arises among the different branches of knowledge, according to the difference of those ideas by which they are initiated and directed; for there is a gradation of ideas, as of ranks in a well ordered state, or of commands in a well regulated army; and thus above all partial forms, there is one universal form of GOOD and FAIR, the καλοκαγαθον of the Platonic philosophy. Hence the expressions of Lord Bacon, who in his great work the Novum Organum, speaks so much and so often of the *lumen siccum*, the pure light, which from a central focus, as it were, diffuses its rays all around, and forms a lucid sphere of knowledge and of truth.

Gradation of ideas.

Metaphysical and physical.

We distinguish ideas into those of essential property, and those of natural existence; in other words, into metaphysical and physical ideas. Metaphysical ideas, or those which relate to the essence of things as possible, are of the highest class. Thus, in accurate language, we say, the *essence* of a circle, not its nature; because, in the conception of forms purely geometrical, there is no expression or implication of their actual existence: and our reasoning upon them is totally independent of the fact, whether any such forms ever existed in nature, or not. Physical ideas are those which we mean to express, when we speak of the *nature* of a thing actually existing and cognizable by our faculties, whether the thing be material or immaterial, bodily or mental. Thus, the laws of memory, the laws of vision, the laws of vegetation, the laws of crystallization, are all physical ideas, dependent for their accuracy, on the more or less careful observation of things actually existing.

Nature.

In speaking of the word Nature, however, we must distinguish its two principal uses, viz. first, that to which we have adverted, and according to which it signifies whatever is requisite to the reality of a thing as existent, such as the nature of an animal or a tree, distinguished from the animal or tree itself: and secondly, the sum total of things, as far as they are objects of our senses. In the first of these two meanings, the word Nature conveys a physical *idea*, in the other only a material or sensible *impression*.

Mere arrangement.

Even natural substances, it is true, may be classed and arranged for various purposes, in a certain order. Such *mere* arrangement, however, is not properly methodical, but rather a preparation toward Method; as the compilation of a dictionary is a preparation for classical study.

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The limits of our present Essay will not allow us to do more than briefly to touch the chief topics of a general dissertation on Method; but enough we trust has here been said, to render intelligible the principles on which our Methodical Encyclopædia must be constructed. We have shewn that a Method, which is at all comprehensive, must be founded on the *relations of things:* that those relations are of two sorts, according as they present themselves to the human mind as *necessary*, or merely as the result of *observation*. The former we have called relations of law, the latter of theory. Where the former alone are in question, the Method is one of necessary connection throughout; where the latter alone, though the connection be considered as one of cause and effect, yet the necessity is less obvious, and the connection itself less close. We have observed, that in the Fine Arts there is a sort of middle Method, inasmuch as the first and higher relations are necessary, the lower only the results of observation. The great principles of all Method we have shown to be two, viz. *Union* and *Progression*. The relations of things cannot be united by accident: they are united by an *idea* either definite or instinctive. Their union, in proportion as it is clear, is also progressive. The state of mind adapted to such progress holds a due mean between a passiveness under external impression, and an excessive activity of mere reflection; and the progress itself follows the path of the idea from which it sets out; requiring, however, a constant wakefulness of mind, to keep it within the due limits of its course. Hence the orbits of thought, so to speak, must differ among themselves as the initiative ideas differ; and of these latter, the great distinctions are into *physical* and *metaphysical*. Such, briefly, are the views by which we have been guided, in our present attempt to methodize the great mass of hnman knowledge.

SECTION II.

ILLUSTRATION OF THE PRECEDING PRINCIPLES.

Introduction. Section II. THE principles which have been exhibited in the preceding section, and in respect to which we claim no other merit, than that of having drawn them from the purest sources of philosophy, ancient and modern, are, we trust, sufficiently plain and intelligible in themselves; but as the most satisfactory mode of proving their accuracy, we proceed to illustrate them by a consideration of some particular studies, pursuits, and opinions; and by a reference to the general history of the human mind.

And first, as to the general importance of Method;—what need have we to dilate on this fertile topic? For it is not solely in the formation of the human understanding, and in the constructions of science and literature, that the employment of Method is indispensably necessary; but its importance is equally felt, and equally acknowledged, in the whole business and economy of active and domestic life. From the cottager's hearth or the workshop of the artisan, to the palace or the arsenal, the first merit, that which admits neither substitute nor equivalent, is, that *every thing is in its place*. Where this charm is wanting, every other merit either loses its name, or becomes an additional ground of accusation and regret. Of one, by whom it is eminently possessed, we say proverbially, that he is like clock-work. The resemblance extends beyond the point of regularity, and yet falls short of the truth. Both do, indeed, at once divide and announce the silent and otherwise indistinguishable lapse of time: but the man of methodical industry and honourable pursuits, does more; he realizes its ideal divisions, and gives a character and individuality to its moments. If the idle are described as killing time, he may be justly said to call it into life and moral being, while he makes it the distinct object not only of the consciousness, but of the conscience. He organizes the hours, and gives them a soul: and to that, the very essence of which is to fleet, and *to have been*, he communicates an imperishable and a spiritual nature. Of the good and faithful servant, whose energies, thus directed, are thus methodized, it is less truly affirmed, that he lives in time, than that time lives in him. His days, months, and years, as the stops and punctual marks in the records of duties performed, will survive the wreck of worlds, and remain extant when time itself shall be no more.

Domestic economy.

Conversation. Let us carry our views a step higher. What is it that first strikes us, and strikes

Introduction. us at once in a man of education, and which, among educated men, so instantly distinguishes the man of superior mind? Not always the weight or novelty of his remarks, nor always the interest of the facts which he communicates; for the subject of conversation may chance to be trivial, and its duration to be short. Still less can any just admiration arise from any peculiarity in his words and phrases; for every man of practical good sense will follow, as far as the matters under consideration will permit him, that golden rule of Cæsar's—*Insolens verbum, tanquam scopulum, evitare.* The true cause of the impression made on us is, that his mind is *methodical.* We perceive this, in the unpremeditated and evidently habitual arrangement of his words, flowing spontaneously and necessarily from the clearness of the leading idea; from which distinctness of mental vision, when men are fully accustomed to it, they obtain a habit of foreseeing at the beginning of every sentence how it is to end, and how all its parts may be brought out in the best and most orderly succession. However irregular and desultory the conversation may happen to be, there is *Method* in the fragments. Section II.

Let us once more take an example which must come "home to every man's business and bosom." Is there not a *Method* in the discharge of all our relative duties? And is not he the truly virtuous and truly happy man, who seizing first and laying hold most firmly of the great first Truth, is guided by that divine light through all the meandring and stormy courses of his existence? To him every relation of life affords a prolific *idea* of duty; by pursuing which into all its practical consequences, he becomes a good servant or a good master, a good subject or a good sovereign, a good son or a good father; a good friend, a good patriot, a good Christian, a good man! Moral conduct.

It cannot be deemed foreign from the purposes of our disquisition, if we are anxious, before we leave this part of the subject, to attract the attention of our readers to the importance of speculative meditation (which never will be fruitful unless it be methodical) even to the *worldly* interests of mankind. We can recall no incident of human history that impresses the imagination more deeply than the moment, when Columbus, on an unknown ocean, first perceived that startling fact, the change of the magnetic needle! How many such instances occur in history, where the *ideas* of nature (presented to chosen minds by a Higher Power than nature herself) suddenly unfold, as it were, in prophetic succession, systematic views destined to produce the most important revolutions in the state of man! The clear spirit of Columbus was doubtless eminently *methodical.* He saw distinctly that great leading idea, which authorised the poor pilot to become "a promiser of kingdoms:" and he pursued the progressive developement of the mighty truth with an unyielding firmness, which taught him "to rejoice in lofty labours." Our readers will perhaps excuse us for Scientific discoveries.

Introduction. quoting, as illustrative of what we have here observed, some lines from an Ode of Section II. Chiabrera's, which in strength of thought and in lofty majesty of poetry, has but "few peers in ancient or in modern song."

COLUMBUS.

Certo, dal cor, ch' alto Destin non scelse,
Son l'imprese magnanime neglette;
Ma le bell' alme alle bell' opre elette;
Sannu gioir nelle fatiche eccelse:
Ne biasmo popolar, frale catena,
Spirto d'onore il suo cammin raffrena.
Cosi lunga stagion per modi indegni
Europa disprezzó l'inclita speme:
Schernendo il volgo (e seco i Regi insieme)
Nudo nocchier promettitor di Regni;
Ma per le sconosciute onde marine
L'invitta prora ei pur sospinse al fine.
Qual uom, che torni al gentil consorte,
Tal ei da sua magion spiegó l'antenne;
L' ocean corse, e i turbini sostenne
Vinse le crude imagini di morte;
Poscia, dell 'ampio mar spenta la guerra,
Scorse la dianzi favolosa Terra.
Allor dal cavo Pin scende veloce
E di grand' Orma il nuovo mondo imprime;
Nè men ratto per l' Aria erge sublime,
Segno del Ciel, insuperabil Croce;
E porse umile esempio, onde adorarla
Debba sua Gente.

CHIABRERA, vol. I.

Mathematics and physics. We do not mean to rest our argument on the general utility or importance of Method. Every science and every art attests the value of the particular principles on which we have above insisted. In mathematics they will, doubtless, be readily admitted; and certainly there are many marked differences between mathematical and physical studies: but in both a previous act and conception of the mind, or what we have called an *initiative*, is indispensably necessary, even to the mere semblance of Method. In mathematics, the definition *makes* the object, and pre-establishes the terms, which alone can occur in the after reasoning. If an existing circle, or what is supposed to be such, be found not to have the radii from the centre to the circumference perfectly equal: it will in no manner affect the mathematician's reasoning on the

Introduction. properties of circles; it will only prove that the figure in question is not a circle according to the previous definition. A mathematical idea, therefore, may be perfect. But the place of a perfect idea cannot be exactly supplied, in the sciences of experiment and observation, by any theory built on generalization. For what shall determine the mind to one point rather than another; within what limits, and from what number of individuals, shall the generalization be made? The theory must still require a prior theory for its own legitimate construction. The physical definition follows and does not precede the reasoning. It is representative, not constitutive, and is indeed little more than an abbreviature of the preceding observation, and the deductions therefrom. But as the observation, though aided by experiment, is necessarily limited and imperfect, the definition must be equally so. The history of theories, and the frequency of their subversion by the discovery of a single new fact, supply the best illustrations of this truth. Section II.

But in experimental philosophy, it may be said, how much do we not owe to accident? Doubtless: but let it not be forgotten, that if the discoveries so made stop there; if they do not excite some master IDEA; if they do not lead to some LAW (in whatever dress of theory or hypothesis the fashions and prejudices of the time may disguise or disfigure it); the discoveries may remain for ages limited in their uses, insecure and unproductive. How many centuries, we might have said millennia, have passed, since the first accidental discovery of the attraction and repulsion of light bodies by rubbed amber, &c. Compare the interval with the progress made within less than a century, after the discovery of the phænomena that led immediately to a theory of ELECTRICITY. That here, as in many other instances, the theory was supported by insecure hypotheses; that by one theorist two heterogeneous fluids were assumed, the vitreous and the resinous; by another, a plus and minus of the same fluid; that a third considered it a mere modification of light; while a fourth composed the electrical aura of oxygen, hydrogen, and caloric: all this does but place the truth we have been insisting on in a stronger and clearer light. For, abstract from all these suppositions, or rather imaginations, that which is common to, and involved in them all; and there will remain neither notional fluid or fluids, nor chemical compounds, nor elementary matter,—but the idea of *two—opposite—forces*, tending to rest by equilibrium. These are the sole factors of the calculus, alike in all the theories: these give the *law* and with it the *Method* of arranging the phænomena. For this reason it may not be rash to anticipate the nearest approaches to a correct system of electricity from those philosophers who since the year 1798 have presented the idea most distinctly as such, rejecting the hypothesis of any material substratum, and contemplating in all electrical Electricity.

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phænomena the operation of a law which reigns through all nature, viz: the law of *polarity*, or the manifestation of one power by opposite forces.

Magnetism.

How great the contrast between electricity and MAGNETISM! From the remotest antiquity, the attraction of iron by the magnet was known, and noticed; but century after century it remained the undisturbed property of poets and orators. The fact of the magnet, and the fable of the Phœnix, stood on the same scale of utility, and by the generality of mankind, the latter was as much credited as the former, and considered far more interesting. In the thirteenth century, however, or perhaps earlier, the *polarity* of the magnet, and its communicability to iron, were discovered. We remain in doubt whether this discovery were accidental, or the result of theory; if the former, the purpose which it soon suggested was so grand and important, that it may well be deemed the proudest trophy ever yet raised by accident in the service of mankind. But still it furnished no genuine *idea;* it led to no *law*, and consequently, to no *Method;* though a variety of phænomena, as startling as they are at present mysterious, have forced on us a presentiment of its intimate connection with other great agencies of nature. We would not be understood to assume the power of predicting to what extent, or in what directions, that connection may hereafter be traced; but amidst the most ingenious hypotheses, that have yet been formed on the subject, we may notice that which, combining the three primary laws of magnetism, electricity, and galvinism,* considers them all as the results of one common power, essential to all material construction in the works of nature. It is perhaps more an operation of the fancy than of the reason, which has suggested that these three material powers are analogous to the three dimensions of space. Hypothesis, be it observed, can never form the ground-work of a true scientific method, unless where the hypothesis is either a true *idea* proposed in an hypothetical form, or at least the symbol of an idea as yet unknown, of a law as yet undiscovered; and in this latter case the hypothesis merely performs the function of an unknown quantity in algebra, and is assumed for the purpose of submitting the phæuomena to a scientific calculus. But to recur to the contrast presented by electricity and magnetism, in the rapid progress of the former, and the stationary condition of the latter: What is the cause of this diversity? Fewer theories, fewer hypotheses have not been advanced in the one than in the other; but the theories and fictions of the electricians contained an *idea*, and all the same idea, which has necessarily led to METHOD; implicit indeed, and only regulative hitherto, but which requires little more than the dismission

* See the experiments of Coulomb, Brugmans, and Goethe. To which may be added, should they be confirmed, the curious observations on Chrystallization, first made in Corsica, and since pursued in France.

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of the imagery to become constitutive, like the ideas of the geometrician. On the contrary, the assumptions of the magnetists (as for instance, the hypothesis that the earth itself is one vast magnet, or that an immense magnet is concealed within it; or that there is a concentric globe within the earth, revolving on its own independent axis) are but repetitions of the same fact or phænomenon, looked at through a magnifying glass; the *reiteration* of the problem, not its solution. This leads to the important consideration, so often dwelt upon, so forcibly urged, so powerfully amplified and explained by our great countryman Bacon, that one fact is often worth a thousand. "*Satis scimus*," says he, "*axiomata rectè inventa, tota agmina operum secum trahere.*" Hence his indignant reprobation of the *vis experimentalis, cæca, stupida, vaga, prærupta!*" Hence his just and earnest exhortations to pursue the *experimenta lucifera*, and those alone; discarding for their sakes, even the *fructifera experimenta*. The natural philosopher, who cannot, or will not see, that it is the "enlightening" fact, which really causes all the others to *be* facts, in any scientific sense—he who has not the head to comprehend, and the soul to reverence this parent experiment—he to whom the εὕρηκα is not an exclamation of joy and rapture, a rich reward for years of toil and patient suffering—to him no auspicious answer will ever be granted by the oracle of nature.

We have said that improgressive arrangement is not Method: and in proof of this we appeal to the notorious fact, that ZOOLOGY, soon after the commencement of the latter half of the last century, was falling abroad, weighed down and crushed as it were by the inordinate number and multiplicity of facts and phænomena apparently separate, without evincing the least promise of systematizing itself by any inward combination of its parts. JOHN HUNTER, who had appeared, at times, almost a stranger to the grand conception, which yet never ceased to work in him, as his genius and governing spirit, rose at length in the horizon of physiology and comparative anatomy. In his printed works, the finest elements of system seem evermore to flit before him, twice or thrice only to have been seized, and after a momentary detention, to have been again suffered to escape. At length, in the astonishing preparations for his museum, he constructed it, for the scientific apprehension, out of the unspoken alphabet of nature. Yet notwithstanding the imperfection in the annunciation of the idea, how exhilarating have been the results! It may, we believe, be affirmed, with safety, that whatever is grandest, in the views of CUVIER, is either a reflection of this light, or a continuation of its rays, well and wisely directed, through fit media, to its appropriate object. Zoology.

From Zoology, or the laws of animal life, to BOTANY, or those of vegetable life, the transition is easy and natural. In this pursuit, how striking is the necessity of a clear Botany.

 idea, as initiative of all Method! How obvious the importance of attention to the conduct of the mind in the exercise of Method itself! The lowest attempt at botanical arrangement consists in an artificial classification of plants, for the preparatory purpose of a nomenclature; but even in this, some *antecedent* must have been contributed by the mind itself; some *purpose* must be in view; or some question at least must have been proposed to nature, grounded, as all questions are, upon *some* idea of the answer. As for instance, the assumption,

"That two great sexes animate the world."

For no man can confidently conceive a fact to be universally true who does not proportionally anticipate its necessity, and who does not believe that necessity to be demonstrable by an insight into its nature, whenever and wherever such insight can be obtained. We acknowledge, we reverence, the obligations of Botany to LINNÆUS, who adopting from Bartholinus and others the sexuality of plants, grounded thereon a scheme of classific and distinctive marks, by which one man's experience may be communicated to others, and the objects safely reasoned on while absent, and recognized as soon as and wherever they occur. He invented an universal character for the language of Botany, chargeable with no greater imperfections than are to be found in the alphabets of every particular language. The first requisites in investigating the works of nature, as in studying the classics, are a proper accidence and dictionary; and for both of these Botany is indebted to the illustrious Swede. But the inherent necessity, the true *idea* of sex, was never fully contemplated by Linnæus, much less that of vegetation itself. Wanting these master-lights, he was not only unable to discern the collateral relations of the vegetable to the mineral and animal worlds, but even in respect to the doctrine which gives name and character to his system, he only avoided Scylla to fall upon Charybdis: and such must be the case of every one, who in this uncertain state of the initiative idea, ventures to expatiate among the subordinate notions. If we adhere to the general notion of sex, as abstracted from the more obvious modes in which the sexual relation manifests itself, we soon meet with whole classes of plants to which it is found inapplicable. If, arbitrarily, we give it indefinite extension, it is dissipated into the barren truism, that all specific products suppose specific *means* of production. Thus a growth and a birth are distinguished by the mere verbal definition, that the latter is a whole in itself, the former not: and when we would apply even this to nature, we are baffled by objects (the flower polypus, &c. &c.) in which each is the other. All that can be done by the most patient and active industry, by the widest and most continuous researches; all that the amplest survey

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of the vegetable realm, brought under immediate contemplation by the most stupendous collections of species and varieties, can suggest; all that minutest dissection and exactest chemical analysis, can unfold; all that varied experiment and the position of plants and their component parts in every conceivable relation to light, heat, and whatever else we distinguish as imponderable substances; to earth, air, water; to the supposed constituents of air and water, separate and in all proportions—in short all that chemical agents and re-agents can disclose or adduce;—all these have been brought, as conscripts, into the field, with the completest accoutrement, in the best discipline, under the ablest commauders. Yet after all that was effected by Liunæus himself, not to mention the labours of Cæsalpinus, Ray, Gesner, Tournefort, and the other heroes who preceded the general adoption of the sexual system, as the basis of artificial arrangement—after all the successive toils and enterprizes of HEDWIG, JUSSIEU, MIRBEL, SMITH, KNIGHT, ELLIS, &c. &c.—what is BOTANY at this present hour? Little more than an enormous nomenclature; a huge catalogue, *bien arrangè*, yearly and monthly augmented, in various editions, each with its own scheme of technical memory and its own conveniencies of reference! The innocent amusement, the healthful occupation, the ornamental accomplishment of *amateurs;* it has yet to expect the devotion and energies of the philosopher. Whether the *idea* which has glanced across some minds, that the harmony between the vegetable and animal world is not a harmony of resemblance, but of contrast, may not lead to a new and more accurate method in this engaging science, it becomes us uot here to determine: but should its objective truth be hereafter demonstrated by induction of facts in an unbroken series of correspondences in nature, we shall then receive it as a *law* of organic existence; and shall thence obtain another splendid proof, that with the knowledge of Law alone dwell power and prophecy, decisive experiment, and scientific Method.

Such, too, is the case with the substances of the LABORATORY, which are assumed to be incapable of decomposition. They are mere exponents of some one law, which the chemical philosopher, whatever may be his theory, is incessantly labouring to discover. The law, indeed, has not yet assumed the form of an idea in his mind; it is what we have called an Instinct; it is a pursuit after unity of principle, through a diversity of forms. Thus as "the lunatic, the lover, and the poet," suggest each other to Shakspeare's Theseus, as soon as his thoughts present him the ONE FORM, of which they are but varieties; so water and flame, the diamond, the charcoal, and the mantling champagne, with its ebullient sparkles, are convoked and fraternized by the theory of the chemist. This is, in truth, the first charm of chemistry, and the secret of the almost universal interest excited by its discoveries. The serious com- Chemistry.

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placency which is afforded by the sense of truth, utility, permanence, and progression, blends with and ennobles the exhilarating surprise and the pleasurable sting of curiosity, which accompany the propounding and the solving of an enigma. It is the sense of a principle of connection given by the mind, and sanctioned by the correspondency of nature. Hence the strong hold which in all ages chemistry has had on the imagination. If in the greatest poets we find nature idealized through the creative power of a profound yet observant meditation, so through the meditative observation of a DAVY, a WOLLASTON, a HATCHETT, or a MURRAY,

> ——— "By some connatural force,
> Powerful at greatest distance to unite
> With secret amity things of like kind,"

we find poetry, as it were, substantiated and realized.

Poetry.

This consideration leads us from the paths of physical science into a region apparently very different. Those who tread the enchanted ground of POETRY, oftentimes do not even suspect that there is such a thing as *Method* to guide their steps. Yet even here we undertake to show that it not only has a necessary existence, but the strictest philosophical application; and that it is founded on the very philosophy which has furnished us with the principles already laid down. It may surprise some of our readers, especially those who have been brought up in schools of foreign taste, to find that we rest our proof of these assertions on one single evidence, and that that evidence is SHAKSPEARE, whose mind they have probably been taught to consider as eminently *immethodical*. In the first place, Shakspeare was not only endowed with great native genius (which indeed he is commonly allowed to have been), but what is less frequently conceded, he had much acquired knowledge. "His information," says Professor WILDE, "was great and extensive, and his reading as great as his knowledge of languages could reach. Considering the bar which his education and circumstances placed in his way, he had done as much to acquire knowledge as even Milton. A thousand instances might be given, of the intimate knowledge that Shakspeare had of facts. I shall mention only one. I do not say, he gives a good account of the Salic law, though a much worse has been given by many antiquaries. But he who reads the archbishop of Canterbury's speech in Henry the Fifth, and who shall afterwards say, that Shakspeare was not a man of great reading and information, and who loved the thing itself, is a person whose opinion I would not ask or trust upon any matter of investigation." Then, was all this reading, all this information, all this knowledge of our great dramatist, a mere *rudis indigestaque moles?* Very far from it. Method,

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we have seen, demands a knowledge of the relations which things bear to each other, or to the observer, or to the state and apprehension of the hearers. In all and each of these was Shakspeare so deeply versed, that in the personages of a play, he seems "to mould his mind as some incorporeal material alternately into all their various forms." * In every one of his various characters we still feel ourselves communing with the same human nature. Every where we find individuality: no where mere portrait. The excellence of his productions consists in a happy union of the universal with the particular. But the universal is an *idea*. Shakspeare, therefore, studied mankind in the *idea* of the human race; and he followed out that idea into all its varieties, by a *Method* which never failed to guide his steps aright. Let us appeal to him, to illustrate by example, the difference between a sterile and an exuberant mind, in respect to what we have ventured to call the Science of Method. On the one hand observe Mrs. Quickley's relation of the circumstances of Sir John Falstaff's debt:

"FALSTAFF. What is the gross sum that I owe thee?

MRS. QUICKLEY. Marry, if thou wert an honest man, thyself and the money too. Thou didst swear to me upon a parcel-gilt goblet, sitting in my dolphin chamber, at the round table, by a sea-coal fire, on Wednesday in Whitsun week, when the prince broke thy head for likening his father to a singing man in Windsor—thou didst swear to me then, as I was washing thy wound, to marry me and make me my lady thy wife. Canst thou deny it? Did not goodwife Keech, the butcher's wife, come in then and call me gossip Quickley?—coming in to borrow a mess of vinegar: telling us she had a good dish of prawns—whereby thou didst desire to eat some—whereby I told thee they were ill for a green wound," &c. &c. &c.

(Henry IV, P. I. Act II. Scene I.)

On the other hand consider the narration given by Hamlet to Horatio, of the occurrences during his proposed transportation to England, and the events that interrupted his voyage. *(Act V. Scene II.)*

HAM. Sir, in my heart there was a kind of fighting
That would not let me sleep: methought I lay
Worse than the mutines in the bilboes. Rashly,
And prais'd be rashness for it——*Let us know,*
Our indiscretion sometimes serves us well,
When our deep plots do fail: and that should teach us
There's a divinity that shapes our ends,
Rough-hew them how we will

HOR. That is most certain.

* ὁ τὴν ἑαυτου ψυχην ὥσει ὕλην τινα ἀσώματον μορφαῖς ποικίλαις μορφώσας.

THEMISTIUS.

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HAM. Up from my cabin,
My sea-gown scarf'd about me, in the dark
Grop'd I to find out them; had my desire;
Finger'd their pocket; and, in fine, withdrew
To my own room again: making so bold,
My fears forgetting manners, to unseal
Their grand commission; where I found, Horatio,
A royal knavery—an exact command,
Larded with many several sorts of reasons,
Importing Denmark's health, and England's too,
With, ho! such bugs and goblins in *my* life,
That on the supervize, no leisure bated,
No, not to stay the grinding of the axe,
My head should be struck off!

HOR. Is't possible?

HAM. Here's the commission.—Read it at more leisure.
I sat me down;
Devis'd a new commission; wrote it fair.
I once did hold it, as our statists do,
A baseness to write fair, and labour'd much
How to forget that learning; but, sir, now
It did me yeoman's service. Wilt thou know
The effect of what I wrote?

HOR. Aye, good my lord.

HAM. An earnest conjuration from the king,
As England was his faithful tributary;
As love between them, like the palm, might flourish;
As peace should still her wheaten garland wear,
And many such like As's of great charge—
That on the view and knowing of these contents
He should the bearers put to sudden death,
No shriving time allowed.

HOR. How was this sealed?

HAM. Why, even in that was heaven ordinant.
I had my father's signet in my purse,
Which was the model of that Danish seal:
Folded the writ up in the form of the other;
Subscribed it; gave't the impression; plac'd it safely,
The changeling never known. Now, the next day
Was our sea-fight; and what to this was sequent,
Thou knowest already.

Introduction. If, overlooking the different value of the matter in these two narrations, we Section II. consider only the form, it must be confessed, that both are *immethodical.* We have asserted that Method results from a balance between the passive impression received from outward things, and the internal activity of the mind in reflecting and generalizing; but neither Hamlet nor the Hostess hold this balance accurately. In Mrs. Quickley, the memory alone is called into action, the objects and events recur in the narration in the same order, and with the same accompaniments, however accidental or impertinent, as they had first occurred to the narrator. The necessity of taking breath, the efforts of recollection, and the abrupt rectification of its failures, produce all her pauses; and constitute most of her connections. But when we look to the Prince of Denmark's recital the case is widely different. Here the events, with the circumstances of time and place, are all stated with equal compression and rapidity; not one introduced which could have been omitted without injury to the intelligibility of the whole process. If any tendency is discoverable, as far as the mere facts are in question, it is to omission: and accordingly, the reader will observe, that the attention of the narrator is called back to one material circumstance, which he was hurrying by, by a direct question from the friend (HOW WAS THIS SEALED?) to whom the story is communicated. But by a trait which is indeed peculiarly characteristic of Hamlet's mind, ever disposed to generalize, and meditative to excess, all the digressions and enlargements consist of reflections, truths, and principles of general and permanent interest, either directly expressed or disguised in playful satire.

Instances of the want of generalization are of no rare occurrence: and the narration of Shakspeare's Hostess differs from those of the ignorant and unthinking in ordinary life, only by its superior humour, the poet's own gift and infusion, not by its want of Method, which is not greater than we often meet with in that class of minds of which she is the dramatic representative. Nor will the excess of generalization and reflection have escaped our observation in real life, though the great poet has more conveniently supplied the illustrations. In attending too exclusively to the relations which the past or passing events and objects bear to general truth, and the moods of his own mind, the most intelligent man is sometimes in danger of overlooking that other relation, in which they are likewise to be placed to the apprehension and sympathies of his hearers. His discourse appears like soliloquy intermixed with dialogue. But the uneducated and unreflecting talker overlooks *all* mental relations, and consequently precludes all Method, that is not purely accidental. Hence,—the nearer the things and incidents in time and place, the more

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distant, disjointed and impertinent to each other, and to any common purpose, will they appear in his narration: and this from the absence of any leading thought in the narrator's own mind. On the contrary, where the habit of Method is present and effective, things the most remote and diverse in time, place, and outward circumstance, are brought into mental contiguity and succession, the more striking as the less expected. But while we would impress the necessity of this habit, the illustrations adduced give proof that in undue preponderance, and when the prerogative of the mind is stretched into despotism, the discourse may degenerate into the wayward, or the fantastical.

Shakspeare needed not to read Horace in order to give his characters that methodical *unity* which the wise Roman so strongly recommends:

> Si quid inexpertum scenæ committis, et audes
> Personam formare novam; servetur ad imum
> Qualis ab incœpto processerit, et sibi constet.

But this was not the only way in which he followed an accurate philosophic Method: we quote the expressions of SCHLEGEL, a foreign critic of great and deserved reputation—"If Shakspeare deserves our admiration for his characters, he is equally deserving of it for his exhibition of *passion*, taking this word in its widest signification, as including every mental condition, every tone from indifference or familiar mirth, to the wildest rage and despair. He gives us the history of minds: *he lays open to us, in a single word, a whole series of preceding conditions.*" This last is a profound and exquisite remark: and it necessarily implies, that Shakspeare contemplated *ideas*, in which alone are involved conditions and consequences *ad infinitum*. Purblind critics, whose mental vision could not reach far enough to comprize the whole dimensions of our poetical Hercules, have busied themselves in measuring and spanning him muscle by muscle, till they fancied they had discovered some disproportion. There are two answers applicable to most of such remarks. First, that Shakspeare understood the true language and external workings of passion better than his critics. He had a higher, and a more ideal, and consequently a more methodical sense of harmony than they. A very slight knowledge of music will enable any one to detect discords in the exquisite harmonies of HAYDN or MOZART; and Bentley has found more false grammar in the PARADISE LOST than ever poor boy was whipped for through all the forms of Eton or Westminster: but to know why the minor note is introduced into the major key, or the nominative case left to seek for its verb, requires an acquaintance with some preliminary steps of the methodical

scale, at the top of which sits the author, and at the bottom the critic. The second answer is, that Shakspeare was pursuing two Methods at once; and besides the psychological* Method, he had also to attend to the poetical. Now the poetical method requires above all things a preponderance of pleasurable feeling: and where the interest of the events and characters and passions is too strong to be continuous without becoming painful, there poetical method requires that there should be, what Schlegel calls "a musical alleviation of our sympathy." The Lydian mode must temper the Dorian. This we call Method.

We said that Shakspeare pursued two methods. Oh! he pursued many, many more—"both oar and sail"—and the guidance of the helm, and the heaving of the lead, and the watchful observation of the stars, and the thunder of his grand artillery. What shall we say of his moral conceptions? Not made up of miserable clap-traps, and the tag-ends of mawkish novels, and endless sermonizing;—but furnishing lessons of profound meditation to frail and fallible human nature. He shows us crime and want of principle clothed not with a spurious greatness of soul; but with a force of intellect which too often imposes but the more easily on the weak, misjudging multitude. He shows us the innocent mind of Othello plunged by its own unsuspecting and therefore unwatchful confidence, in guilt and misery not to be endured. Look at Lear, look at Richard, look in short at every moral picture of this mighty moralist! Whoso does not rise from their attentive perusal "a sadder and a wiser man"—let him never dream that he knows any thing of philosophical Method.

Nay, even in his style, how methodical is our "sweet Shakspeare." Sweetness is indeed its predominant characteristic; and it has a few immethodical luxuriances of wit; and he may occasionally be convicted of words, which convey a volume of thought, when the business of the scene did not absolutely require such deep meditation. But pardoning him these *dulcia vitia*, who ever fashioned the English language, or any language, ancient, or modern, into such variety of appropriate apparel, from "the gorgeous pall of scepter'd tragedy," to the easy dress of flowing pastoral.

More musical than lark to shepherd's ear,
When wheat is green and hawthorn buds appear.

Who, like him, could so methodically suit the very flow and tone of discourse to characters lying so wide apart in rank, and habits, and peculiarities, as Holofernes and

* We beg pardon for the use of this *insolens verbum*; but it is one of which our language stands in great need. We have no single term to express the philosophy of the human mind: and what is worse, the principles of that philosophy are commonly called *metaphysical*, a word of very different meaning.

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Queen Catharine, Falstaff and Lear? When we compare the pure English style of Shakspeare with that of the very best writers of his day, we stand astonished at the *Method*, by which he was directed in the choice of those words and idioms, which are as fresh now as in their first bloom; nay, which are at the present moment at once more energetic, more expressive, more natural, and more elegant, than those of the happiest and most admired living speakers or writers.

But Shakspeare was "not methodical in the structure of his fable." Oh gentle critic! be advised. Do not trust too much to your professional dexterity in the use of the scalping knife and tomahawk. Weapons of diviner mould are wielded by your adversary: and you are meeting him here on his own peculiar ground, the ground of *idea*, of thought, and of inspiration. The very point of this dispute is ideal. The question is one of *unity*: and unity, as we have shown, is wholly the subject of ideal law. There are said to be three great unities which Shakspeare has violated; those of time, place, and action. Now the unities of time and place we will not dispute about. Be ours the poet,

> ——— qui pectus inaniter angit
> Irritat, mulcet, falsis terroribus implet
> Ut magus, et *modo me Thebis, modo ponit Athenis.*

The dramatist who circumscribes himself within that unity of time, which is regulated by a stop-watch, may be exact, but is not methodical; or his method is of the least and lowest class. But

> Where is he living clipt in with the sea,
> That chides the banks of England, Wales, or Scotland?

who can transpose the scenes of Macbeth, and make the seated heart knock at the ribs with the same force as now it does, when the mysterious tale is conducted from the open heath, on which the weird sisters are ushered in with thunder and lightning, to the fated fight of Dunsinane, in which their victim expiates with life, his credulity and his ambition? To the disgrace of the English stage, such attempts have indeed been made on almost all the dramas of Shakspeare. Scarcely a season passes which does not produce some ὕστερον πρότερον of this kind in which the mangled limbs of our great poet are thrown together "in most admired disorder." There was once, a noble author, who by a refined species of murder, cut up the play of Julius Cæsar into two good set tragedies. M. Voltaire, we believe, had the grace to make but one of it; but whether his Brutus be an improvement on the model from which it was taken, we trust, after what we have already said, we shall hardly be expected to discuss.

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Thus we have seen, that Shakspeare's mind, rich in stores of acquired knowledge, commanded all these stores and rendered them disposable, by means of his intimate acquaintance with the great laws of thought, which form and regulate Method. We have seen him exemplifying the opposite faults of Method in two different characters; we have seen that he was himself methodical in the delineation of character, in the display of passion, in the conceptions of moral being, in the adaptations of language, in the connection and admirable intertexture of his ever-interesting fable. Let it not, after this, be said, that Poetry—and under the word Poetry we will now take leave to include all the works of the higher imagination, whether operating by measured sound, or by the harmonies of form and colour, or by words, the more immediate and universal representatives of thought—is not strictly methodical; nay, does not owe its whole charm, and all its beauty, and all its power, to the philosophical principles of Method.

But what of philosophy herself? Shall she be exempted from the laws, which she has imposed on all the rest of the known universe? *Longè absit!* To philosophy properly belongs the EDUCATION of the mind: and all that we have hitherto said may be regarded as an indication (we have room for no more) of the chief laws and regulative principles of that education. Philosophy, the "parent of life," according to the expression of the wise Roman orator; the "mother of good deeds and of good sayings," the "medicine of the mind," is herself wholly conversant with Method. Philosophy.

True it is, that the ancients, as well as the moderns, had their machinery for the extemporaneous coinage of intellect, by means of which the scholar was enabled *to make a figure* on any and all subjects, on any and all occasions. They too had their glittering vapours, which (as the comic poet tells us) fed a host of sophists—

> Θεγάλαι εαι ἀνδράσιν ἀργοῖς
> Αἵπερ γνωμην κ̧ διάλεξιν κ̧ νοῦν ἡμῖν παρέχουσιν,
> Καὶ τερατειαν κ̧ περίλεξιν κ̧ κροῦσιν κ̧ κατάληψιν.
> ΑΡΙΣΤΟΦ. Νεφ. Σκ. δ.

> Great goddesses are they to lazy folks,
> Who pour down on us gifts of fluent speech,
> Sense most sententious, wonderful fine *effect*,
> And how to talk about it and about it,
> Thoughts brisk as bees, and pathos soft and thawing.

But the philosophers held a course very different from that of the sophists. We shall not trouble our readers with a comparative view of many systems; but we shall present to their admiration one mighty ancient, and one illustrious modern, PLATO, and BACON. These two varieties will sufficiently exemplify the species.

Introduction. Plato. Section II.

Of Plato's works, the larger and more valuable portion have all one common end, which comprehends and shines through the particular purpose of each several dialogue; and this is, to establish the sources, to evolve the principles, and to exemplify the art of Method. This is the clue, without which it would be difficult to exculpate the noblest productions of the "divine" philosopher from the charge of being tortuous and labyrinthine in their progress, and unsatisfactory in their ostensible results. The latter indeed appear not seldom to have been drawn, for the purpose of starting a new problem, rather than of solving the one proposed as the subject of previous discussion. But with the clear insight, that the purpose of the writer is not so much to establish any particular truth, as to remove the obstacles, the continuance of which is preclusive of all truth, the whole scheme assumes a different aspect, and justifies itself in all its dimensions. We see, that the Education of the intellect, by awakening the *method* of self-developement, was his proposed object, not any specific information that can be *conveyed into it* from without. He desired not to assist in storing the passive mind with the various sorts of knowledge most in request, as if the human soul were a mere repository, or banqueting room, but to place it in such relations of circumstance as should gradually excite its vegetating and germinating powers to produce new fruits of thought, new conceptions, and imaginations, and ideas. Plato was a poetic philosopher, as Shakspeare was a philosophic poet. In the poetry, as well as in the philosophy, of both, there was a necessary predominance of ideas; but this did not make them regardless of the actual existences around them. They were not visionaries, or mystics; but dwelt in "the sober certainty" of waking knowledge. It is strange, yet characteristic of the spirit that was at work during the latter half of the last century, that the writings of Plato should be accused of estranging the mind from plain experience and substantial *matter-of-fact*, and of debauching it by fictions and generalities. Plato, whose method is inductive throughout, who argues on all subjects not only *from*, but *in* and *by*, inductions of facts! Who warns us indeed against the usurpation of the senses, but far oftener, and with more unmitigated hostility, pursues the assumptions, abstractions, generalities, and verbal legerdemain of the sophists. Strange! but still more strange, that a notion, so groundless, should be entitled to plead in its behalf the authority of Lord Bacon, whose scheme of logic, as applied to the contemplation of nature, is Platonic throughout. It is necessary that we should explain this circumstance at some length, in order to establish by the concurrence of authorities, vulgarly supposed to be contradictory, the truth of a system which we have already maintained on so many other grounds.

Bacon.

What Lord Bacon was to England, Cicero was to Rome—the first and most

Introduction. eloquent advocate of philosophy. It is needless to remind the classical scholar of that Section II almost religious veneration with which the accomplished Roman speaks of Plato' whom, indeed, he calls, in one instance, "*deus ille noster*," and in other places, "the Homer of philosophers;" their "prince;" the "most weighty of all who ever spoke, or ever wrote;" "most wise, most holy, divine." This last appellation, too, it is well known, long remained, even among Christians, as a distinguishing epithet of the great ornament of the Socratic school. Why Bacon should have spoken detractingly of such a man; why he should have stigmatised him with the name of "sophist," and described his philosophy (with the tyrant Dionysius), as "*verba otiosorum senum ad imperitos juvenes*," it is much easier to explain, than to justify, or even to palliate. He was, perhaps, influenced, in part, by the tone given to thinking minds by the Reformation, the founders and fathers of which saw in the Aristotelians, or schoolmen, the antagonists of Protestantism, and in the Italian Platonists (as they conceived) the secret enemies of Christianity itself. In part, too, Bacon may have formed his notions of Plato's doctrines from the absurdities of his mis-interpreters, rather than from an unprejudiced and diligent study of his works.—Be it remembered, however, that this unfairness was not less manifested to his contemporaries; that his treatment of GILBERT was cold, invidious, and unjust; and that he seems to have disdained to learn either the existence or the name of Shakspeare. At this conduct no one can be surprised, who has studied the life of this

——— wisest, brightest, meanest of mankind.

But our present business is not with his weaknesses, or his failings, but with those philosophical principles, which, especially as displayed in the Novum Organum, have deservedly obtained for him the veneration of succeeding ages.

Those who talk superficially about Bacon's philosophy, that is to say, nineteen-twentieths of those who talk about it at all, know little more than his induction, and the application which he makes of his own method, to particular classes of physical facts; applications, which are at least as crude, for the age of Gilbert, Galileo, and Kepler, as were those of Aristotle (whom he so superciliously reprehends) for the age of Philip and Alexander. Or they may perhaps have been struck with his recommendation of tabular collections of particulars; and hence have placed him at the head of a body of men, but too numerous in modern days—the minute philosophers. We need scarcely say, that this is venturing his reputation on a very tottering basis. Let any unprejudiced naturalist turn to Bacon's questions and proposals for the investigation of single problems; to his "Discourse on the Winds;" or to what may almost be called a caricature of his scheme, in the "Method of improving Natural Philosophy,"

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by ROBERT HOOKE* (the history of whose philosophical life is alone a sufficient answer to all such schemes)—and then let him fairly say whether any desirable end could reasonably be hoped for, from this process—whether by this mode of research any important discovery ever was made, or ever could be made? Bacon, indeed, always takes care to tell us, that the sole purpose and object of collecting together these particulars, is to concentrate them, by careful selection, into universals: but so immense is their number, and so various and almost endless the relations in which each is to be separately considered, that the life of an ante-diluvian patriarch would be expended, and his strength and spirits wasted, long before he could commence the process of simplification, or arrive in sight of the law, which was to reward the toils of the over-tasked PSYCHE.†

Had Bacon done no more, than propose these impracticable projects, we should have been far from sharing the sentiments of respect every where attached to his philosophical character. But he has performed a task of infinitely greater importance, by constructing that methodical system, which is so elegantly developed in the Novum Organum. It is this, which we propose to compare with the principles long before

* We refer particularly to pp. 22 to 42 of the above-mentioned work; and we would, above all, notice the following admirable specimen of confused and disorderly minuteness :—" The history of potters, tobacco-pipe-makers, glaziers, glass-grinders, looking-glass-makers or foilers, spectacle-makers and optic-glass makers, makers of counterfeit pearl and precious stones, bugle-makers, lamp-blowers, colour-makers, colour-grinders, glass-painters, enamellers, varnishers, colour-sellers, *painters, limners, picture-drawers, makers of baby heads, of little bowling stones or marbles*, fustian-makers, (query whether *poets* are included in this trade?) music-masters, tinsey-makers, and taggers.—The history of schoolmasters, writing-masters, printers, book-binders, stage-players, dancing-masters, and vaulters, *apothecaries, chirurgeons, seamsters, butchers, barbers, laundresses, and cosmetics!* &c. &c. &c. &c. (the true nature of each of which being exactly determined) WILL HUGELY FACILITATE OUR INQUIRIES IN PHILOSOPHY"!!!

In parallel, or rather in contrast, with the advice of Mr. Robert Hooke, may be fairly placed that of the celebrated Dr. WATTS, which was thought, by Dr. Knox, to be worthy of insertion in the *Elegant* Extracts, vol. ii. p. 456, under the head of

DIRECTIONS CONCERNING OUR IDEAS.

" Furnish yourselves with *a rich variety of Ideas.* Acquaint yourselves with *things* ancient and modern; *things* natural, civil, and religious; *things* of your native land, and of foreign countries; *things* domestic and national; *things* present, past, and future; and above all, be well acquainted with God and yourselves; with animal nature, and the workings of your own spirits. *Such a general acquaintance with things will be of very great advantage.*"

† See the beautiful allegoric tale of Cupid and Psyche in the original of Apuleius. The tasks imposed on the hapless nymph, through the jealousy of her mother-in-law, and the agency by which they are at length self-performed, are noble instances of that hidden wisdom " where more is meant than meets the ear!"

Introduction. enunciated by Plato. In both cases, the inductions are frequently as crude and erroneous, as might readily be anticipated from the infant state of natural history, chemistry, and physiology, in their several ages. In both cases, the proposed applications are often impracticable; but setting aside these considerations, and extracting from each writer that which constitutes his true philosophy, we shall be convinced that it is identical, in regard to the science of Method, and to the grounds and conditions of that science. We do not see, therefore, how we can more appropriately conclude this section of our inquiry, than by a brief statement of our renowned countryman's own principles of Method, conveyed, for the greater part, in his own words: or in what more precise form, we can recapitulate the substance of the doctrines asserted and vindicated in the preceding pages. For we rest our strongest pretensions to approbation on the fact, that we have only re-proclaimed the coinciding precepts of the Athenian Verulam, and the British Plato. Section II.

Their common system. In the first instance, Lord Bacon equally with ourselves, demands, as the motive and guide of every philosophical experiment, what we have ventured to call the intellectual or mental *initiative;* namely, some well-grounded purpose, some distinct impression of the probable results, some self-consistent anticipation, the ground of the "*prudens quæstio*" (the forethoughtful enquiry), which he affirms to be the prior *half* of the knowledge sought, *dimidium scientiæ.* With him, therefore, as with us, an *idea* is an experiment proposed, an experiment is an idea realized. For so he himselfin forms us:—"neque scientiam molimur tam sensu, vel instrumentis, quam *experimentis;* etenim experimentorum longe major est subtilitas, quam sensûs ipsius, licet instrumentis exquisitis adjuti. *Nam de iis loquimur experimentis, quæ, ad intentionem ejus quod quæritur, perité, et secundum artem excogitata et apposita sunt.* Itaque perceptioni sensûs immediatæ et propriæ *non multûm tribuimus:* sed eò rem deducimus, ut *sensus tantûm de experimento, experimentum de re judicet.*" The meaning of this last sentence is intelligible enough; though involved in antithesis, merely because Bacon did not possess, like Shakspeare, a good method in his style. What he means to say is, that we can apprehend, through the organs of sense, only the sensible phænomena produced by the experiment; but by the mental power, in virtue of which we shaped the experiment, we can determine the true *import* of the phænomena.

Now, he had before said, that he was speaking only of those experiments, which were skilfully adapted to the intention, or purpose of him, who conducted the research. But what is it, that forms the intention, or purpose, and adapts thereto the experiment? What Bacon calls *lux intellectûs;* viz. the understanding of the individual man, who makes the experiment. This light, however, as he argues at great length, is obscured

 by *idols*, which are false and spurious notions. His peculiar use of the word idols, is again a proof of faulty method in his style; for it gives a sort of pedantic air to his reasonings; but in truth, he means no more by it, than what Plato means by *opinion*, (δοξα) which the latter calls " a medium between knowledge and ignorance." So, Bacon distinguishes the idols of the mind into various kinds *(idola specûs, tribûs, fori, theatri)*, that is, opinions derived from the passions, prejudices, and peculiar habits of each man's understanding: and as these idols, or opinions, confessedly produce a sort of mental obscurity, or blindness; so, the ancient and the modern master of philosophy both agree in prescribing remedies and operations calculated to remove this disease; to couch the " mind's eye;" and to restore it to the enjoyment of a purer vision. Bacon establishes an unerring criterion between the ideas and the idols of the mind; namely, that the latter are empty notions, but the former are the very seals and impresses of nature; that is to say, they always fit and cohere with those classes of things, to which they belong; as the idea of a circle fits and coheres with all true circles. His words are these: " Non leve quiddam interest inter humanæ mentis *idola*, et divinæ mentis *ideas*, hoc est, inter placita quædam inania, et veras siguaturas atque impressiones factas in creaturis, prout Ratione sanâ ct sicci luminis, quam, docendi causâ, interpretem naturæ vocare consuevimus, inveniuntur." Novum Organum, XXIII. & XXVI.

Some idols, says Bacon, are adventitious to the mind; others innate. And here, we may observe, that he gocs somewhat farther than the mere doctrine of innate ideas, by holding that of innate idols. However, we say not this in disparagement of his *system*, which is clear and correct; nor, on the other hand, do we mean to espouse all its *parts*, which must be left to speak for themselves. What he means by innate idols, he thus illustrates:—not only do the rays of truth, from without, fall obliquely on the mirror of the mind, but that mirror itself is not pure and plain; it discolours, it magnifies, it diminishes, it distorts. Hence, he uses the words *intellectus humanus*, *mens hominis*, &c. in a sense now peculiar, but in his day conformable to the language of the schools, to signify not intellect in general, or mind in its perfection, but the intellect or mind of man, weakened and corrupted, as it is, more or less, in every individual. A necessary consequence of this corruption, is the arrogance, which leads man to take the forms and mechanism of his own reflective faculty, as the measure of nature, and of the Deity. Of all idols, or of all opinions, this is the most difficult to remedy, or extirpate; and therefore, in this view, the intellect of man is more prone to error, than even his senses. Such is the sound and incontrovertible doctrine of Bacon; but herein he does no more, than repeat what both Plato and Heraclitus had long before urged, with most impressive argument. The forms of the reflective faculty are *subjective*; the

truths to be embraced are *objective:* but according to Plato, as well as to Bacon, there can be no hope of any fruitful and secure *Method*, so long as forms, merely subjective, are arbitrarily assumed to be the moulds of objective truth, the seals and impresses of nature.

What then! Does Bacon abandon the hope of rectifying the obliquities of the human intellect; or does he suggest, that they will be remedied by the casual operation of external impressions? Neither of these. He considers, that its weaknesses and imperfections require to be strengthened and made perfect by a higher power; and that this is possible to be done. He supposes, that the intellect of the individual, or *homme particuliere*, may be refined by the intellect of the ideal man, or *homme generale*. He assumes, that as the evidence of the senses is corrected by the judgment, so the evidence of the judgment, beset with idols, may be corrected by the judgment, walking in the light of ideas. It is surely superfluous to urge, that this corrector and purifier of all reasoning, this inextinguishable pole star—

Which never in the ocean waves was wet;

whether it be called, as by Bacon, *lumen siccum*, or as by Plato, νȣς, or φως νοερον, is one and the same light of *Truth*, the indispensable condition of all pure science, contemplative, or experimental. Hence, it will not surprise us, that Plato so often denominates ideas living *laws*, in and by which the mind has its whole true being and permanence; or that Bacon, vicè versâ, names the laws of nature, *ideas;* and represents the great leading facts of science as signatures, impressions, and symbols of those ideas. A distinguishable power self-affirmed, and seen in its unity with the Eternal Essence, is, according to Plato, an IDEA: and the discipline by which the human mind is purified from its idols, and raised to the contemplation of Ideas, and thence to the secure and progressive, investigation of truth and reality, by scientific method, comprehends what the same philosopher so highly extols, under the title of *Dialectic*. According to Lord Bacon, as describing the same truth, applied to natural philosophy, an idea would be defined as—Intuitio, sive inventio, quæ in perceptione sensûs non est (ut quæ puræ et sicci luminis Intellectioni sit propria) idearum divinæ mentis, prout in creaturis, per signaturas suas, sese patefaciant. "That (saith the judicious HOOKER) which doth assign to each thing the kind, that which determineth the force and power, that which doth appoint the form and measure of working, the same we term a LAW."

From all that has been said, it seems clear, that the only difference between Plato and Bacon was, that, to speak in popular language, the one more especially cultivated natural philosophy, the other metaphysics. Plato treated principally of truth, as manifested in the world of intellect; Bacon of the same truth, as manifested in the world

Introduction. of sense; but far from disagreeing, as to the mode of attaining that truth, far from differing in their great views of the *education of the mind*, they both proceeded on the same principles of *unity* and *progression;* and consequently both cultivated alike the *Science of Method*, such as we have here described it. If we are correct in these statements, then may we boast to have solved the great problem of conciliating ancient and modern philosophy. Section II.

Historical view. That the *Method*, of which we have hitherto treated, is not arbitrarily assumed in any, or all of the pursuits, to which we have adverted; nor is peculiar to these in particular, but is founded in the laws and necessary conditions of human existence, is further to be inferred from a general view of the history of the human race. As in the individual, so in the whole community of mankind, our cogitations have an infancy of aimless activity; and a youth of education and advance towards order; and an opening manhood, of high hopes and expectations; and a settled, staid, and sober middle age, of ripened and deliberate judgment.

First period. "The antiquity of time was the youth of the world and of knowledge," said Bacon. In that early age, the *obedience of the will* was first taught to man. He was required to look up, in submission, to that Spirit of Truth, which, after all, we find to be at the head of wisdom. This innocent age was happily prolonged, among those, whose first care was to cultivate the moral sense, and to seek in faith the evidence of things not seen. To them were propounded a Spiritual Creator, and a spiritual worship, and the assured hope of a future and spiritual existence; and therefore they were less curious to watch the motions of the stars, or to become "artificers in brass and iron," or to "handle the harp and the organ." They were less wise in their generation, than the "mighty men of old, the men of renown;" but their ideas were plain, and distinct; they were "just and perfect men;" and they "walked with God;" whilst, of the others "every imagination of the thoughts of the heart was only evil continually." For the latter wilfully chose an opposite *method:* they determined to shape their convictions and deduce their knowledge from *without*, by exclusive observation of outward things, as the only realities. Hence they became rapidly *civilized*. They built cities, and refined on the means of sensual gratification, and the conveniencies of courtly intercourse. They became the great masters of the agreeable, which fraternized readily with cruelty and rapacity: these being, indeed, but alternate moods of the same sensual selfishness. Thus, both before and after the flood, the vicious of mankind receded from all true cultivation, as they hurried towards civilization. Finally, as it was not in their power to make themselves wholly beasts, and to remain without a semblance of religion, and yet, as they were faithful to their original maxim,—

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A new era next appeared, representative of the youth and approaching manhood of Second period. the human intellect: and again Providence, as it were, awakened men to the pursuit of an idealised Method, in the developement of their faculties. Orpheus, Linus, Musæus, and the other mythological bards, or perhaps brotherhoods of bards impersonated under individual names, whether deriving their light, imperfectly and indirectly, from the inspired writings of the Hebrews, or graciously visited, for high and important purposes, by a dawning of truth in their own breasts, began to spiritualise Polytheism, and thereby to prevent it from producing all its natural, barbarising effects. Hence the mysteries and mythological hymns; which, on the one hand, gradually shaped themselves into epic poetry and history, and, on the other, into tragedy and philosophy: whilst to the lifeless statuary of the Egyptians was superadded a Promethean animation; and the ideal in sculpture soon extending itself to painting, and to architecture, the Fine Arts at once shot up to perfection, by a Method founded wholly on a mental initiative, and conducted throughout its progress by the developement of ideas. This rapid advance, in all things which owe their existence and character to the mind's own acts, intellectual or imaginative, forms a singular contrast with the rude and imperfect manner, in which those acts were applied to the investigation of physical laws and phænomena. While Phidias, Apelles, Homer, Demosthenes, Thucydides, and Plato, had, each in his individual sphere, attained almost the summit of conceivable excellence, the natural history and the natural philosophy of the whole world may be said to have lain dormant; especially if we compare them with the efforts which the moderns made in these directions, in the very morning of their strength.

Of the Roman era it is scarcely necessary to speak at large, inasmuch as the Romans. Romans were confessedly mere imitators of the Greeks in every thing relating to science and art. They sustained a very important part in the civil, and military, and ecclesiastical history of mankind; and their devotion to these objects was, in their own eyes, a sufficient apology for their want of originality in what they held to be far inferior pursuits.

> Excudent alii spirantia mollius æra:
> Credo equidem, vivos ducent de marmore vultus:
> Tu regere imperio populos, Romane, memento.

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Still less will it be expected, that we should devote much space to the consideration of those dark ages, which brought the countless hordes of sensual barbarians from their northern forests to meet, in the southern and middle parts of Europe, the spiritualizing influence of Christianity: but one remarkable effect of that influence we cannot suffer to pass unnoticed. We allude to the gradual abolition of domestic slavery, in virtue of a principle essential to Christianity, by which a *person* is eternally differenced from a *thing;* so that the *idea* of a human being necessarily excludes the idea of property, in that being.

Reformation. We come down, then, to the great period of the REFORMATION, which, regarded as an epoch in the education of the human mind, was second to none for its striking and durable effects. The defenders of a simple and spiritual worship, against one which was full of outward forms and ceremonies; the partisans of religious liberty, against the dominion of a visible head over the whole Christian church; and generally speaking, the advocates of the ideal and internal, against the external, or imaginative; maintained a zealous, and in great part of Europe, a prosperous conflict. But the revolution of thought, and its effects on the science of Method, were soon visible beyond the pale of the church or the cloister: and the schoolmen were attacked as warmly in their philosophical, as they had before been in their ecclesiastical character. It is needless to dwell on the various attempts toward introducing into learning a totally new method. That of our illustrious countryman, BACON, was completely successful: and we have already shown, that it was, in truth, the completion of the ideal system, by applying the same method to external nature, which Plato had before applied to intellectual existence.

Modern philosophy. It is only in the union of these two branches of one and the same method, that a complete and genuine philosophy can be said to exist. To this consideration the great mind of Bacon does not seem to have been fully awake; and hence, not only is the general scope of his work directed almost exclusively to the contemplation of physical ideas; but there are occasional expressions, which seem to have misled many of his followers into a belief, that he considered all wisdom and all science, both to begin and to end with the object of the senses. In this gross error are laid the foundations of the modern French school, which has grown up into the monstrous puerilities of CONDILLAC, and CONDORCET; men whose names it would be absolutely ridiculous to mention, in a history of science, if their pupils did not unhappily compensate, in number, what their works want in common sense and intelligibility; and if upon such writers, the French nation did not mainly rest its pretensions to give the law to Europe, in matters of science and philosophy.

SECTION III.

APPLICATION OF THE PRINCIPLES OF METHOD TO THE GENERAL CONCATENATION AND DEVELOPEMENT OF STUDIES.

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WE have already dwelt so much on the general importance of Method—we have recurred to it so frequently—we have placed it in so many various lights, that we ought perhaps to apologise for venturing on one more attempt to illustrate our meaning, partly in the way of simile, and partly of example. Let us, however, imagine an unlettered African, or rude, but musing Indian, poring over an illumined manuscript of the inspired volume; with the vague, yet deep impression, that his fates and fortunes are, in some unknown manner, connected with its contents. Every tint, every group of characters, has its several dream. Say, that after long and dissatisfying toils, he begins to sort, first, the paragraphs that appear to resemble each other; then the lines, the words; nay, that he has at length discovered, that the whole is formed by the recurrence and interchange of a limited number of cyphers, letters, marks and points, which, however, in the very height and utmost perfection of his attainment, he makes twenty-fold more numerous than they are, by classing every different form of the same character, intentional or accidental, as a separate element. And yet the whole is without soul or substance, a talisman of superstition, or a mockery of science; or is employed perhaps, at last, to feather the arrows of death, or to shine and flutter amid the plumes of savage vanity. The poor Indian too truly represents the state of learned and systematic ignorance—arrangement guided by the light of no leading idea; mere orderliness without METHOD!

But see, the friendly missionary arrives! He explains to him the nature of written words, translates them for him into his native sounds, and thence into the thoughts of his heart: how many of these thoughts are then first unfolded into consciousness, which yet the awakening disciple receives not as aliens! Henceforward the book is unsealed for him; the depth is opened; he communes with the *spirit* of the volume, as with a living oracle. The words become transparent: he sees them, as though he saw them not; whilst he mentally devours the meaning they contain. From that moment, his former chimerical and useless arrangement is discarded, and the results of method are to him life and truth.

If some particular studies are yet confessedly deficient in the vivifying power of Method, we much fear that the attempts to bind together the whole body of science

have been, in certain instances, worse than immethodical. A slight glance at the particular department of literature which we have chosen, especially as it has been filled on the Continent; from the memorable combination of deistical talent in the *Dictionnaire Encyclopedique*, to a work, on the same principles, said to be now publishing in France, will demonstrate, that the best interests of mankind have suffered serious injury from this cause; that the fountains of education may be poisoned, where the stream appears to flow on with increasing power and smoothness; and that the *insinuation* of sceptical principles into works of Science, is fraught with the greatest danger to posterity.

To oppose an effectual barrier to the rage for desultory knowledge, on the one hand, and to support that body of independent attachment to the best prinples of *all* knowledge, which happily distinguishes this country, on the other, the ENCYCLOPÆDIA METROPOLITANA has been projected.

We do not undertake, what the most gigantic efforts of man could not atchieve, *an universal Dictionary of Knowledge*, in the most absolute sense of the terms. But estimating the importance of our task rather by the principles of *unity* and *compression*, than by those of variety and extent, we have laboured to build upon what is essential, that which is obviously useful, and upon both whatever is elegant or agreeable in science; and this, we conceive, cannot be well and usefully effected, but by such a philosophical Method, as we have already indicated.

We have shown that this METHOD consists in placing one or more *particular* things or notions, in subordination, either to a pre-conceived *universal* idea, or to some lower form of the latter; some class, order, genus, or species, each of which derives its intellectual significancy, and scientific worth, from being an ascending step toward the universal; from being its representative, or temporary substitute. Without this master-thought, therefore, there can be no true Method: and according as the general conception more or less clearly manifests itself throughout all the particulars, as their connective and bond of unity; according as the light of the idea is freely diffused through, and completely illumines, the aggregate mass, the Method is more or less perfect.

The first pre-conception, or master-thought, on which *our* plan rests, is the *moral origin and tendency* of all true science; in other words, our great objects are to exhibit the Arts and Sciences in their philosophical harmony; to teach Philosophy in union with Morals; and to sustain Morality by Revealed Religion.

There are, as we have before noticed, two sorts of relation, on the due observation of which all Method depends. The first is that, which the ideas or laws of

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the mind bear to each other; the second, that which they bear to the external world: on the former are built the Pure Sciences; on the latter those which we call Mixed and Applied.

Pure sciences.

The *Pure Sciences*, then, represent pure acts of the mind, and those only; whether employed in contemplating the *forms* under which things in their first elements are necessarily viewed and treated by the mind; or in contemplating the substantial *reality* of those things.

Formal and real.

Hence, in the pure sciences, arises the known distinction of *formal* and *real:* and of the first, some teach the elementary forms, which the mind necessarily adopts in the processes of reasoning; and others, those under which alone all particular objects can be grasped and considered by the mind; either as distinguishable in quantity and number, or as occupying parts of space. The *real* sciences, on the other hand, are conversant with the true nature and existence, either of the created universe around us; or of the guiding principles within us, in their various modifications and distinguishing movements; or, lastly, with the real nature and existence of the great Cause of all.

Grammar.

We begin, then, with that class of pure sciences which we have called formal; and of these, the first two that present themselves to us, are *Grammar* and *Logic.* By Grammar we are taught the rules of that speech, which serves as the medium of mental intercourse between man and man; by *Logic,* the mental operations are themselves regulated and bound together, in a certain method or order. As the communication of knowledge is the more immediate object of our present discussion, so we begin with that science by which it is regulated in its forms. Grammar, then, apart from the mere material consideration of the sound of words, or shape of letters, and regarding speech only as a thing significant, teaches that there are certain laws regulating that signification; laws which are immutable in their very nature: for the relation which a noun bore to a verb, or a substantive to an adjective, was the same in the earliest days of *μερόπων ἀνθρώπων,* in the first intelligible conversations of men, as it is now; nor can it ever vary so long as the powers of thought remain the same in the human mind. This, then, is a pure science proceeding from a simple or elementary idea of the form necessary for the conveyance of a single thought, and thence spreading and diffusing itself over all the relations of significant language.

Logic.

Grammar brings us, naturally, to the Science of *Logic,* or the knowledge of those forms which the conceptions of the mind assume in the processes of reasoning. And it is manifest, that this science is no less subject than the former, to fixed laws; for the reasoning power in man can only operate within those limits which Almighty Wisdom has thought fit to prescribe. It is a discursive faculty, moving in a given path, and by

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Mathematics. Under the general term *Mathematics*, are comprised the sciences of *Geometry*, which is conversant about the laws of figure, or limitations of space; and *Arithmetic*, which concerns the laws of number. Now these laws are purely ideal. It is not externally to us that the general notion of a square, or a triangle; of the number three, or the number five, exists; nor do we seek for external proof of the relations of those notions; but on the contrary, by contemplating them, as ideas in the mind, we discover truths which are applicable to external existence.

Metaphysics, Morals, and Theology. The sciences, which we have hitherto noticed, relate to the forms of our mental conceptions; but it is natural for man to seek to comprehend the principles and conditions of real existence, both with regard to the universe in general, with regard to his own internal mover, or conscience, and, above all, with regard to the cause, by which conscience and the whole universe were called into being, and continue to exist, namely, God. Hence, as we advance from form to reality, the sciences of *Metaphysics* and *Morals* first present themselves to view, and these lead us forward to the summit of human knowledge; for at the head of all pure science stands *Theology*, of which the great fountain is revelation. It is obvious, that both Metaphysics and Morals are conversant solely about those relations, which we have called relations of law; for it would be a contradiction to say, that a real existence could be, at the same time, a mere theory or hypothesis. These sciences have, therefore, all the purity and all the certainty, which belong to that which is positive and absolute; and as far as they are distinctly apprehended by the mind, they approach the nearest to that clear intellectual light, which, in the peculiar phraseology of Lord Bacon, is called *lumen siccum*. In the proper philosophical method, the reality of our speculative knowledge, exhibited in the science of Metayhysics, unites itself at last with the reality of our ethical sentiments displayed in that of Morals; and both together are at once lost and consummated in Theology, which rises above the light of reason to that of faith.

Mixed and Applied Sciences. These are all the sciences which embrace solely relations of law: and it is plain that in these, not only the initiative, but every subsequent step, must be an act of the mind alone. But when we descend to the second order of relations, namely those which we bear to the external world, Theory is immediately introduced; new sciences are formed, which in contradistinction from the pure, are called the *mixed* and *applied* Sciences; and in these new considerations relative to Method, necessarily find a place.

Every physical theory is in some measure imperfect, because it is of necessity progressive; and because we can never be assured that we have exhausted the terms, or that some new discovery may not affect the whole scheme of its relations. The discoveries of the ponderability of air, of its compound nature, of the increased weight of the calces, of the gasses in general, of electricity, and more recently the stupendous influences of Galvanism on the successive chemical theorics; are all so many exemplifications of this truth. The doctrines of vortices, of an universal ether, of a two-fold magnetic fluid, &c. are *theories* of gravitation: but the science of Astronomy is founded on the *law* of gravitation, and remains unaffected by the rise and fall of the theories. In the lowest condition of Method, the initiative is supplied by *an hypothesis;* of which we may distinguish two degrees. In the former, a fact of actual experience is taken, and placed experimentally as the common support of certain other facts, as equally present in all: thus, that oxygen is *a* principle of acidification and combustion, is an experienced fact; and became an hypothesis, by the assumption that it is the *sole* principle of acidification and combustion. In the latter, a fact is imagined: as, for instance, an atom or physical point, præternaturally hard, and therefore infrangible in the corpuscular philosophy; or a primitive unalterable figure, in some systems of crystalization.

In all this, we see, that knowledge is a matter not of necessary connection, but of a connection arising from observation, or supposition; that is, it consists not of law, but of theory, or hypothesis. True theory is always in the first and purest sense a *locum tenens* of law; when it is not, it degenerates into hypothesis, and hypothesis melts away into conjecture. Both in law and in theory, there must be a mental antecedent; but in the latter, it may be an image or conception received through the senses, and originating from without; yet even then there is an inspiring passion, or desire, or instinctive feeling of the truth, which is the immediate and proper offspring of the mind. Now, we may consider the facts which are to be reduced to theory, as arranged over the whole surface of a plane circle. If by carrying the power of theory to a near identity with law, we find the centre of the circle, then proceeding toward the circumference, our insight into the whole may be enlarged by new discoveries; it never can be wholly changed. A magnificent example of this has been realized in the science of Astronomy; a recent addition of facts has been effected by the discovery of other planets, and our views have been rendered more distinct by the solution of the apparent irregularities of the moon's motion, and their subsumption under the general law of gravitation. But the Newtonian was not less a system before, than since, the discovery of the Georgium Sidus; not by having ascertained its circumference, but by

Introduction. having found its centre; the living and salient point, from which the method of discovery Section III. diverges, the law in which endless discoveries are contained implicitly, and to which as they afterwards arise, they may be referred in endless succession.

These reasonings, it is hoped, will sufficiently explain the nature of the transition, from the *Pure Sciences* to the *Mixed and Applied Sciences*, and will serve to trace the inseparable connection of the latter with the constitution of the human mind. And as each of these great divisions of knowledge has its own department in the grand moral science of man, it is obvious that a scheme, which, like our own, not only contains each separately, but combines both as indivisible, the one from the other; must present, in the most advantageous point of view, whatever is useful and beautiful in either. In speaking of the mixed and applied sciences, we must be permitted, however, to remark that the word science, is evidently used in a looser and more popular form, than when we denominate mathematics, or metaphysics, a science; for we know not, for instance, the truth of any general result of observation in nosology, as we know that two and two make four, or that a human person cannot be identical with another human person. And in like manner, when the word law, is used with relation to the mixed and applied sciences; as when we speak of any supposed law of vegetation; we use a more popular language than when we speak of a law of the conscience, which is not to be prevaricated. The strictness of ancient philosophy, therefore, refused the name of science to these pursuits: and it might at least be convenient, if in speaking generally of the pure, the mixed, and the applied sciences, we gave them the common name of studies, inasmuch as we study them all alike, but we do not know them all with the same sort of knowledge.

Mixed. Of these, then (be they studies or sciences), we call those *mixed* in which certain ideas of the mind, are applied to the general properties of bodies, solid, fluid, and aerial; to the power of vision, and to the arrangement of the universe; whence we obtain the sciences of *Mechanics, Hydrostatics, Pneumatics, Optics*, and *Astronomy*. It is matter not of certain science, but of observation, that such properties do really exist in bodies, that vision is effected in such or such a manner, and that the universe is disposed in this or that relative position, and subjected to certain movements of its parts. Therefore these sciences may vary, and notoriously have varied; and though Kepler would demonstrate that Euclid *Copernicised*, or had some knowledge of the system afterwards adopted by Copernicus; yet of this there is little proof: and certainly for many ages after Euclid it was the universal opinion, that the earth was the fixed and immoveable centre of the universe. Nor have we here unadvisedly used the word *opinion;* since, as we before showed, it is the ancient expression, signifying a medium

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between knowledge and ignorance: and well did that acute Italian exclaim, *Opinione, regina del mondo!*—for as it is impossible that ignorance, which cannot govern itself, should govern any thing else; so to expect that all the world should be wise enough to submit to the government of wisdom, would be to show that we had followed very little Method in our study either of history, of living men, or even of ourselves.

Applied.

When certain ideas, or images representative of ideas, are applied still more particularly, not to the investigation of the general and permanent properties of all bodies, but of certain changes in those properties, or of properties existing in bodies partially, then we popularly call the studies relative to such matters by the name of *Applied Sciences;* such are *Magnetism, Electricity, Galvinism, Chemistry,* the laws of *Light* and *Heat,* &c. We have already so fully shown the uncertainty of the first principles in these studies, and have so distinctly traced the cause of that uncertainty, in every case, to a want of clearness in the first idea or mental initiative of the science, that it will be unnecessary here to do more than refer to our preceding observations.

Experimental philosophy.

Fine Arts.

We come now to another class of applied sciences, namely, those which are applied to the purposes of pleasure, through the medium of the imagination; and which are commonly called the Fine Arts. These are *Poetry, Painting, Music, Sculpture, Architecture.* We have before said, that the Method to be observed in these, holds a sort of middle place between the method of law, or pure science, and the Method of theory. In regard to the mixed sciences, and to the first class of applied sciences, the mental initiative may have been received from without; but it has escaped some critics, that in the fine arts the mental initiative must necessarily proceed from within. Hence we find them giving, as it were, recipes to form a poet, by placing him in certain directions and positions; as if they thought that every deer-stealer might, if he pleased, become a Shakspeare, or that Shakspeare's mind was made up of the shreds and patches of the books of his day; which by good fortune he happened to read in such an order, that they successively fitted into the scenes of Macbeth, Othello, the Tempest, As you like it, &c. Certainly the fine arts belong to the outward world, for they all operate by the images of sight and sound, and other sensible impressions; and without a delicate tact for these, no man ever was, or could be either a musician, or a poet; nor could he attain to excellence in any one of these arts: but as certainly he must always be a poor and unsuccessful cultivator of the arts, if he is not impelled first by a mighty, inward power, a feeling, *quod nequeo monstrare, et sentio tantum;* nor can he make great advances in his art, if in the course of his progress, the obscure impulse does not gradually become a bright, and clear, and living idea!

Introduction. Section III.

Useful Arts.

Pursuits of utility, we daily find, are capable of being reduced to Method. Thus *Political Economy*, and *Agriculture*, and *Commerce*, and *Manufactures*, are now considered scientifically; or as the more prevalent expression is, philosophically. It may, perhaps, be difficult, at first, to persuade the experimental agriculturist, that he also pursues, or ought to pursue, an ideal Method: nor do we mean by this that he must deal only in ideal sheep and oxen, and in the groves and meads of Fairy Land. But these studies, soberly considered, will be found wholly dependent on the sciences of which we have already treated. It is not, surely, in the country of ARKWRIGHT, that the philosophy of commerce can be thought independent of mechanics: and where DAVY has delivered lectures on agriculture, it would be folly to say that the most philosophic views of chemistry were not conducive to the making our vallies laugh with corn.

Natural history.

We have already spoken of LINNÆUS, the illustrious Swede, to whom the three *kingdoms*, as they are aptly called, of *Natural History*, are so deeply indebted: and if, with all his great talents, he yet failed in establishing the united empire of those three mighty monarchies, on firm laws, and a fixed constitution; we have shewn, that it was only owing to a want of precision in the first ideas of his theory.

Applications of.

Natural history itself becomes a rule for dependent pursuits, such as those of *Medicine* (under which are ***Pharmacy***, and the *Materia Medica)*, and *Surgery*, in which is included *Anatomy*. That in these and the other theoretical studies, so much still remains to be done, ought not to be a subject for regret; but, on the contrary, for a laudable and generous ambition. Yet that ambition should be regulated and moderated by a due consideration of the place, which the particular pursuit in question, holds in the great circle of the sciences; and by observing the only proper *Method* which can be pursued for its improvement. If, in what we have here said, we have done any thing towards the excitement, the regulation, and the assistance of that ambition; if we have faintly sketched an outline of the great laws of Method, which bind together the various branches of human knowledge, we may not improperly indulge a hope that the ensuing work, in its progress, will be found conducive to the promotion of the best interests of mankind.

History and Biography.

Our Plan would not completely meet the views of those to whom such works as the following are eminently useful and agreeable, if besides the philosophic Method already described, we did not present some view of the actual history of mankind. We have therefore devoted a large portion of our labours to the History of the Human Race, on a new, and we trust it will be found an improved system. Biography and history tend to the same points of general instruc-

Introduction. tion, in two ways: the one exhibiting human principles and passions, acting upon a Section III. large scale; the other shewing them as they move in a smaller circle, but enabling us to trace the orbit which they describe with greater precision. The one brings man into contact with society, actuated by the interests which agitate and stimulate him in the various social combinations of his existence; and human nature presents itself in the varied shapes impressed upon it by the different ranks which it occupies. The other brings before us the individual, when he stands alone, his passions asleep, his native impulses under no external excitement; in the undress of one who has retired from the stage, on which he felt he had a part to sustain; and even the monarch, forgetting the pomp and circumstance of his royalty, remembers here only that he is a man. Assuredly the great use of History is to acquaint us with the Nature of Man. This end is best answered by the most faithful portrait; but Biography is a collection of portraits. At the same time there must be some mode of grouping and connecting the individuals, who are themselves the great landmarks in the map of human nature. It has therefore occurred to us, that the most effectual mode of attaining the chief objects of historical knowledge, will be to present History in the form of Biography, chronologically arranged. This will be preceded by a general Introduction on the Uses of History, and on the line which separates its early facts from fable; and it will, in the course of its progress, be interspersed with connecting chapters on the events of large and distinguishing periods of time, as well as on political Geography and Chronology. Thus will the far larger portion of History be conveyed, not only in its most interesting, but in its most philosophical and real form; while the remaining facts will be interwoven in the preliminary and connecting chapters. If in tracing thus the " eventful history" of man, and partioularly of our own country, we should perceive, as we must necessarily do in all that is human, evils and imperfections; these will not be without their uses, in leading us back to the importance of intellectual Method as their grand and sovereign remedy. Hence shall we learn its proper national application, namely, the *education of the mind*, first in the man and citizen, and then, inclusively, in the State itself.

Alphabetic arrangement. Such are our views in the philosophical and historical branches of our work. Of the Miscellaneous or Alphabetical Division we have little to add. But well aware that works of this nature are not solely useful to those who have leisure and inclination to study science in its comprehensiveness, and unity; but are also valuable for daily reference on particular points, suggested by the desires or business of the individual; we could not hold ourselves dispensed from consulting the convenience of a numerous and most respectable class of Readers; while the preceding remarks will go to prove

that for many local and supplementary illustrations of science, no other depository could be furnished.

As the philosophical arrangement is, however, most conducive to the purposes of intellectual research and information, as it will most naturally interest men of science and literature; will present the circle of knowledge in its harmony; will give that unity of design and of elucidation, the want of which we have most deeply felt in other works of a similar kind, where the desired information is divided into innumerable fragments scattered over many volumes, like a mirror broken on the ground, presenting, instead of one, a thousand images, but none entire; this division must of necessity, have that prominence in the prosecution of our design, which our conviction of its importance to the due execution of the plan demands; and every other part of the arrangement must be considered as subordinate to this principal organization. With respect to the whole work, it should be observed, that in what concerns *references* we are guided by principle, not by caprice; nor do we ever recur to them as our only means of escape from an exigency. Throughout the ENCYCLOPÆDIA METROPOLITANA, the philosophical arrangement predominates and regulates; the alphabetical arrangement, and the references, whether to it or from it, are auxiliary. We never refer from the first and second Divisions to the fourth, or from the first to the second, for the explanation of a term, the establishment of a principle, or the demonstration of a proposition. The reference, whenever it occurs, unless it be *retrospective*, is not for the purpose of essential information, but for that which is collateral and subordinate. The theory of the *balance*, for example, is given where it ought to be, in the Treatise on Mechanics; but they who wish to acquaint themselves with the various constructions of balances for the purposes of commerce or philosophy, knowing that these cannot be introduced into a scientific treatise, without destroying the symmetry of its parts by a suspension of the logical order, will naturally turn, whether there be a reference or not, to the alphabetical department of the work. So again, the principles of the *telescope* are given in the treatise on Optics; the varieties of construction in the alphabetical department: the principles of the *thermometer*, when treating of the effects of heat; its varieties of construction in the alphabetical department. Practical detail, and niceties or peculiarities of construction, can seldom be interwoven with propriety among the regular deductions of a methodical treatise: in all cases where they cannot, our general principle, as it comprehends proportion, accuracy, utility, and convenience, demands a reference, whether expressed or not, to the appropriate place for all that is subservient; that is, to the fourth or alphabetical division.

Introduction. Section III.

This final division of our work will bring the whole into unison with the two great impulses of modern times, trade and literature. These, after the dismemberment of the Roman empire, gradually reduced the conquerors and the conquered at once into several nations and a common Christendom. The natural law of increase, and the instincts of family, may produce tribes, and under rare and peculiar circumstances, settlements and neighbourhoods: and conquest may form empires. But without trade and literature, combined, there can be no nation; without commerce and science, no bond of nations. As the one has for its object the wants of the body, real or artificial, the desires for which are for the greater part excited from without; so the other has for its origin, as well as for its object, the wants of the mind, the gratification of which is a natural and necessary condition of *its* growth and sanity. In the pursuits of commerce the man is called into action from without, in order to appropriate the outward world, as far as he can bring it within his reach, to the purposes of his corporeal nature. In his scientific and literary character he is internally excited to various studies and pursuits, the ground-work of which is in himself.

This, again, will conduct us to the distinguishing object of the present undertaking; in endeavouring to explain which we have dwelt long upon general principles; but not too long, if we have established the necessity of what we conceive to be the main characteristic of every just arrangement of knowledge.

Our method embraces the two-fold distinction of human activity to which we have adverted;—the two great directions of man and society, with their several objects and ends. Without advocating the exploded doctrine of *perfectibility,* we cannot but regard all that is human in human nature, and all that in nature is above herself, as together working forward that far deeper and more permanent revolution in the moral world, of which the recent changes in the political world may be regarded as the pioneering whirlwind and storm. But woe to that revolution which is not guided by the historic sense; by the pure and unsophisticated knowledge of the past: and to convey this methodically, so as to aid the progress of the future, has been already announced as the distinguishing claim of the Encyclopædia Metropolitana.

The principles of Method, developed in the preceding Essay, will, it is hoped, render perfectly intelligible the Plan of our whole work, which is comprehended under Four Divisions as follow:

FIRST DIVISION.

PURE SCIENCES. — 2 Vols.

- **FORMAL.**
  - Universal Grammar and Philology; or the forms of Languages.
  - Logic, particular and universal: or the forms of Conceptions and their combinations.
  - Mathematics: (Geometry, Arithmetic, Algebra, &c.) or the forms and constructions of Figure and Number.
- **REAL.**
  - Metaphysics: or the universal principles and conditions of Experience, having for its object the Reality of our speculative knowledge in general.
  - Morals: or the principles and conditions of the coincidence of the individual will with the universal reason, having for its object the Reality of our practical knowledge: (hence, in a lower stage, Politics and Human Law.)
  - Theology: or the union of both in their application to GOD, the Supreme Reality.

SECOND DIVISION.

MIXED AND APPLIED SCIENCES. — 6 Vols.

- **MIXED**
  - Mechanics.
  - Hydrostatics.
  - Pneumatics.
  - Optics.
  - Astronomy.
- **APPLIED.**
  - **I. EXPERIMENTAL PHILOSOPHY.**
    - Magnetism.
    - Electricity, including Galvinism.
    - Chemistry.
    - Light.
    - Heat.
    - Colour.
    - Meteorology.
  - **II. THE FINE ARTS.**
    - Poetry, introduced by Psycology.
    - Painting.
    - Music.
    - Sculpture.
    - Architecture.
  - **III. THE USEFUL ARTS.**
    - Agriculture, introduced by Political Economy.
    - Commerce.
    - Manufactures.
  - **IV. NATURAL HISTORY.**
    - Introduced by Physiology in its widest sense.
    - Inanimate:—Chrystallography, Geology, Mineralogy.
    - Insentient:—Phytonomy, Botany.
    - Animate:—Zoology.
  - **V. APPLICATION OF NATURAL HISTORY.**
    - Anatomy.
    - Surgery.
    - Materia Medica.
    - Pharmacy.
    - Medicine.

THIRD DIVISION.

BIOGRAPHICAL AND HISTORICAL. — 8 Vols.

Biography CHRONOLOGICALLY arranged, interspersed with introductory Chapters of National History, Political Geography and Chronology, and accompanied with correspondent Maps and Charts.

FOURTH DIVISION.

MISCELLANEOUS AND LEXICOGRAPHICAL. — 8 Vols.

Alphabetical, Miscellaneous, and Supplementary:—containing a GAZETTEER or complete Vocabulary of Geography: and a Philosophical and Etymological LEXICON of the English Language, or the History of English Words;—the citations arranged according to the Age of the Works from which they are selected, yet with every attention to the independent beauty or value of the sentences chosen which is consistent with the higher ends of a clear insight into the original and acquired meaning of every word.

The INDEX.—Being a digested and complete Body of Reference to the whole Work; in which the known English name, as well as the scientific name, of every subject of Natural History, will be found in its alphabetical place.

ENCYCLOPÆDIA METROPOLITANA;

OR, THE

UNIVERSAL DICTIONARY OF KNOWLEDGE,

ON AN ORIGINAL PLAN.

First Division.

GRAMMAR.

INTRODUCTORY SECTION.

Grammar. Signification, universal and particular.

GRAMMAR (Fr. *Grammaire*) is a word used to signify both the pure science of universal Grammar common to all languages, and the applied sciences of particular Grammar restricted each to its particular language or dialect.

It is only of Grammar, in the first of these acceptations, that we mean, at present, to treat. In a methodical view of the Pure and Applied Sciences, it is essentially necessary to begin with the former: nor can any particular Grammar be well and thoroughly understood, without some previous knowledge of universal Grammar, as its foundation.

Definition.

Grammar, then, in its most comprehensive sense, may be defined, *the science of the relations of language considered as significant*. We say " of the relations of language," because the knowledge of languages, in as far as it regards the mere acquisition and remembrance of terms, is an affair of the attention and of the memory; whereas to understand the relations which those terms bear to each other, is the business of a science especially directed toward that end. We say too " of language considered as significant;" because language has other properties besides that of signification. Words, for instance, may be made up of longer or shorter sounds, may be delivered with varieties of accentuation, and may be uttered in a softer or louder voice; but these and many other circumstances relative to language, do not properly fall under the science of Grammar, although some of them may be considered as its adjuncts, or dependencies.

Language.

Of the term " *Language*," which we have used in our definition, we must speak more at large. As the word " Grammar," though introduced into English from the French, is derived from the Greek verb γραφω " I write;" so the word " language," which comes immediately to us from the French word *langage*, originates in the Latin *lingua*, " the tongue;" and therefore anciently signified only the use of the tongue in *speech*. A just analogy, however, has extended its meaning to all intentional modes of communicating the movements of the mind: thus we use the expressions, " articulate language," " written language," " the language of gesture," &c.; and this analogy suggests some considerations, which will be found important, in developing the first principles of grammatical science.

Introductory Section.

Man is formed as well internally, as externally, for the communication of thoughts and feelings. He is urged to it by the necessity of receiving, and by the desire of imparting, whatever is useful or pleasant. His wants and wishes cannot be satisfied by individual power: his joys and sorrows cannot be limited to individual sensation. The fountains of his wisdom and of his love spontaneously flow, not only to fertilize the neighbouring soil, but to augment the distant ocean.

But the mind of man which is within him, can only by communicated by objects which are without, by gestures, sounds, characters more or less expressive, and permanent, instruments not merely useful for this particular purpose, but many times pleasing in themselves, or rendered so, by the long continued operation of habit. These, reason adopts, she combines, she arranges; and the result is a language.

Speech.

Speech, or the language of articulate sounds, is the most wonderful, the most delightful of the arts, thus taught by nature and reason. It is also the most perfect. It enables us, as it were, to express things beyond the reach of expression, the infinite range of

Grammar. being, the exquisite fineness of emotion, the intricate subtleties of thought. Of such effect are those shadows of the soul, those living sounds, which we call *words!* Compared with them, how poor are all other monuments of human power, or perseverance, or skill, or genius! They render the mere clown an artist; nations immortal; orators, poets, philosophers divine!

The dialects or systems of speech adopted by various races of men, in different ages and countries, have been, in many respects, strikingly distinguishable. We may remark the copious Arabic, the high sounding Spanish, the broad Dutch, the voluble French, the soft Italian: we may trace minute gradations from the monosyllables of the Chinese, to the long paragraph words of the Sanscrit; or we may rise, still more gradually, in the scale of expression, from the barbarous muttering of a poor Esquimaux in his solitary canoe, to the thunders of Athenian eloquence, and those delightful strains of our own Shakespeare, which are "musical as is Apollo's lute, and a perpetual feast of nectar'd sweets."

Nor is this all: a thousand collateral circumstances tend still farther to diversify the numerous spoken languages of the world. Not only does time produce gradual progress, or sudden change in their forms; but their effect is endlessly modified by combination with other arts of expression, with looks, and actions, with sights and sounds.

Method of study. In this labyrinth of interesting observations, what objects have we to pursue; what clue to guide us? Shall we be content to learn one or two dialects by rote; to burthen the memory without exercising the understanding? Or, if we would rise above this, to a knowledge of their construction, must we draw our general principles from the minute comparison of those numberless particulars, which the longest life would be too short even to contemplate, and which the united wisdom of ages has never attempted to arrange?

The very statement of these questions is a sufficient solution of them. They indicate at once the necessity of assuming some comprehensive principles as the rule and basis of our further enquiries. These first elements of our reasoning must afterwards be followed out into all their concrete forms. The history of language must verify the science; but the science must precede; for such, in the order of nature, is the course of all our knowledge. General notions, vague and indistinct, come first; they form, as it were, the channels into which our daily observations flow; and these observations again correct and strengthen our former notions, and render them sources of clear and abundant knowledge.

Lord Bacon. LORD BACON, indeed, says, that "*that* would be the most noble kind of Grammar, which would be formed, if a man profoundly skilled in many languages, vulgar as well as learned, were to treat of the various properties of each, and to show their several excellencies and defects." But it is obvious, that his lordship here speaks only of the last result of the grammarian's studies; it is previously necessary, not only to learn the words of the languages which are to be arranged and compared; but to acquire the arts of arrangement and comparison.

The first step toward a perfect arrangement is to comprehend the whole subject matter under a general *idea*; and from what we have already said, it is manifest that the idea of speech is included in the still more general idea of language, which comprehends the principles common to speech, with gesture, writing, &c. The various arts to which these principles are capable of application may be considered as branches of one great family; they are all derived from the same source always analogous to, sometimes associated or interwoven with each other; and hence, like the sister graces, they will appear to the greatest advantage together. Introductory Section.

The general idea of language, applicable to all these various modes of its exercise, is, as we have said, a communication of the thoughts and feelings of the mind. But how can we understand the communication, unless we have some idea of the thing communicated? And which shall we consider as the original and shaping power of a word, the sound, or the thought? These questions cannot bear a moment's reflection. If the word were parent to the thought, a parrot or a speaking automaton might be made to understand *gravitation*, as well as Sir Isaac Newton. And yet there are men, in the present day, calling themselves grammarians and philosophers, who have pushed absurdity so far as to assert, that the faculty of reason itself depends wholly on speech! Assuredly to know the powers and employments of the tongue conduces greatly to strengthen and facilitate the operations of the mind; but we cannot understand the former until we have made considerable progress in the knowledge of the latter.

The late Mr. HORNE TOOKE, in his well known work, "The Diversions of Purley," speaks thus:—"The business of the mind (as far as it concerns language) is very simple. It extends no farther than to receive impressions; that is, to have sensations or feelings. What are called its operations are merely the *operations of language*." Let us here ask, What can possibly be meant by "the operations of language," as distinct from those of the mind? Who is language? How does he operate? If my mind, as far as concerns language, do nothing but receive impressions, how comes it to pass that I ever open my lips? And when I speak, how happens it that I utter articulate sounds; that those sounds form words; that those words are arranged in a certain order; and that that order is absolutely essential to my being understood? How does language operate, so as to shape itself into nouns and verbs; and those the very nouns and verbs, which I happen to want; and all the while, without any privity or interference of mine, or any act whatsoever of my mind? Horne Tooke.

It is proper, however, here to observe, that in respect to the general principles here adverted to, Mr. Tooke has neither the merit, nor the demerit, of originality. He is so far a follower of Condillac and the writers of that school, of whose general opinions the following passage may afford a sufficient specimen: and we voluntarily select it from a work published in 1803, by a Member of the French National Institute, and reedited and corrected in 1804 by another Member of the same learned body, at present a peer of France. "We cannot distinguish our sensations," says the author, "but by attaching to them signs which represent and characterise them. This is what made Condillac say, that we cannot think at all without the help of language. I repeat it, without signs there exists neither thought, nor perhaps even, to speak properly, any true sensation. In order to distinguish a sensation, Condillac.

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we must compare it with a different sensation: now their relation cannot be expressed in our mind, unless by an artificial sign, since it is not a direct sensation."*

CONDILLAC, who is here quoted with so much approbation, began to write in 1749. He pretended to found his doctrine on the principles of LOCKE; and we presume it has at last received its final perfection from the hands of M. DESTUTT-TRACY, the noble editor of CABANIS.

It is hardly possible to expose the absurdity of such statements, without descending from the gravity of a serious disquisition. We shall simply analyse the extract which we have just made, applying to its principles (if principles they may be called) a few obvious exemplifications: and, if the result should appear to border too much on the ridiculous, we trust that the imputation of folly will rest with the original authors of a system so perfectly incoherent.

1. "We cannot distinguish our sensations but by attaching to them signs, which represent and characterise them." We might first ask what is a sign? Is it a sensation, or somewhat else? If a sensation, is it direct, or indirect? How do we distinguish one sign from another? What part do signs perform in our mental operations?—and many other such questions; but passing over these difficulties, we will come to our author's own reasoning; and from the principle which he here lays down, it must follow, that if a native of Scotland should see a brook (which in that country is called a *burn*), and should also feel a *burn* occasioned by touching any heated substance, he would not be able to distinguish these sensations, because he would have attached to them the same sign; neither could he distinguish them if he even attached to them different signs, e. g. *rivus* and *ustio*, unless each sign accurately represented the thing signified; so that the one sign should reproduce in him the sight of flowing water, and the other the touch of a heated body.

2. "This is what made Condillac say, that we cannot think at all without the help of language." If Condillac reasoned from such premises, it is no wonder that he came to such a conclusion.

3. "Without signs there exists neither thought, nor perhaps even, to speak properly, true sensation." Signs we have before been told, are things characterising or representing sensations. We now learn that it is on the contrary, the sensations which represent or characterise the signs. We are taught that the portrait is the original, and the man the copy, that without the portrait there would be no man. Some doubt is expressed, whether we might not receive some sort of sensation from striking our heads against a post; but it is argued that this would not be a *true* sensation, that we should not *really* feel the blow, unless we actually cried "post," or read the word "post," which would naturally explain to us the sort of blow we had experienced.

* "Ou ne distingue les sensations qu'en leur attachant des signes, qui les representent & les caractérisent."—Voilà ce qui fait dire à CONDILLAC *qu'on ne pense point sans le secours des langues*—Je le répète, sans signes il n'existe ni pensée, ni peut être meme, à proprement parler, de véritable sensation—Pour distinguer une sensation il faut la comparer avec une sensation differente: or leur rapport ne peut être exprimé *dans notre esprit* que par un signe artificiel, puisque ce n'est pas une sensation directe."—CABANIS. Rapports du Physique & du Moral de l'Homme, vol. i. p. 78.

Introductory section.

4. "In order to distinguish a sensation, we must compare it with another sensation." Here is a new rule to know whether we are alive, and in our senses, or not. If we chance to break our shins, we must not be too hasty in crediting the evidence of that part of our body; we must compare the sensation with some other, as for instance, with that of drinking a glass of champagne, and if we find that they differ, why then we may be assured that they are not the same.

5. "Now, their relation cannot be expressed in our mind, unless by an artificial sign; since it is not a direct sensation." What is meant by a sensation being expressed in the mind, it is not very easy to discover; but the author seems to intimate that a direct sensation may be so expressed, and that it therein differs from the relation between two sensations, which relation he says is not a direct sensation. We presume, that he would rank breaking his shins, or drinking champagne, in the class of direct sensations; these, therefore, may be expressed in the mind, without an artificial sign; and consequently they are not true sensations; for (by proposition 3d), without signs there exists no true sensation; neither can we think at all about them, because (by the same proposition) without signs there is no thought. It is probably meant to be understood that all sensations are direct or indirect. We have seen how the qualities of the former class are explained. Let us next consider what happens with respect to the latter. Some sort of relation probably exists between drinking champagne and breaking the shins, but that relation we are told, cannot be expressed in the mind without an artificial sign. Now as we have never heard of any word or even hieroglyphic to express the particular relation that exists between drinking champagne and breaking the shin, it follows, that no such relation can be expressed in the mind; and consequently (by proposition 4) the separate sensations of breaking the shin and of drinking champagne cannot be distinguished.

It is obvious, that if these ridiculous propositions had been stated plainly and simply, they would never have encountered serious discussion. They have, however, been enveloped in the mystical jargon of the modern ideologists; they have assumed the imposing name of metaphysics; and hence the ignorant multitude have concluded, that there is something in them of profound wisdom.

Causes of modern errors.

Two chief causes may be assigned for the errors of these modern grammarians: first, their rejection of that philosophy of the mind, on which, as we conceive, the philosophy of language depends; and secondly, their confounding historical fact with philosophical principle. The almost unintelligible use of the word *sensation*, in the passages above quoted, and the vague and contradictory meanings, applied by these writers to the word *idea*, sufficiently demonstrate their inattention to the genuine workings of the human mind. In tracing the history of words, they have sometimes shown great ingenuity; but they have erroneously concluded, that because a particular word was once a noun or a verb, it always continues such; forgetting that the identity of the word depends only on its sound, whilst the distinction of the parts of speech relates solely to their signification; and consequently, that the one is a question of the *matter* of language, the other, of its *form*; or perhaps being unable to

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comprehend the ancient philosophical distinction between matter and form, and therefore, concluding that that distinction was frivolous and unmeaning. Thus Mr. Tooke conceiving that our present adverb, preposition, and conjunction, *since*, was anciently the participle, *seen*, or *seeing*, concludes that it has still the same signification. He happens to be mistaken in his fact; for the word 'since' has nothing to do with the verb to see;* but if he had been correct in this, as he really is in many of his etymologies, the inference from it would have been no less illogical. There is no reason, in the nature of language, why one word should not successively fill the office of every part of speech; and, in particular, nothing is more common than for the same word to be both a noun and a verb. Mr. Tooke, therefore, to be consistent, should not have said that "there are only two sorts of words which are necessary for the communication of our thoughts," viz. "nouns and verbs;" but that there is only one sort; which would have been saying in effect there is no such science as Grammar in the world.

Ancient grammarians.

The ancient grammarians, who treated of the Greek and Roman languages, as well as those who in the middle ages cultivated the Arabic and its kindred dialects, and those whose disquisitions on Indian Philology have been laid open to us by recent discoveries, all agree in founding the science of Grammar, on that of the mental operations. Nothing but extreme vanity can lead us to suppose, that all the great men, who have ever considered this subject before ourselves, have been involved in a more than Bœotian mist of ignorance; and that we alone can dispel the cloud by a single "electric flash." The more modest and rational student will confess, with the amiable author of Hermes, that "there is one TRUTH, like one Sun, which has enlightened human intelligence through every age, and saved it from the darkness both of sophistry and error." It may be safely adopted as a general observation, that the man who tells you the whole world was ignorant of any particular subject until he arose to set them right, is himself egregiously in the wrong. The study of Grammar, indeed, like all other studies, is susceptible of gradual improvement; but if we admit that the ancients had a tolerable insight into the powers and operations of the human mind, we must acknowledge that they could not be entirely ignorant of the modes in which those powers and operations were manifested by language. An individual writer may have taken a limited view of the subject; but that view could not be wholly erroneous, if he was adequately versed in the philosophy of the human mind.

Reason.

It would seem, that some ancient writers considered language merely as representing the operations of the reasoning faculty; and they were enabled thus to analyse and explain a great part of its construction. In this system, which was perhaps the most ancient the *syllogism* was considered as the basis of Grammar; logical writers were its chief authorities; its rules were thought applicable only to the graver compositions, such as laws, books of civil institution, history, and treatises of the useful arts and sciences: the more animated compositions of rhetoric and poetry, and the common discourses of daily life, were considered as a kind of barbarous confusion, beyond the pale of grammatical law.

Introductory Section.

Passion.

But man could not forget, that he was a creature of passion, as well as of reason; and seeing that the former was as capable of being reduced to rule as the latter, that it was equally clear in its principles, and equally certain in its operation, he could not but admit its influence on the rules of speech. The syllogism had supplied the two sorts of words, which Mr. Tooke says are alone "necessary for the communication of our thoughts," but in matters of passion the animated *interjection* is quite as necessary as the simple name of a thing or attribute; and in like manner the *imperative* is a verbal form of no less importance, than that which merely indicates, or asserts existence.

Modification and connection.

Again, the mind, whilst it steadily contemplates certain objects, passes rapidly and almost unconsciously over those various relations which serve to modify and connect those objects with other existences. These vague and hasty glances of the mind, these slight and subordinate hints, as it were, give occasion to correspondent distinctions in language. Hence arise whole classes of words called adverbs, conjunctions, prepositions, &c.; and thus have grammarians settled the *Parts of Speech*, which we shall hereafter consider more at large.

Ancients and moderns compared.

Thus far the ancients went, and for the most part went right, in their view of language. Recent authors have rashly called in question the utility of these learned labours. It is not to be denied, that the many new sources of information opened to us in modern times, the numerous dialects, barbarous and polished, which we have the means of studying, the progress of the same language through many successive ages, which we are enabled historically to trace, and in short, the extended sphere of our experimental investigations, in language, may have served to correct some errors and oversights even in our scientific views of Universal Grammar. Let no man ever presume to suppose that his reasoning powers may not be sharpened, his judgment rendered clearer, or his taste more refined by the lessons of experience. The moment that we think there is nothing more to be learned, we give a decisive proof of ignorance. As the moderns however fail most in the philosophy of language, the ancients failed most in its history. They are rarely to be relied on as etymologists: whilst the moderns who have enjoyed so much better opportunities of cultivating this branch of the science, have obtained in it a decided superiority. They have discovered that most of those auxiliary words, which are employed in aiding the construction of nouns and verbs were once nouns and verbs themselves; and that those which appear now void of signification were formerly significant. These observations have in certain instances been extended, with some plausibility, even to the syllables, which are used for purposes of inflection. Considerable ingenuity has been displayed in this sort of investigation by DES BROSSES, COUR DE GEBELIN, TOOKE, and others; and when we come to consider this part of our subject, we shall certainly find them better guides than the ancients, who appear to have treated it with no very reasonable neglect.

It seems to follow from what has here been said,

* *Since* is derived from the Anglo Saxon word *sithe*, which is the same as the German *seit* and English *tide*, signifying *time*; consequently *since* is, literally, *from that time*

Grammar that in order to study Grammar, as a Science, a general survey of the mental faculties should be premised or presumed. These will lead us *first* to a detailed consideration of the parts of speech, both in regard to their separate properties, and also to their syntax or union. Strictly speaking, the pure science of Grammar ends here; for, as VOSSIUS has observed, science is conversant with things eternal and invariable; whereas Grammar, as generally understood, has no immoveable and unvarying essence, but relates to the matter of language, rather than to its form; and hence (as that writer contends) it ought rather to be called an art than a science. We, however, cannot overlook the circumstance that language, as it grows up with man, and forms, as it were, the main instrument of thought, is necessarily so much interwoven with the operations of his mind, that neither can the art be well comprehended without a knowledge of the science, nor can the science be easily developed and rendered fully intelligible without reference to the art. By the *form* of language, as we have already stated, we mean its signification; by the *matter* of language we mean the sound of words in speech, the movement of the body in gesture, and in general the physical and external means employed to effect a communication of the mind. The matter of speech may be considered, generally, as regarding the physical properties common to all language, or particularly with reference to the construction of one or more languages. In the former point of view it is commonly deemed a part of Universal Grammar, and will therefore form the *second* part of our grammatical essay. In the *third* part, we shall endeavour to confirm, and illustrate all our general positions, by reference to various spoken languages, ancient and modern: and in the *fourth*, we shall consider in what manner the invention and practice of written language has affected the science of Grammar. To these we may properly add a *fifth* part, considering language as a source of pleasure in itself, independently of its signification.

Preliminary view of the human mind, with reference to the science of Grammar.

Consciousness. In the mind of man the consciousness of simple existence is the source and necessary condition of all other powers; as in language, the expression of that consciousness by the verb *to be*, is at the root of all other expression.

But we are conscious of different states of existence, in some of which we act, and in others we are acted upon: and thus in language, a verb is a word which signifies *to do*, or *to suffer*, as well as to be. No language, indeed, ever was, or ever could be, formed without such verbs; but the case is different with regard to theories of language, and systems of Grammar. These may be, and have been constructed, on the hypothesis, that the mind of man is a mere passive recipient of mechanical impressions; a something which may be impelled like a foot-ball, but which cannot give to itself, or to any thing else, the slightest impulse. On such a question as this, the only appeal lies to the common sense and daily experience of mankind; and the result of that experience is clearly attested by all languages, living and dead—a species of evidence which is the less to be resisted, because it is not the result of any systematic arrangement whatever.

Every language in the world has grown up from the necessities of those who have used it, and not from intention; from accident, and not from theory; and yet there is among them an universal agreement in their fundamental principles: those principles, then, are indisputably founded on the common constitution of the human mind. Introductory Section.

Feeling. The mind is undoubtedly passive in some respects. If I open my eye to the light, I cannot choose but see; if a sound strikes my ear, I cannot help hearing. These, and many like states of existence, derived from the bodily organs, are called *sensations*; there are other states, in which we are more or less passive, derived from the mind, and commonly called *emotions*. When we come to analyse these latter, we shall easily discover that we are not so entirely passive in their reception, as is often supposed: nevertheless, as we in both cases "suffer," that is to say, are acted upon by external causes, we may not improperly include sensation and emotion as modes of the passive principle, under the common name of *feeling*. The states of sensation, which are agreeable to our nature, we properly call *pleasure*, those of an opposite kind we call *pain*; and the same names are naturally transferred to those emotions of the mind which seem analogous to the respective sensations of the body. Thus the feeling of guilt is called painful, and that of joy pleasant. The pleasurable sensations and emotions, and their real or supposed causes, are all called by the common name of *good*, and their opposites by that of *evil*. The expression of feeling is what constitutes in language the *passive verb*.

Will. As we have called the passive principle, feeling; so we call the active principle *will*, or volition. It is this principle which may truly be called the life of the human mind; it is this which forms and fashions the mind; it is this which impels and governs the man. The conscious being, in his active state, has a power: he says, I *do* this or that: and hence arises the *active verb*. Hence also arises the *pronoun*: for the very idea of an act involves the idea of a *cause*; and it has been clearly enough shown by different writers, that if the idea of a cause did not exist within the mind, it could never be suggested from without.

The will, in its growth, becomes a *moral* energy, that is, it impels us to good, as good, and consequently to the greater good rather than to the less. To choose the greater good is to do *right*, to choose the less good is to do *wrong*. Let philosophers argue, as they please, on liberty and necessity; let them reconcile, as they can, those high doctrines

> Of Providence, Foreknowledge, Will, and Fate,
> Fix'd Fate, Free Will, Foreknowledge absolute;

still the individual, from the first dawnings of reason, distinguishes right from wrong, and knows that he is a cause of the one, or of the other; and feels that the power which he exercises as a cause, is a talent for which he is responsible. Thus is formed *Conscience*, the light and guide of life. We have not now to discuss at length the nature and effects of this precious faculty: other and fitter occasions may be found for that investigation; but we cannot avoid noticing, that as the ideas of right and wrong are seated not merely in the mind, but in the first and elementary rudiments of the mind, it is a dangerous and fatal error to repre-

Grammar. sent them as contrivances of language, to say that "Right is no other than the past participle of the Latin verb *regere*," and that "Wrong is merely the past tense of the verb to *wring*." This is part of the history of words: it is no part of their philosophy.

Reason. Neither will nor feeling have in themselves any limit. The stream of conscious being is, in itself, continuous; it exists alike amid the roar of cannon, and in the soft breathing of the vernal air: in the deep, protracted, meditation of a NEWTON, and in the brief glimpse that is caught of

> The snow that falls upon the river,
> A moment white, then lost for ever.

What is it, then, that reduces the chaos of will and feeling first into distinguishable elements, and then into individual masses? It is the forming and shaping power within us. It is the divine faculty, "looking before and after," to which in its perfection, we give the name of *Reason*. Reason, holds, as it were, the balance between the passive and active powers of the mind. It is fed and nourished by the impressions of the one: it grows and moves by the energy of the other. It has several stages or degrees, of which the first is *Conception*.

Conception. By conception, we mean that faculty which enables the mind to apprehend one portion of existence, separately from all others. In other words, the first act, or exercise of the reasoning power is to conceive one object, or thing, *as one*. Hence arises in language the *noun*; for "the noun is the name of a thing." Here it is, that almost all the modern writers on Grammar have erred. They seem to have considered no such power in the mind to be necessary, and no such act to be performed. They seem to have supposed that *things*, or objects, affected the mind as such, by their own power; and that the mind was quite passive in this respect. When we come to examine this fundamental part of their system, we find the greatest possible confusion of terms. According to one, the first elements of thought are *ideas*, another calls them *objects*, a third *sensations*, and so forth. If you ask what is meant by these respective terms, you are still more bewildered. "An idea," says one, "is that which the mind is applied about whilst thinking." A most vague and insignificant expression, then, it must surely be; and yet it has been justly observed, that "vague and insignificant forms of speech and abuse of language have so long passed for mysteries of science; and hard and misapplied words, with little or no meaning, have by prescription such a right to be mistaken for deep learning and height of speculation, that it will not be easy to persuade either those who speak or those who hear them, that they are but the covers of ignorance and hinderance of true knowledge." All this is eminently true of the abuse and misapplication of the word *idea*, which had a perfectly distinct and specific meaning, until it was in an evil hour made "to stand for whatsoever is the object of the understanding when a man thinks," or "whatever is meant by phantasm, notion, species, or whatever it is which the mind can be employed about in thinking"—from that moment the word *idea* became so extremely convenient to persons, who did not much like the trouble of thinking, it served as such a maid of all work, in the family of Lady ALMA, the mind, that nothing was either too high or too low for it. "Seneca was not too heavy, nor Plautus too light;" and persons, who, in the common phrase, "never had two ideas in their lives," would give you "their ideas" on politics or the weather, on the flavour of venison, or the right of universal suffrage, with equal facility and fluency. Introductory Section.

Some of these ideas, it has been said, are simple, and some complex. In the former the mind is passive, in the latter there is an act of the mind combining several simple ideas into one complex one; but this distinction has been altogether denied, in more recent times; and we have been told, that "it is as improper to speak of a complex idea, as it would be to call a constellation a complex star." But be these ideas simple, or complex; be they ideas of sensation, or ideas of reflection; ideas of mode, of substance, or of relation, the great difficulty is to understand in every case, how each idea exists as *one*; how it is bounded, limited, and set out in the mind; and this, we say, cannot be done in any case without an act of the mind, an exercise of the peculiar faculty which we call conception.

What one set of writers say of *ideas*, another set say of *objects*. "An object in general," says Condillac, "is whatever is presented to the senses, or to the mind." Happy definition! But still the question returns; what constitutes one object? What is meant by one presentation? Is it the sensation, or thought, which takes place in a minute, in a second, or in any other portion of time? Is it the impression made on one sense, or on one part of the organ of that sense? Is it the sensation of warmth, for instance, experienced by the whole body; or that of light experienced by the eye? Is it the impression made on the retina by a house, by the door of the house, by the pannel of the door, or the pane of the window? Is it the altitude of the building, or the colour of the brick? These questions are endless, and perfectly insoluble, if that which makes an object one thing to the mind be not an act of the mind itself; but if it be an act of the mind, then it follows, that with regard to the very first materials of our knowledge, the mind is not passive, but exercises some peculiar faculty; which faculty we call conception.

Mons. Condillac, indeed, admits, that objects are not distinguished but by remarking some one or other of them particularly; and this particular remarking he calls *attention*; from whence it may perhaps be concluded, that the difference between him and us is a mere difference of words; and that he means, by attention, nothing more nor less than what we mean by conception. This, however, is an error; for attention, according to him, is a simple faculty, acting only in one mode, and acting *necessarily*, from an external cause. Thus he states, that the cause of attention to sensible objects, is an accidental direction of the organs; manifestly, therefore, according to him, the mind is no less passive in attention than in sensation.

We say, on the contrary, that in conception the mind acts. The word "to conceive," in its origin, affords an easy explanation of the mode of action. This word, which is derived from *con* and *capio*, expresses the action by which we *take up together* a portion of our sensations, as it were water, in some vessel adapted to contain a certain quantity; for we have before observed, that sensation is in itself continuous, as an ocean, without shore, or soundings: it does not divide itself into separate portions, but is divided by the proper faculty

Grammar. of the mind. The faculty of conception, like all other faculties, operates by certain laws, in a certain direction, and in a certain manner, for such is its constitution. It cannot enable us to view things temporal under the form of eternity, to conceive that a certain time occupies a certain space; or that an emotion belongs to the class of sensations; that jealousy, for instance, is red, or green, or blue, or smooth, or rough, or square, or triangular. These laws, which regulate the power of conceiving thoughts, it will be necessary for a while to consider.

Space. The first law that we shall notice, is that of *extension*. We are so constituted, that we cannot conceive certain objects otherwise than as occupying *space*. The faculty of conceiving them, therefore, presupposes in the mind a sense of space; but this sense has again its necessary laws or modes of operation. In other words, we cannot conceive space but as extending in length and breadth and thickness, and bounded by points and lines, and surfaces. It is by applying these laws to certain objects that we conceive them to be more or less extended, and to possess different shapes and forms. To say that we get the idea of space by the sense of sight or touch, is to confound our notions of sense, which imply an existence in space; it is to reverse the order of knowledge; for if the mind were originally unfurnished with a peculiar faculty, enabling, and indeed compelling it to refer the sensations of sight and touch to some part of space, it could no more acquire an idea of space from those sensations, than from the emotions of gratitude or fear. This peculiar faculty, applied to the sensations of sight and touch, of hearing, taste, and smell, enables us to conceive our own bodily existence, and that of the external world. According as we apply it more or less comprehensively, we conceive the existence of objects larger or more minute: and according as we exercise it with more or less care and attention, the external forms and disposition of objects appear to us more or less accurately defined. It is not, therefore, the external object which necessarily gives shape and form to the conception; but the conception, which by its own act embraces a given portion of space, and thus gives shape and form to the external object.

Time. Similar observations may be made on the law of duration, or *time*. To say that time is a complex idea gathered from reflexion on the train of other ideas, is to forget that the very notion of a train is that of a succession in time, and therefore presupposes what it is adduced to prove. There is nothing complex in the nature of time or duration, but it is a form under which we are necessarily forced to contemplate all things external to us, and some things within ourselves. It is a law of our nature, and so far as regards its peculiar objects, is inseparable from the human mind. But again, it is not the lapse of any particular portion of time which necessarily limits the duration of any object of our thoughts, for we can as easily think and speak of a century as of a second: it is the mind which conceives, as one object, the life of a man, or the gleam of the lightning, a long year of toil, or a brief moment of delight.

Number. These then are the laws of *simple* conception. Whatever occupies a certain portion of time, or of space, or of both, we consider as *one* thing, or *one* thought; but things or thoughts succeed each other incessantly, and by dividing sensation into units, we have done no more than to divide the ocean into drops, or the sand into grains. A further law of conception succeeds. This faculty takes a more complex form. We distinguish conceptions by their *number*; and hence, in all languages, the noun has a *plural number* as well as a singular, in signification, and generally in form. But as the plural is derived from the singular, so the power of conceiving many depends on the power of conceiving one. It has been justly observed by Mr. Locke, that "there is no idea more simple than that of unity, or one."—"Every object our senses are employed about," says he, "every idea in our understandings, every thought in our minds brings this idea along with it." Now since this is the case, since no object, no idea, no thought, ever is conceived in our minds without this impression of unity, why should we imagine that any can be so conceived? And if it cannot be conceived without such impression, then must we consider the power by which that impression is produced as essential to the conception. Before we can speak or think of any thing, we must first conceive it to be one. This *one* may be finite or infinite; that is, our conception may be perfect or imperfect—but still, in order to become an element of reason, it must exist, as one, in the mind. Even the conception of *many* exists in the mind as that of *one multitude;* and if that multitude be divided into distinct parts, so as to be numerically reckoned, the number, whatever it may be, is still contemplated as *one number*. Simple conception indeed could never have advanced us beyond the notion of an unit or integer; it is by the aid of the other reasoning faculties, which we shall hereafter notice, that we are enabled to form the complex conceptions of number, and so to build up the whole science of Arithmetic. Introductory Section.

Identity. Conceptions succeed each other indifferently, whether they are like or unlike; but the mind can only number them by classing them, and can only class them by their similarity; which similarity, when complete, is in the contemplation of the mind *Identity*. Much has been said of the source from whence we derive the notion of our own personal identity. Surely if any thing is essential, not only to reason, but to feeling, to will, and even to consciousness, it is this notion. When Descartes invented his famous reasoning, *Cogito, ergo sum*, he clearly assumed his personal identity: and it is utterly impossible for a human being to reason or think at all, without such an assumption. Even in madness, though the actual identity is often confounded, though a man may fancy himself to be Alexander the Great, or even to be the Almighty, he has before his mind an imaginary identity: he thinks and acts as one being, and not as two: and again, in dreams, when we sometimes see ourselves dead, or alive, yet the self which we contemplate is a mere imaginary personage, with whom we have a strong sympathy, as we have with the hero of a romance. The contemplator always seems to think and act as a separate individual, and never loses the deep sense of identity.

Kinds of conception. We are next to enquire into the different *kinds* of conception thus formed; and we shall find that the ancients were right in dividing them into two, namely *substance* and *attribute;* whence arise in language the *substantive* and *adjective*. It must be remembered

that we first conceive, as one thing or one thought, a given portion of sensation, and that those sensations in their simplest form are limited by the laws of time and space; but those laws are always operating on the mind together, though not always with equal force. Sensations which spread over a large extent of space may occupy a short time, and those which continue for a long time may lie within very narrow bounds of space. Many parts of space too may be contemplated in one moment of time, and many portions of time may refer to the same point of space. Our first notion of substance is personal, unless we should prefer saying that the notion of substance is derived from that of *person;* which might perhaps be a more philosophical mode of speaking; though the former more immediately applies to the common arrangements of grammarians. We refer all our states of being to a substance called *self,* to which each man gives the name of *I:* and thus I feel and know that I am a cause of all the active states of my being. By an inevitable necessity of my nature I am led to believe that there must be a cause or causes foreign to me of all the impressions made on me without my own act. With respect to myself the conceptions which are limited by time and space give me the notions of *matter* and *motion* as belonging to me, those which are not so limited give me the notion of *mind.* To external causes therefore, I attribute the same distinctions of character: and hence the most general notion of external substance is that of a cause of the impressions formed in me. But one cause often appears to be common to several different sensations. I therefore conclude that it is one *thing.* I have, for instance, the sensations of heat, and light, and colour, cotemporaneously, and this not once, but often; and I conclude, that there is some common cause of all these sensations, to which cause I give the name of Fire. The notion of *substance* it is said is obscure; it is no otherwise obscure, than as a thinking and sentient being cannot sympathise with an unthinking and insentient one. Obscure as it is said to be by philosophers, it is what the common bulk of mankind consider as the very plainest and clearest of all their notions. A common man is never troubled with any doubts of the existence of the table or chair that he sees before him, any more than he is of his own personal identity. Others again think, that they have a very clear notion of the existence of these external objects or substances: they can easily understand how the mind conceives the cause of a particular sensation of heat, and a particular sensation of light, to be one object, called fire; and contemplates that object as separate from the sensations produced by it: but they cannot understand how the mind should conceive as one thing, or thought, or one object of contemplation, a *common cause* of all similar sensations. Yet it is certain, that men do, and ever have, used words in language expressive of those common causes, and that those words have always had the form of substantives. Much effort has been made to explain this on the theory of abstraction. These notions have been called *abstract ideas* (a very improper use of the word idea at least); and it has been supposed that they were formed by *abstracting,* or taking away from each particular conception, some circumstance of time or place. Now it appears to us, that this is an operation, which is rarely, if ever performed by the mind. Certainly, the greater part of the conceptions represented to be so formed, may be shown to be produced in a totally different manner. Thus the conception of a straight line, and the consequent conception of straightness in general, is certainly not formed by abstracting from various lines, various inequalities; for if it were so, every man would have a different notion of a straight line from every other man, and every man would go on abstracting, and consequently improving his conception of straightness as long as he lived. Whereas, in truth, the idea of a straight line, as soon as it is once steadily contemplated in the mind, is perfect, and is equally so in all minds. This could not be the case, if all minds did not act by some general laws; and since we are so constituted as to be able to reflect on such laws, we may separate those reflections from the general mass of consciousness, as easily as we can separate a particular sensation from the same mass; we may form of each, a *conception,* a thought, as distinct from all other thoughts, as one external object is conceived to be separate from all other external objects. The thought of a general law as single, has no reference to time or space. Even the laws of time and space, are not supposed to be more or less laws, or to have a more or less real existence, at one time, or in one place than in another place, or at a different time. It is indeed objected, that they have no real existence at all; that there is no *truth* but that of opinion, and consequently, that "two persons may contradict each other, and yet both speak truth;" for such are the precise words of Mr. Horne Tooke. (Vol. ii. p. 404.) The same objection may be made with much more force, against the existence of the external world; for the learned and pious Bishop Berkeley has fully shown, that we have no assurance of the reality of matter or motion, but that which depends on our instinctive conception of their existence, as causes of the changes which we experience in ourselves. But as we are utterly unable to believe, that there is no truth in our own existence; and, as we find it hard to imagine, that this "goodly frame, the earth," this most "excellent canopy, the air," this "brave o'er hanging firmament," this "majestical roof fretted with golden fires," are all fictions and non-entities; so it is difficult for us to imagine, that truth and virtue, beauty and wisdom, glory and happiness, are all empty names: we cannot well believe that time and space are mere fictions of our own minds; and yet it is easier to believe this, than to conceive their existence according to laws different from those which we actually experience; it is easier, for instance, to conceive that there is no real existence in space, than that if it exists, a straight line in space is not the shortest that can lie between two given points, or that a figure may be completely bounded by two straight lines, or that the radii of a circle are unequal, or that the three angles of a right lined triangle are greater or less than two right angles. Hence arises the distinction of *subjective* and *objective* truth. The former we consider as existing in ourselves, the latter as existing in objects out of ourselves; the truth of a mere opinion is subjective, the truth of the fact to which that opinion relates is objective; but if all truth were merely subjective, each man's mind would be the only universe, and it would be a solitary universe, without a creator, without time, or space, or matter, or motion, or men, or angels, or heavens, or earth, or virtue, or vice, or beginning, or ending—one wild delu-

Grammar. lusion without even a framer of the monstrous spell! Now since it is utterly impossible to believe this, either deliberately or instinctively, it follows that there is some objective truth, and that what a man *tryeth*, *troweth*, or *trusteth* to (for these are all of the etymological family of the *word* truth) is in itself, more or less, substantial and permanent. But if this be the case with our conception of a stone, why not of a man? And if of the motion of a stone, why not of the thoughts of a man? And if of thoughts bounded by the laws of time and space, of number and identity, of good and evil, why not of those laws themselves? For the purposes of Grammar, it is hardly necessary to press this argument; for language has been made by men, according to their instinctive opinions; and certainly the prevalent opinion has always been, that there is something which the mind contemplates, when it reasons on man in general, as well as when it reasons on Peter or John. It is probable that Sir Isaac Newton had some object before his mind when he argued on light and colours, as well as a lamp-lighter has, when he lights a lamp; or as a country lass has, when she buys a yard of blue or red ribbon at a fair.

Conceptions, then, are either particular, general, or universal.

Particular. In strictness of speech nothing is particular, but that which occupies only one given portion of time, or of space, or of both. Thus the emotion of fear at a certain moment of time; the sensation of warmth at a given moment; and in a certain part of the body; or the sensation of brightness in a particular part of the retina, are all particular conceptions; and it is somewhat remarkable in language, that men (in early ages, and before they had much turned their thoughts to reflection) so entirely confounded the subjective and objective truth, both of sensations and emotions, that they used the same word to denote both. A man, for instance, would say indifferently, "I am *hot*," or "the fire is *hot*." So, in common parlance, we say "the bird *fears* the scarecrow," but Shakspeare says:

> We must not make a scarecrow of the law,
> Setting it up to *fear* the birds of prey.

Nor is it only a simple sensation or emotion, of which we may form a particular conception. We may certainly conceive as one thing, a substance; that is, many sensations or emotions united in one common cause; whether that cause be active as a person, or passive as a thing; for the notion of a person is founded on *self*, as an active being, and that of a thing on the same *self*, as passive.

These, we say, are the only conceptions which, in strictness of speech are absolutely particular; but almost all writers call those particulars, which we find to be identical; thus Peter or John is said to be a particular individual, though the name, Peter, or John, is given to an object which I have seen on many particular occasions, and only know to be identical by reflection and comparison. In like manner, red is the name of a colour impressed on my retina to-day and yesterday, and which I know to be identical: and so the word, to walk, implies an action which I perform frequently and know to be the same on all occasions. We dwell the more on this observation, because it shows that those who strongly contend for the existence of nothing but particular objects, overlook the fact, that what they call particulars are not such in strictness of speech; and that, if the only business of the mind were to receive impressions (as Mr. Tooke says it is), we could never acquire even what they call a particular idea or conception; we could never know that the John of to-day was the same person as the John of yesterday.

Introductory Section.

This latter species of particulars, however, is the first element of language. We invent signs, not to express a single impression, but the same impression often repeated; and these are of three kinds, the simple sensation or simple quality producing it, which we call an adjective; the simple action, which we call a participle; and the person or substance in which the cause of sensation or of action resides, which we call a substantive.

To these particulars we may add the notion of numbers, either distinct or confused; for the notion of many objects or many qualities may still be viewed as a particular notion: and hence arises, not only the plural of nouns, but the singulars which imply plurality, and are commonly called nouns of multitude, as a troop, an army, a crowd.

We have shown that a particular conception is formed by the mind separating and sorting its sensations and emotions according to certain necessary laws; and arranging them in certain forms more or less distinct. Thus a certain form is that of Peter; but the same form applies nearly to John, the same nearly, though with some other difference, to William; and so on. Now, when we contemplate this form as possibly applicable to a variety of particulars, it constitutes what we call a general conception; and these general conceptions, duly ordered and arranged one within the other, form *genera* and species; and of these, more or less distinct, opinion is chiefly formed. General.

But there is yet one higher step in the power of conception, namely, the Universal. This is when we contemplate the *form* itself in which our lower conceptions were cast. Thus, there is a certain law by which the mind can only conceive a straight line in a certain manner, namely, as length, and as partaking in no degree of curvature, nor interrupted, nor distorted in any manner whatsoever. Now, the first line that we actually conceive to be straight, is not exactly so, yet it approaches to the form in the mind sufficiently to make us give it the name of straight. The second, the third, the fourth, and all successive lines, are perhaps equally deficient; and, by comparing them with each other, were there no common standard to refer them to, we should never attain the knowledge of a simple straight line. All the lines which we actually see, have breadth together with their length, all have some curvature or irregularity; but reflection shows us in the mind, a line, which is merely length without breadth, and which lies evenly between its points. Of this, we are able to make a distinct conception, which, when we have once attained to, we find it entirely independent of time or space, always the same, necessarily true in all its relations, equally applicable to all the particulars which fall under it—a law of the mind—in short, what was alone and properly called by the ancients—an *idea*. The higher, the nobler, the purer these ideas are, the more difficult is it for man to conceive them. They are never conceived without meditation and effort; and the deepest meditation, the highest stretch of our faculties, leaves us lost in admiration and awe at the great overpowering idea of our Almighty Father. Universal.

Conceptions present themselves to our minds, either as accompanied, or not accompanied, with a sense of objective reality. If they are not so accompanied, they are mere creatures of the *imagination*: if they are so accompanied, then, if the object producing them is past, they are conceptions of *memory*, and if yet to come, of *expectation*; but, when the object is present, the conception becomes a *perception*, whether it be of an external thing, or of a general notion, or of an idea.

Assertion.

We have hitherto spoken only of the faculty of conception, by which the mind gives its thoughts their separate forms; but we have next to see them put into action, and rendered as it were, living and operative. Thoughts and opinions come to us in the mass; and it is by developing them into their constituent parts, that we ourselves understand them; but in order to communicate them to others, we must pursue the contrary process; we must state the parts, and *assert* their union. *Assertion*, then, is the faculty which we have next to consider: it is, as it were, the uniting and marrying together of two thoughts, and pronouncing them to be one. Hence the word, which expresses that function of the mind, is called, by some writers, the *copula*, or bond; but in common Grammars, the *verb*: and we rather adopt the latter term, because the former may be apt to lead to the erroneous conclusion, that the mind in assertion, passively contemplates two thoughts as united, whereas, it is active in declaring that union, as it were, by its proper authority; an authority, indeed, often exercised, hastily and amiss, but still the proper act of the mind itself. Conception, then, forms nouns, including under that term substantives, adjectives, and even participles; but these nouns lie dead and inoperative to any purpose of reasoning, till they are vivified by the verb, which pronounces their existence to be a truth. Thus *John*, *existing*, *good*, *loving*, are all perfectly intelligible as conceptions of the mind; yet so long as they stand alone, we see not what use is to be made of them in reasoning: but let us introduce the verb, and a truth immediately *flows* from the mind, whence possibly some etymologists might derive *ρημα*, the verb, and *reor*, to think, from *ρέω*, to flow. Thus we say, John exists, John is good, John loves, and each of these assertions at once takes the form of a truth, and becomes, as will be hereafter shown, the germ and seed of other truths in the mind.

Affirmative and negative.

To assertion belong affirmation and negation. We declare, that conceptions exist, or that they do not exist; and the one of these excludes the other. A thing cannot be, and not be at the same time; but as there are certain conceptions, which are the opposites of each other, so affirming the one is denying the other. To say that black is white, is therefore, in common parlance, to utter a gross and palpable untruth.

Moods.

Neither affirmation nor negation, however, is always positive. The mind contemplates some truths as actual, that is to say, it conceives the subjective truth within itself to be certainly agreeing with the objective truth in the nature of things, and therefore pronounces unhesitatingly and distinctly upon its existence; but of other subjective truths it sees no objective counterpart, and therefore pronounces them not actual, but probable, or merely possible. On this distinction, in great measure, depends what is called the *mood* of verbs.

Tenses.

Again, we assert truths either with or without reference to the time in which we speak. When we speak with such reference, that is to say, when we speak of particulars, we are necessarily compelled to distinguish the present from the past and future; and hence the origin of *tenses*. When we assert any thing of ideas, we speak of a truth ever present, and therefore we use the present tense in its purest form. Thus, when we say John is good, we imply a possibility that he might at some other time be bad; and when we say John is writing, we imply a certainty that he was not writing at some previous time, and will not be writing at some future time; but when we say two and two are four, we not only assert a truth of to-day, or of this year, or of this century, but a truth which must be ever present since we cannot conceive it ever to have beginning or ending. This remark is sufficient to show that those grammarians are in error, who make the signification of time a necessary characteristic of the verb.

The mind active in propounding assertion.

In whatever way we assert any thing, the assertion is a declaring of some truth, real or supposed; it is a propounding, or showing forth the existence of the truth, or in the language of logicians, it is enunciating a proposition. This is not done by a peculiar word, as for instance the word *be*; but by the form of the word; for the word *be*, in some of its forms, as, *to be*, and *being*, is a simple conception; and so are the words *love*, *hate*, *walk*, *sing*, and indeed all others which may be used as verbs. Mr. Tooke therefore was very accurate, as far as regards words, in saying that the verb was "a noun, and something more;" but when toward the end of his book, he came to consider what that "something more" was, he found himself entirely at a loss, and was forced to break off abruptly; since the just solution of the difficulty, as we conceive, would have overturned the whole system, which he had laboured throughout two ponderous volumes, to erect: it would have shown the mind of man to be an active intelligence, not only in forming conceptions, but in uttering, declaring, propounding, asserting them to be truths.

This discovery would have been still more fatal to Mr. Tooke's grammatical system, had it been more fully developed; for when we come to ask how, and in what various ways, a truth, or to speak in the phrase of logicians, a *judgment*, is asserted, we shall find that this depends entirely on the different kinds of conceptions; and, as we have already seen, these kinds are produced by different acts of the mind; whereas Mr. Tooke treats them all as of one kind only, and all as received by the mind from passive impression.

Existence and action.

We assert then, either existence, or action. If the former, we either assert it simply of a conception, as "God exists;" or we assert it conjointly of two conceptions, which are of a nature to exist together, as the substance with its attribute, or the whole with all its parts, or the universal with the particular. Thus we say "God is good," "two and two are four," "gratitude is a virtue." If we assert an action, we must consider it either as proceeding from its cause, or as received by its passive object, that is to say, we must employ either the active or the passive verb; and which ever we employ primarily, we must (if such be the nature of the action) add the other secondarily. There are, indeed, actions which rest in their causes; and the verbs expressing these, whether active or passive, in construction, are really of the kind called neuter, or

Grammar. intransitive, such as, "to rejoice," "to sing," and the like.

Deduction. A truth asserted leads to a further truth, by that faculty, which Shakespeare calls "discourse," from the ancient scholastic and accurate term *discursus*. Hence that beautiful and philosophic passage—

> He that made us with such large discourse,
> Looking before and after, gave us not
> That capability and godlike reason,
> To rust in us unused.

This faculty, for want of a better term, we shall call *deduction*. It arises from the comparison of truths: and as that comparison refers to something common to both the truths compared, the consequence or inference to be drawn is always of the nature of a particular, under some universal expressed or understood. Of the forms of deduction, the most perfect is the syllogism; but the whole force of the syllogism depends on the *universal* conception which it involves. In the enthymeme, which is an imperfect syllogism, the universal, though not expressed, is understood. It is, therefore, clear, that the modes by which one truth is deduced from another, imply a power in the mind beyond that of merely receiving impressions. The deduction may be made from hypothetical premises. Hence arises a further explanation of the use of moods in the verb. We assert a truth, not as actual, but as possible, and the consequence which we deduce becomes a contingency, necessarily following from the premises, but not necessarily true, because the premises themselves are not necessarily so.

Review. Thus have we enumerated the three faculties which go to the making up of the reasoning power, and which are conception, assertion, and deduction, answering to the *simplex apprehensio*, *judicium*, and *discursus* of the logicians. All continued exercise of reason resolves itself into a repeated exertion of these faculties; and the only difference is, that the truths produced by one deduction serve to enlarge or improve the conceptions which are employed in framing other assertions and deductions.

Secondary parts of speech. Hitherto we have had occasion to notice only those operations of the mind, as giving birth to the primary parts of speech, the noun and verb, the substantive and adjective, the pronoun and the participle, which are in most cultivated languages distinguished from the adverb, the conjunction, and the preposition, by being subject to inflection or change of form, either in the beginning, the middle, or the end of the words by which they are expressed. This latter circumstance, however, is merely accidental, and with respect to the essential difference of the adverb, conjunction, and preposition, from the other parts of speech before mentioned, we must repeat what we have before stated, that the mind contemplates truths at first in the mass, and then by reflection breaks down that mass into certain portions which again are subdivisible; so that in asserting one truth, we cast as it were a rapid glance over the subordinate branches of which it is composed; as in viewing the whole beauty and proportion of the Apollo Belvidere, we see at once the graceful turn of the head, the animated advance of the arm, and the receding of the opposite foot; or as in contemplating the agonised frame of the Laocoon, the two sons with the folds of the serpents which twine around them, occupy a secondary place in the imagination. When we come to develop these secondary parts of the composition, we find in them the same principles of unity and connection, as in the general outline of the whole groupe: and so it is with the subordinate parts of a sentence; which are, if we may use the expression, truths within truths, assertions within assertions. Thus even the long and flowing sentences of Milton's prose are each reducible either to an assertion, or at most to a deduction, as their ground work; but upon that ground-work are built many other assertions, which are assumed, though not formally stated as such. Each adverb, each conjunction, each preposition, contains such subordinate assertion, and of course involves a conception; it is therefore true, that these parts of speech ultimately resolve themselves into nouns and verbs—ultimately, we say, but in the first glance and motion of the mind, as it were, they only appear in their secondary character, as helps and expletives to the principal words in the sentence.

Introductory section.

Passions. The *passions* must not be overlooked, in considering the mind in its relation to language. It often happens that an abruptness, a transposition, and that which might be called an irregularity, if we referred only to the operations of reason, become appropriate, and even necessary forms of speech, when the mind is under the influence of passion. The reasoning powers are then disturbed and imperfect; the emotions become inordinate, the will obtains a preternatural force. Hence arises the *interjection*, which some grammarians have refused to reckon among the parts of speech; but their refusal is vain: so long as there are men with human passions and affections, there will be interjections in their speech, words which stand out from the rest, very significant of emotion though not of conception, defying all rules of construction and arrangement, because such rules bear reference principally to the power of reason, which is suspended or superseded, whenever passion produces the animated and expressive interjection. Passion, too, has given birth to what we commonly (though not always very appropriately) call the *imperative mood*. When Esau says, "Bless me, even me also, oh, my father!" We feel the earnestness of the prayer, widely different as it is from a command. Again, this same example shows us, that the *vocative case* of the noun is of similar origin. "Oh, my father," is a strong expression of passion; but it is totally dissevered in construction from the enunciation of any truth, and has nothing to do with any operation of reason. Many other forms and modes of speech take their character from passion; as may be particularly observed of the *interrogative*, so often the result of an eager desire to know the very fact, which, it may be, we fear and tremble to assert.

Conclusion. It is to be observed, that all the exercises of all the human faculties may be clear or obscure, distinct or confused. Our very consciousness may be that of mere dotage, our feelings may be blunted, our will wavering and undetermined, our conceptions vague, our assertions doubtful, our deductions uncertain, our passions a chaos. It has been elsewhere said, that "the thousand nameless affections, and vague opinions, and slight accidents which pass by us 'like the idle wind,' are gradations in the ascent from nothingness to infinity; these dreams and shadows, and bubbles of our nature, are a great part of its essence, and the

chief portion of its harmony, and gradually acquire strength and firmness; and pass, by no perceptible steps, into rooted habits and distinctive characteristics." Still the channels in which the stream of mind flows, so long as it has any current, remain always the same: the mental faculties which we exercise, so long as we can exercise any, are subordinated to the same laws, and display themselves in the same manner. Hence speech is, in all nations, necessarily formed on the same principles; and though no one language was ever constructed artificially, yet it is astonishing how distinctly all present the traces of the same mental powers, operating, in the same manner, on materials so exceedingly different.

CHAPTER I.

OF UNIVERSAL GRAMMAR.

The general view which we have taken of the human mind, appeared to us to be indispensible toward a right understanding of what we shall have to say of Grammar, or the science of language; for as we consider language to be a signifying or showing forth of the mind, it would have been impossible for us to have rendered ourselves intelligible, in explaining the laws or modes of signification, had we not first stated what we understood to be the nature of the thing signified.

Gradations of science.

In different languages there are some things accidentally different, and some things essentially the same. It has been owing to accidental circumstances in the history of mankind, for instance, that the name of the Universal Creator, among the Jews, was *Jehovah*; that it is in France *Dieu*, and in English GOD; and that the Latin words *locum tenens* came to be changed into the Italian word *luogotenente*, the French *lieutenant*, and the English word, which we spell like the French, but pronounce *leftenant*. It is also by accident, that the word *luogotenente* signifies, in some parts of Italy, the civil magistrate of a small community; that in France and England the word lieutenant expresses various ranks in the military and marine services; and that in Ireland it is applied to the vice-roy, or chief representative of the sovereign. On the other hand it is owing to causes which exist more or less permanently in human nature, that in the sounds uttered as language by an Esquimaux, a Hottentot, or a Chinese, there are certain qualities common to them with the eloquent voices of a Cicero or a Demosthenes. Though their articulations vary in many respects, they all articulate; and the nations that whistled like birds, or hissed like serpents, never existed but in the inventions of the same sort of travellers, as those who told of Cynocephali and Cyclops, and of men who sheltered their whole body while they slept, by the shade of one enormous foot. How far the laws of sound and gesture are common to mankind, it is not possible, at least it is not easy, to determine *a priori*; these laws, therefore, we cannot consider in the light of pure science; they form *general* Grammar, but not universal.

We come, however, in the contemplation of our subject, to a part of it, which is universally applicable, and universally true. Cicero or Demosthenes, Plato or Newton, Dante or Shakspeare, might express sublimer, bolder, clearer, lovelier thoughts than men of a common stamp, but they could only express them according to the laws by which every human mind must necessarily act in conceiving and uttering thought. Here then we arrive at *Universal Grammar*, at the *pure science*, which places this part of knowledge on an immoveable basis, renders it demonstrable and certain, and connects it with that TRUTH, which is one and uniform through all ages, and which rashness and ignorance perpetually assail, but can never subdue.

Writers.

It is far from our intention to assert, that *Universal Grammar* has been hitherto so successfully cultivated, as to leave to future investigators no hope of improving this science. Its principles have certainly been no where laid down with that happy and lucid order, which has rendered Euclid's Elements, for above two thousand years, a text book in geometry. Much, however, has been done. The ancient Greek and Latin writers have traced all the principal paths of the labyrinth, and elegant edifices of science have been raised in modern times by such writers as SANCTIUS, VOSSIUS, the writers of PORT ROYAL, and the learned and amiable HARRIS. The last of these writers, as being not only most familiar to the English reader, but most rich in ancient authorities confirmatory of his system, we shall follow as our principal, though not sole guide, in the present chapter.

Order of study.

"Those things which are first to nature," says Harris, "are not first to man. Nature begins from causes, and thence descends to effects. Human perceptions first open upon effects, and thence by slow degrees ascend to causes." And this is well illustrated by Ammonius with reference to speech: "Even a child," says he, "knows how to put a sentence together, and to say *Socrates walketh*; but how to resolve this sentence into a noun and a verb, and these again into syllables, and syllables into letters, here he is at a loss." Hence we may see, that by the very constitution of our nature the most complex things are most familiar to us, that the most general laws, by the very reason that they are most general, and most constantly in action, become habitual to us without our reflecting upon, and consequently without our understanding them. We conform to the complex and intricate laws of vision, we judge of distances and magnitudes by the angles which objects subtend, and yet during a great part of our lives we have not the most distant suspicion that any such things as angles exist, or that they are subtended on the retina; nay, ninety-nine men out of a hundred, and probably a much greater proportion, exercise the power of vision throughout their whole lives, without so much as wasting a thought on its laws. So it is, in regard to speech. All men, even the lowest, can speak their mother tongue; yet how many of this multitude can neither write nor read; how many of those who read know nothing even of the grammar of their own language; and how many who have been instructed so far, have never studied Universal Grammar! In this science, as well as in all other things, the observation which we have above made, holds true; namely, that human perceptions open first upon effects, and thence ascend to causes. Men first notice the practice of speech, as the exercise of some natural faculty, which proceeds, as it were, spontaneously from the wish of communicating their thoughts and feelings. By and bye they observe, that this faculty operates partly from sudden impulses, and gives birth to expressions not easily to be analysed into any component parts, as

Grammar. in the ejaculations of Philoctetes, which fill up many lines of the Greek tragedy, representing his sufferings; and that on the other hand, it is in far greater part the result of thought, and distinguishable into portions separately intelligible. Every discourse, however long, consists of *sentences*. These are combinations of speech which are obvious to all persons; and therefore, before we proceed to analyse speech any further, it may be useful to observe the different kinds of sentences; but our analysis must not stop there; for it is equally obvious, that sentences consist of *words*, and that every word has some separate force or meaning. Here, however, the power of dividing speech into significant portions ends; for though words are made of *syllables*, and syllables of *letters*, yet these two last subdivisons relate wholly to the sound, and not to the signification. A syllable or a letter may possibly be significant, as the English pronouns I and Me; but then they become words, and are so to be treated in the construction of a sentence. Words, then, are the primary integers of significant language; but these may be distinguished according to their separate properties and uses, into two or more classes, which grammarians call *parts of speech*. These parts of speech, therefore, we shall consider separately, and after we have thus exhausted the analytic, or distinguishing method of treating our subject, we shall then advert to the synthetic, or the laws by which the parts of speech are combined together, and which grammarians call *syntax*.

§ 1. *Of sentences.*

Sentences. A sentence is a number of words put together, and obtaining from their combination, a particular power of enunciating some truth, real or supposed, absolute or conditional, or else of expressing some distinct passion, together with its object. Sentences, therefore, are of two kinds, according as they are directed to these two different ends.

Enunciative sentences. The *enunciative* sentence obtains its power of expressing fact or opinion, by the connection of the words of which it is composed; for Aristotle observes (what indeed is self-evident), that "of those words which are spoken without connection, there is no one either true or false; as for instance, "man"—" white"—" runneth"—" conquereth." But let us put together only these two words—

"Jesus wept,"

and we have recorded an historical fact most affecting in itself, and furnishing abundant food for deep and interesting meditation.

When we read in SHAKSPEARE:

"The quality of mercy is not strained,"

we immediately perceive the enunciation of a beautiful truth, which is again presented under an expressive form to the imagination by the following lines:

"It droppeth as the gentle rain from heaven
Upon the place beneath."

So when Milton says:

———————— "in the soul
Are many lesser faculties, which serve
Reason, as chief."

A truth respecting our intellectual (as the former did our moral) nature is distinctly asserted.

This kind of sentence may enumerate many particulars, all bearing on one point of time, or referring to one general idea: such is the following picturesque delineation of what presented itself to young Orlando when in pacing through the forest, chewing the cud of sweet and bitter fancy, he threw his eye aside— Chap. I.

"Under an oak, whose boughs were moss'd with age,
And high top bald, of dry antiquity,
A wretched ragged man, o'ergrown with hair,
Lay sleeping on his back; about his neck,
A green and gilded snake had wreath'd itself,
Who, with her head, nimble in threats, approach'd
The opening of his mouth; but suddenly
Seeing Orlando, it unlink'd itself,
And with indented glides, did slip away
Into a bush; under which bush's shade
A lioness, with udders all drawn dry,
Lay couching, head on ground with cat-like watch,
When that the sleeping man should stir."

Such also is the following argumentative sentence in Bishop TAYLOR's Sermon on the Duties of the Tongue, urging the Christian office of administering consolation to the afflicted:

"God hath given us speech, and the endearments of society, and pleasantness of conversation, and powers of seasonable discourse, arguments to allay the sorrow by abating our apprehensions; and taking out the sting, or telling the periods of comfort, or exciting hope, or urging a precept, and reconciling our affections, and reciting promises, or telling stories of the Divine mercy, or changing it into duty, or making the burden less by comparing it with greater, or by proving it to be less than we deserve, and that it is so intended and may become the instrument of virtue."

Interrogative. The enunciative sentence easily becomes *interrogative*. For the same fact which is simply asserted may be stated as beyond the sphere of the speaker's knowledge, or as being doubted by him, and desirable to be known. This is commonly effected in language by a slight transposition of the words, sometimes by a mere change of accentuation. As in Sterne's celebrated sermon, "We trust that we have a good conscience."— "Trust that we have a good conscience?" Again, by transposing the lines above quoted, we make them interrogations.

Is not the quality of mercy strained?
Droppeth it as the gentle rain from heaven?

But it is to be observed, that as some degree of emotion is implied in the very nature of an interrogation, so it is often used by the poets, orators, and others, to give life and animation to their style, although no doubt exists in their mind or that of their hearers; and the matter which is questioned in point of form, is meant to be asserted in point of fact. Thus when the poet says—

——— who to dumb forgetfulness a prey,
This pleasing, anxious being e'er resign'd?

he means positively to assert that no one ever quitted life with indifference. The humourous speech of Falstaff, when personating the king, illustrates our observation.

"Shall the blessed sun of heaven prove a micher, and eat blackberries? A question not to be asked. Shall the son of England prove a thief and take purses? A question to be asked.

Conditional. Again, the enunciative sentence may be *conditional* or contingent; that is, it may be placed in dependence on, or in counterbalance against some other truth; as in Macbeth—

If it were done, when 'tis done, then 'twere well
It were done quickly.——

Or in Hamlet—

——Duller should'st thou be than the fat weed
That rots itself at ease by Lethe's stream,
Wouldst thou not stir in this.——

Or again in Macbeth, where the contingency takes place in spite of obstacles which might be supposed capable of preventing it:—

> Though Birnam wood be come to Dunsinane,
> And thou oppos'd being of no woman born,
> Yet will I try the last.

Passionate sentences.

In all these and similar instances, the enunciation of a truth is the immediate object in view; but another class of sentences owe their form and construction solely to some *passion*, of which they indicate the object. And it is to be observed, that the indication of an *object* of passion is essential to the constituting such sentences as these. Thus, when the Nurse, in Romeo and Juliet, on finding her young lady dead, cries and laments vociferously, and the parents enter, asking "What noise is here? What is the matter?" Her answers, "O lamentable day!" "O heavy day," are not sentences; for though they plainly show the grief with which she is agitated, they do not at all express the cause or object of that grief. But when Hamlet cries—

> Oh! that this too, too solid flesh would melt,
> Thaw and resolve itself into a dew!

we perceive a distinct expression of the wish to be delivered of life, as burthensome to him. The sentence is as complete and grammatical, and much more poetic than if the place of the interjection O! had been supplied by a verb; for instead of an impassioned and beautiful line, it would have been perfectly absurd, if the poet had said:

> *I wish that this too solid flesh would melt!*

We may observe that these passionate sentences, combine quite as readily as the enunciative ones, with dependent sentences, as "O! that I had wings like a dove! Then would I flee away and be at rest;" which implies the same fact as the sentence "If I had wings like a dove, I would flee away," &c.

Active and passive.

Sentences of the passionate kind either express a passive feeling, as admiration and its contrary, or an active volition, as desire and its contrary. Of the former kind, is that passage of the apostle, "O! the depth of the riches both of the wisdom and knowledge of God?" and the line of Milton, comparing the receptacle of the fallen spirits with their former happy seat—

> O! how unlike the place from whence they fell!

Those sentences which express desire and aversion are commonly expressed by the mood called *imperative;* but they as often imply humble supplication or mild intreaty, as authoritative command. Thus the poet describes Adam gently calling on Eve to awake—

> He, with voice
> Mild as when Zephyrus on Flora breathes,
> Her hand soft touching whisper'd thus: *awake*
> My fairest, my espous'd, my latest found,
> Heav'n's last, best gift, my ever new delight,
> *Awake!*

And again, when our first parents offer up in lowly adoration their morning orisons—they say—

> Hail universal Lord be bounteous still
> To give us only good!

But these emotions are widely different from others, expressed in the same form of sentence: as when king Henry says to Hotspur—

> *Send* us your prisoners by the speediest means,
> Or you shall hear from us in such a sort
> As may displease you.

Or when Juliet exclaims

> *Gallop* apace, ye fi'ry footed steeds,
> To Phœbus' mansion!

Or when Macbeth cries to the ghost of Banquo—

> *Avaunt!* and quit my sight! Let the earth hide thee!

Imperfect sentences.

We have already had occasion to notice, that some sentences are simple, and others complex. We have only to add, that instances occur in which a sentence is manifestly left imperfect, and that with great beauty, as in the well known line of Virgil:

> *Quos ego*—sed motos præstat componere fluctus.

And so Satan first addresses Beelzebub, in the opening of the Paradise Lost:

> *If thou be'st he*—but oh! how chang'd, how fallen!

In both these cases, the words, though not in themselves fully and clearly expressive of the thought which we may suppose to be in the speaker's mind, are yet not wholly unconnected, and therefore, show at once that they are parts of sentences which, indeed, it would be easy for the reader to fill up in his own imagination.

Harris.

Mr. HARRIS distinguishes sentences into two classes, as we have done above; only he gives them the names of sentences of assertion, and sentences of volition. Other writers have classed them somewhat differently, but yet with reference to similar principles. Thus Ammonius states that there are four kinds of sentences besides the enunciative, namely, the interrogative, the optative, the deprecatory, and the imperative; but that in the enunciative alone is contained truth or falshood.

Aristotle.

We have observed, that sentences are composed of words, of which latter every one has some meaning; and this agrees with the definition of a sentence given by Aristotle: *φωνὴ συνθετὴ σημαντικὴ, ἧς ἔνια μέρη καθ' αὑτὰ σημαίνει τι.* We may observe also, that these distinctions were familiar to the old grammarians; and hence Priscian observes, that the parts of a sentence must be called parts with reference to the whole, so that in a sentence in which the word *vires* occurs, we must not divide it into two words, *vi* and *res*, though these might be significant in another sentence; because in the former case, they would have no signification with reference to the whole sentence. But again, as sentences are made up of words, there must be some rules for construeting them, and these rules must depend on the species of words which, as we have observed, are commonly called by grammarians, the parts of speech; our next enquiry, therefore, must be, how those species are to be distinguished, or by what rule they are to be distributed into classes.

§ 2. *Of the parts of speech.*

Parts of speech.

Some principles of classification are better than others. It is not sufficient that we comprehend all our notions on a given subject, under certain heads; but we must be prepared to show, why we choose those heads rather than others. If we are right in our notion of pure science, it will guide us to the proper choice, among these various modes of treating the same subject. It will present to us one *idea*, which masters and directs all the others, and will show us how the subordinate ideas proceed from this common root.

Classing of words.

It is, however, necessary first to explain what we mean by different classes of words. Take the following sentence:

Grammar.

> The man that hath no music in himself,
> And is not fill'd with concord of sweet sounds,
> Is fit for treasons—

Here we know that various grammatical writers call the word *the* an article; *man*, *music*, *concord*, and *sounds*, substantives, or nouns substantive; *no*, *sweet*, and *fit*, adjectives, or nouns adjective; *that*, and *himself*, pronouns; *hath* and *is*, verbs; *moved*, a participle; *not*, an adverb; *and*, a conjunction; *in*, *with*, and *for*, prepositions.

Various opinions.

The first question that occurs to us is, whether these classes themselves are all recognised in all languages, and by all grammarians? And a very little experience will show us that they are not. The same thing has happened in Grammar, which has happened in all other sciences. Some authors have divided speech into two parts, some into three, four, and so on to ten or twelve. Others again have made their division depend on the supposed utility of words; others on their variation; others on the external objects to which they refer, and others on the mental operations which they express. On this point, it is worth while to hear what QUINTILIAN says, in the fourth chapter of his first book—"On the number of the parts of speech, there is but little agreement. For the ancients, amongst whom were ARISTOTLE and THEODECTES, laid it down, that there were only verbs and nouns, and combinatives (*convinctiones*) intimating that there was in *verbs* the force of speech, in *nouns* the matter (because what we speak is one thing, and what we speak about is another), and that the union of these was effected by the *combinatives*, which I know most persons call *conjunctions*; but I think the former word answers better to the original Greek συνδεσμος. By degrees the philosophers, and particularly the stoics, augmented the number; and first, they added to the combinative the *article*, then the *preposition*. To the noun they added the *appellative*, then the *pronoun*, and then the *participle*, being of a mixed nature with the verb; and finally to the verb itself, they subjoined the *adverb*. Our (Latin) language does not require articles, and therefore they are scattered among the other parts of speech; but we have added to the others the *interjection*. Some writers of good repute, however, follow the doctrine of the *eight parts of speech*, as ARISTARCHUS, and in our own day PALAMON, who have ranked the *vocable*, or *appellative* under the noun, as one of its species; whilst those who divide it from the noun, make nine parts. Again there are others who divide the vocable from the appellative, calling by the former name all bodies distinguishable by sight and touch, as *a bed*, or *a house*, and by the latter what is not distinguishable by one or both these means, as the *wind*, *heaven*, *virtue*, *God*. These last mentioned authors, too, add what they call *asseverations*, as (the Latin) *Heu!* and *attractations*, as (the Latin) *fasceatim*; but these distinctions I cannot approve. As to the question whether or not the vocable or appellative should be called προσηγορία, and ranked under the noun, as it is a matter of little moment, I leave it to the free judgment of my readers."

Although Quintilian, who only touches on Grammar incidentally, speaks of Aristotle as maintaining that there were three parts of speech, yet VARRO says truly that Aristotle asserted there were *two* parts of speech, the verb and the noun. In fact, Aristotle, in his book περὶ ἑρμηνείας, treats of these two alone; considering that of them is made a perfect sentence, as "Socrates philosophises:" and therefore PRISCIAN says "the parts of speech are, according to the logicians, *two*, viz. the noun and the verb, because these alone, conjoined by their own force, make up a full speech, or sentence; but they called the other parts *syncategorematics*, or consignificants. Priscian, himself, however, maintained that there were eight parts of speech; and he seems to have been implicitly followed for many centuries; but, though it is of little consequence whether we give the name of parts to particular divisions or subdivisions, it is of great importance to determine on what principle speech should be divided and subdivided.

Chap. I.

Recurring, therefore, to the sentence above quoted from Shakspeare, we will enquire how the words can be grammatically distinguished: and many various modes will readily present themselves:

Variable and invariable.

1. It may be observed that some of the words admit of variation, and others do not. Thus *man* may be varied into *man's* and *men*: *hath* into *have*, *hast*, *had*, and *having*: *sweet* into *sweeter*, and *sweetest*, &c. and, on the contrary, the words *the*, *in*, *and*, *not*, &c. cannot be altered. But this is manifestly not an essential distinction, since it does not take place in the same manner in all languages; but, on the contrary, every language is distinguished, more or less, from every other, by peculiar modes of varying its words. Thus the Greek, Hebrew, Sanscrit, and Arabic languages, have a variation in some or all of their nouns to mark the *dual* number, which is unknown to most other tongues. So the Greeks and Romans varied their adjectives by the triple change of gender, number, and case; whereas the English never vary them in any of those ways. If then the distinction of variable and invariable will not answer our purpose, let us look for some one that is more essential.

Affective and discursive.

2. Having considered in the former instance the sound of the word, we shall now take a distinction which arises from its signification. Thus M. BEAUZEE divides the parts of speech into two classes, of which he says "the first includes the natural signs of *sentiment*, the other the arbitrary signs of *ideas*: the former constitute the language of the heart, and may be called *affective*; the latter belong to the language of the understanding, and are *discursive*." It is manifest that the principle of this distinction is universal, because all men must be influenced by sentiment and understanding, and all languages must find some means of distinguishing these different faculties in language. But the question is, whether this distinction is sufficient to account for the different classes of words: and most assuredly it is not; for though there are some words which express only the objects of sentiment, and others which express only the objects of knowledge, yet there are many which express both together, and many which directly express neither. Nor is it always sufficient to use a word of one class in order to convey either an emotion or a truth. These circumstances more frequently depend upon the combination, than upon the distinction of words.

Object and manner.

3. Let us now come to a third distinction, that of the PORT ROYAL Grammarians, who say "the greatest distinction of what passes in our minds, is to say that we may consider in it the *objects* of our thoughts, and

Grammar. the form or *manner* of our thoughts, of which latter the principal is reasoning or judging; but to this must be added the other movements of the soul, as desire, command, interrogation, &c." This again, is a distinction universally applicable to language in point of signification: and when we come to apply it to existing languages, it will be found sufficiently accurate.

Necessary words and abbreviations. 4. But it has been observed, that this may be done with more or less facility and dispatch; and that some words are absolutely *necessary* for the communication of thought, whilst others may be considered as *abbreviations*, in order to make the communication more rapid and easy; as a sledge may have been first constructed to draw along heavy goods, and may have been afterwards placed on wheels to add celerity to the motion. Such is the theory of Mr. HORNE TOOKE, and so far as we are here considering it, that theory is perfectly just.

Principal and accessory. 5. The words which are necessary for communicating the thought in any given sentence with the utmost simplicity, may well be called *principals*, and those which only help to make out the thought more fully and distinctly may be called *accessories*. These are the terms employed by Mr. HARRIS, and consequently his theory so far coincides with that of Mr. Tooke. Mr. Harris, however, adds, that the principals are significant by themselves, and the accessories significant by relation: whereas, Mr. Tooke says that the necessary words are signs of things, and the abbreviations are signs of necessary words. We shall hereafter have occasion to enter more at large into this part of his doctrine. It is sufficient at present for us to observe, that that doctrine does not interfere with the fundamental principle of classification in all Grammars which deserve the name; that is to say, of all which have proceeded on the signification of words, and not merely on their sound.

Noun and verb. Now, that principle, in whatever terms it is cloathed or expressed, is, that the *noun* and the *verb* are the primary parts of speech; and that without them, neither can a truth be enunciated, nor a passion be expressed, in combination with its object. This principle is the most ancient. It boasts the support of the greatest of philosophers, of him, whom for many ages, even Christianity recognised by the title of "the divine," as approaching the nearest of all heathens to the divine light of the Gospel. PLATO, in his Dialogue called the Sophist, having most profoundly and unanswerably argued on the nature of truth, thus speaks of language: "We have in language two kinds of manifestation respecting existence, the one called *nouns*, the other *verbs*. We call the manifestation of action a verb; but that sign of speech which is imposed on the agent himself a noun. Therefore, of nouns alone, uttered in any order, no sentence (or rational speech) can be composed, neither can it be composed of verbs without nouns; thus "goes," "runs," "sleeps," and such other words as signify action, even though they should all be repeated in succession, would not make up a sentence. And again, if any one should say "lion," "stag," "horse," or should repeat the names of all the things which do the actions before-mentioned, still no sentence would be made up by all this enumeration; for, neither in the one way, nor in the other, do the words spoken manifest any real action, or inaction, or declare that any thing exists, or does not exist, until the verbs are mixed with the nouns. Then, at length, the very first interweaving of them together, makes a sentence, however short; thus, if any one should say, "man learns," you would pronounce at once that it was a sentence, though as short a one as possible; for then at last, something is declared which either exists, or has been done, or is doing, or will be done; and the speaker does not merely name things, but limits, and marks out their existence, by interweaving verbs with nouns, and then, at last, we say "he discourses, and does not merely recite words." The only great name that for nearly 2000 years was ever brought into competition with Plato's, was that of his scholar

Aristotle. ARISTOTLE; but Aristotle also, as we have already seen, agreed with Plato, in stating the noun and the verb as the two primary parts of speech, and indeed the only ones necessary to be considered in the formation of a simple sentence. In other parts of his works, looking at the composition of language in a more general point of view, he enumerated *three* parts, viz. the noun, the verb, and the connective; and, finally, in his treatise on Poetry, c. xx. he enumerates two parts of speech as significant, viz. the noun and verb; and two as non-significant, viz. the article and conjunction.

Previous consideration. The doctrine that the noun and verb are the primary parts of speech, is incontestible. APOLLONIUS, the grammarian, calls them the most animated; and all grammarians concede to them, at least, the superiority over all the other parts of speech, in whatever manner they choose to account for their preference. We are not, however, inclined to adopt this, as the *first* step in our methodical arrangement; because we conceive that by approaching to the most general *idea* of speech, we shall find it easier to reconcile the apparent differences, and to correct the real errors of the different grammatical systems. We have already defined speech to be the language of articulate sounds; and language to be any intentional mode of communicating the mind. Our most general idea of speech, therefore, is, that it is any intentional mode of communicating the mind by articulate sounds. Now keeping in view this idea, let us see how it will apply to the doctrines of those grammarians whom we have already mentioned, in respect to the mode of distributing speech into its parts.

Combination of theories. When writers of any eminence advance a particular doctrine, we may generally be persuaded, that it is not wholly destitute of foundation; although, from the natural partiality that men have for their own thoughts, they may probably rank such doctrines higher than they deserve. All the different theories that we have here noticed are true, to a certain degree, and, by combining them together, we may perhaps attain to the best and clearest view of our subject.

In the method which we are disposed to pursue, we should say, that the principle of M. BEAUZEE first merits attention. There are words which are simply *affective*, namely, *interjections*, which express no operation of reason whatever; all other words are *discursive*, inasmuch as they may be employed in expressing the operations of reason. Again, all words which are employed in reasoning must be considered, with reference to the sentence in which they are so employed, either as *principals* or as *accessories*; we say with reference to the sentence in which they are employed; for it is here that a great error is often committed by

grammarians. They seem not to advert to the circumstance that speech is an expression of the mind, when actually engaged in some operation. They treat words as if they were corporeal substances, cast in a mould, for use. Now, the very same words, that are principals in one sentence, may become accessories in the next. The principal words in a sentence are of course *necessary* for the communication of thought; and thus we combine the principle of HARRIS with that of TOOKE. We cannot, however, communicate what we do not comprehend; and in order to comprehend any thought, we must first conceive an object, and then either assert something respecting it, or express some emotion in connection with it. Here, therefore, the theory of the PORT ROYAL grammarians properly finds its place; for they comprehend alike the assertion of a truth and the expression of an emotion under the words "the manner of thinking." With respect to the v[illegible], who divide words, according as they are susceptible of variation, or the contrary, although it is true that such a quality exists in the words of most languages, yet we have shown that it cannot be taken into consideration in treating of Universal Grammar, being a circumstance merely contingent and accidental.

The result, therefore, of the preceding remarks, is, that we consider speech as intended to communicate either passion or reason; when it communicates mere passion, without any precise object, it supplies the part of speech called the *interjection*; when it communicates passion and at the same time indicates an object, it indirectly reasons, and therefore requires the same parts of speech, which are required in reasoning. Now the parts of speech required in reasoning are either such as are necessary to form a simple sentence, or such as serve for accessories, in order to give complexity to sentences; but a simple sentence cannot be formed without a noun and a verb, and is immediately formed by putting a noun and a verb together. The noun and the verb then are the necessary parts of speech, the former serving to name the conception, the latter to supply in reasoning the assertion, or in passion the emotion. There is, however, one observation very important to be made with respect to the necessary parts of speech, namely, that every verb involves a noun; that is to say, we cannot assert a truth, or express an emotion, which truth or emotion may not be considered by the mind as a conception. Thus, if we say "God exists," we excite in the mind the two distinct conceptions of "God" and "Existence," as much as if we said, "God is in existence:" and so if we say "Come Antony," we excite the conception of coming, as well as of Antony; but the difference is, that the words "come" and "exists" are not presented to the hearer *as* mere objects of thought, but as modes of thinking about other objects, viz. "Antony" and "God."

Thus have we fixed the principle on which the noun and the verb are to be reckoned among the parts of speech; and this principle will readily enable us to clear up several difficulties which occur in the subdivision of these classes.

Divisions of grammarians.

First, with respect to nouns; the old grammarians in general divided them into nouns substantive and nouns adjective; but R. Johnson, Harris, Lowth, and others, consider the *substantive* alone as a noun; and Harris ranks the adjective with the verb, under the common name of *attributive*; whilst Tooke, in consequence of his singular notions respecting the mind, asserts that the adjective is literally and truly a substantive. This author also contends that those words which "compose the bulk of every language," and are commonly, though improperly, called abstract nouns, are not even necessary parts of speech, but abbreviations, or signs of other words. The *pronoun* was originally considered as a noun, and afterwards, though treated separately, was still deemed a secondary sort of noun; but Harris distinguishes, in this respect, the pronoun personal from the others, and considers only the former as a noun, ranking the latter, together with the article, among the accessory parts of speech. Lastly, the *participle*, which was originally so called, because it was thought to partake of the nature of a noun and of a verb (to be a noun when it formed the subject of a proposition, and a verb when it formed the predicate), is wholly excluded by Harris from the class of nouns, and referred to that of attributives: whilst Tooke (who, however, does not explain what he means by a verb), calls it the verb adjectived.

Nouns.

Our principle, on the other hand, will bring us back very nearly to the ancient distinction of nouns. For a noun, in our view, is only the *name* of a conception, or object of thought; thus the "sun," a "horse," or a "man," is an object of thought, and as such may have a name, which name is a noun. So "brightness," "strength," "wisdom," "thinking," "moving," "shining," are objects of thought, and have names, and these names also are nouns.

Substantive.

These nouns are considered substantively, when in reasoning upon them, or asserting any thing of them, we make them the subject of the assertion, and consider them as that in which something else exists. Thus "man," as a thought, has its own peculiar relations to other thoughts; so have "wisdom," "thinking," "strength," "moving," "brightness," and "shining," and all these, so considered, become nouns substantive.

Adjective.

But we may also contemplate each of these conceptions only as existing in another object, as thinking or wisdom in a man; strength or moving in a horse; brightness or shining in the sun; and this we say is employing the same noun *adjectively*; because we are forced to adjoin it to the substantive, in which alone we contemplate it as existing. When we say, "a wise man," or a "thinking man," we contemplate wisdom or thought only as existing in that man; so when we say "the shining sun," or "the bright sun," "the strong horse," or "the moving horse," we speak of the conceptions of shining or brightness, motion or rest, only as modes of qualifying our use of the conceptions sun and horse: and when we do this, the name of the qualifying conception, is properly called a *noun adjective*.

Pronoun.

The substantive conceptions, which the mind forms, either represent the person communicating the thought, the person to whom it is communicated, or some other person or thing. Hence the mind forms three classes of conceptions; but a name being given to each of these classes, stands for the class, a noun for many nouns; and hence it is called the *pronoun*; and upon the pronominal substantives depend the pronominal adjectives. The *article*, which has been often treated as a pronoun, represents the exercise of that faculty of the mind by which we distinguish the universal con-

Grammar. ception from the particular. It seems therefore, to be improperly ranked among the principal or necessary parts of speech.

Participle. The *participle* is clearly a noun adjective, which includes the idea of action, and consequently of time; for the "bright sun," and the "shining sun," differ but little in signification, except, that in the latter, the sun is considered as producing brightness by its own act. And if the phrase be varied, and an assertion be introduced, the assertive power depends not at all on the participle, but on the verb, which must necessarily be added, as the sun "*is* bright," the sun "*is* shining."

Verbs. With respect to the other principal or necessary part of speech, the *verb*, it is only material now to remark, that those who confound it with the adjective and the participle, overlook its peculiar function, which is that of asserting; as the function of the noun, is that of naming. As to the separate classes of verbs, the verb substantive, the transitive, the active, the passive, &c. since these have not been treated of by grammarians as separate parts of speech, it will not be necessary to notice them in this part of our work.

Accessory parts. But the great dispute, especially in modern times, has been with respect to the accessory parts of speech, the nature of which has been illustrated by a variety of similes. They have been said to be like stones in the summit or curva of an arch, or like tha springs of a vehicle, or like the flag of a ship, or the hair of a man, or like the nails and cement uniting the wood and stones of an edifice; and hence soma persons have contended that they are only significant by relation; soma that they are not parts of speech; and some that they are not even words but particles.—Thus APULEIUS says, "they ara no more to be considered as parts of speech than tha flag is to be considered a part of the ship, or the hair a part of tha man; or, at least, in the compacting and fitting together of a sentence, they only perform the office of nails, or pitch, or mortar." PRISCIAN, however, an acute and intelligent grammarian, observes, that if these words are not to be considered as parts of speech because they serve to connect together others which are parts, we must say that the muscles and sinews of a man are no parts of a man; and he, therefore, concludes by declaring his opinion, that the noun and verb are the principal and chief parts of speech, but that these others are the subordinate and appendant parts.

Simple sentence. The decision of this and similar questions will be easily made, if we only advert to the mental operations which these accessory words express; and in order to explain this, we must first ask, what words in a sentenca are accessories. This question again is answered by referring to what we hava said of sentences. In a simple sentence, all the words are principals. Thus "Man is fit," contains two nouns, which are the names of two conceptions, viz. "man" and "fitness," and the assertion of their coincidence by the verb "is;" and moreover, since the conception of fitness is regarded as existing not separately but in the other conception, man, the word "fit" is an adjective and "man" is a substantive. The same would be the case if the place of the noun "man" were supplied by the pronoun "*he*," and that of tha adjective "fit," by the participle *suited*.

Complicated sentence. Such is the case when the sentence is simple; but we are next to consider how a simple sentence is rendered complex; and this is no otherwise done than by engrafting on it other sentences; but in these latter the conceptions only are expressed, and the assertive part is assumed or understood. Thus, if referring to the passage before quoted from Shakespeare, we say "Man is fit," we may be asked, What is the fitness or aptitude of which you are speaking? The answer must be "it is *treasonable*." And again if we are asked, What is the man of whom you make these assertions? We may say "he is *unmusical*; and suppressing the assertions in the two secondary sentences we may form of the whole one complex sentence, thus, "unmusical men possess treasonable aptitudes."

Further complication. In this first process of complication we find only words capable of being used as principals, viz. nouns, substantive or adjective; pronouns, participles, and verbs; but suppose we again resolve these into their constituent conceptions and assertions; suppose we ask what do you mean when you speak of a treasonable fitness, or aptitude? We may answer, we mean that the fitness looks to treason; treason is *before* the fitness (as its mark or object), the fitness is *for* treason. Here it is plain that the word "*for*" involves the conception of *foreness* (or objectiveness), and applies that conception to the other conception of treason: but it does so still more rapidly and obscurely, than in the cases before supposed; and hence it is that in this second process of complication we meet with words which are no longer thought significant, and therefore no longer called nouns or verbs, but articles, adverbs, conjunctions, and prepositions; and these words are the more numerous and frequent of occurrence, in proportion as sentences are rendered more complex by subdividing the primary truth into many others. Thus, as the word "treasonable" may be supplied by the words "for treasons," so the word "unmusical" may be supplied first by tha words "hath no music in himself," and secondly, by the words "is not moved with concord of sweet sounds;" both which, and many similar modes of speech, consist of various aggregations of sentences in which the subordinate assertions ara assumed by the mind in the manner already shown.

Change of signification. The words, which, by use, come to be most frequently employed in any particular language for these secondary purposes, often lose their primary signification, and perhaps undergo some little change of sound; from which circumstances a great dispute has arisen among grammarians whether they are *significant* words or not. Thus the preposition *for*, which, as we have shown, conveys the conception of *foreness*, is nothing more than the word *fore* in *foremost*, *before*, *fore and aft*, and the like words and phrases; but by use, and by the slight change which it has undergone, it has come to lose the property of forming a principal part in a sentence. These circumstances, however, it must be observed, are merely accidental; they may happen to the sama conception in one language and not in another; and, therefore, they cannot form a just scientific criterion between the parts of speech; but on the other hand, those parts may, and must, be distinguished by the different operations of mind which they express; and as we have seen that the operations, expressed by tha articles, adverbs, conjunctions, and prepositions, are clearly distinguishable from those expressed by the nouns, pronouns, verbs, and participles, inasmuch as they relate to a subordinate step in

 the analysis of thought; so there can be no great difficulty or impropriety in calling them accessories, with reference to the others, which we call principals.

Etymology of accessory words.

From what we have said, it will not appear strange, that the accessory words should be for the most part traceable to their origin as principals; that is to say, that the parts of speech last mentioned should in general be found to have been once used (with little or no difference of sound) as nouns and verbs. It has been supposed that this was a *new discovery* of Mr. HORNE TOOKE's, and in many parts of his work he seems to have entertained that notion himself; how justly may be seen from the very title of a little treatise, by C. KOERBER, printed at Jena, in 1712, and called "Lexicon Porticularum Ebræarum, vel potius Nominum & Verborum, vulgò pro particulis habitorum." This writer says, in his preface, that his tutor *Danzius* taught that "most, if not all the separate particles, were in their own nature nouns;" that this was indeed a "new and unheard of hypothesis;" but that on investigation the reader would find reason to conclude universally (in respect to the Hebrew language at least) that "all the separate particles are either nouns or verbs." His words are these: "*Particulæ separatæ si non omnes certè pleræsque suâ naturâ sunt nomina*"—"*hanc thesin hactenus novam & inauditam;*" and again, "*Omnes omninò Ebræorum particulas separatas aut nomina esse aut verba.*"

Koerber.

Koerber illustrates his position by comparing the Hebrew particles with radical words, both in that and the cognate languages, particularly in the Arabic. Among the instances which he gives, are the following, viz.

Juxta, near, being the same as *Latus*, side.
Præter, beside or beyond { *Defectus*, deficiency. *Terminus*, boundary.
Inter, between . . . . *Distinctus*, divided.
Post, after . . . . *Tergum*, the back.
Quoque, also . . . . *Adde*, add.
Vel, or . . . . *Elige*, choose.

He even explains the interjection *Lo!* as being identical with the pronoun of the third person; and suggests that the *termination of the accusative case* is a noun, signifying *object*.

Tooke.

Whether or not Mr. Tooke ever saw this little treatise of Koerber's, or any other of similar import, is immaterial. It may, probably, have been a *bonâ fide* discovery, so far as regarded his own reflections, though not one that was entirely new to the world. But he seems to us to have connected with it a very material error in Grammar, namely, that because a word was once a noun, it always remained so, and consequently that adverbs, conjunctions, &c. expressed no new or different operation of the mind, and were not to be considered as separate parts of speech, so far at least as related to their signification. Had Mr. Tooke been as well acquainted with the writings of Plato, as he was with those of the old English and Saxon authors, which he studied with such meritorious industry, he would hardly have fallen into this error; for he would have perceived that speech received its forms from the mind; he would have acknowledged with that great philosopher that "thought and speech are the same; only the internal and silent discourse of the mind with herself, is called by us Διάνοια, thought, or cogitation; but the effusion of the mind, through the lips, in articulate sound, is called Λόγος, or rational speech." It is, therefore, the mind that shapes the sentence into its principal parts and accessories: it is the mind which distributes alike the principal and the accessory parts into subdivisions, according as they are necessary to its own distinguishable operations.

Ancients.

Those ancient grammarians who acknowledged only three parts of speech, viz. the noun, verb, and conjunction, ranked some of the parts which we here call accessories under the principal parts. Thus Apollonius of Alexandria, and Priscian, rank the *adverb* under the verb, and with them agrees Harris, who calls the adverb a secondary attributive; but Alexander Aphrodisiensis, who is followed by Boethius, observes, that it is sometimes more properly referred to the class of nouns; and so Tooke asserts some adverbs to be nouns and some verbs. The *preposition* which was referred by Dionysius and Priscian to the *conjunctions*, is on a similar principle included by Harris with the common conjunction in the class of connectives; and Tooke distributes both prepositions and conjunctions (in most instances rightly, as far as their etymology is concerned) among the verbs and nouns. Lastly, the *article* appears to have most disturbed the grammarians in their arrangements; for Fabius says it was first reckoned among conjunctions; and we have seen that, when Aristotle divided speech into four parts, he separated the article from the conjunction, making of it a class apart from the three other parts of speech. Vossius inclines to rank it among nouns, like a pronoun; but Harris having divided the accessory parts of speech into definitives and connectives, makes the article a branch of the former. Tooke says that our article *the* is the imperative mood of the Anglo-Saxon verb *thean*, to take! Lastly, Scaliger says, the article does not exist in Latin, is superfluous in Greek, and is, in French, the idle instrument of a chattering people.

New principle proposed.

Since in this diversity of opinions, we find no common view of any principle which connects itself with the idea of language before laid down, we are compelled to seek a new division. We say, therefore, that the accessory parts of speech represent operations of the mind, which from their frequent recurrence have become habitual, and from their absolute necessity in modifying other thoughts, must be found more or less in all languages. It is true, that these operations are not performed by all men with the same distinctness, and therefore do not exist among all nations in the same degree of perfection; and lastly, it is true, that in some languages they are expressed by separate words, and in other languages by different inflections of the same word. Hence a close connection is found between the prepositions of one language, and the cases of another; between the auxiliary verbs of one language, and the tenses of another. Hence too, the comparison of adjectives, always effected in Latin by different terminations, is sometimes effected in English by adverbs prefixed to the adjective. In short, numberless illustrations of this remark will easily occur to the recollection of any person at all acquainted with different languages, ancient or modern, barbarous or refined.

Article.

Of the operations, that we have described, one, and that not the least essential, consists in determining whether we view any given conception as an universal, or a particular; and if as a particular, whether as a

certain, or an uncertain one; and if certain, whether of one known class, or another known class; and so furth. Thus there is a certain conception of the mind expressed by the word "man;" but if we employ that expression for the purpose of communicating the conception, it is necessary that those who hear us should know with what degree of particularity it is to be applied; for it would be one thing to say, that, according to our idea of human nature, man is universally benevolent; and another to say, that men in general are so; and a third to say that any individual man, under given circumstances, is so; and a fourth to say, that this or that man is so. Of these different degrees of limitation some may be marked by separate words; and of those words, some may express a conception so distinct and self evident, as to be capable of forming a simple sentence, in which case we should reckon them as pronominal adjectives, among the principal parts of speech; as when we say, "this is good," "that is bad," the words *this* and *that*, are pronominal adjectives. But since we cannot say "the is good," or "a is good," and since these words *the* and *a*, serve no other purpose but to define and particularize some other conception, and do not even perform this function completely, without reference to some further conceptions, we may, in those languages in which they exist, reckon them as a separate part of speech, under the name of the *article*.

Preposition.

The word *preposition* is badly chosen, as Vossius observes, from its use (and even that use not without exception) in the Latin language; nevertheless, it has become sufficiently intelligible to signify a class of words which describe another sort of mental operation. When one object is placed in a certain relation to another object, whether it be a relation of time, of space, of instrumentality, causation, or the like, the conception of that relation serves as a bond to unite them in the secondary parts of a sentence. That expression may form part of a word, as "to *over*leap a fence;" or it may constitute a separate word, as "to leap *over* a fence;" and in the latter instance the word *over* is called a preposition, which we therefore do not hesitate to rank as a separate part of speech.

Conjunction.

As the preposition connects conceptions, the *conjunction* connects assertions; or, as it is commonly expressed, the preposition joins nouns, the conjunction verbs, and consequently sentences. By connecting, in both instances, we mean showing the relations, whether of agreement or disagreement; and these also may be expressed either in the form of the verb, or by means of a separate particle: of which a sentence before quoted affords an illustration—

> Duller should'st thou be than the fat weed,
> *Wouldst* thou not stir in this;—

where, if rendered into the more common expression, "*if* thou wouldst not stir," the relation between stirring in the cause, and being dull, would be expressed by the word *if*, to which we therefore give the name of a conjunction. Hence, it appears, that the conjunction may not improperly be reckoned a distinct part of speech, since it expresses a distinct operation of the mind.

Adverb.

More doubt may perhaps exist as to the *adverb*, a class in which grammarians have often confounded words of very various effect and import, such as interjections and conjunctions. Neither do we, in this instance, any more than in those of the participle and preposition, pay much regard to the etymology of the word adverb; but we take it as a word in common use, and applying to a large class of words which describe operations of the mind very distinguishable from those which we have already considered. The adverb either expresses a conception which serves to modify another conception of quality or action; or else it expresses a conception of time, place, or the like by which the assertion itself is modified; in either case it serves to modify by its own force, and not, like the preposition, as an intermediate bond between other conceptions.

Thus have we distributed words into various classes according to the following table:—

WORDS
- 1. used in enunciative sentences:
  - 1. *principal* words,
    - 1. the *noun*, or name of a conception,
      - 1. primarily,
        - 1. if expressive of substance (the *substantive*),
        - 2. if expressive of quality,
          - 1. without action (the *adjective*),
          - 2. with action (the *participle*),
      - 2. secondarily (the *pronoun*),
    - 2. the *verb*, or expression of an assertion,
  - 2. *accessory* words,
    - 1. defining the extent of a conception as universal or particular (the *article*),
    - 2. expressing the relation of one substantive to another (the *preposition*),
    - 3. Connecting one assertion with another (the *conjunction*),
    - 4. Modifying either a conception of quality or action, or else an assertion (the *adverb*).
- 2. used either in passionate sentences, or as separate expressions of passion (the *interjection*).

The mental operations which these various classes of words represent, are obviously distinct, but it by no means follows from thence that the words themselves are so; that a word which has been employed as a substantive may not also be employed as a conjunction; or that the very sound by which we have expressed an assertion may not be used as a preposition or an interjection. In short, there is no reason why one word should not successively travel through all the different classes which we have here stated; for we must observe, that words do not communicate thought by their separate power and effect only; but infinitely more so by their connection: and consequently the mode of connecting the signs, and not the signs themselves,

determines their place in any given class. The first exercise of the reasoning power, we have seen, is conception; and of all our mental operations, whether relative to the external world, or to the laws of mind itself, conceptions may be formed; and to all the conceptions which we form, names may be given; and those names are nouns; and therefore it is not surprising that all other words, except interjections, should be historically traceable to nouns as their origin; and since reason and passion are so complicated in man, we must not wonder that a connection is often to be found between interjections and nouns; or that the Latin *væ*, probably pronounced in ancient times *wae*, should be the Scottish substantive *wae*, and our *woe*. Surely this affords no proof, or shadow of a proof, that the different uses of the same, or different words, do not depend on the different exercise of the mental faculties; but, on the contrary, it absolutely demonstrates the necessity of some mental operation to distinguish between the different meanings, force, and effect of the same sign, as employed on different occasions.

§ 3. *Of nouns.*

Having thus settled the classes of words, we shall attempt to explain them in order: and first we begin with that which, according to all systems, stands first in importance; that is to say, the *noun*.

The noun.

"It is by the nouns," says Cour de Gebelin, "that we designate all the beings which exist. We render them known instantly by these means, as if they were placed before our eyes. Thus, in the most solitary retreat, in the most profound obscurity, we are able to pass in review the universality of beings, to represent to ourselves our parents, our friends, all that we have most dear, all that has struck us, all that may instruct or amuse us; and in pronouncing their names we may reason on them with our associates. We thus keep a register of all that is, and of all that we know; even of those things which we have not seen, but which have been made known to us by means of their relation to other things already known to us. Let us not be astonished, then, that man, who speaks of every thing, who studies every thing, who takes note of every thing, should have given names to all things that exist, to his body and its different parts, to his soul, to his faculties, to that prodigious number of beings which cover the earth or are hid in its bosom, which fill the waters, and move in the air; that he gives names to the mountains, the rivers, the rocks, the woods, the stars, to his dwellings, to his fields, to the fruits on which he feeds, to the instruments of all kinds with which he executes the greatest labours, to all the beings which compose his society, or, that the memory of those illustrious persons who deserve well of mankind by their benefactions, and their talents, is perpetuated by their names from age to age. Man does more. He gives names to objects not in existence, to multitudes of beings, as if they formed but a single individual, and often to the qualities of objects, in order that he may be able to speak of them in the same manner as he does of objects really existing."

Its origin.

This great power of the *noun* is to be attributed solely to that faculty of the mind by which it is formed: and that power we have called conception. Every act of this power produces one thought, presents to our view one object, more or less distinct. We conceive a certain impression to which we give a name, be it "red" or "white," "John" or "Peter," "man" or "woman," "animal or "vegetable," "virtue" or "vice;" or whatsoever else we can distinguish from the mass of continued consciousness which constitutes our being.

We do not name every impression that we receive, or every act that we perform. In truth, we do not name any one separately and distinctly from all others. It would be useless to do so, in a single instance: it would be impossible to do so, in all. But we name what often occurs to us. We have often a sensation of colour; we call it "white:" we have often a feeling of pleasure; we call it "joyous:" we often see an object which affects us with peculiar sentiments of regard or aversion; we call it "father" or "enemy:" we often meditate on thoughts, which appear to us amiable or the reverse: we call them "benevolence" or hatred." In this manner it is that our catalogue of names is formed.

Each of these thoughts or conceptions has its natural and proper limits; but these we do not always very accurately observe. No man confounds "red" with "white," but he confounds "whitish" with "reddish." A boy does not think his hoop square, but he knows not whether it is circular, or elliptical. Thus it is, that men do not agree in their opinions of many things, to which they nevertheless agree in giving some common names; otherwise it would be impossible for them to communicate to each other any thing like the thoughts or feelings which they respectively entertain.

Classification of nouns.

Every noun, then, is the name of a class of similar, or identical thoughts. Let us see how these classes may themselves be classed. "Many grammarians," says Vossius, "and among them some of the highest celebrity, first distribute the noun into *proper* and *appellative*, and then into *substantive* and *adjective*; but erroneously; since even the proper noun is a substantive, inasmuch as it subsists by itself in speech. But let us seek our method from the schools. Our great Stagirite first divides τὸ ὂν (or that which is) into that which subsists by itself, and is therefore called substance, and that which exists in another as in its subject, and is therefore called attribute. Afterwards he proceeds to distinguish substance into primary and secondary, the primary being an individual, the secondary a genus or species. By parity of reason, therefore, we should divide the noun first into that which subsists by itself in speech, and is called *substantive*, and that which needs the addition of a substantive in speech, and is called *adjective*; and afterwards we should distribute the substantive into that which belongs to a single thing, and is called *proper*, and that which comprehends many, and is commonly called *appellative*." It is to be observed, that some ancient writers gave the name of *noun* only to the substantive proper, and that of *vocable* (vocabulum) to the appellative; which latter has been, in modern times, erroneously called an *abstract* noun.

We adopt the distribution of Vossius. We call both substantives and adjectives nouns; for they are both names of conceptions, and they are nothing more. They do not imply any assertion respecting these conceptions; and herein they are clearly distinguished from verbs. It is true that the adjective agrees with the verb in expressing, not substance, but attribute;

and therefore it is, that Harris, and some other grammarians rank these two classes of words together under the title of attributives. We do not deny that this arrangement is so far correct; but we say that it interferes with the *method* which we conceive it most advisable to pursue, as the most direct and scientific. As Vossius justly postpones the consideration of the classes of substantives, to the distinction between substance and attribute; so we postpone the consideration of the assertion of an attribute, to the consideration of those conceptions both of substance and of attribute, which must necessarily precede all assertion. This we conceive to be strictly the order of science. Language is a communication of the mind; the mind, as far as it is capable of communication, consists of thoughts and feelings. Thoughts are formed by the reasoning power. The reasoning power is divided into three faculties, conception, assertion, and deduction; but conception necessarily precedes assertion, because we cannot assert that any thing exists until we know what that thing is.

The noun, then, is the *name* of a conception: indeed the English word *noun* is nothing but a corrupt pronunciation of the French *nom*, which, like the Italian *nome*, was again a corruption of the Latin *nomen*, and this latter was of common origin with the Greek *ονομα*, and answered exactly to our word *name*. It is of consequence to observe, that the proper function of the noun is *to name*, and nothing more; for *red* is as much the name of a certain colour, as *Peter* is the name of a certain man, or *England* of a certain country; and in like manner *virtue* is as much the name of a certain thought, as a *ship* is the name of a certain thing; all these, therefore, and whatever other words serve to name any conception of the mind are nouns.

Substantive and adjective.

These conceptions, as has been repeatedly shown, are either conceptions of substance, or conceptions of attribute. This distinction, however profound it may be, is nevertheless, and, perhaps, for that very reason, so perfectly obvious in practice, that no man, however ignorant, can possibly confound the kinds of conception to which it relates. No man can imagine, that in the phrase "a white horse," the word "white" does not denote a quality belonging to the "horse;" or that in the phrase "glorious victory," the word "glorious" does not denote a quality belonging to victory. No man, when he says "the sun is shining," thinks of the sun as an attribute of shining; but, on the contrary, he considers "shining" to be an energy, or property, or quality, or attribute of the sun.

Not convertible.

It has been contended that "the substantive and adjective are frequently convertible *without the smallest change of meaning*," and in proof of this, it is asserted that we may indifferently say "a perverse nature, or a natural perversity:" now surely, although we would not assert, that the person advancing such an illustration was altogether of "a perverse nature" we might without offence attribute his opinion, on this particular point, to a little "natural perversity." In the one case, the friends of the person in question would understand us to assert, that his whole mind was tainted with the vices of obstinacy and self-willedness, that he wilfully shut his eyes against the truth, and maintained opinions which he knew to be wrong in literature, in philosophy, in politics, and in religion—a description of his character, which would naturally occasion them to take great offence. In the other case, they would understand us to give him credit for such reading and literary acquirements, as might well have corrected what we look upon as an error; and they could hardly take it amiss that we attributed that error, rather to a slight defect, from which the best natures are not wholly exempt, than to gross ignorance, or total want of understanding. So much for the particular expressions quoted as proof that substantives and adjectives may be convertible without the smallest change of meaning: on the other hand, the well known instance of a "chesnut horse," and a "horse chesnut," affords a ludicrous example of a change of meaning produced by such convertibility. The fact is, that in all such instances, the views taken by the mind are different, according as it regards the one conception, or the other as principal; just as the man who is on the eastern side of the street considers the western to be the opposite side; whilst he who is on the western side thinks the same of the eastern. We may speak of a "religious life," or of "vital religion." In the one case, we are considering the conception of "life" in the largest extent, as that which must necessarily form the basis of our assertion, and which may be differently viewed, according as it is put in connection with the secondary conceptions of religion, irreligion, business, pleasure, or the like: in the other case, we take the conception of "religion" as the most comprehensive object of thought, and then limit it by the conception of life, or vitality. It is objected, that this limitation may as regularly be effected by a substantive as by an adjective; and that "man's life," or "the life of man" is exactly equivalent to "human life"; which we by no means deny; but then it must be observed, that the sentence takes a different form, and instead of simple becomes complex; the introduction of the casual termination *s*, in one instance, and of the preposition *of*, in the other, effecting such complexity. Dr. WALLIS, indeed, in his valuable English Grammar, first published in 1653, treats the genitive "man's" as an adjective. He says, "Adjectivum possessivum fit à quovis substantivo (sive singulari, sive plurali) addito *s* —— ut *man's nature*, the nature of man, natura humana vel hominis; *men's nature*, the nature of men, natura humana vel hominum." But no other grammarian has adopted this notion, and the principle on which it rests, would equally go to prove that all the oblique cases of substantives, in all languages, should be considered as adjectives; for Mr. Tooke has justly observed, that these cases cannot stand alone; although he has not noticed that this is owing to the complexity of the sentences in which they are used.

Depend on different views of a conception.

The last mentioned writer contends, that "the adjective is equally and altogether as much the name of a thing, as the noun substantive." If he means by *thing*, a conception of the mind, he is perfectly right; but if he means by thing, what, probably nineteen-twentieths of his readers suppose him to mean, namely, an external substance, such as "a horse," or "a man," or "the globe of the sun," or "a grain of the light dust of the balance," he is as clearly wrong. "Red" and "white," "soft" and "hard," "good" and "bad," "virtuous" and "wicked" do not represent any *such* things as the latter; but they do represent conceptions of the mind, some of which conceptions may be considered as belonging exclusively to

Grammar. external bodies, others as belonging exclusively to mental existence, and others as common to both. Mr. Tooke says, he has "confuted the account given of the adjective by Messrs. de Port Royal," who "make substance and accident the foundation of the difference between substantive and adjective;" but if so, he has confuted an account given not only by Messrs. de Port Royal, but by *every* grammarian who preceded them from the time of Aristotle; and whatever respect we may entertain for the abilities of Mr. Tooke (which in etymology were doubtless great), we must a little hesitate to think that he alone was right, and so many men of extensive reading, deep reflection, and sound judgment, were all wrong. But how has he confuted this doctrine? Why, truly, by showing that when a conception is not regarded as a substance, it may be regarded as an attribute; and when it is not regarded as an attribute, it may be regarded as a substance. —"There is not any accident whatever," says he, "which has not a grammatical substantive for its sign, *when it is not attributed*; nor is there any substance whatever which may not have a grammatical adjective for its sign, *when there is occasion to attribute it*;" which is pretty much like saying, there is not any captain whatever who may not be degraded, and placed in the ranks; nor any private soldier whatever who may not be raised from the ranks and honoured with a captain's commission; *and therefore there is no difference between a captain and a private soldier*. The premises are incontestible: the only fault is, that they have nothing to do with the conclusion. We trust, that in these remarks we shall not be thought to have treated Mr. Tooke with too much freedom. We are cautious not to imitate his example, in calling the opinions which he controverts "paltry jargon," or in saying of him, as he does of the learned and amiable Harris, that he mistook "fustian for philosophy." These expressions *prove* nothing; but it is necessary to come to some settled opinion on a question so essential to the science of Grammar, as whether there is any, and what distinction between substantives and adjectives; and on this point, we trust, we have satisfactorily vindicated the principle laid down by Aristotle, and adopted by all grammarians from his time to that of Mr. Tooke. The noun substantive, then, is the name of a conception, or thought, considered as possessing a substantial, that is, independent existence; the noun adjective is the name of a conception, or thought, considered as a quality, or attribute of the former.

Substantive. Of these the noun substantive, to which some writers, as has been observed, give exclusively the name of noun, first demands attention: and with respect to it we shall notice first what is essential, and secondly what is accidental.

The noun substantive differs essentially in *kind* and in *gradation*; it differs accidentally in *number*, *gender*, and *relation* to other nouns or to verbs.

Kinds. By a difference of *kind*, we mean that the noun substantive sometimes expresses a conception of *corporeal* impression, and sometimes a conception of *mental* reflection. Conceptions of corporeal impression are necessarily *particular*; those of mental reflection are necessarily *universal*. By mental reflection we do not mean the precise recollection of a given particular corporeal impression; for such recollection we consider to fall under the same class as the original impression itself; but we mean the reflection on colour as colour, on goodness as goodness, on man as man, on being as being, and the like; and thus we come to the ancient definition of the noun, given by CHARISIUS and DIOMEDES, viz. *Pars orationis significans rem corporalem, vel incorporalem.*

Chap. I.

Incorporeal. It is objected, that there is no incorporeal thing existing; and as the noun is the name of a thing, there can be no noun naming that which does not exist. We answer first, we have nothing to do in this place with any metaphysical question as to the real existence of objects answering to our mental conceptions. The only point that we are concerned to prove is, that conceptions of mental reflection exist, as well as conceptions of bodily impression; that distinct thoughts exist, as well as distinct things; for if such thoughts or conceptions exist, they must have names, in order to be communicated, and such names will be the very nouns in question. Now it is a curious remark, which is made by Mr. Tooke, in his second volume, and which indeed had occurred to us many years before the publication of that book, that "the terminating *k* or *g* is the only difference between *think* and *thing*." Possibly that learned etymologist would have been inclined to derive "think" from "thing," rather than "thing" from "think;" and possibly, as an etymologist, that is as an historian of language, he might have been right; but as a philosophical grammarian, he would certainly have been wrong; for, let us ask what it is that language communicates? Not things certainly, but thoughts—thoughts of things, or thoughts of thoughts. Now let us take any noun, for instance the word "*house*." We say, that this is the name of a thing; and we will admit that the person using it had seen the thing, before he used the name; but how came it to be a thing in the contemplation of his mind? How came he to form a conception of it? We shall perhaps be answered, because he saw it. But what is seeing? An affection of the nerves of the eye. Now it never happens, when we see any one thing distinctly, that it equally affects all the nerves of the eye. Therefore, when the "house" was first seen, other things were also seen. What was it that distinguished these different affections of the eye into marks, signs, or thoughts of different things? What was it that made the "house," in particular, a *thing*, in the contemplation of the *thinking principle*. Could such an effect have been produced otherwise than by an act of the thinking principle itself? And if this was an act of the thinking principle, then the thought was parent of the thing, so far, at least, as Grammar can have any thing to do with it, namely, as capable of being known to the mind, and communicable by language. Let us pursue this investigation a little further. The word "house" does not signify a thing only seen at one moment of our lives; let us suppose, then, that we do in fact see the *same* house several times; it must necessarily happen, that we see it under very different circumstances. As we approach to, or recede from it, every step makes it affect the eye differently, both as to form and colour. What is it that still makes us consider the cause of these different impressions as one *thing?* Plainly the thinking principle; so that again, and in a second degree, the thought is parent of the thing; and be it observed, that it is not until after this secondary process has been often times repeated that

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we give the thing a name. Now what are the acts of the thinking principle, by which we form the conception of this external object as one thing? The applying to it certain laws of the mind, which enable us to say that it is "square," or "circular," the referring it to certain laws of our physical organization, which enable us to call it "red," or "white," the comparing it with other objects, so as to determine that it is "high or low," the dividing it into its parts and appendages, the "walls" and the "roof," the "doors" and the "windows," and so forth. Thus, we see, that so simple a thing as a house, cannot be conceived by the mind, unless the mind has first conceived the ideas of "square" or "circular," "red" or "white," "high" or "low."

Ideas.

But these *ideas* are no physical part or portion of the corporeal object which we contemplate; they cannot be separated from it by any physical means; they do not belong to it more than to any other object with which they may happen to be associated; they are therefore incorporeal things, thoughts, conceptions of mental reflection. Hence it follows that the conceptions of incorporeal things are, in the order of nature, *prior to the conceptions of corporeal things.* And hence again it follows, that the former are not the result of any *abstraction* from the latter, but on the contrary, the latter are produced by *combining* together the former. An *abstract idea* is therefore a contradictory term; and consequently an *abstract noun* is an expression which we think it improper to adopt; but an idea, or universal conception, is one of the first and most necessary conceptions of the mind, and consequently nouns expressing such conceptions are no less essential to language than names of corporeal objects. They are also equally intelligible. Ask the most ignorant man his opinions of "sweetness and sourness," "black and white," "virtue and vice," and he will reason on them quite as well as he will on any particular things or persons to which these qualities belong. Does any man ever say that the natural consequence of "victory" is "defeat?" Does he argue that there is no distinction between "red" and "green?" Does he contend that "ingratitude" is the most acceptable return for "benevolence?" Assuredly not. These terms stand for certain conceptions in his mind of which he may have a clear or an indistinct consciousness, just as he may have a clear or an indistinct recollection of any action that he has witnessed, or of any person that he has seen; but still these conceptions are parts of the mind communicable by speech; they bear names, and these names are substantives of the class under consideration.

Certainty.

It is again objected, that there can be no truth or certainty in these thoughts, and consequently no precise meaning in the words by which they are signified; inasmuch as there is no external standard to which they can be referred. But where there are no means of referring to the external standard, it is in fact no standard at all. Now this must happen, in the great majority of cases, with regard to corporeal conceptions. No sooner have I seen "Peter" or "John," than be may take his departure. Shall I then say he is a nonentity? And what has truth or certainty to do with external existence any more than with internal? We do, in fact, attain greater certainty, and are more confidently persuaded of truth, in regard to some mental than we possibly can in regard to any corporeal conceptions. Mathematical demonstration is proverbially clear and unquestionable; but mathematical demonstration is carried on solely by means of ideal conceptions. If men were to trust to physical measurement, aided by the very nicest instruments, they might be employed for ages before they could satisfy themselves that the three angles of a right-lined triangle were universally equal to two right angles.

Chap. I.

Distinctness.

Certain it is, that all mental conceptions are of a nature to be apprehended with very different degrees of distinctness by different minds applying to them different degrees of attention; and it is as certain, that the words expressing them are often used loosely and without much regard to their precise and literal signification. Thus, Mr. Locke has written two volumes, principally relating to the word *idea*; yet it would be exceedingly difficult for any person to state what conception Mr. Locke had of that word: and most certainly he had not the conception which any one philosopher before his time ever attached to it. But this is a mere proof of the abuse of terms, which affords no conclusion at all against their use. If John happens to be called Peter by mistake, this circumstance in no degree affects his personal identity.

Union with particular objects.

Again, it must be observed, that when an universal idea is coupled with a particular object, the idea may exist in more or less intensity and vigour, according to the peculiar nature of the object. "Whiteness," may exist in snow more absolutely than in paper, and in paper more than in ivory. "Virtue" may exist in Peter more eminently than in John. A "square" may be more truly formed in one mechanical instrument than in another. To this circumstance is owing the comparison of adjectives; but it does not affect the nature of substantives. Whiteness is not less a substantive, when considered with reference to ivory or paper, than with reference to snow; and the virtue of John, though less than that of Peter, equally belongs to the universal idea of virtue.

Some confusion may perhaps have arisen from the common use of the word "substantive," as applied to the names of mental and corporeal conceptions. By "substance" we are apt to understand only material or bodily substance, that which we can touch and handle, and weigh and measure; but this is merely a verbal difficulty. "Substantive," in the grammatical sense, means that which is considered as having an independent and separate existence, and of which something may be affirmed or denied "substantively," without reference to any other thing as its basis and necessary support. This notion, then, of independent existence is the real characteristic of all those words which are called nouns substantive: it applies equally to ideas and to bodies, to thoughts and to things.

Not mere denominations.

What we here call *ideas* are those mental conceptions, to which that name was originally given—conceptions, which, in the language of Plato, though they run through the particular objects which participate their nature, are separate from each individual—*μίαν ἰδέαν διὰ πολλῶν, ἑνὸς ἑκάστου κειμένου χωρίς.*—Thus, as he elsewhere says, the idea or mental conception of a circle is different from every visible impression of a circle; for the former is perfectly round, whereas each of the latter has some part or other approaching to a straight line. M. Condillac (who calls ideas, abstract

Grammar. ideas says, that "abstract ideas are only denominations." On this notion Mr. Tooke enlarges at great length. His several chapters on abstraction, which abound with much curious etymology, occupy above 400 quarto pages, in the course of which he is pleased to inform his readers, that "heaven and hell" are "merely participles poetically embodied and substantiated." What practical inference is to be drawn from this statement we know not; but we have carefully endeavoured to understand Mr. Tooke's doctrine, as far as it relates to the grammatical explanation of the (so called) abstract nouns. It appears to us, we own, rather obscure, but perhaps it may be more satisfactory to some of our readers; and therefore we shall state it as distinctly as we are able, in the following propositions:

1. The verb is the noun, and something more. (vol. ii. p. 514.)

2. The adjective is the noun, directed to be joined to another noun. (vol. ii. p. 431.)

3. The participle is the verb adjectived, *i. e.* "it has all that the noun adjective has, and for the same reason, viz. for the purpose of adjection." (vol. ii. p. 468.)

4. The abstract nouns "are generally participles or adjectives used *without* any substantive to which they can be joined." (vol ii. p. 17.)

The result of this seems to be, that when an abstract noun is a participle (as Mr. Tooke says *heaven* is) it is *a noun, and something more, converted into a noun directed to be joined to another noun, but used without any noun to which it can be joined.* How far this mode of reasoning goes to show that there are not in the mind any such ideas, as "whiteness," "strength," "virtue," and the like; or that these words do not serve to communicate any thing but conceptions of solid, tangible, corporeal, substance, in an abbreviated form, must be left to the determination of the judicious reader; for our own part, we cannot see that it tends much to enlighten what may be thought obscure, in the works of the ancient grammarians; still less does it appear to us to cast a doubt on those principles, which the ancients have stated with great clearness and precision.

Harris's arrangement. Before we quit this part of our subject, we should notice, that Harris mentions three sorts or kinds of substances; the *natural*, as "man;" the *artificial*, as "house;" and the *abstract*, as "whiteness." The two former fall under the class of corporeal conceptions, and as no grammatical distinction applies to them in practice, we think it unnecessary to enter particularly into their consideration. The last kind are the same which we have mentioned as denoting ideal or mental conceptions, and to which we think the word, abstract, inapplicable for the reasons already stated.

Gradation of conceptions. After considering the different *kinds* of substantives, we come next to what we have called a difference of *gradation*; and by this we mean that order or arrangement of conceptions which classes them as *genera*, *species*, and *individuals*. Although the ancient writers have in general noticed only these three gradations, yet it is easy to see that they may be multiplied indefinitely. Thus we may say, "animal" is a genus, "man" a species, "Alexander" an individual; but we may also divide the species man into white and black, or king and subject, or Greek and barbarian; or we may make "being" the genus, "created being" the first species, "organised being" the second, "animal" the third, and so downwards, in regular subordination, until we come to the individual. Hence it appears, that the only important distinction of substantives, in this respect, is into words expressing *individual* things, and words expressing classes more or less *general*; a distinction answering to the old grammatical terms *nomen* and *vocabulum*, or nomen *proprium* and nomen *appellativum*; or, in the language of our modern Grammars, nouns *proper* and *common*.

Particular things. It has been truly observed by Mr. Locke, that "it is impossible that every particular thing should have a distinct peculiar name; for the signification and use of words depending on that connection which the mind makes between its *internal operations* and the sounds which it uses as signs of them, it is necessary, in the application of names to things, that the mind should have distinct *conceptions* of the things, and retain also the particular name that belongs to every one, with its peculiar appropriation to that conception. But it is beyond the power of human capacity to frame and retain distinct *conceptions* of all the particular things we meet with; every bird and beast men saw, every tree and plant that affected the senses could not find a place in the most capacious understanding. If it be looked on as an instance of a prodigious memory, that some generals have been able to call every soldier in their army by his proper name, we may easily find a reason why men have never attempted to give names to each sheep in their flock, or crow that flies over their heads, much less to call every leaf of plants or grain of sand that came in their way by a peculiar name." So far Mr. Locke, in which quotation we have only taken the liberty to substitute for the word *ideas*, in one place *internal operations*, and in two others *conceptions*. The reasoning, however, is not affected by this change, and it is such reasoning as must carry conviction to every mind. We also agree fully with this writer, that to name every particular thing, if possible, would be useless for the purpose of communicating thought, unless every man could first teach the whole of his own endless vocabulary to every other man with whom he conversed, or for whose information he wrote. And again, supposing even this possible, it would not conduce at all to science; for as Aristotle has said, "of particular things there is neither definition nor demonstration, and consequently no science, since all definition is in its nature universal."

Proper names. Proper names are therefore comparatively few in number. They serve to denote a very few of the immense multitude of particular objects which fall under our observation. Some of these, indeed, obtain a distinguished celebrity within a small circle; they are

—Talked of far and near at home.

But the poet, the orator, or the historian, may raise them to a prouder eminence. He may render them the symbols or representatives of the classes to which they belong. It is thus that "Alexander" becomes the synonyme of a conqueror, and "Cicero" of an orator.

Even proper names, however, have in general been given to individuals from some quality or action not strictly peculiar to them. Hence the old English rhyme alluded to by Verstegan, in relation to the family name of *Smith*:

> Whence cometh Smith, albe he knight or squire,
> But from the Smith, that *smiteth* at the fire?

And thus we see how words of individual import, as

Grammar. well as conceptions of individual existence, arise from *ideas*, that is from thoughts, not particular, but universal. Nevertheless it must be admitted, that the universal idea is soon lost in the particular application. Few people reflect, that *George* originally signified "a husbandman," or that *Charles* and *Andrew* both signified "manly" or "strong," the former from its Gothic, the latter from its Grecian etymology. These names have now come to indicate individuals; and as even thus a single word is not found to answer the purpose sufficiently, we have the *baptismal* name and *surname*; as the Romans had the *prænomen*, the *cognomen*, and the *ognomen*.

Nouns common. Beside these proper names, all other substantives are *common*, or what Mr. Locke calls *general words*, which he truly says are "the inventions and creatures of the understanding." But the process of the understanding, in inventing and forming these words, he has not accurately traced; which, indeed, is not much to be wondered at; since he proceeds solely on an incorrect, or at least an imperfect, maxim of the schoolmen, viz. *Nihil est intellectu, quod non prius fuit in sensu;* the only rational meaning of which is, that we receive, by means of our senses, the materials upon which intellect operates, or by which it is first excited to the perception of truth; so that the maxim, as has been well observed, ought in its perfect state to stand thus: "Nihil est in intellectu, quod non prius fuit in sensu, *præter ipsum intellectum*." Now the mode in which the understanding proceeds is easily to be discovered from the general *aim* and object of its process, which is to acquire some knowledge that may be useful, not only on one occasion, but on all similar occasions; to know some truth which may not only apply to Peter or John, but to all persons who resemble Peter or John; but this cannot be done, unless I have a common word which implies that resemblance; and the persons in question cannot resemble each other but by relation to some common conception, which does not necessarily belong to any one of them more than to any other. That common conception therefore, supplies the class-word, which renders the truth common. Thus Peter, James, and Andrew may be *slaves;* the conception of *slavery*, therefore, is common to them all, and whatever is universally true of it, is true not only with relation to Peter, James, and Andrew, but to all others who are, or have been, or may be, in the state of life expressed by the word slave. Again, a slave and a free citizen agree in this, that they are *subjects;* a subject and a sovereign in this, that they are *men;* a man and a beast in this, that they are *animals*. Now all these conceptions, to wit, slavery, subjection, human nature, and animal nature, are so many mental conceptions, or *ideas*, and they are regularly subordinated, one to another, in a certain gradation, according as they are viewed by the mind; which view is determined, not by any accidental impression received from the senses, but, on the contrary, by the general truth, of which the understanding is in search. Thus, if I am in search of some truth relative to the state of slavery: I may consider the conception of *slave* as a genus, and divide it into the species of domestic, political, absolute, limited, and the like; or if I wish to reason on animal nature, I may regard *animal* as the genus, and man, beast, bird, fish, &c. as species. In like manner, I may consider an *angle* as a genus, and the acute, the right, and the obtuse angles as species. Chap. I.

Their use in reasoning. The nature and effect of these genera and species may be thus explained: all truth, which is not intuitive, must be discovered by reasoning; but reasoning is reducible, in all cases, to the syllogistic form. Now a syllogism is a combination of propositions: and a proposition asserts either the agreement of a substance with its attribute, or of a genus with its species. The subject of the proposition is one conception, and the predicate is another. Each of these may be represented by a noun substantive; but one of them (if not both) is necessarily *common;* for the assertion that one proper noun is another, *e. g.* that "John is William," is no assertion at all, for any purposes of reasoning.

Genus and species. Omitting, for the present, the consideration of those propositions, which assert the agreement of the substance with its attribute, let us consider those which assert the agreement of a genus with its species, as "that man is an animal," "that an isosceles is a triangle," or the like. If the conception "animal" includes the conception "man," the proposition "man is an animal" is true; and in like manner, if the conception "triangle" includes the conception "isosceles," then the proposition "an isosceles is a triangle" is true. But nobody who understands these conceptions can doubt the truth of the propositions. Why? Because such is the nature of the *idea* "animal," that it includes the idea "man;" that is to say, it not only applies to all the men that we ever have known, but to all that we ever can know, and to many other conceptions besides; and in the same manner we may reason concerning the ideas of "triangle" and "isosceles" respectively.

The genus, therefore, is an *idea* including the species; not as a day includes an hour, or as a mile includes an inch, that is to say, as a given measurable part or portion, but simply as being of more comprehensive application, and therefore embracing all the particulars which the other embraces, and many more. Thus, let us give what definition we please of the idea "man," we shall find that the idea "animal" includes it, and something more. If "man," therefore, be considered as a species, "animal" will be a genus, or conception of a higher order; and it is simply on the principle of one conception including or not including another, that the whole doctrine of syllogistic reasoning depends.

Individual. From what has here been said, it is manifest, that the distinction of genus, species, and individual, is properly logical. The two first classes, however, are necessarily expressed by the nouns which we have called common: whilst the individual may either be expressed by a proper name or by a common noun individualised (as will be hereafter shown) by the help of an article or pronoun. With respect to the individual also, we have to observe that it is not necessarily indivisible; but on the contrary there is a class of nouns called *nouns of multitude*, each of which, though it represents a number of beings definite or indefinite, still represents them as one thing; of this sort are the words "an army," "a regiment," "a troop," "a nation," "a crowd," "a flock." Those writers who have not well comprehended the distinction of genus and species, have sometimes explained the words representing them as mere nouns of multitude, that is to say, "as representatives of many particular things," instead of being

representatives of an idea common to those particular things.

Thus have we shown that the noun, according to its essential distinctions, signifies a conception corporeal or mental, and that it signifies it by a name proper or common: and such was also the doctrine of the ancients; for the same grammarians who defined the noun as "*pars orationis significans rem corporalem vel incorporalem*," add to that definition "*proprié, communiter.*"

But to these essential distinctions are to be added the accidental ones, of which we have next to speak, viz. number, gender, and relation, or case.

Number.

Whatever is accidental may, or may not, be viewed in connection with that which is essential. Thus the conceptions or ideas of *number*, may or may not be viewed in connection with other conceptions as that of "man," or "whiteness," or "sun," or "star;" and if viewed in connection with any one of these, the complex conception may be expressed by a single word, or by two words, as happens in regard to other combinations of ideas; thus as "saint" is a single word, including the conceptions expressed by the two words, "holy" and "man," so the word "horses" includes the conceptions expressed by the words "horse" and "number."

Whence derived.

In order to understand when the conceptions of number can, and when they cannot, be added to other conceptions, we must consider what the former are. For this purpose we cannot, perhaps, refer our readers to a more satisfactory or better authority than Plato's *Epinomis*, sometimes called the Thirteenth Book on Laws; but the whole passage is too long to be extracted, and we should do it injustice were we to exhibit it in an imperfect state. Suffice it to say, that Plato agrees with Mr. Locke in asserting, that "number is the simplest and most universal idea," for *unity* itself is in this sense the origin of all our ideas of number. But the latter philosopher is by no means correct in saying that "its modes are made by addition; for we might as well say that they were made by division, or by subtraction, or by multiplication; since addition is, equally with each of the others, one of the powers of numbers, and presupposes the idea which Mr. Locke imagines it to produce. He says, "by repeating this idea (viz. of unity) in our minds, and *adding the repetitions together*, we come by the complex ideas of the modes of it. Thus by adding one to one we have the complex idea of a couple." Very true, by adding; but not by simply repeating, which is a totally different operation. John is one and Peter is one, and Henry is one; but one is not two, or three. What makes me then acquire the ideas of two or three? Certainly not the bare act of repeating one, one, one; for children, and idiots, who cannot reckon three, can do this: and M. de la Condamine mentions whole tribes of savages, who cannot reckon beyond three, though certainly they could repeat one, two, three, all the day long. There must, then, be something in the nature of the ideas of number without which it would be impossible for us to "add one to one," and of course to obtain "the complex idea of number." Now, this consists in the still more general nature of all ideas, and in that power, which they have, to grow and multiply by contemplation. Thus, if we enumerate John, and Richard, and Henry, and William, and James, and Edward, and so forth, the very slightest attention will show us that there is not merely unity, but *multitude*, or the idea of number in its most indistinct form; but in order to distinguish this multitude into given numbers, as twos, threes, or fours, it will be necessary to refer each conception to some other. Thus these two, John and Richard, are tall; these three, Henry, William, and James, are short; or these three, John, and Richard, and Henry, stand in the first line; these two, William and James, stand in the second; or the first, John, is counted on the thumb; the second, Richard, on the fore finger; the third, Henry, on tha middle finger; the fourth, William, on the finger next beyond the middle; and the fifth, James, on the little finger. This last mode of sorting and classing conceptions has been generally adopted by mankind, whence the Greek word *πεμπάζειν*, "to reckon by fives," was used generally for "to number." Some barbarous tribes never went beyond the use of one hand for this purpose; whereas the more cultivated nations employed both hands; and this latter mode is the origin of our decimal system of arithmetic, and explains why the numeral figures are still called *digits*.

Plural number.

We have observed, that the first conception of number is simply, that it is something beyond, and different from unity; that it is unity repeated, or multitude. Thus far most nations have gone, in expressing, by one word, the combination of number with any given conception; and this variation in the noun is called, by grammarians, the *plural number*. The plural number usually differs from the singular in form, either by the use of a word altogether different, as "pig and swine;" or by a change of articulation, as "man and men;" or by a syllable added, as "horse and horses," "ox and oxen;" but as the variety of these forms proves that no one of them is essentially necessary; so both experience and reflection will show, that no change whatever is necessary, in the noun itself, provided that some other word serves to show us that the noun is used with reference to plurality; thus in English we say "fifty *sheep*," and "fifty *head* of cattle;" and so in Latin the genitive and dative cases singular, and nominative and vocative plural of the first declension, are identical.

Dual number.

The form in which the noun expresses unity of conception, is called the singular number; but it would not be possible for nouns to have a separate inflection for every separate conception of number, that could be combined with them by the mind. Therefore, they cannot have separate forms for the *dual*, *ternal*, *quaternal* numbers, and so on, *ad infinitum;* but for some of these numbers they may. Experience, indeed, has not shown us that they have ever gone beyond the *dual* number in this respect; and that has been done by very few nations. Some grammarians have warmly agitated the question whether the Latin language has, or has not, a dual number; and as this question may serve to illustrate, in some degree, the principles here advanced, we shall advert to it, in that point of view. SCALIGER says, "*Iones non recte fecere, qui dualem numerum a plurali discerpsere: utque iccirco severiores Æolos neque recepere, neque in Latinos transmisere; et nugacitas illa Ionum in multis temporibus verborum personas aliquot non potuit eruere in eo numero, in nominibus autem pauculos casus expressere.*" "The Ionians acted wrong in dividing the dual number from the plural: for this reason, the more severe Eolians neither re-

Grammar.

ceived nor transmitted it to the Latins; and even this Ionian trifling in many tenses of verbs, was unable to make out a few of the persons in the dual number; and in the nouns they expressed a very few cases." Quintilian, however, observes, that there were some writers, in his time, who contended that the dual number, in the third person plural of verbs, was properly marked by the termination *e*; as *consedere*, if two persons sate together, *consederunt*, if more than two; but, adds he, this rule is observed by none of our best writers, "*quia e contrario Devenere locos;*" *et* "*Conticuere omnes;*" *et* "*Consedere Duces*" *aperté nos doceant nihil horum ad duos pertinere. Quid? Non Livius circa initia statim primi libri* "*Tenuere*" *inquit* "*arcem Sabini;*" *et mox* "*in adversum Romani subiere.*" *Sed quem potius ego quam M. Tullium sequar, qui, in Oratore* "*non reprehendo*" *ait* "*scripsere, scripserunt esse verius sentio.*" "On the contrary, the expressions "*Devenere locos*" (Virgil. Æn. 1. & 6.), and "*Conticuere omnes*" (Æn. 2.), and "*Consedere Duces*" (Ovid. Metam. 13.), may clearly teach us that none of these verbs relate merely to two persons or things. Does not Livy, almost at the very beginning of his first book, say, "*Tenuere arcem Sabini;*" and shortly afterward, "*In adversum Romani subiere.*" But what authority need I follow in preference to that of Cicero himself, who, in his book *De Oratore*, says, "I do not blame those who write *scripsere*, but, for my own part, I think *scripserunt* better." Vossius, too, observes, that in the description of Africa by Sallust, contained in his book on the Jugurthine war, we find, in the course of a very few lines, the plurals "*posuere, interiere, habuere, occupavere, miscuere, appellavere, accessere, corrupere, possedere, coegere, addidere, concessere, condidere*, and *fuere*; so that this supposed distinction in the third person of the verb appears to have been quite imaginary. DONATUS, however, a grammarian so popular in the middle ages, that a "Donat" became the common term for an elementary book on Grammar, argues more reasonably on the use of the words *ambo* and *duo*. "*Numeri*," says he, "*sunt duo, singularis, ut hic sapiens, et pluralis, ut hi sapientes. Est et dualis numerus, qui singulariter enunciari non potest, ut hi ambo, hi duo.*" "There are two numbers, the singular, as *hic sapiens*, and the plural, as *hi sapientes*. There is also a dual number which cannot be expressed singularly, as *hi ambo*, both these; *hi duo*, these two." Donatus is certainly right in calling these expressions *duals*, since they relate to the conception of two; but for the same reason he might call the expressions *hi tres, illi tres*, and the like, *ternals*; and so on, of any other numbers. This remark, however, leads to the clear and easy solution of the dispute among the grammarians; since it shows that each party was right in the different view that it took of the subject. It is certain, on the one hand, that the Latins could and did express the conception which was expressed by the Greek dual; but it is equally certain that they did not express it in the same manner. Amongst the Ionian Greeks the idea of *two* was expressed by a word which from long use and habit had come to be employed as the terminating syllable of any noun with which that idea was connected. Amongst the Eolian Greeks, and their Latian successors, the same idea of two was expressed by words which never happened so to coalesce. Scaliger on this, and some other occasions, reasons as if the formation of different dialects were a matter of premeditation and study; and therefore he calls the Ionians triflers, and describes the Eolians as more grave and severe; whereas it is certain that all languages, in their early state, grow up without much meditation or reflection, and that the cultivation and polishing of its language is one of the last results of a nation's civilization. Nor can this be otherwise; for ideas, which are the laws of mind, develope themselves in practice, and guide our mental operations, just as animal laws direct our bodily actions, long before we suspect either of them to exist. We walk, and dance, and ride, according to the laws of gravitation; we swim by the principles of hydrostatics; we form and express thoughts by the laws of conception, assertion, and deduction; but it is not until long after we have submitted to those laws, that we begin to take cognizance of them as distinct objects of thought; for the last operation of the human intellect is that by which it separates itself from outward things, and discovers within its own nature a world of beauty and order, which even more than this wondrous body of man with all its curious apparatus, chemical and mechanical, more than this terraqueous globe with its animal and vegetable and mineral riches, more than the sun "looking from his sole dominion," or even than the countless numbers of the heavenly host peopling interminable space, discovers to our finite comprehension the traces of that Deity, who cannot be more fully revealed but by his own divine word.

Chap. I.

Absolute truth.

Thus it is, that in intellectual, as in moral speculation, our simplest conceptions are most closely connected with that absolute truth, of which Mr. Tooke altogether denies the existence. "Truth," says he, "supposes mankind: for whom, and by whom alone, the *word* is formed. If no man, no truth. There is, therefore, *no such thing* as eternal, immutable, everlasting Truth; unless mankind, such as they are at present, be also eternal, immutable, and everlasting. Two persons may contradict each other, and yet *both* speak truth." This is not only not common sense, but it is very bad logic. The argument runs thus: A man *trowed* or believed something to exist; he used the word "troweth, troth, or truth," to express this belief; therefore *no such thing* existed. Again two men believed that two different things existed; they both used the same word to express the same belief: therefore the belief of both was equally well founded. Turn Mr. Tooke's sentences how we will, they come to this sort of reasoning. How is such a circumstance to be accounted for, in a man of his acuteness? For that he was acute, his single remark "that the verb includes the noun and something more," incontestibly proves. But his extraordinary sophisms arise wholly from his loose and hasty conception of the word *thing*: which as he uses it, corresponds exactly to Mons. Condillac's *object*, and to Mr. Locke's *idea*; and really means *nothing*; that is to say, nothing certain, definite, or intelligible.

Truth of numbers.

That the human mind can embrace ETERNAL TRUTH, in the widest sense of these terms, it would be folly and madness to assert; but that none of the truths which it is formed to comprehend are eternal, is a proposition, to say the least of it, extremely bold. At all events, the circumstance that men, "such as they are at present," may not be able clearly to comprehend a given truth,

is certainly no proof of its falsehood. Suppose a child does not well comprehend that two and two are four, are they the less so? Now, this is the case with all conceptions of number. We begin with unity, we proceed to multitude, we advance to numeration; but the elementary books of arithmetic will teach us, that this last is the introduction to that science by which Newton brought down the old divinities from their starry thrones, and converted lovely Venus and potent Jove into silent monitors of the lapse of time, or friendly guides of the adventurous navigator on a lonely ocean; that science, by which judicial astrology was for ever confuted, and men learnt to gaze unmoved on the comet, which, as they once thought,

——— from his horrid hair
Shook pestilence and war.———

How connected with other truths.

Such being the nature and power of the conceptions of number, let us enquire how, and on what principles it is that they are connected with other conceptions: and here it will be seen that these principles are founded in the essential nature of the noun, as universal and particular; general, specific, and individual; for the principal office of numbers is to apply science to fact, by distributing the genus into its species, and the species into its individuals; number, therefore, is the bond uniting the universal with the particular, the highest genus with the lowest individual, Eternal Truth with momentary sensation. Therefore it is, that Plato says, *εἴπερ ἀριθμὸν ἐκ τῆς ἀνθρωπίνης φύσεως ἐξέλοιμεν οὐκ ἄν ποτε τι φρόνιμοι γενοίμεθα.* "If we were to take out *number* from human nature, we should become void of thought on every subject;" which he again illustrates by observing, that an animal which has not the distinct conceptions of two and three, or of even and odd, and consequently, is quite ignorant of numeration, can never give any account of those things which he perceives by sense and memory.

Distribute genus into species.

"The genus," as Mr. Harris observes, "is found whole and entire in each one of its species." Thus the genus animal is found in the different species, man, horse, and dog: that is to say, a man is an animal, a horse is an animal, and a dog is an animal. By numbering the kinds, we find that the genus though one, is capable of being conceived as many, and therefore we can speak of many animals. Again, "the species may be found whole and entire in the individual." Thus Socrates is a man, Plato is a man, Xenophon is a man; and by applying the conception of number to the species of man, we call them three men. The plural number, therefore, belongs to genera and species: and accordingly we find all languages apply the plural number to words expressing genera and species, that is to say, to the words, which we have called common, or appellative.

Proper names strictly singular.

But the case is totally different with proper names, when strictly used as such; for in that case they are applied to individuals, and the individual is *not* found whole and entire in the genus or species. The conception of *Cæsar* is not found whole and entire in the genus animal, or in the species man, or in the class of Romans, or of conquerors, or of generals, or of soldiers, or of scholars. The word Cæsar, therefore, when used to express the very individual who passed the Rubicon, and who spoke with so much affected liberality in behalf of the traitors engaged in the Catilinarian conspiracy, and who doubted of a future state, and who associated with the debauched and profligate Antony, and who at once flattered and subjugated the Roman people, cannot receive a plural termination; and for this reason, because the particular conception which it expresses cannot be associated with number; since there never was nor ever will be more than one such man; who therefore spoke philosophically and truly, when he said—

For always I am Cæsar.

But if the word Cæsar be used to express a different conception; if it mean something which is also found whole and entire in Alexander, and Attila, and Jenghiz Khan, and Napoleon Buonaparte, then indeed "the Cæsars" is a proper grammatical form of speech; because the noun is no longer a proper name, but an appellative. Then we may reason on the Cæsars, as on a class or species, and what we say of one will be equally true of another; but then the word, though the same in sound, will be very different in signification; and the reason which before prevented our adding to it the plural termination will no longer exist.

How they become plural.

Mr. Harris has mentioned various ways in which a proper name may come to be used as an appellative. The persons indicated by it may, as members of the same family, or from other accidental causes, happen to bear the same name. Hence the expression of "the twelve Cæsars," to designate twelve Roman emperors who successively bore that name. Hence too the *Howards*, *Pelhams*, and *Montagues*, "because a race or family is like a smaller sort of species;" so that the family name extends to the kindred, as the specific name extends to the individuals. Again another cause which contributes to make proper names plural, is the marked character of some individual who bears it, whether for eminent virtue, or for notorious vice, or simply for any thing extraordinary and singular in his conduct or opinions. It is thus that in speaking on the subject of Grammar, we might not improperly say, "these are the opinions of a *Condillac!*" referring to an author of some celebrity; though, as we think, of remarkable inaccuracy in his views of that subject. So the liberality of Horace's patron and friend has made every patron of literature be called a *Mecænas*; the odious cruelties of Nero have made his name a synonyme with the word tyrant: and on the same principle Shylock, when he would express the integrity and acuteness of the supposed young lawyer, exclaims,

A Daniel come to judgment! Yea, a Daniel!

Gender.

Gender, as an accidental distinction of nouns, has given rise to much litigation among grammarians. "Gender," says Vossius, "is properly a distinction of sex; but it is improperly attributed to those things which have not sex, and only follow the nature of things, having sex, in so far as regards the agreement of substantive with adjective. Sex is properly expressed in reference to male and female, as *Pythagoras* and *Theano*; *ager*, a field, therefore, is improperly called masculine; and *herba*, an herb, is improperly called feminine. But animal is neuter, because it is construed neither way." Scaliger says, that the ancients improperly attributed sex to words; and that with respect to the neuter gender, it is absurd to attribute that to gender which is the negation of gender. Neither is it to be borne, says he, that words should be called of the doubtful gender, from the circumstance of their being sometimes used

Grammar. with a masculine and sometimes with a feminine construction. Mr. Harris, however, has, with some ingenuity, endeavoured to assign reasons for the generic distinction of nouns. "Every substance," says he, "is male or female, or both male and female, or neither one nor the other. So that with respect to sexes and their negation, all substances conceivable are comprehended under this fourfold consideration." Hence he proceeds to consider language as if it had been really and intentionally formed with a view to this classification of substances. As to the first and second class, they are manifestly such as must on many occasions require some mode of expression. The third is rare, and its expression would in general be shunned. But as to the fourth it must necessarily include by far the greater portion of the objects of thought. In languages where the natural sexes alone are expressed by terms corresponding to them, very little difficulty occurs in this part of Grammar. In general, every noun denoting a male animal is masculine; every noun denoting a female animal is feminine; and every noun denoting neither the one nor the other is neuter. The only exception to this general rule, is an exception which is founded in the poetical part of our nature; and it happily serves to distinguish the language of imagination from that of reality. The instances to which we allude are those in which the conception of a thing is raised to the dignity of a person, where we dwell with such fondness on our thoughts as to invest them, as it were, with life and action. Virtue stands before us in the enchanting form of a lovely female. Patience appears "gazing on kings' graves, and smiling extremity out of act."—So Shakspeare says,—

The mortal moon hath *her* eclipse endured.

But perhaps we cannot cite a finer instance of this figurative use of gender than that which is so finely employed in Milton's description of Satan—

——His form had yet not lost
All *her* original brightness, nor appear'd
Less than archangel ruin'd.——

But in languages where the mere terminations of words imply, or are supposed to imply, any or all of these distinctions, it is no wonder that much confusion arises in the various modes of explaining a circumstance so foreign to the general laws of thought. "The Greek, Latin, and many of the modern tongues," says Mr. Harris, "have words, some masculine, some feminine (and those too in great multitudes), which have reference to substantives where sex never had existence. To give one instance for many, *mind* is surely neither male nor female; yet is νους, in Greek, masculine; and *mens* in Latin, feminine." This learned grammarian could not but perceive that "in some words these distinctions seemed owing to nothing else than to the mere *casual* structure of the word itself;" but he was of opinion that in other instances might be detected "a more subtle kind of reasoning, which discerned even in things without sex a distant *analogy* to that great distinction which, according to Milton, animates the world!"

Mr. Harris's theory. We are far from asserting that in particular instances some such analogy may not have operated. Indeed it appears to us to be of the nature of that imagination to which we owe the figurative language above mentioned; but it could only have been a rare accident, by no means capable of carrying us far toward the explanation of the principles on which language in general was constructed. Harris, it must be owned, expresses himself modestly enough, observing, "that all such speculations are at best but conjectures, and should therefore be received with candour rather than scrutinised with rigour" "Varro's words, on a subject near akin," says he, "are for their aptness and elegance well worth attending: *Non mediocres enim tenebræ in silvâ ubi hæc captanda, neque eò, quò pervenire volumus semitæ tritæ, neque non in tramitibus quædam objecta, quæ euntem retinere possunt.*" With this allowance, we may therefore notice the general principle for which Harris contends, namely, that "we may conceive such subjects to have been considered as *masculine*, which were conspicuous for the attributes of imparting or communicating, or which were, by nature, active, strong, and efficacious: and that indiscriminately, whether to good or to bad, or which had claim to eminence either laudable or otherwise;" and again, that "the *feminine* were such as were conspicuous for the attributes either of receiving, of containing, of producing, or of bringing forth, or which had more of the passive in their nature than of the active; or which were peculiarly beautiful and amiable, or which had respect to such excesses as were rather feminine than masculine." Hence he thinks it would be reasonable to consider as masculine nouns, the "sun," the "sky," the "ocean," "time," "death," "sleep," and "God;" and as feminines, the "moon," the "earth," a "ship," a "city," a "country," and "virtue." But the question, as respects the science of Grammar, is not whether any or all of these may not occasionally and accidentally be so considered; but whether there be any necessary cause connecting in our minds the conception of sex with any of them. Now, there can be no other such cause than personification, because sex is a personal distinction; but even that cause does not universally apply to any of these conceptions. God, indeed, our creator and preserver, we usually and properly regard as a person; and then the reasoning of Mr. Harris is so far just, that we cannot easily view the Supreme Being as a female; for even in those heathen mythologies which abound with female divinites, the chief and sovereign Deity is always represented as masculine. But Harris himself admits, what indeed the common experience of every day sufficiently proves, that we often contemplate this ineffable conception without any reference to sex, or even to person, calling it "Deity," "Numen," "το θειον." It must be remembered, that personification was more common among the ancients than the moderns. The Greeks actually worshipped Sleep and Death in the form of men: Virtue was pourtrayed before their eyes by the statue of a female. Nor must we forget that many of these personifications have been handed down to us from them by mere tradition and the language of the poets. Thus it is difficult for us, who have seen Fame and Victory so often delineated as females, on ancient medals, and in sculpture, who read of them as such in poetry, and know that Fama and Victoria are nouns of feminine termination; it is difficult for us when we do personify these airy beings, to figure them to ourselves as men, in a different habit and form, with different accompaniments, and expressed by words and sentences of a different cha-

racter and construction. But there are comparatively few things which we personify in our common prose: and when we do so, the change of the form of words from neuter to masculine or feminine, at once and powerfully marks the transition of the mind from cold matter of fact to ardent imagination. This, however, is again an accidental circumstance appertaining to the particular history of the English language, and not to the philosophy of language in general.

Gender of proper names.

There is a curious difference of opinion between SANCTIUS and Harris. The former writer asserts "that proper names of men, cities, rivers, mountains, and the like do not admit of grammatical gender;" "*Nomina propria hominum, urbium, fluviorum, montium, et cætera hujusmodi, genus grammaticum habere non posse:*" whereas the latter author says "both number and gender appertain to words.—Number, in strictness, descends no lower than to the last rank of species: gender, on the contrary, stops not here, but *descends to every individual*, however diversified." This apparent contradiction between two eminent writers is nevertheless easily reconciled. Harris attributes gender to words as significant of the conceptions of the mind. Sanctius, on the other hand, following the authority of Varro and Diomedes, considers grammatical gender as relating only to the termination or construction of words. "Thus," says Varro, "we do not call those words masculine which signify male beings, but those before which are properly placed *hic* and *hi*, and those feminine with which we can say *hæc* and *hæ*." "*Sic itaque ea virilia dicimus, non quæ virum significant, sed quibus præponimus hic et hi; et sic muliebria in quibus dicere possumus hæc et hæ.*" The reason which this author assigns for his doctrine is suitable enough to Grammar as an art, but not as a science. *Grammaticæ propositum non est singularum vocum significationes explicare, sed usum.* "The object of Grammar is not to explain the *significations* of particular words, but their use." Now, though the mere signification of words is not the object of Grammar, the mode of signification is so far from being an immaterial part of that science, that it is its sole foundation. There is no doubt but that the expression or non-expression of the distinction of sex in connection with other conceptions, must affect the relations of language considered as significant, and consequently must fall under the science of Grammar, according to the definition of it which we have adopted. This expression is not essential to all nouns, but it is an accident universally affecting whole classes of nouns, and therefore demanding for its application some rules of Universal Grammar.

Terminations.

Now those rules not only do not depend on the termination or other peculiarity in the sound of words, but even in the Latin language, as Wallis has observed, sex is not so distinguished; for though the termination *um* is neuter, yet the words *scortum, mancipium, amasium*, &c. are applied both to the male and female sex: and so we find it even in proper names, as *Glycerium mea*, which Priscian notes as figurative.

Union of conceptions.

Regarding only the science of Grammar, as dependent on the nature of thought, it is manifest, that those conceptions which are of a nature to coalesce, in reason or fancy, may be considered either distinctly or in absolute union. Thus the conception of "number" and that of "soldier" are absolutely united in the conception of "army" or "regiment," or "troop;" the conception of "royalty" and that of "man" are absolutely united in that of "king;" and so the conception of "sex" and that of "child" are absolutely united in the words "boy" and "girl." This sort of union gives occasion to many classes of words in most languages, as "horse" and "mare," "ram" and "ewe;" "bull" and "cow;" but there is a second class in which the same distinction is expressed by the compound form of the word, as "shepherd" and "shepherdess," "milliner" and "man-milliner;" and lastly, the sexual quality is often expressed by its proper adjective, as the "male and female elephant," the "male and female rhinoceros."

Common gender.

There are some conceptions in which that of sex is tacitly included, but may not be absolutely determinable, or may not require to be determined for the purpose of communicating thought. Thus a "child" is either a "boy" or a "girl;" but if we are reasoning on the education of children generally, many thoughts may occur to us which indifferently and equally relate to boys and girls, and in expressing which we may therefore use the neuter word "child." And perhaps this consideration alone would afford a sufficient answer to those persons who contend, like Hobbes, that the general word "man" is no more than the representation of some one particular man in my memory or imagination: for if the word child in my thoughts represented a boy, it could not represent a girl, and vice versa; whereas we see in practice that it represents the two contrary sexes at the same time, without the least difficulty, and serves the purposes of reasoning quite as well, and oftentimes better than if we had employed different words for the two sexes.

Figurative gender.

Lastly, there are conceptions, which in reality have nothing to do with sex, but which, from various causes, principally depending on imagination or habit, we are apt to consider in connection with notions of sex. Thus the English sailor, who has contracted a sort of affection for the tight vessel in which he has braved the winds and waves; and who sees in her neat trim and gallant tackling the elegance of female apparel, is habitually led to speak of her as a female. Who has not been electrified with the feeling expressed in the old sea-song—

"*She rights, she rights, boys—we're off shore!*"

To a similar cause it is to be attributed that we can hardly think of Britannia as a mailed warrior "an arm'd man for the battle," or as a sea god wielding his trident over the subject waves; but we see her, like another Minerva, great in arts and arms, circling her brows at once with the olive and the laurel, covering the nations with her ægis, and stretching out her spear for their protection. If we speak of her domestic greatness, it is as

The nurse, the teeming womb of royal kings;

If we lament her errors, and her failings, we

Feel for her, as a lover, or a child.

Animated style.

This is the language, not of mere plain unadorned reason, but of reason elevated and sublimed by passion; yet does not this circumstance take it entirely out of the domain of Grammar, viewed as teaching the necessary modes of communicating thought; for passion is a necessary part of our nature, and it necessarily gives a hue and tinge to our conceptions, and forces us to modify accordingly the forms of expression in language. Unhappy is the critic who knows nothing

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of this part of Grammar; he will not only miss some of the finest beauties in the poets, but if he attempt to correct what he thinks faulty, he will display, in the most ridiculous light, his own want of taste. Mr. Harris has finely exemplified this remark, by a quotation from Milton—

At his command th' uprooted hills retired
Each to *his* place; they heard his voice and went
Obsequious: Heav'n *his* wonted face renew'd,
And with fresh flow'rets hill and valley smil'd.

"Here," says Harris, "all things are personified: the hills hear, the valleys smile, and the face of heaven is renewed. Suppose, then, the poet had been necessitated by the laws of his language (or we may add by the correction of the critic) to have said, Each hill retir'd to *its* place. Heaven renewed *its* wonted face—how prosaic and lifeless would these neuters have appeared; how detrimental to the prosopopeia which he was aiming to establish! In this, therefore, he was happy, that the language in which he wrote imposed no such necessity, and he was too wise a writer to impose it on himself! 'Twere to be wished his correctors had been as wise on their parts." That they were not always so wise we have a striking instance in the celebrated Bentley, who has taken upon himself to make a vast number of alterations of this kind in Milton's text. Thus the great poet in his picturesque description of creation, had written

———— "The swan with arched neck
Between *her* white wings mantling proudly, rows
Her state with oary feet—

On which Dr. Bentley has the following note: "The swan *her* white wings! and *her* state! I wonder he should make the swan of the feminine gender, contrary to both Greek and Latin; always Κυκνος, *cygnus*. Rather, therefore, *his* wings, *his* state." This comes of having learnt only the Greek and Latin Grammars, and not knowing, even of these, the true foundations.

Case.

We come now to the expression of the relations of nouns to each other, which is effected by declension, or *case*, if the relation and the conception coalesce in one word, and by a preposition if in different words. By this short statement we shall easily discover our way among the disputes of grammarians relative to the cases of nouns. Declension is commonly used for the variation of case; but Varro considers case as only one mode of declension. His words are these: "of words, as man and horse, there are four kinds of declension; first nominal, as from *equus* comes *equile*; secondly casual, as from *equus* comes *equum*; thirdly argumentative, as from *albus* comes *albius*; and fourthly diminuent, as from *cista* comes *cistula*." We have, however at present, only to do with the second of these modes.

Number of cases.

It was long disputed what number of cases existed in the Latin language. These are thus enumerated and explained by Priscian: "The first case is called the *right*, or *nominative* case; for by this case, naming is effected; as this man is called Homer, and that man Virgil. The reason that it is sometimes called the right or straight case is, that it is first formed naturally by merely laying down the word, and then the other cases formed by flexion from this, are called *oblique*. The next is the *genitive*, which is also called by some the possessive or paternal. The word genitive is either derived from *genus* a race because we signify by it the race to which any one belongs, as "he is of Priam's race," or from *genero* to generate, because from this case are generated many other words and parts of speech, at least it is so in the Greek language. Again it is called possessive, because we signify possession by this case, as "Priam's kingdom," or the kingdom possessed by Priam: whence possessive adjectives may also be construed by this case; for what is "the Priameian kingdom" but "the kingdom of Priam," or "Priam's kingdom?" It is called paternal for a similar reason, because the father's name is thus expressed, as "Priam's son;" and hence patronymic names may be resolved into this case, as "Pelidan Achilles" is the same as Achilles the son of Peleus. The following case is the *dative*, which some term the *commendative*. I give a thing "*to* a man," or I recommend a person "to a man." Fourthly comes the *accusative* or *causative*: I accuse a man, or I (as a cause) make a thing. The fifth case is the *vocative* or *salutatory*, as "O Eneas!" or "Hail Eneas!" The *ablative* is also called the *comparative*; as "I take *from* Hector," or "I am stronger *than* Hector." Each of these cases, moreover, has many other different uses; but they have received their names from their most general and familiar use, as we see happen in many other things."

Chap. I.

Meaning of the word case.

From this enumeration, it is observable that the sort of declension which the ancients called *case*, not only expressed the relation of nouns to each other, but also that which they bore to verbs, as agent or object; and lastly, their use in the expression of passion, without reference either to another noun or to a verb: in order to explain the reasons of which it will be necessary to observe, that the meaning of the word *casus*, which we render case, is, properly, the falling or declining from a perpendicular line. Thus, if the simple notion of the noun be supposed to be expressed by an upright straight line, as in the letter I, the other cases may be supposed to be expressed by lines obliquely declining one way or the other, as in the letter V.

Nominative.

It was long disputed among the ancient grammarians, whether the nominative should, or should not, be called a case. On the one hand it was urged, that conceptions are only expressed by speech, in some one of the forms called cases, including the nominative; and that of these forms, the nominative expressing the agent of the verb active, was the simplest, and was therefore used whenever there was occasion simply to name a thing or person. Thus we should not say, that the name of the person slain by Marcus Brutus, was *Cæsaris*, or *Cæsari*, but *Cæsar*. Those on the contrary, who called it a case, contended that every expression of a conception in speech, was a declension, or falling away from the simple conception in the mind, which taken by itself does not imply either action, or passion, nor relation. Thus, before I can assert any thing whatsoever of Cæsar, I must form the conception or thought of "Cæsar," as a person; but when I put that thought to another, when I mention the wife "of Cæsar," or the friends who were faithful "*to* Cæsar," or those who revolted "*from* Cæsar;" or assert that "Cæsar conquered," or that "Cæsar was killed;" or express a feeling of any sort by the exclamation "O Cæsar"—on these and all such occasions my conception declines from its original simplicity, and consequently my expression should be said to decline, or fall away from the pure noun. They added, moreover, that it was not

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always the simplest form of the noun, but was sometimes more distant from the radical, and therefore more deserving of the appellation of oblique than some other cases; as, for instance, the vocative or oblative, which latter some writers have considered as the primary and original case of the noun.

Since the notion of action implies the notion of an agent, there must be a form of the noun which denotes the agent to every verb in a simple sentence. The action, however, may be represented as proceeding from the agent, or as received by the object. On the former supposition, it becomes a verb active, and the nominative case is the form of the noun which denotes the agent. On the latter supposition, it becomes a verb passive; and the nominative case is the form of the noon which denotes the object. Thus, "Cæsar fights," "Cæsar is killed," are two simple sentences, in both of which Cæsar is the nominative case. In the former, the word Cæsar signifies the agent that fights; in the latter, the same word Cæsar signifies the object that is killed. In both instances the nominative is essential to the completion of the sentence; for when we speak of fighting, as proceeding from an agent, we must necessarily express that agent; and when we speak of being killed, as received by an object, we must express the object. Hence the trivial rule, that the nominative answers to the question who, or what; as "Cæsar fights." Who fights?—Cæsar. "Cæsar is killed." Who is killed?—Cæsar. It is justly observed by Harris, that the character of the nominative may be learned by its verb. The action implied in the verb "fights," shows the nominative "Cæsar" to be an active efficient cause. The suffering implied in the words "is killed," shows the nominative "Cæsar" to be a passive subject. There are some Beings which may be considered in both these lights; as Cæsar is active in the one instance, and passive in the other. But there are others which cannot, except figuratively, be considered otherwise than as passive, and, consequently, can only become nominatives to passive verbs: as we may say, "the house is built;" but we cannot say, "the house builds."

Accusative and ablative.

The nominative is the most essential of all cases; and it has therefore been described as "that case without which there can be no regular and perfect sentence." With respect to those sentences in which we make the positive *it* serve for a nominative, and which the Latins used without any nominative at all, as *pluit*, "it rains;" *tædet me*, "it wearies me," or "I am wearied;" these are imperfect sentences, which we shall hereafter consider separately. In all other instances, although it may not be necessary to express the object to which an action is directed, or the agent from which a suffering proceeds, yet the converse is absolutely necessary: thus, when we say, "William builds," it is not necessary to add "a house," or "a palace;" but if we say "builds a house," or "builds a palace;" it is necessary to prefix the name of the builder.

In order, however, to extend and enlarge a sentence, it often becomes necessary to state the object of a verb active, or the agent of a verb passive. Hence arises the necessity for two other cases, which have been called the accusative and the ablative. When we say there is a necessity for such cases, it will be understood, from what we have before observed, that we do not contend for the necessity of any particular terminations, or inflections, or prepositions, or arrangement of words, to mark these varieties of case; we only mean, that it is necessary, that by some means or other, the noun, which indicates the conception, should be placed in such or such a relation to the verb which constitutes the assertion. It may happen, and, in point of fact, it does happen in some Languages, that there are no inflections of case; but there are means in all Languages of determining when a noun is the object of an active, or the agent of a passive verb. It has, indeed, been disputed, whether the cases of nouns should be reckoned according to the relation in which they stand to other words, or according to the diversity of their inflections; nor are there wanting names of high repute on either side of this question. Sanctius contends, that there is a natural partition of cases, according to the relations which they imply, and, consequently, that there must necessarily be the same number of cases, which he estimates to be six, in all Languages. Vossius objects to this reasoning, and alleges, that if the cases of nouns were to be reckoned by the relations which they bear to other words, they must be endless. This contest, like many others, has arisen from confounding Universal with Particular Grammar. The difference of Inflection, or position, belongs to the latter; that of signification to the former. True it is, that the relations of nouns to other nouns and to verbs are infinite; but yet they are distinguishable into certain great classes; and whether those classes ought or ought not to be called cases is a mere verbal dispute. We shall so designate them, for the sake of convenience; at the same time, it must be understood that our arrangement is not intended to interfere with the Grammar of any particular Language, in which the cases are arranged according to their inflections.

Nouns. Case.

In our sense of the word case, then, the nominative, that is, the agent of the active, or object of the passive verb, may be called the primary case; and the secondary cases are the accusative and the ablative, in so far as they perform the functions above noticed. These two cases, it is to be observed, are respectively convertible with the nominative, by a change of the verb from active to passive; for "James loves John" is convertible with "John is loved by James;" the accusative of the first being the nominative of the second, and the nominative of the first being the ablative of the second.

Dative, &c.

So the matter stands in the simpler combinations of thought; but let us consider what is to be done, if in one and the same sentence we wish to express not only the agent and object of any action, but also the end to which the action is directed; the cause on account of which it happens, or the instrument, mode, and circumstances of its performance. For these purposes, it is necessary that the conception of such end, or cause, or instrument, &c. should be expressed by a noun; and that some means should be adopted to show whether the noun was meant to stand in the relation of end, cause, or instrument, or in any other relation to the verb. It is, as Vossius justly observes, quite impossible that any Language should have separate inflections for all these relations, and therefore some of them are, in most Languages, represented by separate words, or particles, commonly called prepositions; but others are often expressed by inflections, the number and diversity of which vary exceedingly in different Languages. Thus, in the Sanscrit, there are separate

inflections to signify the end, the instrument, the source, and the situation, answering to our prepositions "to," "by," "from," and "in." In the Latin Language, a particular inflection is used to signify the end to which an action is directed, and the case known by that inflection is called the *dative;* because verbs of giving usually require the expression not only of the thing given, but of the person to whom the gift is made, and whose convenience or benefit is the end to which the gift is destined. In order to express the other relations above noticed, the Latin Language avails itself of the accusative or ablative inflection, either alone or with a preposition.

Genitive.

Thus have we noticed three classes or degrees of relation in which the noun may stand to the verb; but it may also be related to another noun, as depending on, or belonging to it. Thus the words "Priam's kingdom," "the son of William," mark a dependence of "son" on "William," and of "kingdom" on "Priam." This relation is expressed by a separate inflection in Greek, Latin, English, and many other Languages; and it is commonly called the *genitive* case. Now the use of the genitive case in nouns substantive differs but little from the use of an adjective. It expresses one conception, as dependent on another, and the expression of the latter serves to individualize and specify the former. The dependent conception is, therefore, in fact, a mere attribute of the other, and consequently the genitive is easily convertible into an adjective. Thus Βασιλέος Σκῆπτρον, *regis sceptrum*, the king's sceptre, are easily converted into Σκῆπτρον Βασιλικὸν, *sceptrum regium*, the kingly sceptre. For the same reason we find that in some Languages, the Chinese for example, the adjective is in no manner distinguished from the genitive or possessive case of a substantive; for it is said, that the word *haò* signifies goodness, and *gin* signifies man; but *haò gin* is a good man, or man of goodness; and *gin hao* is human goodness, or the goodness of man. Hence, too, we see why Wallis considers the English genitive case as a possessive adjective; e. *g.* "the *king's* court," *aula* regia, where he differs from all other English Grammarians, in calling the word "king's" an adjective. On the other hand, Lowth reckons the words *mine* and *thine*, which are usually called adjectives, as the possessive cases of *me* and *thee*. It is, perhaps, from a similar cause that Dr. Jonathan Edwards asserts the Muhhekaneew or Mohegan Indians to have no adjectives at all in their Language; a fact on which Mr. Horne Tooke lays great stress, but which, in reality, proves nothing as to the signification of Language, whatever it may do as to its forms or inflections.

Vocative.

It seems hardly necessary to distinguish the vocative case by any particular inflection. Indeed, we find the terminations of the nominative and accusative equally employed in Latin as exclamatory: and it is said that the Sanscrit Grammarians do not allow the vocative to be a case. Yet, when we are speaking of the different relations in which a noun may stand to other words in a sentence, it is impossible to overlook its use in those sentences where it stands forth prominently as the object addressed or invoked. Thus, in the first Ode of Horace, we find two verses almost wholly occupied with vocatives:

Mecœnas, atavis edite regibus,
O et præsidium, et dulce decus meum!

Nouns Adjective.

These are the only distinct uses of the noun which it appears necessary to consider under the head of relation or case; but we must observe, that the cases, as distinguished in different Languages, either by inflection, or by being joined with certain prepositions, do not by any means agree with the classes of relation here noticed. In the Greek idiom, the genitive termination sometimes answers to an English accusative, as πίνω τοῦ ὕδατος, I drink *water;* sometimes to the Latin ablative, as ἀντὶ ἀγαθῶν ἀποδιδόναι κακὰ, *mala pro* bonis *reddere;* and sometimes to the Latin accusative, as Ἀνὴρ ἀντ' ἀνδρὸς ἦν, *vir contra* virum *est*. The English genitive, "blind *of* an eye," answers to the Latin ablative, oculo *captus*, and to a case in Sanscrit, which expresses the cause or instrument, but neither the location, nor the derivation, although both these latter equally demand the ablative in Latin. The dative equally varies. In Greek it answers sometimes to the Latin Ablative, as σὺν Θεῷ, *cum* Deo; and sometimes to the Latin accusative, as ἐπὶ τῷ κέρδει, *ob* lucrum. So the English vocative is sometimes expressed by a Latin accusative, as *O, cæcas hominum mentes!* "O, blind understandings of men!" and sometimes by a Greek genitive, as τῆς ἀναιδείας! "O, impudence!" Numberless instances of a like kind might be adduced; but these are sufficient to show, that however convenient it may be, in the Grammar of any particular Language, to distinguish the cases of nouns by their terminations, yet this is a method totally inconsistent with those distinctions of signification on which alone Universal Grammar can be founded.

Adjective.

We have said that the noun adjective is the name of a conception or thought, considered as a quality or attribute of another conception. In order to explain this definition, it will be proper to advert to the nature of a simple enunciative sentence or logical proposition, which consists of a subject, a copula, and a predicate. The subject, or that concerning which something is asserted, is always a noun substantive; the predicate may be a noun adjective. Thus, in the sentence "John is tall," the subject is "John," which is also a noun substantive; the predicate is "tall," which is also a noun adjective. Complex sentences are resolvable into more simple ones: and wherever adjectives are used, so as to render a sentence complex, they are always resolvable into the predicate of a logical proposition. Thus, if it be said, that "a wise man is cautious," this sentence is resolvable into the two simple sentences "a man is cautious," and "that man is wise," and in each of these the adjective is the predicate of the proposition.

The corollaries to be drawn from this statement are several. In the first place, whenever the name of a conception is employed as the subject of a proposition, it is not an adjective. Thus, the conception expressed by the words "good" and "goodness" is the same; but if we predicate any thing of this conception; if, for instance, we say, "goodness is amiable," the word goodness must necessarily be a substantive. And this does not depend on the form of the word; for if the idiom of our Language allowed us to say, "good is amiable," or "the good is amiable," the word "good" would be as much a substantive as "goodness."

Hence it follows, that the distinction between a substantive and an adjective does not necessarily depend on any difference between the conceptions which

they express, but between the different modes in which those conceptions are contemplated by the Mind. If we contemplate goodness as a separate idea, if we assert any thing of that idea, if we make it the subject of any proposition, then it is a substantive; but if we predicate it of any thing else, if we consider it only as a quality of that thing, then it is an adjective.

Hence, again, it will follow, that an adjective and a substantive cannot be convertible, without wholly changing the meaning of the proposition in which they are employed. Thus, to say that "envy is criminal," and that "criminality is envious," are two propositions entirely different.

It is equally a rule of Universal and of Particular Grammar, that an adjective cannot stand alone, but must be joined with its substantive; which is, in truth, no more than saying, that a predicate must necessarily refer to some subject. Mr. Tooke, however, controverts this rule, though it is certainly as old as the words adjective and substantive. He objects that the rule equally applies to the oblique cases of nouns substantive, and that therefore "the inability to stand alone in a sentence is not the distinguishing mark of an adjective;" but, though it were not a distinguishing mark, it might yet be a rule common to all adjectives. However, the real intent of the rule is to distinguish adjectives from the substantives with which they are used; and that in the most simple sentences; and with reference not to their form or inflection, but to their signification. Thus, if we say "a *golden* is valuable," the sense is incomplete, and the adjective "golden" requires the addition of a substantive, as, for instance, "ring," to render it intelligible. On the contrary, if we say "*gold* is valuable," the sentence is perfect. Mr. Tooke contends that "the adjectives *golden*, *brazen*, *silken*, uttered by themselves, convey to the hearer's mind, and denote the same things as *gold*, *brass*, and *silk*. The short answer to this is, that it is contrary to common sense and experience to confound these terms together; and nobody ever does so who understands the English Language in the slightest degree. But if we wish to trace the source of Mr. Tooke's error, we must examine more particularly his expressions. First, what does he mean by "uttered by themselves? Words uttered by themselves are like syllables or letters uttered by themselves. They are the mere elements of discourse. Their proper force and effect in rational speech must depend on their connection with each other. Again, what is meant by "denoting the same things?" In so far as they are both of the same origin, there is doubtless a common conception to which they both bear relation; but it does not follow that they both bear the *same* relation to it. A numerous tribe of words derived from, or connected with, this term, *gold*, is to be found in the different European Languages. Is it to be said that they all "convey to the hearer's mind and denote the same things?" Let us see how this can possibly be made out. From (1) the splendour of the rising or setting Sun, was denominated (2) the yellow colour resembling that splendour. From the name of that colour, was derived (3) that of the jaundice, which rendered the whole body yellow, and (4) that of the gall, which produced the jaundice. From yellow also came (5) the name given to the yolk of an egg. And again, from this colour came (6) the name of gold. Gold, being the most precious of metals, gave its name (7) to riches in general; and particularly (8) to money. Hence were denominated all kinds of payments, whether (9) voluntary gifts, or (10) offerings, or (11) tribute, or (12) rent, or (13) fines; as well as (14) debts due on any of these accounts. In process of time, certain Societies were formed and maintained by regular payments from each member, and these Societies received their name (15) from this circumstance. The name was afterwards extended to Societies (16) or Fellowships in general; and it occasioned the peculiar designation of a well-known building (17) in London. Fines in ancient times were applied, in the nature of punishment, to almost all crimes; and hence their name came to signify (18) punishments in general; and particularly a barbarous mutilation (19) often used as a punishment. Lastly, the general term for punishment was naturally applied to the criminality (20) by which the punishment was occasioned.

Nouns Adjective.

We have traced in the margin* these progressive changes of signification, as they are to be found in the Mæso-Gothic; Anglo-Saxon; Alamannic; Lombardian; Precopian; Greek; Latin, old, middle, and barbarous; Suevian; Swedish; Islandic; Russian; German; Dutch; Welsh; Italian; old and modern French, and old and modern English. Every change of application is occasioned by a new operation of the Mind. The sound of the word conveys a new thought, similar indeed to the preceding, and having reference to the same conception, but placing it in a new light. It would be absurd to say, that the thought remained the same through *all* these different uses; and it is equally incorrect to say, that it remains the same after any *one* step. There is as real, though not as great a difference between "gold" and "golden," as there is between "a guilder" and "Guild-hall. If Mr.

* 1. Gr. γίλα. (Hesych.)

2. Suev. *Gel.* Swed. *Gæl.* Dut. *Geel.* Ger *Gelb.* Russ. *Gelto.* Isl. *Gulur.* Lat. *Gilvus, helvus, galbus, galbeus, galbinus.* Ital. *Giallo.* Fr. *Jaulne, jaune.* A. S. *Geolu, geolewe.* Eng. *Yellow.*

3. Ger. *Gel-sucht.* Dut. *Geel-zught.* Russ. *Geltukha.* Fr. *Jaulnisse, jaunisse.* Eng. *Jaundice.*

4. Russ. *Geltchy.* Eng. *Gall.*

5. Russ. *Geltoky.* Fr. *Jaune.* Eng. *Yelk, yolk.*

6. Isl. *Gull.* M. Goth. *Gulth.* Procop. *Goltz.* A. S. *Gold.* Dut. *Goud.* (N. B. Watcher derives *gold* from *gel-od*, yellow substance.)

7. Wel. *Golud.*

8. Ger. *Gelt.* Dut. *Geld*, (hence *guilder*, &c.)

9. Ger. *Gilt.* (Lomb. *Launchild*, a mutual gift.)

10. A. S. *Gyld.* (*Godgyld* and *deofulgyld*, offerings to God and offerings to Devils.)

11. Isl. M. Goth. and A. S *Gild.* (*Danegyld*, tribute to Danes; *Hodegyld*, tax on woods; *Hornegyld*, tax on horned cattle; whence the family name of *Hornyold*, still subsisting.)

12. Isl. *Aukergiæld*, rent of a field. *Gilibar aaker*, a field producing rent.

13. Barb. Lat. *Geldum, gildum.* Isl. *Giaell.* A. S. *Wergeld*, the fine for killing a man.

14. Alaman. *Gult*, a debt. *Gelter*, a debtor or creditor.

15. A. S. *Gild.* Barb. Lat. *Gelda, gilda, gildonia*, (whence Menage derives the expression, *courir le guilledou.*)

16. *Diobol-gelde*, the Devil's fellowship. (See Eccard.) A. S. *Frythgyld*, the Society of confederates. The Dean of *Guild*, an officer well known in Scotland, &c.

17. *Guild*-hall.

18. Alaman. *Gilten*, to suffer punishment.

19. Eng. *Geld, gelding.* Germ. *Geltze, porca castrata.* Isl. *Gælldo*, (*Gielt-fie, aries castratus.*)

20. A. S. *Gylt, agyltan, gyltend, gyltig.* Eng. *Guilt, guilty.*

N. B. It is remarkable that an analogy similar to that which exists in the above articles, 1, 2, 3, and 6, is found in the Latin words *Aurora, Aureus, Aurigo*, and *Aurum*.

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Tooke were right, to *gild* a thing would be to convert it into *gold*: whereas these words, though of the same origin, are so far from denoting the same conceptions, that they are often used in direct opposition to each other. "Is this *gold*?—No, it is only *gilt*." So *gold* and *golden* are not the same. They both, indeed, refer to the same conception; but they refer to it in different ways. In the once instance, the conception (namely gold) is the very thing of which we are speaking; it is the logical subject of the proposition; the mind looks at it, as it were, directly; as when Bassanio says,

> ———— Thou gaudy *gold*,
> Hard food for Midas—I will none of thee.

Whereas, in the other case, it is noticed but incidentally, as a thought passing over, and giving a momentary tinge to another thought, but differing from it as the light in which we view a substance differs from the substance itself. So the same Bassanio, in the same scene, speaking of his mistress's portrait, says,

> ———— here in her hair,
> The painter plays the spider, and hath woven
> A *golden* mesh to intrap the hearts of men.

It is very true that these secondary thoughts, which are expressed by adjectives, may be brought more distinctly before the Mind, and treated as substantives in connection with other substantives. It is thus, that instead of "a virtuous man," we may say "a man of virtue;" but though there appears, in this instance, very little difference of meaning, yet, on analyzing the two expressions, we shall find that a new and distinct operation of the Mind is performed, which operation is here expressed by the word "of." We do not merely, as in the case of the words "virtuous man," contemplate the conception of "man" as a substance, and that of "virtue" as a quality belonging to the individual in question; but we contemplate "man" as having a substantial existence, and "virtue" as having an existence capable of coalescing with man; and further, we contemplate the actual union of these two thoughts, as expressed by the word "of." Slight, therefore, as the difference of meaning is between the words "a man of virtue and a virtuous man," yet the Grammatical difference is not to be overlooked: and the best proof of this will be to consider how totally the style of any author would be altered if we were always to change the genitive case of the substantive into an adjective, and *vice versâ*. Suppose that, instead of the line—

> The quality of Mercy is not strained,

we were to say, "the merciful quality is not a quality of compulsion," we should certainly not augment the force and beauty of the language; and we should as certainly change the flow and current of the thought; we should alter the Grammar without improving the Poetry.

From what has been already said, we may perceive the absurdity of asserting that "adjectives, though convenient abbreviations, are not necessary to Language," and still more, that "the Mohegans have no adjectives in their Language;" for though this latter fact is vouched by "Dr. Jonathan Edwards, D.D. Pastor of a church in Newhaven; and communicated to the Connecticut Society of Arts and Sciences, and published by Josiah Meigs," yet it amounts to nothing else but that the Mohegans cannot distinguish subject from predicate, or substance from quality; and if so, they must be utterly destitute of the faculty of Reason, which we suppose neither Dr. Edwards, nor Mr. Meigs, nor Mr. Tooke, intended to assert.

Nouns Adjective.

Degrees.

It is a common rule, that the adjective should agree with its substantive in gender, number, and case, from whence, perhaps, it might at first sight be inferred, that gender, number, and case, properly belong as well to the adjective as to the substantive. This, however, is not the fact: the adjective simply expresses a quality; but it must of necessity be connected in Language with its substantive, and that connection is effected in many Languages by a similarity of inflection; and as the inflections of the substantive express gender, or number, or case, those of the adjective often follow a similar rule of construction. This construction, it is obvious, is a matter belonging only to Particular, and not to Universal Grammar. It may exist in one Language and not in another; and, in fact, there are Languages (our own for example) in which all these variations are wholly unknown.

Degrees of comparison.

On the contrary, the variation of *degree* is one which belongs, in an especial manner, to certain adjectives, but not at all to substantives; and where there are variations of degree, they may be compared together, whence arise, what are technically called by Grammarians, the Degrees of Comparison.

Substantives cannot be compared, as such, in point of degree; for that would be to suppose that the nature of substantial existence was variable; and that one existing thing was more truly existing than another, which is absurd. "A mountain," says Harris, "cannot be said more to *be*, or to *exist*, than a molehill; but the more and less must be sought for in their quantities. In like manner, when we refer many individuals to one species, the lion A cannot be called more *a lion* than the lion B; but, if more any thing, he is more *fierce*, more *speedy*, or exceeding in some such attribute. So again, in referring many species to one genus, a crocodile is not more *an animal* than a lizard is, nor a tiger more than a cat; but, if any thing, they are more *bulky*, more *strong*, &c.; the excess, as before, being derived from their attributes. So true is that saying of the acute Stagyrite, *οὐκ ἂν ἐπιδέχοιτο ἡ οὐσία τὸ μᾶλλον καὶ τὸ ἧττον*; substance is not susceptible of more and less." Sanctius, referring to this same passage of Aristotle, observes, that we may hence infer that comparatives cannot be drawn from nouns substantive. "Hence," adds he, "they are deceived, who reckon the words *senex*, *juvenis*, *adolescens*, *infans*, &c. as substantives, for they are altogether adjectives. Nor is it to be objected, that Plautus has made from *Pœnus* the comparative *Pœnior*; for he does not there mean to express the substantial existence of the Carthaginian, but his cunning, as if he had said *callidior*; for the Carthaginians were reputed to be a very cunning people. So the writer who used the word *Neronior*, from *Nero*, meant only to signify an excess of cruelty."

As substantives in general admit not of degree; so there are some adjectives which equally exclude either intension or remission. Thus Scaliger justly observes, that the word *medius* can neither be heightened nor lowered in degree; and that the same may be said of *hodiernus*, and of many other adjectives. On this topic Mr. Harris thus expresses himself: "As there are some attributes which admit of comparison,

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so there are others which admit of none. Such, for example, are those which denote that quality of bodies arising from their figure; as when we say a *circular* table, a *quadrangular* court, a *conical* piece of metal, &c. The reason is, that a million of things participating the same figure, participate it equally. To say, therefore, that while A and B are both quadrangular, A is more or less quadrangular than B, is absurd. The same holds true in all attributives denoting definite qualities, whether contiguous or discrete, whether absolute or relative. Thus, the *two-foot* rule A, cannot be more a two-foot rule than any other of the same length. *Twenty* lions cannot be more twenty than twenty flies. If A and B be both *triple* or *quadruple* of C, they cannot be more triple or more quadruple one than the other. The reason of all this is, that there can be no comparison without intension and remission; there can be no intension and remission in things always definite: and such are the attributes which we have last mentioned." This reasoning, which, as far as it goes, is very just, seems, nevertheless, to require some further developement. What is here meant by "things always definite?" Plainly, what we have already called ideas, and those clearly conceived. The idea of a circle, when clearly conceived, is a thing always definite. By the generality of men it *is* clearly conceived; and, consequently, they would think it absurd to say, that one table was more circular than another; but those who have not a distinct idea of a circle would not perceive the absurdity of the expression. To them, circularity would appear capable of intension and remission; and therefore they would conclude, that this quality admitted of comparison as much as sweetness or sourness, hardness or softness, heat or cold. Hence we find in Language such words as *round*, which expresses the idea of circularity in a vague and indistinct manner; and these words are commonly used in the comparative as well as in the positive degree. For the same reason, all words signifying bodily sensation are capable of comparison; for though we agree generally in the meaning which we attribute to them, yet there is no definite idea to which any one of them can be distinctly referred. Men employ the terms "hot, cold, white, black, green," &c. so as to convey to each other's Mind certain general notions, but not to communicate precise and distinct ideas, like those expressed by the words "square," or "triangle." Again, in Moral qualities there is usually the same indistinctness. We say, one man is braver or wiser than another; because we possess no absolute standard of bravery or wisdom. If we possessed such a standard, we should simply say, that each of the two was either brave or not brave, wise or unwise. There is no more common comparison in all Language than between that which is good and that which is better; yet the pure *idea* of goodness presented to us by the Christian Religion excludes all comparison—"There is none good but one, that is GOD."

We have observed that where there are variations of degree, those variations may be compared together. Grammarians have fixed three Degrees of Comparison; the positive, the comparative, and the superlative.

It seems material to observe, that the comparison here referred to is of two kinds. We may either compare a quality, as existing in any given substance, with the same quality as existing in other substances, or we may compare it with some assumed notion of the quality in general.

Nouns Adjectives Degrees.

Positive.

The *positive* is the simple expression of the quality: and Harris says, it is improperly called a Degree of Comparison; but in this he seems to be wrong; for it is that form in which the comparison of equal Degrees of the same quality is expressed, either affirmatively or negatively. Thus we say, in the positive Degree, "Scipio was as brave as Cæsar," "Cicero was not so eloquent as Demosthenes."

Comparative.

The *comparative* expresses the intension or remission of any quality in one substance, compared with the same quality in some one other substance, as, "Cicero was more eloquent than Brutus;" "Anthony was less virtuous than Cicero." Hence it is manifest, that there are, properly speaking, two kinds of the comparative Degree, one expressing the *more*, and the other the *less* of the quality compared. Languages in general have employed a peculiar inflection only to express the former; but the latter is in its nature no less capable of expression; and both belong to those distinctions which constitute Universal Grammar. It is to be remarked, that the comparative, though it excludes the relative positive, does not necessarily include the absolute positive. If we say, "John is wiser than James," we exclude the assertion, that "James is as wise as John;" but we do not necessarily include the assertion either that "John is wise," or that "James is wise." All that may really be intended by the affirmative, is a negation of the negative. It may only be meant to assert that "John is *less unwise* than James."

Superlative.

The *superlative* expresses the intension or remission of a quality in one thing or person, compared with all the others that are contemplated at the same time. There must be more than two objects compared, but the number compared may be indefinite: we may say, Octavius was the most prudent of the Triumvirate; Homer was the most admirable of Poets; Solomon was the wisest of men. In other respects, what we have observed of the comparative, applies equally to the superlative, which may properly be considered as expressing the most or the least of the quality in question, but which does not, any more than the comparative, necessarily include the absolute positive. Of this remark, the common proverb, "Bad is the best," affords a sufficient illustration.

Hitherto, we have only spoken of the comparison of qualities existing in one subject with those existing in another; but the comparison may be made with a general conception of the quality: and here also may be three similar Degrees. Where the quality is supposed to be of the general or average standard, we use the positive; where we mean to express simply an excess beyond that standard, we use the comparative. Thus Virgil says:

Tristior, et lacrymis oculos suffusa nitentes;

and Horace,

Rusticior paullo est.

Lastly, where we mean to express a high Degree of eminence in the quality of which we speak, we use the superlative, as *vir doctissimus, vir fortissimus*, a most learned man, a very brave man; that is to say, not the bravest or most learned of all men that ever existed, or of any given number of men; but a man pos-

sessing the quality of learning or bravery in a degree far beyond the common standard.

It is of small consequence to inquire whether all these forms of speech together are properly named *Degrees of Comparison*, and equally immaterial whether the particular names, *positive*, *comparative*, and *superlative*, are well chosen to designate each Degree. Many eminent Grammarians have contended on these points. Vossius objects to the name *positive*, because the two other Degrees are equally positive, that is, equally *lay down* their respective significations, (whence the Greeks called the superlative *hyperthetic*,) from τιθέναι, to lay down. Not more appropriate, says he, is the name of the *comparative* Degree, since comparison is applied to many words, both nouns and adverbs, which are not of that Degree, as the adjectives, *like, unlike, double*; and among adverbs, *equally*, &c. Moreover, comparison is effected no less by the superlative than by the comparative: for it would be equally a comparison if I were to say, speaking of Varro, Nigidius, and Cicero, "Varro is the most learned of the three;" as if I were to say, speaking of Varro and Nigidius only, "Varro is the more learned of the two." Lastly, the word *superlative* is not well chosen, since it merely signifies preference, or the raising one thing *above* another: and in this sense the comparative itself is a superlative; for in saying, "Varro is more learned than Nigidius," I prefer, or raise Varro above Nigidius in regard to learning.

For similar reasons, Scaliger proposed new names for the three Degrees. The first he called the *aörist*, or indefinite; the second, the *hyperthetic*, or exceeding; and the third, the *acrothetic*, or highest Degree. Quinctilian and others call the positive the *absolute* Degree; others call it the *simple*, and so forth; but none of these names having come into general use, we think it more convenient to hold to those which are commonly received; not considering the choice of a name as very important, compared with the accuracy of a distinction; and that the three variations of adjectives in Degree are essential to Grammar, we have already sufficiently proved.

It is of more consequence to note, that intension and remission not being confined to adjectives, the Degrees of comparison are not confined to them, but are common also to certain verbs, participles, and adverbs; in short, to the whole class of attributives, (as they are called by Harris,) provided that, in signification, they import qualities which may be increased or diminished. Thus, as the adjective "amiable" admits of the comparative and superlative "more amiable," and "most amiable;" so we may use the expressions "more loving," "most loving;" "to love well," "to love better," "to love more," "to love most of all." These indications of Degree, however, have been rarely expressed by inflection, except in adjectives; and this seems to be the true reason why the Degrees of Comparison have often, but inaccurately, been considered by Grammarians as belonging to adjectives alone. It is scarcely worth while to occupy attention with such words as αὐτότατος, used by Aristophanes; or *ipsissimus*, employed by Plautus. Some critics, indeed, have seriously adduced these as examples of comparison in pronouns, as if I could be more I, or He more He, in reality; whereas it is plainly seen, that the Comic writer, by a natural boldness in the use of Language, employs these pronouns in a secondary sense, as if they expressed a quality instead of a substance; but not as if a man could be more or less himself without losing his personal identity.

Kinds of adjectives.

We come now to consider the two great classes into which adjectives may be divided; and these, as we have before observed, depend on their expressing, or not expressing, action. Thus, if we say "a *four-footed* animal," although the quality of being four-footed has reference, in this instance, to action, as its final end; yet, as it does not express action, (for a table or a chair may also be four-footed,) this is an adjective of the first-mentioned kind. On the other hand, if we say "a *moving* animal," we clearly express that action is really taking place: this, therefore, is an adjective of the second kind. Now, of these two kinds, the former are exclusively called *adjectives* by the majority of Grammarians; but the latter are as commonly called *participles*; and we adopt these distinctive terms from an unwillingness to alter the received nomenclature of Grammatical Science; but at the same time, we wish it to be clearly understood, that both the adjective and participle of the common Grammarians fall under the definition which we have above given of the word adjective in its largest sense.

Of the adjective simple, or unmixed with any idea of action, little remains for us to observe; but before we proceed to the consideration of the participle, it may be proper to notice a large class of adjectives, which, though they do not express action, yet bear reference to it. Such are those words expressive of the capability or habit of action, which Mr. Tooke, in his eager desire for singularity, has thought fit to class among the participles. There is great hazard when a writer chooses to treat all his predecessors with contempt, that he may chance to fall into very gross errors himself. Mr. Tooke has confounded, in his new scheme of participles, the verbal adjectives, gerunds, and participles of former writers; and, at the same time, has laid down no clear definition of his own to guide us out of the labyrinth. What is more, he has adopted as participles the verbal adjectives in *bilis*, *ivus*, and *icus*, and excluded those in *ax*, *arius*, *bundus*, *icius*, &c. which seem quite as much entitled to the same distinction.

Upon a full consideration of all these different kinds of adjectives, there seems to be no reason for classing them apart from the simple adjective, and as little for confounding them with the participle.

They ought not to be separated from the simple adjective, because they do, in fact, express only a simple quality; and it is difficult, if not impossible, to draw a line between qualities which are originally derived from action, and qualities not so derived. Let us take, for instance, the word *falsus*, false. No doubt this is derived from *fallo*, which expresses the *act* of failing or deceiving; yet, by a transition of meaning, it comes to signify simply that which is not true. In like manner, many of the words which Mr. Tooke treats as participles have been really introduced into the English Language as simple adjectives, without the least reference to the *action*, which their radicals expressed in other Languages. Take, for instance, the word "palpable." We commonly say, "it is palpably false," "the truth is palpable," &c.; yet, perhaps, few persons, when they use these phrases, entertain any notion of feeling and handling the truth or falsehood in ques-

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tion, though *palpare*, to feel or handle, is the undoubted origin of this word. The same may be said of "ductile," "frail," "sensible," "noble," and many other English adjectives, which have not the slightest pretence to be considered as participles.

If the mere derivation from a verb is to entitle a word to be called a participle, we should have numerous classes both of substantives and adjectives so distinguished; for if *ductilis* be a participle, because it is derived from *duco*, so is *audax*, because it is derived from *audeo; ridiculus*, because it is derived from *rideo;* and a thousand other adjectives. Nay, we may add to this list the substantives derived from verbs, if the mere derivation is to be a test of the Grammatical use. Thus, we may say, that *pistrinum*, a bakehouse, is a participle of *pinso*, to bake; *juramentum*, an oath, of *juro*, to swear; *judicium*, a judgment, of *judico*, to judge, &c.

The truth seems to be, that in this, as in numberless other instances, Mr. Tooke has mistaken the History of Language for its Philosophy. Because the word *noble* is derived from *nosco*, to know, therefore he calls it a participle of that verb! At this rate, all the Parts of speech must become an inextricable mass of confusion; for, Historically speaking, each is derived from the other, and there cannot be any rule which gives any one the precedence. If we look to the signification, all is clear. Either a given adjective expresses action, or it does not. If it does not, it is a simple adjective; and the circumstance of its referring to the habit or capacity for action cannot alter its character. The words "forcible" and "culpable" relate originally to the actions of forcing and blaming; but they relate to them only as the groundwork of an existing quality, and not as being really in action, or as having been so, or to be so, at any given time. These considerations will probably suffice to clear away all the difficulties which Mr. Tooke has raised respecting what he calls the participles of the potential mood active, the potential mood passive, the official mood passive, and the future active. They are all, as used in the English Language, simple substantives, or simple adjectives: and to rank them among participles, would not only be to oppose the great majority of writers who have treated on these subjects, but to confound all reasonable Principles relating to this part of Grammar.

Participle.

We come, then, to that Part of speech which is commonly denominated the *participle*. The origin of this name is well known. *Partem capit a nomine, partem a verbo.* But this is an explanation which is merely applied to the learned Languages. The definition of Vossius is, *participium est vox variabilis per casus significans rem cum tempore.* Here, too, we see nothing of Universal Grammar. The being variable by cases is a mere accident of certain Languages. The signifying a thing, with time, depends indeed on more general Principles, and these it is necessary to examine.

What is meant, in this part of the definition, by "signifying a thing," we need not, perhaps, make matter of dispute. We will assume, that it means, in the language which we have adopted, "naming a conception." The participle simply *names;* it does not *assert*. The words, "loving, moving, reading, thinking," &c. assert nothing respecting these acts; they merely name the acts, or rather they name the conceptions, as *in action*. It is said that the participle is ranked among nouns when it constitutes the *subject* of a logical proposition; and among verbs when it forms the *predicate;* but this is not accurate; a participle, as such, can never form the subject of a proposition. The example given is, *Militat omnis amans*, Πᾶς ὁ ἐρῶν πολεμεῖ; but in this instance *amans* is a mere adjective, agreeing with *homo* understood; and it is the same in the Greek. On the other hand, when the participle is a predicate, as *Socrates est loquens*, it fills the proper office of an adjective; and is not to be treated as a verb, at least in the sense which we have attached to the latter term.

Nouns Adjective. Participle.

The adsignification of time is proper to the participle, inasmuch as time is essential to action. This point, however, Mr. Tooke contests upon the ground, that the Latin participles, present, past, and future, are not confined to the times from which they respectively receive their designations. *Proficiscens* is a participle of the present tense; yet Cicero says, *abfui proficiscens*, thus connecting time present with time past. So *profecturo tibi dedi literas*, connecting the past with the future: and again, *quos spero societate victoriæ tecum copulatos fore;* where *spero* is present, *copulatos* past, and *fore* future. None of these examples, however, prove any thing against the expression of time by the participles, but merely that time is contemplated in various lights by the Mind in one and the same sentence. Thus, in the phrase *abfui proficiscens*, the first word relates to the time of speaking, and the second to the time of acting. The going was present, when the absence (which is now past) was present. Again, *dedi* refers to a time past; but when that time was present, the departure (expressed in *profecturo*) was future. A thousand such cases as these would lead to no inference whatsoever against the expression of time by the participle.

It is necessary to observe, however, that words which express time, express it in two ways, either as simple existence, or as relative to the different portions of duration. Thus, when we say "justice is at all times mercy," the present is a mere expression of existence, a present continuous. So when we say "the Sun *rises* every day," we speak of an act habitually present. It is the nature of the Human Mind to be able thus to contemplate duration; but this in no degree interferes with, still less contradicts, the view which we take of different portions of time, as past, present, and future, with relation to each other. The assertion, for instance, that the Sun rises every day, does not at all clash with the other assertion, that the Sun rises at this moment. In both cases time is referred to; a certain portion of time is designated in the one case, which coincides with the general assertion in the other; and, in fact, the difference between the two assertions does not depend on the verb itself, but on the accompanying words "every day" and "this moment."

In these respects the verb and participle agree. The participle is an adjective so far participating the nature of the verb as to signify action, and it cannot signify action without the capability of also signifying time.

Particular Languages may or may not have separate words adapted by inflection to signify the different portions of time in a participial form. In truth, the notion of time is in all such cases a new element in the compound conception, which compound conception may be expressed by one word or by several. The complexity of conception may go still further. It may include the

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distinctions of active and passive, of absolute and conditional; and, in short, of all those which we shall have to consider when we come to treat of the verb.

Hence we see, that Languages may have as great a variety of participles as they may of moods and tenses; and it does not seem of the nature of Language altogether to exclude participles from the Parts of speech; for Mr Harris is perfectly right in saying, that if we take away the assertion from a verb, there will remain a participle. Of course he is speaking of the signification, and not of the sound, and therefore Mr. Tooke's ridicule of this passage is entirely misplaced. It is an observation as old as Aristotle, that the words "Socrates speaks" are equal in signification to the words "Socrates is speaking;" but it is evident that the assertive part of this sentence consists entirely in the word "is;" which word being taken away, the word "speaking" still expresses a quality of Socrates, and expresses that quality in action, and is therefore a participle. And so it will happen with every verb, as is instanced by Harris in the words γράφει γράφων, "writeth," "writing." Tooke misrepresents Harris as saying, that, by removing ει and *eth*, he takes away the assertion; whence he concludes, that Harris supposed the assertion to be implied in those syllables; but Harris says nothing about taking away ει and *eth*. He says what is very true, that the words γράφει and *writeth* imply assertions, and that in the words γράφων and *writing*, the assertion is taken away, and yet there remain the same time and the same attribute; which expressions of time and attribute, without assertion, constitute a participle.

It has been laid down as a rule by some writers, that there can be no participles but such as are derived from verbs; and hence they deny that such words as *togatus*, *galeatus*, &c. are to be called participles. Augustinus Saturnius, who treats particularly on this point, calls them, by way of distinction, *participials*. It is manifest, however, that this is a distinction altogether nugatory, in regard to Universal Grammar. When Othello says

My demerits may speak *unbonnetted*,

he uses exactly the same form of speech, as if he had said *uncovered*, and the one word is as truly a participle as the other; although there may be no authority for the use of the verb "to bonnet." Uncovered and unbonnetted equally express a quality, with reference to an action of past time, *viz.* the removing the cover or bonnet from the head; and it is by this signification, and not by their etymology, that the Part of speech to which they belong is to be determined.

We must not be surprised to find, that participles of different classes pass into each other. Many active participles come to have a passive signification. The word *evidens*, which was originally active, is found with a passive meaning, from whence our common adjective, *evident*, is derived. This is a circumstance not peculiar to participles; for when we come to treat more at large of those transitions of meaning, which are the groundwork of Etymological Science, it will be found that they apply to every Part of speech indifferently. Men cannot always find a separate term to express each distinct shade of thought, and they naturally avail themselves of those expressions which come the nearest to their meaning.

From what has before been said on the subject of comparison, it is clear that participles, as well as other adjectives, when they express qualities capable of intension and remission, may admit the three Degrees of comparison: thus we may say *amantior* as well as *durior*, *amantissimus* as well as *durissimus*. It matters not, that in some Languages the idiom will not allow of expressing the Degrees of comparison by inflection; that, for example, in English we cannot say *lovinger*, or *lovingest*; this is a mere accident of the particular Language, depending principally on circumstances connected with its sound; and it is to be observed, that however barbarous such words as *lovinger* or *lovingest* might sound to the ear, yet they would be perfectly intelligible to the Mind: there would be nothing absurd or contradictory in the combination of the thoughts; for the same combination is effected by the words "more loving," and "most loving;" and in all Languages there must be means more or less concise or circuitous to express such combinations.

Nouns Adjective. Gerund.

We have seen how the conception of a quality considered alone, and rendered the subject of assertion, becomes a noun substantive; and this applies, in Principle, as well to those qualities which are expressed by participles, as to those which are expressed by other adjectives. Whether the same or a different word shall be employed for this purpose is, again, a matter of particular idiom. In English, we use the very same word for both purposes. Thus, "singing," "dancing," &c. may be used in construction as adjectives, or as substantives of the sort commonly called abstract. We may say "a singing man," "a dancing woman;" or we may say, "singing is an accomplishment," "dancing is a recreation," &c. In Latin, the idiom is different: *cantans*, *saltans*, &c. can only be used in the former of these two ways; but, nevertheless, a similar Principle is observable in the use of what are called gerunds and supines.

Gerund.

Scaliger gives the following account of the *gerund*: "From these (participles) our ancestors chose certain tenses, by means of which they might imitate those Greek terms λεκτέον, μαχητέον, &c. but with a more ample and extensive use. These they called *gerunds*, assigning them to three cases, *pugnandi*, *pugnando*, *pugnandum*; of which, the second preserved the power of a participle, but so much the more aptly as the verbs were excelled by the participles. For, as the cause of action is more plainly shown by saying cædens *vulneravi*, than by saying *cecīdi*, and better still by saying *quia* cæderem *vulneravi*, the whole of this is expressed by the gerund cædendo *vulneravi*. Moreover, in many things the form and the end are the same; but the end is partly out of us, as the ship is a thing out of the shipbuilder; and partly within us, in our Minds, as is that which is called an *idea*, by which we are impelled to the external end. Now both of these they very skilfully expressed; for both *pugnandi* and *pugnandum* signify the end. Thus I may say, *pugnandi causâ equum ascendi*, I mounted my horse for the purpose *of fighting*; or *pugnandum est ex equo*, I must fight (or the fighting must be) on horseback." "Hence it appears that these (gerunds) are participles, differing little from other participles, either in nature, or use, or even in form." Again he observes: "some writers have called these gerunds from their use *participial nouns*; for they are neither pure nouns, since they govern a case; nor are they pure participles, since, with a passive voice, they bear an active signification."

The same author thus speaks of the *supine*. "Nearly similar is the explanation to be given of the supines; but these latter express the same meaning more forcibly. Thus, *eo ad pugnandum* signifies a future action; *eo pugnatum* expresses the future so as to be quite absolute." "Hence it signifies activity with actives, and passiveness with passives: *eo factum injuriam*, or *injuria mihi factum itur*; but indeed it always savours, in some degree, of passiveness; for it does not so much mean *eo ut faciam*, as it means *eo ut hoc fiat*: as if one were to say, I am going indeed for the purpose of doing so and so, but I hope it is already done; and like Sosia's speech, *Dictum puta*, 'suppose it said.'" "Since, therefore, the end (or aim) of an action was to be thus signified, the other extreme was not improperly expressed by a different word." Hence Scaliger explains the different use of the supines in *um* and *u*, the latter of which he regards as a sort of ablative case. "There is equally a movement," says he, "from and to an object; and therefore we rightly say *venatu venio*, as we do *venatum vado*." He goes at length into these considerations, opposing in some measure what other Grammarians had said of the supine in *u*; but these questions are beside our present object: and all that is necessary for us here is to show the chain of connection which unites the participle, as an adjective, on the one hand with the noun substantive, and on the other with the gerunds, supines, and infinitive mood.

Pronoun.

Hitherto we have considered the noun only in its primary use, whether as substantive or adjective: we have now to regard it in a secondary light, under the common Grammatical designation of a pronoun.

The name of the pronoun is sufficiently descriptive of its use, which is to stand in the place of another noun. The necessity for such words in Language is obvious; but as it has been well and briefly explained by Mr. Harris, we shall adopt that learned author's words. "Every object which presents itself to the senses, or the intellect, is either then perceived for the first time, or else is recognised as having been perceived before. In the former case it is called an object τῆς πρώτης γνώσεως of the first knowledge or acquaintance; in the latter it is called an object τῆς δευτέρας γνώσεως of the secoud knowledge or acquaintance. Now as all conversation passes between particulars or individuals, these will often happen to be reciprocally objects τῆς πρώτης γνώσεως, that is to say, till that instant unacquainted with each other. What then is to be done? How shall the speaker address the other when he knows not his name? or how explain himself by his own name, of which the other is wholly ignorant? Nouns, as they have been described, cannot answer the purpose. The first expedient upon this occasion seems to have been δεῖξις, that is, *pointing*, or indication by the finger or hand, some traces of which are still to be observed as a part of that action, which naturally attends our speaking. But the authors of Language were not content with this: they invented a race of words to supply this pointing; which words, as they always stood for substantives, or nouns, were characterised by the name of ἀντωνυμία, or pronouns." So far Mr. Harris. His observations, indeed, apply in strictness only to the personal pronoun; but upon similar Principles rests the necessity for the other classes of pronouns, as will easily appear when we come to consider them separately.

As the noun is divided into substantive and adjective, so the pronoun, its representative, exhibits the same diversity. If it be necessary to have a word representing a whole class of substantives, it is equally necessary that the quality which consists in belonging to that class should be represented. If *I*, or *you*, or *he*, be to be expressed, *mine*, or *yours*, or *his*, is to be expressed also.

We begin, therefore, with the pronoun substantive: and of this we shall consider, first, the distinctions which relate to it as a member of a simple proposition; and, secondly, those which relate to it more generally.

Considered as the subject of a simple proposition, we have to notice in the pronoun not only the distinctions of number, gender, and case, which are common to it with the noun, but also the further and peculiar distinction of *person*. The noun substantive being the name of a conception, that is of a thing, or of a person, does not specify whether that thing or person is the speaker, or is spoken of, or spoken to. One of these three characters it must needs sustain: and in the intercourses of speech that character is soon distinguished: and here also the statement of Harris is peculiarly clear and satisfactory.

First person.

"Suppose the parties conversing," says he, "to be wholly unacquainted, neither name nor countenance on either side known: and the subject of the conversation to be the speaker himself. Here, to supply the place of *pointing*, by a word of equal power, they furnished the speaker with the pronoun I. 'I write, I say, I desire,' &c.: and as the speaker is always principal with respect to his own discourse, they called this, for that reason, the pronoun of the *first person*."

Second person.

"Again, suppose the subject of the conversation to be the party addressed. Here, for similar reasons, they invented the pronoun *thou*. 'Thou writest,' 'thou walkest,' &c.; and as the party addressed is next in dignity to the speaker, or at least comes next to him, with reference to the discourse, this pronoun they therefore called the pronoun of the *second person*."

Third person.

"Lastly, suppose the subject of the conversation neither the speaker, nor the party addressed, but some third object, different from both: here they provided another pronoun, *he*, *she*, or *it*, which, in distinction from the former two, was called the pronoun of the *third person*." "And thus it was that pronouns came to be distinguished by their respective persons."

The description of the different persons here given is taken from PRISCIAN, who took it from APOLLONIUS. *Personæ pronominum sunt tres, prima, secunda, tertia. Prima est cum ipsa, quæ loquitur, de se pronuntiat; secunda, cum de eâ pronuntiat ad quam directo sermone loquitur; tertia, cum de eâ quæ nec loquitur, nec ad se directum accipit sermonem*, l. xii. p. 940. THEODORE GAZA gives the same distinctions: Πρῶτον (πρόσωπον, sc.) ᾧ περὶ ἑαυτοῦ φράζει ὁ λέγων· δεύτερον ᾧ περὶ τοῦ, πρὸς ὃν ὁ λόγος. τρίτον ᾧ περὶ ἑτέρου. *Gaz. Gram.* l. iv. p. 152.

This account of persons is far preferable to the common one, which makes the first the speaker, the second the party addressed, and the third the subject; for though the first and second be, as commonly described, one the speaker, the other the party addressed: yet, till they become subjects of the discourse, they

Grammar. have no existence. Again, as to the third person's being the subject, this is a character which it shares in common with both the other persons, and which can never, therefore, be called a peculiarity of its own. To explain by an instance or two: When Æneas begins the narrative of his adventures, the second person immediately appears, because he makes Dido, whom he addresses, the immediate subject of his discourse.

Infandum, Regina, jubes renovare dolorem.

From henceforward for 1500 verses (though she be all that time the party addressed) we hear nothing further of this second person, a variety of other subjects filling up the narrative. In the mean time the first person may be seen every where, because the speaker is every where himself the subject: they were, indeed, events, as he says,

——— *Quæque ipse miserrima vidi,*
Et quorum pars magna fui.

Not that the second person does not often occur in the course of this narrative; but then it is always by a figure of speech, when those who, by their absence, are, in fact, so many third persons, are converted into second persons, by being introduced as present.

When we read Euclid, we find neither first person nor second in any part of his whole Work. The reason is, that neither the speaker nor the party addressed (in which light we may always view the writer and his reader) can possibly become the subject of Pure Mathematics.

It follows, from what has here been said, that the pronoun is strictly a necessary part of speech; for though, as standing in the place of other nouns, it may be considered a mere abbreviation of discourse, yet circumstances often occur in which such abbreviations become indispensable. It is clear that discourse could not be intelligibly carried on where the parties were not known to each other by name, and did not also know by name each individual of whom they might speak, unless there were some means of distinguishing them otherwise than by their separate and individual names, which means are really supplied by the pronoun.

It has been observed, that notwithstanding the separate characteristics of each person, there may be a coalescence of the pronouns of different persons; but this is subject to certain restrictions. The pronoun of the first or second person may easily coalesce with the third; but the first and second cannot coalesce with each other. For example, we may say, (and the difference of idiom in different Languages does not affect these expressions,) "I am he," or, "thou art he;" or, as in the text, "art *thou he* that should come, or do we look for another?" But we cannot say, "I am thou," nor "thou art I:" the reason is, there is no absurdity for the speaker to be the subject also of the discourse; as when we say, "I am he;" or for the person addressed, as when we say, "thou art he;" but for the same person, in the same circumstances, to be at once the speaker and the party addressed is impossible; and, consequently, so is the coalescence of the first and second person.

Number. Since the pronoun stands in the place of a noun, and since *number*, as we have seen, is a conception which may be combined in general with nouns, it follows that the pronoun may have the distinctions of number; nor, indeed, is it easy to conceive a Language so constructed as to have pronouns without such a distinction. As to the first person, it is clear that there may be many speakers at once of the same sentiment, or, what comes to the same thing, one may deliver the common sentiment of many, and in their name; for the same reason, therefore, that the pronoun *I* is necessary, the pronoun *we* is so too. Again, the singular *thou* has the plural *you*, because a speech may be spoken to many, as well as to one: and the singular *he* has the plural *they*, because the subject of discourse often includes many things or persons at once.

Nouns. Pronouns. Number. Gender.

Gender. The pronoun is also susceptible of the distinction of gender, because the noun which it represents is so. A difference, however, has been said to exist in this respect between the pronouns of different persons: and the reasoning thereon is plausible. It is certainly true that the pronouns of the first and second person, both in the dead and living Languages, have no distinct inflection expressing their gender; and the reason for this is alleged to be that the speaker and hearer being generally present to each other, it would have been superfluous to have marked a distinction by art, which from nature, and even dress, was commonly apparent on both sides. *Demonstratio ipsa*, says Priscian, *secum genus ostendit*. However, it is by no means true that the pronouns of the first and second person have no gender. They have not, indeed, in any known Language, inflections distinguishing them in point of gender, but they always take, in construction, the gender of the noun which they represent. Thus Dido,

——— *cui me moribundam deseris hospes?*

And Mercury addressing Æneas,

——— *Tu nunc Carthaginis altæ*
Fundamenta locas, pulchramque uxorius urbem
Exstruis?

It is agreed on all hands that the pronouns of the third person must almost of necessity receive the distinctions of gender in all Languages. These pronouns are called in Arabic the pronoun of the absentee, and, in fact, they usually refer to persons or things which being absent require to be distinguished, as to gender, &c. by some expression in the discourse. It is further to be observed, that the pronouns of the first and second person apply only to certain known and present individuals; whereas, the pronouns of the third person may, in the course of one and the same speech, refer to a great diversity of objects, requiring to be distinguished by their respective genders. "The utility of this distinction," says Harris, "may be better found in supposing it away." Suppose, for example, we should read in History these words: *he caused him to destroy him*—and that we were to be informed that the *he*, which is here thrice repeated, stood each time for something different, that is to say, for a man, for a woman, and for a city, whose names were *Alexander*, *Thais*, and *Persepolis*. Taking the pronoun in this manner, divested of its gender, how would it appear which was destroyed, which was the destroyer, and which was the cause of the destruction? But there are no such doubts when we hear the genders distinguished; when, instead of the ambiguous sentence, "*He* caused *him* to destroy *him*," we are told, with the proper distinction, that "*She* caused *him* to destroy *it*." Then we know with certainty what before we knew not, *viz*. that the pro-

moter was the woman; that her instrument was the hero; and that the subject of their cruelty was the unfortunate city.

Case is a distinction which we have already observed to be not essential to the noun, but only accidental. It therefore is to be ranked among the accidents of the pronoun; yet, so frequent is the occasion to use pronouns, that many of them, especially those which are particularly denominated personal, have the variations of case, even in Languages which vary their nouns in this respect very little or not at all. When a person speaks of himself as the performer of any action, he seems naturally led to adopt a different phraseology from that which he employs in speaking of the action as done toward him; and hence the difference between *I* and *me*, *thou* and *thee*, runs throughout far the greater number of known Languages. After all, Universal Grammar only furnishes the reason for this difference, when it exists, but does not prove its existence to be necessary. There may be Languages of which the pronouns have no cases; but where they have cases, the same function is performed by each case in the pronoun as in the noun.

Substantive pronouns have been distinguished, and, as it seems, with sufficient accuracy, into *prepositive* and *subjunctive*. By prepositive are meant all those which are capable of introducing or leading a sentence without having reference, at least for the purposes of construction, to any thing previous. We insert these words, "at least for the purposes of construction," because in truth all but the pronouns of the first and second person must refer to some person or thing previously indicated. When we say, "*he* reigned," or "*she* lived," we presume that the persons included by *he* and *she* are previously known. These pronouns, however, may introduce or lead sentences which do not depend on any previous sentence in point of construction. But it is not so with the other class of pronouns, *viz.* the subjunctive. These cannot introduce an original sentence, but only serve to subjoin one to some other which is previous. The principal subjunctive pronouns in English are *who* and *which*, and sometimes *that*. It does not seem essential to the constitution of a Language that there should always be such pronouns as these; for they may always be resolved into another pronoun and a conjunction; and consequently by such other pronoun and conjunction their place may always be supplied. Let us take the example given by Harris. We will suppose that it is desired to combine into one sentence the two following propositions:

1. "Light is a body."
2. "Light moves rapidly."

Here it is obvious that the use of the noun *light*, in the second proposition, may be supplied by the pronoun *it*, as thus:

"Light is a body:
It moves rapidly."

This slight change, however, leaves the two propositions still distinct: let us then connect them by the conjunction *and*; thus:

"Light is a body;
And it moves rapidly."

Here is a connection of the two propositions, yet still not so much dependence of the latter on the former, not so intimate a union therefore of the parts, as if, for the words "and it," we substitute the subjunctive pronoun *which*; thus:

"Light is a body, *which* moves rapidly."

Accordingly, we see that in the punctuation, which most accurately represents the proper mode of reading the passage, we gradually diminish the interval between the two propositions, from a period to a comma.

Of the nature of the subjunctive pronoun is the interrogative: and therefore we very commonly find the same word performing these two functions. Thus, in English, the subjunctives *who* and *which* are used as interrogatives, though with a remarkable difference in their application. As subjunctives, in modern use at least, *who* is applied to persons, and *which* to things. As interrogatives they are both applied to persons, but *who* indefinitely, and *which* definitely. Thus, the question, "Who will go up with me to Ramoth-gilead?" is indefinitely proposed to all who may hear the question: but when our Saviour says, "*Which of you*, with taking thought, can add one cubit to his stature?" the interrogation is individual, as appears from the *partitive* form of the words "which *of you*;" that is to say, "what one among you all." These applications of particular words are indeed matters of peculiar idiom; but the distinctions of signification to which they relate properly belong to the Science of which we are treating.

The interrogative pronouns are necessarily of a relative nature, and on that account were ranked by the Stoics under the head of the *article*; but as they do in fact stand for, and represent nouns, they are properly called pronouns. On interrogatives in general, Vossius has the following just observation:—"It appears to me, that the matter stands thus: there are two principal classes of words, the noun and the verb; and, therefore, to one or other of these every interrogation must refer. For, if I ask *who*, *which*, *what*, *how many*, I inquire concerning some noun; but if I ask *where*, *whence*, *whither*, *when*, *how often*, I inquire concerning some verb. As, therefore, the words which are subsidiary to the verb are called adverbs, so the words which refer to the noun should be called pronouns."

Of all the substantive pronouns, those only which directly and simply represent the three *persons* of a discourse, as above explained, that is to say, the subject of the discourse, whether that be the speaker, the person spoken to, or the person or thing spoken of; these three classes alone, we say, are properly called pronouns *personal*. Some Grammarians seem to have supposed, that all but the personal pronouns of the first and second person were to be considered as belonging to the third person. This, however, is inaccurate, at least with respect to the relatives, *who*, *which*, *that*, as may be observed in those lines of the old song:

What! you that liked!
And I, that loved!
Shall we begin to wrangle?

Where the relative *that* is of the second person in the first line, and of the first person in the second line: and if translated into Latin it must be rendered, not *tu quæ amabat*, and *ego qui amabat*, but *tu quæ amabas*, and *ego qui amabam*.

We shall not here go into a detailed consideration of the various distinctions which different authors have

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made in the other classes of pronouns, the demonstrative, the distributive, &c. It may suffice to say, that their number and variety in any one Language must, in a great measure, depend on the classification of conceptions, which had become habitual among the early formers of that particular Language. Thus we cannot in English express, without periphrasis, the Latin pronouns *qualis*, *quantus*, &c. any more than we can the adverbs *quoties*, *qualiter*, &c. Nor must it be forgotten that many of these pronouns pass into different classes according as they are used in particular passages, *Sunt ex istis*, says Vossius, *quæ pro diverso, vel usu vel respectu, ad diversas pertineant classes*.

Adjective pronouns.

This latter remark applies not only to the various uses of substantive pronouns, but to their transitions from adjective to substantive. Almost all pronouns, except the first and second personals, are clearly adjectives in origin; but we cannot admit that they continue to be such when they stand by themselves, or, as Lowth rather singularly expresses it, "seem to stand by themselves." It is true, that in such cases, they often have "some substantive belonging to them, either referred to or understood;" but this only proves that they are pronouns. Whether we say "*this* is good," "*it* is good," or "*he* is good," there is always some noun referred to, or understood: and the words *it* and *he* "seem to stand by themselves," just as much as the word "*this*" does. So in the phrases "*one* is apt to think," and "*I* am apt to think," the words *one* and *I* equally "seem to stand alone," that is to say, they equally do stand alone. They perform the function of *naming* an object, so far as it is necessary to be named; and they name it not as a quality of another object, but as possessing a substantive existence in itself. The words *this*, *that*, *who*, *which*, *all*, *none*, and many of a similar kind, are therefore (in our view of them) substantive pronouns when they stand alone, but adjective pronouns when they are joined to a noun substantive. When Antony says

> *This*—*this* was the unkindest cut of all,

we consider the word *this* to be a substantive pronoun. It may, indeed, be explained by transposition, as if it were, "this cut was the unkindest of all;" but such is not the order of the thoughts: and, in fact, the particular wound inflicted by Brutus had been before described at some length, but the noun *cut* had not been used: and supposing that, for dramatic effect, the line had been broken off at the word "was," it would have been impossible to say that the pronoun *this* had any specific reference to this particular noun *cut*, as we may easily perceive by so reading the passage.

> See, what a rent the envious Casca made!
> Through this the well-beloved Brutus stabb'd;
> And as he pluck'd his cursed steel away,
> Mark how the blood of Cæsar follow'd it,
> As rushing out of doors, to be resolv'd,
> If Brutus so unkindly knock'd, or no;
> For Brutus, as you know, was Cæsar's angel.
> Judge, O ye gods, how dearly Cæsar lov'd him!
> This—this was——

If the passage had thus broken off, the pronoun *this* would have rather seemed to refer to the whole narrative of the share which Brutus had taken in the transaction; that narrative presenting to the Mind one complete and definite conception.

Nouns. Pronouns. Adjective.

A passage in *Othello* will further illustrate our meaning. Iago pretends to caution Othello against suffering his mind to encourage any suspicion against his wife's honour:

> ——— O beware, my lord, of jealousy!
> It is a green-eyed monster which doth make
> The meat it feeds on, &c. &c.

After he has pursued this strain of reasoning for some time, Othello, interrupting him, exclaims with surprise,

> ——— Why, why is *this*?

Evidently meaning, Why do you act thus? Why do you talk of jealousy to me, who am not at all disposed to be jealous? The word *this* cannot here be said to refer to any one noun that precedes, or to any one noun that follows it; and it is therefore most manifestly used with the force and effect of a substantive.

On the contrary, it is clearly used as an adjective in a subsequent passage, where Othello, speaking of Iago, says—

> ——— *This* honest creature, doubtless,
> Sees and knows more, much more than he unfolds.

Whether the same or different words shall be employed to express the substantival and adjectival form of pronouns is matter of idiom. Thus, a Language may, or may not, have different forms for the personal and possessive pronouns. Lowth considers the word *mine* as the possessive case of the personal I; but the English substantive mine (if a substantive it be) answers to the Latin *meus*, which is certainly an adjective. On the other hand, the Latin *mi*, which is commonly called the vocative singular of *meus*, seems to be the same word with *mihi*, the dative case of *Ego*; for it is used in connection with plurals as well as singulars, and with masculines, feminines, and neuters indiscriminately. Thus we have in Plautus, *mi homines*; and in Petronius, *mi hospites*; and in Apuleius, *mi sidus*, *mi parens*, *mi herilis*, (sc. *filia*,) *mi conjux*, &c.; and in a passage of Tibullus, the different manuscripts have, some *mi dulcis anus*, and some *mihi dulcis anus*; in all which instances, the dative *mihi* seems to be intended to be used in that manner which Grammarians often, though incorrectly, call redundant; and describe, as adopted, *nullâ necessitatis, sed potius festivitatis causâ*.

There are many other idioms relative to the use of pronouns which it is not here necessary to consider, such as the combination of the adjective *own* and the substantive *self* with the pronouns *my*, *thy*, &c. in English; and the subjoining the syllables *met*, *cunque*, &c. to certain pronouns in Latin, as *ipsemet*, *quicunque*, &c. which are usually accompanied with some corresponding change in the force of the original pronouns.

The qualities from which different classes of pronouns take their common Grammatical designations, as *distributive*, *definitive*, &c. may in general be viewed as existing in the objects, and both the object and the quality may be set forth together, as in common substantives and adjectives. Thus the quality of *alternation*, if we may so speak, is expressed in English by the word *either*, and the quality of *diversity* by the word *other*, and these may doubtless be united with their proper substantives in the same manner as any other adjective may. Thus we say, "take *either* horse," "choose *another* man;" and in these and

Grammar. similar passages the words *either* and *other* are to be considered as pronominal adjectives.

The connection between the pronoun and the article has always been admitted to be very close and intimate; and therefore many authors rank some of these pronouns, especially the definitives, among the articles. Harris is of that opinion, and he cites in support of it the authority of several ancient Grammarians. We do not pretend to decide very dogmatically on this point; but, upon the whole, we are disposed to follow the great majority of writers, in confining the designation of article to those words which perform the simple function of individualizing conceptions; nor can we think it right to reject altogether the pronominal adjectives, which must be the case if we were to adopt Harris's criterion: "the genuine pronoun always stands by itself, assuming the power of a noun, and supplying its place; the genuine article never stands by itself, but appears at all times associated to something else, requiring a noun for its support as much as attributives or adjectives." It does not appear to us correct to say that the pronominal adjectives do not stand for other nouns. They seem to stand for the names of various different conceptions which are principally used for the purpose of distributing our conceptions. The words *this* and *that*, for instance, adjectively used, answer to the adjectives *near* and *distant*.

After all, it might, perhaps, have been better if the personal pronouns alone had received the name of pronoun; and if the words which we are now considering had been arranged in a class between the personals and the article, for they seem to hold a middle place between both; but as we consider it safest not to disturb a long settled order of things, we extend the name of pronoun to all these different classes.

Numerals. There is one set of words which seems to belong to the class of definitive pronouns, but which yet demands a consideration apart. We mean the *numerals*. We have heretofore shown the fundamental importance of the conceptions of number. Those conceptions must have names, and when the names are used to express the mere *ideas* of number, as when we say, "*one* and *one* are *two*," they may be considered as nouns; in the same manner as the words *line*, *point*, *angle*, which are also names of ideas, are considered. But when these nouns are used with an express or tacit reference to some other noun, they become pronouns, either substantive or adjective. When we say, "*two* men are wiser than one," or "*many* men are wiser than one," the numeral "two" seems as much a pronoun adjective as the word "many." And again, if speaking of men, we say, "two are wiser than one," the word two appears to be a pronoun substantive.

Numerals are commonly divided into *cardinal* and *ordinal*: we have hitherto spoken of the former, that is to say, of the names given to our distinct ideas of number, simply as distinguishing them from each other, as one, two, three, &c.; but these same conceptions, viewed with reference to order, form in the Mind a class of secondary conceptions, which are treated as qualities of the substances to which they belong. Hence originate such words as *first*, *second*, *third*, *fourth*, &c. These may be called pronominal adjectives. The ordinal numbers are in general derived from the cardinal numbers, but not necessarily so; for in many, perhaps in most Languages, the words *first* and *second* have no etymological affinity to the words *one* and *two*. In English, the word *first* is properly *fore'st*, or *foremost*, and is connected with the prepositions *for* and *before*; just as our comparative and superlative *further* and *furthest*, improperly written, in modern times, *farther* and *farthest*, are derived from *forth*. Of the numerals, and of definitive pronouns in general, we shall have occasion to speak again when we treat of the article, which is in fact only the definitive pronoun adjective in a new and peculiar form.

§ 4. *Of verbs.*

Verbs. The *verb* expresses that faculty of the Human Mind by which we assert that any thing exists or does not exist: and as all existence is either contemplated by the Mind simply as existence, or as existence in one of its two distinguishable states—action or passion, therefore the common definition of the verb is sufficiently accurate, *viz.* "that the verb is a word which signifies to do, to suffer, or to be." Yet we must observe that the essence of the verb does not consist in the mere signification or *naming* of existence, or of action, or of passion; because so far as that goes the verb is a mere noun; but what Mr. Tooke has observed is strictly true in Language, *viz.* that "the verb is a noun and something more." He has not been pleased to tell his readers what that *something more* really is: and he affects a sort of mystery respecting it, which is peculiarly out of place in a Work of Science; but nothing can be more obvious or less controvertible than that this something more, which is the true characteristic of the verb, is the power of *assertion*.

It is by this peculiarity alone that the verb is distinguished from the noun, as a very few familiar instances will demonstrate. It often happens in Language that the very same identical word, the same in orthography, in pronunciation, and in accent, is both noun and verb. How then can we determine when it is one, and when it is the other. Very simply, and very infallibly. When it involves an assertion it is a verb; when it does not it is a noun. The word *love*, in English, is one of the words which we have just described. It is impossible to tell, *à priori*, whether it will be a noun or a verb in any particular discourse. We must wait to see how it is used, and then all doubt will vanish. Thus it is a noun in those exquisite lines—

> ——— *Love* is not *love*,
> Which alters when it alteration finds,
> Or bends with the remover to remove,
> Oh no! It is an ever fixed mark,
> That looks on tempests, and is never shaken.

And again, it is a verb, in the speech of the crafty Richard to his unsuspecting brother—

> ——— I do *love* thee so,
> That I will shortly send thy soul to heaven.

Objections. Against the doctrine that assertion is the peculiar office of verbs, various objections have been urged.

First, it has been said that we may assert, without the express use of verbs: and this is true; but then the assertion is an act of the Mind, not expressed, but, as Grammarians say, *understood*. The verb is wanting; but its place is not supplied by any other Part of speech, such as a noun, pronoun, conjunction, or the like. Now, whether any particular operation of the Mind may or may not be understood, without being expressed in speech, is pretty much a matter of habit, and there-

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fore forms the peculiar idioms of different Languages; but in Universal Grammar we have to regard the operation of the Mind itself, whether expressed by one or more words, or to be collected from inflection, relative position, accentuation, or any other mode of signification.

Let us consider a few examples. In the Hebrew Language the verb is often omitted. Thus in the 3rd chapter of Exodus, (ver. 2.) "the bush burned with fire, and the bush not consumed," *i. e. was* not consumed. Again, (ver. 4.) "God called unto him out of the bush, and said, Moses, Moses! And he said, here I," *i. e.* here *am* I. And again, (ver. 6.) "Moreover he said, I the God of thy father, the God of Abraham, the God of Isaac, and the God of Jacob," *i. e.* I *am* the God of thy father, &c. So it is in the Greek Language. Thus in St. Mark's Gospel, chapter the 10th, ver. 18, *οὐδεὶς ἀγαθὸς εἰ μὴ εἷς ὁ Θεός*, "No one good, except one, God," *i. e.* "No one *is* good," &c. Again, in St. Luke's, 6th chapter, verses 20 and 21, *Μακάριοι οἱ πτωχοί, Μακάριοι οἱ πεινῶντες νῦν, Μακάριοι οἱ κλαίοντες νῦν*—"Blessed the poor, blessed the hungry, blessed the weepers," *i. e.* Blessed *are* the poor, blessed *are* the hungry, blessed *are* the weepers. The same idiom occurs in Latin. Thus in the parallel passages to those above cited, *Nemo bonus, nisi unus Deus, i. e. Nemo* est *bonus*, &c. And again, *Beati pauperes, beati qui nunc esuritis, beati qui nunc fletis, i. e. Beati* estis *pauperes*, &c. The French Language also admits a similar phraseology: thus,

> *Heureux celui, qui dès ses jeunes ans*
> *S'est tenu loin du conseil des méchans;*

i. e. heureux est *celui.*

Nor is our own Language a stranger to the same construction. Thus in Milton's beautiful description of our first parents:

> ———————— In their looks divine,
> The image of their glorious Maker shone,
> Truth, wisdom, sanctitude severe and pure,
> Severe but in true filial freedom placed,
> Whence true authority in men; though both
> Not equal, as their sex not equal seem'd;
> For contemplation he, and valour form'd;
> For softness she, and sweet attractive grace.

i. e. whence true authority *is* in men; both *were* not equal; he *was* form'd for contemplation; she *was* form'd for softness, &c.

Now, in all these cases, the Mind performs the act of asserting; in the words of Plato it manifests some action, and declares that something exists; and this manifestation or declaration is not contained in the nouns themselves, which do nothing more than name the conception; thus, when we say *nemo bonus*, the assertion is neither included in *nemo*, nor in *bonus*, for these are mere *names* of conceptions. *Nemo* is the subject; *bonus* is the predicate; but neither of them includes the copula. The two terms are not connected by any thing which either of them contains, but their connection is inferred by the Mind from their juxtaposition. But the question which we have here to consider, does not relate to verbs not expressed, but to verbs expressed; and universally where the verb is expressed, it imports assertion either simple or modified, either direct or implied.

A second objection to that account of the verb which we adopt is, that *connection* and not *assertion* is the distinguishing characteristic of verbs. It is true that the verb connects; but it does more, it declares the co-existence of the connected conceptions as parts of one assertion. The conjunction also connects, but it does not predicate one thing of another, or make up one proposition of two distinct terms. Thus, if we say "he is good," the conceptions expressed by the words *he* and *good*, that is to say, the conceptions of a particular man and of goodness, are not only connected, but the one is asserted to exist in the other, and to be a quality belonging to it. Otherwise is it in the Speech of the Duke of Buckingham wishing happiness and honour to his Sovereign Henry VIII.

Verbs.

> ———————— May he live
> Longer than I have time to tell his years!
> Ever belov'd, and loving may his rule be!
> And when old Time shall lead him to his end,
> *Goodness* and *he* fill up one monument!

Here the same conceptions, *viz.* those of a particular man and of goodness, are connected, but the one is not asserted of the other, and they make up no intelligible meaning when taken together, without the further aid of a verb. We cannot assert without connecting our thoughts; for to assert is to declare some one thing of some other thing, which cannot be done without connecting those things together in the Mind; and therefore it is that connection is always one characteristic of the verb: but it is a secondary characteristic, being involved in its more important function; that of asserting, declaring, or manifesting real existence.

Thirdly, the verb being ranked with the adjective and participle, under the general head of attributives, it has by some been considered that *attribution*, that is to say, the expression of a quality, or the denoting of the predicate in a proposition, is the proper function of a verb: but again we must remark, that this is but an accidental circumstance applying to some verbs, and applying to them not as verbs, but in regard to the nouns which they involve. Thus, when we say, "Cicero spoke," the verb spoke includes the name of an act, *viz.* speech, or speaking, which, at a certain time, belonged to Cicero, and which is predicated of him as having so belonged; but this name is a noun, and if expressed simply in connection with Cicero, as Cicero speech, or Cicero speaking, it produces no intelligible meaning: and therefore, in order to convert it into a verb, a power of assertion must be given to it, which is done either by a distinct word, as "Cicero *was* speaking," or, by a peculiar inflection of the same word, as "Cicero spoke." "All those attributives," says Harris, "which have this complex power of denoting both an attribute and an assertion, make the species of words which Grammarians call verbs. If we resolve this complex power into its distinct parts, and take the attribute alone, without the assertion, then have we participles."—From this statement it is manifest that the assertion is that which constitutes the true characteristic of the verb; and that the attribute which it expresses is not essential to it, but may appear under a different form, and constitute another Part of speech.

To be significant of time, or, as it has been expressed, to be *nota rei sub tempore*, is still less the characteristic of the verb, than those other circumstances are which we have been considering; for existence may be contemplated without any reference to

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the lapse of time, as when we say "two and two *are* four." We cannot, indeed, assert any thing without a declaration of existence, and the existence of all *individual* things is referable to time. Time, therefore, is a necessary adjunct of all *such* assertion, and consequently of the verbs by which it is effected; but even in these instances the signification of time is but secondary: it is the assertion, that is, the manifestation, or declaration that the truth is so, or so, which constitutes the appropriate function of the verb.

One more objection which we shall notice is, that the infinitive mood asserts nothing, and consequently that assertion cannot be essential to verbs. To which we reply, that the infinitive is not properly a verb, but rather, as some of the ancient Grammarians called it, Ὄνομα ῥηματικὸν, a verbal noun; or Ὄνομα ῥήματος, the verb's noun. Hence it follows, that in English we may often use indifferently the participial noun, or the infinitive, as "singing," or "to sing;" "parting," or "to part," &c.

> ———*Parting* is such sweet sorrow,
> That I could say good night, till it were morrow.

Where the sense would be unaltered if it were expressed thus:

> ———*To part* is such sweet sorrow.

Thus, too, in the Latin Language, Priscian remarks, that currere *est* cursus, and scribere *est* scriptura, and legere *est* lectio: and he enforces this remark by observing of infinitives, *itaque frequenter et nominibus adjunguntur, et aliis casualibus, more nominum;* ut Persius:

> *Sed pulchrum est digito monstrari et dicier hic est.*

The Stoics, indeed, as Harris informs us, "had this infinitive in such esteem, that they held this alone to be the genuine ῥῆμα, or verb, a name which they denied to all the other modes. Their reasoning was, they considered the true verbal character to be contained simple and unmixed in the infinitive only. Thus, the infinitives περιπατεῖν, *ambulare*, 'to walk,' mean simply that energy and nothing more. The other modes, besides expressing this energy, superadd other affections which respect persons and circumstances. Thus, *ambulo* and *ambula* mean not simply to walk, but mean 'I walk,' and 'walk thou,' and hence they are all of them resolvable into the infinitive, as their prototype, together with some sentence or word expressive of their proper character. *Ambulo*, 'I walk,' that is, *indico me ambulare*, 'I declare myself to walk;' *ambula*, 'walk thou,' that is, *impero te ambulare*, 'I command thee to walk;' and so with the modes of every other species. Take away, therefore, the assertion, the command, or whatever else gives a character to one of these modes, and there remains nothing more than the mere infinitive, which, as Priscian says, *significat ipsam rem quam continet verbum.*" To all this reasoning it is sufficient to answer, that if the Stoics refused the appellation of ῥῆμα to all moods but the infinitive, they clearly did not mean by the word ῥῆμα that distinction which is commonly designated by the term *verb:* and in truth it appears that they meant by it the predicate of a proposition, and nothing more: thus Ammonius says, πᾶσαν φωνὴν κατηγορούμενον ὅρον ἐν προτάσει ποιοῦσαν ῬΗΜΑ καλεῖσθαι, "that every word forming the predicate in a proposition was called a verb." In the view that we have taken of Grammar, the predicate of a proposition must, on the contrary, be considered to be a noun, either by itself, or else as involved in a verb; whereas the copula of the proposition is the true verb, either alone or combined with the predicate. In the sentence, "Socrates teaches," the copula, that is to say, the essential part of the verb, is involved in the word "teaches." In the sentence, "Socrates is teaching," it is expressed separately by the word "is;" and conversely in the word "teaches," the predicate is expressed in combination with the copula; and in the word "teaching" it is expressed alone.

Verbs. Classes.

Different kinds of verbs.

What has been already said will easily lead us to a division of verbs into their different *kinds;* for they either express the simple copula of a logical proposition, or they express the copula in connection with a predicate. In the former case, the verb is called by Grammarians a *verb substantive*, and simply affirms existence; such is the verb *to be*, in its purest form. In the other case, the verb expresses being, together with some attribute of action or passion; and as the *name* of such attribute is properly a noun, all such verbs include a noun. We have said that the verb *to be*, in its purest form, is the verb substantive; by which we mean that verb, when it merely answers the purpose of asserting, and has a separate subject and predicate, as "Socrates *is* wise," "Socrates *is* reading," &c. Other words as well as the word *is* may be used in the same manner, if it becomes idiomatical to give them this simple effect: such was the use in Greek of the verbs ὑπάρχει, πέλει, γίγνεται, &c.; and on the other hand, the verb substantive *is* may be used more emphatically to assert existence, as "God *is*," *i. e.* "God exists," or "is existing."

Verb substantive.

The nature of the verb substantive is thus explained by Harris: "Previously to every possible attribute, whatever a thing may be, whether black or white, square or round, wise or eloquent, writing or thinking, it must first of necessity exist, before it can possibly be any thing else. For existence may be considered as an universal genus, to which all things, of all kinds, are at all times to be referred. The verbs, therefore, which denote it, claim precedence of all others, as being essential to the very being of every proposition in which they may still be found either expressed or by implication; expressed, as when we say 'the sun *is* bright;' by implication, as when we say 'the sun rises,' which means, when resolved, 'the sun *is* rising.' Now all existence is either absolute or qualified; absolute, as when we say 'B *is;*' qualified, as when we say 'B is an animal;' 'B is round,' 'black,' &c. With respect to this difference, the verb *is* can by itself express absolute existence, but never the qualified without subjoining the particular form; because the forms of existence being in number infinite, if the particular form be not expressed we cannot know which is intended. And hence it follows, that when *is* only serves to subjoin some such form, it has little more force than that of a mere assertion. It is under the same character that it becomes *a latent part in every other verb*, by expressing that assertion which is one of their essentials."

Verbs of action.

Beside the verb substantive, all other verbs imply action, and these are commonly distinguished into *active*, *passive*, and *neuter*. It is matter of idiom whether these different classes shall be expressed by different inflections or not; but the distinction of the

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classes themselves is in the nature of the Human Mind, and must therefore have some correspondent expression in Language. Active and passive verbs agree in this, that they reciprocally suppose a separate agent and object, whilst the neuter verb supposes an action terminating with the agent. In the active verb the action is considered as passing from the agent to the object, and consequently the object takes the lead in the sentence, as, "John loves Mary:" in the passive verb the action is considered as received by the object from the agent, and consequently the object takes the lead in the sentence, as, "Mary is loved by John." This difference, as we have already had occasion to advert to it in treating of cases, needs no further explanation here. The neuter verb includes all those numerous classes of action which terminate in themselves, as, "to sleep," "to walk," "to stand." Some persons reckon the verb substantive among neuters; but it seems better to distinguish it altogether as we have done from verbs of action, and to treat the neuters as a branch of the latter. It will be observed that by action we do not mean simply motion, but also rest, or the privation of motion. Thus, "to stop," "to cease," "to die," are not less acts than "to walk," "to fly," "to live," "to wound," or "to kill:" in short, whatever imports any diversity in the states or modifications of being; and we need not repeat, that the verb does not merely *name* those states, but asserts them to be really existing at some period of time.

Other distinctions.

Various other distinctions of verbs occur in Grammatical Works, but they seem all to be merely subordinate to those which we have noticed, or else explanatory of them. Thus the verbs *transitive* and *intransitive* are, in other words, active and neuter; for the verb active is considered as *passing over* from the agent to the object, whilst the neuter is considered as not passing over. Those who speak of actives-intransitive, seem to confound the true distinction between the active and neuter; thus they call the verb to *sleep* a neuter, and to *walk*, an active intransitive, probably because more Physical activity is shown in walking than in sleeping; but it is not the quantity or degree of action that makes the difference between these classes of verbs, but the simple consideration whether they have or have not a separate object. When we say a separate object, we do not mean an object necessarily distinct from the agent; for there is a class of verbs called *reflectives*, in some Languages, in which the agent is its own object; but these verbs are truly actives. When a person says, *Je me flatte*, "I flatter myself," the verb *flatte* expresses an action as proceeding from the agent *Je* to the object *me*. So in the Latin, *Egomet mî ignosco*, "I pardon myself," *ignosco* expresses an action as proceeding from the agent *Ego* to the object *mihi*. An accurate examination of the operations of the Mind in such cases will convince us that we really distinguish the *self*, or Being, with whom the action originates, and in whom it terminates, into two parts, or at least view it in two lights. The Being which flatters or pardons is viewed as active, the Being which is flattered or pardoned is viewed as passive. This power of self-contemplation is the origin of the ancient fable of Narcissus; it is the foundation of that Moral rule which the Philosophers of antiquity considered to be divine,

> ——*E cœlo descendit γνῶθι σεαυτόν.*

And Socrates very finely distinguishes between the Physical and Moral power of contemplation by remarking, that the eye, which sees everything else, cannot see itself; whereas, there is no created object which the Human Mind can or ought so much and so profoundly to contemplate as its own existence and energies.

Verbs. Classes.

It is material to observe, that the quality of neuter or active is not necessarily appropriated to any particular verb; but that a neuter, by a slight change of signification, may often pass into an active, and *vice versâ*. Thus the Latin verb *abstineo*, "I abstain," is commonly used as a neuter; but even in the best writers we find it employed as an active: Cicero says, *abstinere manus;* and Livy says, *Romano bello fortuna Alexandrum abstinuit.* We cannot translate these passages literally into English, "to *abstain* the hands," and "Fortune *abstained* Alexander from a Roman war;" but the reason of this is, that the active or neuter use of particular verbs is a mere matter of idiom. In English, as in most other Languages, custom has confined certain verbs to the one class, and certain others to the other class; but there is generally a number of verbs which are used both in an active and neuter signification, the construction alone determining of which kind they are.

It is again noticeable, that verbs usually neuter have often one particular construction in which they assume an active form. This happens where the accusative which follows the verb is in substance the very same conception which the verb itself expresses, as "to live a life;" or where it forms a species of which that conception is the genus, as "to dance a minuet," that is, to dance a dance of the species called a minuet. For a similar reason we use such expressions as "to walk a mile," "to ride a race," "to swear an oath." It is only by a bold Poetic license, that Timon, addressing the courtezans, says:

> ——I know you'll swear, terribly swear,
> Into strong shudders, and to heav'nly agues,
> Th' immortal Gods that hear you.

The expression "to swear the Gods," is employing a neuter verb in an active sense unknown to the general idiom of the English Language, and only justified by that energy of feeling with which the all-powerful Poet has invested the dramatic character of Timon.

In the distinctions of verbs, as in most other parts of Grammar, we find Grammarians continually confounding signification with form. Thus they say there are five classes of verbs in the Latin Language: 1. Verbs ending in o, which also admit or; these they call *active*. 2. Verbs ending in o, which do not admit *or;* these they call *neuter*. 3. Verbs ending in or, which are also used in *o;* these they call *passive*. 4. 5. Verbs ending in *or*, which are not used in *o;* these they call *common*, or *deponent*.

Vossius justly blames this division; but his own method is not wholly free from censure; for though he properly begins with the triple distinction of signification, according as the verbs express doing, suffering, or being, he proceeds to subjoin to this a fourfold distinction in point of form, observing that verbs are either biform, (ending in o and *or*,) and these are *active* and *passive;* or else they are uniform, ending in o only if *neuter*, and in *or* only if *deponent*, or *common*. By the word deponent are meant those which have *laid aside*

Grammar.

the passive signification properly belonging to the termination *or*; as in Virgil,

Pictis bellantur Amazones armis;

so io Plautus,

Adeunt, consistunt, copulantur dexteras.

By the word common, are meant those which, though used actively by some writers, retain also a passive signification as employed by other writers of great weight and authority. Thus *complector* is generally used with an active eignification; but it is passive in the Speech of Cicero for Roscius—*Quo uno maleficio, scelera omnia* complexa *esse videantur*. The *middle* verb io Greek has cometimes the effect of the Latin deponent, that is to say, it has a passive form with an active signification; but in other instances it is rather of the nature of a *reflective* verb, producing a sort of mixed sense between the active and the passive. "The mixed sense," says KUETER, "consists in this, that the action of such *middle* verbs does not pass over to another object, bnt is reflected back on the agent, so that the same Being becomes both ageot and patient; and this, whether he directly euffer any thing from himself, or order, direct, or permit it to be done to him by another." Thus ἐπείγειν in the active is to urge or impel another; but ἐπείγεσθαι in the middle form is to urge or impel one'e self, that ie, to make haste. Heoce it happens that the same word in the active and middle forms has two distinct, and, in some measure, contrary senses, as δανεῖσαι is to leod; but δανείσασθαι is to borrow: and it is remarkable that our common English verb *borrow* anciently eignified both to lend and to give a pledge for that which was lent, and heoce to be plighted or married to a person. Thus Wachter says, *Borg, mutuum, auf borg geben, mutuo dare, auf borg nemen, mutuo accipere. Proprie quidem est mutuo datum, a borgen mutuo dare; mox etiam mutuo acceptum, quia dare et accipere sunt correlata et in notione debiti et crediti conveniunt.* Again, *Borgen, mutuo dare, dare in creditum. Belgis borgen, Anglis borrow. Ab hoc significatu habent Anglosaxones borgiend, fœnerator.* And further, *Borgen, mutuo accipere, accipere in creditum. Anglosax. borgan, borgian.* The old Scottish Ballad speaking of Tam Lane, or Tom Linn, who was carried away by the Fairies, and married to a Lady of the Fairy Court, says:

She that has *borrowed* young Tam Lane
Has gotten a stately groom.

Thus we see that the Principle, which in one Language gives different meanings to the same form of epeech, founde in other Languagee a distinction of meaning between different forme of the same word.

We have thought it necessary to take this short notice of the classes nf verbs last mentioned, both because the terms *deponent*, *common*, *middle*, &c. are of frequent occurrence in Grammatical Writers; and more particularly because some of the very best Grammarians have eodeavoured to unite io one common system these distinctions of form, with the distinctions of signification, an attempt which canoot but be prejudicial to scientific clearness and accuracy; inasmuch as it confounds Universal Grammar with Particular, and thus forms a system which properly belongs to neither.

There are again other distinctions which relate indeed to the signification of verbs; but which do not interfere with the primary Grammatical classes; and rather belong to the richness of a Language, than to its necessary cooetruction. Such are the Latin *inceptive* verbs io *sco*, as *albesco*, *tumesco*, the Greek verbs of *habit*, in ιζω, φιλιππίζω; the Hebrew verbs called by some writers *intensive*, and many others in most Languages.

Verbs. Classes.

Verbs of this kind are generally derived from other verbs, but sometimes from oouns, as *calesco*, *horresco*, *splendesco*, from the verbs *caleo*, *horreo*, and *splendeo*; *noctesco* aod *ruresco*, from the nouns *nox* and *rus*. Of the Latin verbs in *sco*, it has been disputed whether they can or cannot properly admit the expression of past time; but Vossius satisfactorily proves that they may, by adverting to their proper signification, which ie not merely inchoative but also continuative. "Hence," saye he, "as the Philosophers teach that all motion is produced by succession, there must be iu it a beginning, a middle, and an end; and it is one thing to have perfected the beginning, another to have proceeded to the middle, and another to have reached the end; and he who says that he did at a certain time begio a movemeot, only means to assert that such beginning was perfected, and not the whole motion." Many varioue classes of verbs may be things distinguiehed by various ehades of derivative signification. They do oot simply assert the conceptioo involved in them to exist, but to exiet under some particular modificatioo. Thus we have seen that the Latio verbs in *sco*, imply the *inchoation* and *continuation* of an action. Verbs in *to*, *so*, *xo*, and *co*, are called *frequentatives*, or *iteratives*; as *pensito*, from *pendo*; *tracto*, from *traho*; *vendito* from *vendo*; but it has been observed, that they ofteo imply, in a secondary sense, not the repetition of an action so much as its greater violence; and may therefore be called *intensive* or *augmentative*. Thus, *rapto*, derived from *rapio*, is used by Virgil to signify not only the repeated, but the violent dragging of Hector's body in triumph round Troy—

Ter circum Iliacos raptaverat Hectora muros.

On the other hand, they are sometimes taken to signify a weaker degree of the same action; as TURNEBUS observes—"There are maoy words which, by learoed Grammariens, are reckoned to be of a frequentative form, and which plainly exhibit the appearance of that form; but which if they are narrowly inspected, and if we observe the manner in which they are used by the best authors, should rather be called *desideratives*. I will enumerate a few of them, which may afford to the studious sufficieot specimens to direct their search for others of the same kiod." "*Capto* is not, 'I take frequently,' but 'I endeavour to take,' as *capto cœnam, capto benevolentiam*. *Vendito* is not, 'I sell frequently,' but 'I desire to sell;' as in Cicero (*De Arusp. Resp.*) *atque ei sese, cui totus venierat, etiam vobis inspectantibus* venditaret, that is, *se ei vendere vellet*: and so in Plautus, *lingua venditaria* is not 'a tongue which sells' bnt 'which wishes to sell,' as the Parasite says his own was. *Dormito* is not 'I sleep often,' but 'I am nodding, or napping,' as in Plautus (*Amphitr.*) *te dormitare dicebas*: and so io the Gospel of St. Matthew (chap. xxv. v. 5. ἐνύσταξαν πᾶσαι καὶ ἐκάθευδον, 'they all slumbered and slept,') the word ἐνύσταξαν is elegantly rendered by the tranelator *dormitarunt*; because they who are ready to fall asleep cannot keep their heads upright. *Ostento* is not 'I show frequently,' but 'I wish to ehow.' *Munito* is used by

Grammar. Cicero (*Pro Roscio*) in the sense of *munire cupio*. In fine, there are many other words which might be cited; but it is sufficient to have pointed out the class, as it were, and to have afforded a specimen of them to the studious."

Slight shades of distinction are to be observed in the use of these and similar words: nor does the same termination always express the same modification of the original thought. Thus the termination *so* in *viso*, has a desiderative force, in *pulso*, a frequentative, for the former is I go to see, the latter is I knock or push frequently; and in like manner *verso*, as used by Horace, is I turn over frequently:

> ——— *Vos exemplaria Græca*
> *Nocturnâ versate manu, versate diurnâ.*

Of the termination *sso*, different commentators speak differently. Thus Virgil:

> *Haud mora, continuo matris præcepta facessit.*

On which passage Servius observes, that *facesso* is a frequentative verb, inasmuch as there were many victims sacrificed; on the other hand, Nonius and Donatus both explain *facesso* to signify simply the same as *facio;* but in reality it has an intensive force, and signifies more than the simple verb, though not necessarily a repetition of the same act. Thus, in the passage just cited from Virgil, *facessit* obviously means setting about the business that was commanded, with diligence and anxiety. The termination *co* is noted as having in general a weakening force; for *claudico* is I halt a little, and the difference between *nigrantem colorem*, and *nigricantem colorem*, if any, is that the latter is less strongly inclining to black. Critics have observed a difference between those verbs which express only the simple desire to do an act, and those which express together with the desire the actual engagement in it: the latter kind they call *desideratives;* but the former they distinguish as merely *meditatives*. Thus *facesso*, as we have seen, is a desiderative; but first of the verbs in *rio* are meditatives; for *esurio* rather implies a negation of the act of eating, and is only I hunger, or have a desire to eat, without any gratification of that desire. But here, too, we perceive that the termination is not a sure guide to the use of the word, for *scaturio* and *ligurio* imply the performance of the respective actions, and not merely the desire or meditation of them; as in Horace:

> *Si quis eum servum patinam qui tollere jussus*
> *Semesos pisces tepidumque* ligurierit *jus,*
> *In cruce suffigat.*

Lastly, the termination *lo* or *llo*, generally serves to diminish, as *murmurillo*, I murmur gently, from *murmuro; sorbillo*, I sip drop by drop, from *sorbeo; cantillo*, I hum a tune, or sing in an under voice, from *canto*, and the like.

In most Languages there are *negative* or *oppositive* verbs, as *volo* and *nolo* in Latin; to *do* and *undo* in English; *fier* and *mefier* in French, &c. There are also in various Languages, as in Persian, Sanscrit, &c. *causal* verbs formed by a peculiar inflection, whereas in some other Languages the simple and causative meaning are found in the same word. Thus it is probable that our verbs to *lie* and to *lay*, though recently distinguished in use, and indeed supposed to be derived from two different Anglo-Saxon roots, were both of the same origin; for Wachter explains the ancient German word *lage, situs, sedes, campus;* and observes that it agrees with the Latin *locus*, hence *ligen* in the first sense is to *lie*, or occupy a certain *lage;* and *legen* in the secondary sense is to cause to lie, to cause to occupy a *lage*. In like manner our common verbs to *fell* and to *fall* are the same. "To fall timber" is an expression still used in many parts of England, and it signifies to fell, or cause to fall. So we say to *bleed* a person, for to make him bleed. Verbs. Moods.

The words which we have been considering, as distinguished by Grammarians into so many classes of verbs, inceptive, desiderative, frequentative, negative, causal, &c. are all derivatives; and derivative words are, in fact, compounds; that is, they unite the name of one conception with that which serves as the name of another, as the name *albus*, white, is united with the termination *esco*, which serves as the name of growth; so that *albesco* is, literally, I *grow white*. But we have seen that what is effected in one Language by the derivative verb is effected in another by the simple verb. The thought expressed is, in both cases, the same: but the mode of expression varies; and the variations are properly matter of Particular, and not of Universal Grammar.

After having thus reviewed the different *kinds* of verbs, we come to the considerations which regard all these kinds alike, and which are usually ranked by Grammarians under the heads of mood, tense, person, number, and, in some Languages, gender.

The *Mood* of a verb is that manner in which its assertive power is exhibited, and which depends on the state of Mind in which the speaker may be placed with relation to the assertion. Hence Grammarians have sometimes defined the mood to be a certain inclination of the Mind shown in speech. Thus PRISCIAN says, *Modi sunt diversæ inclinationes animi, quas varia consequitur declinatio verbi.* The latter circumstance, however, belongs not to Universal Grammar. Whether the different moods have or have not different forms of declension, or conjugation, depends on the idiom of the particular Language; but whatever variations the verb may have in point of form, it must necessarily be susceptible of those varieties, in point of signification, which properly belong to its assertive power. Mood.

Grammarians differ widely as to the number, and no less as to the names of the moods. SCALIGER says, that mood is not necessary to verbs; and SANCTIUS contends that it does not relate to the nature of the verb, and therefore is not an attribute of verbs: *non attingit verbi naturam, ideo verborum attributum non est;* on which passage PERIZONIUS very justly observes, that great as the merit of Sanctius was in many parts of his Work yet he had in others, and particularly in what regarded the moods of verbs, been misled by an excessive desire of novelty and change. It is very true, as observed by Sanctius, that the great mass of Grammatical writers are so extremely discordant in their opinions respecting this part of the Science of which they treat, that they have left us scarcely any thing on it which can be said to be established by general consent. Some make only three modes, others four, five, six, and even eight. Again, some call these affections of the verb moods; others call them divisions, qualities, states, species, &c.; and as to the various appellations of each mood we have the personative and impersonative, the indicative, declarative, definitive, *modus fini-*

Grammar. *endi*, *modus fatendi*, the rogative, interrogative, requisitive, percontative, assertive, enunciative, vocative, precative, deprecative, responsive, concessive, permissive, promissive, adhortative, optative, dubitative, imperative, mandative, conjunctive, subjunctive, adjunctive, potential, participial, infinitive, and probably many others.

In this confusion of terms and of notions, it is absolutely necessary to adopt some distinct Principle which may guide us through the labyrinth; and that Principle, we apprehend, will be easily and intelligibly supplied by adverting to the peculiar function of the verb itself, namely, assertion. It must be observed, that we use this term, in its largest sense, for the manifestation of some distinct perception or volition; and we consider, that in every such manifestation an assertion is either expressed or implied. Portia, addressing Brutus, says,

————————— Dear, my lord,
Make me acquainted with your cause of grief.

And again, she says,

————————— Upon my knees
I charge you, by my once commended beauty,
By all your vows of love, and that great vow
Which did incorporate and make us one,
That you unfold to me, yourself, your half,
Why you are heavy.

In both these instances she asserts her earnest demand to be made acquainted with the secret cause of that trouble which she perceived to exist in her husband's Mind. In the one instance, however, the demand is expressly asserted by the words "I *charge* you that you unfold:" in the other it is implied, with no less clearness, by the words "*make* me acquainted." Whether, therefore, the assertion be express or implied, the verb is that part of the sentence by which it is manifested; the verb animates the sentence, connects the passion with its object, or the object with its predicate.

Again, Cæsar in describing Cassius, first asserts positively what he had observed in his outward appearance, and then hypothetically what might be supposed to pass in his Mind:

Yon Cassius has a lean and hungry look;—
Seldom he smiles, and smiles in such a sort,
As if he *mock'd* himself, and scorn'd his spirit,
That could be mov'd to smile at any thing.

And so, referring to Antony's expression, "fear him not," Cæsar asserts positively that he does not fear him, but puts a case hypothetically, in which he might do so:

————————— I fear him not;
Yet *if* my name were liable to fear,
I do not know the man I *should avoid*,
So much as that spare Cassius.

Having thus explained what we mean by the term assertion, we proceed to apply that principle to the doctrine of moods.

Assertion, then, takes place either in an enunciative sentence, or in a passionate sentence: in the former it is express; in the latter it is implied. By express assertion a truth is enunciated, absolutely if the sentence be simple, but conditionally, in the dependent branch of a sentence which is complex. By implied assertion, in like manner, a passion is connected with the object either absolutely or conditionally: in the one case the desire or aversion is positive, in the other it is qualified by some consideration of circumstances. These four kinds of assertion supply us with four correspondent moods of the verb, namely, the *indicative*, the *conjunctive*, the *imperative*, and the *optative*. It has been contended, that there are two moods in which assertion does not take place, namely, the *interrogative* and the *infinitive*; but these we are not inclined to reckon as separate moods, for reasons which will hereafter be stated. Of the four other moods we proceed to take notice in the order above-mentioned. Verbs. Mood.

Indicative. If we simply declare or indicate something to be or not to be, this constitutes the mood called by most Grammarians the *indicative*, but by some the declarative, enunciative, &c. Thus, "I love," "I walk," "he died," "we shall rejoice," are all simple assertions of fact, some of which do, and some do not relate to passions of the Mind, but which do not necessarily imply any passion in the enunciation. Some of them too may in reality be contingent, or doubtful, and may be dependent on the truth or falsehood of other assertions; but as they are not so enunciated, but on the contrary are declared positively and simply, they belong to the indicative mood. It is to be observed that the indicative, from its very nature, is capable of being united with the conjunctive, as well as of standing alone. An assertion does not necessarily become the less positive for being coupled with another, although that other may be doubtful or contingent.

Conjunctive. When a fact is asserted not as actual but merely as possible, or contingent, the form of words by which such assertion is expressed in any particular Language, may perhaps be the same as if the assertion were more positive; yet the context will show, that the verb is no longer in the indicative mood. The mood adapted to such contingent assertion has received various appellations, of which we consider the *conjunctive* to be the most appropriate, inasmuch as the contingency is usually marked by a conjunction (such as *if*, *though*, *that*, *except*, *until*, &c.) which connects the dependent sentence with its principal.

There are various methods of thus connecting sentences; but they may be distinguished into two great classes. In one class an uncertain sentence is connected with a certain one; in the other, both sentences are uncertain; that is to say, in the former case, a conjunctive is dependent on an indicative; in the latter, both sentences are conjunctive. Some Grammarians make this distinction the ground of a distinction of moods, calling the contingent assertion, in the first case, *subjunctive*, because it is subjoined to the indicative; and in the other case *potential*, because it states a potential, and not an actual existence. It seems, however, unnecessary thus to multiply moods; first, because no Language (that we know of) has assigned separate forms to the potential and subjunctive; and, secondly, because if we were to proceed this length, there is no reason why we should not go much further, and call every possible variation of contingency a separate mood. Of these we shall here notice some instances easily distinguishable in point of Principle.

1. Ut jugulent *homines surgunt de nocte latrones*.

Here *jugulent* is in the conjunctive, as indicating the *end* and *object* of the rising.

2. Peter said unto him, though *I should die* with thee, yet will I not deny thee.

Here "*I should die*" is mentioned as a *motive* to denial, but an insufficient one.

> 3. *Si fractus* illabatur *orbis,*
> *Impavidum ferient ruinæ.*

Here, in like manner, *illabatur* is in the conjunctive, as expressing a fact which might be the *cause* of fear to ordinary minds, but which *is not* so to the just and steadfast-minded man; and the conjunction *si* in the one case is equivalent to *though* in the other, both of them having the proper force of our expression "even if."

> 4. Except a man *be born* of water, and of the Spirit, he cannot enter into the kingdom of God.

Here the conjunctive *be born*, is placed in opposition to the indicative "cannot enter;" so that if the one be in the negative, the other must be so too, and *vice versâ*; for the implication is, that if a man be born of water and of the Spirit, he can enter into the kingdom of God. Accordingly, the Greek conjunctions in this and the preceding example are directly opposed to each other: in No. 3, the word used in the Greek text is Κἂν, that is, Καὶ ἐὰν; but in No. 4 it is ἐὰν μή.

> 5. *Cæmentis licet* occupes
> *Tyrrhenum omne tuis et mare Apulicum,*
> ——— *non animum metu*
> *Non mortis laqueis* expedies *caput.*

Here the condition differs from that of No. 2, in being a fact of present time; and on the other hand the indicative *non expedies* differs from the indicative *feries* in No. 3, by being in the negative.

> 6. The sceptre shall not depart from Judah, nor a lawgiver from between his feet, until Shiloh *come*.

Here both the facts are future, but the conditional one is the *term* or *boundary* of the other.

> 7. ——— *tacitus pasci si posset Corvus,* haberet
> *Plus dapis.*

In all the preceding instances one assertion is absolute; but here it is neither asserted that the Crow can feed in silence nor that it has more food; both parts of the sentence, therefore, are contingent, and, consequently, both are in the conjunctive mood.

> 8. *If it were done*, when 'tis done, then *'twere well*
> It were done quickly.

Here is also one contingent, namely, *'twere well*, depending on another contingent, *if it were done;* and on each we see a further contingency also depends.

These eight examples are sufficient to show that the varieties of contingent assertions are too various to be considered and treated as so many distinct moods of the verb. The first six are of the kind called, by some writers, *subjunctive;* the last two are of the kind called, in contradistinction from the subjunctive, *potential;* but as they are all equally conjunctive, it suffices to give them that name; and, indeed, it is a more correct and systematic distribution of the Grammatical nomenclature so to do; for the proper correlative to the term indicative is not subjunctive or potential, but some term which comprehends them both; as, for instance, the term conjunctive. The indicative asserts simply: the conjunctive asserts with modification: if the one is a mood, so is the other; but if the conjunctive is a mood, then its subdivisions cannot be properly so called; but they should rather be called sub-moods, if it were necessary to give them any peculiar denomination.

The effect of passion is to break in upon and disturb the regular processes of reasoning. Reasoning is conducted by express assertion, absolute or conditional. Passion goes at once to its object, assuming it as the consequence of an implied assertion. Thus, if the fact be that *I desire* a person *to go* to any place, it is not necessary that I should distinctly state my *desire* in the indicative, and his *going* in the conjunctive; but by the natural impulse of my feelings—feelings which Language conveys as clearly as it does the more gradual processes of thought—I say in a mood different both from the indicative and the conjunctive—*go!* Now, this mood, from its frequent use in giving commands to inferiors, has been called the *imperative*, and that name, as being the most general, we shall adopt. Some writers have distinguished from the imperative, the precative, the deprecative, the permissive, the adhortative, &c.; but so far as Language is concerned, these are but different applications of the same mood: the operation is the same in communicating the object of the passion and implying the assertion that such passion exists. A few examples may serve to explain our meaning:

> 1. *Let there be* light, said God; and forthwith light
> Ethereal, first of things, quintessence pure,
> Sprang from the deep; and from her native east
> To journey through the airy gloom began. *Milton.*
>
> 2. ——————— Fear and piety,
> Religion to the Gods, peace, justice, truth,
> Domestic awe, night rest, and neighbourhood,
> Instruction, manners, mysteries, and trades,
> Degrees, observances, customs, and laws,
> *Decline* to your confounding contraries!
> And *let* confusion *live!* *Shakspeare.*
>
> 3. *Help* me Lysander! *help* me! *Do* thy best,
> To pluck this crawling serpent from my breast!
> Ay me for pity. What a dream was here! *Id.*
>
> 4. Go, but be mod'rate in your food!
> A chicken too might do me good. *Gay.*

In the first of these examples, we have an instance of the highest *imperative*, that which proceeds from the Almighty Power, to whose command all things created and uncreated are subject; and who, in Milton's fine paraphrase of the first chapter of Genesis, is described as calling into existence the hitherto uncreated essence of light. The second example is *deprecative*, or rather *imprecative*, in which Timon calls down on his worthless fellow-citizens the natural consequences of their profligacy. The third is *precative*, in which the deserted Hermia, waking from a terrific dream, calls for help from her faithless lover Lysander. The last is *permissive*, in which the old dying fox, after a long harangue to dissuade the younger members of his community from pursuing their usual trade of rapine, at length permits them to go out on a similar excursion.

Now, in all these varieties of the imperative mood, the Grammatical process, both of thought and expression, is the same. In all of them the assertion of desire or aversion on the part of the speaker is clearly implied. The sense is, "I *command* that there be light"—"I *wish* that confusion may prevail"—"I *pray* you to help me"—"I *permit* you to go;" but it is unnecessary to express those various assertions, because they are all implied in the imperative moods, and without those moods they could not be so implied. The *imperative* animates the passionate sentence, as the *indicative* or *conjunctive* animates the enunciative

sentence. It converts the name of an object of passioo, or will, into a manifestation that such object exists; just as the indicative or conjunctive converts the name of an object of perception or thought into an assertion that it is really existing. The original text, "God said let there be light, and there was light," affords a plain example of this operation in both ways. The conceptions in both, are two; namely, *existence* and *light*. Each of these, without the verb, would remain a mere noun. The word "light" does so remain; but "existence," hy becoming a verb, exhibits itself first in the imperative as ao object of volition, and then io the indicative as an object of perception. Io the one case it implies an assertion of the Divine Will that light should exist; in the other it expresses an assertion that light did exist.

Optative.

We should not be inclioed to separate the optative mood from the imperative, were it not that various Languages, and particularly the Greek, distinguish it by a separate iofiection. The difference between those two moods appears to be rather a differeoce of degree than of kind; for we cannot agree with Scaliger, who says, (lib. iv. c. 144.) *differunt, quòd imperativus respicit personam inferiorem, optativus potentiorem:* "they differ in this, that the imperative regards an inferior person, the optative a superior." This differeoce is altogether accidental. Moreover, it makes no provisioo for the common case of wishes expressed between equals; and again, how are we to determine whether a request is addressed to a person in one character rather than another? Or why should we not have moods to designate the different degrees of superiority and inferiority? The fact seems to be, that the more distant and indirect union of the will with its object, bas given rise, in some Languages, to a peculiar form of the verb, generally called the optative mood. Yet eveo this distinction does oot appear to be very accurately observed in practice, for we sometimes see the optative used where the Imperative might have been more naturally expected. Tbus, in the *Electra* of Sopbocles, wben Orestes is forcing Ægisthus into the palace, to kill him in the apartmeot in which he had murdered Agamemnon, he says to his reluctant victim,

> Χωροῖς ἂν εἴσω ξὺν τάχει· λόγων γὰρ οὐ
> Νῦν ἐστὶν ἀγὼν, ἀλλὰ σῆς ψυχῆς πέρι.
> Go in, without delay, for now the strife
> Is not for useless words, but for thy life.

Where the optative χωροῖς undoohtedly expresses a pretty strong volition that Ægistbus should do what he was equally unwilling to perform.

The common distinction between the optative and the imperative is nearly expressed by the English use of the auxiliaries "may" and "let." Thus, the followiog passage in the *Hymn to Sabrina* is an example of the optative:

> Virgin daughter of Locrine
> Sprung of old Anchises' line,
> *May* thy brimmed waves, for this,
> Their full tribute never miss,
> From a thousand petty rills
> That tumble down the snowy hills!
> Summer drouth, or singed air,
> Never scorch thy tresses fair!
> Nor wet October's torrent flood
> Thy molten crystal fill with mud!
> *May* thy billows roll ashore
> The beryl, and the golden ore!
> *May* thy lofty head be crown'd
> With many a tow'r and terrace round!

These are matters not within the power or cootrol of the speaker, and which he, therefore, can only wish. On the contrary, when the speaker can command the execution of his wishes, he uses the word *let*, as the King, iu *Hamlet*,

> *Let* all the battlements their ord'nance fire—
> ——————— Give me the cups,
> And *let* the kettle to the trumpet speak,
> The trumpet to the cannoneer without,
> The cannon to the heavens.

It is observed by Vossius, that the Latin optative is no other than the conjunctive; and, indeed, the form is the same io both; for we say, *utinam amem*, or *cùm amem; utinam amarem*, or *cùm amarem; utinam amaverim*, or *cùm amaverim; utinam amavissem*, or *cùm amavissem*. Aod so in the passive voice, *utinam amarer*, or *cùm amarer; utinam amer*, or *cùm amer; utinam amatus sim*, or *cùm amatus sim*, &c. The mood, however, is not to be determined by the form, but by the signification; for it often happens that particular Languages do oot possess distinct forms for the different moods: and where they do so, the form of one mood is frequeotly used with the force of another. This takes place eveo in the Greek Language, which possesses the richest abundance of Inflectioos io its verbs. The Greek indicative is often used for the subjunctive and optative, and *that* through almost all its tenses, as VIGER has shown at large iu his celebrated Treatise *On Greek idioms:* and in return the optative, especially in the Attic dialect, is used for the indicative.

Interrogative.

Many authors cootend for a mood wbich they call interrogative: and it must be admitted that the act of the Mind, in asking, is different from that which it performs in indicating, or stating conditionally, or commanding or wishing. Yet it is unnecessary to constitute, on that accouot, a separate mood of the verh; for the Interrogative is no other than the indicative, with such accentuation or transposition of words, as to show the doubt of the speaker, and sometimes with an interrogative particle prefixed. The question is, as it were, the answer anticipated; but the answer, if complete, must necessarily be in the indicative mood, and, consequeotly, so must the question be. Thus: "Did Brutus kill Cæsar?"—"Brutus did kill Cæsar." "How maoy years reigned Augustus?"—"Augustus reigned forty-four years." Varro, Indeed, speaks of the moods of asking and answering as different, but this is true only with reference to the whole state of Miod expressed in the respective sentences, and not with reference to the particular form of the verb, which in both instances must necessarily be the indicative mood. Hence Apollonius says, Ἡγε ἂν προκειμένη ὁριστικὴ ἔγκλισις τὴν ἐγκειμένην κατάφασιν ἀποβάλλουσα, μεθίσταται τοῦ καλεῖσθαι ὁριστική,—ἀναπληρωθεῖσα δὲ τὴν καταφάσεως, ὑποστρέφει εἰς τὸ εἶναι ὁριστική—"the indicative mood, of which we speak, by laying aside that assertion, which hy its nature it implies, quits the name of indicative; when it reassumes the assertion, it returns again to its indicative character."

Infinitive.

It ooly remains to consider that which, as Vossius observes, not only the *semidoctum vulgus*, but eveo some of the *scientissimi*, have called the *infinitive* mood. We, however, are so far from ranking it among the moods, that we do not acknowledge it to he a verh at all; hut consider it, as we have already stated, to be more properly called a verbal noun.

Two principal grounds are alleged for reckoning the infinitive among moods, first that it is expressive of time, and, secondly, that it governs nouns, in construction, like a verb. As to the first of these reasons, it can only be valid in the opinion of those who adopt the definition of a verb, as being *nota rei sub tempore*, which definition we have already shown to be inexact. Time is an element which enters in many ways into our conceptions, but the Parts of Speech are not determined by the nature of the conceptions expressed, but by the manner of expressing them; and, as we have often repeated, there are two principal modes of expression, that is to say, *naming* our conceptions, and *asserting*, or manifesting their existence. Now the infinitives, "to love," *aimer*, *amare*, "to have loved," *avoir aimé*, *amavisse*, assert nothing by themselves, either as to the conception of love, or as to the conception of time in which the action of loving took place, they express both only in the way of notation, or naming, and not in the way of declaration; and therefore, in so far as either of those conceptions is concerned, the infinitive must remain in the class of nouns. As to construction, it is clear that this is merely a question of Particular Grammar. To say generally, that the infinitive governs a noun which follows or precedes it, is only to say, that it causes such noun to be in some case; but this is also effected by another noun; and therefore the mere circumstance of a change of case is in itself no test of the nominal or verbal character of the infinitive. The particular case in which the governed noun is placed remains to be considered, and *that* is to be ascertained, not by its termination, or inflection, or accompanying particle, but by its signification. Now, as to its signification, if the governed noun be not the object or the agent of some action or existence asserted, the case in which it is does not imply that the governing word is a verb. Hence the phrase "I desire the sight of thee," is exactly similar to the phrase "I desire to see thee." The words "sight" and "see," neither of them assert that the action of seeing takes place, and consequently the words "thee" and "of thee" are not either of them the agent or the object of any such assertion: and we cannot conceive any reason, in the signification of the words, which should have prevented the Latin idiom from being *cupio videre tui*, as well as *cupio visum tui*; for, in fact, *videre* and *visum* are alike *names* of the action of seeing; they alike express the *object* of the verb *cupio*; in other words, they are nouns, and it is matter of idiom whether the relation which they bear to the following noun should be expressed by the termination *s* or *ui*.

We have before observed, that Priscian says *currere* is *cursus*: and we have shown that, in English, "to part" is "parting;" there are, therefore, three kinds of verbal nouns, which in various idioms are differently interchangeable, namely, those which are called by various writers the infinitive mood, the abstract noun, and the participle (including the gerund and the supine.) This will appear from a comparison of the idioms of almost all Languages. We are told, that in Gaelic the present participle and the verbal noun are the same; and again, that the infinitive is formed by the dative of the present participle. In the Ethiopic Grammar, Ludolf says, *Infinitivus sæpissimè nominascit*; and again, *cum affixis ba et la, Latinè per gerundia in do et dum exprimi potest*. In Bengalese, too, the infinitive answers to the verbal noun; and the first gerund supplies the place of the English infinitive, when two verbs come together. From these and many similar observations we may infer, that there are various classes of nouns substantive and adjective derived from (or rather connected with) all verbs; but that such nouns relate solely to the noun, which, as we have stated, is involved in every verb, and not to the part of the verb on which its verbal character essentially depends. These nouns may be thus classed:

1. Verbal adjectives, (commonly so called,) which express the conception in the form of an attribute, as the Latin verbals in *bilis*, &c. of which Mr. Tooke makes a class of participles, and which do not involve the notion of time.

2. Participles, (commonly so called,) which agree with the former, except that they involve the notion of time.

3. Abstract nouns, (commonly so called,) which express the conception in the form of a substantive, as the Latin nouns in *io*, &c. which do not involve the notion of time.

4. Infinitives, (commonly called infinitive moods,) which agree with the former, except that they involve the notion of time.

Now it happens in most Languages, that distinct forms are wanting for some of these four classes of nouns, or that the forms are reciprocally used for each other. Hence "he learns *to sing*," or "he learns *singing*," are used in English indifferently; and so "he *learns* singing," and "he *is* singing," are equally consistent with our idiom.

We have said that the infinitive involves the notion of time; and this we conceive is the proper distinction between *currere* and *cursus*, when they are distinguishable; for we may say *festinat currere*, but not (in the same sense at least) *festinat cursum*. It is only when *currere* does not involve the notion of time that the remarks of Priscian become strictly accurate; and when this happens, then, in fact, the word *currere* belongs to the third, and not to the fourth class of words above-mentioned.

In respect to the expression of time by infinitives, a distinction is to be observed analogous to the distinction which we have before noticed between the verb substantive and the verb of action. If an individual fact is meant to be referred to, then, as this fact must necessarily occur at some given time, the time in question is expressed by the infinitive; and it is then only that we give it the name of infinitive. Thus, Θελω, θέλω φιλῆσαι means, I desire to love *at this moment*; whereas χαλεπὸν τὰ μὴ φιλῆσαι means, the state of not loving is hard *at all times*. In the former case, φιλῆσαι is strictly an infinitive, and should not be rendered into Latin by the accusative *amorem*, but by *amare*. In the latter case μὴ φιλῆσαι is strictly equivalent to a noun of action, and consequently is used, in the Greek idiom, with the article τὰ.

Whether we call infinitives nouns, or verbs, the propriety of the name infinitive is very evident from the observation of Vossius: *Ut finitum est nomen, tum Philosophus, tum plurativus Philosophi; quippe illo unus, hoc multi significantur: at contra infinitum est sui, quia utriusque est numeri, item Græcum ἐκεῖνά, quo et ille et illi denotatur; sic finitum verbum est audio, ac facio, ut quo certus numerus designatur; infinita autem sunt audire, agere, ut quæ deficiant numeris ac personis, et*

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undique sunt indefinita ac indeterminata. "As the noun *Philosophus* is finite, both in the singular and in the plural *Philosophi*, since the former signifies one person, and the other many; but on the other hand the word *sui* is infinitive, because it is both singular and plural; and in like manner the Greek word *ἑαυτὰ* is infinitive, because it denotes both *him* and *them;* so the verbs *audio* and *facio* are finite, as designing a certain number; but *audire* and *agere*, which express no certain number or person, and are in every way indefinite and indeterminate, are called infinitives."

It is to be observed, that the Latin nouns in *io* seem properly to have been definites; that is to say, that they originally signified only a certain number of acts, and not action in general, as *actio* meant a singular exercise of the active power, and *actiones* several such exercises; but in a secondary use of the word *actio*, it came to be employed for such exercise generally; and in this secondary use it is properly an infinitive, and coefficient with *agere*. The Greeks, it is well known, though they did not give their infinitive moods the terminations of case, like other nouns, yet distinguished them by the articles of the different cases; as *τὸ γράφειν, τοῦ γράφειν, ἐν τῷ γράφειν.* This construction is unknown to the Latin; for we cannot say *hoc amare, hujus amare*, &c. nor *ad amare, ab amare*, the place of which latter phrases is supplied by the gerunds, as *ad amandum, ab amando.* And again: in English, it is only by a forced imitation of the Greek idiom, totally unsuitable to the genius of our Language, that Spenser says—

> For not to have been dipp'd in Lethe's lake
> Could save the son of Thetis *from to die.*

And this Hellenism is the less excusable, as we have actually an infinitive which admits of being used with the preposition: for the proper and intelligible English construction would have been—

> Could save the son of Thetis *from dying;*

whereas the usual opposition between the prepositions "from" and "to" renders the phraseology of the Poet intolerably harsh and inconsistent. Nor does it appear that Harris, who seems to approve of this line of Spenser, is much more accurate in another example, *viz.* "he did it to be rich;" where, he says, we must supply by an ellipsis the preposition *for*, as "he did it for to be rich." Certainly this is a Provincial way of speaking, but it is a mere rustic pleonasm. In French, *pour s'enrichir* is proper, because the infinitive *s'enrichir* has not in itself the objective mark; but in English, where that mark is supplied by the preposition *to*, a similar mark in the word *for* is altogether superfluous.

We have thought it necessary to dwell the longer on the consideration of the infinitive, because in rejecting it not only from the moods but from the verbs, we certainly deviate, more than we are generally disposed to do, from the path pursued by the great majority of Grammatical writers. Yet that this deviation is justified by high authority, we have before shown, in stating that many of the Ancients (and those, as Harris says, "the best Grammarians") have called the Infinitive *ὄνομα ῥηματικὸν*, or *ὄνομα ῥήματος*: and with these agrees Priscian, in the following passage, "*A constructione quoque vim* REI *verborum, id est,* NOMINIS, *quod significat ipsam rem, habere* INFINITIVUM *possumus dignoscere*" "From the construction, too, we may perceive, that the infinitive has the force of the *thing*, of the verb, that is to say, of the *noun*, which signifies the thing itself." What is here called the thing, (or substance,) of the verb, is what we have called the conception, the mere name of which is a noun. Thus, "I die" expresses the conception of dying, but it not only names that conception, it asserts the thing to exist, with reference to a certain person; whereas "to die" expresses the conception, that is to say, names the thing, and does nothing more: it does not manifest the existence of the thing as an object either of perception or volition; it does not assert that any person is dying, or has died, or will die, or may die; neither does it evince any desire that such an event should occur, either positively or conditionally. "Take away the assertion, the command, or whatever else gives a character to any one of the other modes," says Harris, "and there remains nothing more than the infinitive." Take away from the other modes, say we, whatever gives them the verbal character, and there remains the noun. Whether we call this noun a verbal noun, or a participial noun, or simply an infinitive, is immaterial; provided we clearly understand that it belongs not to the class of verbs, but to that of nouns, and that its nature does not depend on its form; since, in English, the words *death, to die*, and *dying*, may all be used as Infinitives; and, when so used, are generally convertible into each other, without any change of meaning. Lastly, we may observe, that as the participle is a verbal adjective, so the infinitive is a verbal substantive. The former can supply only the predicate of a proposition, as "I am walking;" the latter may form the subject, as "walking is pleasant," "to walk is pleasant;" in which two latter sentences the words "walking" and "to walk" are both Infinitives, and must be translated into Latin by the word *ambulare*, and not by the word *ambulans.* This consideration renders it the more remarkable, that Harris should incline to rank the infinitive among the moods of the verb, since he himself had classed the verb among attributives, all of which, as he observes, "are, from their very nature, the predicates in a proposition."

Verbs. Tense.

Tense

The second peculiarity of the verb consists in its tenses. The word *Tense* plainly shows that our chief Grammarians, in the early periods of Grammatical study in England, were Frenchmen; for it comes from the Latin *tempus*, through the French thus, *tempus, temps, tems, tense.* Tense, therefore, originally and properly means the expression of time in combination with the assertion of existence; but this must not be taken to be the sole effect of the tense in particular Languages, as we shall presently perceive. In order, however, to comprehend this subject fully, we must begin, as Harris judiciously does, by considering existence according as it is mutable or immutable. We are well aware that, in the proud and insolent ignorance of modern Philosophy, we shall be told that there is no such thing as immutable existence; that men's Minds are made up, as their bodies are, of a certain small dust, which is perpetually whirling about, and taking various forms and arrangements, some of which it pleases every man to call true, and others false; that this latter circumstance, however, is a mere delusion of the individual's mind, *mentis gratissimus error;* that when the man dies, his notions, their truth and their falsehood, their wisdom and their folly, all die with him; and though some truths wear better than others, and keep in fashion for twenty or

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thirty centuries, while the greater part of our notions do not last longer than the small ephemeral insects of the Nile, yet that in the end they all sink into one common Lethe.

——— *animæ quibus altera fato Corpora debentur.*

The opposite Philosophy to this, although stigmatized as "a Metaphysical jargon and a false Morality, which can only be dissipated by Etymology," we feel ourselves constrained to adopt, from the utter repugnance of the former to any thing like common sense or intelligibility. We cannot conceive that the objects of intellection and Science are mutable in any possible number of years, or in any imaginable conjuncture of circumstances. We cannot, for instance, believe that in a square the diagonal ever was, or will be, or can be, commensurable with one of the sides. These two magnitudes are not incommensurable because Euclid happened to think so, or because his doctrine on the subject has prevailed for above two thousand years. Their incommensurability is a truth as independent of that lapse of time, as any two things can possibly be of each other. The opposite to it cannot be conceived by the Human Mind. The existence of this truth, therefore, is justly styled immutable.

Present.

Of such immutable existence the *Present* tense is usually considered the proper exponent, because, in most Languages, it is among the simple forms of the verb, and in particular it has no distinct mark of time about it. There is no reason, *a priori*, that there should not be a separate inflection of the verb to distinguish perpetual, absolute, immutable existence, from that which is predicated with reference to some certain time; but as no Language that we know of has adopted any such form, and as absolute existence is naturally contemplated under the form of a time perpetually present, it is sufficient for us to consider this as one of the uses of the present tense.

The other use of the present tense depends on the nature of mutable existence. Now, mutable objects exist in time. When, therefore, we declare them to exist, that is, whenever we employ a verb active, or passive, or neuter, we must declare them to exist in some time. But time is distinguishable as to its periods into present, past, and future; and as to its continuity into perfect or imperfect; and though the present, from its nature, must be definite and positive, yet the other two periods may be stated indefinitely and with relation to some different time. From these sources, and from the differences of mood already noticed, may be derived all the tenses which appear in use in different Languages. And first, as to the Present, considered as marking a certain portion of time, it is manifest that we may consider as present to us a greater or less portion of time. Time flows on continuously, and has in itself no stops or periods, but the Mind dwells on certain portions, and gives them a distinct expression in Language. The *names* of these portions are various, as an age, a year, a day, an hour, a moment; but the *assertion* of their existence is a collateral incident to the verb. It has been well shown by Mr. Harris that the present time, strictly speaking, is not cognizable by any human faculty; for it is

——— Like the lightning, which doth cease to be,
Ere one can say it lightens.

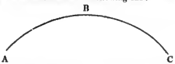
Verb. Tense.

"Let us suppose," says he, "for example, the lines AB——BC——"

"I say, that the point B, is the end of the line A B, and the beginning of the line B C. In the same manner let us suppose A B, B C to represent certain times, and let B be a *now*, or instant, which they include; the first of them is necessarily past time, as being previous to it; the other is necessarily future, as being subsequent." Hence he concludes, that time present has at best but a shadowy and imaginative existence; and, of course, as sensation refers only to time present, that sensible existence is itself altogether imperceptible, eluding the steady grasp of thought, and approaching to absolute nonentity. This will, doubtless, appear strange to the modern Philosophers, who hold that sensible existence is the only existence; but let them meditate on what they mean by the words *now*, or *instant*, or *moment*; let them consider how difficult it is to arrest the fleeting progress of time, and fasten it down to the periods indicated by those terms; and they will, perhaps, perceive that their notions are not quite so clear as they have hitherto fondly imagined.

We will assume, that in the above diagram the *perfect* present is correctly indicated by the point B. At that moment, I open my eyes and I contemplate, at one view, a large theatre crowded with numerous happy faces, with splendour, and beauty, with the diversities of age and sex, and condition, with mirth and gravity, and all the passions, which, though not meant to be brought into public, could not entirely be thrown off and left at home, like an unvalued garment. Or, perchance, I am on a proud hill-top, from whence, at one glimpse, I behold mountains and valleys spread in rich perspective before me, with the near cottages, and the distant town, and, beyond all, the remote and hazy ocean. I see the variegated foliage and the ripening corn, the clouds of heaven sailing high in air, the rustics at their labour, and the little vagrant boy picking daisies at my feet, and delighting in his idleness. Without any time for reflection, without a thought of the successive action of the machinery in this grand landscape, I say, "*I see*" all this, at the present moment, and I enunciate it in the present tense perfect.

But if I wish to express a continuous action, if, for instance, I mean to describe myself as remaining for some time in contemplation of the scenes just described, I am compelled to change my expression, and to adopt the present tense *imperfect*. In that case, I say "I am contemplating," "I am beholding:" and the diagram before drawn will not then so well express the time intended to be described as the following one:



Here, the present time, designated by the letter B, extends indefinitely toward A and C, embracing a

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segment, the whole of which is viewed by the Mind as being at once present to its contemplation, though without any definite boundary on either side. The English Language easily distinguishes this sort of present tense from the other, by the use of the verb *to be* and the participle present; but in most other Languages the present perfect and the present imperfect have one and the same form, and can be distinguished only by the context.

Past.

We have seen that the present imperfect implies something of the past, and something of the future. Modern Philosophy is very well satisfied to pass over all the difficulties which occur in regard to the nature of time. We are told, "that we have our notion of succession and duration from this original, *viz.* from reflection on the train of ideas which we find to appear one after another in our own Minds," and that "time is duration set out by measures." This is surely any thing but reasoning. First, it is assumed that there is a train of ideas which constantly succeed each other in every man's understanding. Each of these ideas then must either occupy an indivisible point of time, or it must have some distinguishable duration. In the former case we cannot at all understand how reflection on many indivisible points should afford us the notion of any continuous quantity. In the latter case there would be no occasion to reflect on a train; for the reflection on a single idea would present to us the notion of duration in itself. But what are these ideas; and how do they march in train? Are they all of equal duration? If so, or if not, what is it that determines the duration of each? Is it not the voluntary act of the Mind?—Again: is there no interval in the train? *Aliquando dormitat Homerus*, was an old remark; and we suspect that it applies even to the most lively and active Minds of the modern Philosophical School. On the hypothesis above stated, it would seem that before a man could have any notion of duration, and consequently of time, he must have formed in his own Mind thoughts of a certain duration; these thoughts must have succeeded each other in a distinguishable order, he must have been fully aware of that succession, and he must afterwards have made it the subject of reflection. But this statement is absurd; for on what is he to reflect? On a succession which would not present any notion of duration unless it involved that notion in the first instance; nor would the succession of any two or more ideas produce a notion of duration if the thoughts themselves, or the interval between them, did not involve it. The truth is, that the idea of duration, or time, is not to be made up out of any other elements, but is an original law and first element of thought in the Human Mind. We perceive duration of time just as we perceive extension of space, because it is one of the necessary forms under which alone we can contemplate existence. Whilst we are contemplating the indivisible moment which constitutes the perfect present, it has already melted into the imperfect present; and if we attempt to seize it again, it has already become the past; its distinction is then fully marked; for the past is presented to us by memory, as the present is by sensation.

The past has its perfect and its imperfect, its definite and its indefinite, its positive and its relative. We may speak of an action which was performed on a given day, at a given hour, and a given minute; as of Cæsar's leaping into the Rubicon, or of the first shot which was fired at the commencement of the Thirty years' War: or we may speak of an action in which a person was occupied, and which was going on at the time to which we refer. Thus the ancient artists inscribed their works with the word *faciebat*, to indicate that they did not put them out of hand, as finished and perfect, but that they had been for some time engaged making them, and would have carried further their attempts toward perfection, had time and circumstances permitted. Thus, too, Syrus in the *Heautontimorumenos*, describing the employment in which he found Antiphila and her servants employed, says,

> *Texentem telam studiose ipsam offendimus;*
> ——————————— *Anus*
> *Subtemen* nebat: *præterea una ancillula*
> *Erat:* ea texebat una.

Verbs. Tense.

Again, we may speak of the past time *definitely*, fixing the epoch when it happened, as,

> That day he *overcame* the Nervii.

Or *indefinitely*, declaring that the act of which we are speaking is past, but not ascertaining whether the time of its performance was near or distant; as,

> Thou art the ruins of the noblest man
> That ever *lived* in the tide of times.

Lastly, the past time may be mentioned simply as past at the present moment, or as past at some time preceding the present; and these two tenses may be reciprocally distinguished as *positive* and *relative*. Thus, in the positive, Macbeth says,

> I *have liv'd* long enough; my way of life
> Is fallen into the sere, the yellow leaf.

In the relative, Thyrsis, (the attendant Spirit,) in the Masque of *Comus*, says,

> This ev'ning late, by then the chewing flocks
> *Had ta'en* their supper on the sav'ry herb
> Of knot-grass dew-besprent, and were in fold,
> I sate me down to watch.

Future.

As the past time exists in memory, so the *Future* exists in imagination. Such is the nature of Man, or he would be unable to attain "that large discourse, looking *before* and *after*," which the Poet truly assigns to him. The conception of duration may be supposed to exist in a Being which had only the perception of the present and the past; but to render that conception operative and useful, to convert it into an accurate idea of *time*, it is necessary that the notion of futurity should be superadded. It is a mistake to say that the present impression is distinguished from the memory of what is past by superior vividness and strength. It often happens that things present

> ——— Pass by us, like the idle wind
> Which we regard not;

whilst objects of memory so fully occupy our attention, that, like Hamlet, we think we see them "In the Mind's eye." Still we see them (whilst we possess our reasoning faculties) not as present, but as past, with a specific difference of perception. The perception of the future, as such, is also specifically different from either of the others. Reason and reflection alone could not explain to us the necessity of such a distinction, because it is an element of Reason, so far as that faculty applies to events occurring in time. It would be as correct to say, that by reasoning on the nature of light and colours, we come to discover the existence of red

and green, as to say, that by reasoning on duration, we come to discover that there is a past, a present, and a future.

When we treat of past, present, and future, we treat of them with reference to some particular moment; for as time is perpetually flowing on, that which was future yesterday is to day present, and that which was present yesterday is to day past. The particular moment which thus characterises the time, is that in which the speaker or writer is addressing himself to his hearers or readers. We have seen, however, that that moment is not always referred to as indivisible, but sometimes as capable of extension and indefinite continuance. So it was observed to be in the present and past; and so it is in the future. A person may say, "I *shall mount* my horse;" and he may say, "I *shall be* an hour *riding* from London to Richmond." In the former instance the tense may be called the future *perfect*; in the latter the future *imperfect*. Again, the future may be *definite*; as, "I shall mount at six o'clock;" or *indefinite*, as, "I shall ride some time in the course of the day." Lastly, it may be *positive*, considering the act only as future at the moment of speaking, which is the case with all the preceding examples, or *relative*, considering the act as not to take place till after some other which is also future. Thus, a person may say, "I *shall have mounted* my horse before the clock has struck;" or, "I *shall have been riding* an hour when I reach the next milestone."

These distinctions refer properly to time. There are others which refer to the contingency of the act, or to its frequency and habitual performance; these seem to draw their distinctive character properly from the mood, or kind of verb, and therefore, we think them not so much tenses as modifications of the tenses already named. Somewhat more of doubt may, perhaps, be allowable with respect to those forms of speech which imply either the immediate intention to begin an act, or its recent completion. Of the first class are "I am about to write," "I was beginning to write," "I shall begin to write:" and of the second class, *Je viens d'écrire*, "I have just written;" *Je venois d'écrire*, "I had just written;" Ἔσομαι γεγραφώς, "I shall have done writing." Yet though these forms of speech serve to mark given periods of time, and therefore may be called tenses; yet they seem to go somewhat further, by including other notions not strictly referable to time. At all events, there must be a limit to the combinations which are distinguished as tenses. Time is capable of endless divisions, and Language would be infinitely minute in all its ramifications, if it provided a separate inflection for all those separate modifications of thought. It is true that idioms vary in nothing more than in the varieties of tense, for which they provide. Some are very meagre; others luxuriant; some are strictly confined to differences of time; others mix up, with these, a variety of other considerations. Thus the English Language marks a distinction unknown, we believe, to any other Language, between the future of choice and the future of necessity: and what is remarkable, that distinction varies with the different persons of the tense. "I shall go" implies no particular volition, nor indeed any thing but the certainty of the event. "I will go" implies absolute volition. On the other hand, "you will go" implies no volition of any person, but "you shall go" implies the volition of the speaker. It is a striking proof how much nicety and difficulty there is in the peculiar use of the tenses of verbs, that scarcely a single Scottish writer, however eminent, will be found to have accurately observed the distinctions of "shall" and "will" throughout all his compositions. The reason is, that the writers in question have from infancy become accustomed to the Scottish idiom, and idiom is much less a matter of reasoning than of habit. A critical examination of the idioms regarded as most elegant, will show them to abound with the same pleonasms and ellipses, which are commonly considered as marks of rusticity in the Language of the people. The English idiom above-mentioned, however, is of very simple explication. It refers primarily to the will of the speaker. If, therefore, he says, "I will," it is to be understood that so far as his power extends, the action is to be performed; but if he says "I shall," inasmuch as he indicates no volition of his own, nothing further is to be inferred but the futurity of the action. Again, if he says, "you shall go," "he shall go," he intimates a necessity; for the original meaning of *shall* is that which is necessary, and must, or at least, ought to be done, from the Mœso-Gothic *skal*.* But this necessity, being declared by the speaker, relates to his will alone. Thus, in Coriolanus:

SICINIUS. ——————— It is a mind
That *shall remain* a poison where it is,
Not poison any further.
CORIOLANUS. *Shall* remain?
Hear you this Triton of the minnows? Mark you
His absolute *shall*?

On the other hand, when the speaker says, "you *will* go," "he *will* go," he intimates no will of his own; and, therefore, nothing is understood but the futurity of the action. The proper force and effect, therefore, of the two English futures may be thus expressed:

1. Future compulsory. "I will go," *i. e.* it is *my will* to go. "Thou shalt go," *i. e.* it is *my will* to compel thee to go. "He shall go," *i. e.* it is *my will* to compel him to go.

2. Future not compulsory. "I shall go," *i. e.* there is some cause compelling me to go, *independently of my will*. "Thou wilt go," *i. e.* there is some cause compelling thee to go, *independently of my will*. "He will go," *i. e.* there is some cause compelling him to go, *independently of my will*.

The same reasoning of course applies to the plural number as to the singular; and, consequently, "we will go," "ye shall go," "they shall go," belong to the first kind of future; and "we shall go," "ye will go," "they will go," belong to the second. What we have here called the future compulsory has sometimes a merely permissive force, sometimes a promissive, and sometimes it is used in the manner of an imperative mood, as "Thou shalt not steal," "Thou shalt do no murder," for "steal not," "murder not;" and this idiom is found both in the Greek and Latin: Ἔσεσθε οὖν ὑμεῖς τέλειοι, Ye shall be therefore perfect, *i. e.* Be ye therefore perfect, St. Matt. ch. v. ver. 48. And so Horace: *Inter cuncta* leges *et* percunctabere *doctos*. Lib. i. Epist. 18.

To return from this digression, we may observe, that though various circumstances, of the nature of

* See JUNIUS *ad vocem*. Also WACHTER, *schuld*, *schuldig*.

those which we have already pointed out, do, in fact, enter into the composition of tenses in various Languages; yet they do not properly belong to the scientific division of tenses in Universal Grammar, which ought to regard only distinctions of time, and not even those beyond a certain degree of minuteness and complexity. Where the divisions of time are very minute or complex, their expression rather forms a sentence than a word. It is more than the Mind can easily grasp or communicate in one combined form, and which therefore to be understood requires to be analyzed into different words.

In a subject which has undergone such various treatment by Grammarians, as the distribution of tenses, we are far from arrogating to our own method any very superior merit; still less do we recommend the name which we have given to each tense as the best calculated to express its distinctive character. Instead of the perfect and imperfect, some writers use the terms *absolute* and *continuous*; and what we have called positive and relative, corresponds nearly with the *perfectum* and the *plusquam perfectum*, the *futurum*, and *paulo post futurum*.

The arrangement proposed by the learned Mr. Harris, though differing considerably from that which we have suggested, is, we must acknowledge, entitled to great attention: and, therefore, without going into all his reasonings in favour of it, (for which we refer to the 7th chapter of the 1st book of *Hermes*,) we think it right to state its general outline.

"Tenses," he observes, "are used to mark present, past, and future time, either *indefinitely*, without reference to any beginning, middle, or end; or else *definitely*, in reference to such distinctions.

"If indefinitely, then have we three tenses, called *aorists*, (so called from the Greek ἀόριστον, undefined, or unlimited,) *viz.*, an aorist of the present, an aorist of the past, and an aorist of the future.

"If definitely, then have we nine other tenses, *viz.*, three to mark the beginnings of the present, past, and future respectively, three to denote their middles, and three to denote their ends.

"The first three of these nine tenses we call the inceptive present, the inceptive past, and the inceptive future: the next three, the middle present, the middle past, and the middle future; and the last three the completive present, the completive past, and the completive future.

"And thus there are in all twelve tenses, of which three denote time absolutely, and nine denote time under its respective distinctions."

1. Denoting time absolutely and indefinitely:
    1. Aorist of the present, *γράφω*, *scribo*, I write;
    2. Aorist of the past, *ἔγραψα*, *scripsi*, I wrote;
    3. Aorist of the future, *γράψω*, *scribam*, I shall write.
2. Denoting time under the respective distinctions of inception, continuity, and completion.
    1. Denoting inception:
        1. Inceptive present, *μέλλω γράφειν*, *scripturus sum*, I am about to write:
        2. Inceptive past, *ἔμελλον γράφειν*, *scripturus eram*, I was beginning to write;
        3. Inceptive future, *μελλήσω γράφειν*, *scripturus ero*, I shall be beginning to write.
    2. Denoting continuance:
        1. Extended present, *τυγχάνω γράφων*, *scribo*, or *scribens sum*, I am writing:
        2. Extended past, *ἔγραφον*, or *ἐτύγχανον*, *γράφων*, *scribebam*, I was writing;
        3. Extended future, *ἔσομαι γράφων*, *scribens ero*, I shall be writing.
    3. Denoting completion:
        1. Completive present, *γέγραφα*, *scripsi*, I have written;
        2. Completive past, *ἐγεγράφειν*, *scripseram*, I had done writing;
        3. Completive future, *ἔσομαι γεγραφώς*, *scripsero*, I shall have done writing.

Whatever arrangement we adopt, we shall certainly not find it fully followed out in many Languages; for while some have great varieties of inflection or construction to express the different times, others have fewer; and yet it may happen that the idiom, which upon the whole is the least rich in tenses, is more minute than all the others in some one particular distinction.

On the combination of tense with mood, much judicious criticism is to be found in various Grammarians, and particularly in the Work last quoted, the *Hermes* of Mr. Harris, who has collected not only his own observations, but those of the Philosophers of successive Ages; herein evincing a modesty the more admirable, when it is contrasted with the too prevalent vanity of the present day, by which every Tyro in Science and Literature is led to believe himself a luminary arising to enlighten and vivify a benighted world. These self-complacent gentlemen often succeed in drawing round themselves a little circle of admirers; and in that case their contempt of all who preceded them in their own particular line of study is usually unbounded. It may, perhaps, be useful to observe, that such overweening presumption, as it proceeds on a great mistake in point of fact, so it indicates a narrowness of mind extremely inconsistent with true genius, or the power of permanently benefiting and delighting mankind. Let us hear Milton, that noble ornament of modern Poetry, speaking of his predecessors, even the most ancient:

> —— O sad virgin, that thy power
> Might raise MUSÆUS from his bower.

And elsewhere:

> ————————Nor sometimes forget
> Those other two equal'd with me in fate
> (*So were I equal'd with them in renown!*)
> Blind THAMYRIS, and blind MÆONIDES.

And again:

> Æolian charms, and Dorian lyric odes,
> And his who gave them breath, but *higher sung*,
> Blind Melesigenes, thence HOMER call'd.

On the other hand, we are certainly taught a very

different mode of estimating ancient and modern Poets by the too well known Philosopher of Sans Souci.

Ah! dans ces jours, où notre heureux destin
Nous a fourni, pour effacer Homère,
Un Apollon plus vif, et plus brillant;
Comment peut-on, en possédant Voltaire,
Avec dédain, regretter un instant
Ce vieux bavard?

It would be somewhat curious to form a list of the modern writers who have been characterised by their admirers, or by themselves (which is still more frequently the case) as being absolute *inventors* in the different branches of Science and Literature: and the best commentary on such a list would be another, somewhat more difficult indeed to make out, which should contain the discoveries, or even improvements, for which the World is really indebted to these, its supposed enlighteners and guides. In Grammar, we have been told that a certain writer of recent date dispelled, "by a single electric flash of genius," the obscurity which hung over the whole Science. It is the duty of the Encyclopædist to correct such errors in point of fact, and to expose such absurdity in point of opinion. In Physical Science there may be discoveries which go to alter much of our general reasoning on all subjects connected with those discoveries. Substances altogether unknown, organizations never before suspected to exist, may be rendered obvious by experiment. But in the Sciences which depend on a knowledge of the Human Mind, it is altogether weak and absurd to suppose that any such cause of sudden and total improvement can exist. By industry and attention, we may, perhaps, be enabled to methodize some portions of every such Science better, or even to correct, in some degree, their general arrangement; but we cannot possibly find in them any one topic which has not been admirably handled by some Philosopher, ancient or modern; and as to the great leading systematic Principles on which they respectively depend, these will generally be found to have been established from the highest antiquity. The illustrations of Particular Grammar, it is true, are of the nature of Physical Science, for they depend on the comparison of numerous Dialects, ancient and modern, some of which are to this day unknown to the civilized and studious World, and others remain in a great measure buried in the dust of antiquarian records. The Etymologist, therefore, may possibly discover some facts affording an important clue to discoveries beyond the attainment of Plato or Aristotle; as, for instance, those which may hereafter explain the confusion of Languages, or the dispersion of the different families of mankind over the face of the Earth; nor are we at all inclined to undervalue the Etymological studies of modern writers, and particularly of the late Mr. Tooke; but it is material to observe, that whatever they are, they belong less to the Philosophy of Language than to its History. Again, that part of Grammar which relates universally to what we have called the *matter* of Language, that is, to the construction and use of the organs employed to effect a communication of the Mind, is evidently Physical, and of course follows the common laws of Physical Science. In this, therefore, we may possibly look for discoveries, affecting in a very great measure the whole system of such communication. In this view, the formation of a Common Alphabet for all nations, or of a Real Character, or even of an Universal Language, is not beyond the bounds of rational hope or expectation, and, if ever attained, may be the result of some great, and perhaps sudden improvement in this part of Grammatical Science; nor while we are speaking on this subject, should we neglect the opportunity of paying a deserved tribute of respect to the memory of that excellent man, Bishop Wilkins, whose *Essay towards a Real Character, and a Philosophical Language*, first published in 1668, is beyond compare the most ingenious Work of the kind which has ever fallen under our observation. But the Pure Science of Grammar, however it may lend its aid to any of the discoveries here spoken of, cannot receive from them any great or important improvement; for its Principles, as we have abundantly shown, are founded on the operations of the Human Mind, and certainly the Human Mind was understood, and all its principal functions developed and explained by the Philosophers of ancient Greece and Rome, with far more clearness, depth, and precision, than they have been by any writer in France or England within the last fifty years. The ancient Grammarians were formed in the Schools of ancient Philosophy, and were themselves Philosophers of no mean rank. Such a person was Apollonius of Alexandria, surnamed Δύσκολος, or "the difficult," whose four books περὶ Συντάξεως, "on Syntax," are considered to be the most Philosophical of any extant on the Greek Language. He himself says he composed them, μετὰ πάσης ἀκριβείας, "with all possible accuracy." Priscian, who professes to make him his chief guide, says of his Dissertations, *quid Apollonii scrupulosis quæstionibus enucleatius possit inveniri?* The celebrated Theodore Gaza confesses that he owes to him almost every thing. The learned Thomas Linacer follows him, as it were, step by step. And lastly, Harris, who quotes him liberally throughout the whole of *Hermes*, declares him to be "one of the acutest authors that ever wrote on the subject of Grammar." In thus tracing the literary genealogy of Grammatical authorities, we at once prove their present title to respect, and show that it could not have subsisted through so many centuries, if it had not been originally founded on superior talent and ability. When, therefore, we find an author like Apollonius employing much learning on the illustration of the tenses, and their combination with the different moods, we are not to be persuaded that such speculations are wholly trifling or useless to those who would obtain a perfect acquaintance with the Science of Grammar.

Now Apollonius observing on the connection which we have already noticed between the future tense and the imperative mood, satisfactorily explains why in most Languages there is not a distinct form for the future tense of that mood. The reason is that all imperatives are in their nature futures; for thus argues Apollonius; Ἐπὶ γὰρ μὴ γινομένοις ἢ μὴ γεγονόσιν ἡ Προστάξις· τὰ δὲ μὴ γινόμενα ἢ μὴ γεγονότα, ἐπιτηδειότητα δὲ ἔχοντα εἰς τὸ ἔσεσθαι Μέλλοντος ἐστι. "A *command* has respect to those things which either are not doing or have not yet been done. But those things which being not now doing, or having not yet been done, have a natural aptitude to exist hereafter, may be properly said to appertain to the *future*." And again he says, Ἅπαντα τὰ προστακτικὰ ἐγκειμένην ἔχει τὴν τοῦ μέλλοντος διάθεσιν —σχεδὸν γὰρ ἐν ἴσῳ ἐστὶ τὸ, ὁ τυραννοκτονήσας τιμάσθω, τῷ τιμηθήσεται, κατὰ τὴν χρόνου ἔννοιαν· τῇ ἐγκλίσει

διηλλαχὸς, καθὸ τὸ μὲν προστακτικὸν, τὸ δὲ ὁριστικόν. "All *imperatives* have a disposition within them which regards the *future*. With regard to time, therefore, it is the same thing to say, *Let him that kills a tyrant be honoured*, as to say, *He that kills a tyrant shall be honoured*; the difference being only in the mood, inasmuch as the one is *imperative*, the other *indicative*." So Priscian shows the connection of the imperative with the future.—*Imperativus verò præsens, et futurum (tempus) naturali quâdam necessitate videtur posse accipere. Ea enim imperamus, quæ vel in præsenti statim volumus fieri, sine aliquâ dilatione, vel in futuro.* "The imperative (mood) seems to receive the present and the future (tense) by a certain natural necessity; for, we command those things which we wish to be done, either immediately at present, without any delay, or in future." From this reasoning, it is plain that the present tense of the imperative mood is a present inceptive, looking necessarily to a continuance or completion in futurity. It is really present only to the speaker, but as to the person addressed, it is a future, either immediate or prospective. Thus, when Lear cautions Kent not to interfere between him and his anger to Cordelia, the will and the act are closely conjoined:

> *Come* not between the dragon and his wrath!

But when he imprecates curses on his unnatural and cruel daughters, the object of his prayer is one which cannot take effect till a far distant period, and which may continue for a long course of years:

> ———— If she must teem,
> *Create* her child of spleen, that it may live
> And be a thwart, disnatur'd torment to her;
> *Let it stamp* wrinkles on her brow of youth,
> With cadent tears *fret* channels in her cheeks,
> *Turn* all her mother's pains and benefits
> To laughter and contempt.

In the nature of things there is no reason why any particular idiom should not have a distinct form of imperative for the proximate and distant future; except that in general usage, the gradations might be so minute as not easily to be distinguishable; and that as some degree of futurity is necessarily implied in every present command, any fixed barrier, separating the nearer from the more distant, and assigning one form of tense to the one, and another to the other, must be purely arbitrary.

From what has been said it might perhaps be inferred that the imperative mood could not in any case admit of combination with a past time; but this would be incorrect, for the Mind can throw itself forward, as it were, into futurity, and so command an action to be past. We cannot by our will alter that which is past at the moment of our speaking, but we can command that at a future moment it shall have been done: and it is thus that Apollonius distinguishes between the imperatives present, and the imperatives past in Greek. Thus in explaining the different force of σκαπτέτω τὰς ἀμπέλους, "set about digging the vines," and σκαψάτω τὰς ἀμπέλους, "get the vines dug," he says the first is spoken εἰς παράτασιν, by way of extension or allowance of time for the work, the other εἰς συντελείωσιν, with a view to immediate completion. And elsewhere explaining the difference of these tenses γράφε and γράψον, he says of the latter οὐ μόνον τὸ μὴ γενόμενον προστάσσει ἀλλὰ καὶ τὸ γινόμενον ἐν παρατάσει ἀπαγορεύει, "it not only commands something which has not yet been done; but it forbids also that which is now doing in a slow and tedious progress." Therefore, if a person is writing slowly, to say to him, γράφε, "write," would be unmeaning; for that he is already doing: but to say, γράψον, "get your writing done," would be, in fact, to forbid that tedious mode of writing which he was pursuing. In this explanation of the imperative past tenses, Apollonius is followed by Priscian, who says, *Apud Græcos etiam præteriti temporis sunt imperativa; quamvis ipsa quoque ad futuri temporis sensum pertineant; ut* ἀνεῴχθήτω πύλη, *aperta sit porta. Videtur enim imperare ut in futuro tempore sit præteritum; ut si dicam, aperi nunc portam, ut crastino sit aperta.* "The Greeks possessed even imperatives of past time; although these also belong to a sense of future time; as, ἀνεῴχθήτω πύλη, 'let the door be opened:' for this expression seems to command that at a future time the action may be past; as if I were to say, open the door now, in order that it may be open to-morrow." And the inference which he draws from this reasoning is not less remarkable nor less correct. *Ergo nos quoque possumus in passivis, vel in aliis passivam declinationem habentibus, uti præterito tempore imperativi, conjungentes participium præteriti cum verbo imperativo præsentis, vel futuri temporis; ut amatus sit, vel esto,* πεφιλήσθω: *doctus sit, vel esto,* δεδιδάχθω; *clausus sit, vel esto,* κεκλείσθω. "Therefore, even in passives, or in words having a passive conjugation, we may use a past tense imperative, by joining the participle past with the imperative verb of present or future time; as *amatus sit*, or *esto*, πεφιλήσθω; *doctus sit*, or *esto*, δεδιδάχθω; *clausus sit*, or *esto*, κεκλείσθω." It is objected that these are not tenses but combinations of words; to which Vossius justly replies that such combinations are uniformly admitted to be tenses in the indicative and subjunctive moods; and consequently they may be so in the imperative. Either, therefore, says he, we should always reject those periphrastic modes of expression from among the tenses, or we should allow this diversity of tense to the imperative. In many Languages, and particularly in the English, to adopt the former alternative would be to say, that our Language was almost wholly destitute of tenses; but we, who have all along regarded Grammatical distinctions principally with reference to signification, must certainly admit, that the modification of the assertion, in regard to time, whether it be effected by a change of accentuation or quantity in the syllable, or by a syllable prefixed, interposed, or adjoined; or, lastly, by some combination of distinct words, is to be regarded as a tense. We are not ignorant that, in all our English compound tenses, the auxiliary verb originally performed a more leading part in the combination, and the verb now considered as principal was used in the infinitive, being regarded, in the common Grammatical phrase, as "the latter of two verbs." Thus Chaucer,

> Quoth then Cresside *wol ye don* o thing?

That is, "*will you do* one thing?"

And so,

> Thou shouldest never out this grove pace,
> That thou ne *shouldest dien* of mine hond.

That is, "shouldest die."

Grammar.

But to the general purposes of Grammatical Science it is of little import how the tense came to be originally formed. It suffices, that at present the former verb acts merely as auxiliary to the latter, which indeed, in modern use, has even laid aside its infinitive termination, in order to coalesce, as it were, more intimately with the other element of the tense thus formed by their combination.

It is true that all our auxiliaries do not simply signify time. Indeed none of them do so properly; for *have*, the auxiliary of past time, properly signifies possession; because we cannot properly be said to possess an act until it is past; so, *will* implies futurity, because volition regards only that which is not yet in being. In like manner, *may*, *can*, *must*, &c. do not in themselves imply time, except with reference to the conjunctive mood. Hence Vossius has observed, that what is commonly called the present conjunctive has in some instances a future import; as, when Cicero says, in one of his Epistles to Atticus, *Est mihi præcipua causa manendi; de quâ utinam aliquando tecum loquar.* "I have a particular reason for staying here, concerning which I hope I may some time or other talk to you;" where *utinam loquar*, "I hope I may talk," relates entirely to a future time. It is needless here to follow the numerous and minute remarks of many learned critics on the mixed or variable times which are expressed by all the conjunctive tenses. Suffice it to say, that the combination of any mood which implies contingency or futurity, with a tense, referring to present or past time, must necessarily affect the expression of time, and, consequently, that in this respect, the tenses of the indicative must differ from the analogous tenses in any other mood. As, therefore, in nouns, the term gender, originally used to express the mere distinction of sex, has been applied in use to distinguish large classes of words from each other, with reference only to their terminations; so in verbs the word tense, originally meaning the expression of time alone, has been also used in most Grammars to express that idea in combination with the others which we have noticed.

Person.

We come next to a quality usually attributed to the verb, but certainly not necessary to be combined with it in the same word, namely, *Person*. The difference of person peculiarly belongs to the pronoun, and has been sufficiently explained in treating of that Part of speech. In many Languages the person is necessarily expressed by a pronoun. This is universally the case in the Chinese, for the verb being alike in all the persons, it would be impossible to distinguish one from the other without the addition of some other word. The three persons singular of the present tense run thus:

Ngo Ngai, *I love*;
Ni Ngai, *Thou lovest*;
Ta Ngai, *He loves*.

And the same occurs in the other tenses, and in the plural number.

In English we find it partially the case; for though in the singular we have three distinctions of person in the present, as "I love," "thou lovest," "he loves," and two in the past, as "I loved," "thou lovedst," yet in all other parts (with the exception of the irregular *to be*) the verb remains unaltered. Nor does this arise from any peculiarity in the original genius of our Language, for the more ancient Dialects from which it is derived, abounded with personal terminations. Now those terminations, it is very manifest, were, in their origin, nothing more than the pronouns themselves, which, in process of time, coalesced with the expression of conception, assertion, and time, and so formed words, signifying at once all these different circumstances, together with the additional distinction of person.

Verbs Person.

The English Language is chiefly derived from two sources, the Anglo-Saxon and the Latin, of which the former is related to the Mœso-Gothic, and the latter to the Greek: and it is remarkable that all these four Dialects bear a great resemblance in the manner in which they express the persons of the verb; as will appear by inspection of the following Table of the present tense:

		Gothic.	Saxon.	Latin.	Greek.
Sing.	1st person	a	e	eo	o
	2d	ais	est	es	eis
	3d	aith	nth	et	ei
Dual.	1st person	ain	..	..	omen
	2d	os	..	..	eton
	3d	ats	..	..	eton
Plural.	1st person	am	ath	emus	omen
	2d	aith	ath	etis	ete
	3d	and	ath	ent	onti*

The similarity continues through the other tenses; and in all it is manifest, that the personal termination is the personal pronoun. We mention this circumstance, connected rather with the Etymology than with the Philosophy of Language, partly in illustration of the general doctrine of personality in verbs, and partly to account for some circumstances which have given occasion for dispute, on this subject, among Grammatical writers. Thus, for instance, we see why, in the Greek and Latin Tongues, the two principal pronouns, that is to say, those of the first and second person, *ego* and *tu*, are never used but for emphasis, or else, where the verb is omitted. For the former reason, Virgil says,

Nos *patriam fugimus*, TU *Tityre, lentus in umbrâ*,
Formosam resonare doces Amaryllida sylvas.

For the latter reason, Juvenal thus expresses himself:

Semper ego auditor tantum, nunquamne reponam?

It was necessary for Virgil to express, emphatically, the opposition between the different lot of the two shepherds: and, therefore, though this opposition would, to a certain degree, have been manifested by the mere words *patriam fugimus*, and *doces resonare*; yet, for Poetic effect, it became necessary to add the emphatic words *nos* and *tu*. In like manner, the Atys of Catullus exclaims, in the extremity of passionate regret,

Ego gymnasi fui flos, ego eram decus olei!

In the line above quoted from Juvenal, we see that there is a necessity to express *ego* before *auditor*, because the verb *ero* is wanting; but there is no necessity for expressing it before *reponam*, because it is involved in the termination of that word. The same thing, indeed, is true of the third person, so far as respects merely the pronoun; for the verbal termination *et*, *at*, or *it*, is undoubtedly the same as the pronoun *id*, or *iste*; and therefore the pronoun of the third

* *Onti* is the more ancient, *ousi* the more modern termination of this person, in Greek.

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person is never expressed but for the sake of distinction or emphasis any more than those of the first and second persons. Thus, Virgil says,

> ——— *Amplexus nati* CYTHEREA *petivit;*
> *Arma sub adversâ posuit radiantia quercu.*
> ILLE *Deæ donis et tanto lætus honore*
> *Expleri nequit, atque oculos per singula volvit,*
> *Miraturque, interque manus et brachia versat.*

Here ILLE is necessarily expressed to distinguish the agent of the verb *nequit* from *Cytherea*, the agent of the verb *petivit;* but that distinction being once made, the verbs *volvit*, *miratur*, and *versat*, are employed without a nominative expressed.

Again, the same author says,

> *Arcades his oris genus a Pallante profectum*
> *Delegere locum, et posuere in montibus urbem.*
> HI *bellum assiduè ducunt cum gente Latinâ;*
> HOS *castris adhibe socios.*

Where HI in the nominative, and HOS in the accusative, are used emphatically; and the former without necessity, so far as mere intelligibility is concerned; for the verb *ducunt* alone would have sufficiently indicated that the Arcadians were the persons who warred against the Latins.

Impersonals.

Some verbs are called *impersonal*, a name which only seems to mean that they are not usually conjugated with distinction of persons, but remain always in the form of the third person. If they had no other peculiarity than that from which their name is derived, it might not be necessary to notice them in a Treatise on Universal Grammar; but, in truth, they are constructed on a Principle different from that which has been already explained in reference to person. The impersonals are of two kinds, active and neuter. By active we mean those which require an object, as "it grieves me," "it pains me," *miseret me, decet me*, &c.; by neuter we mean those verbs of which the action terminates in itself, as "it rains," "it snows," "it is hot," "it is cold;" the Latin *pluit*, the French *il fait chaud*, the Italian *fa freddo*, the German *es donnert, es friert*, &c. In all these instances the verb contains a mere assertion of the existence of the conception; but does not indicate any agent. These verbs have been sometimes explained as agreeing with a nominative implied in them: thus *pudet* is said to be a verb agreeing with the implied nominative *pudor*, as if the meaning were, "shame shames me;" but this is perhaps rather a formal than a substantial explanation. *Pudet* in reality contains, and does not merely imply the noun *pudor:* it expresses the same conception as the noun, and asserts its existence. It is therefore rather of the nature of a verb substantive, than of a verb active; and though, in some Idioms, a nominative is expressed, yet in reality that nominative is superfluous, or, at most, is only introduced to keep up the general analogy of the Language. The nominative *it* in the English Language, and *il* in French, have no distinct reference to any conception. They are pronouns, which do not stand for any noun. If any one should say, "It rains," we cannot, as in the common case, where a distinct nominative is expressed, ask "what rains?" for the answer would only be *it;* and if we were then to ask, "what is *it?*" we must be left without any answer. Hence, in translation, the nominative *it* is often lost. We do not say, in Latin, Hoc *pluit;* nor in Greek, ΤΟΥΤΟ χρῆ; nor in Italian, EGLI *fa freddo*. The proper notion of an Impersonal verb, therefore, is, that it expresses action without expressing an agent. Many such forms exist in Language. The French on *dit*, is of the nature of an impersonal; so are the English "they say;" the Italian *si dice;* the Spanish *se cuenta;* the English "methinks;" the German *mich dünkt;* the Portuguese *basta, parece, convem, sucede*, &c.

Verbs. Number.

Where the object of an impersonal is expressed, as "it grieves *me*," the sense may be rendered by a passive verb, of which that object is the nominative, as, "I am grieved;" and, on the other hand, the Latin Language admits of passive impersonals, followed by a dative or ablative, which are equivalent to personal verbs active: as in Livy, *Romam frequenter migratum est a parentibus raptarum;* for *parentes raptarum migraverunt*. Where the impersonal is the former of two verbs, (according to the common mode of speaking,) the latter being in the infinitive mood, the proper construction is to regard the infinitive as a noun forming the nominative to the verb, which, consequently, is not an impersonal, but a personal. Thus, in the sentence, *Dulce et decorum est pro patriâ mori;* when rendered into English, "It is sweet and seemly to die for our Country;" the nominative *it* does not properly render the verb *is* an impersonal, because it relates to a definite conception, which is afterwards expressed, and which renders the verb personal. Hence, in all such sentences, the word *it* is superfluous, and may be got rid of by mere transposition; thus, "to die for one's Country is sweet and seemly;" or, it may be said to answer to the emphatic word *that*, if the sentence were turned as follows: "To die for one's Country, *that* is sweet and seemly."

It has been contended that many of the Latin impersonals are not really so, because they may be used as personals. Thus Horace repeatedly uses *decet* in the plural, as,

> ——— *Tristia mæstum*
> *Vultum verba decent.*

So Ovid,

> *Nec dominam notæ dedecuere comæ.*

In these instances, however, the verbs really become personals: and as we have before seen that the same verb is often of different kinds, being sometimes used as an active, and sometimes as a neuter, so there is nothing but the idiom of a particular Language to prevent the same verb from being used sometimes as a personal, and sometimes as an impersonal. The impersonal neuter may, in like manner, be used as an active; for, as Scaliger has observed, we may say *pluit sanguinem et lapides*, and, indeed, *pluit* is even used with a nominative, (Gen. xix. v. 24.) *Dominus pluit super Sodomam et Gomorrham sulphur et ignem.*

The expression of *Number* is another accidental quality of the verb, which belongs to it not as a verb, but in so far as it may be combined with the expression of person. It is, therefore, like the same quality in the adjective, a mere method of connecting it in construction with the noun substantive, or pronoun which forms its nominative. Accordingly, it applies to verbs in the same manner as it does to nouns and pronouns. When they admit a dual number, as in Sanscrit, Arabic, and Greek, the verb admits the same; when they do not, it has only a singular and a plural. Indeed, the matter could not well be otherwise, since we have Number.

Grammar. seen that the personal terminations of the verb are really the pronouns themselves coalescing with it. The verb is equally said to be in the singular or plural, whether it has or has not distinct terminations appropriated to those different numbers; we call "I love" singular, and "we love" plural; but it is manifest, that in all such instances the expression of number exists only in the pronoun, and is imputed to the verb by Grammarians quite gratuitously. These are questions of Particular Grammar: all that can be laid down on the subject, as a rule of Universal Grammar, is, that as on the one hand there is nothing in the peculiar nature of the verb which involves the idea of number, so there is nothing in the idea of number which can prevent it from being combined with the verb, where the genius of the Language permits such a union.

Gender. Since the verb, by means of its connection with the pronoun, admits person and number, there is no reason why it should not also admit *Gender;* and, in fact, this distinction obtains in the Arabic, the Ethiopic, and some other Languages. It is, however, rare; and as gender properly belongs only to nouns, or pronouns substantive, with respect to which it has been already discussed, we need not here pursue the investigation.

Comparison. Some writers contend, that the verb, as expressing an attribute, is capable of comparison; nor does it appear that this can be gainsaid, if we regard only the attributive nature of the verb. There are, indeed, certain attributes, as has been already observed, which are not intensive; and those of course cannot admit degrees of comparison; neither can the assertive power be compared: for the verb must either assert a thing to exist or not to exist. On the other hand, verbs may be compounded with conceptions implying comparison, as "to outdo," "to overtake," *subesse*, *superesse*, &c. They may too, in general, be compared by means of the adverbs of comparison, *more, most, less, least*, &c.; but we are not aware that it has been attempted, in any Language, to combine in one and the same word the assertive power with the comparative. It is not easy to conceive any form of verb which in itself would express the degrees of comparison; and the reason probably is, that though the mere qualities of substance may be simply intensive, yet actions are intensive in various modes, as well as in various degrees. Of different substances, concerning which whiteness can be predicated, some may be more and some less white; but of different Beings concerning which the act of walking may be predicated, all equally walk, though one walks more, another less; one faster, another slower, &c.; and so of mental action, several persons love, but one loves more warmly, another more violently, another more purely; so that there is not in actions, as there is in qualities, a simple scale of elevation and depression; and, consequently, the mere comparison of more and less would not answer all the purposes of Language, as applied to the verb, though it does as applied to the adjective. For this reason participles, when they are compared, lose their participial power; for *sapientior* and *potentior* do not express acts, but habits, or fixed qualities, and therefore answer to the English adjectives "wiser" and "more powerful."

Thus have we seen, that though the proper force and effect of the verb—that on which its essential character depends, is assertion, yet it is capable of uniting therewith, and in fact does so unite, not only the conception, which Priscian calls the *res* of the verb, but the expression of mood, tense, person, number, and even gender. "Observe," says the President De Brosses, "how, in one single word, so loaded with accessory ideas, every thing is marked, every idea has its member, and the analogical formulas are preserved throughout on the plan first laid down." Elsewhere he adds, "All this composition is the work, not of a deeply-meditated combination, nor of a well-reasoned Philosophy, but of the Metaphysics of instinct." The Goths, the Saxons, the Greeks, and the Latins, in forming the schemes of conjugation above noticed, were probably impelled by Principles in the Human Mind, the very existence of which they hardly suspected. Similar Principles have operated, but with endless diversity of application, in the formation of all the various Dialects which have been spoken in ancient and modern times, by nations the most barbarous and the most civilized; and it is the developement and explication of these ever-operative Principles which forms the proper object of the science of Universal Grammar.

§ 5. *Of articles.*

Articles. Having explained the uses of the principal Parts of speech, we come now to consider the accessories. The principal Parts, as we have already stated, are those which are necessary for communicating thought in a simple sentence: and the communication of thought requires the *naming* of some conception, and the *assertion* of its existence as an object either of perception or of volition. Conceptions are named by the *noun:* they are asserted to exist by the *verb;* but it often becomes desirable to modify either the name, or the assertion, or the union of both. How is this to be done? We have seen certain modifications incorporated with the noun by its cases, and numbers, and genders; with the verb by its moods, tenses, and persons; with the adjective by its degrees of comparison; and with the participle, gerund, supine, and infinitive, by their marks of time, relation, &c. The same, or similar effects, may be produced by separate words; and what must those separate words be? Nouns, or verbs, which, appearing in subordinate characters, are no longer to be considered as such.

We wish to modify a conception; how can we do it but by another conception? We wish to modify an assertion; how can we do it but by another assertion? It is therefore plain, that the accessory words must have had originally the character of principals; that is to say, they must have been either nouns or verbs. This is a truth extremely obvious in itself; and of which it clearly appears, that many Grammarians have been fully aware; but there is another truth, which seems to have been less apprehended, namely, that these subordinate and accessory words act a very different part from that which they sustained as principals in a sentence. The Mind dwells on them more slightly; they express a more transient operation of the intellect. In process of time some of them come to lose their original meaning, and to be significant only as modifying other nouns and verbs. It cannot be denied that this is a fact. It cannot be denied that the words *and, the, with*, and the like have no distinct meaning, at present, in our Language, except that which depends on their association and connection

Grammar. with other words. The Etymologist may succeed, or he may not succeed, in his attempts to trace these non-significant words to the significant words from which they are derived; but whether he be successful or unsuccessful, the fact will be no less certain, that in their secondary use they lose their primary character and signification; they are no longer nouns or verbs, but inferior Parts of speech.

Particles. These inferior Parts of speech have been called *particles:* and, as such, are sometimes distinguished from *words*, and sometimes treated only as a separate class of words. To explain and account for them seems to have given much trouble to many Grammatical and Philosophical writers; and after all, the subject has been often left in a state of great confusion. LOCKE, in his IId volume, has a short and somewhat vague chapter on particles, from which we may infer that he considered nouns to be the names of thoughts, or, as he expresses it, of ideas. All other words, he thought, served to connect ideas. The principal of these (which we call the verb) he calls the mark of affirming or denying; and he says, "the words whereby the Mind signifies what connection it gives to the several affirmations and negations that it unites in one continued reasoning or narration are called *particles*." Elsewhere he says of these particles, "they are not truly by themselves names of any ideas;" and again, "they are all marks of some action or intimation of the Mind, and *therefore*, to understand them rightly, the several views, postures, stands, turns, limitations, and exceptions, and several other *thoughts* of the Mind, for which we have either none or very deficient names, are diligently to be studied." The confusion which occurs in these passages between *ideas, thoughts*, and *actions of the Mind*, leaves Locke's real meaning very much in the dark; but it seems as if he thought that the particles (in some instances, at least) could not be derived from nouns, inasmuch as they signified some thoughts, which had never been expressed by means of nouns.

HOOGEVEEN states the general doctrine of particles very briefly. He says, *particulas in suâ infantiâ fuisse vel verba, vel nomina, vel ex nominibus formata adverbia.* "The particles were, in their infancy, either verbs or nouns, or adverbs formed from nouns." *Ipsæ verò*, QUATENUS PARTICULÆ, *per se solæ spectatæ, nihil significant.* "They themselves, *as particles*, considered alone, signify nothing." And again, in defining the particle, he says, *particulam esse voculam, ex nomine vel verbo natam.* "The particle is a small word derived from a noun or a verb." Had Mr. Tooke properly reflected on these passages, which he quotes from Hoogeveen, he would have found them to contain all that was valuable in his own system, without the errors into which he has fallen.

The term particle is, perhaps, not well chosen to include the inferior Parts of speech; nor do Grammarians agree as to the extent of its signification. Locke only describes it as including "prepositions and conjunctions, &c.;" leaving it to his reader's judgment to determine what classes of words fall under the *et cætera:* SCALIGER says, *ut omittam particulas minores, cujusmodi sunt præpositiones, conjunctiones, interjectiones:* and Hoogeveen, as we see above, seems to distinguish the particle from the adverb; whilst other Grammarians include in it all indeclinable words, and even the Article, which in Greek is declinable. It is not, however, necessary, that we should adopt either this, or any other generic term, to express the Parts of speech of which it remains for us to treat; but we shall proceed to consider them separately, in succession; and first we shall treat of the Article. Articles.

Office of the Article. The proper office of this Part of speech is to reduce a noun-substantive from a general to a particular signification. We have already observed, in speaking of nouns, that by far the greater part of them must be what Mr. Locke calls general terms, that is to say, names common to many conceptions. We cannot give a distinct name to every distinct object that we perceive, or to every distinct thought, which passes through the Mind; nor are these thoughts, or even these objects so entirely distinct to human conception as many persons are apt to imagine. If I see a horse to-day, and another horse to-morrow, the conceptions which I form of these different objects are indeed different in some respects; but in others they agree. The one horse may be black, and the other white; but they are both quadrupeds. The word *horse* is a noun, expressing the conception which I form of the points in which they agree. But this word applies to a class of conceptions, and it is necessary that I should possess some means of expressing the individuals of that class. Now those means are afforded by adding the Article to the noun. To illustrate what we mean, let us take a general term; for instance, the word *Man*. The conception expressed by this word alone, is one which exists in several other conceptions, as in that which I form of "Peter," or of "James," or of "John." *Peter*, therefore, is a word expressing the general conception, "Man," together with something peculiar to a certain individual; and the same may be said of James and John; but it must frequently happen, that the proper name Peter, or James, or John, is unknown to us. How, then, are we to express our conception of any one of them? To each the term "Man" belongs; but it belongs to each equally; and therefore it does not distinguish the individual from his class, or one individual from another. If, therefore, we use this term "Man," we must also employ some other means of showing that we mean by it *this*, or *that* man; or at least some *one* man, as distinguished from the conception of "Man" in general. Now, these means are afforded by the Article; and they are afforded in two different ways: we either speak of the general term simply, as applicable to a notion of individuality, or else with relation to some particular circumstance which we know belongs only to an individual. In the former case we may be said to enumerate, in the latter to demonstrate, the person or thing intended. In the one we say positively "*a* man," in the other we say relatively "*the* man."

Two classes. Hence arise two classes of Articles. They have been called the *indefinite* and the *definite;* but it has been justly observed by HARRIS, that they both define, only the latter defines more perfectly than the former. It would, perhaps, be more appropriate to call the one *positive*, and the other *relative*, or the one *numeral*, and the other *demonstrative*. We shall adopt the first two of these designations, merely for convenience; but we consider the names by which it may be thought fit to

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designate the different classes of words, as comparatively unimportant. The most material object with us is to establish the classification itself on clear and intelligible principles.

Whether necessary.

Grammarians have disputed whether the Article be, or be not a *necessary* Part of speech. Before this question can be properly answered, it must be clearly stated. Mr. Tooke says, "in all Languages there are only two sorts of words which are necessary for the communication of our thoughts; and these are, 1. noun, and 2. verb;" and he adds, that he uses the words noun and verb "in their common acceptation." It would seem from this, that he meant to describe the Article as unnecessary; for in common acceptation it is certainly not considered to be identical, either with the noun, or with the verb. However, he afterwards describes it as "necessary for the communication of thought," and even "denies its absence from the Latin, or from any other Language." We have already adverted to the doctrines of the ancient writers, who considered the noun and the verb as the only, or, at least, as the principal and more distinguished Parts of speech; but they who reasoned thus, either included the Article among the *syncatagoremata*, that is, consignificant words, or else denied its necessity, and even its existence, in some Languages, particularly in the Latin. *Noster sermo*, says QUINCTILIAN, *Articulos non desiderat. Articulos*, says PRISCIAN, *quibus nos caremus—Articulos integros in nostrâ non invenimus Linguâ.* And so SCALIGER, *Articulus nobis nullus, et Græcis superfluus.* And VOSSIUS, *Articulum, quem Fabio teste Latinus sermo non desiderat, imò, me judice, planè ignorat.* From these authorities, and indeed from a very slight inspection of the Language itself, it is clear, that the Latin had no separate words answering to the Articles of the English and other Languages; nor is it less clear, that the Greek had only the relative Article ὁ, ἡ, τὸ, and was entirely destitute of our positive Article. Mr. Tooke is undoubtedly right in inferring, from the necessity of general terms, the necessity of the Article; if we thereby understand the necessity of some means to apply general terms to their individual instances. He is, however, wrong in supposing that this purpose is always effected either by a distinct word, or by some prefix or termination added to words: nor is the ingenious, but fanciful COUR DE GEBELIN less erroneous in asserting that the Article was supplied in Latin by the termination; for the termination in no manner whatsoever defined whether the word was to be taken in a more or less general acceptation. It indicated the case, the number, and the Grammatical gender; but it did nothing else. *Homo* signified "Man" in general, or "a man," or "the man" before spoken of; and the termination afforded no help toward determining in which of these three senses the word was to be taken in any particular passage. This was to be discovered in Latin, as in some other Languages, merely by the context. If, therefore, the question, whether the Article be necessary, mean whether a separate class of words performing the function of the Article be necessary, it must be resolved in the negative; because no such class is to be found in the Latin and some other Languages. If, on the other hand, it mean whether in all Languages there must be some mode of performing the function of the Article, it must be answered affirmatively; and this is a question which, as it relates to the operations of the Mind, properly falls within the scope of pure Grammatical Science.

Articles.

Gradations of conception.

Even though a particular Language may have no class of words called Articles, the persons speaking that Language must certainly distinguish, in their conceptions, the general from the individual. In treating of the noun, we have already spoken of the different gradations of conception; but it is necessary that we should here advert again to the grounds of this distinction. The inattentive observer of internal objects believes that their forms are always impressed distinctly on the eye; and that every superficies is bounded by a visible outline. A more reflecting and more accurate Philosophy teaches us, that even in contemplating the objects which we most admire, Imagination does much more than mere sensible impression toward supplying us with a knowledge of their forms; and, that, in a sense not merely Poetical,

We half create the wondrous world we see.

In like manner, the inattentive observer of the operations of Mind, as they relate to Language, is apt to suppose that all his thoughts or conceptions are definite and distinct; and consequently, that the words which serve to name these thoughts are so too; but this is far from being the case. Let us consider each of the three classes of conception before noticed, *viz.*, the conception of a *particular* object, that of a *general* notion applicable to many particulars, and that of an *idea* or *universal* truth. The first and last of these are in themselves perfectly definite. No man can have two distinct ideas of "virtue," considered absolutely and in the primary signification of the word: and the same may be said of "squareness," "power," "duration," "space," "wisdom," &c., &c. In like manner we cannot have two distinct conceptions of a particular person or thing, and therefore, when we know its proper name, as "George," "Louis," "London," "Paris," "Alexander," "Bucephalus," "Europe," "Guildhall," &c., &c., it is unnecessary to prefix thereto any other word for the sake of more clearly showing the individuality of our conception.

Hence we see the reason why neither *Proper names* nor *universal terms* do of necessity require to be used with an Article, either positive or relative. The idiom of a particular Language may, indeed, sanction such a construction; but this depends on separate considerations, to which we shall hereafter advert. Generally speaking, such idioms as the following cannot be necessary to intelligibility in any Language: "*the* George reigns in *the* England," or "*a* Guildhall is situated in *a* London;" or, "*the* virtue produces *the* happiness;" or "*an* Alexander aimed at *a* glory;" and the reason is obvious; because it is not necessary to define or distinguish, in such sentences, one George from another George, one England from another England, one virtue from another virtue, &c.

General terms.

But the remaining class of conceptions, though general in their nature, serve to communicate the greater part of our knowledge respecting particular objects. We have often no other conception of the individual than that he belongs to such or such a species. We know the man only by his profession, the soldier only by his regiment, the officer only by his rank. Hence

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the great use of *general terms* in all Languages; and hence too, the necessity for individualizing them, either tacitly in the Mind, or expressly in Language. When this process of individualization is effected by a separate word, we call that word an Article; and thus we say, that it is necessary to add the Article "a" or "the" to the general term "man," in order to designate an individual of the human species.

It is to be observed, that, in a secondary sense, all words of the other two classes may be considered and treated as *general terms*; and, consequently, may require the use of the Article to individualize them. For, first, the idea expressed by an *universal term*, such as "virtue," "truth," and the like, may be considered as existing separately in each subordinate conception of quality, action, &c. in which it is involved. If we speak of virtue simply, as opposed to vice, or in any other manner which regards the pure idea of virtue, without any modification, it is an universal term which needs not the aid of an Article; but if we speak of those subordinate ideas, such as justice, prudence, temperance, fortitude, in each of which the higher idea of virtue is involved, as the conception of Man is in the conception of Peter or John, we may consider the word virtue, in a secondary sense, as applicable to each of them separately, and therefore may call each "*a* virtue," or "*the* virtue." And not only does this apply to subordinate conceptions of the same kind and nature as their superior, but sometimes to others, in which that superior is equally involved. The conception of injustice is of the same kind and nature as the conception of vice. They are both *ideas*, both universal, both regard *qualities* of the Mind; but the conception of an unjust *action* partakes, though in a remoter degree, of both these ideas, and therefore it is sometimes called "an injustice," or "a vice." Thus Hamlet, on Horatio's saying that he is not acquainted with Osrick, replies, "Thy state is the more gracious, for 'tis *a vice* to know him." And so Bassanio, urging the Duke to wrest the law to his authority, exclaims,

To do a great right, do a little wrong.

It is only in this secondary sense that such words as virtue and vice, right and wrong, can be employed in the plural number; and hence arises in all Languages a vast class of general terms, which unhappily are but too often perverted in use. The idea of *crime* does not always agree with our conceptions of *crimes*; and we often find an opposition between the notions of *right* and *rights*, *honour* and *honours*.

Secondly, a *Proper name*, which, in its primary sense, designates only an individual man, may be made to stand for a conception common to many other individuals; because we can suppose, however contrary it may be to fact, that there is a class of men, each possessing those qualities and powers which make up all that we know of a certain individual. Thus the word SHAKSPEARE primarily means that wonderful Poet who wrote *Hamlet* and the *Midsummer Night's Dream*, who could portray the characters of *Othello* and *Falstaff*, *Richard II.* and *Richard III.*, and who as much excelled every writer of his day in the sweetness and facility of his language, as he did in richness of imagination and in profound knowledge of the human heart. It is in vain to *expect* another Being so endowed to arise before the return of the fancied Platonic year; and yet we may *suppose* a whole club of such dramatists, like the "cluster of wits" in Queen Anne's time; we may imagine one from every Country under heaven; and therefore we may talk of "a French Shakspeare," or "a German Shakspeare," "the Shakspeare of Tennessee," or "the Shakspeare of Tombuctoo." Articles.

Articles whence derived.

The words which answer the purpose of individualizing general terms, in the two modes above described, were originally pronominal adjectives. In some instances they have undergone a change of form, by becoming Articles; in others, they remain unchanged. The French *le* and *un*, are the Latin *ille* and *unus*; the English *the* and *a* are the Anglo-Saxon *that* and *ane*. Hence, it is not surprising, that many Grammarians comprehend, under a common designation, the demonstrative pronoun and the Article. Such was the doctrine of the Stoics, some of whom gave to both these kinds of words the common name of *Article*, calling our pronoun the definite Article; and our Article, the indefinite Article; whilst others considered both as *pronouns*, and only denominated our Articles, Articular pronouns. *Articulis autem pronomina connumerantes*, says Priscian, *finitos, et Articulos appellabant; ipsos autem Articulos, infinitos Articulos dicebant; vel ut alii dicunt, Articulos connumerabant pronominibus, et Articularia eos pronomina vocabant.*

Difference from a pronoun.

There are, however, some marked differences between the pronominal adjective and the Article, which, we think, justify us in considering the latter as a separate Part of speech.

In our own Language, the same words which act as pronominal adjectives may also be used substantively; and, in particular, the words *that* and *one* are sometimes to be considered as substantive pronouns, as when we say, "*that* which I love," "*one* whom I respect;" but we cannot, in like manner, say, "*the* which I love," "*a* whom I respect." This distinction, however, depends on the idiom of the English Language, and, therefore, will not afford a discriminating characteristic between the separate Parts of speech in Universal Grammar.

The case is different, when we come to consider the manner in which the pronominal adjective and the Article respectively affect the meaning of a general term. They both individualize it: but the Article performs this function simply; the pronominal adjective does more; it marks some special opposition between different individuals. When we say, "the man is good," there is no opposition implied in the word "the," although there may be in each of the other words. We may say, for instance,

1. "The *man* is good; but the *boy* is bad."
2. "The man *is* good; but he *was* bad."
3. "The man is *good*; but he is not *wise*."

On the contrary, when we say "*that* man is good," we imply no opposition to the other words in the sentence, but only to the word "that." We intimate not only that there is a particular individual who is good, but also that there is some *other*, who is not good. This distinction is strongly marked in Latin by the pronominal adjectives *hic* and *ille*; as when Ovid says,

——— *dissimiles* HIC *vir, et* ILLE *puer.*

Where the English Article *the* is used, the Latins, who have no such Article, do not supply its place by the pronominal adjective, but use the noun alone, as

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"Blessed is *the* man that walketh not in the counsel of the ungodly." *Beatus vir, qui non abiit in consilio impiorum;* and not *Beatus* ILLE *vir.*

It is manifest, that the act of the Mind is very different in the two cases of which we have spoken. Simply to individualize, is a more transient operation than to individualize and at the same time to cootrast. Hence, the word *the* is less susceptible of accentuation than the word *that.* It resembles, in this respect, those Greek pronouns which are called *enclitic.* When the oblique cases of the personal pronouns, in that Language, were used by way of cootradistinction, they were strongly accented, and were called by Grammarians ὀρθοτονουμέναι, *uprightly accented;* but when they were merely subjoined tu verbs, without any emphasis being placed on them, they were eolled Ἐγκλιτικαὶ, that is, *leaning*, or *inclining*. Thus the Greeks had, in the first person, Ἐμοῦ, Ἐμοὶ, Ἐμὲ, for contradistinction, and Μοῦ, Μοὶ, Μὲ, for enclitics; whence Apollonius proposes, instead of the common readiug, in the beginning of the *Iliad*—

> Παῖδα δέ μοι λύσαιτε—

tu read

> Παῖδα δ' ἐμοὶ λύσαιτε.—

For it is plain, argues he, that a distinction is intended hy the Poet between the words Ὑμῖν and Ἐμοὶ; and therefore the enclitic μοὶ is improper. The Principle in the Human Mind, which converts the contradistinctive prooouu into an enclitic, is no other than the eager desire of hastening toward the object of its wishes—

> *Semper ad eventum festinat;*

and the same Principle it is, which converts the demonstrative pronoun into an Article. Instead of "this horse," or "thăt horse," we say "thĕ hurse:" shurtening the Article in pronunciation, because we dwell but little upon it in thought. In the Anglo-Saxoo Language, the word *thæt* appears to have been shortened into *the;* and we have retained the longer word for our pronoun, whilst we use the shorter for our Article.

The.

When Mr. Tooke asserts that the word *the* is the imperative of the verb *thean*, it does not appear that he throws any great additional light on the subject. It may, however, be eurious to observe how he wrests an etymology, to support his theory. "*That*," says he, "in the Anglo-Saxon *theæt*, i. e. *thead*, *theat*, means *taken*, *assumed*." Now, the i. e. here plays a notable part. The fact is, that there is a Saxon verb *thean*, which properly means "to do," or "prosper." "Ill mote he *the*," in old English, is, "Ill may he do," or "prosper." And there is a Saxon pronoun *thæt*, answering to our "*that.*" It is not very clear that these two words have any other connection than what Mr. Tooke ingeniously supplies by *id est*. The Gothic verb *thihan*, which Mr. Tooke also cites on this occasion, (vol. ii. p. 59,) is our verb to *take;* and seems to form o third element in this etymological medley. We are not much advanced in the knowledge of Articles, by being told that the verbs to do, to prusper, or to take, have some similarity in sound to the pronoun *that;* and yet this is all we learn from Mr. Tooke. As to the verb "to *the*," it seems to be the origin of our old English word *thews;* as in Hamlet—

> Nature's crescent does not grow alone
> In *thews* and bulk.

And so Falstaff says,

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> Care I for the limbs, the *thews*, the stature, bulk, and big semblance of a man?

Again, the word *that*, in old German, signifies an "act" or "deed," and is derived, by WACHTER, from the verb *thun*, which is nothing but our old English *doen*, to do. It is possible that all these words may have some etymological affinity to each other; but if the cooncction were more clearly made out than it really is, it would throw but little light on the true Grammatical force of our Article.

Much of the general reasoning which we have applied to the relative Article *the*, is equally applicable to the numeral Article *a*, or *an*. In French, the word *un*, "one," is spelt in the same way as the Article *un*, "a, or an," but it is pronounced more slightly. In English the word has been not only abbreviated in point of quantity, but changed in articulation, from "one" to "a." The mental operation, however, is the same in both instances. The conception of *one* is expressed, not in opposition to that of *two*, *three*, or any other conception of number, but as distinguished from *all* the other individuals of the same class. A.

In the Scottish Dialect, *ane* was retained as an Article to a late period; thus NICOL BURNE, in his "Disputation," A. D. 1581, says, "Tertullian provis, that Christ had *ane* treu body, and treu blude." And on the other hand, in the old English, the numeral pronoun one was sometimes abbreviated to o, as we read in Chaucer—

> Sithe thus of two contraries is o lore;

and so in the more ancient MS. Poem of the *Man in the Moon*—

> He hath his o foot his other to foren;

but it was still accented as a separate word; whereas the Article *a* (as we have before observed of the other Article *the*) is passed over hastily in pronunciation, as a mere prefix to the general term, which it serves to individualize. Again, the numeral pronoun *one* (like the relative *that*) is capable of being used alone, which the Article *a* or *an* is not. We may say, "one seeks fame, another riches, and a third, the wisest of the three, content;" but if we use the Article, we must add its substantive, as "*a man* should seek content, rather than fame or riches."

Since it has appeared that all Languages do not employ separate words to perform the office of the Article, Not superfluous.
it may be thought that those words when so employed in any Language are always superfluous; but this would be a great error. Articles add much to the clearness, the strength, and the beauty of a Language: and to be perfectly furnished with them it is necessary to possess both positive and relative Articles. The Latin Language had neither: the Greek had only the latter of the two; but most of the modern European Languages have both. It follows, that in this respect the Latin was less perfect than the Greek, and the Greek than either the French or the English; and Scaliger was, therefore, wrong in denying the use of this Part of speech altogether: *Articulus*, says he, *nobis nullus, et Græcis superfluus;* and his sarcasm on the French nation was somewhat misapplied, when he called the Article *otiosum loquacissimæ gentis instrumentum.*

Yet it must be allowed, that in many European Languages, and in none more frequently than in the French, Sometimes so used.

Grammar. instances occur in which the Article is employed superfluously. This circumstance is, for the most part, attributable to an elliptical mode of speech, which is sufficiently capricious. In English, we generally prefix the relative Article to the names of our rivers, but seldom to those of our mountains. We say, "*the* Thames," "*the* Tweed;" *i. e.* the *river* Thames, the *river* Tweed; but we never say *a* Thames, *a* Tweed: nor do we say *the* Snowdon, *the* Skiddaw; or, *a* Snowdon, *a* Skiddaw. In French, the superfluous use of the relative Article is very frequent; but it is to be explained on the same Principle of ellipsis. *Il seroit à souhaiter*, says Condillac, *qu'on supprimât l'Article toutes les fois que les noms sont suffisamment déterminés par la nature de la chose, ou par les circonstances; le discours en seroit plus vif. Mais la grande habitude, que nous nous en sommes faite, ne le permet pas: et ce n'est que dans des proverbes plus anciens que cette habitude, que nous nous faisons un loi de la supprimer. On dit: Pauvreté n'est pas vice, au lieu de dire, La pauvreté n'est pas un vice.* "It is to be wished that the Article were suppressed whenever the noun is sufficiently determined by the nature of the thing, or by the circumstances; the style would thereby be rendered the more lively. But the great habit that we have acquired of using it, does not permit this change; and it is only in old proverbs, more ancient than this habit, that we make a rule of suppressing it. We say, *Pauvreté n'est pas vice*, instead of saying, *La pauvreté n'est pas un vice.*" It is here to be observed, that the proverbial expression, which Condillac seems to recommend, is as much defective as the common expression which he blames is redundant. The Article *la* before *pauvreté* is superfluous, and originates in an ellipsis of some word answering to "state" or "condition;" so that "the poverty," means "the condition of poverty:" but, on the other hand, the word *vice* properly demands the Article *un;* for it is not meant to deny that poverty is the *idea* of vice, which nobody would have asserted; but to deny that poverty is *one* of those states which necessarily include the idea of vice. The most accurate and Philosophical mode of expressing this sentence would therefore be, if the idiom of the Language permitted it, *Pauvreté n'est pas un vice;* answering exactly to the English idiom in such phrases.

As the French often employ the Article redundantly with an universal term, and with the names of places, so the Italians employ it with the names of persons: *Il Tasso, La Catalani*, meaning "the famous Poet Tasso," "the celebrated singer Catalani." It is obvious that these expressions are to be accounted for on the same Principle of ellipsis already explained. The Article in all such cases does not in reality serve to modify the Proper name expressed, but the general term understood.

Special effect. There is a particular use of the relative Article, with a general term, which, as it tends to individualize, in a special and peculiar manner, should not be passed without notice. Certain individuals, having obtained celebrity for their peculiar excellences, have been denominated from this circumstance, as ὁ ποιητὴς, the Poet, means Homer; ὁ ῥήτωρ, the Orator, Demosthenes; ὁ θεολόγος, the Theologian, St. Gregory Nazianzen; ὁ γεωγράφος, the Geographer, Strabo; ὁ Δειπνοσοφίστης, Athenæus, author of the Work entitled *The Feast of the Sophists;* but this is no more than we daily practise, when we speak of "the King," "the Queen," "the Prince Regent," meaning the King of England, the Queen of England, and the Prince Regent of England; just as we hear in private families and narrow circles of society, of "the captain," "the doctor," "the parson," "the squire," &c. the particular application of which general terms is settled, as it were, by a common understanding among the parties; since each of the individuals thus honourably distinguished has his little sphere of celebrity, and

Is talk'd of, far and near, at home.

Plurima ejusdem farinæ, says Viger, *ubique obvia*.

Other distinctions. We do not think it necessary to enter at length into those distinctions of the Article, which do not coincide with our definition of this Part of speech. Such is the distinction often found in the Greek Grammarians between the prepositive and subjunctive Articles. The prepositive, *viz.* ὁ, ἡ, τὸ, is what we have called the relative Article: the subjunctive, *viz.* ὅς, ἥ, ὅ, is what we have called the subjunctive pronoun. The latter, it is manifest, has no effect whatever in individualizing a general term, because it is only employed in a dependent sentence, with reference to a term which must have been individualized in the prior or leading sentence.

The learned Hickes, in that invaluable Work the *Thesaurus Linguarum Septentrionalium*, suggests that the Anglo-Saxon *sum*, which answers nearly to the Latin *quidam*, should be considered as an indefinite Article. It appears to us rather to belong to the class of pronouns; yet in this and some other instances the two classes of words approach very nearly together;

And thin partitions do their bounds divide.

§ 6. *Of Adverbs.*

Adverbs. Before we enter on the consideration of the preposition and conjunction, we find it convenient to treat of the Adverb, which, in our Language, and probably in most others, furnishes the greater part of the words employed in the other two classes. Mr. Tooke modestly observes, that "neither Harris, *nor any other Grammarian*, seems to have any clear notion of the nature and character of the Adverb;" and then he proceeds to give us his own notions, not of the Adverb in general, but of a number of Adverbs in particular, from which, and from what he had before said of the conjunctions and prepositions, he leaves his reader to collect that knowledge which, in his opinion, no Grammarian beside himself had ever acquired. As this does not appear to be a very fair way of treating the Grammatical student, we shall endeavour to pursue a more satisfactory method, even at the hazard of adopting, from the ancient Grammarians, some of those notions which appear to Mr. Tooke so obscure.

Definition. The Adverb was originally so called, because it was added to the verb, to modify its force and meaning; hence the Greek writers defined it thus: Ἐπίῤῥημα ἐστι μέρος λόγου ἄκλιτον, ἐπὶ τὸ ῥῆμα τὴν ἀναφορὰν ἔχον.—"The Adverb is an indeclinable Part of speech, having relation to the verb." The question of its being indeclinable or not, is unimportant in our present investigation, since this circumstance depends on the idiom of a particular Language; but the relation which the Adverb bears to the verb depends on the Science of Universal Grammar: and this relation is stated by most of the ancient Grammarians as the peculiar property of the Adverb. Donatus makes it the only characteristic of this Part

Grammar.

of speech: *Adverbium est pars orationis, quæ adjecta verbo significationem ejus aut complet, aut mutat, aut minuit.* "The Adverb is a part of speech, which being added to the verb, either completes, or diminishes, or alters its signification." Vossius, however, observes, that the Adverb is added not only to verbs, but to nouns and participles; and consequently, that its name must be understood to have been given to it, not from the use to which it is always applied, but from that for which it most generally serves. *Non solis adjicitur verbis, sed etiam nominibus et participiis: nomen igitur accepit non ex eo quod semper, sed quod plurimum fit.* By the word, nouns, Vossius, as he afterwards explains it, means adjectives, both nominal, pronominal, and participial. "We say," adds he, "*benè disserens*, as well as *benè dicere*, and *benè doctus*." And so we may say, *prorsùs meus, propemodùm suus, et magis nostras*, as well as, *prorsùs amicus, propemodùm liber, magis Romanus, &c.*

For want of a clear and intelligible definition of the Adverb, some writers have undoubtedly exposed themselves to the sarcasm of Tooke, who thus translates a sentence of Servius: *Omnis pars orationis*, "every word," *quando desinit esse quod est*, "when a Grammarian knows not what to make of it," *migrat in Adverbium*, "he calls an Adverb." And, indeed, among the twenty-one classes of Adverbs which are enumerated by Charisius, there are some which ought rather to be called interjections, as the pretended Adverb of invocation, *Heus!* that of answering, *item*, that of wishing, *utinam*, and that of showing, *ecce*. Nay, even nouns and pronouns were sometimes reckoned among Adverbs; as *quanti* datur, *eras* Romæ, &c. &c.

It is impossible to avoid these errors, unless we first establish a definition of the Adverb, to which, as a test, the various classes of words comprehended by different Grammarians under this common designation may be applied. We are aware of Scaliger's remark, *Nihil infelicius Grammatico definitore;* but the task which we have undertaken obliges us to state, as clearly as we can, what we consider to be the function properly distinguishing the Adverb from all other Parts of speech. The Adverb, then, is *a word added to a perfect sentence, for the purpose of modifying primarily an adjective, or a verb, or secondarily another Adverb.* We shall first consider the purpose for which it is used, then the sentence to which it is added, and, lastly, the sort of word which may be so employed.

Modification.

I. It is used to modify an adjective, or a verb, or another Adverb. All these words, it is well known, are called by Harris *attributives:* and therefore he aptly denominates the Adverb "an attributive of a secondary order," or "an attributive of an attributive." Harris, indeed, argues that the Greek word Ἐπιῤῥῆμα is of the same force and meaning as these phrases, inasmuch as the word Ῥῆμα is used by many writers to signify not only what is commonly called a verb, but also what are called adjectives, participles, &c. Thus Ammonius says, *κατὰ τὸ τοτὸ σημαινόμενον, τὸ μὲν* ΚΑΛΟΣ, *καὶ* ΔΙΚΑΙΟΣ, *καὶ ὅσα τοιαῦτα* ῬΗΜΑΤΑ *λέγεσθαι, καὶ οὐκ* ὈΝΟΜΑΤΑ.—"According to this signification, (that is of denoting the attributes of substance and the predicates in propositions,) the words *fair, just*, and the like, are called *verbs* and not *nouns*." And so Priscian, speaking of the Stoics, says, *Participium connumerantes verbis*, participiale verbum *vocant*. "Reckoning the participle among verbs, they call it a *participial verb*." Whatever may be thought of this reasoning, it clearly corroborates the fact, that the Adverb is employed to modify the adjective and the verb. On the other hand, the Adverb is not employed to modify the substantive; because that is the function of the adjective, or of the article. Let us then consider, first, the Parts of speech which are primarily modified by the Adverb.

Adverbs.

Of adjectives.

1. The *adjective*. Under this term we comprehend the adjective simple, or proper, the participle, or participial adjective, and the pronominal adjective. It is manifest that all the attributes which these various classes of words express are capable of modification. Thus, a house which is "lofty," may be "*surprisingly* lofty," or "*very* lofty," or "*moderately* lofty;" or some one may assert that it is "*not* lofty." And in like manner we may speak of "a *remarkably* intelligent youth," "an *over* indulgent parent," "a *truly* affectionate friend." So, when we use a participle, or a pronominal adjective, we may modify it by the aid of an Adverb, as "*much* obliged," "*greatly* indebted," "*wholly* yours," "*absolutely* mine," "*nobly* born," "*well* bred," "*highly* gifted," "*universally* respected," "*little* moved," "*less* affected," "not *so* energetic," "*equally* judicious," "*how* admirable!" "*thus* far," "*no* further." In all these instances, it is obvious, that the attribute expressed by the adjective undergoes some modification from the Adverb. In truth, we form a double conception, as, first a conception of *loftiness* with reference to the house, and secondly a conception of *surprise* with reference to the loftiness; so that the sentence "the house is surprisingly lofty" resolves itself into these other two sentences, "the house is lofty" and "the loftiness is surprising." Mr. Harris, therefore, had great reason to call the Adverb an attributive of an attributive; for, in the latter of these two sentences, we find the word "surprising" represents an attribute of that loftiness, which, in the prior sentence, was considered as an attribute of the house. It is not the house altogether which excites surprise, but only its quality of *loftiness*. A house may be both lofty and surprising, without being surprisingly lofty.

Comparative.

The instances which we have hitherto noticed, may be called those of positive modification. When we say a house is "surprisingly lofty," we do not compare its loftiness with that of any other house; but if we have occasion to make that comparison, we resort to another class of Adverbs, and say it is "*more* lofty," or "*less* lofty," or "*equally* lofty," or "*as* lofty," or "the *most* lofty," or "the *least* lofty;" in short, we exercise that mental operation which has been already described in treating of the comparison of adjectives; only the degrees of comparison are expressed by Adverbs, instead of being incorporated in the same word with the attribute compared. Nor is this all. We may compare different attributes of the same substance, as well as different substances in regard to the same attribute. We may consider the house as being "*more* lofty than convenient;" or as being "*equally* convenient and lofty." It is manifest, that in all cases of comparative modification, the Adverb cannot be employed simply or singly. It is then of a *relative* nature, being necessarily joined in construction, either with some other word, or inflection of a word in the same sentence; which words, or inflections, when they serve to modify adjectives or verbs, we also consider to be of the nature of Adverbs.

Grammar. Of verbs.

2. The *verb*. It must be remembered, that the verb asserts or manifests existence, either simply or together with some attribute of action or passion. The Adverb, therefore, may either modify the attribute involved in the verb, or it may modify the mere assertion of existence. When it modifies the attribute, its operation is exactly similar to what we have described, in regard to the adjective. "He runs swiftly" is of the same import as "he is running swiftly;" and the word *swiftly* modifies the verb *runs*, and the participle *running*, in the very same manner. The case is somewhat different when the Adverb modifies the assertion of existence; and this it does whenever it expresses any limitation of the time, place, circumstances, or actual occurrence of the fact. Thus the words, "now," "then," "when," "always," "never," &c., modify the assertion in point of time. If I say that a certain event "happens *now*," my assertion is limited to the present time; if I say it "happened *yesterday*," the assertion is limited to a certain time past. The assertion, that it "*always* happens," contradicts the opposite assertion, that it does "*not always* happen," and *a fortiori* the assertion that it "*never* happens." So, with respect to place, the assertion that a fact occurred *here*, or *there*, is no assertion, with regard to what may have happened *elsewhere*. Again, the occurrence of any event may be certain or doubtful, actual or contingent; and we may therefore say, "it will *perhaps* happen," "it may *possibly* take place," "it is *certainly* the case," "it *really* occurred," &c. As to the variety of circumstances attending different transactions, which may be expressed by Adverbs, they are beyond enumeration. The event in question may occur *aboard*, or *ashore*, *aloft*, or *below*, *abroad*, or *at home*, the ship may be cut *adrift*; the army may be *afoot*; it may be marching *homewards*, the battle may cease *awhile*, it may be begun *anew*, it may terminate *successfully*, &c., &c., &c.

Secondary use.

Such being the primary uses of the Adverb, it is easy to conceive that the secondary use is similar. As the adjective modifies the substantive, and the Adverb modifies the adjective, so may a second Adverb be applied to the former with the same power of modification. As the word *admirably* may be prefixed to *good*, so may *very* be prefixed to them both together; and we may say "a *very admirably* good discourse;" in which, and the like instances, the analysis is similar to what we have before stated. The discourse is good, the goodness is admirable, the admiration is extreme.

Sentences modified.

II. We have next to consider the sort of sentence to which an Adverb is added, and the manner in which the addition is effected.

First, we say, the Adverb is added to a perfect sentence; and by a perfect sentence we here mean one which either enunciates some truth, or expresses some passion with its object. Therefore, even to a simple imperative the Adverb may be added, since a perfect sense is expressed without it, and its addition only serves to modify the verb. Thus the word "fly!" is, in effect, a perfect sentence, for it implies an agent and an act, and it couples the conception of the act of flying with the conception of the person addressed, if not in the perception of the speaker, at least in his volition. To this sentence, therefore, an Adverb may be added consistently with our definition, and we may say "fly *quickly*!" After this explanation of the passionate sentence, it is scarcely necessary to explain the enunciative. When the verb expresses action or passion, there can be no difficulty: thus, when Macbeth says,

Adverbs.

After life's fitful fever he sleeps *well*.

there can be no difficulty in understanding that the Adverb *well* modifies the verb *sleeps*. A question, however may arise where the verb merely expresses existence; as, in the line just quoted, if the expression had been "he is well," it might be questioned whether the word *well* was an Adverb or an adjective. A similar remark may be made on such expressions as "he is *asleep*," "he is *awake*," &c. It is true that in the English Language these and many other such words have an Adverbial form, and cannot be employed in immediate connection with substantives, as "a well man," an "asleep man," or "an awake man:" yet where they thus form the predicates of verbs, they are in effect adjectives. "He is well" corresponds exactly with "he is healthy"—"he is asleep" with "he is sleeping"—"he is awake" with "he is waking:" and in a question of Universal Grammar, the idiomatic form of the words cannot at all decide the question.

When we say the sentence must be perfect, we mean it must be perfect in the Mind; in expression a part or even the whole of it may be understood. A *part* is understood when the Mind evidently supplies what is necessary to complete the sentence, as in the animated lines of WALTER SCOTT—

———On Stanley!—On!—
Were the last words of Marmion.

Here the Adverb *on* manifestly refers to some verb understood in the Mind, such as "march," "drive," "rush," or the like. The verb is suppressed, because it is indifferent to the speaker: the Adverb is expressed, because it is of the utmost importance—because to the thoughts and feelings of the dying warrior the mode of getting at the enemy was totally immaterial; but to get at them by some means or other was his most eager wish. The *whole* of the sentence is understood, when the adverb is responsive: as, "Will you come? Yes."—"When will you come? Presently."—"How often did he come? Once."—For these answers mean, "I will come *certainly*"—"I will come *presently*"—"He came *once*."—And consequently the Adverbs, *yes*, *presently*, and *once*, are to be taken as modifying the verb "will come" and "did come," respectively.

Words employed.

III. We have next to inquire what sorts of words may be employed, as Adverbs, to modify adjectives and verbs: and in reality the proper answer is—all sorts. For the expression of Servius, though ridiculed by Tooke, is literally true: *Omnis pars orationis migrat in Adverbium*. "Every Part of speech is capable of being converted into an Adverb."

Adjectives.

I. From what has already been said, it is manifest that an *adjective* may be used Adverbially. Let us suppose that it is necessary to enunciate these three propositions successively.

1. A certain quantity exists.
2. That quantity is large.
3. That largeness is sufficient.

We have here three conceptions, *viz.*, *quantity*, *largeness*, and *sufficiency*. The first is only considered as a substance; the second is considered as an attribute in one instance, and as a substance in the other; and the

third is only considered as an attribute. Now, if we unite these three sentences in one, and say there is "a sufficiently large quantity," we, in fact, convert the adjective "sufficient" into an Adverb. In some instances this difference in the employment of the word, is attended with a correspondent change in the form; as in English the adjective *sufficient* is changed into the adverb *sufficiently;* but this neither prevails in all Languages nor in all Adverbs of the same Language; and is, indeed, a circumstance often appearing to be perfectly accidental, or capricious. Again, the adjectives thus employed sometimes remain unchanged in form, but lose in practice their adjectival use, either partially or altogether. These circumstances, it is true, depend on the idioms of particular Languages; but it is not the less important to notice some of them, because there is no more common source of error among Grammarians, than the confounding of what is universal in Language with what is particular, the Scientific rule with the accidental exception. This will appear from many instances in the class of words now under our consideration, namely, the adjectives proper, when used as Adverbs; and in order to consider them the more distinctly, we shall notice first the simple, and then the compound words.

Much, very, enough, fain, lief, scarce, stark, and several other words, more or less frequently employed as Adverbs, were originally simple, uncompounded adjectives. They have all some peculiarities in their use, the notice of which may serve to illustrate the present investigation.

Much. *Much* is employed Adverbially before a participle, or after a verb; and, though in modern use, we do not give it the regular adjectival construction, as "a much quantity," "a much portion," &c.; yet, this was anciently and still is provincially done with its derivative *muchel, muckle,* or *mickle.* Mr. Tooke, who says that this word *much* has "exceedingly gravelled all our Etymologists," derives it from the Anglo-Saxon verb *mawan,* "to mow," of which, says he, the regular præterperfect is *mow,* and the past participle *mowen.* "Omit the participial termination *en,*" continues he, "and there will remain *mow,* which means simply that which is mown; and, as the hay, &c. which was mown, was put together in a heap, hence, figuratively, *mowe* was used in Anglo-Saxon to denote any heap; and this participle, or substantive, call it which you please—for however classed, it is still the same word, and has the same signification—was pronounced, and therefore written *ma, mo,* &c., which, being regularly compared gave *ma, maer, maest, mo, more, most,* &c.; and *much* is merely the diminutive of *mo,* passing through the gradual changes of *mokel, mykel, mochill, muchell, moche, much.*" Such is the substance of an etymological disquisition, in the course of which Mr. Tooke takes upon him to speak with great contempt of Junius, Wormius, Skinner, and Johnson, and pretends to remove all those difficulties which have so "exceedingly gravelled" other Etymologists! The leading Principle in this disquisition is a very extraordinary one. Mr. Tooke assumes that in the formation of Language, the conceptions of distinct action must necessarily have obtained a name before those of quality. Indeed it is not very clear that he conceives mankind ever to have acquired conceptions of quality at all. However, the fundamental assumption is perfectly arbitrary: it cannot possibly be supported by History, and we do not see the least ground for it in any rational system of Philosophy. We may observe, that the reasoning relative to the words *more* and *most* would be at least equally satisfactory if its order were exactly reversed, and the premises made the conclusion. These words *more* and *most,* we might say, are the comparative and superlative of the old word *mo,* which was an adjective signifying "much:" when much of any thing, therefore, was heaped together, it was called *mo;* and consequently a *mowe* was a "heap;" but as hay, when it is cut down, is, in the very act of cutting, heaped together, to cut hay was called to *mow,* and the hay that was cut was said to be *mowed.* These opposite trains of reasoning agree in this, that names must necessarily be supposed to have been given to the conceptions of the Human Mind, in some one certain order, that is to say, either proceeding from the more general to the more particular, or the contrary. We do not know that this can be positively asserted; but, if it may be so, still we should incline against Mr. Tooke's Etymology. According to him, our rude ancestors could not have known whether a thing was much or little, until after they had invented the art of making hay, had regularly conjugated their verbs, added the participial termination *en,* taken it away again, and compounded the word (thus unnecessarily prolonged and curtailed) with a syllable implying diminution; and after all they could never alter the signification of the word; but if they talked of much money, or much wisdom, much acuteness, or much absurdity, the word much would only signify a heap of hay! So much for his theory: as to his facts, we believe it would be exceedingly difficult to discover where or when *ma* was used for a hay-mow, or a barley-mow; and when we come to derive *mokel, muchel,* or *michil,* from *mo,* we shall be "exceedingly gravelled" to account for the unlucky *k* and *ch* which happen to be inserted before the syllable said to be expressive of diminution.

That there may be some affinity between *mo* and *much* is possible; but it is very improbable that *much* should be an abbreviation of *muchel.* On the contrary *muchil* is, in all probability, derived from *much.* At least, it is certain, that we find *much,* or *mich,* as early as we do *muchil.* Wachter, speaking of these words, says, *Simplicissimum est* MICH *quod in antiquissimis dialectis ponitur pro magno et multo.* "The most simple is *mich,* which, in the most *ancient* dialects, signifies great and much." Thus, in the old Persian, *mih* was great, *mihter* greater, *mihtras* greatest; whence the Sun was called *mithras.* The aspirate *h* was easily converted into the guttural *ch,* and the palative *k* or *g.* Hence the Greek μεγ, in μέγαν; and the Latin *mag,* in *magnus, magister,* &c.; and as that which is great is usually powerful, we have an infinite number of words from this radical signifying power, as the Mæso-Gothic and Anglo-Saxon *magan,* to be able, which supplies our auxiliaries *may* and *might,* the old German *machen,* and Anglo-Saxon *makan,* to make, &c., &c. Again Wachter, speaking of the ancient word *mich,* says, *Postea invaluit* MICHEL, *eodem sensu.* "Afterwards *michel* came into use, in the same sense." Hence the Gothic *mikils,* the Anglo-Saxon *micel,* the Alamannic *mihhil,* the Islandic *mikill,* and, possibly, the Greek μεγάλη. Nor does it at all appear that the

Grammar. final syllable *el* or *le* is meant to express diminution; *muchel* is no more the diminutive of "much," than *handle* of "hand," or *spindle* of "spin;" but *much* and *muchel* are used *eodem sensu*, and so were anciently *lite* and *litel*.

It is at least certain, that *much* is to be found in English as early as *muchel*, and that these two words seem to be used indifferently by our most ancient writers. The modern English Language is founded on the Anglo-Normannic, of which the two earliest specimens referred to by HICKES are the *Life of St. Margaret* and the *Description of Cokaygne*.* In the former of these we find—

> Tho he couthe of wisdom he hatede *muche* sunne.
> * * * * * *
> And yeld here servise, ofte mid *muchele* wowe.

In the latter,

> Undir heuen his lond iwisse
> Of so *mochil* ioi ant blisse
> * * * *
> The yung monkes, eueriche dai,
> Aftir met goth to plai:
> Nis ther hauk no fule so swifte,
> Bettir fleing bi the lifte,
> Than the monkes heigh of mode,
> With har sleuis ant har hode,
> Whan the abbot seeth ham flee
> That he holt for *moch* glee.

The date of these Poems is not positively fixed, but they were certainly anterior to Edward I. That monarch died in 1307; and among the Harleian MSS. (No. 2253, fol. 72.) is an Elegy, apparently written immediately after his death, and consequently before the time of Chaucer or Gower, in which are these lines:

> The pope to is chambre wende
> For dol ne mighte he speke na more
> But after cardinals he sende
> That *muche* couthen of Cristes lore.

With respect to the two great Poets themselves, CHAUCER and GOWER, the former seems generally to prefer the word *much*; but the latter uses it indifferently with *muchel*. Thus, in *The Testament of Love* (book ii.) "*Moche* folk at ones mowen not togider *moche* thereof have;" *i. e.* "Many persons at once should not have too *much* thereof, *viz.* of riches." And again (book iii.) "Opinion is while a thinge is in non certayne, as thus: yf the son be so *mokel* as men wenen." "Opinion is while a thing remains in uncertainty, as whether the sun be so large as men suppose." In the Romance of *Kyng Alisaunder*, which was probably subsequent to the time of Chaucer, we find— Adverbs.

> With *muche* ost he is comyng
> * * *
> Dieu mercy, to *mychel* harme
> Many knighth there gan hym arme.

In that of *Octovian Imperator*, about the same age,

> Ther n'as nother old ne yynge
> So *mochell* of strength.

And in the *Lyfe of Ipomydon*, (Harl. MSS. 2252.) also of the same period,

> Hye and low louyd hym alle,
> *Moche* honoure to hym was falle.

From all these authorities, it is very clear that *much* is the name of a conception of greatness in quantity, quality, number or power; and that when this conception is viewed as the attribute of any substance, the word *much* is an adjective; when as the modification of an act or quality, it is an Adverb.

Very is correctly stated by Mr. Tooke to be the Latin adjective *verus*, "true," changed, in old French and old English, into *veray*, which, in modern French, is *vrai*. The adjectival use of this word still remains in the Liturgy of the Church of England, "*very* God of *very* God." Chaucer uses it as an adjective both in the positive and comparative degree. Thus, in his translation of Boethius, *On the Consolation of Philosophy* (b. iv.) "It is clere and open that thilke sentence of Plato is *very* and sothe." And again, (b. iii.) "which that is a *more verie* thinge." From the same word *veray* we have our compound adverbs *verily* and *verament*, of which the latter, though now obsolete, was once in Poetical use. Thus, in the above-quoted Romance of *Kyng Alisaunder*, published by Mr. Weber, from MSS. in the Lincoln's Inn and Bodleian Libraries: Very.

> By the steorres and by the firmament
> He him taughte *verrament*.
> * * * * *
> Ther ros soche cry *verrement*
> No scholde mon yhere the thondur dent.

That an adjective primarily signifying "true," should, in a secondary sense, form an Adverb expressing eminence of degree, as applied to all other qualities, is not surprising; for a thing that is *very* good or bad, may be said κατ' ἐξοχην, to be *truly* good or bad. The Italians express the same modification of qualities by *molto*, "much," the French by *fort*, "strong," the Latins by *multum*, "much," and *valdè*, from *validus*, "strong:" and our ancestors by a variety of attributes, as *swythe*, *sothfast*, *right*, *full*, *strong*, *well*, &c.

Swythe may possibly be the adjective *swift*; but is Swythe. more probably from the Gothic *sive*, sicut; as *sooth* is from the Gothic *so*, hæc. We still use *sooth* Poetically for "truth;" as in Lear—

> In good *sooth*, or in sincere verity.

And in Macbeth—

> If I say *sooth*, I must report they were
> As cannons overcharged.

* The *Description of Cokaygne* is a rude, satirical Poem, probably written about the year 1200, in ridicule of the Monastic life: and it is curious, as affording the etymology of the modern term *cockney*.

From the Latin, *coquina*, a kitchen, came the French words *coquin* and *cocagne*. *Coquin* was originally *coquinus*, an attendant in the kitchen, a turnspit, and thence came to signify any other mean, worthless person. *Cocagne* was the luxury of the kitchen. Hence, to this day, among the amusements of the common people in France, at public feasts and rejoicings, it is usual to erect a mast called the *Mât de cocagne*, at the top of which are placed roast meats, and other delicacies, as prizes for those who can most quickly reach them by climbing. The land of Cocagne, therefore, is an imaginary land of luxury, which the author of the Poem above quoted places "far in see bi west Spayngne," *i. e.* "far in the sea to the westward of Spain," the supposed situation of the great island Atlantis, the Hesperian Gardens, and other fancied scenes of happiness, beyond the reach of navigation, as then practised. The metropolis of England being considered, by the rude inhabitants of the country parts, as a seat of mere luxury and idleness, afterwards received, in contempt, this name of *cokayné*, corrupted by them into *cokeney*, as appears by a scoffing rhyme of one of the old barons—

> Were I in my castle of Bungay,
> Beside the river of Waveney,
> I would not care for the king of *Cokeney*.

And it is somewhat amusing to trace in the satirical *Description of Cokaygne*, the origin of the puerile story of roasted pigs running about the streets of London, crying "come eat me."

> The gees irostid on the spitte,
> Fleegh to that abbai God hit wot,
> And gredith gees, al hote, al hot.

Grammar. In the *Geste of Kyng Horn*, (Harl. MSS. 2253. fol. 83.) which WARTON says is the most ancient English metrical Romance, we find—

Tueye feren he hadda
That he with him ladde
Alle riche menne sones
Ant alle *swythe* feyre gomes.

We must observe that Warton was a very inaccurate transcriber, and therefore is not to be relied on as authority for any minote peculiarities of diction or orthography; but we have, in general, corrected his quotations, by the original manuscripts, and cite them from the latter, with such variation only, as is necessary to reoder them legible in the English character, changing the Saxon *th* and *w* for modern letters, and filling up the contractions, which would ooly serve to puzzle a mere English reader.

Io *Kyng Alisaunder*, this word often occurs, as

He smot the hors, and in he leop:
Hit was *swithe* brod and deop.

So in the Ballad on the defeat of the French by the Flemings, at the Battle of Bruges, A. D. 1301, (Harl. MSS. 2253. fol. 73. b.)—

Sire Jakes de Seint Poul yherde hou it was,
Sixtene hundred of horsmen asembleda othe gras,
He wende toward Bruges, pas pur pas,
With *swithe* gret mounde.

Sothfast. *Sothfast* is the same adjective *sooth*, compounded (as in the word *stedfast*) with fast, *i. e.* firm. Hence it was similar both in meaning and use to *swythe*. In a sort of Dramatic Poem, probably of the XIIIth or XIVth century, on *Christ's Descent into Hell*, (Harl. MSS. 2253. f. 55. b.) are these lines:

And so wes seyde to Habraham,
That wes *sothfast* holy man.

Again, in the *Pricke of Conscience* (see Warton, v. I. p. 258.) it is used adjectively—

Thou mercyfull and gracious god is,
Thou rightwis, and thou *sothfast*.

Right. *Right*, the Latin *rectus*, we still use adverbially in the titles "right honourable," "right reverend," "right worshipful," &c. The ancient usage was more general.

In the *Geste of Kyng Horn*—

Athulf quoth he, *ryht* anon,
Thou shalt with me to boure gone.

In the Romance of *Syr Launfal*—

Her manteles were of grene felwet
Ybordured with gold *ryght* well ysette.

In Chaucer's *Clerke's Tale*—

Ther ys *right* at the west syde of Ytaly,
Down at the rote of Vesulus the colde
A lusty plaine.

Full, sometimes used Adverbially at the present day, was much more frequently so in former times.

Iu Chaucer's *Franklein's Tale*—

Listenythe of a knyght of tyme *full* olde.

In the MS. Poem of *St. Jerome* (Cotton MSS. Calig. A. 2.)—

Seynte Jerome was a *full* good clerke.

In the *Geste of Kyng Horn* it is superadded to another adjective used adverbially—

The leuerokes that beth cuth
Lightith adun to manis muth
Idight in stu *ful swithe* wel.

Adverbs. *Strong*, which we only use as an adjective at present, seems to have been anciently adopted in the Norman-Saxon Adverbially, as a translation of the French *fort* and the Latin *valdè*.

Thus, in the *Geste of Kyng Horn*—

Horn, quoth heo, wellonge
Y haue loued the *stronge*.

Well. *Well* is derived from the Anglo-Saxon substantive *wela*, "well-being," or "felicity." In that Language the Adverb was *wel*; in the Mæso-Gothic it was *waila*; in the Alamannic *wuola*; in the Islandic *vel*; in the Dutch *wel*; and in the German *wohl*. Of the substantive use of this word, we have an instance in the *Description of Cokaygne*—

Ther nis lond undir heuenriche
Of *wel*, of godnis, hit iliche.

In the present day we rarely use it to modify adjectives proper, or numerals, but those constructions are common in the old writers. We have just quoted the instance of *wellonge*, *i. e.* very long. In the Ballad on the Battle of Bruges, before mentioned, we have *wel muchele*, *i. e.* very great:

Sire Jakes ascapede by a coynte gyn
Out at one posterne ther the men sold wyn
Out of the fyhte hom to ys yn
In *wel muchele* drede.

In *Syr Launfal*—

With Artour ther was a bacheler
And hadde ybe *well* many a yer
Launfal for soth he hyght.

Again, in the *Description of Cokaygne*—

Wendith meklich hom to drink
Ant geth to har collacione
A *wel* fair processione.

In the *Prologue* to the *Canterbury Tales*—

That night was come into that hostelry
Wel nine and twenty in a company.

Chaucer also has the compound *weleful*, *i. e.* full of felicity. "O *weleful* were mankind if thilke loue that gouerneth the heuen gouerned their corages."

Enough. The word *enough* is explained in BAILEY's Dictionary only by the adjective "sufficient." It is, indeed, used adjectively after the verb to be, as "that is enough," *i. e.* "that is sufficient;" but we cannot employ it as we do the word "sufficient" in immediate connection with a substantive; we cannot say "an enough qoantity," as we do "a sufficient quantity." For this no other reason can be given than established usage,

Quem penes arbitrium est, et jus, et norma loquendi.

This same adjective is used Adverbially, without any change of form; but again, custom obliges us to place it after the adjective which it modifies, and not before it, as is usual with other Adverbs. We say "very large," "pretty large," "too large," "sufficiently large," but we must say, "large enough." The accidental variation of arrangement, however, in no degree affects the Grammatical character of the word, which is decided by its signification and use, not by its form or position. The Etymologists have thrown little light on this word. Mr. TOOKE supposes it to be "the past participle *genoged*, *multiplicatum*, manifold, of the verb *genogan*, *multiplicare*." It may, perhaps, be doubted, whether there ever was such a word as *genoged*, with the signification of *multiplied*; but if

Grammar. there was, how does this circumstance explain the Grammatical use of our present Adverb *enough?* What has the conception of sofficiency, conveyed by this Adverb, to do with moltiplication, any more than with division. A single thrust through the body may be quite *enough* to dispatch a man, and if it be not, he will hardly wish it *multiplied.* Dr. JOHNSON's observations also on this word are rather singular. "It is not easy," says he, "to determine whether this word be an adjective or Adverb," as if it must, of necessity, be always one, or always the other; and yet he afterwards says, (which is equally erroneous,) that "after the verb *to have*, or *to be*, it may be accounted a substantive." Add to this his suggestion, that when *enough* is an adjective, "*enow* is its plural!!!"—although, in his Grammar, he had said, that English adjectives were indeclinable, "having neither case, gender, nor *number*"—and of course no plural. JUNIUS says, *inductus orthographiâ, quam præclaræ antiquitatis monumentum nobis exhibet, libens deduxerim* ENOUGH *à Gothico* GANAH; *et* GANAH *a* γανόω, *lætitiâ afficio, voluptatem affero; quod nihil æquè miseros mortales exhilaret, quàm rerum omnium satietas.* "Induced by the orthography which the monument of illustrious antiquity (the *Codex Argenteus* of Upsal) exhibits, I should willingly derive *enough* from the Gothic *ganah;* and *ganah* from γανόω, 'I exhilarate or give pleasure;' since nothing so much exhilarates miserable mortals, as to have enough of every thing." Lastly, the Rev. Mr. LEMON, in his English Orthography, derives *enough* from ἱκανὸς, "sufficient in quantity or quality," and adds, "indeed our word *enough* undoubtedly wears a very Gothic appearance; but still is derived from the Greek." Of such etymologies, and such reasoning on them, it is time to cry *enough!* The plain fact is, that the word *enough* is the Anglo-Saxon word *genoh*, or *yenogh*, having precisely its present meaning; and that this word had some affinity with the Mœso-Gothic *ganah*, the Frankish *ginuagi*, the old German *ginuoh* and *kanuht*, the modern German *genug*, and the Dutch *genoeg*, all words of the same signification, and all descended, as WACHTER conjectures, from a more ancient Teutonic word, *nog*, which HELVIGIUS derives from the Hebrew *anag*, "to delight." However this may be, these words are connected with a great number of others, all bearing some relation, more or less distinct, to the conception of "sufficiency," as the German *genuge*, "plenty," *genugen*, *vergnügen*, *gnugthun*, "to satisfy;" *genau*, "exact," &c. &c.; nor is there any reason to believe that our rude ancestors could not form a conception of what was "enough," quite as easily as a conception of what was "multiplied," and give a name to the former as easily as to the latter. Now, such *name*, when used substantively, would be a noun substantive; when used as the attribute predicated directly or indirectly of any substance, it would be an adjective; and when used to modify the conception of any attribute, it would be the Adverb *enough*, which we are at present examining.

Fain. *Fain*, says Mr. Tooke, is a participle: and then he gives three examples, in each of which it has merely the force of an adjective proper, which it still retains in the Scottish name of a well-known tune, "I'll mak ye be *fain* to follow me," *i. e.* "I'll make you be *glad* to follow me." This word is used substantively in *Kyng Alisaunder:*

Adverbs.

Now quyk, sire, and snel,
Do ryng alle thy bellis,
And do thy seolf thyn *fayn*
Thy folk al to ordeyne.

Lief. *Lief* also, Mr. Tooke contends, is a participle. It is not so; because it expresses no particular action, but an habitual quality. Participles often make this transition. Thus, the word "inoocent" is, literally, "doing no harm;" yet, in common parlance, it expresses a certain Moral state of being, a freedom from guilt. It would be as rational to say that *love* was a participle, as *lief*, for they are both equally connected with the Anglo-Saxon verb *lufian*, "to love." The general conception which prevails through these and a great number of derivative words in the Northern Languages, is found in the old German *lieb*, which Wachter explains to be *bonum, quod omnes appetunt, sive sit honestum et naturæ conveniens, sive delectabile tantum.* "Good, that which all desire, whether as being honourable, and well suited to the nature of Man, or as merely delightful." Hence *lieb, amatus, carus, dilectus, amicus;* in which seoses, he says, it occurs in all the Dialects. Thus the passage in St. Mark's Gospel, "Thou art my beloved son," is rendered in the Gothic, *Thu is sunus meins sa liuba.* Mr. Tooke properly says, it "always means *beloved*." but *beloved* differs, in modern use, from *loved;* for as we do not use the verb to belove, but, to love; so beloved, though a participle in form, has the force and effect of an adjective proper. *Leove* is thus used in the Poem on *Christ's Descent into Hell*, where Eve says to Christ,

Knou me Louerd icham Eue
Ich ant Adam the were so *leoue*.

In the comparative, it occurs in the *Prologue* to *Kyng Alisaunder*, where the Poet says, there are many persons

That hadde *levere* a ribaudye
Than to here of God other of Seinte Marie.

GOWER has it in the superlative:

Three pointes, which I fynde
Ben *levest* unto mans kynde
The first of hem it is delite
The two ben worship and profite.

In the *Romant of the Rose*, it is used for the beloved person:

His *leefe* a rosen chapelet
Had made and on his heed it set.

It is also found in composition, as *leflich*, which is the modern word "lovely," *leofman*, which is Shakspeare's word "leman," *leofsum*, "amiable," &c. In short, the word *leof*, in all its forms, is no other than the word *love*, which our ancestors used adjectively, whilst we use it only as a substantive and as a verb. No one thinks of saying that the substantive *love* is formed by adding to the verb *love* the participial termination *ed*, and then taking it away again; nor is there any greater reason for supposing this operation to take place with the adjective.

Scarce. Stark. *Scarce* and *stark* are admitted by TOOKE to be adjectives, and their Adverbial use is equally well established. *Stark*, indeed, is now seldom used as an adjective, and only in combination with a very few adjectives as an Adverb; but these are merely the accidents of idiom. There are, as has been already observed, several other simple adjectives which, either in ancient or modern use, are employed as Adverbs;

but we have already specified instances enough of these, and must now proceed to the compounds.

Adverbs in *ly*.

The first and most numerous class are those terminating in *ly*, the greater part of which are only employed at present as Adverbs; while the same words, in a simple form, without the termination, are used adjectively. Thus we have in modern use the adjectives "wise," "grateful," "judicious," and the Adverbs "wisely," "gratefully," "judiciously." Hence some persons, from an injudicious desire of precision, apply what they suppose to be a distinctive mark of the Adverb to words which do not require it, such as *well* and *ill*; for which they say *welly* and *illy*. Welly, indeed, is provincial in the North of England, in the peculiar sense of *well nigh* as *fully* is in Scotland, in connection with comparative adjectives, as, "fully more," "fully better," &c. *Ly* is an abbreviation of the adjective *like*; and the words wisely, gratefully, judiciously, &c. are the compound adjectives wiselike, gratefullike, judiciouslike, &c. The termination *lyk* or *lich* is common in old English. Thus, in *Kyng Alisaunder*, we have the adjectives *eorthliche*, (earthly, mortal,) *ferliche*, (strange, wonderful,) and the Adverbs *gentiliche*, (gently,) *sikerlyk*, (securely, certainly,) *theofliche*, (like a thief,) *quykliche*, (quickly,) *stilliche*, (quietly,) *skarschliche*, (scarcely,) *aperteliche*, (openly.)

So, in *Syr Launfal*,

> He gaf gyftys *largelyche*,
> Gold, and siluer, and clodes ryche.

And again, in the same Poem—

> The lady was brygt as blosme on brere,
> With eyen gray, with *lovelych* chere.

This word *lovelych* is the identical word *leflich* before mentioned, and whieb occurs in one of the most ancient love-songs now existing in English, composed probably about the year 1200. The song begins, "Blow Northerne Wynd," and the lover describes his mistress

> With lokkes *lefliche* and longe.

Chaucer writes our word early, *erliche*; as in the *Knight's Tale*.

> An tellin her *erliche* and late.

In the *Description of Cokaygne* we have already seen the Adverb *meklich* (meekly.) In the *Geste of Kyng Horn* we find *evenliche* (evenly, straightly) used as an Adverb:

> Thou art fair & eke strong,
> & eke *evenliche* long.

This termination, therefore, is not less pure and distinguishable in the old English than it is, as Mr. Tooke observes, in the sister Languages—German, Dutch, Danish, and Swedish, in which it is written *lich*, *lyk*, *lig*, *liga*. In the Anglo-Saxon we find it used both adjectively and Adverbially, as in the translation of Bede's *Ecclesiastical History*, (book iii. c. 3.) "*tha lifigendan stanas thære cyricean, of* eorthlicum *setlum, to thom* heofonlicum *timbre, gebær!*" "the living stones of the Church, from *earthly* seats, to the *heavenly* building, it bore." And again, (*loc. cit.*) "*tha cyricean* wundorlice *heold & rihte*:" "the Church he *wondrously* held and ruled." The simple adjective "like" is, in the Anglo-Saxon, *lic*, which also signifies "the body." In Mæso-Gothic *leiks* is "like;" and *leik* is the body: whence the Scottish word *lyke-wake*, corrupted into *late-wake*, signifies "the watching of a dead body." That the *name* of the conception of "body" should be transferred to the conception of "likeness," is not at all surprising; for what is so like any person or thing as the very body of that thing, or of that person? Hence, Shakspeare, meaning to intimate that the use of the Drama is to represent the exact likeness of living manners, says, it is "to show the very age and *body* of the time, its form, and pressure;" as if he had said, "the Drama holds up a mirror to the present time, exhibits its age of manhood or decrepitude, represents its very *body*, the shape which it bears, and the impression which it produces on the mind of the observer, as a seal does on wax, or a statue on the plaster from which a cast is to be taken." Neither is it surprising that the adjective "like" should enter into composition with a great number of other adjectives; for if any attribute could not be exactly predicated of a particular substance, something like that attribute might be so; if a person or thing could not be said to possess exactly a certain quality, it might be said to possess a quality similar, or nearly the same; if it was not great it might be greatlike; if not good, goodlike, &c. Thus the pronominal adjectives *such*, *each*, *which*, were formed from compounds literally signifying so-like, one-like, and what-like.

1. In the Mæso-Gothic *swa* is "so," and *swa leik* is "such." In the Anglo-Saxon it is contracted to *swylc*, in the Old English to *swylke* aud *swiche*, and thence to *sich* and *such*. And the same is found in the cognate Languages: in the old Frankish and Alamannic, it is *solich*, *sulich*; in the Dutch, *zulk*; in the Swedish, *slyk*; and in the modern German, *solche*.

In the Romance of *Richard Coer de Lion*, we have

> Kyng Alysaundre ne Charlemayn
> Hadde neuer *swylke* a route.

And Chaucer says,

> In *swiche* a gise as I you tellen shal.

2. The words *ilk* and *ilka* are to be found in our old writers, and still exist in the Scottish Dialect. *Ilk* was sometimes written *iliche*, and has been abbreviated to *each*. The following lines occur in a satirical Poem entitled *Syr Peni*, or *Norracio de Domino Denario*: (MS. Cotton. Galb. E. 9.)

> Dukes, erles, and *ilk* barowne
> To serue him er thai ful boune
> Both biday and nyght.

In another part of the same Poem are these lines:

> He may by both heuyn and hell
> And *ilka* thing that es to sell
> In erth has he *swilk* grace;

where we see *swilk* used for "such," and *ilka* for "every," as it is by Burns, in his *Twa Dogs*—

> His honest sonsie, baws'nt face
> Ay gat him friends in *ilka* place.

3. *Which* is, in the Anglo-Saxon, *hwilc*; in the Mæso-Gothic, *hweleiks*; from *hwas*, or *hwe*, "whom," and *leiks*, "like." In the Alamannic it is *huuielich*; in the Danish, *huilk*; in the Dutch *welke*; in the German, *welche*. The word *whilk*, anciently written *quhilk*, was common in Scotland to a late period, and perhaps still exists in some remote parts of the Country. It is uniformly used in the Work of Nicol Burne, before quoted: as "I micht produce monie siclyk places, *quhilk* I never hard zit cited be zou;" that is, "I might produce many such places, (of Scripture,) which I never heard yet cited by you."

Grammar.

4. Agreeing with these is the old English *thilke*, still retained in the Wiltshire Dialect, and pronounced *thik*, for "that." Thus SPENSER, in his *May*, says,

> Our blonket liveries been all too sad,
> For *thilke* same season, when all is ycled
> In pleasance.

CHAUCER, in his translation of Boethius, says, "Certes yet liveth in good point *thilk* precious honour of mankind."

And in the Poem on *Christ's Descent into Hell* are these lines:

> The smale fendes that bueth nout stronge
> He shulen among men yonge
> *Thilke* that nulleth ageyne hem stonde
> Ichulle he habben hem in honde.

That is, "the small fiends that are not strong shall go among mankind, and *those* persons who will not stand against them, I am willing they should have in hand."

Thus have we traced a substantive (signifying *body*) through its transitions, first into an adjective proper, (*like*,) thence as part of the compound adjectives proper and pronominal, (*lovelike* and *solike*,) and, lastly, into the termination (*ly*,) which we still use both in adjectives and Adverbs, though with idiomatic differences in respect to particular words, some being only considered as belonging to the one class, and some to the other. Thus, *goodly*, though not much used in the present day, and rather as an Adverb than an adjective, is employed by SHAKSPEARE in the latter character, through all its degrees of comparison:

1. In *Hamlet*—

> I saw him once, he was a *goodly* king.

2. In *All's Well that Ends Well*—

> If he were honester he were much *goodlier*.

3. In *King Henry VIII.*—

> She is the *goodliest* woman that ever lay by man.

So the word *kindly* is commonly considered to be an Adverb, but BURNS uses it as an adjective, in *Poor Maillie's Elegy*:

> Thro' a' the toun she trotted by him;
> A lang half-mile she cou'd descry him;
> Wi' *kindly* bleat, when she did spy him,
> She ran wi' speed.

On the other hand, the word *lonely* is treated in the English Dialect as an adjective; but BURNS, in the same Poem, employs it Adverbially:

> Our bardie, *lanely*, keeps the Spence
> Sin' Maillie's dead.

Godly, *lovely*, *portly*, and some other such words, are employed exclusively, in modern times, as adjectives; but it is observable that *godly* has obtained by custom a different meaning from the identical adjective *godlike*. We have, too, some of these words in one form of composition, and not in its correspoodent compound. Thus we say *ungainly* for awkward; though the word *gainly*, formerly in use, has become obsolete. Dr. HENRY MORE, a very learned writer of the XVIIth century, says, "She laid her child, as *gainly* as she could, in some fresh leaves and grass." (*Conj. Cabal.*)

Prefix *a*.

A mistake similar to that which we have noticed in regard to the termination *ly*, also prevails with reference to the prefix *a*, which is considered by some persons as necessary to distinguish Adverbs from their adjectives, as *aloud* from *loud*; but the Poets, who commonly judge of Language more correctly, by a delicacy of feeling, than Pedants do, by the narrow rules with which they are conversant, adhere to no such distinction. Thus MILTON describes the "civil suited morn"—

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> ——kercheft in a comely cloud
> While rocking winds are piping *loud*—

not "loudly," nor "aloud." In fact, this prefix is of different origin in different Adverbs, and is more or less essential in modern use, according to the diversity of its original signification.

1. It is corrupted from the Saxon participial prefix *ge* or *ye*; as *adrift*, that is, *driven*.

2. It stands in the place of the prepositions *in* or *on*; as *alive*, anciently written *on lyve*, *i. e.* in *life*, or in a *living* state.

3. It was formerly expressed by the preposition *of*; as *anew*, anciently written *of new*, as we now say *of late*.

4. It is the positive article *a*; as *awhile*, *i. e.* a time.

5. It is part of the pronominal adjective all; as *alone*, anciently written *all one*, *i. e.* absolutely one.

6. It is the French preposition *à*, as *adieu*, which, however, is rather to be ranked among interjections.

7. It appears to be merely superfluous, as *alike*, anciently written *iliche*, for like.

We shall consider the participles, substantives, &c. hereafter; for the present, we mean to direct our attention only to those Adverbs with the prefix *a* which appear to be directly formed from adjectives proper, as, *aloud*, from "loud;" *anew*, from "new;" *abroad*, from "broad."

Aloud. Anew. Abroad.

Aloud, *anew*, and *abroad* were anciently written "on loud," "of new," and "on broad," corresponding to the expressions still current, "on high," "of late," &c. Thus, in the Poetical History of *Sir William Wallace*, the Scottish author of which seems to have lived not long after our great English Poet Chaucer, we read,

> *On loud* he speir'd what art thou?

GAWIN DOUGLAS, another Scottish Poet, in his spirited translation of the *Æneid*, which was completed in 1513, has these lines:

> The battellis were adjonit now *of new*.

And again—

> ———— his baner quhite as floure
> In sing of battell did *on brede* display.

It may be thought that the expressions "of new," "on broad," "on loud," and the like, are elliptical; and that a substantive is always understood, as "of new *beginning*," "on broad *expanse*," &c.; but what we mean by a substantive understood, is a conception present to the Mind, though not expressed in Language. Now, in the Adverbs "anew" and "abroad," or, in their equivalent phrases "of new," "on broad," there are no conceptions present to the Mind but those of *newness* and *breadth*, except that of the connection between these conceptions and the verbs which they are intended to modify. The words *new* and *broad*, therefore, notwithstanding their adjectival form, are rather to be considered as substantives. They name the respective conceptions, not as attributes of a fancied "beginning" or "expanse," but as general terms, which may serve, with the aid of a preposition, to indicate some circumstance or modification of the action expressed in the verb. The PORT-ROYAL Grammarians observe, that the greater part of Adverbs are only intended to express, in one word, what could not other-

Grammar. wise be marked except by a preposition and a noun substantive, as *sapienter* for *cum sapientiâ*; *hodie* for *in hoc die*; and this observation applies to the class before us. To display a banner "broadly," "on broad," or "abroad," is to display it "in breadth;" to begin a battle "newly," "of new," or "anew," is to begin it "with newness," compared with the former beginning. And this force of the expression may frequently be illustrated by comparing it with its converse, as "on high" with "the earth," "abroad" with "at home." Nor should we hesitate to explain thus even the plural ὑψίστοις in the angelic doxology (*St. Luke*, ch. ii. 14.) δόξα ἐν ὑψίστοις Θεῷ, καὶ ἐπὶ γῆς εἰρήνη—"Glory to God in the highest, and on Earth peace;" for, as δόξα is opposed to εἰρήνη, so is ἐν ὑψίστοις to ἐπὶ γῆς; and it signifies "in the heights," or rather in "the heights of heights;" as in the 148th *Psalm*, "Praise Him ye heavens of heavens." Again, it may receive further illustration from some equivalent modes of expression, as, *at large*, written by Chaucer, *at thi large*:

Thou walkest now at Thebes *at thi large*.

LONGLAND, in his celebrated *Vision of Piers Ploukman*, written about 1350, instead of the Adverb *alone*, uses the expression *mine one*:

And thus I went wher walking *mine one*.

A mode of expression not dissimilar to *my 'lane*, which is still used with the same meaning in some parts of Scotland.

Askew. Askance. Awry. It may be doubted whether the words *askew*, *askance*, and *awry* are taken immediately from adjectives or from participles. In respect to the first, Mr. Tooke seems to have quoted Gower erroneously:

And with that worde all sodenly
She passeth as it were *askie*,
All cleane out of the ladies sight.

Askie is "into the sky," *tenues vanescit in auras*, and does not appear to bear any relation to *askew*, which is connected with our word *skewer*, or as the vulgar, more consistently with its etymology, pronounce it, *skiver*, an instrument used to "twist" and "wrest" meat into a shape fit for the table; from the Danish *skiæv*, wry, crooked, oblique, and *skiæver*, to "twist," to "wrest," or force awry. Our word *shy* is probably of this origin. *Shy* is in German *scheu*, whence the verb *scheuen*; in Frankish *scuwan*, in Dutch *schuwen*, in Italian *schifare*, in French *esquiver*, and in English to *eschew*, all having reference probably to the Greek Σκαιὸς, the "left hand," inasmuch as the left hand has always been the mark of inferiority, that which was turned from, or eschewed: "the sheep were on the right hand and the goats on the left." The word *scoff* seems to be of the same origin. Thus, in *Kyng Alisaunder*, we find,

Alisaunder lookid *askof*
As he no gef nought thereof;

where it seems difficult to determine whether we should understand, "Alexander look'd scoffingly," or "Alexander look'd askew."

Participles. 2. It is not only the adjective proper which serves to modify other adjectives or verbs. The *participle* performs the same office, and in the same manner. An Adverb may be said to be derived from a participle, when it expresses a quality or circumstance produced by the action which the participle denotes. Thus *adrift* is an Adverb, which may be said to be directly or indirectly taken from the past participle of the verb *drifan*, to drive. To be turned *adrift*, is to be put in the state of a vessel *driven* about by the winds and waves, without a pilot or a helm. This conception of *driving*, considered absolutely, forms the substantive *drift*, which we apply Physically to the snow driven along the ground by the wind, or to the sand driven along the channel of a river by the stream. Intellectually, it is applied to the tendency of the arguments in a train of reasoning: as in Shakspeare

Adverbs.

What is the course and *drift* of your compact?

That is to say, whither do they *drive*? The word *adrift*, therefore, may have originally been *ydrived*, as Mr. Tooke seems to suppose; or it may have been on *drift*, that is, "in the state of *driving*;" but in either case, it presents the notion of a state or quality produced by action. *Aghast* seems to be *aghasted*, that is affrighted, as one who has seen a *ghost*. It is from the Anglo-Saxon *gast*, a ghost. *Ago* is the participle *ygo*, gone; as in Chaucer:

A clerke ther was of Oxenforde also
That unto logike hadde long *ygo*.

Asunder certainly bears some sort of reference to the participle of the verb *sondrian*, which may also have some connection with the substantive *sond*, the sand; but it is also to be observed that, in many of the Northern Dialects, the general conception of *separation*, or being apart from other things, is expressed by words of this radical. In the *Codex Argenteus* we have *sundro siponiam seinaim*," apart from his disciples"—hi the warth *sundro*, "when he was alone"—*afiddia* sundro, "he went apart." In the Anglo-Saxon *Sunder spræc* is "a private or separate conference." *Sundor land* is any separate and distinct tract of land possessing peculiar privileges, (whence the modern name of Sunderland,) *sundor gyfe*, a privilege, or peculiar grant.—So the Pharisees are called *sundor halgan* as affecting a singular and peculiar sanctity. Considering, therefore, that this word *sunder*, or, as we express it, *sundry*, has so distinct an adjectival force, it seems rather more probable that the word *asunder* was originally formed from the adjective than from the participle, and was probably expressed *on sunder*, from "sunder," as *on newe*, from "new."

Afret was the participle of the verb to *fret*, or to *freight*. Thus in the *Romant of the Rose*—

For round environ her crounet
Was fulle of rich stonis *afret*;

which may either mean in *fret*-work, or *freighted*, loaded with jewels. The former, *viz.* fretwork, seems to be taken from the act of gnawing or eating, as "a moth *fretting* a garment," whence "eating cares," *edaces curæ*, are said to *fret* the mind: and Chaucer has

The sow *fretting* the chyld right in the cradil.

There are two old German verbs *fressen* and *fretten*. The former is our verb to *fret*, the Mæso-Gothic *fretan*, Anglo-Saxon *frætan*, Dutch *vreten*, Frankish *frezan*, all signifying to eat or devour. The other verb is from the old German word *fret*, a load or burden, restrained in French to the lading of a ship, whence our substantive *freight*. In a very rude specimen of the satirical talents of the XIIIth century, (Harl. MSS. No. 2253. fol. 124. b.) the author, reviling the ribalds, or idle, disorderly persons of his day, says,

The duel huem *afretye*
Rau other aroste.

Atwist is evidently from the past participle of the verb *twisan*, which Mr. Tooke properly deduces from *twy*, two; but it is somewhat extraordinary that this very instance should not have shown him the error of his notion, that words in their Grammatical transitions from one Part of speech to another, undergo no change of signification. And it is the more remarkable, because Wallis, of whose Grammar Tooke speaks with some respect, has given three curious stanzas of his own composing, on the word *twist*, with a view of showing the variety of significations which may be expressed by English words of similar origin:

1.

When a twister, a twisting will twist him a twist,
For the twisting his twist he three twines doth intwist;
But if one of the twists of the twist doth untwist,
The twine that untwisteth, untwisteth the twist.

2.

Untwirling the twine that untwisteth between,
He twirls with his twister the two in a twine;
Then twice having twisted the twines of the twine,
He twitcheth the twine he had twined in twain.

3.

The twain, that in twining before in the twine
As twins were intwisted, he now doth untwine,
'Twixt the twain intertwisting a twine more between
He twirling his twister makes a twist of the twine.

The proof that these words, alliterative as they are in sound, and identical in origin, do nevertheless express a great variety of conceptions, is very ingeniously given, by exhibiting them in a Latin translation, in which the same care is taken to avoid similitude of expression, as in the former case to observe it.

1.

Quum restiarius aliquis, conficiendis torquendo funibus jam occupatus, vult sibi funem tortilem contorquendo conficere; quò hunc sibi tortilem funem torquendo conficiat, tria contortu apta filamenta complicando invicem associat; verùm si ex contortis illis in fune filamentis unum fortè se explicando complicationi eximat; hoc ita se explicando dissocians filamentum, funem torsione factum detorquendo resolvit.

2.

Ille autem celeriter evolvendo retexens intermedium illud quod se explicando dissociaverat filamentum, versorio suo torsionis instrumento, duo reliqua celeri volvens turbine contorquet, funiculum ex binis filamentis inde conficiens. Tum verò, quum jam secundâ vice torquendo convolverat funiculi bi-chordis bina filamenta; quem ex binis filamentis torquendo concinnaverat funiculum raptim divellendo dirimit.

3.

Tandem, quæ torquendo pridem in funiculo bimembri filamenta duo, tanquam gemellos, unâ consociaverat torquendo, jam detorquendo dissociat; et binis illis filamentum adhuc aliud intermedium interserendo consocians, versorium ille suum gyro celeri fortiter versando, ex funiculo bimembri plurimembrem torquendo conficit funem.

The participles hitherto mentioned have the form of past time; but we also, though less frequently, see those which have the form of present time used in like manner Adverbially; as "stark *staring* mad," "*roaring* drunk," and, in Shakspeare, more elegantly, "loving jealous."

——— I would have thee gone,
And yet no further than a wanton's bird,
Who lets it hop a little from her hand,
Like a poor prisoner in his twisted gyves,
And with a silk thread pulls it back again,
So *loving jealous* of his liberty.

But in all these cases the specific notion of time does not attach to the participle. When it becomes an Adverb, it loses that property; because it either modifies a verb, and then the time is expressed in the verb itself, or it modifies an adjective, and then there is no expression of time necessary.

Numerals.

3. The *numeral* pronouns supply a class of Adverbs, which are not very abundant in any Language. Verbs of action represent conceptions which may be often repeated. If it be meant to limit the action to a single instance, the conception of the number *one* must be expressed, and so of any other number, and to this is added, either expressly, or, at least, in the Mind, the conception of time. Thus we say, "he marched *six times* through Spain;" "he conquered more than *twenty times* in pitched battles;" "he was *twelve times* crowned with laurel." In most Languages, it is unnecessary to express the conception of time in connection with the lower numbers, the numerals themselves supplying an inflection, by which that conception is perfectly understood. Thus are produced our Adverbs *once*, *twice*, *thrice*, which are no other than the old genitives, *onis*, *twyis*, *threyis*. The Latin Language is more felicitous in this respect; it has *decies*, *vicies*, *centies*, and *millies* to express ten times, twenty times, a hundred times, and a thousand times.

In a Poem of the time of Henry VI. entitled, "*How the wyse man taght hys son*," (Harl. MSS. 1596.) is the line

For and thy wyfe may *onys* aspye,

In *Kyng Alisaunder*,

Twyes is somer in that londe.
* * * *
Ye haveth him *twyes* overcome.

With respect to the Adverb, *once*, however, it is to be noted, that as *one* is not always opposed to *two* or *three*, or any specific number, but sometimes merely to *many*; so *once* does not always signify "at one time," as opposed to two, three, or any other number of times, but merely "at some time" different from the present. Thus, when Wordsworth says of Venice,

Once did she hold the gorgeous East in fee,

he means to contrast the greatness of a former time with the degradation of the present. As if he had said, although at this present time she lies so low, there was *one* other period, at least, in her History, which presented a far different picture. At *that* time she was rich and great, famous and powerful—

——— *Now* lies she there,
And none so poor to do her reverence.

Nor is this signification confined to the time past. *Once* equally means some uncertain time as applied to the future. Thus, in the *Merry Wives of Windsor*—

I pray thee, *once* to night, give my sweet Nan this ring.

Nearly the same effect is given in Latin to the Adverb *olim*, which means some one point of time, either past or future; and seems to have the same connection with the relative article, as our word *once* has with the positive; for *olim* appears to be derived from *olle*, which the early Romans used for *ille*, and which, in the plural, was written *oloe*, as in the Royal Law: *Si parentis puer verberit, ast* OLOE *ploransint.*

The numerals hitherto spoken of are those called cardinal; but the ordinals also supply a certain class of Adverbs, as *thirdly*, *fourthly*, *fifthly*, &c. which are formed from the adjectives *third*, *fourth*, *fifth*, &c. by adding the termination *ly*, before explained. In the Latin Language, the correspondent words *tertiò*, *quartò*, &c. are manifestly the adjectives *tertius*, *quartus*, &c. with the termination of the ablative case. In English, too, we use the adjective *first*, Adverbially, without any alteration. It is a circumstance worthy of note in the History of Language, that the first two of the ordinal numbers generally appear not to be taken from the names of the cardinal numbers; thus we do not say in English the *oneth*, the *twoth*, nor in Latin *unitus*, *duitus*, nor in Greek ἑνοτὸν, δυιτὸν; but in these Languages respectively, *first*, *second*, *primus*, *secundus*, πρῶτον, δεύτερον; and when we look to the etymology of these words, we shall be inclined to suspect that they are in their origin simpler, and therefore, perhaps, earlier than the adjectives taken from the ordinal numbers. The word *first* is manifestly the superlative of *fore*, the first, being, of course, the *for-est*, or that which is *before* all others. Of this word, however, we shall have occasion to speak more at length when we come to consider the preposition and conjunction *for*. The Latin *primus* is in like manner the superlative of the old word *pri*. Scaliger, speaking of the word *primus*, says, *superlativum est; nam* PRI *vetus vox fuit*, sicut NI: postea latiore vocali fusæ sunt NE, PRÆ, *unde Adverbium*, PRIDEM; *comparativum*, PRIUS; *superlativum*, PRIMOM; *nam ab Adverbio, pridem, primum qui ducunt, errant*. And elsewhere, *ex* DI *factum est* DE; *sicut ex* PRI, PRÆ; *et sicut ex* NI, NE. Vossius observes, that *præ* was connected with an old adjective *præs*, present, that is, *before* the persons assembled; for when the names were called over at the public meetings, each individual answered *præs*. The Greek πρῶτον is in like manner the superlative of πρῶ, which is found in various shapes, but most simply in the preposition πρὸ, answering exactly to the Latin *præ*, before, either with regard to time or place, and secondarily as to order, or what we call *preference*. The word πρῶ, indeed, is used for the *first* dawn of day; but this appears to be merely a contraction from πρωΐ, which, however, is undoubtedly connected with πρὸ; nor can there be much doubt that the three radicals to which we have alluded, *viz*. *pri*, *pro*, and *for*, have all one common origin.

Demonstrative pronouns.

The *demonstrative* pronouns, with which we rank the *subjunctive*, form, in most Languages, a large class of Adverbs, the construction of which is elliptical. The words *here* and *there*, *hence* and *thence*, *hic* and *illic*, *hinc* and *illinc*, for instance, are manifestly in their origin demonstrative pronouns, equivalent to the words *this* and *that*; but, by use, they have come to signify "at *this* place," "at *that* place;" "from *this* place," "from *that* place;" the substantive "place" being clearly understood by the Mind. Neither can it be doubted that the Latin Adverbs *quum* and *quo* are the subjunctive pronoun *qui*, with the terminations of the accusative and ablative case; which word *qui* is probably the same in origin with the Gothic *hwo*, the Saxon *hwa*, the Scottish *quha*, and the English *who*.

It happens, that the English Language is not perfectly systematic in regard to the pronouns which it has adopted for Adverbial purposes; and the same may be said of most other Languages. We have the simple Adverbs just mentioned, which form three distinct classes, with reference to place, distinguishing the place where we are, from another definite place, and supplying an interrogative for the place which we know not, which interrogative is also a subjunctive.

The first of these is *here*, the second *there*, and the third *where*. It happens too, with regard to place, that each of these three forms has three varieties to express "at a place," "to a place," and "from a place;" and all these are variously compounded with several other words or particles, *fore*, *ever*, *soever*, &c. Some of the words which form Adverbs of place, also become Adverbs of time, manner, cause, &c.; but these latter ideas have some few Adverbs which are peculiar to themselves, agreeing, nevertheless, in principle and derivation, with the Adverbs of place. Hence may be formed the following Table of the simple Adverbs of this kind:

Place	here	there	where?
	hence	thence	whence?
	hither	thither	whither?
Time		then	when?
Manner		thus	how?
Cause			why?

The three classes into which we have distributed these Adverbs, have not always been thus accurately distinguished. In our old Language, we shall find the prepositive forms *here* and *there* often interchanged with the subjunctive or interrogative form *where*; yet it is clearly evident that these distinctions must have always existed in point of signification, however inaccurately or imperfectly expressed.

Here.

The word *here* is not only used in its simple form, but in a variety of compounds, as, *hereafter*, *hereabout*, *hereat*, *hereby*, *herein*, *hereinto*, *hereof*, *hereon*, *hereupon*, *hereto*, *hereunto*, *heretofore*, *herewith*, *heirfoir*, *heirintill*, &c. In the simple form it is principally confined to the signification of "this place;" whereas, in the compounds it generally signifies "this time," "this thing," "this event," or the like. The cognate word *hier*, in German, does not follow exactly the same variations of meaning. Both in its simple and compound forms it principally refers to place, as *hieran*, *hieraus*, *hierdurch*, *hierein*, *hierinnen*, *hierober*, *hierunter*, &c.; and so, *heran*, *herbey*, *herein*, &c.; though some compounds are more general in their application, as, *hierum*, *hiervon*, *hierzu*. In both Languages, however, it is manifest that the word *here*, *hier*, (or *her*, intrinsically signifies no more than the word *this*; and that the other significations, such as "place," "time," "event," "reason," or the like, are supplied by the Mind, according to the context. The words *heirfoir* and *heirintill*, being of the old Scottish Dialect, now obsolete, it may be proper to explain them by some instances. In the Scottish Act of Parliament, A. D. 1493, the King (James IV.) recites the inconveniences of alienating the Royal domains, thus, "Sen it is leuit and permittit be the constitutiounis and ordinancis of lawis ciuile and canon, that personnis constitute in youtheid and tender age quhilks ar greitlie dampnageit and skaithit in thair heritage be imprudent alienatiounis, &c. may at thair perfectioun of age mak reuocatioun, &c.; *Heirfoir*, we, James, be the grace of God, King of Scottis, &c. reuoks, reducis, cassis, and annuls, all infeftments," &c. In this example, the word *heirfoir* is simply "for this," the word "cause" or "reason" being understood. Again, in the Act of 1554, "like

Grammar. as and all the hieast and maist vailyeable thing is of the premissis had bene expressit *heirintill*," where the word *heirintill* signifies " in this," or " within *this*," *viz.* " writing," or " statute." We cite the words of these Acts from the careful copies of the original documents lately printed by order of Government, which present a very valuable record of the state of the Scottish Language from 1424 to 1592; and in which collection we also find many other Adverbial compounds of the word *heir*, as *heirof, heirupone, heirtofoir, heirafter, heiranent*, &c. in all which, *heir* signifies *this*, although, in some instances, it is applied exclusively to place; in others, to time; and, in a third class, to this time, this place, this thing, &c. indifferently.

The pronouns are among the simplest, and probably the most ancient words in all Languages; and hence we must not be surprised to find some difficulty in tracing the pronominal Adverbs to their proper origin. However, it can hardly be doubted that the elements of the word *here* are to be discovered in *he* and *er*, which occur in many of the Northern Languages, as signifying this person or these persons, this thing or these things, so that the radical conception is what we express by the word *this*. First, the element *he* occurs, in Anglo-Saxon and old English, in the words signifying he, she, they, and their respective cases. The Anglo-Saxon pronoun personal is *he, heo, hi*, he, she, they; and the very word *here* occurs for the genitive plural, as *heom* does for them. The same, or similar words are frequent in old English writers. In the *Vision of Piers Ploukman*—

> Hermets on a heape with hoked staues
> Wenten to Walsingham, and *her* wenches after.
> * * * * * *
> Cokes and *her* knaues crydee, hote pyes, hote:

That is, " *their* wenches," and " *their* knaves," or " boys."

In Chaucer's *Parson's tale*, " Certes this vertue makith folk vndertake hard and greuous things hy *her* own will;" that is, " *their* own." In an ancient Ballad, probably of the XIIIth century, beginning " In Mayhit muryeth," (Harl. MSS. 2253. fo. 71.)—

> Ynot non so freoh flour
> Ase ledies that beth bryht in bour,
> With loue who mihte *hem* bynde:

That is, " I know no flower so fresh as ladies who are bright in bowers, to those who may bind *them* with love." In a dialogue between a body and a spirit, of the same date, (Ibid. fo. 57.) " *he* wolleth" occurs for " they will." This word was sometimes written *heo*, as, in a satirical Poem against the Ecclesiastical lawyers, (Ibid. fo. 71.)—

> *Heo* shulen in helle on an hok
> Honge there fore.

And sometimes *hi*, as in another manuscript in the Harleian collection, (No. 2277. fo. 195.)—

> Tho *hi* dude *here* pelrynage in holie stedes faste,
> So that among the Sarazyns ynome *hi* were atte laste:

That is, " *they* did *their* pilgrimage, so that *they were* taken at last."

In the *Lai le frain*, which is a translation from the Norman-French of the celebrated Poetess Marie, we have *he* and *hye*, for " she;" and *him* for " her:"

> The maiden abode no lengore,
> Bot yede hir to the chirche dore;

> O Lord, *he* seyd, Jesu Crist, &c.
> * * * *
> *Hye* loked vp, and by hir seighe
> An asche, by hir, fair and heighe.
> * * * *
> A litel maiden childe ich founde,
> In the holwe assche therout,
> And a pel *hem* about.

Adverbs. The other element, *er*, is found in the modern German *er*, he, and in the Islendic *er*, am, is, and who; as in the *Edda* of Snorro, *Feyma heiter su kona* ER *ofram* ER *svo sem ungar meyar eru*. " Feyma is called the woman *who* modest *is*, as the young maidens are." In the Frankish and Alamannic the demonstrative and relative pronouns of the third person are *er, her*, and *ir*. Thus, in the Frankish of OTFRID the Monk, Ex *gibot then uuinton*, " *He* commanded the winds:" in that of TATIAN, ER *quam in sin eigan*, " *He* came to his own." In the Alamannic of Isidore, *Dhaz* ER *Jhesus uuardh chinennt*, " That *he* Jesus was named." These two elements, then, *viz. he* and *er*, are identical in signification; and are only redoubled for the sake of emphasis, which is a habit common to Barbarous nations, and to the illiterate in all Countries. Hence it is, that the French have their *ce-ci* and *ce-la*, and even *ce-lui-ci* and *ce-lui-la*; and that our own rustics commonly say *this here, that there, thick there*, &c. From this source undoubtedly come the Gothic, Anglo-Saxon, Danish, and Islandic *her*, the Frankish and Alamannic *hier, hiar, hiera*, the modern German and Dutch *hier*, and the English *here*, all used to signify, " at *this* place," although the simple and radical meaning of them all is simply " *this*." The various explanations which are given to the Adverb *here* by Dr. Johnson only serve to show that the conception of a distinct and particular place is no necessary constituent in the meaning of the word. Thus *here* is opposed to a future time, as well as to a different place, by BACON, in his advice to Villiers: " you shall be happy *here* and more happy *hereafter*;" which might be paraphrased " in *this* life and in a life after *this*"—" in *this* world, and in a wurld after *this*"—" in *this* state of existence, and in a state of existence after *this*," always retaining, however, the conception expressed by the word *this*. So when the words *here* and *there* are explained hy Johnson " dispersedly; in one place and another;" as in another extract from Bacon: " I would have in the heath some thickets made only of sweet-brier, and honey-suckle, and some wild vine amongst; and the ground set with violets; for these are sweet, and prosper in the shade; and these to be in the heath *here* and *there*, not in order." The words *here* and *there* are still to be explained *this* and *that*; for the Imagination forms conceptions of places separate from each other, although quite indeterminately as to any specific external situation, and even as to number, except that the place signified hy the word *here* is an Imagination separate from that expressed by the word *there*. The indistinct process of the Imagination, therefore, in the passage above cited, may be explained by supposing an individual carelessly wandering over the ground which is to he ornamented, and occasionally stopping to say, I will have a thicket planted in *this* place and another in *that* place. The same expression occurs in a beautiful Sonnet by Shakspeare—

> Alas! 'tis true, I have gone *here* and *there*;

M

Grammar. which corresponds with the expression "ranged," in the preceding verses—

As easie might I from my selfe depart,
As from my soule, which in thy brest doth lye:
That is my home of loue. If I haue *rang'd*,
Like him that trauels, I returne againe.

Here and *there* are doubtless used indefinitely in such phrases; but not more indefinitely than the pronouns *this* and *that* might themselves be used, as in the Song,

This way, or *that* way, or which way you will;

and in DRAYTON's pleasing description of a winter evening's chat with his friend—

Now talk of *this*, and then discours'd of *that*,
Spoke our own verses twixt ourselves, &c.

Nay, even the pronoun personal is sometimes used with the same uncertainty of application; as in Chaucer's spirited description of a tournament, in the *Knyght's Tale*—

He rolleth under foote, as dothe a ball,
He foyneth on his feet with a tronchoun,
And *he* hurleth with his horse adoun,
He through the body is hurt and sith ytake.

In none of which instances is there any certain antecedent to the word *he*; and yet it stands first for one man, then for another, then for a third, and lastly for a fourth.

Hence and *hither* may be considered as cases of the word *here*; but perhaps it would be more accurate to treat these three words as different compounds of the element *he*, with *er*, *an*, and *der*. *Hence* is the Anglo-Saxon *heonan*, and the Frankish *hina*. It seems to be connected with the Islandic *han*, he, and *hin*, it; and with the syllabic *hin*, which, in various German compounds, signifies "from *this* place," "from *this* time," "at *this* time," "to that place," &c.; and which is used alone to signify any thing that is "gone *hence*;" "lost," or "annihilated;" as in the *Leonore* of BÜRGER—

O mutter, mutter, hin *ist* hin!
Verlohren ist verlohren!

So they say *er ist hin* for "he is dead;" *hinrichten* is to execute justice on any one, to put him to death; *hindag* is "this day;" *hinfort*, "henceforth," "from this time forth;" which is also expressed *forthin*. *Immerhin* is an exclamation answering to our "let it go," and meaning "be it ever *thus*, I care not;" as, *er mag immerhin schreyen*, "he may bawl as long as he likes." So *hinauf* and *hinab*, "above and below;" *hinein* and *hinous*, "within and without," mean respectively above *this* place, below *this* place, within *this* place, out of *this* place. *Hinfahren* is to go away, to go from *this* place; and, in the Frankish, *hinofahrt* is "death." Our English word hence, in old writings, is *hen*, *han*, *hin*, and *hennes*. In the Romance of *The Seuyn Sages*, we find,

A fend he is, in kinde of man;
Binde him, sire, and lede *han*.

Chaucer, in the *Knyght's Tale*, says,

The fires whiche on min auter brenne
Shal declaren er that thou go *henne*
This auenture of loue.

So in *Christ's Descent into Hell*—

Bring vs of this lothe lond
Louerd *henne* into thyn hond.

In the Scottish Act of Parliament, A. D. 1438, "that all the kinge's liegis be vnharmyt & vnscaithit of the said house & of thaim that inhabits theirin fra *hyn* furth." Adverbs.

Hither is the Anglo-Saxon and Gothic *hidre*. In the old English too it was often written with a *d*; as in Chaucer's *Monk's Tale*—

And if you list to herken *hiderward*.

So in two manuscript Poems in the British Museum, (Harl. MSS. 2253. fo. 64. and fo. 124.)—

Herketh *hideward*, and beoth stille.
* * * *
Herkneth *hideward* horsmen
A tidyng ichou telle.

And, in the Poem on *Christ's Descent into Hell*, Satan says,

Ne may non me worse do,
Then ich haue had *hiderto*.

There. *There*, *thence*, *thither*, are manifestly constructed on the same principles, and applied in the same manner as here, hence, and hither; and as we suppose the first element of *here* to be *he*, so we suppose the first element of *there* to be *the*, which, in the Anglo-Saxon, was prefixed as an article to substantives in all cases, and in both numbers; and which appears in various Dialects under the forms of *thei*, *thy*, *tho*, *tha*, all relating to the pronoun *that*. *Thei* is the Gothic conjunction "that." *Thy* in the old English compound *forthy*, signifies "for that," *viz.* cause. *Tho* is explained by Junius, *qui*, *illi*, and *tunc*, *viz.* "that person," in the plural; and "that place" used Adverbially; and he adds, that the Anglo-Saxon *tha* admits all these significations.

Tho, for "then," (see Warton, vol. i. p. 161.)—

The messengers *tho* home went.

Tho, for "when," (Harl. MSS. 2253. fol. 37)—

Tho Jhesu was to hell ygan.

Tha, for "those," (*The Seuyn Sages*, v. 3901)—

Al *tha* wordes ful well he knew,
He was so ferd him changed hew.

Thae, for "those." See the second volume of *The Antiquary*, (one of the recent Novels which so accurately delineate the manners and Language of Scotland,) p. 297—

Thae's your landward and burrowstown notions.

Tho, for "those," (Harl. MSS. 2253. fo. 55, 56.)—

Parmafey ich hold myne
All *tho* that bueth her yune.

There seems to be compounded of *the* and *er*; as *here*, of *he* and *er*; but however this may be, *there* manifestly agrees with the German *der*, which is a demonstrative and relative pronoun, as well as an article, and consequently answers to our *the*, *this*, and *who*. In like manner, the Anglo-Saxon *thære* or *thær* formed the genitive of the article, and also the demonstrative and relative adverb; as in the 4th chapter of *Joshua*, "Nyman twelf stanas on middan *thære* ea, *thær* tha sacerdas stodon, & habban forth mid eow, to eowre wicstowe, & wurpan hig *thær*." "Take twelve stones from (the) midst (of) *the* water, *where* the priests stood; and have (them) forth with you, to your abiding-place, and cast them (down) *there*;" in which passage we see *thære* and *thær*, answering to *the*, *where*, and *there* successively. So in the old English, *there* is

often used in two connected sentences, for *there* and *where*; as in Chaucer's *Wife of Bath's Tale*—

> *There* as wont to walken was an elfe,
> *There* walketh now the limitour himself.

It might not unreasonably be surmised, that where the operations of the Mind are so distinct, as those indicated by a demonstrative and a subjunctive pronoun or Adverb are, they would necessarily require expressions equally different; but a careful attention to the History of Language will show us that it differs very widely in this respect from its Philosophy. It is for want of having sufficiently considered this circumstance that we find Grammarians so often at a loss to account for different idioms, and giving reasons for them which are purely imaginary, not to say absurd. It is, no doubt, a great excellence in a Language, to mark, by distinct expressions, the distinct operations of the Mind, and the more nicely this is done, the more accurate and expressive does a Language become; but this is generally the result of time and of an undefinable sense of inconvenience, which induces men to inflect and vary words, as it were, insensibly, and to assign to the various inflections, though of similar origin, different effects. In no Language, however, has this Principle been carried into full operation; and hence we see the different meanings of a word, and the different Parts of speech which it constitutes, passing into each other by gradations, which, at first sight, it is not always easy to explain. Thus, in Greek, the subjunctive pronoun, or, as some call it, the subjunctive article, ὅς, is sometimes said to be used for the prepositive ὁ; sometimes for τίν interrogatively; and sometimes for ἀυτός. Again, Ὅστις sometimes answers to the Latin relative *quis*, and sometimes to *quisquis*. The Adverb Ὅπου, besides the common signification "where," answers to "whither;" and in argument, to "since;" and in description, to "in this place," or "in that place." So, ὅτε, "when," signifies also "since," like the Latin *cum*: and the examples of this kind are infinite. We shall not, therefore, be surprised to find considerable diversity from the modern idiom in the following, and many similar instances:

Ther is used for *the*, *that* or *them*; as, in *The Seuyn Sages*, *therewhile* for *the* while:

> *Therwhile*, sire, that I tolde this tale,
> Thi sone mighte tholie dethes bale.

GAWIN DOUGLAS has "*thare aboue*" for "above *that*;" and "*tharon*" for "on *them*."

In the old Scottish Dialect *thir* was used for *these*, or *them*; as in the Act of 1424, "*thir* ar taxis ordaynt throu the counsaile of Parliament." So in DUNBAR's *Goldin Terge*, written about a century afterward—

> Full lustiely *thir* ladyis all in feir
> Enterit into this park of maist pleseir.
> * * * * *
> And every ane of *thir* in grene arrayt
> And harp and lute full mirreyly they playt.

In the same Dialect we find *thairto* and *thairfra*, *thairfoir* and *thairefter*, *tharapone*, *thairuntill*, &c.

Chaucer uses *therto* in the sense of "moreover," or "in addition to *that*," as in the *Rime of Sir Thopas*—

> He couthe hunt at the wildé dere
> And ride an hauking forby the riuere
> With grey goshauke on honde:
> *Therto* he was a good archere.

Therefore, which, in modern times, is commonly used conjunctively, occurs in a rude old English Poem before quoted, (Harl. MSS. 2253. fo. 71.)—

> Heo shulen in helle on an hok
> Honge *there fore*.

In short, comparing the different authorities, ancient and modern, we find that the word *there*, however variously spelled, did not originally relate to place exclusively, but was equally applied to time, to persons, and to events: and the same may be said of *thence* and *thither*. *Thenceforth*, which we use with reference to time, agrees with the old Scottish phrase *fra thin* furth, as in the following passage in the Act of 1503, which is, on many accounts, worthy of notice:

> "It is statute and ordanit that *fra thin furth* na baroun, frehaldar, nor vassal, quhilkis ar within ane hundreth merks of this extent that now is, be compellit to cum personaly to the parliament, but gif it be that our souerane Lord write speciale for thame. And sa (sal) no be unlawit for their persons, and thai send thair procuratours to answer for thame, with the barouns of the schire, or the maist famous personis. And all that ar aboue the extent of ane hundreth merks to cum to the parliament, vnder the pane of the auld vnlaw."

Thither was, in the Anglo-Saxon and old English, *thider*, as in the Poem often quoted, (Harl. MSS. 2253. fo. 55.)—

> God for his moder loue,
> Let us neuer *thider* come.

And as they had *hiderward* for "hitherward," or "toward this place," so they had *thederwart* for "thitherward," or "toward that place:" as in the ludicrous Poem called *The Huntyng of the Hare*:

> Thei toke no hede *thederwart*,
> But euery dogge on oder start.

Where, *whence*, and *whither*.—These words have also a similar analogy, together with this further peculiarity, that they serve indifferently for interrogatives and subjunctives. Thus in the interrogative: Where.

> They continually say unto me, *where* is thy God?—Psal. xlii. 3.
> And he said, Hagar, Sarai's maid, *whence* camest thou; and *whither* wilt thou go?—Gen. xvi. 8.

And again in the subjunctive—

> Let no man know *where* ye be.—Jer. xxxvi. 19.
> I wist not *whence* they were.—Josh. ii. 4.
> He went out, not knowing *whither* he went.—Heb. xi. 8.

We have already seen that the subjunctive force of the word *where* was not peculiar to it, but was sometimes expressed by the word *there*. We do not find this to be the case in English with the interrogative force of the same word; but in Greek the relative pronoun τίν is also an interrogative; as in St. Mark's Gospel, ch. ii. ver. 6, 7: Ἦσαν δέ ΤΙΝΕΣ τῶν γραμματέων ἐκεῖ καθήμενοι καὶ διαλογιζόμενοι ἐν ταῖς καρδίαις ἀυτῶν· ΤΙ' οὗτος οὕτω λαλεῖ βλασφημίας; ΤΙΣ δύναται ἀφιέναι ἁμαρτίας, εἰ μὴ εἷς ὁ Θεός;—" But there were *certain* of the Scribes sitting there, and reasoning in their hearts, *why* doth this man thus speak blasphemies? *Who* can forgive sins, but God only?"—Hence it is clear, that the interrogative effect of a word does not require a peculiar form, any more than the subjunctive. So the Latin *quidam*, which means "a certain person," and *aliquis*, which means "some one," are reciprocally connected with the interrogative *quis*, and the subjunctive *qui*. SCALIGER was of opinion that the Latin *quis* and *qui* were the Greek καὶ ὅς and καὶ ὁ: and TOOKE, probably thinking to improve on this etymology, has only gone further in error. He says, "As *ut* (originally written *uti*) is nothing but ὅτι; so is *quod* (anciently written *quodde*) merely καὶ ὅτι.—

Quodde, *tuas laudes culpas nil proficis hilum.*
LUCILIUS.

"*Qu* in Latin being sounded not as the English, but as the French pronounce *qu*, that is, as the Greek K; *καὶ*, by a change of the character, not of the sound, became the Latin *que*, used only enclitically indeed in modern Latin. Hence *καὶ ὅτι* became in Latin *qu'otti, quoddi, quodde, quod.*"—The only foundation for all these conjectures seems to be, that in the very nature of a subjunctive pronoun something equivalent to a conjunction is implied; and as to the assertions respecting the Roman pronunciation they are perfectly gratuitous. It is not very probable that the ancient pronunciation of *qu* was the same as of K; on the contrary, it more probably resembled that of χ, or rather of the Gothic ⊙, which our Anglo-Saxon ancestors expressed by *hw*, the old Scottish writers by *quh*, and we by *wh*. Scaliger and Tooke forgot, that if their explanation might be thought to account for the subjunctive pronoun, or conjunction, it left the interrogative pronouns and Adverbs quite unexplained; and the fact seems to be, that the Latin Language originally agreed with the Gothic and other Northern Languages in employing the articulation marked by the Æolic digamma, where the softer Greek Dialects omitted that articulation; thus the Greek *οἶνον* was the Latin *vinum* and Gothic *wein*; the Greek *οἶ* was the Latin *væ* and Gothic *wai*; and lastly, the Greek aspirated pronouns *ἡ, ὁ*, were the Latin *quæ, quo*, and the Gothic *hwa, hwo*.

It is manifest that *where* did not originally refer to *place* alone, any more than *here* or *there* did; but, like those words, was originally a pronoun signifying *this* or *that*; for in its composite forms it often signifies no more than those pronouns, the substantive to which it refers being usually expressed, but sometimes understood. Thus we have *whereabout*, for "about *which* business"—

Let no man know any thing of the business *whereabout* I send thee.—1 Sam. xxi. 2.

Whereto, for "to *which* thing"—

It shall prosper in the thing *whereto* I sent it.—Isaiah lv. 11.

Whereby, for "by *which* name"—

There is none other name under heaven given among men *whereby* we must be saved.—Acts iv. 12.

Wherefore, for "for *which* cause"—

What is the cause *wherefore* ye are come?—Acts x. 21.

All these compounds may be employed interrogatively, (and indeed the subjunctive use of some of them has at present become rather obsolete,) but in this form also they are not necessarily significant of "place."—Thus *whereby* is used for "by *what* means?"—

Zacharias said unto the angel, *whereby* shall I know this?—Luke i. 18.

Wherefore, for "for *what* reason?"—

Now he is dead *wherefore* should I fast?—2 Sam. xii. 23.

It is to be observed, however, that there are certain Adverbs compounded with *where*, which cannot be used interrogatively, such as *whereas, wherever, wheresoever*; but the reason is that in these, as well as in *whensoever, whithersoever*, &c. the pronouns *as* and *so*, and the word *ever*, necessarily give them a relative force and effect:

Have ye not spoken a lying divination, *whereas* ye say, The Lord saith it?—Ezek. xiii. 7.

Ye have the poor with you always: and *whensoever* ye will ye may do them good.—Mark xiv. 7.

The Lord preserved David *whithersoever* he went.—2 Sam. viii. 6.

It would be impossible to express these passages interrogatively, "whereas say ye?" "whensoever will ye?" "whithersoever did he go?" not on account of the meaning of the words "where," "when," or "whither," but of the others with which they are compounded.

In these compounds, the particles or words *as* and *so* seem to have been originally used superfluously, as the particle or word *that* was in many similar combinations. Hence, on the one hand, we have *where* for *whereas*; and on the other, we have *where* and *that* for *where*: and, in like manner, we find many such expressions as *how that, which that*, &c. *Where* for *whereas*, occurs in the preambles of many old Statutes. In a remarkable document existing among the Rolls of Parliament, A. D. 1461, we find it so used. The document to which we refer is called *Cedula formam actûs in se continens*, and was exhibited in the first Parliament summoned by King Edward IV. After reciting many alleged crimes, on the part of Henry VI. and his followers, it contains a judgment, or law of attainder, against the latter, and of forfeiture of the Duchy of Lancaster against Henry. Of the recitals, some are introduced by the word *forasmoch*, and others by the word *where*: thus, "*Forasmoch* as Henry Duc of Somersett purpossing ymaginyng & compassing, of extreme & insatiate malice & violence to distroy the Right Noble and famous Prynce of wurthy memorie Richard late Duc of Yorke, Fader to our Liege & Soverayne Lord Kyng Edward the fourth, & in his lyf very King, in right, of the Reame of England, &c. and also Thomas Courteney late Erle of Devonshire, &c. &c. (naming various persons) with grete despite & cruell violence horrible & unmanly Tyrannye murdred the seid right noble Prynce Duc of York; and *where* also Henry Duc of Excestre, Henry Duc of Somersett, &c. &c. (naming the same and other persons) reared warre ayenst the same King Edward thir right wise true & naturall liege Lord, &c. *It be declared* and adjudged by the assent & advis of the Lordes Spirituelx & Temporelx & Commyns," &c. &c. In the more ancient Parliamentary records, which were in French or Latin, preambles of this kind were introduced by the old French word *come*, or by the Latin *cum*, both which words are the ancient *quom* from *qui*, who.

Where that, in CHAUCER's *Knyght's Tale* (see Harl. MSS. 7335.)—

Duk Thesens him leet out of prison
Frely to goon *wheer that* him list al;

and in DUNBAR's *Goldin Terge*—

Full lustily thir ladies, all in feir,
Enterit into this park of maist pleseir,
Quhair that I lay heilit with leivs rank.
* * * * *
Then crap I throw the brenches & drew neir
Qhuar that I was richt suddenly affrayit.

How that (Harl. MSS. 7333. fol. 147. b.)—

How that the foule fende asseylithe the soule.

Which that (Harl. MSS. 7333. fol. 203.)—

Mysing vpon the restlees besinesse
Which that this troubly worlde hath ay in honde.

Grammar.

So is, in like manner, compounded with *where*, *who*, *what*; as in the English *whereso* and *whoso*, and the Scottish *quha sa*, which mean respectively "wheresoever" and "whosoever:"

1. And redde *whereso* thou be, or ellis songe.
CHAUCER. *Troilus*.

2. ——————— He inclos'd
Knowledge of good and evil in this tree,
That *whoso* eats thereof forthwith attains
Wisdom. MILTON. *Par. Lost*.

3. It is ordanyt, that all men busk thame to be archaris fra thai be xii yeris of eilde. And *quha sa* vsis not the said archary the lorde of the lande sal raiss of him a wedder.
Scottish Act of Parl. 1424.

Nor is it extraordinary that the words *that*, *so*, and *as* should be used in a similar manner; for, as Mr. Tooke has justly observed, "*as* is an article, and means the same as *it*, *that*, or *which*." And again, "the German *so*, and the English *so*, though in one Language it is called an Adverb, and in the other an article, or a pronoun, are yet both of them derived from the Gothic article *so* or *sa*, and have, in both Languages, retained the original meaning, *viz. it, that*." But on these words we shall presently have occasion to make some further remarks.

Where is also used with the pronominal adjectives *any*, *every*, *no*, but still adverbially, as in the common expressions *anywhere*, *everywhere*, *nowhere*; and being thus limited to some determinate signification in respect of place, it is neither subjunctive nor interrogative:

Those subterraneous waters were universal, as a dissolution of the exterior earth could not be made *anywhere* but it would fall into waters. BURNET. *Theory of the Earth*.

'Tis *nowhere* to be found, or *everywhere*. POPE.

In the old English it was even used with a simple adjective, as *wide-wher*.

And thus I went *wide-wher* walking mine one.
LONGLAND. *Piers Plow.*

Whence is sometimes found, in the old English, unnecessarily cumulated, as it were, with *thence*; nor is this any thing more than we have already observed to be common in the formation of pronouns and pronominal Adverbs in all Languages, as *ce* and *ceci* in French, *ita* and *itaque* in Latin, &c.

Thus, in the Romance of *Syr Ypotis* (see Warton, vol. i. p. 208)—

The emperour, with milde chere,
Askede him *whethence* he come were.

And the same may be observed of *thedence* in the Romance of *Alisaunder* (see Warton, vol. i. p. 309)—

Thedince so ondrace with his ost.

In the West of England, to this day, we find that the country people use for *hence* and *thence*, the words *hereance* and *thereance*, which are manifestly similar and unnecessary cumulations of expression.

Whither is confounded with *ward* in our old writers as well as *hither* and *thither*; but though the latter two are noticed by Johnson, the first is not so:

——— A puissant and mighty pow'r
Of gallow-glasses and stout kernes
Is marching *hitherward* in proud array.
SHAKSPEARE. *Hen. VI.*

By quick instinctive motion, up I sprung
As *thitherward* endeavouring. MILTON. *Par. Lost.*

Who so wolde myghte ride
Whiderwardis so they wold. *Romance of K. Alisaunder.*

Adverbs.

From what has been said, it is abundantly clear that the Adverbs *here*, *there*, *where*, *hence*, *thence*, *whence*, *hither*, *thither*, and *whither*, although in their modern and uncompounded use they principally express a conception of "place," yet did not really include the name of any such conception in their original signification, but were the mere pronouns *he*, *this*, and *what*, diversely compounded, and assigned by use to separate and distinct significations.

Then. When.

The very same is to be observed of the Adverbs *Then* and *When*, which we have above noted, as principally signifying time. We have not, indeed, the word *Hen* for "at this time," though it occurs in old English for *hence*, i. e. from this place. Thus, in the scoffing Ballad made on the defeat of Henry III. at *Lewes*, in 1264, and which, from its tenour, must have been composed very soon after the event, we find the following lines:

He hath robbed Engelond the mores ant the fenne
The gold ant the selver ant yboren *henne*.

Hann, in the Islandic, is "he," and *hun* is "she;" and STIERNHELM, (*Gloss. Ulph. Goth.* p. 85,) speaking of the Gothic word *hana*, as in *hana hrukida*, "the cock crew," (Matth. xxvi. 74), says, *Omnis avis mascula dicitur* HANA, *ab* HAN, *ille*, *et fœmina* HÖNA, *ab* HON, *illa*; "every male bird is called *hana*, from *han*, he; and every female bird *höna*, from *hon*, she." Hence we may infer that the element *en* was compounded in some of the Northern Dialects, as we have already seen that *er* was, *viz.* with *he*, *the*, and *who*, producing *hen*, *then*, and *when*, as well as *here*, *there*, and *where*, all of them originally pronouns, and all used in a restricted sense by an ellipsis of the words time, place, &c. as Adverbs.

In the Gothic, *Than* is both "then" and "when," and *yuthan* is used for "now." *Than* is also used for *autem*, δὲ, "but;" and it is manifestly nothing more than the article or pronoun *thana*, or *thanei*, answering to the Greek τὸν or ὃν, as *Seimon* THANA *hailanan Zeloten*, Σίμωνα TO'N καλούμενον Ζηλωτήν, "Simon, *who* (was) called Zelotes," (Luke vi. 15); THANEI *wildedun*, "ON ἤθελον, "*whom* they would," (Matth. xxvii. 15.) *Thon*, for "those," is still used in many parts of Scotland; *thynfurth* we have seen in the old Dialect of that Country, for "thenceforth," which, in the Parliamentary Articles of 1461 above quoted, is written "thensforth:" and as *henne* was used in old English for "hence," so *thenne* was used for *thence*, i. e. from that place; as in *Christ's Descent into Hell*:

Nas non so holy prophete,
Seththe Adam & Eue the appel ete,
Ant he were at this worldes syne,
That he ne moste to helle pyne:
Ne shulde he neuer *thenne* come,
Nere Jesu Crist Godes sone.

When is the Gothic *hwan*, which is used for the Latin *quando*, *quoniam*, *quantum*, *quam*, and is manifestly the same as *hwana*, *quem*, "whom;" as HWANA *sokeith*, "*whom* seek ye?" (John xviii. 4.) As the Gothic *than* and *hwan*, and the old English *there* and *where* were often used convertibly, so were *then* and *when*; and in the Harleian MSS. (No. 2253. fo. 55. b.) we find *the* for when:

The he com there, tho seide he.

Why.

It will not be necessary to use much argument in proof of the identity of origin between *Why* and the

words before mentioned, *where*, *when*, &c.; it is manifestly only another form of the pronoun *who*. In modern usage we do not oppose *thy* (in the sense of *this cause*) to *why*; but this mode of expression occurs in the old words *forthy* and *withthy*. *Forthy* occurs in the Scottish Act of 1424, in the two senses of "because" and "therefore." So in BARBOUR's *Bruce*—

> But God that most is of all might
> Preserved thame in his forsight
> To venge the harm and the contrair
> That those fell folk and pautener
> Did to simple folk and worthy,
> That couth not help themselven; *forthy*
> They were like to the Maccabeis.

The same author seems to use *nought for thy* in the sense of "nevertheless," as

> And *nought for thy*, thocht they be feil,
> God may richt weil our werdes deil.
> * * * * *
> And *not for thy* thair faes then were
> Ay twa for ane that they had there.

So he uses *with thy* for "provided," or "on this condition"—

> And I sal be in your helping
> *With thy* ye give me all the lond
> That ye have nou into your hond.

In all which instances *thy* is simply *this*, *viz*. cause, reason, or condition, all which substantives are understood by the sort of ellipsis already explained.

How. *How* is simply the pronoun *who*, or *hwa*, sometimes written in old English *ho*; as in the Harleian MS. No. 2277. fo. 1.—

> Seinte Marie day in Leynte, among
> Alle other dayes gode
> Is ryt forto holde heghe
> *Ho* so him vnderstode.

And as we have seen the pronoun *that*, and the Adverb *as*, used convertibly, so we find *hou* in the old Scottish Dialect used where we should emply *so*, or *as*; e. g. *housone*, for "so soon as"—

> That *housone* ony truble, questioun, or causis happynnis to be movit—than incontinent it salbe lesum, &c.
> *Scottish Acts*, A. D. 1554.

We have thus traced, at some length, the English Adverbs of place, time, &c. which are in truth no other than the demonstrative and subjunctive pronouns, appropriated by custom to certain distinct significations; but though the particular applications are matter of mere idiom, and vary, as we have seen, considerably in the same Country at different periods; yet in most, if not all Languages, the same general Principle is to be traced. In most, if not all, the words which are employed as Adverbs of time, place, manner, and cause, are pronouns with little or no variation of form.

In Latin, from the pronouns *is*, *ea*, *id*, come the Adverbs *ibi*, *alibi*, *ibidem*, *inde*, *proinde*, *ita*, *itaque*, *ideo*, *iccirco*, *eo*, *adeo*, *eorsum*, *uspiam*, *nusquam*, &c. From *hic*, *hæc*, *hoc*, come *hinc*, *huc*, *adhuc*, *huccine*, *horsum*, *hodie*, *antehac*, *posthac*, *hacpropter*, &c. From *ille*, *illa*, *illud*, come *illic*, *illico*, *illuc*, *illinc*, *olim*, &c. From *qui*, *quæ*, *quod*, come *quo*, *quoque*, *quam*, *quando*, *quia*, *quamvis*, *quare*, *quin*, *quidem*, *cum*, *cur*, and probably *ubi*, *alicubi*, *ubivis*, &c.

Ibi, says MARTINIUS, in his *Lexicon Philologicum*, A. D. 1655, is from *is*, as αὐτόθι from αὐτος; and *ibidem* is from *ibi* and *idem*. The same author observes, that *huc* was anciently written *hoc*, as in the VIIIth *Æneid*, Hoc *tunc ignipotens*, &c. To which VOSSIUS adds, that *ad huc* meant *ad hoc*, (subanditur *tempus*;) and that they also used *hac* for *hæc*. Whence *antehac* and *posthac* signified respectively *ante hæc* (*tempora*) and *post hæc* (*tempora*.) GIFFANIUS, in his Index to Lucretius, observes, that for *hinc* and *illinc*, the Ancients used *him* and *illim*. VOSSIUS notices the ancient *quor*, for *cur*, as *quoi* for *cui*; *quoique* for *cuique*; *quoiusque* for *cujusque*; and *quoiquam* for *cuiquam*.

Ubi appears to have been formerly *cuibi*, or *cubi*, for so it is found in the compound *alicubi*; but *cuibi* must have been written in the most ancient Latin *quoibi*; for, in the Laws of the Twelve Tables, we find *quoi*, *quoius*, *quoium*, and *quom*, instead of the more modern *cui*, *cujus*, *cujum*, and *cum*. *Ibi* and *ubi*, therefore, were merely *is* and *qui* compounded with the particle *bi*, which was, perhaps, of similar origin with the Gothic *bi* and the English *by*. We must not omit, however, to notice that the distinction between the relative and interrogative force of the word *ubi* was accurately marked by the accent. UBI *interrogativum*, says Martinius, *penultimam acuit*, *ut*, UBI *est Pamphilus*? *Relativum gravatur*, *ut*, *Sævus* UBI *Æacidæ telo jacet Hector*. *Sic*, UNDE, QUANDO, *et similia interrogativa penultimam acuunt*, *relativa gravant*. It was also repeated for the sake of emphasis, as *ubi ubi*, for *ubicunque*; an idiom similar to that of the Anglo-Saxon *tha tha*, *quamprimum*, *thær thær*, *quo in loco*, &c.

It is needless to trace the pronominal Adverbs in Greek; but it may be somewhat curious to observe the same Principle in the Persian Language, in which the pronouns are *een*, this; *aun*, that; *ke*, who; *che*, which.

From *een*, "this," are derived *eenjâ*, "here," *eensû*, "hither."

From *aun*, "that;" *ânjâ*, "there;" *ânsû*, "thither;" *angâh*, "then."

From *ke*, "who;" *cu* or *cujâ*, "where," "whither."

From *che*, "which;" *chun*, "how, or when?" *chend*, "how many?" *chera*, "wherefore?" *hemchun*, "so as," &c. (See Sir William Jones's *Persian Grammar*; and compare pages 32 and 33 with 93, 94, 95, and 96.)

So. Also. The pronominal Adverbs which we have just considered serve principally to modify the verb; for when we say "this is *here*, and that is *there*," the words *here* and *there* serve to modify the assertion; and the same may be observed of the phrases "to come *hither*," "to go *thither*," &c.: but there are some other Adverbs which are derived from pronouns, and of which the principal use is to modify adjectives. Such are the words *so*, *as*, *than*, &c. We have already noticed the pronominal origin of *so* and *as*, which are both synonymous with *it* or *that*. *As*, in the German, is written *es*, and forms the pronoun *it*. *That*, in the Scottish colloquial Dialect, is sometimes used for *so*, as in *The Antiquary*, (vol. ii. p. 281,) "*that* muckle," for "so much." These words *so* and *as* had respectively their compounds *all-so* and *all-es*, which latter was the old English *als*. *So* and *also* are the Scottish *swa* and *alswa*, which occur in the Act of 1424. *Richtswa* occurs in the Act of 1478; and *swa furth*, i. e. "so on," in that of 1491.

Alles was formerly used where we should use *also*, as in the Romance of the *Kyng of Tars*, (see Warton, v. i. p. 191)—

> And *alles* I swere withouten fayle.

Mr. Tooke has correctly explained this word *alles*, *als*,

Grammar. to be *all-as*, and to correspond with the words *all that*, as in the following instance:

> Glidis away undir the fomy seis
> *Als* swift *as* ganse, or fedderit arrowis fleis.
> GAWIN DOUGLAS.

i. e. "glides away with *all that* swiftness *that* arrows fly with."

So in ROBERT DE BRUNNE, an English writer: (circ. A. D. 1300:)

> Richard *als* suithe did raise his engyns.

In the Scottish Act of Parliament, 1493, *alswell*, for "as well," or "all as well." *Als*, in the sense of *also*, very frequently occurs in our old writers. Thus, in BARBOUR's *Bruce*, which was written about A. D. 1375, we have,

> And the gode Lord *als* of Douglas.
> * * * * * *
> He might have seen, that had been there,
> A folk, that merry was & glad,
> For their vict'ry; and *als* they had
> A lord so sweet & debonair.

Again, in the before-mentioned English Poem, entitled *The Pricke of Conscience*—

> And *als* he yaf him a fre wille.

Else. It would seem that the word *elles*, or *els*, is sometimes to be considered as identical with *alles*, or *als*; and sometimes to be derived from the old German *el*, alius, alienus, peregrinus, which WACHTER calls *Vox Celtica et primitiva, quæ Græcis effertur* ἀλλος, *et Latinis alius*. HENISCHIUS, in his Thesaurus of the German Language, explains *el* alius, *jemand el*, alter, quispiam, somebody *else*, *niemand el*, nemo alius, nobody *else*. The compounds and derivatives of this word are found in all the Northern Languages, as in Welsh, *aliwn* alius, *alon* alieni, *alltad* alienigena, *alltudo* in exilium pellere, *allwlad* alienigena, &c.; in Gothic *aljath* alio, aliorsum, *aljathro* aliunde, *aljakunja* alienigena, &c.; in Frankish *allasuara* alio; in Alamannic *allesuanan* aliunde; in Anglo-Saxon *elles* alias, alioquin, *elles-hwær* aliorsum, *altheodig* externus, peregrinus, *eltheodisce men* peregrini, *elreordig* barbarus; in Islandic *ella* aliue; in provincial German *al-fanz* aliena loquens, *el gotze*, idolum peregrinum, *ellend* terra aliena, *büff-el* bosperegrinus. To which we may add the Scottish *elritch*, strange, of a foreign Country, for *ritch* is from *ryk*, a kingdom, or dominion.

Mr. Tooke derives this word *else* from *a-lesan*, an Anglo-Saxon verb, of which he says it is the imperative, and that it signifies *dimitte hoc*, or *hoc dimisso*. The derivation is not very probable; but he expresses the most violent indignation at its having been questioned by some anonymous critic; as if an error in conjectural etymology were a matter of moral turpitude, and inferred absolute infamy to a man's character. In reality few errors can be more innocent—a circumstance peculiarly fortunate to Mr. Tooke; for among many ingenious conjectures he has certainly ventured on some that are perfectly erroneous.

Than. *Than* has been already explained under the word *then*; for it seems to have escaped the notice of most English Grammarians that these two words are perfectly identical, and indeed have not been generally distinguished in use much more than a century. Thus in Shakspeare's *Sonnets* (A. D. 1609)—

> 'Tis better to be vile *then* vile esteemed,
> When not to be receiues reproach of being;

and in Milton's *Paradise Lost* (edit. 1669)—

Adverbs.

> ——— Native of heav'n! for other place
> None can *then* heav'n such glorious shapes contain.

So we have *thane* for at that time in the Harleian MS. No. 7333. f. 14. b.:—"Thie balade made Geffrey Chaunciers the laureall poete of Albion, and sent it to his souerain lorde Kynge Richard the Secounde, *thane* being in his castell of Windesore."

Thus. *Thus*, which is similar to *so*, is the word *this*. As in the Ist Sermon of LATIMER, A. D. 1562: "He hath lain *this* long at great costes and charges, and canne not once haue hys matter come to the hearynge."

Verbs. 5. If there be a doubt whether any one particular class of words can be used Adverbially, that doubt must apply to the *Verbs*. In English, the words to which this doubt applies are either of uncertain etymology, or else their use is rather conjunctional or interjectional than Adverbial.

Yet. *Yet* has been considered as the Imperative mood of the Anglo-Saxon verb *gytan*, or *getan*, to *get*; but it is not very evident how this imperative can be applied to the different senses in which the word *yet* is used. It is differently written in our old manuscripts, *gyt*, *yite*, *yet*, *yut*, *yit*, but generally with the Saxon letter which answers to our *g* or *y*, (consonant,) and which, from the similarity of its form to *z*, is printed as that letter in old Scottish books.

It sometimes relates simply to time, and would seem to be connected with the Gothic *ya*, now, as "is he not *yet* arrived?" *i. e.* is he not arrived at *this* late hour?—Where it is to be observed that the correspondent word in French is *encore*, which clearly expresses the conception of time; for *encore* is the Italian *ancora*, which Menage derives (perhaps not quite correctly) from *hanc horam*; but which is certainly from the Latin *hora*, the hour, or time. In this sense, *yet* is used nearly in the same manner as the adjective Adverb *still*, as

> Hs yef of the holy cross sum that ther *yut* is.
> ROBERT OF GLOUCESTER, 296.

Sometimes *yet* has the force of *moreover*—

> *Gyt* he presented him the spere. WARTON, i. 94.
> *Yite* I do yow mo to witte. *Harl. MS.* 913.

Sometimes of *also*—

> The slear of himselfe *yet* sawe I there.
> CHAUCER. *Kn. Tale.*

Sometimes of *nevertheless*—

> Alas that he *yet* shulde dye. *Elegy on Edw. I.*

So in the striking passage of *Macbeth*—

> Though Birnam wood be come to Dunsinane,
> And thou oppos'd, being of no woman born,
> *Yet* will I try the last.

Where *yet* is used for also, moreover, or nevertheless, it is properly to be considered as a conjunction; but the distinction between a conjunction and a relative Adverb is not always easy to be drawn.

Yes. *Yes* and *No*, if considered as Adverbs, must be taken to modify the verb contained in the interrogative sentence to which they form the answer. They are commonly ranked by Grammarians as belonging to this Part of speech; but perhaps it might be more proper to consider them as interjections. Whether or not in English they are verbs, may be doubted. The French word *oui* undoubtedly is the participle "heard;" the Italian *si* is probably *sit*, "be it so; and Mr. Tooke

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labours to derive our *yes* from the French *ayez*, "have it," "enjoy it." This is not the happiest of his etymologies, at least it is not one of the best supported; for he quotes CHAUCER's *Romant of the Rose* very much at random, in support of his conjecture:

> And after, on the daunce went
> LARGESSE, that set al her entent
> For to ben honorable and fre;
> Of Alexander's kynne was she;
> Her most joye was ywis,
> When that she yafe, and sayd HAVE THIS.

Where Mr. Tooke says, "Which might, with equal propriety, have been translated—

> When she gave, and said YES."

The most frigid critic could not well have missed the spirit of his author more completely. *Largesse*, or liberality, is personified; like another Timon, scattering her gifts on all sides, and not waiting for any demand to which she might answer yes. So we find, from the admirable Scenes with Lucullus and Lucius, that Timon had been in the habit of surprising them with unexpected presents:

LUCULLUS. One of Lord Timon's men?—A gift, I warrant. Why, this hits right: I dreamt of a silver bason and ewer to-night. Flaminius, honest Flaminius, you are very respectively welcome, sir. (Fill me some wine.) And how does that honourable, complete, and free-hearted gentleman of Athens, thy very bountiful good lord and master?

FLAM. His health is well, sir.

LUCUL. I am right glad his health is well, sir—and what hast thou there, under thy cloak, pretty Flaminius?

* * * * * *

SERV. May it please your honour, my lord hath sent—

LUCIUS. Ha! What hath he sent? I am so much endeared to that lord: he is ever a sending. How shall I thank him, thinkst thou?—And what hath he sent now?

In like manner Largesse set all her pleasure in free, spontaneous, and unexpected acts of bounty, with the munificence of a mighty monarch, another Alexander, surprising those whom she benefited by the sudden exclamation, "Have this!"

If our *yes* were derived from *ayez*, we should find the latter word used in that sense, in some of the French Dialects; but this circumstance nowhere occurs; and it can hardly be doubted, but that *yes* includes, or is derived from the word *yea*. Junius, indeed, explains *yes* as a contraction of *yea is*: which etymology, if right, affords an explanation of what Tooke calls Sir Thomas More's "ridiculous distinction" between *yea* and *yes*. More says, that if a question be framed affirmatively, the answer, if affirmative also, should be by the word *yea*; if framed negatively, by the word *yes*. Thus he supposes one person to ask Tyndal the translator, if his book *is* worthy to be burned, and another to ask him if his book *is not* worthy to be burned. To the first, he says, the answer should be *yea*, and to the other *yes*; and he appeals for this distinction to the then common use and practice, in which a man of such eminence in the profession of the Law, and of such frequent attendance about the King's person and Court, could hardly be mistaken. If More then was right, *yea* meant simply "true," or "so," *i. e.* "it is as you *say*;" but *yes* signified "true it *is*," or "so it *is*," rejecting the negative which had been introduced into the question; in other words it signified, "it is as you *mean*, but not as you say;" for the questioner, in both cases, is understood to intend the same assertion, though the expressions are opposite.

It is not very clear that the word *ayez* was used in French before *yes* was used in English; since it appears to be a corruption of *avez*; which was taken from *havez*, or *habez*, part of the very ancient verb *haben*, of which the radical *hab*, in the sense of our word *have*, was common to the Latin with all the Gothic Languages; for the Latin verb was *habere*, the Mæso-Gothic *haban*, the Anglo-Saxon *habban* and *hæbban*, the Frankish, Alamannic, and modern German *haben*, the Islandic *hafa*, the Danish *haffne*, the Swedish *hafwa*, the Dutch *hebben*; and it even seems to have been used in one Dialect of the Greek Language, for Hesychius and Phavorinus prove that ἅβειν was used for ἔχειν, particularly by the Pamphylians, and from this root an infinity of nouns are derived in the Northern Languages. It would therefore require some diligence of investigation, to discover at what period in the History of the Frankish, or French Language, the distinctive *b* or *v* of the radical word was dropped in the imperative *ayez*; and it could not have been long after that period, if at all, that the imperative was converted, by common use, into an Adverb among the French; and again, at a much later period that this Adverb was adopted from the Norman-French into the Norman-Saxon, whence it must have descended to the modern English; not one of the steps in which progress has Mr. Tooke attempted to verify; and if he had, in all probability his labour would not have led to any confirmation of his conjectural etymology of the word *yes*. Adverbs.

Again he suggests that *Yes* and *Yea* are of very different origin, the one being from the French verb *avoir*, the other from some Northern verb (he does not exactly determine which) that signifies "to own." Now verbs also of this signification are very numerous, as well as the adjectives and substantives derived from them. Thus the Gothic verb is *aigan*, the Anglo-Saxon *agan*, whence our verb to *owe* is derived; the Islandic *eiga*, the Swedish *æga*, the Alamannic *eigan*, and with these probably the Greek ἔχειν has some affinity. Nor is the adjective less general, with the sense of *own*, proprius. In Gothic it is *aigin*, in Anglo-Saxon *agen*, whence the old Scottish *awin*, and old English *owen*, the Alamannic *eigan*, the Danish *eget*, the Islandic *eyga*, and the Dutch *eygen*. It does not, however, happen in these Languages generally, that the affirmative Adverb, or interjection, has the form of any part of the verb, or indeed much resemblance to it. Our *yea* is undoubtedly the Gothic *ya*, *yai*, which, with very little change, pervades most of the Northern Dialects, being in Welsh *ie*, in Armoric, Dutch, German, and Swedish, *ja*, (where the *j* is pronounced as *y*,) and in Anglo-Saxon *ia*, *ya*, *yæ*, *yea*. Of this word *ya*, the origin is much doubted by etymologists. Some derive it from the Hebrew *Jah*, Jehovah; but as we cannot think that the Hebrews would ever have profaned the name of the Almighty, by thus introducing it into their most common and trivial discourse; so it is still less probable that the nations, who knew not Jehovah, should have done so, except from imitation of the Hebrews; and this etymology, if true, would present a singular contradiction to the words of CHRIST in the Gothic translation of the Gospels. Our Saviour commands His disciples not to swear at all; but, in their common discourse, to use simple affirmations or negations. Whereas, on the hypothesis above mentioned, the Gothic text *siy waurd izwar ya, ya*, (Matth. v. 37,) Yea.

Grammar. ought to be rendered, "let your word be, by Jehovah! by Jehovah!" It seems most probable that *ya* was originally of similar origio with the Latio word *sic*, which was used for the same purpose. Thus, in Terence, we find—*Itane ais Phanium relictam solam?* SIC. *Daturne illa hodie Pamphilo nuptum?* SIC EST. *Quid narras?* SIC EST FACTUM. In which three different examples, we see the affirmative Adverb gradually brought back, as it were, to its prooominal origin; for the last answer might as well have been *ita est factum*, or *id est factum*.

The Latin *sic*, so, and *si*, if, were manifestly of similar origin with *se*, himself, which io the dative is *si-bi*, and with the verb *sit*, which was anciently written *si-et*.

In the Gothic, we shall, in like manner, perceive a coonection between *ya* and the pronouns and Adverbs of pronomioal origin, *so*, *it*, *this*, and *that*:

Ya-ins——(ille) "*this* man,"
Ya-ind——(illuc) "to *that* place,"
Ya-than——(forsan) "it may be *so*,"
Ya-u——(si) "be *it*, that,"
Yu——(jam) "at *this* time."

Besides the mere expression of acquiescence in a question or demand, *yea* has, in its moderu use, a particular force which answers to the Latin *imo*; ond *imo*, it is to be observed, is really the pronoun *im*, which occurs constantly for *eum* in the remaining fragments of the Laws of the Twelve Tables; as, *si IM aliquips occisit, joure cæsus esto*, where MACROBIUS says: *ab eo quod est IS, non EUM, casu accusativo, sed IM dixerunt*. In this sense of the word *yea*, MILTON says,

——They durst abide
Jehovah thund'ring out of Sion, thron'd
Between the cherubim—*yea*, often plac'd
Within His Sanctuary itself their shrines.

It is somewhat remarkable, in the English idiom, that the word *nay* (the antipodes, as one would think, of *yea*) is used in the very same sense as that which we have just described. Thus DRYDEN says, "This allay of Ovid's writings is sufficiently recompensed by his other excellencies; *nay*, this very fault is not without its beauties." What is still more singular, BEN JONSON uses hoth *yea* and *nay* with the same augmentative force io one and the same sentence: "A good man always profits by his endeavour; *yea*, when he is absent; *nay*, when dead, by his example and memory." In all these passages, *yea* seems still to bear its relation to the pronoun *this*; for the meaniog is, "they durst abide Jehovah thundering out of Sion; *this* they did and ofteo more." "A good man profits by his endeavours; *this* he does when preseot, and even when absent:" and the word *nay* only serves still further to complete the same sense; for, in the instances above quoted, the meaning is, "the allay of Ovid's writings is accompanied by other excellences: *this* is the case, and *not* only this, but the very fault has its beauties." "A good man profits us by his endeavours when absent: *this* he does, and *not* only this, but even when he is dead, we profit by his example and his memory."

There is still one more use of *yea*, which confirms our view of its import; as in the 3d chapter of Genesis —"*Yea?* Hath God said, ye shall not eat of every tree in the garden?" Here the word *yea* has an ioterrogative force; and means "is this *so*?" Do you say yea to *this*—namely, that God hath forbiddeo you to eat of every tree?

Adverbs.

In fine, the conception always expressed by *yea* is that of true and affirmative existence. Hence Dr. HAMMOND, explaining the passage "all the promises of God in him are *yea* and amen," (2 Cor. i. 20,) says, "that is, they are VERIFIED, *which is the importance of yea*; and confirmed, which is meant by amen." Now the conception of positive existence, as applied to a particular thing or event, is expressed by the words "this is;" and if there be an ellipsis of either word, the same conception may be expressed by the other word. In this view of the subject, it seems not unreasonable to conclude that the word *ya* may have been originally either a pronoun, or o part of the verb of existence; and it is to be remembered, that in many, perhaps in all Languages, the verb of existence is merely expressed by a pronoun.

Ay. *Ay* appears to be merely *yea*, a little varied in pronunciation. Dr. JOHNSON, indeed, suggests that it may be derived from the Latin *aio*; but words io general are not transferred from one Language to another, so as to come into common use, without leaving some traces of their gradual progress. The Latin terms which have been incorporated with our colloquial discourse, have been received either through the mediom of the French, or else have been technical terms, chiefly of the Law; and in either case it is geoerally easy to discover the gradations by which they have come to form a part of our Language. Now there is no such proof of the transition of the Latin verb *aio* into the English *ay*, hut much to render it improbable. *Ay* has some slight differences of application from *yea*, as *yea* has from *yes*; but this is no more remarkable than the different force and effect which, as we have already seen, is given in different cases to the same word, *yea*. Thus, in the following passage from Shakspeare's *Henry VI.*, *ay* expresses somewhat more of passionate and proud reproof, than if the word *yea* were employed:

Remember it; and let it make thee crest-fall'n;
Ay, and abate this thy abortive pride.

As *yea* appears to have been corrupted into *ay*, so was *ay* into *I*; but without any variatioo of meaning:

Hath Romeo slain himself? Say thou but *I*;
And that bare *vowel*, *I*, shall poison more
Than the death-darting *eye* of cockatrice.
Romeo and Juliet.

The other Adverb *aye*, always, (for it is a totally different word,) we shall hove occasion to consider it hereafter.

Nay. No. *Nay* and *no* have some differences in use, but they are probably the same word io origin. JUNIUS indeed suggests, thot *nay* is from the two Saxon words *ne-ia*, "nut yes;" but there is no proof that the Saxons, or any other nation, ever used this strange periphrasis to express o conception which is so universal and primary in the Human Mind; being, as it were, the bound oud limit of oll other conceptions. The following are the remarks of the President DE BROSSES on this subject: "Man, in order to communicate his perceptions, has occasion to express, not only existing objects, and the manner of their existence, but also io what manner they do *not* exist. And so with regard to feelings, he has occasion to make known whether they are agreeable to his will, or *not* agreeable to it. It is necessary

then, that besides the different radicals serving to express positive ideas, and different classes of objects, he should have another radical, which may serve to express a negative idea; appropriated merely to indicate, that what he describes is not in what he wishes to describe. One single radical will always suffice for that affect, to whatever object it may be applied. Negation being an absolute and privative sensation, a mere counter-assertion, it is quite enough that we have *one vocal sign*, one organic articulation, to advertise the hearer, that what we say is not in the subject of which we speak. The negative feeling being one which contains in itself a positive and contrary volition, it is not difficult for a man to express it by a gesture, or, what is the same thing, by a single movement of the organ of speech." The learned President proceeds to show, that in the formation of many Languages, mankind had chosen the nasal articulation for the expression of what he calls the *sentiment negatif*. This is at least so far true, that the general conception of negation is expressed in the Latin, and most of the Northern Languages, by the syllables *na ne, ni, no*, &c. *Ne*, says WACHTER, *particula negandi vetustissima, a Scythis in Persiâ, Greciâ, et Septemtrione proseminata ; quæ Persis effertur* NEH, *Græci νὴ et νὲ in compositis, sicut Latinis* NE and NI, *Gothis*, NI, NIH, NE ; *Anglo-Saxonibus* NA, NE ; *Francis et Alamannis* NI ; *Anglis* NO ; *Suecis* NEY ; *Sorab.* NE ; in *compositis*. He also justly observes of the letter *n*, that in many compounds it is an abbreviation of *ne, ni*, &c. and as such has a negative power; as in the German *nichts, niemand, niemal, nimmer*, and many others, of which the list might be extended to an immense length, were we to include all the European Languages. Nor is it only in the distinct compounds, such as *ever, never, one, none, volo, nolo, ullus, nullus*, &c., that this effect is discernible, but also in some terms which conversely express positive and negative conceptions, as *light, night, lux, nox*, &c. Without entering deeply into those Metaphysical speculations on the τὸ ὂν and the τὸ μὴ ὂν, for which Mr. Tooke so much ridicules Lord MONBODDO, and without pretending to decide the disputed points respecting positive and negative ideas, positive and negative quantities, and the like, it is sufficient for us to observe, that every child, in the first glimmering of Reason, must necessarily form a conception of negation; and that it does in fact acquire, among its first articulate sounds, the sound which expresses that conception. The child has as distinct a conception that its nurse is *not* present, or that its food is *not* agreeable to its palate, as it has of the opposite circumstances. It may perhaps be urged, that this negative conception is in its very nature adjectival; that it can only be applied in the manner of an attribute to some other conception which is of a substantive nature. *Il est impossible*, says DE BROSSES, *de former un Nom absolument privatif; c'est à dire une locution, qui ne contienne pas une idée vraiment positive*. Be it so; but at least the adjectival conception may be applied, in the manner of all other conceptions of the same class, to modify substantives, adjectives, verbs, and Adverbs; thus we may apply the negative words or particles *no, not*, and *un*, to modify the substantive *man*, the verb *is*, the adjective *wise*, or the Adverb *always*, in the following phrase:

No man is always wise.
Man *is not* wise always.
Man is always *unwise*.
Man is *never* wise (i. e. *always not* wise.)

Whether there be any thing in the nature of the nasal organ, which peculiarly fits it, as De Brosses supposes, for the expression of conceptions of doubt and privation, may, perhaps, be reasonably questioned. Negative terms are found in many Languages to which this remark certainly cannot apply. However, the negatives in Latin and in the Gothic Languages, generally have the nasal articulation variously combined; nor do these various combinations necessarily give a distinct force to the word. The Latin *ne, non*, and *nec*, were anciently confounded, and so were the English *ne, no, not, nor*. In a fragment of the Laws of Numa Pompilius, preserved by FULVIUS URSINUS, we find *nei* for *ne*.

Sei Hominem folminis occisit, im supera genua NEI *tolito.*

Again, in a fragment of the first Tribunician Law, *nec* is used for *ne*—

Sei quis aliuta faxsit eum pequnia familiaq sacer estod : sei quis im occisit paricida NEC *estod.*

Again, in the Laws of the Twelve Tables—

Patris familias quei en do testato moritor quoique souos heres NEC *escit.*

In old English *ne* was used for *not* and for *nor*.

1. For *not* in the Harleian MS. 2253. fo. 70. b.—

Ne mai no lewed lued libben in londe.

2. For *not* in the Prophecy of *Thomas de Essedoune*, in the same volume, fo. 127—

Whenne shall this be?
Nouther in thine tyme, *ne* in myn.

No was used in the same two senses.

1. For *not* in the Romance of *Alisaunder*—

Alisaunder and his folk alle
No had noght passed theo halvendall.

2. For *nor*, in the *Description of Cokaygne*—

Ther nis halle, bure, *no* bench.

On the other hand, *nor*, in the old Scottish Dialect, was used for *than* :

The fell strong traytour Donald Owyr
Mair falset had *nor* udir four. — DUNBAR.

Compleitly, mair sweitly
Scho fridound flat and schairp,
Nor Muses, that uses
To pin Apollo's harp.
ALEX. MONTGOMERY, circ. 1597.

The particle *ne*, which forms part of our modern words, none, never, &c., was anciently incorporated with many verbs, as, *I not*, for "I ne wot," or "know not;" *I nuste*, for "I ne wist;" *I nabbe*, for "I ne have;" *I nulle*, for "I ne will;" *I nolde*, for "I ne would;" *it nis, it nas, it nere*, for "it ne is," "it ne was," "it ne were."

The bors vanisheth I *not* in what manere.
CHAUCER. *Squ. Tale.*

In all this wurhliche won
A burde of blod and of bon
Neuer yete *ynuste* non
Lussomore in londe.
Harl. MS. 2253. fo. 72.

Uch a srewe wol hire shrude
Tha he *nabbe* nout a smok, &c. — *Ibid.* fo. 61. b.

I *nul* soffre that no more. — *Ibid.* fo. 55. b.

Whil God wes on erthe
And wondrede wyde
What was the reson
Why he *nolde* ryde?
For he *nolde* no grome
To go by ys syde.

Harl. MS. 2253. fol. 124. b.

Ther *nis* londe vnder heuenriche.

Id. No. 913.

—— that he *nas* wenemyd anon.

Lyf of Seint Patrik.

Wymmen were the best thing
That shup our heye heune kyng
Yef feole false *nere*. *Harl. MS.* 2253. fol. 71.

Double negative.

. It is sufficient for the general purposes of communicating thought, that the negative conception should be once expressed in a simple sentence; but we generally find it redoubled in old English, a circumstance derived from the Anglo-Saxon idiom, as, *Ne om ic na Crist*, "I am not the Christ." (John i. 20.) The same idiom prevails in the modern French, although it was not always observed in that Language at an earlier period. In the XVIth century they said, *l'habit* NE *faict le moyne*: at present the same proverb is expressed thus, *l'habit* NE *fait* PAS *le moine*. It is difficult to account for the reduplication of the negative upon any other Principle than that of the eager desire, which we commonly see in Barbarous and ignorant People, to give utterance to their strong feelings and imperfect conceptions, and which usually leads to much tautology in their discourse. This genuine result of Barbarism, however, has been sometimes mistaken for a proof of extraordinary learning; and critics have dignified it with the title of an *Archaism*, a *Hellenism*, or some such pompous appellation. "The editor of Chaucer," says HICKES, "knowing nothing of antiquity, asserts that the Poet imitated the *Greeks* in using two negatives to express negation more vehemently; whereas Chaucer was entirely ignorant of the Greek Language, and only used the two negatives according to the prevailing custom of his own times, when the Language had not yet lost its Saxonisms, as, "I NE said *none* ill." In the Saxon writers, indeed, *three* and even *four* successive negatives are sometimes to be found, as, NE *yeseah* NÆFRE NAN *man God*; "*no* man ever saw God." (John i. 18.) And again, NE NAN NE *dorste of tham dæge hyne* NAN thing mare *axiyean*; "and no man durst from that day forth ask him any more questions." (Matth. xxii. 46.) It is to be observed, however, that some of the best of those writers, and particularly the Royal translator of Bede's *Ecclesiastical History*, generally employ but a single negative; and such also is the uniform style of that venerable monument of Gothic literature, the *Codex Argenteus*.

Ado.

There are some Adverbs which have a very obvious affinity with verbs, such as *ado*, *together*, &c. but which it would, nevertheless, be somewhat difficult to trace directly to any particular part of the verb. *Ado* is well known in English from the name of the popular drama, *Much Ado about Nothing*. In the Scottish Dialect too it is very ancient. In the Preface to Gawin Douglas's translation of the *Æneid* we find the expression, "it has nathing *ado* therwith."

Together.

Together has a manifest relation to the verb *gather*, which, however, we now use with some diversity of meaning. The Adverb and the verb rather seem to refer to some common origin, which does not exist in English, but appears in a more simple form in Dutch, in which *gade* is a consort, as *een duyf en haare gade*, "a dove and her mate;" *gadeloos*, matchless; *gadelyk*, sortable, &c. The word *gathering*, which was formerly used in English for a meeting, or assemblage, has fallen into disuse; but was anciently in very general acceptation; as in BARBOUR—

Adverbs.

And the kyng than a parlament
Gart sett therafter hastily
And thider summond he in by
The barouns of hys roialtie
And to the lord the Bruce sent he
Bidding to come to that *gathering*.

In the Scottish Acts of 1592 the word *togidder* occurs; but in more recent compositions it is spelt, as it is in fact pronounced in Scotland, *thegither*. Thus in the well-known Song descriptive of the connubial affection of an old married couple:

John Anderson, my jo, John, we clomb the hill *thegither*.

In some of the old Romances the words *to* and *geder* are written separately, as if the latter were considered as the plural of *gede*, answering to the Dutch *gade*. (See Warton, i. 100)—

To geder schal sit at the mete.

The correspondent expression *in fere* is, in like manner, derived from the Anglo-Saxon *foera*, and old English *fere*, a companion; as in the *Geste of King Horn*—

Tueye *feren* he hadde
That he with him ladde.

The Scottish Dialect employed the verb to *effeir*, and the participle *effeiring*, thus in the Act of 1587, "Ordanis lettrez to be direct heiropone, gif neid beis, in forme as *effeiris*:" and again, "The elvand, the pund trois, & the stane proportional & *effeiring*." Barbour uses the word with some slight difference in the signification:

Sheriffs & baillies made he then
And all kind other officeirs
That for to govern land *effeirs*.

Another expression nearly correspondent to *together* was the Adverbial phrase *all samyn*, or *in samyn*, answering to the Latin *insimul*, and to the French *ensemble*. GAWIN DOUGLAS employs both *togidder* and *all samyn* in the same passage:

Togidder with the principallis of younkeris
The nobir senatouris & pure officiaris
All samyn kest encense.

In the Romance of *Syr Launfal*—

To daunce they wente alle *yn same*.

In that of *Octouian Imperator*—

The emperour with barouns *yn same*
Rood to Parys.

BARBOUR employs the double Adverb *twasum samyn*, *i. e.* two together:

That was in an euill place,
That so strait and so narrow was,
That *twasum samen* might not ride.

The word *samen* is the English pronoun *same*: it is now probably obsolete in Scotland, but was the legal language of 1592, as appears by the Acts of that year, and also by ALEXANDER MONTGOMERY'S Tale of the *Cherrie and the Slae*, composed about the same time:

Lyk as befoir we did submit
Sae we repeit the *samyn* sit.

Substantives.

6. The last class of separate words which we have to notice as used Adverbially are substantives. It is

Grammar.

manifest that substantives may be used in the formation of compound words to express the attributes of attributes. Thus *stone*, in its primary sense, is a substantive, and *blind* is an adjective; but in the compound *stone-blind*, the former part of the word modifies the latter, as much as if we were to say "a *stony*, or *stonelike* blindness." In like manner, substantives standing alone may be taken Adverbially, as modifying either a verb or an adjective. The latter mode is the less common in modern English, but it occurs not unfrequently in the older Dialects: the former mode is common in most Languages. The Adverbial use of the substantive to modify a verb, somewhat resembles the *ablative absolute* of the Latin Grammarians. It expresses a conception simply, without asserting it to exist or not to exist. The construction is consequently elliptical, and the sense may always be more fully expressed by adding the assertion. Thus, in the text "I will sing praise to my God *while* I have my being," (*Psal.* civ. 33.) the word *while*, which was originally a substantive signifying *time*, becomes an Adverb, by the absolute mode of expressing it. The passage is literally "I will sing praise to my God, *time* I have my being," *i. e.* "during the time;" and the three following propositions are included in the whole passage as co-existent:

I will sing;
I shall have my being;
Time will endure.

Nothing but use and the convenience of discourse has assigned their *peculiar* Adverbial force to substantives thus employed. The conception of time, for instance, may be employed, as in the above case, simply to mark continuance, or to mark continuance from a certain point, or to a certain point. Thus in the text "There was a great earthquake, such as was not *since* men were upon the earth, so mighty an earthquake and so great;" (Revel. xvi. 18.) the word *since*, which is also a noun signifying *time*, may be rendered "from the time that." And again, in the text "I will not leave thee *until* I have done that which I have spoken to thee of," (Genes. xxviii. 15.) the word *until* may be rendered "to the time that." *Until*, indeed, is not a noun signifying time, as *while* and *since* are; but the word *while* is often used for it in our provincial Dialects, and occurs in many of our old compositions. Thus in the Scottish Act of Parliament, 1587, the enactment is ordained to last "Ay, and *quhill* His Hienes nixt parliament." So in Alexander Montgomery:

Cum se now, in me now
The butterflie and candill
And as scho flies *quhyl* scho be fyrt.

Of *until* and *since* we shall speak more particularly among the prepositions. The substantives used as Adverbs of time in English are *while*, *tide*, *sithe*, *time*, and *season*.

While.

While is the Gothic and Anglo-Saxon *hwila*, and Alamannic *uuila*, time, or a certain space of time, which seems to be of the same origin as our *wheel*, in the Anglo-Saxon *hweol*, Danish and Swedish *hiul*, Islandic *hiool*, and Dutch *wiel*, which are derived, by J. Davies, from the Welsh *chwyl*, turning, and seem to have some affinity with the Latin *volvo*, and Gothic *walwyan*, to roll; nor is there any more apt or more common symbol of time than the continual rolling of a wheel. Be this as it may, we find the word *while* in English and *weile* in German used substantively for a space of time, as in German *es ist eine gute weile*, "it is a good while," or "a long time." So in the relation of the meeting of Joseph with his father Jacob, (Gen. xlvi. 29.) "he fell on his neck, and wept on his neck *a good while*." We have seen this word used in the two senses of "whilst" and "until:" it is also used in the Scottish Dialect for "sometimes," as in the well-known anecdote of an English traveller, who had been confined at a village in Scotland several days together by the rain, and who, at length, losing his patience, asked the landlord pettishly, "What! does it rain here always?" To which the other replied with a smile, "Hoot, na! it snaws *whyles*." The word *awhile* is commonly used Adverbially for "a short time;" as Samuel said to Saul, "Stand thou still *awhile* that I may show thee the word of God" (1 Sam. ix. 27.) The same idiom occurs in the *Goldin Terge* of Dunbar:

Adverbs.

Acquentance new embrasit me *a quhyle*,
And favourt me till men micht gae a myle,
Syne tuk hir lef, I saw hir nevir mair.

In a very ancient English Love-song *whyle* is used in this sense without the article. (Harl. MSS. 2253. fol. 63. b.)

Betere is tholien *whyle* sore
Then mournen euermore.

It is somewhat remarkable that though in the German Language the substantive *weile* is not used Adverbially in the same senses as *while* is in English, yet it has the same Adverbial, or rather conjunctional sense that we give in matters of reasoning to *since*, which word, as we have observed, also signifies "time." Thus the German *weil* implies the consequence or dependence of one fact on another, as Weil *ers verlanget*, *so soll ers haben*; "since he desires it, he shall have it."

The compounds of *while* still in use, such as *meanwhile*, *erewhile*, require no explanation. They plainly express the conception of time, and signify "in the mean*time*," "some*time* before," &c. *Erewhile* was anciently written *whilere*, and so we find in the different old Dialects *whilom* and *umquhill*, which both agree with the old word *sometime* for "formerly." "The whiles" occurs in old writings for *meanwhile*; as in *Kyng Alisaunder*—

Alisaundre is in his lond
And hath sone a newe sonde.
From a cité in the Est
That nul no Phelippes heste.
Thider he wendith with gret pres,
This stordy cities for to dres.
The whiles, herith a cas.
A riche baroun in Grece was, &c.

Whiles was used at no great distance of time where we now use while or whilst; as in Shakspeare's *Much Ado about Nothing*—

What we have we prize not to the worth
Whiles we enjoy it.

The same idiom also prevailed in Scotland—

The bramble grows althoht it be obscure,
Quhylis mountane cederis tholes the bousteous winds,
And myld plebyan spirits may lief secure,
Quhylis michty tempestis toss imperial myndis.
Montgomery.

Mr. Tooke conceives that *whilst* and *amidst* are mere corruptions, and that we should write them as formerly, *whiles*, and *amiddes*; but it would seem that there was some particular reason for the final *t*, because in the common Scottish Dialect of the present day it is found

Grammar. in the word *alongst*. Possibly the expressions originally were "*on long is it; on mid is it;*" "*while is it.*"

In the *Morale Proverbes of Crystyne*, printed by Caxton, A. D. 1478, we find the expression *long saison* for "a long while," or "a long time:"

> A temperat man cold from hast assured
> May not lightly *long saison* be miscured.

So in the *Dictes and Sayings of Philosophers*, printed 1477, "There was *that season* in my company a worshipful gentleman called Lewis de Bretaylles."

Stound. *Stound*, which is from stond, occurs adverbially in the sense of time; as in *Octouian Imperator*—

> Men blamede the bochere *oft stoundys*
> For his sone.

This, which we should now express oftentimes, was anciently expressed also *ofte sithes*; as in CHAUCER'S *Troilus and Creseida*—

> And such he was I proued *ofte sithes*.

Sumwhile occurs in *Kyng Alisaunder*—

> There woned *sumwhile* Kyng Appolyn.

In the *Lay le Freine*, published by Mr. Weber, we find *therwhiles:*

> The abbesse hir in conseyl toke
> To tellen hir hye nought forsoke
> Hou hye was founden in al thing
> And tok hir the cloth and the ring
> And bad hir kepe it in that stede;
> And *therwiles* she lived so sche dede.

The Scottish *umquhill* appears in the Act of 1455, "James *umquhill* Erle of Dowglas." In the Act of 1540 we find both "*umquhile* James Coluile," and "Archibald *sumtime* Erle of Anguis." *Sumtyme*, answering to *olim*, occurs in MONTGOMERY'S *Cherrie and Slae:*

> Then furth I drew that double dart
> Qhuilk *sumtyme* schot his mother.

Tide. Our word *tide* is connected with the word *sithe* before mentioned by the German *zeit*, (pronounced tseit,) for on the one hand it is *tseil*, tide, dropping the initial *s*; and on the other it is *tseil*, sithe, dropping the initial *t*; and in both cases changing the final *t* into its approximate articulation, *viz.* in the one instance *d*, in the other *th*. We do not use *tide* in modern English Adverbially; but in German the word *seit* is used in the sense of "since," or "from that time." In the different Northern Languages this word appears in various forms, and with many analogous significations. In the Alamannic Glossaries we find *citi*, "times;" whence probably comes the Latin *cito*, quickly. In the Frankish, *vonna alten* ZITIN, "from old *times*;" *tho sih thin* EIT *bibrahta*, "when the *time* was brought near;" in modern Dutch, *in voorige* TYDEN, "in former times;" *by onzen* TYD, "in our time," &c. The hours of the day are called, in Frankish, *citi* and *ziti*; and in Anglo-Saxon, *tida*; as in Gloss. Keron. *fora einera* ZITI, before one o'clock; and in the Anglo-Saxon Gospels, *hu ne synt twelf* TIDA *thæs dæges?* "are there not twelve *hours* in the day?" In modern German they say *welche* ZEIT? for "what's o'clock?" *hochzeit*, a marriage festival, or any other festival; in which latter sense the expression runs through a great variety of Dialects, as the Frankish *hoho ziti*, the Alamannic *hohzil*, the Swedish *hogtyd* and *hogtyds dag*, the Dutch *hooghtiid*, the Anglo-Saxon *heah-tid*, and the old English *high tide* and *hock-tide*. In German, too, the separate words *hohe zeit* are used as we use "high time; as, *es ist hohe* ZEIT, "it is high time" (that such a thing were done.) So they say *bey zeit*, as we do Adverbially *betimes*; *bey* ZEITEN *wieder commen*, is "to come back in good *time*," *von* ZEIT ZU ZEIT, "from *time to time*," *essenszeit*, "dinner-time," &c. In this last sense, where we say church-time, the Dutch say *kerk-tyd*; and where we say *bed-time*, our Saxon ancestors said *bed-tid*. So *underntid* was the hour of nine o'clock in the forenoon, when the *undernsang* was sung in churches, and when individuals were accustomed to take the meal called in Gothic *undaurnimat*, and in Anglo-Saxon simply *undern*. Hence, in the Romance of *Syr Launfal*— Adverbs.

> In hys chamber he hyld hym stille
> All that vndern *tyde*.

The German *zeit* is also a season or "time of the year;" *vier zeiten*, "the four seasons." The Dutch *tyd* is "opportunity," "convenient time," "leisure," "sufficient *time*." Of the same origin are our *noon-tide*, *Whitsuntide*, and the *tide*, or periodical *time* of the sea's ebb and flow.

> Let him hear the cry in the morning, and the shouting at *noon-tide*.—Jer. xx. 16.

> *Noon-tide* repast, or afternoon repose.
> MILTON. *Par. Lost.*

> And behold, at *evening-tide* trouble; and before the morning he is not!—Isaiah, xvii. 14.

In the Romance of *Kyng Alisaunder*, *long tydes* means a long while (several days, as it should seem by the context)—

> They reste heom *longe tydes*
> And wel ofte on ryver rydes.

Hence our verb to *betide*, or happen at a certain *time*, which, by BARBOUR, is written simply *tide*—

> But ye trusted unto lawtie,
> As simple folk, but malwitie,
> And wist not what shuld after *tide*.

Hence the substantive *tidings*, what happens at a certain *time*, and, in a secondary sense, what is reported to have happened.

Hence, too, the adjective *tidy*, of which the first sense is seasonable, happening in due *time*—

> If weather be fair and *tidie*, thy grain
> Make speedilie carriage for fear of a rain.
> TUSSER.

So the Islandic *tidugur*, *tempestivus*; the German adverb *zeitig*, maturely, in good *time*; (answering to the Scottish *timeous*, and *timeously*;) the German substantive *unzeit*, an inconvenient *time*, with its adjective *unzeitig*, unseasonable; *unzeitige geburt*, "an *untimely* birth," and of the same construction as our *untidy*.

Thus we have seen in different Languages the connection and interchanged use of those substantives which furnish a large class of the Adverbs of time. There is another class also relating to time, derived from a source common to most of the Northern Languages, *viz.* the Adverbs *ever* and *aye*, with the compounds of the former, as *evermore*, *never*, *nevermore*, &c. *Ever* is the Latin *ævum*; as *aye* is the Greek αἰών; and that *ævum* and αἰών are the same word no one can doubt, who remembers that in the Latin of the early Ages *æ* was written *ai*, and *um* was written *om*; and that the modern Latin *v* was the Æolic digamma ϝ, or our *w* which, in fact, is the abbreviated articulation of the vowel sound *oo*, as our *y* is the abbreviated articulation Ever. Aye.

Grammar.

of our vowel sound *ee*. Thus the ancient Romans would have written *ævum* aiʒom or aiʒon; for we find *rocon* for *rogum* in the Laws of Numa Pompilius. VELIUS LONGUS says, *quæ nos per Æ, antiqui per* AI *scriptitaverunt:* and MARIUS VICTORINUS to the same effect, *æ syllabam quidem, more Græcorum, per ai scribunt.* *Om* for *um* occurs constantly in the Laws of the Twelve Tables, as *devortiom, corom, finiom,* &c. In the fragments of the Laws of Numa Pompilius we read *acnom* for *agnum*.

Pelex Asam Junonis nei tagito. Sei tagit, Junoni crinebos demiseis ACNOM *feminam cædito.*

The Æolic digamma is described by DIONYSIUS of Halicarnassus, in the 1st book of his *Antiquities*, where he says that the ancient Grecians used a letter, which was *ὥσπερ γάμμα διτταῖς ἐπὶ μίαν ὀρθὴν ἐπιζευγνύμενον ταῖς πλαγίαις*, "like a gamma with two (horizontal) lines united to one perpendicular;" and the examples which he gives are *Fελένη* for Ἑλένη, *Fάναξ* for ἄναξ, *Fοῖκος* for οἶκος, and *Fανὴρ* for ἀνήρ. The Æolians employed this letter to express a sort of aspiration either at the beginning or in the middle of words; and as they said *oϝις* (or owis) for ὄϊς, and *ωϝον* (or owon) for ὤιον, so it is probable that they said *αιϝων* for αἰών. In ancient Latin inscriptions the F is inverted, as DIꟻAI for *Divæ*.

The Latins not only introduced the articulation *w*, in order to separate two vowels, but also the aspiration *h*, as in *cohors* for *cōors*, from *cōorior; ahenеus* for *æneus, mihi* for *mii*, &c.

If it be thought necessary to seek a common radical for these words *ævum* and αἰὼν, it may probably be found in the ancient *av* or *ab*, which seems to have very generally signified the flowing of a river; which, like the rolling of a wheel, has been in all times considered as a symbol of time. *Etiam hodiernis Persis*, says BAXTER, in his *Glossarium Antiquitatum Britannicarum, (ad voc.* ABALLABA,) AB *pro aqua est, quam et veteres nostri* AV, SAV, *et* TAV *appellavere;* and again, *(ad voc.* ABONA,) *nomen suum sortita est ab ipso flumine, quod Britannis plurativo numero dicitur* AVON, *et antiquâ scripturâ* ABON. Hence, *aben*, in old German, is to fall, to decline, and *der abend* is the evening, the falling or declining of the Sun: and the Helvetic Swiss, as PICTORIUS asserts, use the verb with reference to the decline of life, as *ich aben fast*, "I decline, or draw fast to my end."

However this may be, there can be little doubt but that the Anglo-Saxon *æfre*, whence our Adverb *ever* is lineally descended, was of the same origin with the Latin substantives *ævum* and *ætas*, which latter is only a derivative of the former, being written in the Laws of the Twelve Tables *ævitas*.

Æfr, *ever*, *e'er* are used to denote *time* in its general continuity; and consequently to denote eternal duration, of which we have no other, or at least no better conception, than of *time*, in continuity unlimited. The same contraction *e'er*, spelt in Gothic and old Scotch *air*, in Anglo-Saxon *ær*, in Frankish *er*, and in modern English *ere*, denotes *time*, in its inception, or the time immediately preceding the event or period of which we speak; and this word, in its compounds, *erliche*, *early*, also signifies *time* incipient, but not prior to the period in question. In general it may be regarded as a rule in etymology, that where the simple and compound word have two meanings apparently opposite, they both refer to a third meaning, in which those opposites concur; for of opposites, as Aristotle has observed, there is the same Science: we reason in the same *manner*, though to contrary *results*, on positive and negative quantity, on lights and shades, on vice and virtue. There can be no doubt that *erliche* is derived from *er*. It signifies a conception, *like* the conception expressed by *er;* but for that very reason it differs from *er;* because, according to the scholastic rule, *simile non est idem;* yet, on the other hand, as similarity approaches to identity, and as the limits are not always accurately distinguishable or distinguished, it is not always easy to decide, whether in Language, two terms like *er* and *early*, do or do not absolutely exclude each other's meaning; or even whether one word, like *er*, may not embrace two meanings, excluding each other in their different application to facts. Thus, in the Gospel of St. Mark, are the two following passages: *Καὶ πρωῒ ἔννυχον λίαν ἀναστὰς ἐξῆλθε* (ch. i. ver. 35.)—*Καὶ λίαν πρωῒ τῆς μιᾶς σαββάτων ἔρχονται ἐπὶ τὸ μνημεῖον ἀνατείλαντος τοῦ ἡλίου.* (ch. xvi. ver. 2.) It is plain that the exact points of time here spoken of, with relation to the diurnal revolution of the Earth, are different; and if we assume a moment immediately preceding the elevation of any part of the Sun's disk above the visible horizon, the time referred to in the first passage will be *before* such moment, and that referred to in the latter will be *after* it; and consequently the conception of the one will be as opposite to the conception of the other in this respect, as *before* is to *after*. Nevertheless, they are both expressed in the Gothic translation by the word *air*, the first being AIR *uhtwon usstandands*, the other, *filu* AIR *this dagis:* in the first instance, the Anglo-Saxon version has *swithe* ÆR *arisende:* in the second, the Anglo-Saxon has *swythe* ÆR *dæge;* and the Frankish, ER *themo liohte;* and comparing together these different uses of the words *air, ær, er;* it is impossible not to perceive that they sometimes stand for our word *ere*, and sometimes for our *early*. In the modern English version, the two passages are correctly distinguished thus: "in the morning, rising up a great while *before* day, he went out"—and "very *early* in the morning, the first day of the week, they came unto the sepulchre, at the rising of the Sun."

Adverbs.

Ere.

In our ancient writers, *or* is frequently used in a similar sense with *ere;* but it may be doubted, whether this be the same word differently spelt, or a contraction of *before*. However this be, we find it both alone, and followed by *ere* and *ever;* which may possibly be a mere reduplication for the sake of greater emphasis, as we have already seen in various examples.

The various uses of these words, *air, er, or, ar, ere, or ere*, and *or ever*, will appear from the following quotations:

BARBOUR, in his introductory verses, uses *air:*

> ——— Old stories that men redis
> Represents to thame the deidis
> Of stalwart folk that lived *air*.
> * * * *
> He shald that arbitry disclair
> Of thir twa that I tald of *air*.

In the metrical *Chronicle of England*, composed in the reign of Edward II., (see RITSON'S *Metrical Romances*, v. iii. p. 337,) we find *er*,

> This lond was cleped Albyon
> *Er* then Broyt from Troye com.

Grammar. In *Octouian Imperator*, *ere* and *er*—

> They that were *ere* than agaste
> Tho hadde game.
> * * * * *
> That day Clement was made a knyght
> For his *er* dedes wys and wyght.

In *Richard Coer de Lion*, *or*—

> He it is, my dedly foo:
> He schal abeyen it, *or* he goo.

In *Kyng Alisaunder*, *ar* and *or*—

> No schal he twyes see the sonne
> *Ar* he have him perforce ywonne
> * * * * *
> For Alisaunder wol *or* night
> Broke the castel doun ryght.

In *Macbeth*, *or ere*—

> ——— The deadman's knell
> Is there scarce asked for whom; and good men's lives
> Expire before the flowers in their caps,
> Dying *or ere* they sicken.

In the Book of *Daniel* (ch. vi. ver. 24) *or ever*—

The lions brake all their bones in pieces, *or ever* they came to the bottom of the den.

Erliche and *erst*, the compounds of *ær*, form first adjectives, and then Adverbs, both retaining an exclusive reference to *time*. The Adverb *erliche* occurs in CHAUCER's *Knyght's Tale*:

> And fallen her *erliche* and late.

Erst is the superlative of *ær*, being the Anglo-Saxon *ærista*, primus; and it is used in the senses of early *time*, past or future, *i. e.* "formerly," "soon."

In the Romance of *Sir Guy* (see Warton, i. 170.) it means "at any former time," "before:"

> Suche one had he neuer *erst* seene.

In SPENSER's *Fairy Queen*, "at *erst*" is used for "at the earliest future time," "as soon as possible:"

> Sir Knight, if knight thou be,
> Abandon this forestalled place *at erst*.

Erewhile and *whilere*, are the same compound in two different forms, but with a single meaning, *viz.* "a time preceding the present, usually at no great distance," as in SHAKSPEARE's *As You Like It*:

> That young swain that you saw here but *erewhile*;

and in the *Tempest*—

> Let us be jocund. Will you troul the catch
> You taught me but *whilere*.

Of the other compounds from *ær*, *viz. erelong*, *erenow*; and of those from *ever*, as, *evermore*, *never*, *nevermore*, *forever*, &c. it is unnecessary to speak.

Ayforth is used by Barbour, as a derivative from *aye*, ever, always;

> To put in writ a sothfast story
> That it last *ayforth* in memory.

Of the same origin with the Saxon *æfr* and Latin *ævum*, seem to be the Gothic *aiw* and *aiwa*; the Danish *evig*; the Dutch *eewig* and *eeuwe*; the German *ewig*; the Frankish and Alamannic *ewo* and *ewic*; the old Danish or Runic *æfi*, *æfisaga*, *æfintyr*, &c.

Eft. Whether we ought to refer to the same origin the Anglo-Saxon *æft* and old English *eft*, may perhaps be doubted; but the fact of their common origin seems not improbable. The words *æft* and *æfr* certainly resemble each other in sound, and both relate to a common conception, *viz.* that of *time*.

CHAUCER uses *eft* in the sense of a second *time*: Adverbs.

> Were I unbound, also mote I the,
> I wolde neuer *eft* come in the snare.
> * * * * *
> For thee have I begon a gamen plaie
> Which that I neuer doen shal *eft* for other
> Altho he were a thousand fold my brother.

GAWIN DOUGLAS uses it a little differently, in the sense of "a short *time* afterwards." Thus, in describing the snake, which, after devouring the offerings on the altar, glided back into the earth, (*Æn.* 5. l. 92.) he says,

> And but mair harm in the graif enterit *eft*.

In this latter sense the word *eft* is used by SPENSER:

> *Eft*, through the thick they heard one rudely rush,
> With noise whereof, he from his lofty steed
> Down fell to ground, and crept into a bush.

SPENSER also uses in this latter sense the compound *eftsoons*:

> *Eftsoons* the nymphs which now had flowers their fill
> Run all in haste to see that silver brood.

Upon the whole, it appears that the Adverbs which relate to *time* generally, are all traceable with more or less distinctness to nouns, that is, to names anciently given in various Dialects to the general conception of time. The case is still plainer when we come to the particular divisions of time, such as morning, evening, day, night, week, month, year.

Tomorrow. *Tomorrow*, our Adverb, which answers to the Latin *cras*, signifying the next day to that on which we speak, is simply "the morning," and in the present Scottish Dialect is expressed "the moro;" as, "wul ye gang til the kirk the morn?"—"will you go to church tomorrow?" *Morwe* and *dawe*, in old English, meant morning and day, from the old German *morg* and *tag*, the final *g* being of an obscure sound between our *y* and *w*. The morning is in Gothic *maurgin*, Alamannic *morgan*, Isl. *morgun*, Danish and Dutch *mergen*, modern German *morgen*, and Anglo-Saxon *mærigen*, *mergen*, *morgen*. WACHTER says, that in the ancient computation of time the evening being reckoned first, the *morning* came from that circumstance to signify the future day. Whether this was the reason or not, the fact is certain that most of the Northern Nations did so use the word morning; and hence we have the expressions *amorwe*, *amorrow*, *on morrow*, *by the morrowe*, *tomorrow*.

LYDGATE has "the *morwe*" for "the morning," in his Poem on the *Virgin Mary* (Harl. MSS. 2255. fol. 88.)—

> Atween midnight and the fresh *morwe* gray.

CHAUCER, in the same sense uses *morrow*—

> The merrie lark the messager of daie
> Salueth in her songe the *morrow* gray.

ROBERT OF GLOUCESTER uses *amorwe*, for "on the following morning"—

> The the kynges men nuste *amorwe* wer he was bicome.

In the *Proces of the Sevyn Sages* it is used in the same sense—

> *Amorewe* themperour gan rise
> And clothed him in riche gise.

So CHAUCER, in the *Knyght's Tale*—

> And thus thei been departed till *amorrow*

In *Octouian Imperator* we find *amorn* for the next day—

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Amorn the emperoure, yn ire,
Sente aboute, in hys empyre,
After many a ryche syre
To deme her dome.
The folk tho cam from each a schyre
Ryght ynto Rome.

In *Richard Coer de Lion* occur, in this sense, *on morwe* and *on the morwe*—

On morwe they begunne to ryde
With her boost to Taburet.
* * * *
On the morwe, withouten fayle,
The cyte they gunne for to assayle.

CHAUCER seems to use *on morrow* for "in the morning," as opposed to "in the evening," in the *Plowman's Tale*—

To worship God men would whate,
Both on even and *on morrow*,
Such harlotry men would hate.

So he says, "*morrow* milke" for "morning milk":—

An anelace and gipsere all of silke
Hing at his girdle white as *morrow* milke;

and in the same Prologue—

Well loued he *by the morrow* a soppe in wine.

Our present word *morning* seems to have been formed as a participle from a verb to *morrow*, or to *morwen*, whence we have the old words *morrowing* and *morwening*.

In DUNBAR's *Goldin Terge*—

Sweit were the vapouris saft the *morrowing*.
Hailsum the vail, depaynt with flowris ying;

and in CHAUCER's *Troilus and Cressida*—

Bright was the sonne and clere the *morwening*.

The word *morn* too seems to have been brought into common use in Scotland, at least before the year 1449, since it occurs in the Act of Parliament of that year:

The first of the thre tymis begynnande on the *morn* nixt after the sheref court.

As we have morrow from *morgen*, so we have *sorrow* from *sorgen*, the modern German word being derived from the old German *sorg*. *mæror*, *tristitia*, which doubtless originates in the Frankish *ser* and our *sore*. In one of the Harleian Manuscripts (No. 2251. f. 298.) we find

To tell my *sorwe* my wittes bien all bare;

and in *Octouian Imperator*—

There was many a wepyng eye,
And greet *sorwe* of ham that hyt seye.

The origin of the prefix *to* in our modern words *tomorrow*, *tonight*, *today*, &c. will be considered when we come to the prepositions. As to the distinction adopted in modern times between the two words *morrow* and *morning*, it is no more than what occurs in a variety of cases; as in the instance just mentioned of *sore* and *sorrow*; where the former word, at least in its substantive sense, is applied to a bodily disease, the latter only to a mental affliction.

Today. The Adverb *today* is of the same class with tomorrow. Anciently we had the Adverb *aday* for "in the daytime;" as in *Syr Launfal* (Cotton. MSS. Calig. A. 2. fol. 39.)—

Aday whan hyt is tygt.

Of which expression we at present retain a trace in the colloquial phrase *now adays*. In the same Poem the substantive *days* is written *dawes*. The opening lines are

Le doughty Artour's *dawes*
That held Engelond yn good lawes.

Adverbs.

The substantive name of the conception, *Day*, was easily converted into a verb, as in the very old Pastoral Ballad (Harl. MSS. 2253. fol. 71. b.)—

In May hit muryeth when hit *dawes*.

The present participle of this verb occurs in the old Scottish Song, the tune of which is said to have been played by the troops of King Robert Bruce, in marching to battle:

Landlady count the lawing
The day is near the *dawing*
The cocks are at the crawing.

But the participle is written *dawening*, as from the verb *dawen*, or *dawn*, in *Kyng Alisaunder*:

In the cole *dawenyng*
Wende we forth in al thyng;
Then mowe we, God hit wote,
Resten our bestis in the hote.

In the time of Shakspeare, the substantive *dawning* appears to have been most commun; as in *King Henry V.*—

Alas poor Harry of England he longs not
For the *Dawning*, as we do;

and in *Cymbeline*—

Swift, swift, ye dragons of the night! that *dawning*
May bare its raven eye—

In more modern times the substantive use has come to be confined to the word *dawn*.

The Adverb *tonight* presents in itself nothing remarkable; but it suggests an observation on the Latin Adverb of the same signification. Cicero uses the expression *noctu an interdiu*, "by night or by day;" but that neither this nor the ablative termination is necessary to give the noun an Adverbial force, is evident from the circumstance, that in the Laws of the Twelve Tables, the simple nominative *nox* is used for "by night:" Tonight.

Que nox furtum faxsit, sei im aliquips occisit, joure caesos estod.

The Adverb *anights* was formerly in common use; as in SHAKSPEARE's *As You Like It*—

CLOWN—I remember when I was in love, I broke my sword upon a stone; and bid him take that for coming *anights* to Jane Smile.

And, in like manner, *an even* was sometimes used for "in the evening;" as in *The Sevyn Sages*—

An even late the emperour
Was browt to bed with honour.

The substantive *e'en* for *even* is still retained in the common salutation of the Scottish peasantry, "*gude e'en*;" but as we have changed *morrow* to *morning*, so we have *even* to *evening*. The Germans, on the contrary, retain *morgen* and *abend*. These circumstances appear to be perfectly accidental; for whilst we have adopted the participial termination in these two instances, we have unaccountably rejected it from the word *dawning*.

The common people, in many parts of the country, still use the Adverbial expressions *to week*, *to month*, and *to year*, which are otherwise obsolete. Some copies of Chaucer have this last expression in the IIId Book of the *Troilus and Cressida* (v. 242.)—

Whan I the saw so languishing *to yere*.

But this is possibly an error of the transcriber.

Some Adverbs of time, which are probably derived from substantives, are also Adverbs of place; but, in general, we mean to consider the Adverbs of place

Grammar. among the prepositions, since the same words are almost invariably employed for these two purposes. Thus we equally say "John walks *before*," in which phrase "before" is an adverb; and "John walks *before Peter*," in which phrase "before" is a preposition. So we say "Peter walks *behind*," or "Peter walks *behind John*:" and a similar observation applies to the words *about*, *above*, *below*, &c.; hence, the old jest, that a man beating his wife in an upper chamber is a man of perfect integrity, "because he is *above*, doing a bad action;" or (with a slight variation of expression) because he is *above doing a bad action*."

Deal. The substantive *Deal* is often employed in the nature of an adverb of quantity. Thus in Saint Mark's gospel, c. x. v. 48.

He cried the more *a great deal*, thou Son of David, have mercy on me.

In ancient times, this word, deal, entered into numerous adverbs and adverbial phrases. We had *halvendall*, *thriddendale*, *somdele*, *everydeale*, *a full great dele*, *a thousand deale*, &c.

In *Kyng Alisaunder* we have both *halvendall* and *thriddendale*.

Alisaunder and his folk alle,
No'hadde nought passed theo *halvendall*.
* * * *
The knightes sloden on heighe brymme
And lepen into the cees arme:
That was bothe reuthe and harme.
Swithe wightlych hy bigynne
The *thriddendale*, and fair swimme.

In CHAUCER, very frequently, somedele—

A goodwife also there was, beside Bathe;
But she was *somedele* defe, and that was scathe.
* * * *
The rule of Sainct Maure, and of Saint Benet,
Bicause that it was old and *somdele* streit,
This ilke monke did letten old things passe.

In the *Romaunt of the Rose*, he thus uses a thousand dele, and euerydeale in the same passage:

Richesse a robe of purple on had
Na trow not that I lie or mad,
For in this world is none it liche,
Ne *by a thousand deale* so riche,
Ne none so faire, for it full weale
With orfraies laied was *euerydeale*.

Again in the Prologue to the *Canterbury Tales*, describing the Physician, he says,

He kept his Pacient *a full great dell*,
In houres, by his majike naturell.

Our word *deal* is the Anglo-Saxon substantive *dæl*, and Gothic *dail*, the Dutch *deel*, the Frankish and Alamannic *teil*, and modern German *theil*, all which signify a part or division. It is the same word with our *dale*, because in hilly regions "the dales" form the great natural divisions of the country; and it is also the Gaelic *dal*, a farm, or division of land occupied by one tenant. As the verbs *dailjan*, *dælan*, *deelen*, *teilen*, and *theilen*, corresponding with the abovementioned substantives in the different northern dialects, all mean to divide, so some others signifying to divide by cutting, are reasonably believed to be of the same origin, particularly the barbarous Latin *taliare*, which is the origin of the Italian *tagliare*, and the French *tailler*, from which last come our English substantive *taylor* (tailleur) and the proper names *Telfair* (taille-fer), *Tallboys* (taille-bois), *Talbot* (taille-bote), &c.

Adverbs.

Afyn. The old adverb *afyn*, appears to be the French substantive *fin* "the end;" but in the following passage from *Syr Launfal* (Cotton MSS. Calig. A. 2. fol. 35. *b*.) it seems to be used as an adverb of quantity in the sense of "sufficiently," "to a sufficient end."

Mete and drink they hadde *afyn*,
Pyement, Clare, and Reynysh wyn,
And elles greet wondyr hyt wer.

Trop. *Trop*, which in modern French is used to signify the excess of quantity or quality, answering to our adverb *too*, was in old French used for the adverb *much*, as "ceste aide eust este moult grant, et *trop* plus que aides de fait de monnoye." It is the Italian adverb *troppo*, from the old barbarous Latin substantives *troppus* and *troppa*, which last was a corruption of *turba*. In the Alamannic laws (Tit. 72. s. 1.), "*si, in* TROPPO *de jumentis illam ductricem aliquis involaverit*," "if in a *troop* of beasts of burden, any person steals the leading animal." From troppus came the French *troupeau*, as from *turba* came the old French *torbe*, and old English *turbe*; as "TORBE *dez cercieles*," "a *turbe* of teles." (See the ancient manuscript entitled *Femina*, quoted by Hickes, v. 1. p. 154.)

Guise, wise, way, gate. The substantives *guise*, or *wise*, *way*, and *gate*, furnish a variety of adverbs principally relating to the manner of doing an action.

Guise and *wise* are the same word, being both identical with the Anglo-Saxon *wise*, Dutch *wyse*, German *weise*, Italian and Spanish *guisa*, and French *guise*; the mode or manner of a thing's existence, that by which it shows itself to us, or makes itself known. Hence we find the German verb *weisen*, Dutch *wyzen*, and Swedish *wysa*, to show, and the German verb *wissen*, Frankish and Alamannic *wizzen*, Gothic and Anglo-Saxon *witan*, Dutch *weten*, Swedish *weta*, and Islandic *vita* to know: and probably from this source are the Latin *video* and *visus*.

CHAUCER, who followed chiefly the French pronunciation, uses the word *guise*—

In swiche a *guise* as I you tellen shal.

BARBOUR, whose dialect was more purely Gothic, says *wise*.

He sware that he shuld vengeaunce ta
Of Bruce, that had presumed sa,
Against him for to brawl or rise,
Or to conspire on sic a *wise*.

Hence we have the adverbs *likewise*, *otherwise*, (which is the Alamannic *andarwis*), &c. which are sometimes expressed in a substantive form, as "in like *wise*," "in no *wise*"—"in what *wise*"—on this *wise*."

In likewyse it is statut be the haill parliament.
Scottish Act, A. D. 1424.

Whosoever shall not receive the kingdom of God as a little child shall *in no wise* enter therein.
St. Luke, c. xviii. v. 16.

For therein is taught how and *in what wyse*
Men vertues shulde use and vices despise.
MS. circa, A. D. 1480.

When Sir Edward the mighty king
Had *on this wise* don his liking
Of John the Baliol——
BARBOUR. *Book* I. v. 180.

Grammar. The modern English adjective *righteous*, and the Scottish legal term *wrongaus*, with their derivative adverbs, are originally from this source. Righteous is the Anglo-Saxon *rightwise*, and it is used by Barbour, for right, lawful—

And that he cam to make homage
To him as to his *rightwis* kyng
• • • •
Lawtie to love is no folly
Through lawtie live men *rightwisely*.

In the Scottish acts, A. D. 1425, occurs *wrangwisly*—

Swa that the causis litigiousis and pleyis be not *wrangwisly* prolongit.

Gate is identical with *gait*, and means *going*; hence the gate of a city or dwelling is that through which men *ga*; the *gait* of an individual is his manner of *going*; and in the old adverbs *algates*, the word *gates*, means the modes of *going* on.

O thou, my love, O thou my hate
For the mote I be dede *algate*.
GOWER, *Confessio Amantis*.

Barbour has "how *gate*"—"this *gate*"—"many *gates*," &c.

He told him hailly all his state,
And what he was, and als *how gate*
The Clifford held his heritage.
• • • •
This gate lived they, in sic thirlage,
Baith puir and thay of hie peerage.
• • • •
For knowlege of manie estates,
May whiles avail ful *many gates*.

It is not surprising, that *way* should be used adverbially in the same sense as *gate*; since they both originally signify a passage, or road by which we reach our destined object. In modern English, indeed, we apply the adverb *always* to time, but this is evidently a secondary meaning. In the old Scottish writers we meet with the phrases "on woman ways"—"on Buchan ways," &c.

In some satirical verses by one CLERK, a contemporary of Dunbar's, are these lines, ridiculing the affected dress of a great man's servant—

With velvet bord about his threid-bare coit,
On woman ways weil tyit about his waist

In the verses of ALEXANDER SCOT, in praise of the month of May:—

In May men of amours suld gae
To serve their ladies and nae mae
Sen thair relief in ladies lyes,
For sum may cum in favour sae
To kiss their luve *on Buchan ways*.

Our common adverb *away* seems to have been formerly written *on way*, and thence *owai*, as in the *Seuyn Sages*, v. 1161.

The maister was *owai* inome
The Emprour was to chaumbre icome.

In Italian the simple noun *via* is used, as *andate via*, go away.—So in Launcelot Gobbo's laughable soliloquy, in the *Merchant of Venice*—

Certainly, my conscience will serve me to run from this Jew my master. The fiend is at mine elbow, and tempts me—*via!* says the fiend—*away!* says the fiend!

Kind. *Kind* which we now use only for "sort" or "species," was formerly *nature*, a signification which it long retained in the English idiom: as in Hamlet's answer to the king:— Adverbs.

KING. But now, my cousin Hamlet—and my son—
HAMLET. A little more than kin, and less than kind.

That is—"cousin and son! a close affinity, indeed!—something more than a common relationship, and yet something repugnant to nature—as he afterwards intimates to the queen.

QUEEN. Have you forgot me?
HAMLET. No, by the rood, not so:
You are the Queen—your husband's brother's wife;
And—would it were not so—you are my mother.

Kin and *kind*, though thus used in contradistinction by Shakspeare, were originally the same word, and doubtless of the same origin with the Greek γενος, and Latin *genus*, connected with which are many large classes of words in most of the northern languages. In the sense of "sort" or "species," it gave occasion to the Scottish phrases *allkin*, or *all kind, no kind, what kind*, &c. In the modern colloquial dialect of that country the expression "*allkin kinds* of things" is not uncommon. The other expressions occur frequently in BARBOUR:—

But God that is of maist poustie
Reserved to his majestie
For to knaw in his prescience,
Of *all kind* time the first movence.
• • • •
But thay would upon *no kind* wise,
Ishe, to assail them in fighting,
Till cured wer the nobil king.
• • • •
The King Robert wist he was there,
And *what kind* chiftains with him were.

Besides the kind, or nature of an action, we may advert to a variety of circumstances expressed by abstract nouns, as wonder, ease, need, abundance, order, chance, fellowship, &c. &c. and all these nouns may take an adverbial construction.

The word *wonder*, has been used as an adverb, in different forms, as *wonder, wonderly, wondrously*. Wonder.

Thus CHAUCER in the *Romaunt* of the *Rose*—

Such light sprang out of the stone,
That Richesse *wondir* bright shone,
Bothe her hedde and all her face,
And eke about her all the place.

BARBOUR uses *wondirly*—

But *wondirly* hard things befel
To him, or he to state was brought.

"Eath," says JUNIUS, "idem est cum *easie* facilis;" and easie he derives from the Gothic *azets*, whence also the French *aise*. Ease.

In that early romance, the *Geste of Kyng Horn*, we find the word *ethe* for easily.

The Kyng hade to fewe
Ageyn so monie schrewe.
So fele myghten *ethe*
Bringe thre to dethe.

That is, "the King (Allof, the father of Horn) had too few (supporters) against so many enemies. So many might *easily* bring three (persons) to death."

JOHN DE TREVISA, one of our earliest English prose writers, has the following passage in his translation of a Latin sermon of Radulf Bishop of Armagh, about A. D. 1387.

"In my tyme in the Universite of Oxenford, were thritty thousand scolers at ones, and now beth *unnethe* sixe thousand."

Grammar.

If this were a solitary instance of the word, we might perhaps suppose it to be of the same origin as *beneath* and to signify "*under* six thousand;" but numberless other instances show that it means "*hardly* six thousand." Thus in CHAUCER's Canterbury Tales

> The glorious sceptre and real majeste
> That hadde the King Nabuchodonosor,
> With tonge *unethes* may descrivid be.

So in the old ballads of *The Huntyng of the Hare*—

> Sum thei fond leyd on the grownd,
> Al thei wer wel ny swonand,
> *Unethe* thei had heir lyfe.

Need.

Need is from the Gothic *nauth*, Anglo-Saxon *neod*, Alamannic *not*, Danish *noed*, Dutch *nood*, all implying necessity, hard compulsion, or want. Hence our colloquial adverb *needs*, in the proverb "*needs* must, when the devil drives." The Dutch proverb says, *nood breekt* wet, for "necessity has no law."

In BARBOUR we find *needlings*—

> ——they that were arrested then
> Were of their taking wonder wo;
> But *needlings* it behoved so.

Abundance.

From the substantive, abundance, we have in modern English use only the adverb *abundantly*; but the idea has perhaps in other times and countries given origin to more than one adverb.

Mr. TOOKE contends that the word *asseth*, in Chaucer's Romaunt of the Rose, signifies *enough*, sufficient, in the following passage, applied to a miser—

> Yet never shal make rychesse,
> *Asseth* unto hys gredynesse.

Where URRY explains *asseth* to mean *assent*; and interprets the passage thus; "riches (here personified as a deity) shall not assent to the miser's greediness;" whereas Mr. Tooke more probably understands it to signify, "that riches will never give sufficiency, or content to the miser's greediness;" in conformity with the preceding lines—

> Rychesse ryche ne maketh nought,
> Hym that on treasour sette his thought;
> For rychesse stonte in suffysaunce.

It remains to be considered whether *asseth*, in this sense, comes from the French *assez*, or from the Gothic word *azets*, above noticed. Tooke's argument of *forth* from *fors*, as *asseth* from *assez*, is conclusive neither way; for as forth does not come from fors, so possibly asseth may not come from assez. Our law-term *assets* is certainly the French adverb reconverted into a noun, and it shows the origin of the word to have been the Latin *ad satis*.

M. COUR DE GEBELIN ingeniously traces another French adverb to a source signifying abundance. *Souvent*, often, he says, is from the Italian *sovente*, and that from the Latin *sæpe*, which he derives from the Hebrew *shepo* abundance; and supposes that the English *sheep* may be from the same source, as implying that in which the wealth of early ages almost exclusively consisted; much in the same manner as *pecunia* is derived from *pecus*.

Order. Chance.

The modern adverb *orderly* is expressed by GAWIN DOUGLAS, *perordoure*.

> He had do schaw the credence that they brocht,
> *Perordoure* alhale thare answere faland nocht.

And in another place, he says—

> Rowpand attanis adew, quhen all is done,
> Ilkane *perordoure*.—

Adverbs. Chance.

The opposite idea, that of chance, has been expressed adverbially in various ways, as perchance, percase, casually, peradventure, perhaps, mayhap, haply.

SHAKSPEARE uses perchance, with a remarkable diversity in the two following passages:—

> VIOLA. *Perchance* he is not drown'd:—what think you, Sailors?
> CAPT. It is *per chance*, that you yourself were sav'd.

Why Mr. Tooke derives *chance* from *escheoir*, rather than from *cheoir*, the old French verb corrupted from *cadere*, it is not easy to discover.

BACON uses *percase*—

> A virtuous man will be virtuous *in solitudine* and not only *in theatro*; though *percase* it will be more strong by glory and fame.

This word, Tooke observes, was anciently written *parcas*, and it certainly was formed by the two French words *par cas*, answering to the Latin *per casum*.

In the Scottish dialect we find *in case* used anciently in the sense of *lest*, and the same use continues provincially to this day.

Thus ALEXANDER MONTGOMERY says—

> He hit the yron quhyle it was het,
> *In case* it sould grow cauld.

Peradventure, (anciently *peraunter* and *paraunter*,) is the French *par aventure*.

In the romance of *The Lyfe of Ipomydon*, we find *paraunter*.

> Tomorrow when I the duke see,
> *Paraunter* in suche plyte I may bee,
> That I wille the bataille take.

It also occurs in the forms of *inaunter*, *inaventure*, *be adventure*, &c.

Thus GAWIN DOUGLAS—

> Quhen thyne allane musing as thou sal ga,
> *Be aventure* besyde ane water bra.

Perhaps, *mayhap*, *haply*, as well as the adjective *happy*, and its compounds, are from the word *happen*, anciently written *hap*, which was used both as a verb and as a noun.

GOWER uses *hap* as a substantive:—

> The *happes* over mannes hede
> Ben honged with a tender threde.

In the ballad of *Octouian Imperator* we find the substantive *unhap*.

> He slogh the xii. dusepers of Fraunce,
> Thys was *unhap* and hard chaunce,
> To all Crystendome.

In CHAUCER we find *uphap* for perhaps, or upon hap:—

> Thou seekest rewarde of folkes smale wordes, and of vayne praysynges. Trewely, therein thou lesest the guerdon of vertue, and lesest the grettest valoure of conscyence, and *uphap* thy renome everlastyng.
>
> *Test. of Love.*

Mr. Tooke seems to be in error in reckoning the anomalous expression *hab nab* among the derivatives from *hap*: it is rather from *hab*, the root of the Latin *habeo*, and of the verbs *haban*, *haben*, &c. to be found in all the Teutonic languages.

LILLY in his *Euphues* employs this expression adverbially—

Grammar.

Philautus determined, *hab nab*, to send his letters.

In the present day it is used rather as an interjection on challenging a person to drink a glass of wine: and seems to have been originally a mark of discarding ceremony. Hab! ne hab! Have it or not, as you will; a form of speaking not unlike the vulgar *will he, nill he*, as in Hamlet:—

Clown. Give me leave. Here lies the water. Good. Here stands the man. Good. If the man go to this water and drown himself, it is *will he, nill he*, he goes; but if the water come to him, and drown him, he drowns not himself.

Fellowship.

In speaking of the adverb *together*, we have already noticed the substantive *fere*, and the pronoun *same* as used to imply fellowship. But it may be worth while to trace these words still further. In Lye's Junius we find "*Fere*, vet. Angl. socius, D. S. fera:" and the word is retained, in the same sense, in the admirable and well known Scottish song of *Auld lang syne*—

An' gie's a hand, my trusty *feir*,
An' here's a hand to thine.
An' we'll tak a right gude-willie waght
For auld lang syne.

In the romance of *Octouian Imperator*, we find *in fere* used for *in company*—

Clement flevgh, and hys wyf *yn fere*
Into Gascoyne as ye mowe here.

In the ballad of "*A contrauerse bytwene a Louer and a Jaye*," printed by Wynkyn de Worde, are these lines—

The foules to here
Was myne entente,
Synrynge *in fere*
On bowes bente.

Barbour, describing the three traitors who attacked King Robert Bruce, has these lines—

—— he perceived that in hy
By their *effeir*, and their having,
That they lov'd him in no kind thing.

And again—

He said, you ought to shame, pardie,
Since I am one, and ye are three,
For to shoot at me *upon feir*.

As we have seen *together* and *in same* used synonymously in the same passage; so we may find passages which employ *in fere* and *in same* synonymously. Thus in the romance of *Richard Coer de Lion*—

To Westemenstre they wente *in fere*,
Lordyngs and Ladyys that ther were—
And aftyr mete, in hyyng
Spak Kyng Henry our kyng,
To the Kyng that sat *in same*,
Leve Sire, what is thy name?

In same corresponds exactly with the French *ensemble*, as may be observed in the instance before quoted, and also in the *Lay le Fraine*, which was evidently a close translation from the French—

Le Codre and her mother there
Ys same unto the bour gan fare.

Sameness. Self.

From close association, to identity, the transition is easy. As *fere* is connected with *same*, so is same with *self*: and perhaps it may not be hazarding too much to say that same is to be found in the substantive form in the Greek σωμα, body; and self in the German *seele*, Dutch *ziel*, Swedish *siæl*, Islandic *sal*, Anglo-Saxon *saul*, Alamannic *sela*, and Gothic *saiwala*, the soul.

Adverbs.

We have traced the adverbial termination *ly* to the substantive *leik*, body; and therefore it is not surprising to find *ilke* (which is only the word *leik* in another form) employed as we now use the word same.

Thus Chaucer—

This *ilke* worthy knight had been also
Sometime with the Lord of Palatie.

So in the Scottish mode of designating the principal family of a name, "Macpherson of that ilk," is Macpherson of the same, or Macpherson of Macpherson.

Wachter observes that the terminating particle *sam* in German, which is our *some*, is synonymous with *lich* (our *ly*); and that the German writers use promiscuously *friedsam* and *friedlich*, for peaceful. So in old English we find *loathly* and *loathsome*, *lovely*, and *lovesome*. The Goths and Germans both compound *sam* with *leik*, but in the inverse order, the former using *samaleiko*, the latter *gleichsam*, adverbially for "as, like as, almost."

The following words and particles, in various languages, seem to be connected with our English *same* and *some*.

In Greek, besides the substantive σωμα, we find the preposition συν, or συμ, and (as the sibilant articulation easily passes into the rough breathing) the adjective ὁμος, and the adverbs ὅμα and ἅμα, with the compounds of all these; σύμμαχος, one who fights in the *same* cause; συμπάθεια, a feeling of the *same* kind, συμφωνια, an agreement of the same sounds; ὅμοιος, like, or approaching to the *same*; ὁμουσιος, of the *same* essence, ὁμοιουσιος, of an essence like, or approaching to the *same*; ἁμάδιον, the adverbial noun of ἅμα, ἁμαδρυάδες, Hamadryades, nymphs who were born and perished at the *same* time with the trees.

In Persian, the particle *hem*, agreeing nearly with the Greek ἅμα in sound, and entirely in sense, forms, when prefixed to nouns, a class of compounds implying society and intimacy, as *hemdshiyan* of the *same* nest; *hemâheng* of the *same* inclination; *hemkhâbeh* of the same sleeping place.

It might, perhaps, be thought too great a refinement of speculation to suggest that in Latin *homo* was connected with the Greek ὅμος, as *sum*, *sim*, *similis*, *simul*, *semel*, were with συμ; but that these latter agree in origin with our English word *same* cannot reasonably be doubted.

In Mœso-Gothic we find *sums*, unus, aliquis, quidam; *sama*, ipsum; *saman*, simul, unà, pariter; *samalaud*, æqualis; *samaleiko*, similiter; *samaleikos*, convenientia, &c.

In Anglo-Saxon, *samman*, to collect; *sibsum*, pacific; *langsum*, tedious; *sam-wyrkan*, to co-operate, &c.

In Frankish, *liepsam*, lovely; *leidsam*, loathsome; *sama*, as, in like manner, *zisamane*, together.

In Islandic, *samfara*, a society; *samlag*, marriage.

In Danish, *samle*.

In Dutch, *samen*, *tzamen*, together; *samt*, with; *tzamenbinden*, to bind together; *tzamenhang*, a series, or connection; and numberless other compounds with *tzamen*.

In German, *sammt*, with; *sammtlich*, altogether,

Grammar. *sammlung*, a collection; *zusammen*, together; *zusammenbinden*, to bind together; and numberless other compounds with *zusammen*; also *langsam*, slow; *langsamkeit*, slowness.

In French, *ensemble* together; *rassembler* to collect together, &c.

In modern as well as ancient provincial Scottish and English the termination *some* is used in many compounds, otherwise obsolete, as *winsome*, *grusome*, *lissome*, *foursum*, *threttiesum*, *lovesome*, *wilsome*, &c.

Lissome, a Wiltshire word, is an abbreviation of *lithesome*, from the Anglo-Saxon *lithe*, pliant, and *lith*, a joint; Islandic *leda*, to bend; old Scottish *lidder*, flexible.

Alexander Scot, in his "Justing and Debate," has—

> Thou art mair large of *lyth* and lim,
> Nor I am be sic thrie.

Gawin Douglas—

> His smottrit habit over his schulderis *lidder*,
> Hang pevagely kayt with ane knot togidder.

David Lindsay uses the word *threttiesum*—

> Thir currish coffes that sails owre sune,
> And *threttiesum* about a pack.

Alexander Scot has *foursum*—

> For were ye *foursum* in a flock,
> I compt ye not a leik.

Lindsay also has *wilsome*—

> He leaves his saul nae gude commend,
> But walks a *wilsome* way, I wiss.

Alexander Montgomery uses *lovesum*—

> Quha wald haif tyr't to heir that tune,
> Quhilk birds corroborate ay abune,
> With lays of *lovesum* larks?

In the ancient MS. No. 2253, of the Harleian collection, we find *lossom*—

> The mone mandeth her bleo,
> The lilie is *lossom* to seo.

And in another poem of the same collection—

> With *lossum* chere he on me loh.

In the romance of *Syr Launfal*—

> Sche had a croune upon her molde
> Of ryche stones and of golde,
> That *lossom* lemede lygt.

In the ancient ballad, "Blow Northerne Wynd," which was probably composed about A. D. 1200.—

> A burde of blod and of bon,
> Never yete y nuste non,
> *Lussomore* in londe.

Self may be traced in like manner through various dialects, as the Mœso-Gothic *silba*, self. Frankish and Alamannic *selbo*, self. Anglo-Saxon *sylf*, self. Islandic *sialf*, self. German *selbst*, self or same, which Wachter explains as *selbist* ipsissimus. And so in compounds, as the Anglo-Saxon *sylf-myrth*, and German *selbst mord*, self murder; the Islandic *sialfvitringur*, self-taught; the Alamannic *selpuuillin*, self-willed. It is not to be doubted but that *self* or *selb* is allied to *seld* or *selt*; and that both are from the more radical *sel*, or *sol*, implying individuality.

In the Anglo-Saxon we find *seld*, rarus, with its comparative *seldor*, superlative *seldost*, and compounds, as *seldhwænne*, &c. In the Alamannic and Frankish *seltkalauffa*, is raro occurrentia, *seltsan*, insolitum; in Swedish *sællsam*, rare. Adverbs.

We find Shakspeare using both *self* and *seld*, in modes now obsolete; thus—

> If I might in intreaties find success,
> As *seld* I have the chance.
>
> *Troilus and Cressida.*

> ——————*seld shown* Flamens
> Do press among the popular throngs.——
>
> *Coriolanus.*

> Being over full of *self-affairs*, my mind
> Did lose it.——
>
> *Midsummer Night's Dream.*

> I would not have your free and noble nature,
> Out of *self-bounty*, be abused.——
>
> *Othello.*

Shakspeare's compound *seld-shown*, is similar in form to the old word *selcouth*, which we have disused though we retain *uncouth*.

Selkouthe occurs in *Kyng Alisaunder*:—

> Thise men han *selkouthe* wyues
> And childern bot ones in all her lyues.

And again, shortly afterward—

> Now is this a *selkouthe* game.

Selcouth and *uncouth* are both from *couth*, which seems to have some obscure connection with the Gothic *qithan*, and Anglo-Saxon *cwethan*, to say; but in signification it means knew, or known.

Thus Chaucer says of the Squire's Yeoman—

> Of wood craft wel *couth* he al the usage.

Selcouth therefore is "seldom known," and *uncouth*, "unknown;" and the latter word shortened to *unco'*, forms in the Scottish dialect an adverb signifying "extremely," "prodigiously," "strangely," as in Burns's "Address to the *unco'* guid, or rigidly righteous." *Unco's* also are used, in the same dialect, substantively, for "news," and also for "strangers;" and in old English uncouth is used as an adjective for "foreign" or "strange." Thus the romance of *Richard Coer de Lion* describes Henry receiving the King of Antioch, and his daughter, on their arrival in England—

> Kyng Henry lyght in hyyng,
> And grette fayr that *uncouth* kyng,
> And that fayr lady alsoo,
> Welcome be ye me alle too.

The French adverb *guere*, or *gueres*, furnishes a remarkable instance of a substantive used adverbially, if the derivation of this word by M. Cour de Gebelin, (who explains it as synonymous with our word *wares*,) be correct. Gueres.

The *Dictionnaire de l'Academie*, thus describes the word, in its adverbial form—

> Guere, gueres, adv. Pas beaucoup, peu. Il ne s'employe jamais qu'avec la negative · *il n'y a guere de gens tout à-fait desinteressez : il n'y a gueres de bonne foy dans le monde.* On le met quelquefois dans le sens de *presque point* : et alors on le joint tousjours avec *que*. *Il n'y a guere que luy qui fust capable de faire cela*, c'est à dire, il n'y a presque que luy.

Gueres, then, adverbially used, signifies, according to the academicians, "not much," "but little," "almost none." Certainly, these meanings are at a great distance from *wares*, goods, or merchandize; and yet it is highly probable that M. Cour de Gebelin's derivation is correct.

In the first place, it is not *gueres* alone, but *ne gueres*

Grammar.

which signifies *pas beaucoup*. We have already spoken of double negatives properly so called; and we may further observe, that in some idioms three, or even four negatives are often accumulated on each other without altering the general effect of the sentence. In the following example from Chaucer, there are four in succession :—

> He *never* yet *no* vilanie *ne* sayde,
> In alle his lif unto *no* manere wight.

Instances of a like kind are to be met with both in Greek and Latin writers. "Notandum est," says Viger, "plures negationes interdum vehementius negare; ut Οὐδὲ μὴν ἔγωγε μὴ οὐχὶ τοῦτο ποιήσαιμι, *Neque ego id unquam fecerim*. Quod Tullius ipse, cum alibi, tum etiam de Finibus iii. cap. 15, imitatus est, dum ait; *Quanquam negent, nec virtutes, nec vitia crescere*." And Hoogeveen adds, "Scitè admodùm *quatuor* negativa conjungit Plato, in Parm. prope finem. Ὅτι τἆλλα τῶν μὴ ὄντων οὐδενὶ οὐδαμῇ οὐδαμῶς οὐδεμίαν κοινωνίαν ἔχει. *Quoniam alia cum eorum, quæ non sunt, aliquo nullibi ullo modo aliquod commercium habent*. But the union of *ne* and *gueres* depends on totally different principles; and *gueres* has only come to be considered as a negative particle in French, from the long habit of using it only in conjunction with a negative. The same is the case with *pas* and *point*; *pas* being the Latin *passus*, a step; and *point*, the Latin *punctum*, a point. *Ne pas*, therefore, is literally not a step, *ne point*, not a point; and it is remarkable that we use the word *jot*, signifying the Greek *iota*, or Hebrew *jod*, (which last is little more than a *point* in writing,) nearly in the same manner. So in Shakspeare—

> This nor hurts him, nor profits you, *a jot*.

And in like manner we use, not a whit, or no whit, as in Hooker—

> The motive cause of doing it is not in ourselves, but carrieth us, as if the wind should drive a feather in the air, we *no whit* furthering that whereby we are driven.

Whit, then, or *jot*, may as well be called a negative as *gueres*, or *pas* or *point*.

Secondly, *ne gueres* is literally "no abundance;" and M. de Gebelin traces three gradations of meaning in the word *gueres*, viz. 1. Exchange; 2. Things exchangeable, or commodities; and 3. Abundance of commodities, or abundance simply. Upon this last transition we may observe that the vulgar in the present day use the word *lots* exactly in the same way; for they not only say "lots of goods," but "lots of fun." As to the preceding steps in the derivation, Wachter considers the Teutonic *waren*, custodire, to be connected with the Greek ὄριον, custodia, and also with the barbarous Latin *warenna*, a warren.

Junius explains *ware*, merx, mercimonium, A. S. *wara*, B. *waere*, Su. *wara*; and, he adds, "potest vox desumpta videri ex *waeren*, cum curâ custodire; quod mercimonia sollicitè custodiantur." Menage in his *Origines de la Langue Francoise*, says of the word *guerite*. "On prononçoit anciennement *garite*. Les Espagnols disent aussi *garita*. Il y a apparence qu'ils ont pris ce mot de nous, et que nous l'avons pris de l'Allemand, ou du Flaman, *waeren*, ou *bewaren*, qui signifie *garder, sauver, conserver*." It is scarcely necessary to add that the French initial *gu* is the Saxon *w*; as in *guespe*, wasp; *Guillaume*, William; *guichet*, wicket, &c.

Adverbs.

Lastly, we may observe that the old French adverb *nagueres*, (explained in the *Dictionnaire de l'Academie* to signify *il n'y a pas long temps*,) is only another form of this same substantive, *Wares*, restricted to the signification of time; as in the example, *Cet homme qui nagueres estoit les delices de la cour*. "This man who, not long since, was the delight of the court." In this sentence, *nagueres* is an abbreviation of the sentence, *Il n'y a gueres de tems*, "there is but a little time."

Parts of the body.

Various parts of the body afford (as Tooke observes particularly of the hand and foot,) a variety of allusions, and adverbial expressions in all languages. Thus we have *headlong, chiefly topsy-turvy, vis-à-vis, face to face, at eye, by eye, at the eye, near hand, hond habbing, maintenant, à genoux, aside, aback, astride, a foot, on foot, foot-to-foot, foot-hot, pedetentim*, &c.

Head.

To begin with the *head*. We will not insist on the derivation of the Latin *apud*, from *caput*, according to some etymologists; but *caput* was certainly the original of the Italian *capo*, which is our *cape*, (or headland,) and the French *chef*, "On a fait *chef* de *capo*," says Menage, "comme *chen* de *cane*, qu'un a depuis prononcé *chien*." Hence from caput comes *pocket-handkerchief*, thus, *chef* the head, *couvre-chef, kever-chefe, kerchief*, a napkin for covering the head or neck; *handkerchief*, the same napkin carried in the hand; and *pocket handkerchief* put into the pocket; whence the Scotch often say pocket-napkin.

In the romance of *Richard Cuer de Lion*.

> The *kever-chefes* he toke on honde,
> And thought in that ylke while,
> To slee the Lyon with some gulle.

De son chef is an adverbial phrase in French, answering to our colloquial expression, of his own head. "On dit aussi *de son chef*," says the Dictionnaire de l'Academie, "pour dire de luy mesme, de son mouvement, de son authorité. *Il a fait cela de son chef, sans en avoir ordre*. *Chef*, the head, being taken for the whole person, the tenant *in chief* was one who held lands of another directly in his own person. "All tenures being derived, or supposed to be derived from the king," says Blackstone, "those that held immediately under him, in right of his crown and dignity were called tenants *in capite*, or in chief." From the same origin is the old law term *chivage*, or *chevage*. "Villeines," says Sir Edward Coke, use to pay their lords an acknowledgement of their bondage, for their several *heads*; and thereupon it is called *chevage, chevagium* of the French word *chef*, as it were the service of the head. Of which Bracton saith *chivagium* dicitur recognitio, in signum subjectionis et dominii de *capite* suo." Thus our *poll*-tax, and *polling* at elections, are from the *poll* taken for the whole head, and *that* for the person. The adjective *chief* is sometimes used by the poets adverbially, but we more commonly say *chiefly*. *Chiefest* is used by Milton—

> But first, and *chiefest*, with thee bring,
> Him that yon soars on golden wing,
> Guiding the fiery-wheeled throne,
> The cherub, Contemplation.

The French had an old adverb *derechef*, for une autre fois, de nouveau; but it is now obsolete. Menage says, "il vient de *derecapo* composé de ces trois mots, *de re capo*."

Of the word *headlong*, Johnson says, (with little

consideration of grammatical principle,) "it is often doubtful whether this word be adjective or adverb;" and thereupon he cites from SHAKSPEARE an instance on which one would think no grammarian could have doubted for a moment—

> I'll look no more
> Lest my brain turn, and the deficient sight
> Topple down *headlong*.

Headlong here applies most emphatically to the verb *topple*, in the manner specified by Donatus; *adjecta verbo significationem ejus complet*.

Eye.

The modern adverb, "*visibly*," supplies the place of several adverbial phrases, relating to the *eye*, in ancient authors.

Thus CHAUCER uses *at eye*—

> This maiest thou understand and seen *at eye*.

GOWER has *at the eye*—

> The thing so open is *at the eye*
> That every man it may behold.

In the romance of *King Alisaunder*, we find *by eyghe*—

> Theo two barouns he knew *by eyghe*.

Foot.

The *foot* supplies various adverbs and adverbial phrases, as a foot, *pedetentim*, and the remarkable expression *foot hot*, occurring frequently in Chaucer, Gower, Gawin Douglas, &c.

A foot is obviously the same as *on foot*, which occurs in the tale of *The Seuyn Sages*, in rather a singular passage—

> A childe thai had bytwix tham two
> The fayrest that *on fote* myght go.

The Latin adverb *pedetentim*, is thus explained by VOSSIUS.

Pedetentim, quasi *pede tentando*. Cato dierum dictarum de Consulatu suo, *Eam ego viam pedetentim tentabam*. Est apud Charisium in II.

"*Foothot*," says Mr. Tooke, "means *immediately*, *instantaneously*," and so far he is undoubtedly right; but whether *hot*, means, as he supposes, *heated*, or as WARTON suggests, *hit* against the ground, that is, *stamped*, may be matter of doubt. "In the twinkling of an eye," "in the space of a look," "at a glance," are expressions used to express the shortest possible lapse of time: and "a stamp of the foot," may well be supposed to convey a similar idea of brief duration.

DUNBAR, in his *Goldin Terge*, has the following lines:—

> And suddenlie, *in the space of a luke*,
> All was hyne went, ther was but wilderness;
> Ther was nae mair but bird, and bank, and bruke.
> *In twinckling of an ee*, to schip they went.

E vestigio is a well known Latin phrase for *confestim*, *properanter*, &c.; thus Cicero, giving an account of the assassination of Marcellus says, "*e vestigio*, eò sum profectus." By a similar analogy we say one misfortune treads *upon the heels* of another: and thus in *Timon of Athens*, the Poet answers the Painter's question:—

> PAIN. Sir, when comes your book forth?
> POET. Upon *the heels* of my presentment.

In this sense the French colloquially use *aux trousses*, or *à ses trousses*. Thus in the *Dictionnaire de l'Academie*:—

Aux trousses. Façon de parler du style familier, pour dire à la poursuite. *Je lui mettray un prevost aux trousses, à ses trousses*. On dit aussi *estre aux trousses de quelqu'un*, pour dire estre toujours à sa suite, soit pour l'espionner soit de quelque autre maniere qui l'incommode.

The good old Bishop LATIMER, who, it must be confessed, was more remarkable for piety in his sermons, than for elegance of style, uses "in their *tailes*."

I will be a suter to your Grace that ye wil geve your Bishops charge, ere they goo home, vpon theyr allegiaunce, to loke better to theyr flocke—and send your visitours *in their tailes*.

Fote hot is generally accompanied with some other expression serving still more clearly to shew the idea of quickness, which it is meant to convey.

Thus GOWER—

> And *forthwith*, all *anone*, fote hote,
> He stale the cowe.

So CHAUCER—

> And Custaunce han they taken *anon*, fote hot.

And GAWIN DOUGLAS—

> *The self*, *stound*, amid the preis, fute hote,
> Lucagus enteris into his chariote.

The same idea is expressed by the phrase *in a trice*, for which Gower uses *as who saith treis*.

> All sodenly, *as who saith treis*,
> Wher that he stode in his paleis,
> He toke him from the men's sight;
> Was none of them so ware, that might
> Set eie wher he becom.

It is therefore probable that *a trice* meant no more than the time of crying "thrice!"—a common signal for starting in a race, launching a vessel, &c. after once, and twice have been called out as notes of preparation.

For *near*, in point of time, we find *nearhand* used, though rather as a preposition than an adverb, in the Scottish Act of Parliament, A. D. 1429, "gif it be *nere hande* the Witsonday or Martynmes, the seysing salbe gevin to the party contrary." This is not very unlike the French use of *maintenant* for "now." *Hondhabbing*, which is a more exact translation of *maintenant*, is used in a different sense in the abovementioned tale of *The Seuyn Sages*:— Hand.

> Th' Emperour saide, I fond hire to rent,
> Hire her and hire face ischent;
> And who is founde *hond-habbing*
> Hit nis non nede of witnessing.

Hondhabbing, or *hand-habend*, is a law term of Saxon origin, corresponding with the Norman term *mainour*, or *manner*; and they are both applied to a thief taken *flagrante delicto*, with the goods stolen in his hand: see *Leg. Hen.* I. c. 59; *Bracton*, lib. 3. tract. 2. cap. 8. &c. "One mode of prosecution, by the common law, without any previous finding by a jury," says JACOB, "was when a thief was taken with the *mainour*, that is with the thing stolen upon him, *in manu*." The French adverb *maintenant*, which is literally the same as *hond-habbing*, being formed of *main*, the hand; and *tenant*, holding; has come to be restricted by use to the signification of "now;" that is to say, the time, which we hold, as it were, by the hand; opposed to that which we have suffered to escape; for the word *maintenant*, "now," is used in contradistinction to *autrefois*, "formerly."

We use the expression *at hand*, as the French do *à la main*, and the Germans, *bey der hand*, to signify a

Grammar. thing that is near, or within reach. Thus Shakspeare in the first part of Henry IV.

> Gadshill. What, ho! Chamberlain!
> Cham. *At hand*, quoth pickpurse.
> Gadshill. That's even as fair as *at hand*, quoth the Chamberlain.

In Latin we find *ad manum*, and *sub manu*, differing from each other, if at all, only by slight shades of meaning, as they both do from *in promptu*, and *ex tempore*, which two latter we have naturalised as a substantive and adjective; for we call an unpremeditated epigram "an *impromptu*;" and an unpremeditated oration "an *extempore* speech." "*Ad manum* esse," says Stephanus, "est aliquid ita in promptu esse, ut quasi manu teneatur." Thus Livy, "adde quod Romanis *ad manum* domi supplementum esset. We find the phrase *sub manu* employed by Plancus in a letter to Cicero, "Vocontii *sub manu* ut essent, per quorum loca mihi fideliter pateret iter." This, as Manutius observes, is a Grecian mode of speaking; for Lucian says, "Ὅτι ἂν πρωτον ὑπο τὴν χεῖρα ἔλθῃ, "quod primum sub manu venerit." The Greeks also have the adverb προχείρως, answering nearly to our phrase "out of hand." In some parts of the West of England, the adjective *handy* is used as an adverb or preposition with reference to place, as "he lives *handy* Warminster;" or, "he lives *handy*."

The Grecians used διὰ χειρῶν, as Cicero does *de manu in manum*, which the French have literally copied in their phrase *de main en main*, and we in ours, "from hand to hand." The Dictionnaire de l'Academie exemplifies this by the following sentence, "C'est une tradition que nos ancestres nous ont laissée *de main en main*."

We say, "to have a work *in hand*;" the Germans say, "*unter den handen* haben." We also say "to take it *in hand*;" they say "*vor die hand* nehmen."

En un tourne-main is an adverbial phrase in French, to signify a very brief space of time, not longer than is necessary to turn the hand: "*c'est un esprit inconstant: il change en un tourne-main*."

The allusion to the *hand* seems to be altogether superfluous, in our adverbs *beforehand* and *behindhand*.

Thus in Arbuthnot's *History of John Bull*—

> When the lawyers brought extravagant bills, Sir Roger used to bargain *beforehand* to cut off a quarter of a yard in any part of the bill.

The substantive use of the word *forehand* is more emphatic; as in King Henry the Fifth's fine soliloquy—

> And but for ceremony, such a wretch,
> Winding up days with toil, and nights with sleep,
> Had the *forehand* and vantage of a king.

Dr. Johnson accuses Shakspeare of a licentious use of the adverb *behind hand*, as an adjective; but the truth is that the mighty poet knew and felt the powers of the English language much better than his critic.—

> ——and these thy offices
> So rarely kind, are as interpreters
> Of my *behind hand* slackness.
>
> *Winter's Tale.*

Face, front. The *face*, and *front*, or forehead, furnish many adverbs and adverbial phrases in various languages. Shakspeare has a front for "in front," "indirect opposition to the face," as in Falstaff's inimitable narrative of his pretended combat:—

> These four came all a front, and mainly thrust at me. I made me no more ado, but took all their seven points in my target, thus. Adverbs.

The Latin *primâ facie* has become naturalised in English style; so that we even speak of "a *prima facie* case." Quintilian has *primâ fronte*, lib. vii. c. 2. "dura primâ fronte quæsto." In the same sense we say "at the first *blush*, this question appears difficult." We have also copied the Greek adverbial phrase πρόσωπον προς πρόσωπον, in our "face to face;" as in St. Paul's First Epistle to the Corinthians, ch. xiii. ver. 12. "For now we see through a glass, darkly; but then *face to face*;" whence the French adverb and preposition *vis-à-vis* is also taken; for *vis* in old French was a face. Thus Marot says in his Temple of Cupid:—

> Car en ce lieu un grand prince je vis,
> Et une dame excellente de Vis.

The Italian adverb *dirimpetto*, expresses the same idea of direct opposition, but refers to the breast, *petto* (from the Latin *pectus*) instead of the face. Our adverb *abreast* is employed in a different sense, which however is not very happily explained by Dr. Johnson. He says— Breast.

> Abreast, *adv.* [see Breast.] Side by side; in such a position that the breasts may bear against the same line.

And then he quotes, as an illustration of this idea, the animated exhortation of Ulysses to Achilles—

> ——— Take the instant way;
> For honour travels in a strait so narrow,
> That *one* but goes *abreast*.

Surely Ulysses did not mean to advise Achilles to advance "side by side" with any other warrior; but on the contrary to keep the path in which but one could travel, and particularly not to suffer Ajax to advance "in the same line" with himself.

In petto has been adopted as an adverb from the Italian language into the English; but only in a figurative sense. We say, "I have a scheme *in petto* to attain this object;" that is, I have it in reserve, unknown to my adversary.

The French sometimes use *à genoux*, "on the knees," in a figurative sense. "*Je vous le demande à genoux*," says the Dictionnaire de l' Academie, "signifie aussi, demander avec un grand empressement." Knees.

Our wellknown adverbs *aside*, *aback*, *ahead*, &c. scarcely need further notice, than merely to show their analogy to the class of adverbs and adverbial phrases of which we are now treating. Side.

Clerk, the Scottish poet, in his satire on Pride, describing the dress of a proud serving-man, mentions—

> His hat on *syde* set up for ony haist.

And so Gawin Douglas—

> Now bendis he up his bardoun with ane mynt,
> *On syde* he bradis for to eschew the dynt.

In Falconer's Marine Dictionary we find the following explanation of the technical meaning of the adverb *aback*, as applied to the sails of a ship:— Back.

> A back, *coeffé*, the situation of the sails when their surfaces are flatted against the masts by the force of the wind. The sails are said to be *taken aback* when they are brought into this situation either by a sudden change of the wind, or by an alteration in the ship's course: they are *laid aback* to effect an immediate retreat without turning to the right or left.

The simple noun *back* is also used adverbially, of which Dr. Johnson has given a variety of examples, diversifying their supposed signification according to the context; as, 1. to the place from which one came; 2. backward from the present station; 3. behind, not coming forward; 4. toward things past; 5. again, in return; and 6. again, a second time; but in reality the force of the word *back* is in all these instances very nearly, if not altogether identical; having the same analogy to the time, place, or other circumstances spoken of, as the back has to the other parts of the body, in their general position.

Lo! the Lord hath kept thee *back* from honour.
Numbers xxiv. 11.

But where they are, and why they come not *back*,
Is now the labour of my thoughts.
MILTON.

In the first instance, bonour is represented, as it were, *before* the person; but he is prevented from advancing toward it; in the other instance, the individuals in question have gone *forward*, and are expected to return; and in both cases the situation expressed by the adverb *back* is that in which the *backs* of the persons were originally placed.

Where we use the simple substantive *back*, adverbially, the adverb *arere* from the French *arriere*, was formerly employed; as in *Richard Coer de Lion*—

Kyng Richard bethought hym thoo,
And gan to crye, "Turne *arere*,
Every man with his banere."

From *back* we form the compound adverb *backwards*, as from *fore* we do *forwards*; and these words, *backwards* and *forwards* are directly opposed to each other in signification, as they are in etymology.

Topsy-turvy. Upside-down.

Topsyturvy, and *upsidedown*, are adverbs perfectly familiar and intelligible in modern colloquial usage; but somewhat obscured by the learned labours of etymologists. SKINNER suggests that *topsyturvy* is "quasi *tops in turves*, i. e. vertices seu capita in cespite." LYE says, "*Topsy-turvy*, inverso ordine. Haud scio an sint à *top*, fastigium, et Isl. *tyrva*, obruere. BARBOUR uses the phrase *top o'er tail.*

And when the king his hounds has seen,
These men assailyie their master sa,
They lap to one, and can him ta
Right by the neck full sturdily,
Till *top o'er tail* they gart him fly.

Upsidedown is so written by SPENCER, RALEGH, and other writers of the age of Queen Elizabeth; but some older authors write it *upsodown*, which Tooke (for what reason does not appear) considers the more proper form of the word.

In the romance of the *The Seuyn Sages* we meet with this phrase several times repeated—

Bitwene the adder and the grehound,
The cradel turnd *up so doun* on ground.
• • • •
The cradel and the child thai found,
Up so doun upon the ground.
• • • •
Of the adder he fond mani tronsoun,
And the cradel *up so doun.*

So GOWER:—

——If the lawe be forelore,
Withouten execucion,
It maketh a londe turne *up so downe*.

Correspondent with the English upsidedown, or upsodown are the Italian *sossopra*, and the French *sans dessus dessous*, which the Dictionnaire de l'Academie thus explains—

Adverbs.

SANS DESSUS DESSOUS. Façon de parler du style familière, qui signifie qu'une chose est tellement bouleversée, qu'on ne reconnoist plus ni le dessus, ni le dessous. On dit aussi *sans devant derrière*, pour dire qu'on ne reconnoist plus ce qui doit estre derrière, ni ce qui doit estre devant."

MENAGE says, "il faut escrire *sens dessus dessous*, comme on escrit *en tout sens, de ce sens là*, &c. *Sens* c'est-à-dire face, visage, situation, posture, &c.

Others again say it should be written *c'en dessus dessous*, as being taken from the old phrase *ce que dessus dessous*, used by COMMINES the historian, "De tous costez ay veu la maison de Bourgogne honnorée, et puis tout en un coup choir *ce que dessus dessous.*"

Ahead is principally used as a sea-term; as in DRYDEN—

And now the mighty Centaur seems to lead,
And now the speedy Dolphin gets *ahead.*

Prefix, a.

Our mariners, indeed, appear to have had a special affection for this prefix, *a*; for they have a vast variety of adverbial expressions, in which it is employed, as *aboard, ashore, ahull, apeek, atrip, aweigh, abaft, aloft, afloat, astern, alee, aloof, alongside, alongshore, amidships, athwartships*, &c. &c. all of which are fully explained in the work before referred to, FALCONER's *Marine Dictionary*. Many other adverbs there are, ancient and modern, beginning with the same prefix, besides those already noticed; as *aswon, alive, afire, ablaze, aloud, asleep, aroume, alove, abroad, alength*, &c.

In the romance of *Amis and Amiloun*, occcurs *aswon* for "in a swoon." Aswon.

He loked opon his scholder bare,
And seighe his grimly wounde thare,
As Amorauut gan him say.
He fel *aswon* to the grounde,
And oft he seyd, Allas, that stounde,
That euer he bode that day.

So in the romance of *The Seuyn Sages*—

The Leuedi when sche herde this,
Aswone sche fil adoun I wis.

In *Octouian Imperator*, we have *on lyue*, for "alive." Alive.

Her sone bygan to the and thryue,
And wax the fayryste chylde *on lyue.*

So in CHAUCER's *Troilus*—

By God, quoth he, that wol I tel as bliue,
For prouder woman is there none *on liue.*

So likewise in a MS. ballad written about the time of HENRY VI. entitled "*How a Merchande dyd hys Wyfe betray.*"—

Y thanke hyt God, for so y may,
That evyr y skapyd *on lyve* away.

In "*A mery Geste of the Frere and the Boye*, emprynted at London, by Wynkyn de Worde," we find "*thy lyve*," used for "thy life." v. 86.

That shall last the, all *thy lyve.*

CHAUCER has "*hir live*," for "in their lives."—

They were ful glad to excusen hem, ful blive,
Of thing the which they neuer agilt *hir live.*

In another passage he extends this adverbial phrase to a greater length "*time of al here lyues.*"

Ne neuer shal, *time of al here lyues.*

The adverb *blive*, which occurs above, is thus noticed

In Lye's Junius, "*belief, belife, belive, blive*, confestim, protinus, statim, extemplo: a Norm. Saxonico *bilive*, de quo nihil certi habeo quod dicam." There seems little doubt, however, but that it is from the substantive, "life," and signifies in a quick and *lively* manner.

It occurs in the romance of *The Seuin Sages.*

> His owin Lady he toke *bylive*,
> And gaf the Knyght until his wine.

The same adverb is found in the ballad on the *Battle of Bruges*, beforementioned.

> Thenne seide the Kyng Phelip lustneth nou to me,
> Myn Eorles ant my Barouns gentil ant fre,
> Goth faccheth me the traytours y bounde to my kne,
> Hastifliche ant *blyue*.

Afire, ablaze. Gawin Douglas uses *in fyre*, for our modern adverbs *a fire*:—

> Turnus seyes the Troianis in grete yre,
> And al thare schippis and nauy set *in fire*.

In like manner, Gower employs the expression *on blaze* for our colloquial adverb *ablaze*:—

> That casten fire and flam aboute
> Both at mouth and at nase,
> So that thei setten all *on blase*.

Alond. In the romance of *Octouian Imperator* is *alond*, for "to land."

> The Kyng of Masydonye com ryde
> With hys ost *alond* that tyde,
> *   *   *   *
> The Kyng of Greece herde that cry,
> *To lond* he rowede ryght hastyly.

Asleep. In *Amis and Amiloun* also occurs *in slepe*, for the modern "asleep."

> The Knight that was so hende and fre,
> Wel fair he leyd him vnder a tre,
> And fell *in slepe* that tide.

Aroume. In *Richard Coer de Lion, aroume* for "aside."

> Alle that was ther tho hym beheeld,
> Hou he rod as he wer wood,
> *Aroume* he hovyd, and withstood.

Alove. Chaucer, in the *Testament of Love*, uses the adverb *alove*, for *in love*.—

> Wo is hym that is *alove*!

Alength. We have before mentioned *on brede*, which is used by Chaucer and Douglas for *abroad*, or *on breadth*; similar to this is the adverb *alengthe*, used by Nicolls, whose work was published in 1550, under the following title. "The hystory writtone by Thucides the Athenyan, translated oute of Frenche into the Englysh language, by Thomas Nicolls, Citezeine and Goldesmyth, of London." In fo. 118, a, is the following passage, "They dyd take a greate piece of timber and made it hollowe—afterwardes they fastened yt wyth yrone at bothe endes, and also *alengthe*."

Home. The substantive *home* is used adverbially in English both in its simple sense of a place of residence, as "to go home;" and in the figurative meaning of completion, which Shakspeare seems particularly fond of giving to it—

> No further halting. Satisfy me *home*
> What is become of her
>
> *Cymbeline.*
>
> ——It confirms me *home*,
> This is Pisanio's deed.
>
> *Ibid.*

> He charges *home* my unprovided body.
>
> *Lear.*
>
> Wear thy good rapier bare, and put it *home*.
>
> *Othello.*

In the simple sense of a dwelling, our adverb *home*, answers to the Greek adverb *οἴκαδε*, and to the Latin accusative *domum*, as the word *heim* does in the German compound *heimgehen*, "to go to our place of residence." But though the nouns *house* and *home*, may in certain cases be applied indifferently to the same edifice, yet we not only do not use the word *house* adverbially, as we do *home*, but we affix a different idea to it when used substantively, with the preposition "to." This peculiarity of idiom cannot be better exemplified than by a circumstance which occurred to a German nobleman, who not long since visited London. *Nach hause gehen*, in German, and *aller à la maison*, in French, both signify "to go home," the foreigner, therefore, returning from a visit, thought that he could not err in ordering his coachman to go "to *the house*;" but as the latter had been accustomed to drive some of his former masters to "the House of Commons," which alone he knew by the distinctive name of "the house," he accordingly proceeded thither, instead of conveying the nobleman to his own residence.

Domi. As "*home*" answers to the Latin *domum*, so "*at home*," answers to *domi*; for as Vossius observes of "domi focique," in Terence, (Eun. Act IV. scen. 7,) "dubium non est quin sint genitivi adverbialiter positivi." Donatus, indeed, goes further; for he calls not only these genitives, but even accusatives and ablatives, adverbs. "*Romæ Romam, Româ*," says he, "sunt adverbia loci, quæ imprudentes putant nomina. In loco, ut sum *Romæ*; de loco, ut *Româ* venio; ad locum, ut *Romam* pergo. And with this very learned grammarian agrees Servius. Diomedes, in like manner, calls *vili* and *carò* "æstimationis adverbia;" and others call *forte, fortuna, nihil, casu, militiæ, belli*, &c., adverbs; which doctrine is strenuously resisted by Vossius in his first book *De Analogiâ*. It is not here necessary to examine this dispute very minutely; but we may observe that the distinction between an adverb, and a genitive case used adverbially is not made out by Vossius with that clearness for which his grammatical writings in general are remarkable. It may be allowed that where a noun substantive or adjective is joined with another, either expressed or necessarily understood, it should rather be considered as making a part of an adverbial phrase than as an adverb. Thus *sponte sua, domi suæ*, or *mane primo*, may be regarded respectively as clauses in a sentence; but *sponte*, or *domi*, or *mane*, alone may be called adverbs; and such is the distinction drawn by that excellent grammarian Priscian.

Palam. Clam. Of the Latin adverbs, *palam*, and *clam*, Mr. Tooke quotes, with some approbation, the etymology given by *M. L'Eveque*, who derives them from the Sclavonic *pole*, "the earth," and *kolami*, "wooden stakes." This derivation seems farfetched; yet it is not impossible that some affinity may have existed between the radical sounds of the Sclavonic and ancient Latin languages. Certain it is, that *clam* was originally written *calim*, as in the law of the Twelve Tables, *quei* CALIM *endo urbe nox coit coiverit kapital estod.* This was the law against secret societies which Por-

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cius Latro charged Catiline with having violated. *Calim*, " secretly, obscurely," had evidently a relation to *caligo*, obscurity, or cloudy darkness, and *caligo* may possibly have been derived from *cala*, a wooden log or stake, which thrown moist on the fire would produce a thick smoke :—

> ———lacrimoso non sine fumo,
> Udos cum foliis ramos urente camino.

The word cala is thus explained by Servius ; " *calas* dicebant majores nostri *fustes*, quos portabant servi sequentes dominos ad prœlium. Unde etiam *calones* dicebantur Nam consuetudo erat militis Romani, ut ipse sibi arma portaret et vallum. Vallum autem dicebant calas. Sic Lucilius,

> Scinde calam, ot calcas :

i. e. O puer, frange fustes, et fac focum." The derivation of *palam* it is not so easy to trace. It signifies " openly, publicly," as in Virgil—

> Ipsa *palam* fari omnipotens Saturnia jussit.

And it may have some relation to the verb *palo* " to wander about," as in Sulpicius.

> Sic nostri palare senes dicuntur.

Or to *Pales* the rural goddess invoked by Virgil,

> ———Te quoque magna Pales.

Or to *palus*, a marsh, or *palus*, a pale or stake.

Possibly all these words, though differing in the quantity of their first syllable, which in some is short and in others long, may have had an indistinct connection ; but be this as it may, we can scarcely doubt that the adverb *palam* was derived from some noun in the old Latin language, and was indeed that noun in an antiquated form. It must be observed too that it was not used merely as an adverb, but as a preposition. Thus Livy says " *palam* populo." Cicero " *palam* hoc ordine," and Horace " te *palam*," which last example proves the error of Calepin and others, who thought that *coram* was " in presence of one person," and *palam* " in presence of many." *Clam* also was used as a preposition, as in Terence, " Hæc *clam me* omnia ;" nor was this all ; for it was sometimes used adjectivally, by the same author. " Si sperat fore *clam*:" in which manner also the correspondent Greek adverb *κρυβδὴν*, was sometimes employed, as by Demosthenes, *Οὐ γὰρ ἐι κρύβδην ἐστὶν ἡ ψῆφος, λήσει τοὺς θεούς.* " Suffragium, etsi obscurum est, Deus tamen latere non potest."

Compound adverbs.

We have noticed the adverbial force of substantives used in the formation of compound adjectives ; particularly of the substantive *stone*, which in forming the compound adjective *stone-blind* serves to modify the adjective, *blind*. The English language is not very rich in compounds ; yet some of this kind occur particularly in our old writers, and in the proverbial and trivial expressions of the vulgar. Thus *bolt-upright*, is as upright and straight as a *bolt*, the old word for an arrow. So Shakspeare uses the compounds *death-practised*, for " practised in death ;" *tongue-tied* for " restrained from speaking ;" *wreckfull*, for " full of wrecks," &c.

> With this ungracious paper strike the sight
> Of the *death-practised* Duke.
>
> *Lear.*
>
> My *tongue-tied* muse in manners holds her still.
>
> *Sonnet* 85.
>
> Against the *wreckfull* siege of batt'ring days.
>
> *Sonnet* 65.

Adverbs.

Chapman, the most poetical of all translators of Homer, abounds in such epithets, as *gold*-helm'd, *mind*-master, *town*-guard, *force*full, *oar*-bound, &c.

> Mars, most strong ; *gold-helm'd ;* making chariots crack ;
> Never without a shield cast on thy back ;
> *Mind*-master, *town*-guard, with darts never driven ;
> Strong-handed, all-arm'd, fort and fence of heaven ;
> Father of victory !
>
> *Hymn to Mars.*
>
> Alcides, *force-fullest* of all the brood
> Of men !
>
> *Hymn to Hercules.*
>
> Chuse two and fifty youths, of all the best
> To use an oar ; all which see straight imprest,
> And in their *oar-bound* seats.
>
> *Odyss.* b. 8.

In the ballad of *The Huntyng of the Hare*, is *ston-styll.*—

> Jac Wade has a dogge wyll pull,
> He hymselvue wyll take a Bull
> And holde hym *ston-styll.*

In the Scottish Act of Parliament, A. D. 1587, entitled " Mesuris and wechtis and the just quantitie thereof," the word *rewl-richt*, (i. e. as straight as a ruler) occurs in the directions for making the Firlot measure. " That the mouth be reyngit about with a circle of girth of irne, inwith and outwith, haveing a croce irne bar passing ovir fra the ane syd to the wther, thrie squarit ane edge doun and a plane syde quhilk sall gang *rewll richt* with the edge of the firlot."

Adverbs themselves may be in like manner compounded. " Ut in aliis classibus," says Vossius, " ita quoque in adverbiis, compositorum alia fiunt è duobus, ut *perdiu*, *abhinc*, alia è pluribus ut *forsitan*. Nam, ut *forsit* ex *fors* et *sit*, quasi *forte sit* ; ac *forsan* ex fors et *an*, quod et in *fortassean* ; ita forsitan ex tribus istis *fors*, *sit*, *an*. And thus it is in English. We have *together* formed of *to* and *gather* ; and we have *altogether* formed of *all*, *to*, and *gather*. So in French *tout à fait*, " altogether," from *tout*, *à*, and *fait* ; in Italian *nondimeno*, " nevertheless," from *non*, *di*, and *meno*, &c. ; in German *vielleicht*, " perhaps," from *viel* much, and *leicht* easily ; *nimmermehr*, " nevermore," from *nie*, *immer*, and *mehr*, &c.

In forming compounds of this nature, all parts of speech (except interjections) are employed. " *Nulla est vocum classis*, says Vossius, " *ex quâ non adverbium componatur*." Thus a composite adverb may be formed in any of the following ways :—

1. From a pronoun and substantive, as *quare* from *quâ* and *re*.

2. From an adjective and substantive, as *postridie*, from *postero* and *die*.

3. From an adverb, substantive, and adjective, as *nudiustertius* from *nunc*, *dies*, and *tertius*.

4. From a substantive and verb, as *pedetentim* from *pede* and *tentare*.

5. From a participle and substantive, as *perendie* from *peremptâ* and *die*.

6. From an adverb and adjective, as *nimirum*, from *ne* and *mirum*.

7. From a preposition and substantive, as *obviam*, from *ob* and *viam*.

8. From a pronoun and adverb, as *alibi*, from *alio* and *ibi*.

9. From a pronoun and preposition, as *adhuc*, from *ad* and *hoc*.

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10. From two verbs, as *scilicet*, from *scire* and *licet*.

11. From two adverbs, as *etiamnum*, from *etiam* and *nunc*.

12. From an adverb and a verb, as *deinceps*, from *dein* and *capio*.

13. From a preposition and adverb, as *abhinc*, from *ab* and *hinc*.

14. From a conjunction and adverb, as *etiam* from *et* and *jam*.

Vossius, not improperly, ranks among compound adverbs those which are formed from other words, by the addition of an adverbial particle, like our prefix, *a*, or termination, *ly*; as *tantisper*, from *tantus* and *per*; *quandoque*, from *quando* and *que*, &c. So we find not only *scienter*, from *sciens* and *ter*, but even *Catiliniter*, from *Catilina*; not only *jucundè*, from *jucundus*, but *Tullianè*, from *Tullius*.

It may be worth while to examine more particularly some of these compounds.

Quare.

To begin with the first, *quare*. This adverb, considered in its origin and derivatives, will aptly illustrate the transition from a distinctly significant phrase, to an indistinctly significant, or consignificant; or, as it has even been termed, insignificant word or particle The entire phrase is *quâ de re*, as in Plautus :—

> An. Nimia nos socordia tenuit.
>
> Ag. *Quâ de re, obsecro* ?
>
> An. Quia jam non dudum ante lucem ad ædem Veneris venimus.

Quâ de re, shortened, in familiar discourse, to *quâ re* signified "for what thing"—"for what cause"—"wherefore," or, as it is expressed in the Scottish idiom, "what for;" as "*what for* did you not come, when you were called?" i. e. why did you not come? The separate words *qua* and *re*, having by long use been melted together in pronunciation, as *quare*, this latter word, in old French, became *quar*, and in more modern French *car*; but the last mentioned word, even in the 16th century, had travelled so far from its source, that the learned H. Stephanus did not recognise in it the Latin *quare*, but thought it was derived from the Greek γάρ. Menage has justly corrected this error in his *Origines de la langue Francoise*, under the word *car*. "Henry Estienne et autres," says he, "le deriuent de γάρ. Il vient de *quare*, et c'est pourquoy vous trouuerez escrit *quar* dans les anciens liures. On prononçoit, il n'y a pas encore long-temps, *care, cando, canobrem, canquan* au lieu de *quare, quando, quamobrem, quamquam*."

Dies.

It is at first sight as difficult to trace the Italian adverb *oggi*, the French adverb *jadis*, and the English substantive *journalist* all to the Latin *dies*; and yet no etymologies are more certain than these three.

Hodie.

From *hoc die*, by the mere rapidity of pronunciation, came the Latin *hodie*; and this, by an imperfect attempt on the part of the barbarians, to imitate the Roman articulation, was easily changed into *hoggi*, pronounced as an Englishman would pronounce *hodje*.

Thus Annibal Caro, in his verses on the death of Francesco Molza, a. d. 1544.

> E questo è'l monte ond'è c'*hoggi* si scorga,
> La gloria de le muse——

and the modern Italians have softened this word into *oggi*.

Adverbs. Journal.

From *dies* also the Romans formed the adjectives *diurnus* and *diurnalis*, "daily;" and these in the corrupt Latinity of the later ages, received secondary meanings. "*Diurnum* pro *die* dixit infima Latinitas," says Salmasius, "et *diurnale* mensuram agri quæ uno *die* posset arari." The Italians from *diurnum*, in the secondary sense of a *day*, made *giorno*, which the French shortened into *jour*; and *diurnale*, in like manner, produced *giornale, journal, journalist*.

Jadis.

Again from the Latin *dies* came the adverb *diu*, and from *jam dies* came *jamdiu*. The Italians altered *jam* into *gia*, and the French into *ja*, and hence *jamdiu* became *jadis*.

Hora.

Hora, another Latin word in constant use to mark the lapse of time, has also undergone very considerable changes. In the Norman French of the 13th century, we find the word *onkore*. It occurs in a letter from *Perres de Mounfart*, (Peter de Montford,) 1 Rym. *Fœd*. p. i. fo. 339, ed. 1816, giving an account of some successes which he had obtained over the Welsh, in 1256. He first states the occurrences of the Thursday next after St. Matthew's day; and then continues, "E *onkore* le lundi siwant;" where the word *onkore* is what was anciently written in French *ancore* and now *encore*. In Italian it is now spelt *ancora*, and was formerly *anch'ora*, "iterum," "again," "once more."

Encore.

Menage, and Cour de Gebelin, derive it from the Latin phrase *in hanc horam*, or hanc horam; but this is not quite satisfactory. Ὦρος, as we learn from Herodotus, was an Egyptian name of the sun, the great measurer of time; from whence, probably, the Greek ὥρα came to signify "time," in general, or a certain portion of time, as "a season," a day," an "hour." And so in Latin "*Hora*," says R. Stephanus, "Tempus significat, h. e. quamcunque æternitatis partem, sive annum, sive diurnum, sive nocturnum spatium complectens. Item partes ipsæ, in quas distinctus est dies, similiter *horæ* vocantur." From the Latin *hora* came the Italian *hora, ora, or*, which was used not only as a substantive signifying a certain portion of the day, but as an adverb signifying "now" "at this hour," "at this point of time."

Thus Petrarca—

> Ma ben veggi' *hor*, si come al popol tutto,
> Favola fui gran tempo.

Hence it was redoubled, with a relative force connecting different parts of a sentence, and signifying "at one time," and "at another time;" as "now," in the following line of Pope—

> *Now* high, *now* low, *now* master up, *now* miss.

Thus Machiavelli in the first book of his *Istorie Fiorentine*, says, "Vedendosi l'Imperatore assalire da tante parti, per aver meno nemici, cominciò, *ora* con i Vandali, *ora* con i Franchi, a fare accordi:" that is, he began to make treaties, *at this time*, with the Vandals; *at that time* with the Franks. Hence also *ora* was used conjunctionally, as connecting one link in a chain of reasoning with all that has preceded it; agreeing also in this respect with our word "now." Thus in South's sermons: "The other great and undoing mischief, which befals men is by their being misrepresented. *Now*, by calling evil good, a man is misrepresented," where the word "*now*" may be para-

phrased, "at this point of my discourse;" as "I have *already* shown you the major proposition, namely, that all misrepresentation is mischief; *now, at this period of my discourse,* I show you the minor proposition, namely, that to call good evil is misrepresentation; and *after* I have shown you the major, and minor, you can easily come to the conclusion yourselves, namely, that to call good evil is to do mischief. Hence the authors of the *Dictionnaire de l'Academie,* say, "*Or* est une particule qui sert à lier une proposition à une autre, comme la mineure d'un argument à la majeure. *Le sage est heureux: or est il que Socrate est sage, donc Socrate est heureux.*" In *ancora,* therefore, the word *ora* itself includes the meaning of *in hanc horam;* and to this is prefixed the Italian adverb *anche* "also," which seems to be a corruption of the Latin *etiamque;* as in Boccacio, "*Anche* dite voi, che voi vi sforzerete, e di che?" *Encore,* therefore, is literally "also now;" "we have heard the song lately, let us *also* hear it *now;*" "we have heard it *once,* let us hear it *again.*" *Hora* also appears in the French *alors* from the Italian *allora,* which is the Latin *ad illam horam;* and in the Spanish *agora* or *ahora,* which is the Latin *hâc horâ.* The French *desormais* is *de hora magis;* we find it written in the abovementioned letter of *Perres de Mounfort, desoremes,* "Pour quei je vous pre e requer—ke hom mette coosul comeot la terre seit *desoremes* defendue."

Qorest. We had formerly a remarkable adverb from this source in our old law French, viz. *qorest,* for so it is written in the Statute of Westminster, 22 Edw. IV. A. D. 1482, "en temps del victorious reigne oostre dit Seignur le Roy *qorest.*" This was a corruption of *qui* or *est,* "who now is." If the word *qorest* had remained in use, and its etymology had been unknown, it might perhaps have prevented an ingenious legal objection, said to have been taken in behalf of a prisoner, who was indicted on a statute passed in the reign of George II. but was not brought to trial until that of George III. when it was argued (in arrest of judgment, or otherwise,) that the indictment charged the prisoner with violating a statute alleged to have been passed in the reign of "our lord the king *that now is,*" whereas in fact no such statute had been passed in that reign. Whether this was a real occurrence, or a fiction, it served to supply the humourous genius of Foote with another jest which also turned on the peculiar use of the adverbs employed. He introduced a character boasting of the skill with which he had escaped from a charge of perjury "We were indicted," says he, "for committing perjury *now,* but we proved that we committed it *then.* If they had indicted us for committing perjury *now and then,* it would have gone hard with us." This adverbial phrase, "*now and then,*" is perfectly idiomatical in English, and there is perhaps no expression exactly corresponding to it, in any other language. The Italian *talvolta,* and the French *de tems en tems,* are somewhat similar to it in signification, but with neither of them is it quite identical.

Adverbial phrases. From what has been said of compound adverbs, it will have been seen, that the greater part of them were originally, short phrases, or clauses added to a perfect sentence, for the purpose of modifying the adjective, or verb, which it contained. The office of such a phrase is, therefore, exactly the same as the office of an adverb, and thence we call it, as Mr. Tooke does, an *adverbial phrase.* Two corollaries follow from this remark, both of which we have seen illustrated in the preceding examples; first, that a distinctly significant adverbial phrase may degenerate, in length of time, to an indistinctly significant adverb; and, secondly, that the adverbs of one language, or idiom, may be supplied by analogous adverbial phrases in another. Adverbs.

For the nonce. An adverbial phrase, which occurs frequently in our old writers, has greatly puzzled most of their commentators—the phrase "*for the nonce.*" As it is used by Shakspeare in two instances, it would seem merely to signify "for the occasion," "to serve the present turn."

I have cases of buckram *for the nonce,* to immask our noted outward garments.

First Part of Hen. IV.

———When in your motion you are hot,
And that he calls for drink, I'll have prepared him
A chalice *for the nonce.*

Hamlet.

Yet, perhaps, even here, a sort of ironical sentiment of admiration at the *importance* of the occasion is meant to be expressed, implying really a contempt for the parties concerned; and this is more clearly the meaning in another instance.

This is a riddling merchant *for the nonce!*

First Part of Hen. VI.,

Admiration appears to be expressed, but not ironically, by Chaucer, in the *Romant of the Rose.*—

But he were konning *for the nones,*
That coud devise all the stones
That in that circle shewen clere,
It is a wonder thing to here.

In the *Canterbury Tales,* on the contrary, he seems to use it with some mixture of the ridiculous:—

The miller was a stout carle *for the nones,*
Full big he was of braune and eke of bones.

And again, the Host, ironically praising the Monk, says to him—

———As to my dome,
Thou art a maister when thou art at home—
And therewithal of braune and eke of bones,
A right wel faring person *for the nones.*

In describing the Cook, it is doubtful whether he means to express any thing more than that this personage was engaged for the purpose of exercising his art in case of need:—

A coke they had with them *for the nones,*
To boil the chickens with the maribones.

Here the phrase might perhaps have been supplied, had the rhyme permitted it, with the other phrase of "*for nede,*" which Chaucer elsewhere uses:—

The stone so clere was and so bright,
That al so sone as it was night,
Men mighte seen to go *for nede,*
A mile or two in length and brede.

Lidgate evidently uses *for the nones,* in the simple sense of "for the purpose."

———Her yoong sonne she tooke,
Tender and greene both of flesh and bones,
To certaine men ordained *for the nones,*
Fro point to point, in all manner thing,
To execute the bidding of the king.

Grammar.

However the writers already quoted may differ in their application of this phrase, still there is no doubt but that they all understood it, and all applied it according to the just analogies of language; but this was not the case with SPENSER, who in the following passage applies it in a manner wholly arbitrary and licentious:

> ———— I saw a wolf
> Nursing two whelps: I saw her little ones
> In wanton dalliance the teat to crave,
> While she her neck wreathed from them *for the nonce.*

Mr. Tooke justly observes that Spenser is no authority for the right use of the English language. The reason is not to be sought in any want of genius, taste, knowledge, or feeling; for in all these this great poet deservedly ranked high; but he had adopted (probably from his great and deserved admiration of Chaucer) the erroneous ambition of writing in an antiquated dialect, and hence his language was often that of no age, ancient, or modern.

In the ballad of "*The Huntyng of the Hare*," we meet with "in the nownes," which seems to be used in a sense rather different, and not very intelligible; though probably of the same origin with "for the nonce."—

> The course Y wold that ye had sene,
> *In the nownes* ye had me the coppe gene;
> For therof had Y nede.

The derivation of the word *nonce*, or *nones*, is as obscure as the exact meaning of the phrase appears doubtful. "NONCE, n. s. (says Dr. JOHNSON.) The original of this word is uncertain. SKINNER imagines it to come from *own* or *once*; or from *nutz*, German, need or use." These two derivations may both be thrown aside as mere conjectures, destitute not only of proof but of probability.

TYRWHIT suggests as its origin the Latin *pro nunc*; but *pro nunc* is hardly to be called a Latin phrase: and from *pro nunc* to *for the nunc*, and then *for the nonce*, are barely possible transitions. JUNIUS says it may be from the French word *noiance*, "atque ita *for the nonce* tantundem significabit Anglis ac si dicerent quin mihi sic libet, vel ob hoc solùm, ut ei incommodem." But this meaning does not seem applicable to any examples of the phrase now extant. *Nonce*, the French denomination of the Pope's *nuncio*, may possibly have led to a phrase of somewhat similar import, for as the nuncio had often powers little short of royal, he must have appeared to the common people as a sort of king or prince; and as we say, "this is a dish for a king," so they might say this is "a cook for the nonce,"—"a cook for the nuncio." He is "a stout churl fit to wait upon the nuncio,"—"a stout carle for the nonce." There is a curious passage in BALE's Acts of English Votaries, (A. D. 1550,) retailing the scandal of a former writer on Thomas a Becket, which seems to give some colour to this explanation. "In the towne of Stafforde was a lustye minion, a trulle *for the nonce*, a pece *for a prince*. Betwixte this wanton damsel, or primerose pearlesse, and Becket the chancellor, went store of presentes, and of loue tokens plenty."

It must be confessed, however, that this explanation will not suit several of the instances which occur in old writers; and it is besides observable that the more ancient orthography was *nones*, from which *nonce* was probably a corruption. Now *nones* is the name of a fixed time of the day, viz. the ninth hour, at which time a certain religious service was always performed. "NONE," says the *Dictionnaire de l'Académie*, "se dit aussi de celles des sept heures canoniales, qui se chantent, ou qui se recitent après Sexte. (L'Ecriture dit que N. S. fut crucifié à *Sexte*, et qu'il rendit l'Esprit à l'heure de *None*.) *Où en estes vous de votre Breviaire? J'en suis à None.* Après *Sexte* on dit *None*, et puis *vespres*." Hence it is possible that a proverb may have originated among the clergy and clerical students, then a very numerous body, that such a one was always ready *for the nones*; and this may have been metaphorically applied to any thing done in due time, and with a special regard to any fixed purpose. It is somewhat in favour of this etymology, that our word *noon*, mid-day, anciently written *none*, is believed to be of the same origin. "NOON, n. s." says JOHNSON, "*non*, Saxon; *nawn*, Welsh; *none*, Erse. Supposed to be derived from *nono*, Latin, the ninth hour, at which their *cœna*, or chief meal, was eaten; whence the other nations called the time of their dinner or chief meal, though earlier in the day, by the same name.

Adverbs.

Anon.

Mr. Tyrwhit endeavoured to help his derivation of *for the nones*, from *pro nunc*, by deriving *anon* from *ad nunc*; but *anon* is probably, as suggested by JUNIUS, *in one*, (minute, understood.) It occurs in the ballad on the *Battle of Bruges.*—

> Tho the kyng of Fraunce yherde this, *anon*
> Assemblede he is dousse pers eueruchon.

So CHAUCER, according to the Harleian MS. No. 1758, fo. 68.

> Our oost vp on his stiropes stood *a non.*

It is somewhat differently written in *Syr Launfal.*

> Wha they had sowpere the day was gon,
> They wente to bedde, and that *anoon.*

So in the Harleian MS. 7333, fo. 150.

> And *a noon* the knyght cride to his seruantis, &c.

LIDGATE also writes it in the same manner, Harl. MS. 2278, fo. 45.

> Wherupon the kyng gan caste *anoon.*

In "*The Proces of the Seuyn Sages*," the MS. of which is transcribed in the Scottish orthography, it is written *onane.*

> The sext maister rase vp *onane*,
> The fairest man of tham ilkane.

And in like manner GAWIN DOUGLAS—

> Thus sayand scho the bing ascendis *on ane.*

For the maistry

To revert to the phrase "*for the nones*," it is in form, though not in signification, like another phrase, "*for the maistry*," which occurs in CHAUCER:—

> A monke there was, fayre *for the maistry.*

This phrase is also found in the rude old ballad of the *Mon in the Mone*, and seems to signify "in a masterlike manner," "in a superior degree:"—

> We shule preye the hayward hom to vr hous,
> And maken hym at heyse *for the maystry.*

There is also some analogy to the phrase "for the nones," in the Latin *pro tempore*, the French *à propos*, the Italian *a posto*, &c.

French adverbs with à.

The French have many adverbial phrases beginning with à, such as *à l'écart, à l'encan, à l'abri*, &c.; the origin of some of which is sufficiently obvious, but of others less so.

A l'écart.

A l'écart, answers to our adverb *aside*; as, *il le mena à l'écart, sous prétexte de promenade*, " he drew him *aside* under pretence of a walk." *Ecart* is also used in French as a substantive, to which we have no single word corresponding, as *son cheval fit un écart*, " his horse *started aside*." This word was formerly written *escart*, and may probably have been more anciently *escarpt*, answering nearly to our expression " a *sharp* turn." Thus MENAGE derives *escarpé* (as *un rocher escarpé*,) from the German *scharf*, formerly *scarf*, in English, *sharp*; and in Anglo-Saxon *scearp*: and elsewhere speaking of the word *escarpins*, he says, it is taken from the Italian *scarpini*, " d'où nous auons fait *escarpins*, en mettant, à nostre ordinaire, un *e* deuant l'*s*.

A l'encan.

The French *à l'encan*, is a mere corruption of the Italian *all'incanto*, " by auction;" and *incanto* is so called from *cantare*, the price offered for the article being *cried out aloud*, or (as our sailors say) *sung out*; whence this mode of sale is called in Scotland " public *roup*," agreeing with the German *rufen* to call aloud; Swedish *roop*, clamour; Gothic *hropjan*, to cry out; Dutch *roepen*, to call; *roep*, a call, &c. In the north of England, too, *roopy* is hoarse, (from crying out,) and a cold, (which makes a person hoarse,) is called a *roop*. In certain parts of the country a sale by auction is termed " a sale at public *outcry*."

A l'abri.

A l'abri is a French phrase of which the *Dictionnaire de l'Academie* gives the following explanation. A L'ABRI, façon de parler adv. à convert, *se mettre à l'abri de la pluye, du vent, du mauvais temps, de la tempeste*." And again, " ABRI, s. m. lieu où l'on se peut mettre à couvert du vent, de la pluye, de l'ardeur du soleil," &c. " ABRI, se dit aussi fig. de quelque lieu que ce soit où l'on est en seureté, et généralement de tout ce qui nous met hors de danger." " On dit fig. *se mettre à l'abri de la persécution*." On the origin of this word etymologists differ, and the way in which they differ serves to illustrate the true and false genius of etymology. PIERRE PITHOU, a very learned old French lawyer, in his valuable treatise on the Counts of Champagne, derives the name of the country of *Brie*, in France, from *abri*; and that from *arbre*, because that which is under cover of a tree is *à l'abri*, protected from the rain, wind, and sun; and MENAGE, catching at this ingenious notion, fills up from his own imagination the steps by which the supposed derivation has proceeded. From the Latin *arbor*, pronounced *albor*, and thence *alberus*, says he, came the Italian *albero*; and from *alberus* came *albericus, albricus*, which the Spaniards pronounced *abrigo*, and which COVARRUVIAS explains *reparo contra las inclemencias del cielo, particolarmente contra el frio*. Now, the fault of this reasoning is, that it is almost entirely conjectural; and conjectural etymology is like conjectural criticism, which ought only to be indulged in very sparingly, and under the control of a most sound and experienced judgment. There is no doubt but that BENTLEY was a man of prodigious learning; but a more ridiculous book was never published than his edition of Milton's Paradise Lost, in consequence of the absurd latitude of conjectural criticism, which he allowed himself in the notes. Among Etymologists some most ingenious men, such as COUR DE GEBELIN and WHITER, may be taxed with this infirmity, nor is MENAGE entirely exempt from it, though his work contains abundance of sound information on language. The other and more judicious derivation of *abri* is from the Latin *apricus*. *Aperio* was to lay open, as in LIVY, " quum calescente sole dispulsa nebula *aperuisset* diem," and in PLINY, " Quem (florem) noctu comprimens, *aperire* incipit solis ortu." Hence, (as Servius observes), *Aperilis*, or *Aprilis*, was the month which *opened* the earth in spring. The old Latins, in like manner, called places open to the sun *aperica*, " Aprica loca dicuntur," says SALMASIUS, quæ opportunè Solem accipiunt, quasi *aperica*, quòd Soli aperta sint, nam *apericum* veteres dixere." Hence Virgil by this epistle describes old men fond of *sunning* themselves.—

Aprici meminisse senes.

And in like manner he applies the same epithet to the sea-birds delighting to *sun* themselves on the open rock in summer time :—

Ex procul in pelago saxum—
—*apricis* statio tutissima mergis.

Now, those places which were distinguished as open to the sun, were generally *sheltered* from cold biting winds; and it was with reference to this circumstance that they were so called; for *apricus* was a word of the winter or spring, but not of the summer. " Est sciendum," says R. Stephanus, " apricum non dici nisi respectu frigoris. Nam in æstivo calore nihil propriè *apricum* dicitur." Whatever, therefore, was *sheltered* either from cold, wind, rain, or even from the extreme heat of the sun, came to be called *apricum*, and this word shortened, as is common in French words derived from the Latin, formed *apri*, or *abri*.

At random.

Some of the French adverbial phrases beginning with *à*, have been adopted, as words, into the English language. Such are our colloquial adverbs *alamode*, and *apropos*. Others have furnished us with adverbial phrases, such as, *at random*. The substantive *randonnée* still remains in French as a term of the chace. " *Le lievre fut pris à la troisieme randonnée*," and the word *randon* was formerly in use. MENAGE says, " RANDON. S'enfuir *à grand randon*: l'origine de ce mot ne m'est pas connue. Du substantif *randon*, on a fait le verbe *randonner*, pour s'enfuir rapidement." From the French *randonner* came the verb to *randy*, used in the west of England in the peculiar sense of taking the part of a candidate at an election in a noisy and riotous manner. The adjective *randy* is also used in the north of England, and in Scotland, by the vulgar, to signify riotous, noisy, obstreperous. See GROSE's Provincial Glossary. The word *randon* was early introduced into the English language; for it occurs in the *Description of Cokaygne*.

The monkes ligtith nogt adun
Ac furre fleeth *in a randun*.

In the romance of *Richard Coer de Lion*, we find " *with gret randoun*."—

Hys brothir come to that bekyr,
Upon a stede, *with gret randoun*,
He thoughte to bere Kyng Richard doun.

BARBOUR uses the expression " *into a randoun*."—

Sir Aymer then, but mair abade,
With all the folk he with him had,
Ished enforcedly to the fight,
And rode *into a randoun* right.

Hickes derives *randon* from the Frankish *rent dun* a torrent, compounded of *rennan*, "to run," and *dun* "down." In the abovementioned *Description of Cokaygne*, the word *rent* occurs signifying the running of a stream.—

Ther beth iiii. willis in the abbei
Of tracle and halwei
Of baum and ek piement
Euer ernend to rigt *rent*.

The Gothic and its derivative languages often use *rennen* and *rinnen* in the sense of flowing; and to this origin Wachter is inclined to attribute the name of the *Rhine*. "Hue etiam," says he, "spectat, multorum judicio, *Rhenus*"

Spick and span.

Spick and Span is an adverbial expression, which at present has descended to the vulgar, but which was currently used by many of our best authors from Chaucer to Swift. Mr. Tooke has rather dogmatically laid it down, with some contempt for those who may differ from him in opinion, that the proper signification of *spick and span new*, is "*shining new from the warehouse*." The way in which he makes this out is rather curious. *Spyker*, he says, is a warehouse in Dutch, and *spange* is any thing shining in German. The Dutch use the phrase *spick-spelder-nieuw*; and the Germans use the phrase *spannen*; and therefore by taking *spick* from the Dutch and *span* from the German, we may ascertain the meaning of the English *spick and span*. We cannot say, that this appears to us a very satisfactory mode of illustrating our own language. Dr. Johnson (though no great etymologist) seems in this instance to have proceeded more rationally, in looking to the English words *spike* and *span* as likely to throw some light on the subject. We doubt, however, whether he is altogether right in saying that *spick and span new* is a metaphor originally taken from cloth, and signifying "newly extended on the spikes or tenters." Perhaps it will be found that the two expressions *span new*, and *spick and span new* are of different origin. It is true, that *spannan* in Anglo-Saxon is to stretch, and from thence comes our verb to *span*; the participle of this latter, however, is not *span* but *spanned*; as in Shakspeare—

———My life is *spann'd* already,
I am the shadow of poor Buckingham.

But the word *span*, *spon*, or *spun*, was the participle of the verb to *spin*; as in the memorable old distich of the friends of equality—

When Adam delved and Eve *span*,
Who was then the gentleman?

Span-newe, therefore was litterally *newly spun*; and so it appears to have been used by Chaucer—

———————Troylus
Was never ful to speke of this matere,
And for to praysen unto Pandarus,
The bounte of his righte lady dere;
This tale was aye *span-newe* to begyne.

It is still more clear that such is the meaning of *spon-neowe*, in the romance of *Kyng Alisaunder*; where the king instead of punishing the Persian who attempted his life, sends him away with honours and rewards—

Richeliche he doth him schrede,
In *spon-neowe* knyghtis wede,
And sette him on an hygh corsour,
And gaf him muche of his tresour.

Spick and Span, or more probably *Spike and Span*, was more probably taken from the lances in use in former times, of which the *spike* was made of iron, and the *span* or part grasped in the hand, was made of wood. Of course a lance which was new, both in *spike* and *span*, was considered as most valuable.

The idea of newness is expressed in various ways by the people of different countries, as by the Dutch *spick-spelder-nieuw*, according to Mr. Tooke, "new from the warehouse and the loom;" by the Germans *span-neu*, *spannagel-neu*, *funkelneu*, and *funkelnagelneu*, by the Danes *funckelnye*, and by the Swedes (as Mr. Tooke says) *spitt-spangandny*. Adelung does not agree with Tooke in considering *span* in *span-neu*, and *span-nagel-neu*, to signify *shining*; but he thinks its meaning doubtful. He however elsewhere observes that *spinnen* (in the past tense *ich spann*, and by the vulgar *ich sponn*) is a very old word, being found in the Gothic, Anglo-Saxon, Frankish, Islandic, Swedish and English, and being derived, as he is inclined to think, from the Greek σπαειν. This, therefore, may perhaps have been the origin of *span-neu* in German as in English; while *spann-nagel-neu* may have been derived from *spanne*, "a span," the measure of the outstretched fingers, and "*nagel*," the finger-nail; so that it would imply newness "in part and whole," "in span and nail," *ad unguem*.

The labour of the smith appears to have suggested the metaphors of *funkel neu*, i. e. sparkling new; as it certainly did our *fire-new*, and *brand-new*, or *brent-new*. Fire new. Brand new.

Thus Shakspeare—

Despight thy victor sword, and *fire-new* fortune.
Lear.

Your *fire-new* stamp of honor is scarce current.
Richard III.

And Burns, in his incomparable tale of *Tam o'Shanter*—

Warlocks and witches in a dance—
Nae cotillion *brent-new* frae France,
But horn pipes, jigs, strathspeys, and reels,
Put life and mettle in their heels.

The adverb, or adverbial expression, *pell-mell* is rather curiously treated by Johnson, who designates it a *noun substantive*, and in proof of his assertion cites two passages, in which it has an *adverbial*, and one in which it has an *adjectival* construction. "Pell-mell," says he, "n. s. [*pesle mesle*, Fr.] confusedly; tumultuously; one among other." Pell-mell.

When we have dash'd them to the ground,
Then defie each other; and *pell-mell*
Make work upon ourselves.
Shakspeare's King John.

——Never yet did insurrection want
Such moody beggars, starving for a time,
Of *pellmell* havock and confusion.
Henry IV.

He knew when to fall on, *pellmell*,
To fall back, and retreat, as well.
Hudibras.

So much for Johnson and his examples. How long "confusedly," or "tumultuously," have been

Grammar. nouns substantive, we know not; but clear it is, that in such phrases as "to make work on ourselves, pell-mell," or "to fall on, pell-mell, the word *pell-mell*, performs the function of an adverb in modifying the verbs "make," and "fall," respectively; and it is equally clear, that in the phrase "a time of pell-mell havock," the same word performs the function of an adjective, in qualifying the substantive "havock."

Johnson is certainly right in deriving this word from the French *pesle mesle*, now written *pêle-mêle*; but the previous history of the word it is not quite so easy to trace.

The origin of the latter syllable, however, is much clearer, than that of the former. We will begin with the Greek verb μισγω to mix, which is supposed to agree with the Hebrew *misseeh*. The roots μισγ and μιξ, appear to be the same (just as our English verb *ask*, pronounced by the vulgar *ax*, is the Anglo-Saxon *acs* in the verb *acsian*,) and they also appear in the simpler form μιγ; large classes of words involving the same idea, being derived from each of these sources, as Μισγάγκεια, Μισγητεύω, Μισγοδείη, Μισγονόμον, Μιξαιθρία, Μιξέλληνες, Μιξις, Μιξοδίη, Μιγα, Μιγαν, Πολυμιγής, Ἀργυρομιγής, &c.; but the most remarkable, with reference to our present inquiry, is Μισγόλαν a tumult, "nimirum," says the Lexicographer, "ex conflaxu et *mistione* multitudinis."

The same roots, *misc*, and *mix*, are of early date in the Latin language, appearing in the words *misceo*, *mixtus*, *mixtio*, *mixtir*, &c.

The Frankish *miskenti*, mixing; and *duruhmiste*, thoroughly-mixt; are of this origin, as are the Armoric *misgu*, and many Sclavonic words in various dialects.

In the modern German, we have *mischen*, to mix; *mischung*, mixture, &c. In the modern Italian, are *mischiare*, *mischiato*, *mischianza*, *mischiamento*, *mischio*, *mischiatura*, and likewise *mistura*, *mistione*, *mistianza*, *mistiato*, all conveying the idea of mixture. It is also observable that the older Italian writers use *mischia* and *mistia*, for a quarrel. In Spanish we have *mixto*, *mistion*, *mistura*, *misturar*, all signifying to mix. In Portuguese *mestura*, *mesturar*, *mesturado*, *mestico*, *misturada*, *mixto*. In French *mixte*, *mixtion*, *mixtioner*. In English to *mix*, *mixture*, &c.

But we must now revert to the Latin. In that language derivatives in *ellus*, *illus*, *olus*, *ulus*, &c. were common; as *misellus*, from *miser*; *tenellus*, from *tener*; *lapillus*, from *lapis*; *sciolus*, from *scio*; *tepidulus*, from *tepeo*. In like manner we find, in the times of classical antiquity, *miscellus*, and *miscellio*, from misceo; as "vites *miscellæ*," explained to be "quæ in omni agro conveniunt, quod cæteris ubique utiliter *miscentur*." CATO, de R. R. cap. vi. and so "Uvæ *miscellæ*," "quibus præliganeum vinum fit, quod operarii bibant." Ibid. cap. xxiii. According to FESTUS, "*Miscelliones* appellantur, qui non certæ sunt sententiæ, sed variorum *mixtorumque* judiciorum."

JOANNES DE JANUA, a writer of the middle ages, uses this word *miscellio* for one "qui novit artem *miscendi* varios cibos."

In the barbarous Latin of those times we also find *misculare*, of which we have the following instances—

Nulla persona audeat pillum *misculare* cum aliquâ lanâ."
Stat. Riper. cap. 225.

Aurum vel argentum adulterasse, vel *misculasse*.
Edict. Pistense, cap. 23.

Adverbs.

De istis rapinis nihil vos debeatis *misculare*.
Hincmar.

And hence was formed *misculatio*:—

Per ipsam *misculationem*.
Placit. sub Carol. Mag.

From *miscel* or *miscul*, to *mell*, or *mel*, the transition was easy; thus, 1. *miscel*; 2. *miscl*; 3. *mescl*; 4. *misel*; 5. *mesel*; 6. *misl*; 7. *mesl*; 8. *medl*: 9. *mell* or *mel*; of all which there remain instances in the barbarous Latin of the middle ages, forming words which convey the idea of *mixture*, under various modifications; such as *mixt* corn, *mixture* or community of goods, a *mixt* colour, a disease which rendered the skin of a *mixt* colour; a mob of people confusedly *mixt* together; a disturbance, or fight, in which the parties were confusedly *mixt*, &c.

1. *Miscel*, as "Miscella," and "Miscelantia."

Miscella, mixt and confused riots, or disturbances, "Si *miscella* in villâ fortè facta erunt." Charta Theob. Com. Campaniæ, A. D. 1200.

Miscelantia (erroneously written *miscedantia*) is also a *mixing* of people in a tumult—

Quod aliqua persona non debeat currere, cum armis, ad aliquam rixam, *miscelantiam*, vel rumorem.
Stat. crimin. Riper. cap. 175.

2. *Miscl*, as "miscla," and "misclantia."

Miscla occurs in the same sense as *miscella*, and in the same charter.

Misclantia, is used, like *miscelantia*, to signify a riot "Consules teneantur denuntiare omnes rixas et *misclantias*." Stat. Mantuæ, c. 17.

3. *Mescl*, as "mescla," "mesclania," "mesclelana," "mesclatus."

Mescla, in an account dated A. D. 1322, is used to signify "*mixt* grain."

Mesclania, in a charter of the year 1244, occurs, with the same sense of "*mixt* grain."

Mesclelana seems doubtful; it may be of the same meaning as *mesclania*; or it may be intended to signify "*mixt* wool." "Pannorum falsorum et falsæ *Mesclelanæ*." Chart. Libert. Cast. Nov. de Arrio, A. D. 1356.

Mesclatus pannus," in a charter of the year 1329, is "cloth of a mixt colour."

4. *Misel* as "misellus," and "misellaria."

Misellus, a leper. This word occurs in very old writings, particularly in a charter of the year 1165, and also in Matthew of Paris, under the year 1254. Some authors suppose it to be the classical word *misellus*, used by Cicero as a diminutive of *miser*; but this would not account for its peculiar application to the disease of leprosy, whereas its agreement with the old French *mesel* and the English *measles* in referring to a disease which gives the skin a *mixed* colour, leaves little doubt that its origin is from *miscellus*; more especially when we consider that though we, for convenience, here arrange the barbarous Latin words with the classical, before we come to the Italian and Old French, yet in reality the two latter were in most instances the medium of corruption, which led to the Latin of the middle ages.

Misellaria. An hospital built for lepers. This word is common in old records, particularly in a charter of the year 1245.

5. *Mesel*, as "Meselia," and "Meselaria."

Meselia is explained "bonorum mobilium communitas inter conjuges." Regist. Parlam. Paris, A. D. 1267.

Meselaria is the same word as *miselaria* already noticed. "Lego pro remedio animæ meæ Centum Libras Turonenses Monasteriis, et Ecclesiis, Hospitalibus, *meselariis*, capellanis et pauperibus in civitate Tolosanâ." Chart. A. D. 1281.

6. *Misl*, as, "mislata."

Mislata is used for "a tumult." Vide Concil. Lillebon. A. D. 1083.

7. *Mesl*, as, "meslea,"—"mesleia,"—"mesleare," "mesleiare,"—"mesleta."

Meslea, "a tumult." Chart. A. D. 1293.

Mesleia this word also occurs for "a tumult," in many ancient charters, e. gr. "Si homo episcopi fecerit *mesleiam* in terrâ comitis." Chart. A. D. 1206. So in Chart. A. D. 1207, 1224, &c. In one instance it is erroneously written *merleiam*.

Mesleare, "to mix"—"monetas prohibitas cum bonâ monetâ *mesleare*." Edict. Phil. Pulchr. A. D. 1329.

Mesleiare, "to quarrel," "to "raise disturbances"—"si infra claustrum serviens rixando, vel *mesleiando* aliquem percusserit." Chart. A. D. 1206.

Mesleta is used for riots or tumults in the old Sicilian Constitutions.

8. *Medl*, as "medletum."

Medletum an affray or tumultuous quarrel. "Cognoscere de *medletis*," to hold plea of affrays, or tumultuous quarrels. GLANVIL, l. i. c. 2.

9. *Mell*, or *mel*, as "melleia," "meleia," "melleya," "meleare," "melleta," "melliator."

Melleia often occurs for a tumult, particularly in Stat. Eccl. Meldens.

Meleia is used indifferently with *mesleia* for a tumult, in the charter of 1206, before quoted, as "si ad *mesleiam* applegiatus sit—si ad *meleiam* aplegiatus non fuerit."

Calida melleia, or *calida melleya*, a tumult while the blood is warm—this word occurs in many instances. Vide Tabular. S. Genov. Paris, A. D. 1241. Chart. Phil. iii. Reg. Franc. Chart. A. D. 1352, &c.

Meleare to riot or make disturbance "rixando vel *meleando*." Chart. A. D. 1206.

Melleta occurs in the old laws of Scotland for "affrays."—"Ad vicecomites etiam pertinet, propter defectum dominorum, cognoscere de *melletis*, de verberibus, et de plagis." Regiam majestatem, l. i. c. 3. s. 7.

Melliatores are common brawlers; Stat. Coll. Corunb. A. D. 1380.

In the modern languages, we find numerous analogies to the words already quoted from the barbarous Latin.

From *misculare*, come the Italian *mescolare*; *mescolamento*, *mescolante*, *mescolanza*, *mescolata*, *mescolatamente*, *mescolato*, and *mescolatura*. Also the Spanish *mescla*, *mesclar*, *mesclado*, *mescladura*, and the Portuguese *mesclar*, *mesclado*.

It is also worth while to note the Italian *misléa*, which, like the barbarous Latin *mesleia* signifies a tumult or conflict "onde si cominciò una grande zuffa e *misléa*." Giov. Villani.

The various dialects of the French language, however, will more clearly point out the connection of these terms with our present adverb.

In a charter of BERNARD DE LA TOUR, in the provincial dialect of Auvergne, A. D. 1270, *mescla*, is used, like the barbarous Latin *miscla*, for a tumult. "E si l a *mescla*, e om l trai glasi nudament, per la *mescla*." "And if there be a riot, and men draw their swords nakedly during the riot."

Meselaigne, like *mesclania* above cited, signifies "mixt grain," "une quarte de mesclaigne." Reg. cens. Dom. de Nereux, A. D. 1418.

Meslée is used for a crowd, or *mixed* and confused number of persons, in a letter written A. D. 1479, "une *meslée* de gens, qui estoient assemblez au lieu de Semur."

The *Dictionnaire de l'Academie* says of this word, "il se dit proprement d'un combat opiniastré, ou deux troupes de gens se *meslent*, l'espée à la main, l'une contre l'autre. *Rude meslée, sanglante meslée, se jetter dans la meslée.* Il se dit aussi d'une batterie de plusieurs particuliers: *il y a une grande bagarre, une grand meslée, dans la rue. Il a perdu son chapeau dans la meslée.* Il se dit aussi d'une contestation aigre entre plusieurs personnes. *Comme je vis que la dispute s'echauffoit, je me tiray de la meslée.*"

But though the substantive *meslée* is thus chiefly confined to quarrelling or fighting, the verb *mesler* is applied to almost any sort of mixture, as *mesler des grains ensemble, mesler des couleurs, mesler l'eau avec le vin*, &c. &c.; in short it is, as MENAGE justly observes, merely the Italian verb *mescolare* abbreviated.

The old adjectives *meslius*, *meslieux*, take their meaning from the substantive *meslée*: the nouns *mesil* and *mesel*, take theirs from the verb *mesler*.

Meslius is an old French word for quarrelsome, riotous; as in *Le Doctrinal*:—

Li hom qui par coustume est *meslius*.

Meslieux has the same signification, or rather is the same word varying only in orthography. "Icellui Guerard, qui estoit homme merveilleux *meslieux* et rioteux." M. S. Letter, A. D. 1432.

Mesil is "mixt grain." "Le carge de *mesil* xiii. den." Pedag. Bapal.

Mesel, a leper, leprous. "Oindre le visage du Seigneur, qui estoit *mesel*," M. S. Letter, A. D. 1408. "Li *mesel* ne poeat estre heirs a nului." Anc. Const. Normand. This severity of the law against lepers was not peculiar to Normandy. Great part of the romance of *Amis and Amiloun* turns upon this circumstance; and Mr. Weber, the learned editor of that romance, has collected in the notes some curious information respecting the laws relative to lepers; especially from a MS. in the French Royal Library (No. 8407,) where it is said "que home ne pot sa femme lessier que por fornication, et por lepre non, et *mesel* se poent marier." The fate of "False Creseide," as related by CHAUCER, also illustrates this subject; and CHAUCER employs the word *mesel* for a leper.

Mezellerie is an old French word for a hospital of lepers.

Mézeline is described in Restant's Dictionary, as "sorte d'étoffe *mêlée* de soie et de laine."

Mesteil is doubtless of a similar origin. It is said in the *Dictionnaire de l'Academie*, to be "Froment et seigle *meslez* ensemble."

Meller agrees with *meleure* abovementioned, in signifying to riot, quarrel, or coose to quarrel. Thus in a letter dated A. D. 1427, is the following passage:—"Pour ce que icellui Wairon, qui estoit parent au suppliant, l'avoit *mellé* eovers le Seigneur Du Bos."

Melleys is the same as *meslius*, or *meslieux*; e. gr. in a letter, A. D. 1375, "Jehan Fenin, qui estoit homs rioteux, et felons, et *melleys*."

Mellif agrees in significatioo with the preceding. "Si aucun des dits chappelains est *mellif* ne rioteux." Chart. Joan. Duc. Brit. A. D. 1433.

Fioally *mellée*, or *mêlée*, is the same as *meslée*, the Italian *mistéa*, the Auvergnese *mesclu*, and the barbarous Latin *meleia*, *melleia*, *meslea*, *miscla*, and *miscella*, signifying a closehanded battle, or tumult, io which the different parties are confusedly *mixed* together, and fight, as we say, *pellmell*.

Thus in the *Roman De Vacces*—

> Tel vient sain à *mellée* qui au departir saigne.

Hcoce *caude mellée* was the literal translation of *calida melleia*. Philipe de Beaumanoir says "quant *caudes mellées* sourdent entre gentilshommes."

This latter term was early adopted into the jurisprudence of Scotland. The following passage occurs io the laws of King Robert II. A. D. 1372, "homicidium ex calore iracundiæ, videlicet *chaudemelle*." In the English law, as Glanvill had written *medletum*, where the Scottish lawyers had writteo *melletum*, so for *chaudemelle* was writteo *chaudemedley*, which has since been corrupted into *chancemedley*.

Indeed our *meddle* and *medley* are only the Freoch *mesler* and *meslée*, changing, as Sir Edward Coke observes, the *s* into a *d*.

Upon the whole, then, it is clear that the syllable *mell* in the adverb *pellmell* is derived from the Greek *μιγω*, and signifies a *melée*, or *mixt* contest.

But what is the signification of *pell*; or has it any signification?

It might perhops appear at first sight not improbable to derive this syllable from *pela*, *pellir*, *pelain*, or *pila*.

Pela is a barbarous Latin word, from the old Latin *pala*, a stake, and it is the origin of a word spelt very variously in French, *paelle*, *paele*, *pelle*, *pêle*, used in modern language for a shovel, either of wood or iron, but probably in more ancient times for a plaio wooden stake.

Pellir is used in a charter of the year 1411, for to drive away with such a stake or shovel "pela cogere."

Pelain is explained by CARPENTIER, the continuator of DU CANGE, to signify "clades, strages, *defaite*, *deroute*, in Gest. Brit. apud Marteo, tom. iii. anecd. col. 1465.—

> Ceci leur fist a crespelain
> Ou il les mist en tel *pelain*.

Pila is "a ball," from whence comes our word a *pile* or heap, and *pilagium*, which is used in a record of the *Chambre des Comptes* at Paris, A. D. 1310, and which CARPENTIER explains "servitii genus, messem uempe, seu fœnum in *pilam* sive struem ordinare."

The word *pillocellus* occurs in a MS. of the year 1354, and is explained by CARPENTIER, pila lusoria; bot in the passage cited by him, it seems more probably to signify a *racquet*, and may therefore possibly have been written *pillomellus*; perhaps in the French of that day *pile-malle*, from *pila* and *malleus*.

According to these etymologies *pellmell* must either have signified a contest with *staves*, or a contest followed by *defeat*; or else it must have been a metaphor wholly borrowed from the tennis court; but these are at best ingenious conjectures, and we are inclined to think that *pell* was merely added to *mell*, for the sake of the sound, and to strengthen the conceptioo of confusion already expressed in the word *melée*, by describing a "confusioo worse confounded."

Certain it is, that this principle of the iteration of sound, with a trifling variety of articulation, in order to augmeot the force of the expression, enters very largely into the formation of words and phrases, in all countries, especially among the commoo people, and more particularly where the conceptioo to be expressed, though accompanied with strong feeling, is in itself vague, obscure, and confused. Whatever, therefore, may be thought of the applicatioo of this principle to the adverb *pellmell*, it is of great consequence to the proper understanding of grammar, that the principle itself should be carefully considered; nor is it any objection to such consideration, that the practice in qoestion originates with the vulgar and ignorant. On the contrary, this very circumstance throws an additional light on the science of language; for it is not only in the formation of such words as the one under consideration; but in the general frame and construction of all languages, that we may find reason to attribute a great iofluence to the strong feelings and imperfect conceptioos of the ignorant, the vulgar, and the barbarian: and moreover, even in the class of expressions which we are more particularly examining, there is a force and a suitableness, which eveotually makes them force their way upwards in society, until they become equally familiar and intelligible to high and low, to the coarse and to the refined. This is owing chiefly to orators and poets, who (if they are truly such) will not address themselves solely to morbous sensibilities, or pedantic judgments, and therefore will not ask whether an expression has been branded as obsolete or trivial by the magisterial asterisk of a lexicographer; but whether it will carry convictioo and enthosiasm to the mind of the hearer or reader. The great LUTHER somewhere recommends to one who would know all the powers and energies of the German language to listen to it as spokeo by the mother in the house, and the dealer io the market. BURNS, the delightful poet BURNS, would never have attained that immortality which is iosured by his "Twa Dogs," and his "Tam o'Shanter," if he had confioed himself to such book-language as the "Verses to Miss C——, a very young Lady."

> Beauteous rose-bud, young and gay,
> Blooming on thy early May,
> Never may'st thou, lovely flow'r,
> Chilly shrink in sleety show'r!

&c. &c. all in the same strain.

That the alliterative formation of words by the vulgar is oot confined to England or France, but is natural to such persons in all countries, we may learn from a curious little story which occurs in ETON'S

Grammar. *Survey of the Turkish Empire.* "An Arab, who had let out his camel to a man, to travel to Damascus, complained to a kadi, on the road, that the camel was overloaded. The other bribed the kadi; 'What has he loaded it with?' asks the cadi. The Arab answers, 'With *cahué* (coffee) and *mahué*;' i. e. coffee *et cætera*, (changing the first letter into *m* makes a kind of gibberish word, which signifies *et cætera.*) '*Sugar* and *mugar*, *pots* and *mots*, *sacks* and *macks*,' thus going through every article the camel was loaded with. 'In short,' concludes the complainant, 'he has loaded it *twice as much* as he ought.'—'Then,' says the kadi, 'let him load the *cahué*, and leave the *mahué*, the *sugar*, and leave the *mugar*, the *pots*, and leave the *mots*, the *sacks*, and leave the *macks*;' and so on, to the end of all the articles enumerated; and as the poor Arab had told every article, and only added *et cætera*, according to the Arab custom, the camel took up the same loading as it had before."

The learned and laborious ADELUNG has collected several instances of words similar to our *pell mell* in form, and probably in the mode of their original construction, both in German and other languages; such as the German *mischmasch*, (answering to BURNS's *mixtie-maxtie*, to the Low Saxon and Danish *miskmask*, and to the French *micmac*,) also *schnicksnack*, *wischwasch*, *zick-zack*, *wirr-warr*, *fick-facken*; the Low Saxon *hink-hanken*, *tick-tacken*, &c.

To these we may add in English *chit-chat*, *ding-dong*, *dingle-dangle*, *fiddle-faddle*, *giff-gaff*, *handy-dandy*, *helter-skelter*, *hum-drum*, *hurly-burly*, *knick-knack*, *namby-pamby*, *pit-a-pat*, *prittle-prattle*, *riff-raff*, *see-saw*, *skimble-skamble*, *slip-slop*, *snip-snap*, *tag-rag*, *tittle-tattle*, &c.

There are also many expressions, which if not formed by mere alliteration, seem to be retained in use chiefly by that quality in their construction, such as *rigmarole*, *hocus-pocus*, *hugger-mugger*, &c.

"It is a property," says ADELUNG, "of the common, or vulgar German language, and of its cognate dialects, to form a kind of intensive or frequentative words, by a repetition of the same sound." And elsewhere he observes, that "in the Low Saxon dialect, particularly, this is customary," and that "in doubling the syllable they generally change its vowel;" but in High German such words are rare.

Misch-masch. *Mischmash*, in German, is a heap of things thrown together without taste or order; from *mischen*, to mix. The French in borrowing from it their word *micmac*, have given it the secondary sense of *intentional* confusion and obscurity.

The *Dictionnaire de l'Academie* defines *micmac*, "intrigue, maniganee, pratique secrete pour mesnager quelque interest illicite. *Il y eut bien du micmac dans cette affaire; on ne connoist rien à tout ce micmac.*" It is, like most words of this kind, stigmatised as a *low* expression.

Schick-schnack. *Schnickschnack* is a kind of strange, foolish chattering, or jargon, from *schnack*, *chattering*. In Dutch *sink* is to sob, and *snack* is a droll, chattering fellow. These words are doubtless formed by imitation of the sound, described, as the common word *sniggering*, for suppressed laughter, as in English. To the same origin are we to ascribe the provincial word *sneck*, the latch of a door; whence a *sneck-up*, was a thievish vagabond, who watched his opportunity to lift the *sneck up*, and steal into a house for the sake of pilfering. Thus Sir Toby Belch calls Malvolio, in derision, "*sneck-up*;" Falstaff says of Prince Henry, "the prince is a Jack, a *sneck-up*;" and Mr. STEVENS has collected many other instances of this cant term of reproach from various old plays.

Wisch-wasch. *Wischwash* seems to differ but little from the former, being derived from *waschen* or *schwassen*, to babble or talk idly.

Zic *Zickzack*, is the origin of the French *ziczac*, and of our *zigzag*; and they all signify a line continued backwards and forwards from point to point. Its origin is clearly the German *zacken*, a point, or pointed substance, as the *points* in the branches of a stag's horn; and so *eiszacken*, an icicle or *pointed* piece of ice. And this word agrees with the Dutch *tak*, a bough; the Swedish *tagg*, the Islandic *taggar*, the French *dague*, and *daguet*, and the English *tack*, *tag*, *dag*, *jag*, &c. Our word *tack*, is used for a small *pointed* nail, for fastening things together with nails; and also for the action of a ship in going from *point* to *point*.

Our old word *takil*, and the Welsh *tacel*, a *pointed* arrow, was a derivative of *tack*. Hence CHAUCER,

The *takil* smote, and in it went.

In Islandic, *tag* is the *point* of a lance.

This word *tag*, Mr. Tooke says, is in English the participle of *tian* vincire; but he is wrong, it is a *point* of metal put to the end of a string, and to *tag* with rhyme, is to *point* a line with rhyme. *Dag* is the very same word, it was an ancient name for a *dagger* or short *pointed* sword, called in French *dague*, whence the old French verb *daguer* was to stab with the *point* of the dagger; and *daguet* was a young stag (called by Shakspeare a pricket) when the *points* of his horns first begin to shoot. In Italian and Spanish, a *pointed* short sword or dagger is *daga*, in Dutch *dagge*, in German *degen*. Skinner calls a *pointed* piece of cloth a *dag*, from the Anglo-Saxon *dag*, sparsum pendens, and in Dutch the *pointed* end of a rope is called *een endye dagg'*. GROSE says a *pointed* spade is called in Norfolk and Essex a *dag*-prick. With *dag* also agrees *jagg* which signifies a point, and *jagged*, cut into points.

Wirr-warr. *Wirrwarr* is a confusion of many things *whirling* round, as it were, in confused circles, and clashing together. LESSING seems to have adopted it from the Low Saxon dialect. "Salmasius macht über diese stelle einen trefflichen *Wirrwarr*." "Salmasius makes, on this place, a fierce confusion." *Wirren*, the origin of this word is our *whirl*, and the Latin *gyrare*: in its first sense signifying to turn round in a circle, and thence to confuse, or disturb the state and order of things. Thus in the Frankish of OTFRID *unio er is allaz uuirrit* quomodo omnia perturbet.

Fick-facken. *Fickfacken* is a trivial word in Low Saxon, signifying to run about idly here and there without any particular object, or to employ one's self in idle tricks. Adelung supposes it to come from *facheln*, as if it were a metaphor taken from the motion of a fan; but it seems rather from *fach*, which is probably the same in origin as our *pack*, meaning a portion, quantity, division, &c. "Das schlagt nicht in mein *fach*."—"That does not fall to my lot, it is no business of mine, I have nothing to do with it." The Gothic *fahan*, and Frankish *fahen*, are explained by WACHTER

Grammar. "capere, quocunque modo, manu, mente, ambitu, spatio," and he considers it to be the same word which in other dialects is pronounced *fangen*. From *fahen* come *fæhig* capax, *anfahen*, ordiri *auffahen* excipere, *empfahen* vel *entfahen* accipere, *umfahen*, amplecti, *unterfahen* conari, and lastly the above word *fach* which is the Anglo-Saxon *fæc*, as in *stowe fæc*, spatium loci, *lytel-fæc* modicum temporis, *twegra dæga fæc* biduum. In Lower Saxon it answers in composition with numerals to the Latin *plex*, and our *fold*, as *einfach*, simplex, *zweyfach*, duplex, *vielfach*, multiplex, &c. In the Scottish dialect the word *feck* is still retained in the sense of quantity; as "will it rain to day? There'll be nae *feck*." The verb *fa'* seems also to be the old *fahen* or *unterfahen*, as in Burns—

> A king may mak' a belted knight,
> A lord, a duke, an' a' that;
> But an honest man's abune his might;
> Gude faith, he manna *fa'* that.

Hinkhanken, in Low Saxon, is to go hopping along lamely, with one leg shorter than the other, from *hinken* to halt.

Ticktacken, in the same dialect, is to touch gently and often, from *ticken* which is connected with the Gothic *tekan*, and the old Latin *tigere*, to touch. From this comes *trictrac* (backgammon) which Menage says the French anciently pronounced *tictac*.

Besides the preceding, several other words of a similar construction are cursorily mentioned by Adelung; as the Low Saxon *titeltateln*, *wibbelwabbeln*, *tiesketauske* or *zieskezaaske*; the Swedish *pickpack*, *willerwalla*, *dingldangl*, and the Islandic *fimbulfambe*, to which we may add the French *charivari*.

We now come to the English words of this kind.—

Chit-chat. *Chitchat*, Dr. Johnson says is "corrupted by reduplication from *chat*, and is a word only used in ludicrous conversation; as in the *Spectator*, No. 560, "I am a member of a female society, who call ourselves the *chitchat* club." It is true that most of these words are originally trivial, and many of them ludicrous; but when they find their way into books of such classic celebrity in our language as the Spectator, it is surely necessary that the student of language should understand by what means they got there, upon what principles they were formed, and to what class of words they are properly to be referred in grammatical arrangement. Now the only part of this word which was originally significant is *chat*; but even of *chat* the origin is unsatisfactorily explained by Johnson, who, though entitled to the highest praise for industry as a lexicographer, was perfectly ignorant of the history of the English and other modern languages. Thus he suggests that *chat* may be from "the French *achat*, a purchase, or cheapening, on account of the prate usually produced in a bargain." He might as reasonably have derived it from the French *chat*, a cat; because many old women chatter to their cats. Long after the word *chat* was in common use in England, the French word, now spelt *achat*, was spelt *achapt*, and the verb *acheter* was *achepter*, or *achapter*, being derived, as some suppose, from the barbarous Latin *adcaptare*; but at all events agreeing with the German *kaufen*, Dutch *koopen*, Scottish *to coff*, Anglo Saxon *ceapan*, or *aceapan* to buy, *ceap*, cheap, *ceapman* a dealer or chapman, *ceapstow*, forum mercatorum, Chepstow, in Wales. Hence many names of places in England, as *Chipping Norton*, *Chipping Ongar*, *Chippenham*, &c. Skinner derives *chat* and *chatter* from the French *caqueter*; but this latter, as well as the Italian *chiacchierare*, and *chiacchillare*, and the Latin *cachinnare*, may rather be compared with our verb *cackle*; whereas *chatter*, as when the teeth *chatter* with the cold, is more analogous to the German *zittern*, to tremble; with which also agrees our word *twitter*. All these, however, are instances of the onomatopœia, or formation of significant words by the mere imitation of sound. The insignificant syllable *chit* was subsequently prefixed to *chat*, as we suppose *pell* to have been to *mell*, *zick* to *zack*, and *tick* to *tack*, from an indistinct wish to give it an intensive, or frequentative force. Adverbs.

Ding-dong, "a word (says Johnson) by which the sound of bells is imitated." It is singular, that the learned lexicographer should call this word a noun substantive, and cite as an example, from Shakspeare— Ding-dong.

> Let us all ring Fancy's knell.
> *Ding, dong*, bell!

In this instance, ding-dong is manifestly a mere interjection. It is, however, sometimes used adverbially, as when it is said "they went to fighting *ding-dong*." To *ding*, is to strike or beat, from the Anglo-Saxon *dyngian*; and *dong*, *dung*, or *dang*, are used in different dialects as the past participle of this verb. Thus there is an old Scottish song in praise of the town of Dunse, entitled "Dunse *dings* a'," i.e. Dunse beats or excels all other places. There is also a song entitled "Jenny *dang* the weaver;" that is, she beat or overcame the weaver. A Yorkshire lad, who had come to London as a servant, was one day asked by his master what had occasioned some water to be spilt on the carpet. He replied, in his provincial dialect, "I *dung* doon t'urn;" meaning, I accidentally knocked down the tea urn.

Mr. Tooke justly observes, that the substantive *dung*, manure, is this participle *dung*: and he quotes, among other authorities, Sir Thomas More, who spells it *dong*. "All other thynges in respecte of it I repeite (as Saienct Paule saith) for *dong*." Ding-dong therefore is no more than "strike stroke."

Dingle-dangle. *Dingle-dangle* expresses in English, as it is said to do in Swedish, a swinging or oscillating motion, from the verb *dangle*, which Skinner supposes to have been originally *hangle*, from *hang*. If so, it was probably formed, like the words we have been considering, *hangle-dangle*. In a modern comedy an uncle reproaching his extravagant nephew, says, "I shall see thee go off, just at twelve o'clock, *dingle-dangle*."

Fiddle-faddle. *Fiddle-faddle*. Dr. Johnson quotes this word both in its substantival and adjectival use.—

> She said that their grandfather had a horse shot at Edgehill, and their uncle was at the siege of Buda, with abundance of *fiddle-faddle* of the same nature.
>
> *Spectator*, No. 299.

> She was a troublesome, *fiddle-faddle*, old woman.
>
> *Arbuthnot.*

The history of this word is curious. According to Cicero, faith between man and man was called *fides*, from *fio* to be. "Fundamentum est autem justitiæ *fides*; id est dictorum conventorumque constantia et

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veritas. Ex quo (quanquam hoc videbitur fortasse cuipiam durius, tamen ut audeamus imitari Stoicos, qui studiosè exquirunt unde verba sint ducta) credamus, quia *fiat* quod dictum est, appellatam *fidem*." Again, a harp was called *fides*, according to FESTUS, on account of the truth of its tones, "*fides*, genus citharæ, dicta quòd tantum inter se chordæ ejus, quantum inter homines *fides*, concordent." The diminutive of *fides* gave *fidicula*, which our Anglo-Saxon ancestors called *fithele*, the Germans *fidel*, and we *fiddle*. In modern times, however, the more dignified name of this instrument in German is *violine*, and in English *violin*; and some degree of contempt is attached to the word *fiddle*, both as a noun and a verb: in its primary sense it expresses an inferior instrument and a vulgar performance; in its secondary sense, to *fiddle*, is in the words of Dr. Johnson, "to trifle, to shift the hands often and do nothing; like a fellow that plays upon a fiddle." To convey this latter idea, the more forcibly, the word is repeated, with the mere change of a vowel. SKINNER seems anxious to discover some separate meaning for the word *faddle*, which he thinks may be from the French *fade*, and Latin *fatuus*; or from the German *faden*, a thread; so that "a *fiddle-faddle* person" would be either a *fiddle-foolish* person, or a *fiddlestring* person; which etymologies are equally superfluous and inappropriate.

Giffe-gaffe.

Giffe-gaffe, is formed from the Anglo-Saxon *gifan*, to give; as *ding dong* is from *dingan*. This expression, now obsolete, occurs in one of Bishop LATIMER's Sermons, published in 1562. "Somewhat was geuen to them before, and they must neades geue somewhat againe; for *giffe-gaffe* was a good felow."

Handy-dandy.

Handydandy. This word also, it pleases Dr. JOHNSON to call a noun-substantive. It may be so used, no doubt; but in the instance which he cites from SHAKSPEARE, it is an interjection.—

> See how yond justice rails upon yond simple thief! Hark in thine ear! Change places, and *handy dandy!* which is the justice? which is the thief?"
>
> *Lear.*

Helter-skelter.

Helter-skelter, Dr. Johnson who admits this to be an adverb, explains it, "in a hurry, without order, tumultuously." In fact, it combines these notions with something of inconsiderate eagerness, whether occasioned by fear, as when a troop of men are said to fly *helter-skelter*, or by a desire to reach a particular object, as when Pistol hastens to carry to Sir John Falstaff the glad tidings of Prince Henry's accession to the throne:—

> Sir John, I am thy Pistol, and thy friend;
> And *helter-skelter* have I rode to England,
> And tidings do I bring——

SKINNER, in his anxiety to make sense of every part of this expression has given two etymologies which make nonsense of the whole. He thinks it may either be derived from the Anglo-Saxon *heolster sceado*, "the darkness of hell;" or from the Dutch *heel-ter-schetter*, which be thinks is "all dispersed or shattered to pieces." The real origin of the word, however, is obscure. If we suppose the principal meaning to be in the first part, it may possibly come from the Islandic *hilldr* pugna; if in the latter part, it may be from the German *schalten*, to thrust forward; or from *skale*, which in the dialect of the north of England, means "to scatter and throw abroad as molebills are when levelled;" or from *skeyl* which in the same dialect is to push on one side, to overturn.

Adverbs.

Humdrum.

Humdrum. It seems to be admitted that there is no origin for this word, but the interjection *hum!* which is explained to be "a sound implying doubt or deliberation;" it forms, however, first an adjective, and then an adverb; as "I was talking with an old, *humdrum* fellow," *Spectator*; and again—

> Shall we, quoth she, stand still, *hum-drum*;
> And see stout bruin overthrown?
>
> *Hudibras.*

Hurly-burly.

Hurlyburly. Dr. Johnson has recorded an absurd etymology of this word, from the names of two families, *Hurleigh* and *Burleigh*. The word *hurl*, or *hurley*, signifies a tumult, from the French *hurler*, to howl like wolves or dogs; and to this the word *burly* appears to have been added, as a mere reduplication.

> When the *hurly-burly's* done,
> When the battle's lost and won—
> That will be ere set of sun.
>
> *Macbeth.*
>
> Methinks, I see this *hurly* all on foot.
>
> *K. John.*
>
> He, in the same *hurl*, murdering such as he thought would withstand his desire, was chosen king.
>
> *Knolles.*

Knick-knack.

Knickknack. In this word, which is chiefly used as a substantive, the syllable *knick* is only prefixed to knack for the sake of the sound, and to give a slight degree of intensity to the meaning. The word *knack* is reasonably enough derived from the Anglo-Saxon *cnawan*, to know; and is explained "a little machine, a petty contrivance, a toy."

> Knaves, who in full assemblies have the *knack*
> Of turning lies to truth, and white to black.
>
> *Dryden.*
>
> ——When I was young, I was wont
> To load my she with *knacks*. I would have ransack'd
> The Pedlar's silken treasury, and have pour'd it
> To her acceptance.
>
> *Winter's Tale.*

Namby-pamby.

Namby-pamby. This word seems to be of modern fabrication, and is particularly intended to describe that style of poetry which affects the infantine simplicity of the nursery. It would perhaps be difficult to trace any part of it to a significant origin.

Pit-a-pat.

Pit-a-pat. This expression also Dr. Johnson calls a substantive; and gives the following example—

> A lion meets him, and the fox's heart
> Went *pit-a-pat*.
>
> *L'Estrange.*

Here *pit-a-pat* is clearly an adverb; as it is in the *Beggar's Opera*.

> As when a good housewife sees a rat
> In her trap, in the morning taken,
> With pleasure her heart goes *pit-a-pat*,
> In revenge for the loss of her bacon.

This expression is not derived from the French *pas-a-pas*, (with which it has nothing to do, either in meaning or etymology,) nor "from the French *patte-patte*," which it is apprehended, never was a French phrase; but the verb to *pat* is "to strike lightly, to tap," and a *pat* is "a light quick blow, a tap;" the word being, no doubt, made from the sound. It is

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true that Casaubon learnedly deduces it from the Greek 'Ἀπαντᾶν;' but this is an etymology, which we need not trouble ourselves to refute. *Pot* marks the strong blow, in the beating of the heart; and *pit* is prefixed to it, to express the weaker blow, which forms the alternation.

Pritle-prattle.

Prittle-prattle. As *prattle* is a diminutive of *prat*, agreeing with the Dutch *praten*, and possibly derived from the Latin *prædicare*; so *prittle* prefixed to *prattle* makes a further diminutive, and is particularly applied to the early attempts of children to talk.

Riff-raff.

Riff-raff. We have the verb to *raff*, to huddle up, and take away hastily without distinction. CAREW says, " their causes and effects I thus *raff* up together;" and a *rafe* or *raff*, in the provincial dialect of the midland counties of England, is " a low fellow," probably from comparison with dirt and other matters thus carelessly swept away. To the word *raff*, in this signification *riff* being prefixed, augments the feeling of contempt, whilst it applies the expression more loosely to a whole class of people. *Raff* is no doubt connected with *reave*, of which *rafte* is the old past tense:—

> O trust, O faith, O depe assurance!
> Who hath me *rafte* Creseyde?

And Mr. TOOKE does not err much in saying, that *riff-raff* is identical with *rof*, the past participle of the Anglo-Saxon *reafian*; but he is entirely mistaken in ascribing the adjective *rough* to the same origin; for *rough* is the German *rauh* from *ragen*, eminere, prominere; whereas *reafian* agrees with the German *raffen* and *rappen* the classic Latin *rapere*, the barbarous Latin *reffore*, &c.

See-saw. The significant syllable here is *saw*, and the word *see saw* is meant to express a motion similar to that of sawing; *see* being merely prefixed for the sake of adding force to it. Pope uses it as a noun, and Arbuthnot forms a verb from it.—

> His wit all *see-saw*, between that and this.
>
> POPE.

> Sometimes they were like to pull John over; then it went all of a sudden again on John's side: so they went *see-sawing* up and down.
>
> ARBUTHNOT.

Skimble-skamble.

Skimble-skamble, is formed, as Johnson observes, " by reduplication from *scamble*. Thus SHAKSPEARE makes Hotspur ridicule the pretended prodigies and portents of Glendower—

> A couching lion, and a ramping cat,
> And such a deal of *skimble skamble* stuff,
> As puts me from my faith.

Scrobbling, scrambling, scambling, shambling, are all words expressive of an awkward, struggling, or shuffling motion.

Slipslop.

Slipslop. This is, in like manner, said by Johnson to be formed by reduplication of *slop*. He expounds it " bad liquor;" but since the days of Fielding it has come generally to signify the incorrect and ungrammatical language of chambermaids, from the character of Mrs. Slipslop, in Tom Jones.

Snip-snap.

Snip-snap. Tart dialogue, in which each party *snaps*, as it were, at the other's argument before it is finished.—

> Dennis and dissonance, and captious art,
> And *snip-snap* short, and interruption smart.
>
> POPE.

Adverbs.

Tag-rag.

Tag-rag. This word is in signification very similar to *riff-raff*. Dr. Johnson does not make a separate word of it, but places it among his examples of the use of the word *tag*, which, he says, signifies any thing paltry and mean; but why *tag* should have that signification, it is not easy to guess; certainly not from the etymology which he gives of it; for he derives it from the Islandic, *tag*, the point of a lance. The leading conception in the compound *tag-rag* is undoubtedly that expressed by the word *rag*; and *tag* seems to be prefixed to it merely for the sound. Casca speaking with the utmost contempt of the Roman populace, whom he calls " the rabblement," and the " common herd," and ridicules for their " chopped hands," and " sweaty nightcaps," goes on to speak thus of their conduct towards Cæsar:—

> If the *tag-rag* people did not clap him and hiss him, according as he pleased and displeased them, as they use to do the players in the theatre, I am no true man.

Tittle-tattle.

Tittle-tattle. This is properly described by Dr. Johnson, " a word formed from *tattle* by reduplication. Idle talk, prattle, empty gabble."

> Of every idle *tittletattle* that went about, Jack was suspected for the author.
>
> ARBUTHNOT'S *Hist. of J. Bull.*

> You are full in your *tittletattlings* of Cupid.
>
> SIR P. SIDNEY.

We have sufficiently shown that this mode of forming words is common to many languages; that it is of considerable antiquity in our own language; and that, so early at least as the age of Queen Elizabeth, words so formed were adopted into the style of the best authors; not indeed as conveying any distinctness of impression, or dignity of sentiment, but as appropriate and suitable to the subject before them, and to the feelings with which they wished it to be regarded.

Rigmarole.

The pleasure derived from alliteration is one of the earliest and simplest of the mere pleasures of sound in language. Hence alliteration appears to have preceded rhyme, in the rude attempts at poetry, which were made by our Saxon ancestors; and even after rhyme was introduced into English verse, the ballads and popular poems of the day were full of alliterative expressions. In one of those poems already quoted, (Harl. MS. 2253, fo. 124,) we find an expression, which seems to be the origin of our trivial word *rigmorole*. The poem in question begins thus:—

> Of rybauds y ryme
> And *rede o my rolle.*

That is, " of ribalds (or idle, disorderly persons,) I rhyme, and read out of my roll." The accounts, records, and other long and tedious writings of that day were usually preserved on rolls; therefore a " *read-o'-my-roll*" story would be an apt expression for a long, tedious story: and the vulgar would easily corrupt *read o' my roll* into *rigmarole*.

Hocus-pocus.

Hocus-pocus, is a vague word for juggling and cheating.

Thus BUTLER says—

> For Justice, though she's painted blind,
> Is to the weaker side inclin'd,
> Like charity; else right and wrong
> Could never hold it out so long;

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And, like blind Fortune, with a sleight,
Conveys men's interest and right,
From Stiles's pocket into Nokes',
As easily as *Hocus-pocus*.

The expression "is corrupted," as Dr. Johnson says, "from some words that had once a meaning and which cannot now be discovered." The suggestion of TILLOTSON is probably the right one. At the time of the Reformation, many jests, and some of them grossly profane, were made on the rites of the Roman Catholic church; and the priests who celebrated the holy mysteries were treated as no better than jugglers. Thus, in a Scottish poem of that period, beginning "The Paip, that Pagane full of pride," we find the following passage—

Thay sillie Freiris, mony Yeiris,
With babbling bleirit our ee.
Hay Trix! Tryme go Trix!
Under the greenwod Trie.

The words *hoc est corpus*, employed with reference to the doctrine of transubstantiation, were very likely to have been turned into ridicule by the opponents of that doctrine, and from *hoc est corpus*, corrupted by vulgar pronunciation, may have been formed *hocus pocus*. JUNIUS derives the expression from the Welsh word, *hoced*, a trick, and the English word *poke*, a bag; but it is neither probable that a juggler's bag would obtain the mixt Welsh and English name *hoced-poke*, nor, if it did, that the Latin termination *us* would be substituted for the second, and added to the third syllable.

SKINNER, with more learning than judgment, derives *hocus-pocus* from *quassare* and *fodicare*. "Totum enim istiusmodi artificum mysterium," says he, "in eo consistit, ut pilas vel sphærulas, in vasculis seu pyxidibus *quassent*, et digitis quàm celerrimè motis, res immissas surripiant." From *quassare*, he derives the French *hocher*, and from *fodicare* the French *pocher*; which, he says, is "digito extrudere et quasi effodere;" but though *hoche-poche* in French might possibly convey the idea of shaking a bag and thrusting the fingers into it, we have not met with that word so used; still less can we suppose it to have been Latinised, in termination, if derived from this origin. The French *hocher*, to shake, is the Dutch *hutsen*, or *hutselen*, from whence come our *huddle* and *hustle*. The Dutch have the word *hutspot*, for a dish made of meat cut into small pieces, and shaken in the pot, with vegetables, &c. whilst it is dressing. The French also have *hochepot*, and the Scotch have *hotch-potch*, with the same meaning. The French *hochepot*, signifying some kind of cookery, is used by Chaucer; and it was adopted in a figurative sense into the terms of our law, at least as early as the year 1474; for at that time Sir THOMAS LITTLETON wrote his Commentaries, in the third book of which, (sect. 267,) occurs this passage, "En cel case le baron ne le feme avera riens, pur lour purpartie de le dit remnant, sinon que ils voile mitter lour terres, dones en frankmarriage, en *hotchpot* ovesque le remnant de la terre."—"Et ili semble, que cest parol, *hotchpot*, est, en English, a *pudding*; car en tiel *pudding* nest communement mies un chose tantsolement, mes un chose ovesque auters choses ensemble." COKE, however, observes, "in English we use to say, *hodgepodge*." But as none of these derivations from *hutsen* or *hocher* have any relation, in point of meaning, to *hocus-pocus*, so neither can they at all serve to explain the manner in which that word acquired the Latin termination *us*; which circumstance becomes perfectly intelligible, if we adopt Tillotson's suggestion as true.

Adverbs.

Hugger-mugger.

Hugger-mugger. This word implies a clandestine way of doing things, as in the following example from L'ESTRANGE's fables: "There's a distinction betwixt what is done openly and barefaced, and a thing that's done in *huggermugger*, under a seal of secrecy and concealment." Johnson explains it "secrecy, *bye-place*;" but it does not appear to have so much to do with the *place* where, as with the *manner* in which things are concealed; and it seems to allude to *hugging* things up close to prevent their being seen. The conjectural etymologies of this expression are exceedingly various. SKINNER derives it from the Dutch *hugghen*, which, he says, signifies to observe, and the Danish *morcker*, darkness; an etymology alike improbable and inappropriate. JOHNSON says it is "corrupted perhaps from *hug er morcker*, a hug in the dark," in what language *hug er morcker* has this signification he does not mention, nor does any phrase correspondent to the English *hugger-mugger*, appear to have ever become proverbial in any other language. The Spanish affords the nearest approach, to the separate parts of this expression; for *hogar* is a chimney corner, and *muger* is a woman; and if we could suppose *hugger mugger* to be taken from that language it might refer to the notion of a woman cowering in the chimney corner; but as nothing can be more delusive than to be guided in etymology by mere similarity of sound, we may safely reject this derivation of the phrase in question. Some persons have supposed *hugger-mugger* to be derived from the old English word *hoker*; because Sir THOMAS MORE, (it is said,) uses the word *hoker-moker*; but it is not very clear that he meant by it what we mean by *hugger-mugger*; and if he did, no great stress is to be laid on a casual variation of orthography in that age, when spelling had nothing like fixed rules. The word *hoker*, had no reference in point of meaning, to the idea conveyed by the word *hugger-mugger*; for it signified peevish, froward, and was probably taken from the French *hocher la tête* to shake the head at any thing in sign of contempt.

Thus CHAUCER in the Reve's Tale, describing the Miller's Wife:—

She was as digne as water in a diche,
And as full of *hoker* and of besmare,
As though that a Lady should her spare
What for her kinred, and her nortelry
That she had lerned in the nonnery.

And the same idea is still more fully expressed in the *Lay le Freine*:

Than was the leuedi of the hous,
A proude dame, and an envious,
Hokerfulliche missegging,
Squeymous and eke scorning.

The last etymology that we shall mention is from the Dutch title, *Hoog Moogende*, (High Mightinesses,) given to the States General, and much ridiculed by some of our English writers; as in *Hudibras*—

But I have sent him for a token
To your Low-country *Hogen Mogen*

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It has been supposed that *hugger-mugger*, corrupted from *Hogen Mogen*, was meant in derision of the secret transactions of their Mightinesses; but, it is probable that the former word was known in English before the latter; and upon the whole it seems most probable that *hugger* is a mere intensive form of *hug*, and that *mugger* is a reduplication of sound with a slight variation, which, as we have already seen, is so common in cases of this kind.

The same disposition toward alliteration appears in some of our quaint proverbial phrases, where the words are distinct, as in "*tit* for *tat*;" and also in some passages of our comic writers. Thus in the *Taming of a Shrew*, Petruchio, in his feigned anger against the Tailor, exclaims—

> What's this? a sleeve? 'tis like a demi-cannon,
> What! up and down! carv'd like an apple tart!
> Here's *snip* and *nip*, and cut, and *slish* and *slash*!

So Parson Evans says to his friend, Justice Shallow:—

> It were a goot motion, if we leave our *pribbles* and *prabbles*, and desire a marriage between Master Abraham and Mistress Anne Page.

Adverbial phrases.

We have observed that the primary use of the adverb is to modify adjectives or verbs, and its secondary use to modify adverbs. The same may be said of adverbial phrases, and generally of whatever stands in the place of an adverb. Thus we may say "this happened *afterwards*," or "this happened *long* afterwards," or "this happened *many days* afterwards," or "this happened *not many days* afterwards." In the first case the adverb *afterwards* modifies the verb "happened;" in all the other cases the same adverb *afterwards* is modified, first, by the adjective *long* used adverbially, then by the adjective and substantive *many days* forming an adverbial phrase, or standing in the place of an adverb; and lastly, by the adverb, adjective, and substantive, *not many days*, which in like manner may be said to form an adverbial phrase, or to stand in the place of an adverb. So in Lord Berners's translation of Froissart, executed by command of King Henry VIII. and printed in his reign, the following passage occurs, fo. cxcix. *b*. "Nowe the Duke of Berrey commaundeth me the contrary; for he chargeth me *incontynent his letters sene*, that I shulde reyse the syege." In this passage *incontynent* is an adverb modifying the verb *reyse*; and *the letters sene* is a phrase, (similar in construction to the Latin ablative absolute, as it is termed, *visis epistolis*,) which modifies the adverb *incontynent*, a word at that time used where we should say *immediately*.

Thus in the romance of *The Foure Sonnes of Aimon*, printed in 1554, we find—

> Now up Ogyer, and you Duke Naymes, light on horseback *incontinent*.

Adverbial phrases are in another point of view material to the consideration of adverbs properly so called. By comparing different languages we not only find, that a certain phrase in one language corresponds to a different phrase in another language; but that phrases in the one correspond to words in the other. Thus in comparing the French with the Italian we not only find such expressions as *à chaudes larmes*, answering to *a dirotte lagrime*, or *tout-à-coup*, to *di primo lancio*; or *à gorge deployée*, to *alla smascellata*; but we also find *à tatons* rendered by *tentone*, *à peu près*, by *quasi*, &c. &c.

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Recapitulation.

We have now exhausted the considerations arising out of our definition of the adverb. We said, *first*, that an adverb was *a word used for the purpose of modification*; and we showed how it modified primarily an adjective or a verb, and secondarily another adverb. *Secondly*, we said, that for this purpose it was *added to a perfect sentence*," and we distinguished between a sentence perfect both in the mind and expression of the speaker, and a sentence perfect in the conception, but broken short in the utterance. And *thirdly*, we explained what *sort of word* might be used for the purpose of such modification. Under this head we showed that the adverb might be a simple or compound word, and we instanced adjectives, participles present and past; pronouns, numerical and demonstrative; verbs and substantives, all of which have been used as adverbs, and indeed constitute the mass of the words commonly known by that designation. We showed also that compound adverbs might be formed of all the other parts of speech; and, lastly, we noticed a variety of adverbial phrases, or words derived from such phrases, which, in the construction of sentences, supply the place, and perform the function of adverbs. In the course of these investigations it has been rendered most manifest that phrases often become words, and that of words it is the *use* and not the *form*, which entitles them to be considered as adverbs. If a substantive be employed adverbially it is equally an adverb whether it have or have not previously undergone any inflection. *Nox*, in the passage quoted from the laws of the Twelve Tables, is as much an adverb as *noctu*, quoted from Cicero.

Other writers.

It may be proper, however, before we close the chapter of adverbs to advert to some few considerations, which though they have no particular reference to any part of the definition above given, have occupied much of the attention paid by other writers to this part of speech.

Classification.

In works professedly treating of grammar, it has not been uncommon to distribute adverbs into classes according to their signification. Thus the very learned and admirable Hickes, (a name never to be mentioned without veneration,) enumerates in the Anglo-Saxon language no less than 28 different kinds of adverbs; viz. 1. of time; 2. place; 3. exhorting; 4. dissuading; 5. excepting; 6. denying; 7. affirming; 8. wishing; 9. doubting; 10. diversity; 11. distance; 12. quantity; 13. separation; 14. situation; 15. transition; 16. comparison; 17. augmentation; 18. remission; 19. congregating; 20. quality; 21. manner; 22. likeness; 23. opposition; 24. order; 25. demonstrating; 26. interrogating; 27. number; and 28. cause. It is almost needless to observe that this sort of enumeration is infinite; for there is scarcely a conception of the human mind which may not be applied adverbially, and even form a class of adverbs. Harris has only spoken particularly of adverbs of intension, remission, comparison, time, place, motion, and interrogation; but he has quoted a passage from Theodore Gaza, which is more to the purpose; for that acute grammarian justly observes that the readiest way to reduce the infinitude of adverbs, (considered according to the conceptions signified by them,) is to refer them by classes to the ten logical

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predicaments, existence, quality, quantity, relation, &c. &c.

Such a classification, however, though it may be useful to the memory, is no essential part of the office of a grammarian, because there is no difference in grammatical use between an adverb of one of these classes, and an adverb of another such class; between an adverb of time, for instance, and an adverb of place; an adverb of quantity, and an adverb of quality; or if any such difference exist in a particular language, it depends on the idiomatical peculiarities of that language, and not on any essential principles of universal grammar.

Confounded with other words.

A more important consideration is this, that adverbs are often confounded with other parts of speech, by writers of no mean reputation; and this happens in two ways; for 1st. the whole class of adverbs may be confounded with other classes; or 2dly. particular words, whether adverbs, or others, may be confounded with classes to which they do not belong.

Ben Jonson says, "Prepositions are a peculiar kind of adverbs, and ought to be referred thither." Caramuel says, "Interjectio posset ad adverbium reduci; sed quia majoribus nostris placuit illam distinguere non est cur in re tam tenui hæreamus."—"Interjectiones," says Vossius, "à Græcis ad adverbia referuntur, atque eos sequitur etiam Boethius." It is clear from the definition of an adverb, which we have given, that a preposition can no more be considered as a peculiar kind of adverb, than a substantive can be considered as a peculiar kind of adjective or verb; for the proper function of the preposition is to modify a conception of substance; and the proper function of the adverb is to modify a conception of attribute, either alone, or combined with an assertion; but the part of speech which names a conception of substance is the noun substantive; the part of speech which names a conception of attribute is a noun adjective; and the part of speech which asserts is the verb.

Again, as to *interjections*, they do not serve to modify either noun or verb; but on the contrary are interjected, as it were, between different nouns or verbs, and as Vossius says, "citra verbi opem, sententiam complent;" for though, as we have said, the interjection may, both in signification and construction, supply the place of a verb, in certain instances; as in the passage, "O! that I had wings like a dove," where the interjection O! supplies the place of the verb "I wish;" yet this, in no respect, modifies the signification of the verb "had," but merely affects its construction in the sentence.

If, indeed, with certain of the Greek philosophers, we were to admit only three parts of speech, the noun, the verb, and the combinative, it might at first sight appear somewhat doubtful under which head the words which we have termed adverbs, should properly fall; for some of them, as we have seen, are in origin nouns, and others verbs; but in that case we ought not to look so much to their origin, as to their use; and, therefore, we should class them among verbs; for by verbs the philosophers, here alluded to, really meant what Harris calls attributives; and the adverb is, as he has justly said, the attributive of an attributive.

It adds something to the confusion of the classes of words, if they are placed out of their common and natural order, in any system, as where the adverb is treated of before the participle, which was done by Donatus Servius, and some others; or after the preposition, which was the order of Priscian, who therein followed Apollonius. We trust it will be found in the sequel, that the order which we have adopted from Diomedes and Vossius, is the most natural and the best, namely 1. adverb, 2. preposition, 3. conjunction, and 4. interjection.

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From the consideration of classes of words, we come to that of words singly; and among these we find frequent instances of the confusion before alluded to; adverbs are treated as being other parts of speech; and other parts of speech are treated as being adverbs.

It is not surprising, that where a noun retains its form unchanged, the adverbial character, which it acquires in construction, should be sometimes overlooked. Among the adverbs which we have cited, some e. gr. *wonder*, are now used only as substantives; others e. gr. *right*, *full*, &c. are now rarely used but as adjectives; and as substantives and adjectives respectively they would probably be treated by all those persons, who do not reflect that it is the use of a word in a particular sentence that determines the part of speech to which, in that sentence, it belongs. We have seen Dr. Johnson, a scholar certainly of great acquirements, designating as nouns substantive, such words as *pell-mell*, *ding-dong*, *handy-dandy*, *pit-a-pat*, and *see-saw*, when in the very examples which he quoted they were used as adverbs; and this is the more remarkable because he designates other words, of the very same formation and use, adverbs; e. gr. *helter-skelter*, which certainly approaches as nearly to *pell-mell*, in its grammatical use, as it does in the mode of its formation, and in its general import.

On the other hand, the term adverb is that which almost all grammarians apply to an indeclinable word when they either are at a loss to ascertain its proper use, or do not give themselves time to reflect on the matter. The acute and ingenious De Brosses calls the French *chez* an adverb, which is most manifestly a preposition, for *chez moi*, and *apud me*, are phrases exactly similar in construction. Even the learned Vossius calls the Latin *mecastor* an adverb, and R. Stephanus terms it "*jurandi adverbium*." Now *mecastor* is from the Greek μα, and *Castor*, the name of a deity, and it is literally, "by Castor," an oath used as a common expletive in conversation. Thus we find in Terence, "Salve, *mecastor*, Parmeno;" where *mecastor* cannot by any ingenuity be made to modify the verb *salve*, or indeed any other word; but is truly and properly an interjection, which all words of the same kind must be, such as *Gadso!* which though Mr. Tooke distinctly calls an oath, yet he preposterously reckons among the adverbs. *Gadso!* and *'Odso!* were abbreviations of "by God it is so;" or "is it so, by God?" for men happily shrink from their own profaneness, and rather reduce their words to unmeaning exclamations, than advert seriously to their original import. As to the obscene Italian expression to which Tooke alludes, it had probably nothing to do with the interjection Gadso, however it may have furnished a hint to the unpolished satire of Ben Jonson, in the passage quoted from one of his plays.

Grammar. CHARISIUS, out of the twenty-one classes of adverbs, that he enumerates, mentions three, which are clearly interjections; namely those which he calls adverbs of wishing, as *utinam*; of answering, as *hem!* and of showing as *ecce!* This last mentioned word is sometimes used redundantly with the similar word *en*, as in APULEIUS, "En, ecce, prolatam coram exhibeo," where Vossius, reckoning it among adverbs, nevertheless adds, with his accustomed sagacity, "nisi hæc (adverbia demonstrativa) potiùs interjectioni accenseotur." Mr. TOOKE, however, falls into the common error, and enumerates among adverbs the plain interjection *lo!* which is (as he himself observes) the imperative of the verb to *look*. This consideration alone should have taught him that *lo!* could not be used with an adverbial construction; and the same may be said of *halt!* and *fie!* which he nevertheless includes in his list of adverbs. *Halt!* is the imperative of the German *halten*, and was probably transmitted to us directly from the French, who borrowed it from the Italians, and they from the Germans.

The Anglo-Saxon *healdan*, and our verb to *hold*, are indeed the same verb with the German *halten*, but from them we could never have formed in the imperative *halt!* terminating with a *t*; although our old writers used *halt*, as the past tense of those verbs. Had such been our derivation of the present word, halt! it would probably have been more extensive in its application; but its confinement to the purposes of the military art, shows that it was received from a foreign nation, with that distinct application.

As to *fie!* the imperative of the Gothic and Anglo-Saxon verb *fian*, to hate; from whence comes *fiand*, the *fiend*, the enemy of mankind, it is surely as genuine an interjection as *proh!* or *væ!* or any other word of that class.

Mr. TOOKE too, calls "*prithee*" an adverb. It is the phrase, "I pray thee," shortened, and used as an interjection; and it never did or could serve as an adverb in modifying either a verb, an adjective, or another adverb. By a similar error some ancient writers reckoned the verb *amabo* among adverbs, but CÆLIUS CALCAGNINUS expunged it from that class; and rightly so, as Vossius remarks.

Thus, too, DONATUS called *quæso* an adverb. The truth is that such verbs as *quæso* and *amabo*, thrown into a sentence interjectionally, and not connected with any other word in the construction of the sentence do not differ, as to grammatical principle, from pure interjections, and therefore may be referred to that part of speech; but cannot be regarded as adverbs without great impropriety.

The interjections *heus!* and *utinam*, have also been reckoned among adverbs: and even the pronouns compounded with a preposition, as *mecum*, *nobiscum*, and the like, the error of which is ably pointed out by VOSSIUS in his first book *De Analogiâ*, cap. 2.

There is, perhaps, some nicety in determining whether certain words are more properly to be reckoned adverbs or conjunctions. Thus *primò*, *deinde*, *denique*, and such like words, are called adverbs, and sometimes not improperly so; but when they serve to combine together sentences, and to show the relation of the verbs to each other, they ought to be deemed conjunctions. In this class we are inclined to place such words as *nevertheless*, which Dr. JOHNSON, and after him TOOKE, call an adverb. Prepositions.

Thus in the following passage from Lord BACON:—

Many of our men were gone to land, and our ships ready to depart; *nevertheless* the admiral with such ships only as could suddenly be put in readiness made forth towards them.

Nevertheless answers exactly to *yet*, which is distinctly stated to be a conjunction both by Johnson and Tooke. Nay Johnson, in explaining the word *yet*, thus expresses himself—

YET *conjunct* (gyt, get, geta, Saxon.) *Nevertheless*, notwithstanding, however.

And in the sentence above quoted the sense would be exactly the same, whether we should say—

Though many of our men were gone to land, the admiral put forth.

Or—

Many of our men were gone to land, *yet* the admiral put forth.

Or—

Many of our men were gone to land, *nevertheless* the admiral put forth.

Upon the whole, it will be seen, in these and similar instances, that the conjunction is an adverb *and something more*. It is an adverb, inasmuch as it serves to modify the verb, with which it is immediately connected; but it is something more, inasmuch as it shows a relation between that verb and another, and connects together the sentences to which those verbs belong.

§ *Of Prepositions.*

We now come to a class of words, best known in Name. modern times by the name of *prepositions*, though they have by some writers been more appropriately termed *adnomina*, or *adnouns*. As our object, however, is to change as little as possible received terms and modes of reasoning, we shall adopt the generic word *preposition*, for the part of speech, which we have at present to consider.

In the Greek and Latin languages, the words thus Errors distinguished were most commonly (though with respecting. some exceptions) *placed* immediately *before* the substantives to which they referred; and they were subject to few variations in point of form. These circumstances, as will presently be shown, were merely accidental or idiomatical, but they were unfortunately selected by some grammarians as essential to the preposition; and hence originated the well-known definition *præpositio est pars orationis invariabilis, quæ præponitur aliis dictionibus.* Some of the Greek grammarians, considering that prepositions connected words, as conjunctions did sentences, ranked both the preposition and conjunction under the common head of Σὺνδεσμος, or the *connective*, and the stoics adding this circumstance to the ordinary position of the preposition, in a sentence, called this part of speech Συνδεσμος Προθετικος. Another accidental peculiarity of most of the words which were used as prepositions, in Greek and Latin, as well as in some modern languages, was that their original and peculiar meaning had, in process of time, become obscure; and from hence some persons were led to

Grammar. think that these words had no signification of their own. The learned Harris gives the following definition, "*A preposition is a part of speech devoid itself of signification, but so formed as to unite two words, that are significant, and that refuse to coalesce, or unite of themselves.* Campanella also says of the preposition *per se non significant*; and Hoogeveen says, "*Per se posita et solitoria nihil significat.*" Under the same impression, the Port Royal grammarians say, "*On a eu recours, dans toutes les langues, à une autre invention, qui a été d'inventer de petits mots pour être mis avant les noms, ce qui les a fait appeller prépositions.* And M. de Brosses says, "*Je n'ai pas trouvé qu'il fut possible d'assigner la cause de leur origine; tellement que j'en crois la formation purement arbitraire.*"

Now, in all this there was certainly much inaccuracy of reasoning. As to the position of these sort of words in a sentence, even in Latin, the preposition *tenus* was always placed *after* the noun which it governed; so Plautus uses *ergo*, after a pronoun, as in *mederga*, for *ergo me*; and *cum* is employed in like manner in the common expressions *mecum*, *tecum*, *nobiscum*, *vobiscum*. These and other examples of a like kind induced some authors to make a class of *postpositive prepositions*. "Dantur etiam," says Caramuel, "*Postpositiones*, quæ *præpositiones postpositivæ* solent dici;" but there are languages in which all the prepositions, if we may so speak, are postpositive.

Dr. Jault, speaking of the Turkish and Hungarian tongues, says, "Les *prépositions* de ces deux langues, aussi bien que de la *Georgienne*, se mettoient toujours *après* leur regime." And Halhed in his grammar of the Bengal language, says, "the noun *in regimine*, with a preposition, should properly be in the possessive case, and *prior* in position."

It is not surprising that Mr. Tooke should ridicule these *postpositive* prepositions, and *nonsignificant* words which communicate signification to other words; but unfortunately he only substitutes worse errors of his own, when he asserts that prepositions are always names of real objects, and do not shew different operations of the mind.

The real character and office of the preposition have been stated with a nearer approach to accuracy by Bishop Wilkins and Vossius; but neither of them seems to have given a full and satisfactory definition of this part of speech. Wilkins says, "Prepositions are such particles whose proper office it is to join integral with integral on the same side of the copula, signifying some respect of cause, place, time, or other circumstance, either positively or privately." Vossius says, *præpositio* est vox per quam adjungitur *verbo* nomen, locum, tempus, aut caussam significans, seu positivè seu privativè."

It suited Wilkins's scheme of universal grammar to call the preposition a *particle*, but however appropriate this may be to a theoretical view of language, such as it never did, and probably never will exist, it does not suit our view of those philosophical principles on which the actual use of speech among men depends. On the other hand, as Wilkins includes under the term *integral* both the noun and the verb, he is in this respect more accurate than Vossius, for the preposition does not merely join a noun to a verb, but sometimes to another noun.

Definition. We, therefore, with that diffidence which becomes all persons who endeavour in any degree to clear the path of science, shall propose the following definition of a preposition: *a preposition is a word employed in a complex sentence to express the relation in which a substantive stands to a verb, or to another substantive.* Prepositions.

Saul was *before* David.

He speaks *concerning* the law.

The Duke *of* Wellington liberated Spain.

Cæsar, *with* his army, extinguished freedom *in* Rome.

Justice is nobler *than* unlicensed force.

In these examples the same function is performed in the construction of the respective sentences, by the words *before*, *concerning*, *of*, *with*, and *in*; but it is performed in somewhat a different manner.

1. The preposition *before*, expresses the relation of *priority*, in which the substantive *Saul*, stands to the substantive *David*, the mere verb of existence intervening.

2. The preposition *of* expresses the relation of *appurtenance*, in which the substantive *duke*, stands to the substantive *Wellington*, no verb intervening.

3. The preposition *concerning*, expresses the relation of *subject* to *action*, in which relation the substantive *law* stands to the verb *speaks*.

4. The preposition *with*, expresses the relation of *means* to *action*, in which the substantive *army*, stands to the verb, *extinguished*.

5. The preposition *in*, expresses the relation of *place* in which the substantive *Rome*, stands to the same verb, *extinguished*.

I. We say, that the preposition is always employed in a *complex* sentence; for as the noun and verb make up one proposition, and the noun, verb and adverb two, so the noun, verb, and preposition, with the noun which follows, or is governed by the preposition, make up three propositions. Thus "John walks *before*," is a sentence involving these two propositions— Complexity of sentence.

John is walking.

John is before.

But "John walks *before* Peter," is a sentence involving these three propositions—

John is walking.

John is *before*.

Peter is *behind*.

In like manner the sentence "the Duke of Wellington conquered," may be resolved into these three propositions—

The Duke conquered.

He *belonged* to a certain town.

The town (to which he *belonged*) was Wellington.

And thus we may always resolve a sentence into its separate propositions, by expressing in a distinct form the conception implied by the preposition, and connecting it successively with the two terms related to each other.

II. The origin and use of prepositions may best be considered, by adverting to the three different modes in which the particular relation of a substantive to a verb, or to another substantive, may be expressed in language, namely, by a *combination of words*, by a Origin and use.

single word, or by the *declension* of a word.

A combination of words constitutes a *phrase*, or clause in a sentence, which may be introduced solely to express the relation conveyed in a different language, or mode of writing, by a single preposition. Thus in the letter which Hotspur reads in King Henry IV. part 1. "I could be well contented to be there *in respect of* the love I bear your house," the words "*in respect of* the love," may be rendered in Latin "*propter* amorem;" or may be turned in English "*for* the love."

Let us, therefore, first consider how phrases of this kind are formed.

Substantival phrases. 1. We may place under the head of *substantival* phrases, all those in which the conception of the relation meant to be expressed is given in the form of a substantive. Such are the phrases, "in respect of," "per rispetto di," "in consideration of," "à cause de," "per mancanza di," &c. &c.

Now these words *respect*, *rispetto*, *consideration*, *cause*, and *mancanza*, retain in English, French, and Italian, respectively, their separate use as substantives; and the same may be said of the expression more common in Scotland than in England, "in *place* of;" but the phrase corresponding to this last, viz. "*instead* of," exhibits a noun, which, in the sense of "place," has become obsolete. Accordingly, Dr. Johnson, in his Dictionary, has the following articles:—

Stead. STEAD, n. *s.* 1. Place. Obsolete.

> Fly, therefore, fly this fearful *stead* anon,
> Lest thy fool hardize work thy sad confusion.
> *Fairy Queen.*

Instead *of*. Prep. [a word formed by the coalition of *in* and *stead* place.]

I. In room of; place of.

> Vary the form of speech, and *instead* of the word church make it a question in politics, whether the monument be in danger.
> SWIFT.

Here, we see, is some little confusion; inasmuch as Johnson has not very clearly explained whether he considers the two words *in* and *stead*, or the three words *in*, *stead*, and *of*, to have coalesced into one word, and formed one preposition. It may, therefore, be more advisable to call all such expressions prepositional phrases.

It is easy to conceive, that the noun *stead* might have been used alone, with the same force and effect as we now use the whole phrase *instead of*; for, in fact, the word *statt*, which is only a variety of pronunciation, is so used in the German language, as *statt meiner*, "instead of me:" and in a manner not very dissimilar, we ourselves use the Latin noun *vice*, especially in the official notices of appointment to rank or office, as, "X. Y. to be captain by purchase, *vice* T. B. promoted."

Cause. "*Because of*," answers to the French prepositional phrase, *à cause de*, and to the Italian *per rispetto di*. Dr. Johnson says of the word *because*, "it has, in some sort, the force of a preposition; but because it is compounded of a noun, has *of* after it."—

> Infancy demands aliment such as lengthens fibres without breaking, *because of* the state of accretion."
> ARBUTHNOT, *on Aliment.*

Fault. The substantive *faute* in French is employed in the formation of a prepositional phrase, both with and without the preposition *à* preceding it; as "il est mort, *faute* de secours,"—"*à faute de* lui rendre foi et hommage, il fera saisir le bien." So in low colloquial English, we use the expression "for fault of;" as in the *Merry Wives of Windsor*—

> QUICKLY. Peter Simple, you say your name is?
> SIMPLE. Ay, *for fault of* a better.

And in Italian the substantive *mancanza*, is employed in a similar phrase: "non fu già fatto, che *per mancanza di* fede, o di memoria." (*Lettere di* G. DELLA CASA.)

Spite. *In the spite of* is a prepositional phrase occurring in Bishop LATIMER's sermons:—

> A gentlewoman came to me, and tolde me that a great man kepeth certayne landes of hers from her, and wyll be her tenaunte *in the spyte of* her tethe.

This phrase is shortened by some of the poets to *spite of*. Thus ROWE—

> For thy lov'd sake, *spite of* my boding fears,
> I'll meet the danger which ambition brings.

The substantive *spite* signifies malice, rancour, hatred, malignity, malevolence; but the prepositional phrases "*spite of*," "*in spite of*," and "*in the spite of*," are often used, as JOHNSON observes, without any malignity of meaning; for words, in the course of time, obtain, in some instances, a greater latitude, and in others a closer restriction, of meaning; and in the present case there is a transition from the idea of that opposition which arises from malignity, to the more comprehensive idea of forcible opposition in general.

It is somewhat doubtful whether the substantive *despite*, and the prepositional phrases, *despite of*, and *in despite of*, are not of different origin from the preceding. *Spite* is certainly connected with the Dutch *spyt*, spite, vexation; and in that language are the phrases *my te spyt* "in spite of me," and *spyt zyn bakkus*, in spite of his teeth; but the Dutch *spyt* enters into the composition of several other words, as *spytig*, spiteful, fretful, vexatious, *spytigheyd*, fretfulness, *spytiglik*, spitefully; and they say *dat is spytig*, for "that is vexatious," "that is a pity." The notion conveyed by all these words is analogous to the sense of being pricked or wounded by a pointed instrument, and it is doubtless connected with our word *spit*, and with the German *spitze*, which signifies any substance terminating in a sharp point. Hence *spiz*, according to WACHTER, is "acutus, acuminatus;" *spizzi stechun*, in Frankish, is pointed stakes, "Dicitur allegoricè," adds Wachter, "de ingenio *acuto*, sed callido, *maligno*, et ad decipiendum nato. Inde *spiz-kopf* caput astutum, *spizbube*, fur vafer," &c.

Despite. *Despite*, on the other hand, is from the French *depit*, formerly spelt *despit*, which MENAGE derives from *dispectus*, (he must mean *despectus*,) despised. From *despectus* was formed the Italian *dispetto*, as in the prepositional phrase *per dispetto di*, "in contempt of." Thus BOCCACIO says, "Che ne dobbiam fare altro, se non torgli que' panni, ed impiccarlo, *per dispetto degli* Orsini, a una di queste querce." The French *depit* or *despit*, is explained in the *Dictionnaire de l'Academie*, "fascherie, chagrin meslé de colère;" and it is added, "On dit, *en depit de luy*, pour dire malgré luy;" but in an earlier period of the French language

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the prevalent idea conveyed by the word *despit* was not anger, but *contempt*. Thus in a poem on the Game of Chess, the earliest on the subject now extant, having been transcribed in the 13th century, (MS. Cotton. Cleop. b. ix. 1.) we find the following passage:—

> Mes vne gentz sount ke *endespit*,
> Vnt les giuspartiz, e prisent petit,
> Pur ceo q' poi enseiuent ou nient.

i. e. "but there is one kind of people who have *in contempt* games (of chess) and prize them little; because they know little or nothing about them."

And from the lines immediately following it appears that the obsolete verb *despire* was exactly our verb "to despise."

> Mes ceo net pas a dreit ingement,
> De *despire* ceo du't neu seit la uerité.

i. e. "But this is not (according) to right judgment, to *despise* that of which one knows not the truth."

Shakespeare appears to have felt the true meaning of the word *despite*, as implying, from its Latin origin, contempt, when he makes Coriolanus exclaim to the tribune, Sicinius—

> Thou wretch! *despite* o'erwhelm thee!

Gré.

The French substantive *gré*, gave rise to our obsolete preposition *maugre*, (for so it is spelt in Bishop Latimer's sermons,) and it will be worth while, first, to trace the growth of this substantive from the Latin adjective *gratus*, and then to observe how it was employed in various prepositional phrases, and those phrases ultimately melted down into a single word, so as to form a clear and genuine preposition.

From the classical Latin adjective *gratus*, agreeable, were formed the barbarous Latin substantives *gratus*, and *gradus*, signifying that which is *agreeable* to a person, or conformable to his free will; as in the following instances:—

Idem feodum a manu monachorum alienare non possumus, nisi *grato* et voluntate Ducis Burgundiæ.
Chart. A. D. 1197.

Tu qui meus es, quomodo teneas hoc quod ego non dedi tibi extra meo *grato*?
Vet. Chart. ap. Besliam, p. 392.

Ipse autem de suo *gradu* respondit quod in illud scriptum non intraret.
Capit. Carol. Calv. tit. 24.

From these substantives come the barbarous Latin verbs *grato* et *grator*, "to agree or grant freely," and the adverb *gratanter*, "willingly."

From the same source came also the Italian substantive *grado*, free will, approbation, thankfulness, as in Dante:—

> Ma poichè pur al mondo fu rivolta,
> Contra suo *grado*, e contra buon usanza,
> Non fu dal vel del cuor giammai disciolta.

And in Boccacio—

Niuna ragion vuole, che *grado* si senta del non ricevuto beneficio.

So we find *a grado*, and *a grande grado*, used in an adverbial manner, for "agreeably," "very agreeably."

Tanto bene, e sì *a grado* cominciò a servire ad Egano, che egli gli pose amore.
Boccacio

Fatto era, quanto egli aveva comandato, *a grande grado* e piacere di santa Chiesa.
M. Villani.

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Di grado, and *di proprio grado*, are also used, in an adverbial manner, for "willingly," "spontaneously."

Che difendesse la sua franchezza, e libertà, e che non si mettesse *di grado* in servitudine; perocchè maggior vituperio è sostenere servitudine *di proprio grado*, che per forza.
Volgar. Pist. Senec. 95.

From the Italian *grado* proceeded the old French *greit*, *grez*, and *gré*.

Car ilh s'estoient tos bin wardeis, sans avoir mal *greit* de nulle des parties.
Hemricurtius, *de bel. Leod.* c. 38.

> Tos furent lié de sa venue;
> *Grez*, et mercez lui out rendue.
> MS. *Poeme: Guer de Troie.*

Gré, in more modern French, is explained "bonne, franche volonté, qu'on a de faire quelque chose;" as "il y est allé de son *gré*, de son plein *gré*;" "ils ont contracté ensemble de *gré* à *gré*;" "il le fera bon gré, *mal gré*. *Savoir gré*, is "to be satisfied with" a person's conduct, to be obliged to him for it: *lui savoir un gré infini*, "to be infinitely obliged to him." Thus, in a letter written by order of the King of France, in 1814, to the author of certain political works, it is said, "Sa majesté vous *sachant un gré infini* de la manière dont vous avez pris, dans des temps difficiles, la défense de ses justes droits," &c. and these phrases appear to be imitated from the Italian *so grado*, as in Boccacio—

Signori, di ciò, che ieranera vi fu fatto, *so io grado* alla fortuna.

Faire gré, in old French, was to do what is *agreeable* to right and justice, as to satisfy a debt, a tax, or a reckoning.

Se il avient que uns hom fesist semonrre un autre pardevant le justiche por dete, et cil, de qui on se clameroit, ne seroit mie de le quemugne, si connoissoit le dete, il seroit tantost à 2 sols et demi, et se li convarroit *faire son gré* s'il avoit de coi; et s'il desconnoissoit la dete, il en demoureroit quites.
Usat. MSS. Civ. Ambian.

Icellui Guillaume compta et *fist gré* à l'oste de l'escot de lui, et de ses compaignons.
MS. Letter, A. D. 1395.

This expression is imitated by Chaucer in his *Merchant's Second Tale*, v. 1326.

> And be myght be take he shuld *do me gre*

From the substantive *gré* came the old French *gréer*, to agree to, grant, or approve:—

Toutes les choses dessus dittes il *gréerent*, roerent, ratefierent, et accorderent.
Chart. A. D. 1323.

In the same sense was used *agréer*, whence came the barbarous Latin *agreamentum*, "an agreement," which Rastall whimsically expounds *aggregatio mentium*.

From *gré* came also the old French word *engrés* for willing, ready, well disposed.

> Soions *engrés*, soions engrant,
> De lui servir et jour et nuit!
> *MS. Mirac. B. M. V.* lib. 2.

The word *grex*, anciently used in Valentia for a marriage gift *freely* made by the husband to the wife,

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appears to agree with the old French *greit.* (See FONTANELLA *de pact. Nupt.* f. 2. cl. 7. gl. 1.)

GAWIN DOUGLAS has adopted the French *gré* into the Scottish language, in the sense of a prize; as—

> The bull was the price and *gre* of thare dereyne.

But Junius erroneously derives this from the French *degré*, the origin of which is the Latin *gradus*, a step.

Having thus traced the simple word *gratus*, "agreeable," through its derivatives, we have next to view it compounded with *malé*, "badly."

Malo grato is used by MATTHEW of PARIS, in two passages, with a slight difference of construction. Under the date of the year 1245, he says, "Libertatem ecclesiæ, quam ipse nunquam auxit, sed magnifici antecessores ejus *malograto* suo, stabilierunt." Under 1252, he thus relates the quarrel of King John with his brother "Cui ait electus—Domino Deo vos commendo At Rex, et ego te diabolo vivo. Et ego te *molo grato* Dei et ejus sanctorum," &c." Here the adjective and substantive appear to be used separately; but they are combined into one word in *malegratibus dentium* which occurs in a MS. of the year 1350. "Galteronus ira motus dicit ad supplicantem plurima verba injuriosa, quod *malegratibus dentium* ipsius supplicantis, ipse bene solveret simbolum suum."

So, in Italian we find, *a mal mio grado, o mio mal grado, a mal grado di lui, mal mio grado, mal sua grado, malgrada di voi*, &c.

> La casa oscura, e muta, e molto trista me ritiene, riceve, *a mal mio grado.*
>
> BOCCACIO.

> Il dì sequente passarono il fosso, *a mal grado* della forza de' Pisani.
>
> M. VILLANI.

> Che chi possendo star cadde tra via
> Degno è, che *mal suo grado* a terra giaccia.
>
> PETRARCA.

In like manner *mal* and *gré* are combined in French. These two words appear to be used as an adjective and substantive in the *Roman de Rou.*

> Guert out si le conseil troublé,
> Que puis n'i out home escouté,
> Qui de faire pais ait parlé,
> Qui des plus riches n'ait *mal gré.*

But they seem rather to form a compound substantive, in the following passage of a MS. letter dated A. D. 1401.

> Guillemette Quesnel jeune femme non mariée, pour ce qu'elle estoit ensainte, et grosse d'enfant—doubtant le *malgré* de ses amis, &c.

Malgré became *maugré* by the general tendency of the French to corrupt *al* into *au*, as *alter auter, autre; ultra, outre;* thus *mau* is used for *mal* in the old proverb, "à *mau* chat, *mau* rat," meaning "two knaves well met." So in the compounds *maudire*, to curse; *maudisson*, a curse, opposed to *benisson*, a blessing; as in the Scottish dialect *malison* is to *benison; maumené* ill used *mauffait*, a *goblin; mougréer*, to revile, rail upon, and show ill will to.

CHAUCER frequently uses *maugre* as a preposition. Thus in the *Knight's Tale*:—

> And I will loue her *maugre* all thy might.

In BARBOUR we find the same word spelt *magre*.—

> Through him I trow my land to win,
> *Magre* the Clifford, and his kin.

Prepositions.

Lastly, in Bishop LATIMER's Sermons, it is spelt *mauger*.—

> God worketh wonderfully, he hath preserued it *mauger* theyr heartes.

Time. Term.

The English substantive, *time*, and the French *temps* are used in prepositional phrases, more or less, ample or abbreviated. Thus, in the statute 1 Ric. III. c. 7., which was enacted A. D. 1483, and remains on record both in the French and English languages of that day, we have "the meane tyme" where we should now use "in the mean time," "all plees *the meane tyme* to cesse;" in the French copy "toutz plees *le meane temps* de cesser." In another part the phrase is fuller "*en* le mesme temps toutz pleez cessent;" "*in* the same tyme all plees cesse." And elsewhere we have "*al temps* de le dit fine levez;" "*at the tyme* of the seid fyne levied."

But in another passage, the words *tyme* and *temps* are respectively used without either preposition or article preceding them, "saving to every persone such right, &c. as they have to or in the seid londes, &c. *tyme of* such fyne ingrossed."—"Sauvant a chascune persone autielx droit, &c. queux ils ount *au* ou en les ditz terres *temps* dutiel fine engrosse." The word *term* is also used in the same absolute way, in the first chapter of the statutes made in this year, (the earliest statutes on record in the English language,) "ne leses *à terme* de vie ou des ans, ne annuiteez grauntez à ascune persone ou personez pur leur service *pur terme* de leur vies," which in the English MS. copy runs thus, "Nor leses *terme of* lyff or of yeres, nor annuites graunted to eny personne or persones for their service, *terme of* their lyfes."

Tour.

From the French substantive *tour* comes the old word *entour*, which is used both as part of the prepositional phrase *à l'entour de*, and also alone, as the mere preposition "about."

An ode of RONSARD, imitated from Anacreon, begins thus—

> Le petit enfant, Amour,
> Cueilloit des fleurs, *à l'entour*
> *D'une ruche*, où les avettes,
> Font leur petites logettes.

In the letter of PERRES DE MOUNFORT, before quoted, we have *entour*, where in modern French *environ* would be used, the former preposition having become obsolete though the verbs *entourer* and *environner*, are alike in use. "Defendimes le givez del ewe de Osk—jekes au Samadi *entour* oure de midy." "We defended the fords of the river Esk, until Saturday *about* the hour of noon."

Adjectival phrases.

2. *Adjectives* may be used in the same sort of prepositional phrases. Thus MILTON, in his "*Essay on the reason of Church Government*," says, "If the course of judicature to a political censorship seem either tedious or too contentious, much more may it to the discipline of the church, whose definitive decrees are to be speedy, but the execution of rigour slow, *contrary to* what in legal proceedings is most usual."

This adjective, *contrary*, we find used prepositionally in the Scottish acts of Parliament, both in the phrase "in *contror* the command," and also in the separate word "*contrare*," as in the act of 1554,

 " *contrare* the priuilegis of oure crowne." In the latter instance it answers precisely to the French preposition *contre*, and therefore is equally entitled to be ranked in that class. In old French there was also the preposition *encontre*, which now exists only as a substantive, signifying an adventure: nor is the verb *encontrer* at present in use, though the substantive *rencontre*, and the verb *rencontrer* both are so; and though in English we retain *encounter* and *rencounter*, both as substantives and as verbs. It is probably from *rencounter* that we originally took the expression of *running counter to*; as in LOCKE—

He thinks it brave at his first setting out to signalize himself in *running counter* to all the rules of virtue.

Where, as the words *counter to*, perform the function of a preposition, they may justly be described as a prepositional phrase.

Salvus. The Latin adjective *salvus*, when placed in the ablative case absolute, may be considered as used prepositionally, and has in fact given rise to the Italian *salvo*, the French *sauf*, and the old English *saufe*, all which may be regarded as real prepositions.

CICERO, in a letter to P. Lentulus, the proconsul, describing his success in a debate against the tribunes of the people, thus speaks—

Quod ad popularem rationem attinet, hoc videmur esse consecuti, ut ne quid agi cum populo, aut *salvis* auspiciis, aut *salvis* legibus, aut denique *sine* vi, possit.

where we see, that in the construction of the sentence, *salvis* and *sine*, have the very same effect; for *agere salvis auspiciis*, and *agere salvis legibus*, and *agere sine vi*, describe three modes of action, in which the relation of the substantives *auspiciis*, *legibus*, and *vi*, to the verb *agere* is expressed by an intervening word, in the nature of a preposition.

In the *vocabolario degli Academici della Crusca*, we find *salvo* thus described, " SALVO. Avverb. che talora si adopera in forza di *prepositione*; e vale eccettuato, fuorchè, se non," and among other examples given is the following, " Rendégli la signoria di Lombardia, *salvo* la Marca Trivigiana."

In the *Dictionnaire de l'Academie Francoise*, it is said, " SAUF se met quelque fois par manière de *préposition*, et signifie sans blesser, sans intéresser, sans donner atteinte; *sauf votre honneur*," &c. And again, " SAUF signifie quelquefois hormis, excepté, à la reserve de; *il luy a cedé tout son bien, sauf ses rentes*."

GOWER has adopted this word *sauf* into English poetry with a conjunctional force :—

Saufe only, that I crie and bidde,
I am in tristesse all amidde.

Long. The word *long* is employed in English prepositionally, as we shall presently show; but not always in its adjectival sense. The English adjective *long*, is from the Latin adjective *longus*, signifying length either of space or time. It does not appear that *longus* was ever employed prepositionally, although it may perhaps be justly said that *longè* was so, in such phrases as " *longè* gentium," which CICERO employs in writing to Atticus.—

Scribendum aliquid ad te fuit—non quo me aliquid juvare posses, quippe res est in manibus; tu autem abes, *longè* gentium."

The Italian *lungo*, however, which is only this same adjective *longus* differently pronounced, is universally reckoned among prepositions.

LUNGO. *Preposiz. Rasente, Accosto; e si usa per lo più col quarto caso. Lat. juxta, prope.*

Vocab. Della Crusca.

Già eravam dalla selva rimossi—
Quando 'ncontrammo d'anime una schiera,
Che venia *lungo* l'argine.

DANTE.

The French use *long* substantively in the prepositional phrases " *le long de*," " *du long de*," and " *au long de*," and this both with respect to space and time; as *il a jeuné tout le long du carême*; *allez tout du long de l'eau*, &c.

They also appear to have formed their adverb and preposition *loin*, formerly written *loing*, from *loinquo*, a corruption of the Italian *longinquo*, which was the Latin adjective *longinquus*, derived from *longus*, as *propinquus* was from the old word *propus*, mentioned by VOSSIUS.

In old and modern English we have the following words, which it will be convenient to consider together, *endlong*, *along*, to *belong*, and to *long*. Mr. TOOKE treats of them at some length; but not satisfactorily. *Along*, to which he ascribes only one origin, appears to have had two, viz. *on long*, i. e. on length; and *gelang*, i. e. belonging, or appertaining to. When Mr. TOOKE observes that the Anglo-Saxon *lengian* is " to make *long*," he merely proves that *long* in *longus* and *leng* in *lengian*, were originally the same word, which is by no means extraordinary; for the radicals *leng*, *lang*, *lag*, *lauk*, are found in most of the northern dialects, expressing a variety of conceptions all connected either with the idea of *length*, or else with the more general idea of *position*; for *lagen*, " to lay," and *langen*, " to stretch out," appear to have been words of the same or similar origin. Hence we have

1. The Gothic *lagg*, Anglo-Saxon *lang*, *leng*, *long*, Frankish and Alamannic *lang*, *lanc*; modern German and Scottish *lang*, Islandic *langr*, all signifying that which is extended in *length*, either of space or time.

2. The Alamannic *alangaz*, *alonges*, *et alongi*, totum, ex integro; " Dictio figurata," says WACHTER, " quâ *longus* ponitur pro *non-interruptus*, quia integrum continuo simile est.

3. The Frankish *gilengen*, to prolong.

4. The German *langsam*, slow, tedious from *length* of time.

5. The Frankish *langen*, to draw or stretch out in length; *lang*, plaustrum; the German *belangen*, trahere in forum, accusare, &c.

6. The German *verlangen*, desiderare, and the English " to *long* for." " Sensu," says WACHTER, " a trahentibus desumpto quia desideria trahunt, et desiderantes trahuntur in rem, eamque vicissim attrahunt. Utrumque sanè habet suos funiculos, et desiderium quo trahimus trahimurque, et res concupita quæ trahit."

7. The German *gelangen* to attain to that which we have *longed* for, which we have been *long* in seeking, and which *at length* we have got.

8. The German *anlangen*, and *belangen*, pertinere, as in the phrases cited by WACHTER, *was mich belangt*, *was mich anlangt*, quod ad me spectat.

From a similar source were probably derived our *lank*, *lag*, *linger*, &c.

Grammar.

That *lagen* and *langen*, or *lengan*, should have been merely different modes of pronouncing the same word, will surprise no one who has observed the frequent instances, in which the letter *n* was used by some Gothic tribes, and omitted by others, in words of precisely the same origin and import: thus we have the Gothic *munds*, and English *mouth*: the Latin *dens*, and English *tooth*, &c. &c. The Anglo-Saxon verb *langan* or *lengan* had therefore two senses; one being to make *long*, the other to be *laid* on to, connected with, or dependent on: and the diversity of its application has produced a corresponding difference in the use of the more modern words which are traceable to its origin.

1. One class presents either the literal or metaphorical conception of that which is stretched out in length: and to this class belong the old English preposition *endelong* and Scottish *endlang*, signifying extension in length from end to end; the modern preposition *along*, as used in the same sense; the same word *along* formerly used as we now use *long* in the phrase "all night long;" and, lastly, the verb, to *long* for; that is, to stretch out the mind after an object.

2. The other class signifies connection, or dependence. Hence, to *belong* to (the German *anlangen* or *belangen* abovenoticed) is to be *holden*, as a house metaphorically is by its owner, or to be *bound*, as a son is in the figurative bonds of relationship to his family. Hence also the now obsolete phrase *along of*, or *long of*, implying to be caused by the person or thing specified.

A few examples will illustrate what we have here said.

Thus Dunbar, in his *Goldin Terge*, uses *endlang*.

> Ladys to daunce full sobirly assayit,
> *Endlang* the trotting river so they may it.

And so Gawin Douglas, in many places, *e. gr.*

> Bot than the women al for drede & affray,
> Fled here and there *endlang* the coist away.

In the romance of *Richard Coer de Lion* the word *endelang* is used adverbially in describing an engine employed by that monarch at the siege of Acre:

> Ovyrtwart & *endelang*,
> With strenges of wyr the stones hang.

The same word occurs in the Scottish Acts of Parliament, v. ii. p. 19. A. D. 1430; "strekande *endlang* the coste."

Gower and Chaucer use *endelonge*.

> She slough them in a sodeine rage
> *Endelonge* the borde, as thei ben set.
>
> Gower.

> This lady rometh by the clyffe to play
> With her meyné *endlonge* the stronde.
>
> Chaucer.

Tooke justly derives our modern *along* from *on long*, or *on length*; which last expression is used by Chaucer, in the *Testament of Love*. "And these wordes said, she streyght her *on length*, (i. e. she stretched herself *along*) and rested awhile." But Tooke erroneously supposes that our most ancient English writers only used the word *along* in the sense of the Anglo-Saxon *gelang*; i. e. "operâ, causâ, *impulsu*, culpâ cujusvis;" and he therefore improperly accuses Gower of using *alonge* for *endlonge* in the following line,—

> I tary forth the night *alonge*.

The force of the word *alonge* is here the same as that of *long*, in Milton's beautiful lines—

Preposi-tions.

> See there the olive grove of Academe,
> Plato's retirement, where the attic bird
> Trills her thick warbled notes, the summer *long*.
>
> *Paradise Regained.*

And Gower is not singular in using *along*, to signify length of time; for we meet with the following passage in Robert De Brunne:—

> Here I salle the gyue alle myn heritage,
> And als *along* as I lyue to be in thin ostage.

To *along* was in like manner used, where we now use to *long*; as in Gower,

> This worthy Jason sore *alongeth*
> To see the strange regions.

The meaning of this verb, to *long*, is well illustrated by Tooke from the Anglo-Saxon "*Langath* the awuht Adam up to Gode," *i. e. longeth* you, *lengtheneth* you, *stretcheth* you, up to God.

The preposition *along*, in the sense of *on length*, is now commonly used, as in the following passage from Milton's *Lycidas*

> So Lycidas sank low, but mounted high,
> Where other groves, and other streams *along*,
> With nectar pure, his oozy locks he laves.

The modern Scottish dialect, for *along*, in this sense, uses *alongst*; as we say *amongst*, *amidst*, *whilst*, for *among*, *amid*, *while*; and so we find *alongest* and *alongst* in old English.

> Phormyo —— was constrayned to cause his people to be soubdenly embarqued, and to sayle *alongest by* the lande.
>
> Nicoll's *Thucidides.*

> They toke their waye towards the sea, *alongest* the sayd ryuer.
>
> *Ibid.*

> The Turks did keep strait watch and ward in all their ports thereabout *alongst* the sea-coast.
>
> Knolles. *Hist. Turks.*

It is somewhat remarkable, that Johnson, in citing this last sentence, should call *alongst* an adverb; since it is manifestly a preposition governing (as the common grammarians say) the noun "sea-coast;" and the sense is, "the Turks watched the coast *on its length*," or "the Turks watched throughout *the length* of the coast."

Amidst, which is a word exactly of the same nature as *along* or *alongst*, Johnson properly calls a preposition; and with the same propriety explains to signify "in the midst"—

> ———————— Of the fruit
> Of each tree in the garden we may eat;
> But of the fruit of this fair tree *amidst*
> The garden, God hath said, ye shall not eat.
>
> Milton.

Here it is equally manifest that the preposition, *amidst*, is nothing more than the noun *mid* or *middle* (from the Latin *medius*) with the superlative termination *est*, and the corrupted prefix *a*; and that the whole sense would be "in the middest (or middlemost) part of the garden."

To return to the preposition, *along*, in the sense of "on length," we may observe, that it is identical with the adverb *along* in the common exclamations "Go along!"—"Get along!"—that is, "*éloignez vous*;"—"*abi in longinquum*;"—"remove yourself to some *distance* from this spot." In like manner

must we explain the adverb *along* in the phrase "to go along with." Thus SHAKSPEARE—

> I your commission will forthwith dispatch,
> And he to England shall *along* with you.
>
> *Hamlet.*

The verb, *to belong*, must be differently explained: it is obvious, that this verb implies *length* or *distance*, if at all, only in a very indirect, and indistinct manner; but refers more distinctly to the notions of *connection* and *dependence* already mentioned: and the same must be said of the word *along* prepositionally used by old writers to signify the relation of an effect to its cause. In this sense it was followed by *upon*, *on*, and latterly *of*, as the Anglo-Saxon *gelang* was by *aet*. Thus, in the instance cited by LYE, "*aet* the ys ure lyfe *gelang*;" "on thee does our life depend.

> But thus this maiden had wronge
> Which was *upon* the kinge *alonge*. GOWER.
>
> ——— your loue al fully graunted is
> To Troylus, & therto trouth yplight,
> That but it were *on* him *alonge*, ye nolde
> Him neuer falsee. CHAUCER.
>
> It is *long of* yourself; for you were the party that commended him to me. ARCHB. ABBOT'S *Narrative*.

Near. The adjectives, *near* and *nigh*, are commonly used in English as prepositions, as is the corresponding Italian adjective *vicino*. The Latin *propè*, which answers to our preposition, near, is the adverbial form of the old adjective *propus*, before alluded to. The French *près* is the Italian *presso* from the Latin participle *pressus*. These words scarcely need illustration: we may however observe, with Mr. TYRWHIT, that *next* is the superlative of *nigh*, as the Saxon *hext* was of *high*. This critic was also right in his remark, (which Tooke unnecessarily censures,) that in modern use we more commonly employ *next*, to signify the "nighest following" than the "nighest preceding;" though, in fact, it means simply the nighest. These, however, are matters of mere idiom.

Opposite, &c. Although the word *opposite* be in its Latin original *(oppositus)* a participle, yet it was first adopted into the English language as an adjective, and then employed colloquially as a preposition. Thus we say, "opposite Somerset House"—"opposite the Horse Guards." In like manner, many other adjectives are used prepositionally, as "to walk *round* London," &c. Tooke therefore properly enumerates among prepositions, *round* and *around*, "whose place" he says, "is supplied in the Anglo-Saxon by *hweil* and *onhweil*; in the Danish and Swedish, by *om-kring*; in Dutch, by *om-ring*; and in Latin, by *circum*; n Gr. *κερκος*, of which *circulus* is the diminutive." *Hweil*, it will be observed, is our substantive *wheel*, and is probably connected with our verb *to whirl*; and it is remarkable, that this same *hweil* forms our adverb *while*, and substantive "*a while*," a time; for the continued motion of time has been often typified by a wheel; and by a similar analogy, the year was called in Latin, *annus*, from *annulus*, a ring; as the Greeks termed it *ενιαυτον*, from its revolving into itself.

Mesne. The use of the adjective *mesne*, though not strictly prepositional in the following passage, may yet serve in some measure to illustrate the subject of which we are speaking.

> Contrarie lawe it is, if after the exigent awarded, the appeale doe abate for insufficiencie, or for *that*, *that* he *that* is outlawed was imprisoned *mesne* betweene the awarding of the exigent and the outlawrie pronounced. STAUNFORD, *on Prerogative*.

3. *Participles* being merely adjectives involving the notion of action as in existence, it is naturally to be inferred, that they may be used as we have seen the pure adjectives used, to perform the function of a preposition. We have already had occasion to notice the Latin ablative case absolute, in the instance of "salvis auspiciis," where we showed that the adjective *salvis* had in reality the force and effect of a preposition; and this became still more obvious in considering the old word *saufe*, which is only the same adjective transmitted from the Latin language through the French into English. The case is not altered, when we find the participle *saving*, or the old Scottish *saufande*, employed in the same manner. Thus, in the Act of 1455, we find "*saufande* the poynts quhilks ar neidful for the conservacion of the treaty." So we say in colloquial language "*barring* accidents." In the Scottish Act of 1456, the participle *belangande* occurs with the same prepositional construction. "As to the thirde artikill, *belangande*, the sending to France." In the Act of 1524, we meet with the expression "*enduring* the time of his office;" where, in modern English, we should use *during*. In legal phraseology the ablative absolute *durante vitâ*, is rendered "*for* and *during* the term of his natural life;" where, as the word *during* and the word *for* are used with exactly the same force in the sentence, it is plain, that if *for* be a preposition, *during* is one also

Participial phrases.

It happens, however, that our lexicographers have only acknowledged those participles to be prepositions which are most frequently so employed; such as *touching* and *concerning*, which are thus noticed by Dr. JOHNSON:—

"TOUCHING, *prep.* [This word is originally a participle of *touch*.] With respect, regard, or relation to."

> *Touching* things which belong to discipline, the church hath authority to make canons and decrees, even as we read in the apostles times it did. HOOKER, book iii.

"CONCERNING, *prep.* [from *concern*: this word, originally a participle, has before a noun the force of a preposition.] *Relating* to, with relation to."

> There is not any thing more subject to errour, than the true judgment *concerning* the power and forces of an estate. BACON.

Many other participles, however, might be pointed out in various languages, which are plainly used as prepositions, and some of them so recognised by grammarians. Thus COUR DE GEBELIN ranks among prepositions the present participles *pendant*, *durant*, *touchant*, *moyennant*, *nonobstant*, *suivant*, and the past participles, *attendu*, *vû*, and *hormis*. So we use *pending*, *during*, *hanging*, *living*, *failing*, *considering*, *omitting*, *regarding*, *respecting*, and anciently *moiening*.

> At whose instigacion and stiring, I have me applied, *moiening* the helpe of God, to reduce and translate it. R. COPLAND.

The participle *hanging* is used in one of our earliest English statutes, as we now use *pending*, and the French *pendant*: and corresponding to the ablative absolute *pendente lite*. "The said accompt to be ij or iij yere *hanging*," Stat. 1. Rich. III. c. 14.

Verbal phrases. 4. *Verbs*, either singly, or in combination with other words, supply the place of prepositions, and sometimes come to be considered as such. Thus, as we have seen the adjective *sauf* and the participle *saufande*, used prepositionally, so we find the imperative of the verb *save* employed for the same purpose.

Grammar.

Dr. JOHNSON, by oversight, as it should seem, calls this word an adverb! TOOKE, in his Chapter on prepositions, more correctly mentions it thus—

"SAVE. The imperative of the verb. This prepositive manner of using the imperative of the verb *to save* afforded Chaucer's Sompnour no bad *equivoque* against his adversary the Friar.

> God *save* you all, SAVE this cursed Frere."

Here the construction is "Save (set aside or except) this Friar; and then I hope that God will save (deliver from evil) all the rest of you."

So in the *Squire's Tale*.

> This strange Knight that came thus sodenly
> All armed, *saue* his hedde—

That is, the Knight was entirely armed, but when you say entirely, you must save (or except) his head.

The words "save and except" are often used synonymously in many of our legal instruments: we shall not therefore be surprised to find *except* reckoned by Dr. Johnson among prepositions—

"EXCEPT. *preposit.* [from the verb.] This word, long taken as a preposition or conjunction, is originally the participle passive of the verb, which, like most others, had for its participle two terminations, *except* or *excepted*. *All* except *one*, is *all*, *one* excepted. Except may be, according to the *Teutonick* idiom, the imperative mood: *all*, except *one*; that is, all but one, which you must except."

"1. Exclusively of; without inclusion of.

> Richard *except*, those, whom we fight against,
> Had rather have us win than him they follow."
> SHAKSPEARE, *Rich. III.*

For *except* were anciently used *out-take outtak* and *outtaken*.

> Which euery kynde made die
> That upon middle erthe stoode
> *Outtake* Noe and his bloode. GOWER.
>
> But yron was there none, ne stele
> For all was golde men myght se
> *Out-take* the fethers and the tre CHAUCER.
>
> And schortly euery thing that doith repare
> In firth or feild, flude, forest, erth or are,
> *Out-tak* the mery Nychtyngale Philomene.
> G. DOUGLAS.
>
> But none of them it might beare
> Upon his worde to yeue answere
> *Outtaken* one, whiche was a knight. GOWER.

Tooke has quoted from BEN JONSON the preposition *outcept*, which he says is "the imperative of a miscoined verb, whimsically composed of *out* and *capere*, instead of *ex* and *capere*." But this is probably no more than a miscoinage of Ben Jonson's coarse and pedantic wit, putting into the mouth of one of his characters such language as never was spoken. The passage is from his *Tale of a Tub*:

> "I'ld play him 'gaine a Knight or a good Squire, or Gentleman of any other countie i'the kingdome—*outcept* Kent; for there they landed all Gentlemen."

Very similar to the use of the imperatives *except* and *save*, as prepositions, is the colloquial expression, "let alone," in use among the Irish Peasantry. Thus in Miss EDGEWORTH's tale of *Ormond*, Moriarty Carroll says: "It might happen to any man, *let alone* gentleman:"—The sense of which expression nearly answers to the Latin *ne dicam*; but in the construction it is "let alone gentleman, speak not of that class of society; for it is not only to them, but to any man that such an accident might happen."

Prepositions.

Mr. Tooke says with some plausibility that the French preposition *avec* is only a contraction of *avez que*, have that; but we must observe that in old French we find it written, *oveke*, *ove* &c.; as in the letter of Sir Perres DE MONTFORT (A. D. 1256) before quoted; and therefore it may possibly be of a different origin.

Most of the verbs and participles, which we have noticed; together with many others of a like nature, are acknowledged by grammarians in general to be prepositions, without any change of form or even of accentuation; but there are other prepositional phrases which, occurring frequently in conversation, lead to abbreviations and ellipses, and thus ultimately leave a single word which performs the function of a preposition. In order to illustrate what is here meant, we shall begin with those words which retain the same sense both in the form of prepositions and in that of nouns or verbs: and afterwards we shall notice those prepositions in which the original meaning of the noun or verb from which they are derived, has become obsolete, or is to be traced only by analogy.

Abbreviated forms.

The substantive *Term* has been already noticed as employed prepositionally in our old Statutes: nor was this a mere legal technicality: in an old poem, entitled *Tytus and Gesyppus*, published in the beginning of the sixteenth century, we find the following lines:

Where not obsolete.

> Tytus his wedynge rynge forthe than dyd take,
> And put it on the fynger of his wyfe,
> Grauntynge to be her husbonde *terme* of lyfe.

Here the full construction in modern language would be "granting to be her husband, *during the term* of her life;" but the noun being used absolutely becomes a sort of preposition; and if this mode of speaking had obtained in general use, the word *term* would no doubt have been reckoned by modern grammarians among our prepositions.

We have already said that the same might have happened with the word *stead*, in English; as it has with the same word, pronounced *statt* in German. The Germans too use our noun *craft*, (which with them means strength) as a preposition; as *kraft seines Amtes* "by the power of his office." So they say "*Laut des briefes*," the word *laut* (our loud) being the substantive "sound." *Laut des briefes*, then, is originally "according to the *sound* of the letter," and in its modern sense "according to the *purport* of the letter;" as we say an act "*sounds* to folly:" and so CHAUCER

> *Sowning* in moral vertue was his speche,
> And gladly wolde he lerne and gladly teche.
> *Prol. viii.*

The Germans likewise use the prepositions *diesseits*, and *jenseits*, literally "this side," and "yon side." A similar use is colloquially made, (particularly in the West of England) of our common nouns *outside* and *inside*: and the former is used by COLERIDGE in his *Christabel*.

> *Outside* of her kennel, the mastiff old
> Lay fast asleep in the moonlight cold.

No difficulty whatever can occur in the explanation of words, beginning with the prefix *a*, or *be*, most of which we have already noticed in their adverbial use; such as *along*, *amidst*, *around*, *across*, *astride*, *aboard*, *below*, *beside*. In all these instances, the nouns or verbs

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are in common use: and it is clear that in employing any one of these words to express the relation in which another substantive is placed, we give the name of that relation, considered as a separate conception of the mind; in other words we employ a *noun* in a secondary use, as a preposition.

It is unnecessary to explain the nouns *round, cross, stride, board, low,* and *side;* only let it be observed that these nouns become prepositions, not by the addition of the prefix *a* or *be* (for that is merely an accident of idiom, and applies equally to the same words when used as adverbs) but their prepositional force depends on their expressing a relation of some substantive, as "around the tree," "across the street," "astride the horse," "aboard the ship," "below the hill," "beside the church."

Less obvious.

There is a little, and but a very little more difficulty in such prepositions as *athwart, against, among, about, behind, between, betwixt, beyond, beneath.* At first sight, it is clear, that they are of the same family with the preceding; and a very short investigation suffices to trace them to the nouns or verbs, of which they are only a slightly varied form. It must be remembered that throughout this part of our disquisitions, when we speak of nouns *and verbs,* we only conform to the established usage; for, as we have already shown, the essence of the verb being *to assert,* it is really the *noun* involved in the verb, which furnishes the *name* of the particular conception to be found in the adverb, preposition, or any other secondary part of speech.

Athwart.

Thus the preposition *athwart* is derived says Johnson from *a* and *thwart.*

> "Themistocles made Xerxes post out of Grecia, by giving out a purpose to break his bridge *athwart* the Hellespont."
>
> BACON. *Essays.*

Here the bridge is not in fact *asserted* to have borne the relation of *thwartness* to the Hellespont, or to have been *thwart* with regard to the Hellespont, but the assertion is supposed and the *name* of the conception only is expressed. However, it seems very immaterial, whether we derive the preposition *athwart* from the adjective *thwart* or from the verb *thwart*; both which we happen to have in our language:

> Mov'd contrary with *thwart* obliquities. MILTON.
>
> ——————Swift as a shooting star
> In autumn *thwarts* the night IDEM.

Equally immaterial would it have been, whether our preposition had happened to be written *thwart* or *athwart;* for as we have frequently observed, it is not the sound of the word, but its manner of signification, which determines what part of speech it is to be deemed. There is a conception of obliquity, and thence of harshness, perversity, &c. &c., which in the various northern dialects is expressed by this word *thwart* and similar articulations, as *thwur, twer, dwar, zwar, swer,* of which it may be worth while to notice some instances:

1. *Thwar, thwur, thweor, thver.*

Anglo-Saxon, *thwur,* oblique: *thwur, thweor, thwyr, thwurh, thweorlice,* perverse; *thweorscipe, thwyrnisse,* perverseness; *thweorian, thwyrian,* to thwart or oppose.

Gothic, *thwains,* angry, thwarting.
Runic, *thver,* contrary, rebellious.
Islandic, *thverskytningr,* a contrary wind.

2. *Twer, tver.*

Islandic, *tuer,* transverse.
Swedish, *twert.*
Danish, *tverer, tvert, tver.*
Old German, *twerch,* the dwarfs supposed to be a perverse race of beings.
Armoric, *gitwerch,* the pigmies.

3. *Dwar, dwer.*

Armoric *duerh,* athwart, *duerahen* oblique
Swedish, *dwerg,* the dwarfs.
Dutch, *dwars,* transverse, *dwarsdryven,* to thwart, *dwarsdryven,* a cross-grained fellow, *dwarslyk,* cross-wise, *dwarsstraat,* a cross street, *dwerg,* a dwarf, &c.
Islandic, *dwergur,* the dwarfs, *dwergmal,* the echo or voice of the dwarfs.
Anglo-Saxon, *dwerg, dweorh,* the dwarfs.

4. *Zwer, swer.*

German, *zwerch,* oblique. *Zwerg,* a dwarf.
Gothic, *tuzweryan,* to faulter.
Dutch, *zwerven,* to swerve, *zwerver,* a wanderer.
English, swerve.

Prepositions.

We may observe that the same analogy which applies to the word *thwart* applies also to the word *across;* for in English we use it adjectively to signify perverse and peevish, and our old writers also employ it as a preposition:

> Betwixt the midst and these, the Gods assign'd
> Two habitable seats to human kind:
> And *cross* their limits cut a sloping way.
>
> DRYDEN'S *Virgil.*

Against.

Against seems to be derived merely from the verb *ga.* In the Anglo-Saxon it is *ongeon* and *ongegen* in the German *entgegen*; in the Dutch *jegens* and *tegens.* TOOKE observes that instead of this preposition the Danes use the analogous words *mod* and *imod* from their verb *moder* to meet; and the Swedes *emot* from the verb *mota* of the same meaning: both which verbs agree with the Gothic *motyan,* the Dutch *moeten* and the English *meet;* to which might be added our *mate* in the old word *folc-mote,* the modern *ward-mate,* &c.

Among.

Among is accurately explained by MINSHEU; being from the word *meng,* the root of many terms in the northern dialects signifying to *mingle* or mix.

Dutch, *mengen, mengelen,* to mix.
Anglo-Saxon, *maengan,* the same.
German, *mengen,* to mix, *menge,* a mixed quantity.
English, *monger,* as in Cheese*monger,* Iron*manger,* &c.

Mr. Tooke says *among* is always pronounced *amung:* we do not happen to recollect any instance of this in rhyme, which would be one mode of testing his accuracy of observation: and we apprehend that such a pronunciation is by no means universal, nor even common.

Amonges is used adjectivally by CHAUCER.

> "Yf thou castest thy seedes in the feldes, thou shuldst haue in mynde, that the yeres bens *amonges,* otherwhyle plentuous, and otherwhyle bareyn."

Gower uses *amonge* adverbially—

> And tho she toke hir childe in honde
> And yafe it souke; and euer *amonge*
> She wepte—

He also uses *amongest* and *emonge* prepositionally—

> I stonde as one *amongest* all
> Which am oute of hir grace fall.
> * * * * *
> The Kyng with all his hole entent
> Then at laste hem axeth this,
> What kynge men tellen that he is
> *Emonge* the folke—

Grammar. In the ballad on the *Battle of Bruges* (A. D. 1301,) we find both *amonges* and *among* used as prepositions.

The kyng of Fraunce made statua newe,
In the lond of Flaundres *among* false ant trewe,
That the comun of Bruges ful sore con arewe,
Ant seiden *amonges* hem.

In the *Seuyn Sages*, it is written *omang*;—

Lene he was and also lang,
And most gentil man tham *omang*.

In the Scottish Acts of Parliament, we find *amangis*.—

That thai ressaue and admitt *amangis* thame Maister Williame Lundy. *Sc. Acts.* A. D. 1567.

In an old English Letter of the year 1258 it is *amanges*.

To halden *amanges* yew ine hord. 1 *Fed.* 378.

The word *meynt* appears identical with this preposition, being merely the participle of the same Anglo-Saxon verb, *mængan*, to mingle.

Warme milke she put also therto
With honey *meynt*. GOWER.

For euer of loue the sickenesse
Is *meynt* with swete & bitternesse. CHAUCER.

Tooke observes that the Danes use, instead of *among*, the prepositions *mellem* and *iblandt*. *Mellem* is from the Danish *melerer*, French *mêler*, Italian *mescolare*, from which source also came the old English *ymell*.

Herdest thou ever slike a song er now?
Lo! what a complin is *ymell* hem alle.
CHAUCER.

The Danish *iblandt* and Swedish *ibland* are from the verbs *iblander*, and *blanda*, to blend.

About. This word is evidently of similar origin with the French *bout*, the *butt*, limit, or end, of any thing; which MENAGE supposes to be derived from an old Celtic word *bod*, and which occurs again in the German *boden*, and in the English *bottom*, *bottomless*, &c. About, is directly from the Anglo-Saxon *onboda*, *onbuta*; and it means on the extremities or limits of any thing, round *about* it.

Behind. On the *hind-part*. In the Gothic Gospels we read *gang hindar mik Satana*, "Get thee behind me, Satan!" *Matth.* c. viii. v. 33. In the Armoric, *hinter* is behind. In the Anglo-Saxon, *hindan* is the same. In the modern German, *hinten* and *hinter* are behind. In the Gothic, *hindar*, that which is left behind. Hence also the English, to *hinder*, *hind-most*, the *hind-wheel*, *hind-quarter*, *hinderling*, &c.

Between, Betwixt. By twain, by twice.

By twene the waiwe of wode and wroth,
In to his doughter chambre he goth. GOWER.

Bytuene Mersh and Aueril, when spray beginneth to springe,
The lutel foul hath hyre wyl on hyre lud to synge.
Harl. MSS. No. 2253. fol. 63.

This was the forward pleinly for t'endite,
Bitwixen Theseus and him Arcite. CHAUCER.

This latter word, it will be observed, very closely resembles the German *zwischen*, between, from *zwey*, two; as *zwischen fünf und sechs*, "between five and six."

Euery man to other will seyne,
That *bytwyx* you is somme synne.
Romance of the Lyfe of Ipomydon.

Thy wife and thou mote hange fer atwynne,
For that *bytwyt* you shall be no synne. CHAUCER.

In the year 1420 we find it written *betwyx* and *betwene*. (9 Rymer, 916.)

Sir PHILIP SIDNEY uses *betweene* as an adjective:

His authoritie hauing bin abused by those great lords, who in those *betweene* times of raigning had brought in the worst kinde of oligarchie. *Arcadia.* Prepositions.

In the old English we find from this same source the adverbs *a twayne* and *otuynne*.

With his axe he smote it *atwayne*.
See WHARTON, v. i. p. 156.

He fondred the Sarazyns *otuynne*.
ROBERT DE BRUNNE.

Beyond. This word seems to be of the same origin as the preposition, against; being from the verb *gan*, *gangan*, or *gongan*, to go. Hence says Mr. Tooke, "*beyond* any place" means "be passed that place," or "be that place passed." It might perhaps be more correctly explained, "that place *being* passed;" for as we have before observed, the preposition does not *assert*, which is the function of the verb; but merely *names* a conception, which is the function of a noun.

Beneath. Beneath is by the *nether*, that is, lower part. In the old English it is written *binethen*.

Here kirtel, here pilche of ermine,
Here keuerchefs of silk, here smok o line,
Al togidere, with both fest,
Sche to rent *binethen* here brest.
Rom. of the Seuyn Sages.

Niden and *nider*, with their derivatives, are found in many northern dialects, signifying that which is below, or inferior.

German, *nieder*, below.

Swedish, *nedre*, *neder*.

Danish, *ned*.

Dutch, *neder*, down; *Nederland*, the Low Countries, or Netherlands; *beneden*, beneath; *benedenwaards*, downwards, &c.

Anglo-Saxon, *nither*, below

Armoric, *nithane*, under.

Frankish, *nidana*, beneath.

To this same source Mr. Tooke traces the preposition *under*, as being originally *an neder*.

Where obsolete. Hitherto we have spoken of words used as prepositions, and also as nouns or verbs in the same, or nearly the same signification; and in these we have proceeded from the more to the less obvious. There is no absolute line to be drawn in matters of this kind between that which is discoverable at first sight, or on a short reflection, and that which it requires some study to make out; because the different capacities, and the different experience, of different men, must influence the degrees in this scale But we may proceed by almost imperceptible degrees from that which almost all men think clear and self evident, to that which almost all will admit to be involved in obscurity, and yet the analogical principle, discreetly used, will give us scarcely less confidence in the latter than in the earlier stages of this progress.

With. Following this clue, we come to the preposition *with*, which will probably be found rather more obscure in its derivation than any of the words hitherto examined. There are no less than three etymologies, to which it has been thought necessary to resort, in order to account for the different uses of this one preposition:—

1. The Gothic verb *withan*, to bind, or join together.
2. The Gothic preposition *withra*, toward, or against

3. The Anglo-Saxon verb *wyrthan*, (or rather the Gothic *wisan*) to be.

We are inclined to regard the first and second of these etymologies, though at first sight so widely different in signification, as originally the same. When any two visible objects are nearly *connected*, in local situation, they must appear to be placed in *apposition* to each other, if both be viewed from a distant point; but if one be viewed from the other, it will appear to be placed in *opposition*. Now, the preposition *with*, both in Anglo-Saxon and in English, expresses these different relations of *apposition* and *opposition*: it is therefore probable, that the original radix of the word, (so far as these two significations are concerned,) expressed the idea common to both, namely, the idea of *connection*. To exemplify this observation, let us suppose that John and Andrew are seen at the distance of half a mile by Peter; they appear to be close together, to be *joined* with, or bound to each other; but on approaching them he finds that there is a considerable interval between them, and the one either stands *opposite to* the other, or comes *toward* him, or stands *against* him resisting, or draws *back* from him. Now all these conceptions of being joined with, standing opposite to, coming toward, resisting, and drawing back from, with others of a like kind, will be found to be expressed in different Teutonic dialects by words obviously related to our preposition *with*. This will appear more at large as we separately examine the above stated etymologies.

1. The idea of *connection*, or joining together, was expressed by the Mæso-Gothic verb, *withan*, of which the past tense, *gawath*, occurs in the following passage of the *Codex Argenteus*. *Thata Goth gawath, Manna ni skaidai*, "What God hath *joined together*, let not man put asunder." (St. *Mark*, x. 9.) Hence, as a particular kind of weed is called *bind*weed, because it twists round and binds together other plants; so a particular kind of tree (the willow) was called the *with*-tree, or *withy*-tree, (in old German, *weide*-baum, or *wette*-baum); because its tender twigs were used to *with*, (that is, to bind together,) many objects in rustic economy. The twigs so used for binding were also called *withs*, or wythes: and a *with* or *wythe* was a term given to any thing that bound either the body or the mind.

Mortimer, in his *Husbandry*, speaks of the tree:—

Birch is of use for ox-yoaks, hoops, screws; *wythes* for faggots.

Lord Bacon uses this word to signify the twig:—

An Irish rebel put up a petition, that he might be hanged in a *with*, and not a halter; because it had been so used with former rebels.

The two words *with* and *halter* are simply *binder* and *holder*; but one, it appears, had appropriated the former, to a binder made of willow twigs; and the latter, to a holder made of hemp.

King Charles employs the same word metaphorically:—

These cords and *wythes* will hold men's consciences, when force attends and twists them.

In different Anglo-Saxon glossaries, we find *withig*, the willow; *withthe*, a hoop, or band; *cynewiththe*, the diadem, the king's band, or "golden round," as Shakspeare calls it.

In an Alamannic glossary, "ubi recensentur res pistrini atque horrei," says Junius, "*with* expositor *torta*."

"Danis quoque," says the same author, "*widde* est *copula viminea*; potissimum tamen, ut videtur, copula ex salignis viminibus contexta, contortave."

In Dutch, the willow is called *wiede, wiide, weyde, wiie*.

Our word *willow* itself originally conveyed the same notion of binding; it being derived from the Anglo-Saxon *wilig*, which came from the verb *wilan*, as *withig* did from the verb *withan*; and both *withan* and *wilan* signified to *bind*.

It is little to be doubted, but that our verb *wed*, to marry, is radically the same as *with*; and means simply, to join, or bind together.

Wed seems to have been opposed to *shed*; the former signifying to join together, the latter to separate. *Shed* is still used in the Scottish dialect, in the particular sense of separating or dividing the hair on the forehead; as in the old ballad—

Janet hath kilted her green kirtle,
A wee abune her knee;
And she hath *shed* her yellow hair,
A wee abune her bree.

It is obviously derived from the Gothic *skaidan*, referred to in the above quotation from the *Codex Argenteus*; and *skaidan* is identical with the Anglo-Saxon *sceadan*, the German *scheiden*, and Dutch *scheyden*, to separate, or put asunder.

Moreover, as there were two verbs, *withan* and *wilan*, signifying to join, so there were two analogous verbs, *skaidan* and *schalen*, signifying to separate. Of *skaidan* we have already spoken: *schalen* is still extant in German, in the sense of separating the outer coat, rind, peel, or shell, of a thing from that which is within: and the substantive *schale* in that language is the *shell* of an egg or nut; it also signifies a small cup or saucer for drinking out of, probably because *shells* were originally used for that purpose. Connected with this word *schale* is our substantive *shell*, which was written in old English *shale*, and is the same word with the *scales* of a fish, meaning that which is *scaled* off, or separated from the main body. Hence, in Scotland, the *kirk-scaling* is the departure of the congregation from church, when they separate in all directions. Our word shell is also the Danish, Swedish, and Icelandic, *skal* or *skel*, the Anglo-Saxon *scyll*, the Dutch *schell*, *schille*, the Italian *scaglia*, and the French *escaille*, or *écaille*. In Dutch also, the tiles of houses are called *schalien*, and in Gothic *skalyos*.

To return to the preposition *with*. Wachter derives the German *weide*, and Frankish *wida*, a willow, from the old verb *wetten*, to bind; "ab usu, quem arbor officiosa præbet colonis et hortulanis in jungendis et alligandis rebus;" and he suggests, that the Latin *vitis*, a vine, is so named from its binding round other trees. *Weiden* also he explains, to bind, and *weid*, *wied*, *wette*, a bond. The Frankish *languuid*, is a waggon-rope. *Wette* also signifies, metaphorically, the law, which *binds*; and this in Dutch is *wet*, whence *wet-boek* is a law-hook; *wetsteller*, *wetmaaker*, *wet-geever*, a legislator; *wethouders*, magistrates; *wet-geleerde*, a lawyer; *wetbreker*, a lawbreaker; *wetloos*, an outlaw; *wettig*, legitimate, &c. The verb *wetten* is not only to bind, but to bind in *wedlock*. "Oritur," says Wachter, "a *wette*, vinculum, copula, ligamen,

unde reliqua, tam verba, quam substantiva, tanquam ex matrice prodierunt."

From all these authorities we may safely conclude, that we have ascertained the proper origin of our common preposition *with*, in the sense of association, e. gr.

> In all thy humours, whether grave or mellow,
> Thou'rt such a touchy, testy, pleasing fellow;
> Hast so much wit and mirth, and spleen about thee,
> There is no living *with* thee, nor without thee.
>
> *Tatler.*

2. It is obvious, that in several of the other uses of this preposition, which Dr. Johnson points out, it really expresses no more than the same conception of joining or binding together, modified by the nature of the objects spoken of. Such are the following:—"in company,"—"in partnership,"—"in appendage,"—"in mutual dealing,"—for I am *joined* with those with whom I am in company;—I am *bound* to one with whom I am in partnership;—a thing is *joined* to that of which it is an appendage;—two persons, who mutually deal together, are *bound* by the laws of honesty to each other;—and so of similar cases. It is remarkable that Johnson himself gives the two following senses of this preposition, in immediate succession.

4. On the side of; for—

> ———— O madness of discourse!
> That cause sets up *with*, and against thyself.
>
> Shakspeare.

5. In opposition to; in competition, or contest—

> ———— I do contest
> As hotly and as nobly *with* thy love,
> As ever against thy valour. Shakspeare.

This illustrates the transition before mentioned, from *apposition* to *opposition*; and hence Johnson says, "*With*, in composition, signifies opposition or privation." Instances of this use of the word in modern English are, *withdraw*, *withhold*, and *withstand*.

Barbour uses *withsay*—

> With right or wrang it have would thay;
> And if ane would thame *withsay*,
> Thay would sa do, that thay suld tine,
> Eyther land or lyfe, or live in pine.

This is in German *widersagen*; as in the old baptismal formula "*widersagestu dem Teufel und allen seinen werken?*" "Dost thou renounce the devil and all his works?" And in this sense *absagen* is also used.

It is observable, that the modern German, which does not use *with*, or any similar preposition in the sense of association, has *wider* to signify opposition, both in the simple form, and in a great number of compounds,—as *wiederhalten*, to resist; *widerlegen*, to refute; *wider-reden*, to reply; *wider-sprechen*, to contradict, &c.; and so *widerschein*, a reflected light; *widersinn*, an absurdity; *widerschall*, an echo, &c.

In the Anglo-Saxon, *with* and *wither*, are both used in the sense of opposition, or reflecting back from; as *withstandian*, to resist; *with-coren*, *wither-coren*, cursed, *with-secyian*, *withersecyian*, to contradict; *with-lædan*, to lead back; *with-scufan*, to repel. In the laws of Canute we find *witersacan*, apostates. In the old English laws we have *withernam*, in the barbarous Latin of that day, *withernamium*, a rescizure. This last word is said to have given an easy victory to an English lawyer in Italy, at an epoch when it was the custom for scholars to offer public challenges for disputation on any given subject. As the party who accepted the challenge had the choice of a subject, our lawyer proposed, as his question, *An averia capta in withernamio replegiari possint;* to which his antagonist, as he did not understand what *withernamium* meant, was unable to give any reply.

In the Islandic, we find both *vid* and *vidur* signifying against. In the Frankish, *wid* and *with* are "against," as *with thenne Divvel*, "against the Devil." But in most of the other Teutonic dialects, when the sense is *contra*, or *retro*, the letter r is found in the word. In the Gothic of Ulfila *withra* signifies both toward and against, as *alla so baurgs usiddya withra Jaisu*, "all the city went out *toward* Jesus." *Saei nist withra izwis, faur izwis ist;* "He that is not *against* us, is for us." and so in the compounds *withrawairthan*, "opposite," "over against;" *withraidya*, "be met;" *withragamotyan*, "to meet." In the Frankish, *uuidrunpiotun*, is to write in reply. In the Alamannic, *uuidartragan*, is to carry back. In the old Salic laws, *widredo* is a repeater of his oath, from *eid*, an oath. In the Lombard laws, *widerboran* is a manumitted slave. This last word is also written *guiderbora*, as in the laws of Luitprand, (circ. A. D. 720.) "Si quis aldiam alienam aut suam ad uxorem tollere voluerit, faciat eam *guiderboram*." Another remarkable instance of the use of *wider* in composition, is in *widrigildum*, which some writers confound with *wergeldum*; but Eccardus accurately distinguishes these words, observing that the latter properly signifies the price, ransom, or value of a *man*; the former, any composition by which a loss is paid *back*, or compensated. *Weregelt* is well known to the old English and Scottish law; (see Fleta, and the *Regiam Majestatem*.) *Weregeltthef* according to Fleta is "Latro, qui redimi potest." Hence Somner derives *wer-geld* from *wer*, a man, and *gelt*, price. On the other hand, Grotius, (in the preface to the Gothic writers,) defines *wedrigeldium* "quod pro talione datur;" and this word is properly derived by Wendelinus from the Teutonic *weder* contra, vicissim, and *gelt*, æstimatio. It is differently written, *widrigilth*, *widrigildum*, *guidrigild*, *wedrigildum*, *wedrigeldum*, *widrigild*.

> *Widrigilth* secundum quod appretiatus fuerit.
>
> *Decr.* Childeb. II. (A.D. 711.)

> Suum *widrigildum* omninò componat.
>
> *Decr.* Ludov. II. (A. D. 879.)

> Si stupri crimine detectæ fuerit componat *guidrigild* suum.
>
> *Capitul.* Arech. Princ. Benevent.

> Juxta quod *widrigild* illius est.
>
> *Capit.* Lothar. Imp. (A. D. 824.)

Perhaps the most remarkable derivation from the word *wither*, or *wider*, now remaining in our language, is *guerdon*; and the more so, as the English etymologists in general have entirely mistaken its origin.

The English word *guerdon* is a mere adoption of the French *guerdon*, of which Menage thus speaks:—"Je croy qu'il vient de *werdung* qui signifie *pretii æstimatio*, et dont les escrivains de la basse Latinité ont fait aussi *werdunia* pour dire la mesme chose. De *guerdon* les Espagnols ont fait *galardon*, et les Italiens *guiderdone*." Skinner cites this; but prefers the derivation of *guerdon* by Mylius from the Dutch *weerderen*, *waerderen*, æstimare, censere; and this from *weerd*, *waerd* dignus, et weerde, valor, pretium.

Grammar.

Junius cites the French *guerdon*, Italian *guiderdone*, Spanish *galardon*, and Welsh *wherth*; "quæ omnia," says he, "valde affinia sunt Teutonico *weerde, weerdiie*." What is meant by *galardon* being "*valde affine*," to *weerde*, we cannot guess; any more than we can tell how the Italians formed *guiderdone* out of *guerdon*; and as to the base Latin *werdunia*, we never happened to meet with that word. The real history of the word *guerdon*, however, may, we apprehend, be very satisfactorily traced, as follows:—

1. *Widerdonum*. This word is correctly explained by Du Cange, "Vox ibrida, a *widar* Teutonico, contra, et *donum* Latino, munus." This mixture of Teutonic particles with Latin substantives or verbs, is a fact, which, properly considered, may cast some light on the true principles of etymology; for those principles have been hitherto but little studied. Thus we find our word *miscreant* to be compounded of the Teutonic *mis* (our verb, to *miss*,) and the Latin *credere*: and the French have many such compounds, e. gr. *mécompte, méconnoitre, mécontent, mésaventure, mésoffrir, mésestimer, médire, méfaire*, &c. *Widerdonum* occurs in the *Tabularium Casauriense*, (A. D. 876.)

Quis tu dedisti mihi, pro memoratâ convenientiâ, *widerdonum*, caballum unum, et argentum solidos centum.

2. *Guiderdone*, or *guidardone*. This is merely the word *widerdonum* with the Italian articulation *gui* for *wi*, as in *guidrigild* and *guiderbora* above noticed, for *widergild* and *widerbora*. The Latin termination *um* was universally changed into *o* by the elder Italians; and in modern writing it is softened still more into *e*; whence we find in the *Vocabolario della Crusca, guiderdone* and *guidardone*, with their derivatives, *guiderdonare, guidardonare, guiderdonato, guiderdonatrice, guidardonatrice, guiderdonamento, guidardonamento*, &c.

E come i falli meritan punizione, così i beneficii meritan *guiderdone*. Boccacio, (*circ.* A. D. 1350.)

E per *guidardone* del vincitore apparecchiò Ghirlande. *Idem.*

3. *Guizardonum*. This is evidently a mere provincial corruption of pronunciation from *guidardonum*.

Item quod nullum munus, *guizardonum*, vel xenia aliqua recipiet. *Statut.* Massil. (*circ.* A. D. 1220.)

4. *Guiardonum*.

Non currant aliquæ pœnæ, usuræ, *guiardona*, vel expensæ. *Statut.* Vercel.

This word is thus explained in an old glossary:—"*Guiardonum*, remuneratio; Ital. *guidardone*, nostris *guerredon*." (Vide *Glossar. Provinc. Lat ex cod. Reg. Paris*, No. 7657.)

5. *Guiardon*, in the Provençal dialect, præmium.

6. *Guerredon*. The old French word above alluded to, which is also found in the *verbal* form, *guerredonner*.

Se Dieu saure le baron,
Iis en auront bon *guerredon*.
Roman D'Athis.

Voulons, pour ce, yceulx *guerredonner*, et poursuir de faveur especial. *Chart.* Phil. VI. (A. D. 1330.)

7. *Guerdon*. In the old French translation of the passage above cited from the Statutes of Marseilles, the words "*Guizardonum* vel xenia," are rendered "*guerdon*, ou estrenne."

In English, *guerdon* is used to signify a just recompense either for good or bad deeds.

He shall by thy revenging hand at once receive the just *guerdon* of all his former villainies. Knolles. *Hist. Turk.*

Prepositions.

Fame is the spur that the clear spirit doth raise,
(That last infirmity of noble mind)
To scorn delights, and live laborious days;
But the fair *guerdon* when we hope to find,
Comes the blind fury, with th' abhorred sheers,
And slits the thin spun life. Milton.

3. Having examined two derivations of our modern preposition *with*, we come to the third, which is thus stated by Mr. Horne Tooke.

"With is also sometimes the imperative of *wyrthan*, to be. Mr. Tyrwhit, in his glossary, (art. But,) has observed truly, that 'by and with are often synonimous.' They are always so, when with is the imperative of *wyrthan*: for by is the imperative of *beon*, to be. He has also in his glossary, (art. With,) said truly, that 'with *meschance*, with *misadventure*, with *sorwe*: 5316, 7797, 6916, 4410, 5890, 5922, are to be considered as parenthetical curses.' For the literal meaning of those phrases is (not God *yeve*, but) be *mischance*, be *misadventure*, be *sorrow*, to him or them, concerning whom these words are spoken. But Mr. Tyrwhit is mistaken when he supposes—'with *evil prefe*, 5829; with *harde grace*, 7810; with *sory grace*, 12810;' to have the same meaning; for in those three instances, with is the imperative of *withan*; nor is any parenthetical curse or wish contained in either of those instances."

There is something ingenious in connecting *with* and *wyrthan*; and it was probably suggested to Mr. Tooke by the analogy which *with*-out and *with*-in bear to the Scottish "*but* and *ben*;" i. e. *be*-out and *be*-in. The Anglo-Saxons also used *with*-yeondan, for beyond; and indeed they employed the separate preposition *with* so loosely, as to afford room for supposing that it was only equivalent to the general expression of existence, *be*: for Hickes, in his *Grammatica Anglo-Saxonica*, explains *with* by the Latin words *juxta, cum, contra, adversus, pro, circa, circiter, erga, a, ab*; and one of his examples is remarkable, as using *with* for *by* in the sense of *near to*. "*With tha sæ Adriaticum, juxta* mare Adriaticum."

Still we are in some doubt whether the Anglo-Saxon *wyrthan* affords the proper solution of this question; since we do not find the *r* ever introduced before the *th* into either the Anglo-Saxon or English preposition: in other words, we do not find *wyrth* used as a preposition in Saxon, or *worth* in English: and though *worth* is certainly used for *be* in the parenthetical curse *wo worth!* and in the parenthetical blessing *well worth!* it is not quite so clear that *with* is thus used in the expressions "*with* meschance, *with* misaventure, or *with* sorwe."

In the vision of *Piers Plowman* we have the verb *worth*, to be. In Chaucer we have *wo worth*, and in *Piers Plowman, well worth*, and *much wo worth*.

And said, mercy madam, your man shall I *worth*.
Piers Plowman.

Wo worth the faire gemme vertulesse!
Wo worth that hearbe also that doth no bote!
Wo worth the beauty that is routhlesse!
Wo worth that might that trede ech under fote!
Chaucer.

Much wo worth the man that misruleth his inwitte!
And *well worth* Piers Plowman that pursueth God in his going!
Piers Plowman.

The Anglo-Saxon *wyrthan, wurthan*, or *weorthan*,

Grammar. and the English *worth*, are from the Gothic *wairthan*; but perhaps the Anglo-Saxon and English *with*, used synonymously with *be*, are rather from the other Gothic verb substantive, *wisan*: for the different Teutonic tribes used three verbs substantive, (as they are called,) viz. *beon*, *wisan*, and *wurthan*; of which we retain traces in the different tenses of our verb, to be; namely, *be*, *was*, and *were*.

By. From the last-mentioned signification of the preposition *with*, the transition is easy to the preposition *by*, which in many of its uses is manifestly nothing more than the imperative *be*. Dr. Johnson, in his usual manner, gives no less than twenty-five senses of this preposition, as denoting the agent, the instrument, the cause, &c., all which is very proper in lexicography, but will assist us little in grammar, without some further analysis. The dictionary maker, moreover, has in general little or nothing to do with those uses of words which have become entirely obsolete; but these may often assist the researches of the grammarian. Perhaps we may trace all the uses of this preposition, and of the analogous words in other Teutonic dialects, to two different origins, namely, the verbs to *be*, and to *big*. When derived from the former, it is a sort of elliptical expression, the word agent, instrument, cause, &c. being understood from the context: when derived from the latter, it signifies proximity. Thus, in the following examples, (quoted by Johnson)—" The grammar of a language is sometimes to be carefully studied *by* a grown man;" —" When Hector fell *by* Pelides' arms;"—" If we give you any thing, we hope to gain *by* you:"—The meaning is, " the grammar is to be studied, there *being* a grown man as the student;"—" Hector fell, there *being* the arms of Pelides, which caused his fall;"—" We hope to gain, there *being* you, to promote our gains." But in the following examples, *by* signifies proximity, either stationary or in passage.

> ——— A spacious plain, whereon
> Were tents of various hue; *by* some, were herds
> Of cattle grazing. MILTON.

> Many beautiful places, standing along the sea-shore, make the town appear much longer than it is, to those that sail *by* it. ADDISON.

SHAKSPEARE puns on the two different uses of the word *by* in the following passage:—

> So thou may'st say the church stands *by* thy tabour, if thy tabour stands *by* the church.

That is, " if you use the word *by* improperly, you may be understood to mean that the church is supported *by means* of your tabour; whereas, the fact merely is, that your tabour happens to be placed *near* the church."

It is in this latter sense of proximity, that we find the word *by* used adverbially and as a substantive, and also (when in composition) adjectivally.

1. Adverbially—

> And in it lies the God of Sleep,
> And snorting *by*,
> We may descry
> The monsters of the deep. DRYDEN.

> ——— I did hear
> The galloping of horse. Who wast't came *by*?
> SHAKSPEARE.

2. As a substantive, in the phrase " by the *by*;" anciently written, " upon the *by*."

> This wolf was forced to make bold, ever and anon, with a sheep, in private, by *the by*. L'ESTRANGE.

Prepositions.

> In this instance there is, upon *the by*, to be noted, the percolation of the verjuice through the wood. BACON.

3. " *By*, in composition," says Johnson, " implies something out of the direct way, and consequently some obscurity, something irregular, something collateral, or private." These are instances of the natural transitions of the mind in the use of words, and the enumeration is only defective in not specifying the first link of the chain; thus, a *by-stander*, one who stands *near*. A *by-road*, a road, which, branching off from the main road, is of course less frequented and comparatively obscure; a *by-end*, an object obscurely connected with the known and ostensible end in view: the *by-play* of an actor, those actions and gestures which are carried on apart from the main business of the scene: a *by-law*, a law apart from the public laws of the state, and affecting only a private body of men: a *by-word*, a word of reproach, used aside as it were, and separately from honest and honourable conversation: a *by-name*, a surname, or nickname, added to or substituted for the original and proper name of the individual: *by past* times, are those times which once were passing *by* us, (as the mariners sailed *by* the town above spoken of by Addison,) but which have now passed by, and are gone. Dr. Johnson says that " this" composition is used at pleasure; but in fact it is very much regulated by custom; for several even of the instances which he quotes would now be considered as antiquated expressions; such are a *by-concernment*, *by-interest*, a *by-name*, *by-respects*, *by-views*, &c.

To this we may add the use of the word *by* in the phrase " *by and by*:" and perhaps in the phrase " *Good by!*"

By and *by* seems to signify a time near to the present, but not immediately following it, and usually refers to some action out of the course of that on which the individual is at the present moment employed. Thus, when the friar wishes to conceal Romeo before he opens the door to the nurse, who is knocking, he says,

> ——— Stand up.
> Run to my study—(*by and by*)—God's will.
> What wilfulness is this! (I come, I come.)

where the passages in parenthesis are addressed to the nurse.

So, Othello, speaking alternately to himself and to Emilia, who is calling for admittance, says,

> (Yes!)—'tis Emilia—(*by and by*) She's dead.

SPENSER uses this expression narratively, to signify proximity of time.

> The noble knight alighted *by and by*
> From lofty steed.

CHAUCER uses it to signify proximity of place.

> And so befel that in the tas they found
> Two yonge knightes ligging *by and by*.

The phrase *Good by!* is commonly supposed to be a mere contraction of the words "*Good be with you*;" but it seems as probably to have been an elliptical phrase for " may good be *by*:" that is, " may good be *near* you, wherever you are!"

There is a singular difference in the use of the

preposition *by* in the sense of proximity, between the English and Scottish dialects. In the former, *by himself* means "alone," "no one else being *by*:" in the latter it signifies "insane," "*beside* himself."

Sitting in some place *by himself*, let him translate into English his former lesson. ASCHAM.

And monie a day was *by himsel'*,
He was sae sairly frighted. BURNS.

In old English *be* and *by* are often used indifferently: e. gr. "Damville *be* right ought to have the leading of the army; but *bycause* thei be cosen germans to the admirall, thei be mistrusted." (See Lodge's *Illustrations*, vol. ii. p. 9.) So in the ballad—"*How a Merchande*," &c.

Bothe *be* daye and *be* nyght

So in MONTGOMERIE'S "*Cherrie and the Slae.*"

I saw nae way quhairby to cum,
Be ony craft to get it clum.

In the description of *Cokaigne*—

Ther beth briddes mani and fail,
That stinteth neuer *bi* har migt,
Miri to sing dai and nigt.

In like manner we find *byfore* and *before*, *bylove* and *belove*, *byeause* and *because*, &c.

By any other cause or matter hadde or made *byfore* the seid fyne levied. *Stat.* 1. RIC. III. c. 7. MS.

He was newly fallen to his fader's herytage, who was so well *byloved* in his royalme. BERNERS'S *Froissart*, fol. lxxxi.

His men murmured and spake of hym otherwyse then they sholde do *bycause* of them of the garyson of Dulcen. *Ibid.*

In the letter of HENRY III. (A. D. 1258.) which is one of the most ancient specimens of English extant, we find *biforen*—

Alswo alse hit is *biforen* iseid.
Fœdera, vol. i. part i. p. 378.

There are several uses of the preposition *by* in old English, which have now become obsolete; as "*by* daies olde," which GOWER uses for the modern phrase "*in* old times."

In the romance of *Sir Guy* we find "*by* twenty mile round about," instead of "for twenty mile round about." King JAMES I. of Scotland uses *by* for *of*.

As Tantalus I travail, ay buteles,
That ever ylike hailith at the well,
Water to draw with buket bottemles—
So *by* myself this tale I may well telle.
King's Quair, canto ii. st. 51.

These and many other uses of the analogous prepositions occur in the Mæso-Gothic, Anglo-Saxon, &c. In Mæso-Gothic the verb *beon* or *bion* is not found, but the preposition *bi* exists both separately and in composition. In its separate use it answers to the Latin *in, pro, cum, contra, secundum, post, de*, and *circa*. As a component particle, it appears in *bigitan*, to find, (or, as we say, to get *at*;) *bifotuns*, fetters; *bihlahyan*, to laugh *at*; *bimaitan*, to cut *around*; *bigaurdans*, girded round about, &c. In Anglo-Saxon, "*be* Petres mæssan," is "*at* Peter-mass;" i. e. "on the festival of St. Peter." "Tha he gehyrde *be* tham Hælende,"—"When he heard *of* the Healer," i. e. of Jesus. "*Be* Wihtgares dæge,"—"*In* the days of Whitgar." "*Be* hyra wærtmum ye hig oncnawath,"—"*By* their fruits shall ye know them." *Be* also enters into the composition of several Saxon prepositions, as *beforan*, before; *betwux*, betwix; *beheonan*, on this side; *beæftan*, or *bæftan*, after; *binnan*, within; *butan*, without; *bufan*, upon. In these and in many other compound words, *be* is evidently the mere root of the verb *beon*, to be. It is, however, sometimes written *bi*, or *big*, as "se bisceop the him *big* sæt,"—"The bishop who sate *by* him," i. e. near him: and in this sense it may be reasonably supposed to have some affinity to the verb *byan*, to inhabit; or *bicgian*, to build; which latter is still retained in the Scottish verb, to *big*; as in the song of *Bessy Bell and Mary Gray*.—

They *bigg'd* a bour on yon burn brae.

From the verb *bicgian* or *byan* comes our local termination in *by*, so frequent in Yorkshire and Lincolnshire, as in *Danby, Manby, Ranby, Belby, Kelby, Welby, Holtby, Boltby, Kirkby, Birkby, Harmby, Barmby, Haxby, Saxby, Romanby, Normanby, Solmonby, Horeby*, &c. The German preposition *bey*, which is rendered by the French *chez*, and the Latin *apud*, may perhaps be in like manner derived, as ADELUNG suggests, from the old verb *bio, bo, bauen*, in the sense of dwelling at, or occupying a certain spot. In the old Prussian language *bo* or *po* was used prepositionally in this sense; and hence the *Borussi* or *Porussi*, the ancient name of the Prussians signified those who dwelt near the Russians, as *Pomeranii*, the Pomeranians, signified those who lived (*Po-Meere*) near the sea. In the Frankish we find *pi*, as *pi hantun*, at hand. In the Alamannie it is written *pii*, as *pii inkange*, near the entrance.

Before. For.

Among the prepositions compounded with *be*, or *by*, we have already noticed *behind, before, between, beyond, beneath*, &c. and we have shown that in the compound word *behind*, the simple word *hind* is a *noun*, that is to say, the name of a conception formed by the mind. There can be little doubt, we apprehend, but that *before* is a preposition of the same nature as *behind*; that is to say, that the words *hind* and *fore* were equally in their origin, nouns. We still use them both adjectivally, even in their separate state.

Resistance in fluids arises from their greater pressing on the *fore* than *hind* part of the bodies moving in them. CHEYNE.

And so they occur in various compound words, as *forewheel, hindwheel, foremon, hinderling*, &c.

As we have said that the preposition *athwart* might have been *thwart*, that the preposition *across* has been actually written *cross*, and that the Germans indifferently use *anstatt* and *statt*, so it is obvious that the preposition *before* would be equally intelligible, and would convey exactly the same meaning if it were written *fore*; for the prefixes *a* and *be* are mere matter of idiom, and do not alter the meaning of the words *thwart, cross, fore*, &c. with which they are united in common use. Accordingly *afore* and *tofore* were formerly used for *before*.

Whosoever should make light of any thing *afore* spoken or written, out of his own house a tree should be taken, and he thereon be hanged. ESDRAS, ch. vi. v. 22.

Darie in a verger ys,
Tofore him mony knyghtis ywis.
Kyng Alisaunder.

And so we still use these expressions in the compound words *aforesaid, aforementioned, heretofore*, &c

Fore, therefore, must be considered as a noun, or the name of a conception. Now of what conception is it the name? This question will be best answered by comparing together several instances of its use. We have, in English, the words *forecastle, foredeck, fore-end, forefinger, forefoot, forehead, foreland, forelock, foremast*, &c. relating to place; and the words *fore-advise, forebode, forecast, forefather, forenoon*, &c. relating to time. It is plain that there is an analogy between these two classes of words; for they both agree in expressing, by the particle *fore*, one common conception, namely, that the thing spoken of is *before* some other thing, either in place, or in time. A *fore*-castle, for instance, is the elevated part of a ship; which, as she moves through the water, goes *before* the main body of the vessel. A *fore*land is a part of the land which projects *before* the rest into the sea. To *fore*advise is to give advice *before* the emergency to which it is applicable, actually occurs. The *fore*-noon is that part of the day, which elapses *before* the sun reaches the meridian.

Now, this conception, so expressed by the particle *fore*, is not the conception of a *real object*, but it is the conception of a *relation* existing between two objects. We may give it the name of *foreness* or of *precession*, or any other name that may be thought more suitable; but the conception itself must unavoidably be formed by all men. A savage, when in presence of his enemy, not only apprehends that such enemy exists, but that he is *before* him. The same savage, when he perceives the sun rising, not only knows that a certain portion of the day is elapsing, but that such portion is *before* the noon. In order to know these two facts, however, he must necessarily be able to form a conception, in the one case, of a *relation of place*, and in the other case, of a *relation of time*. But the relations of place and those of time, are, in many instances, if not identical, at least so closely analogous, as to be expressed in most languages by the same term: and thus, in most languages, we find that the word, which implies priority of time, expresses also precession in space; which is the case with the word in question, *fore*.

Other analogies again coincide with these. The person that is chief in *dignity*, rank, or order, is usally said to precede or go *before* his inferiors; and the final *cause, motive*, or *end* is placed *before* the mind when deliberating on an act to be done.

Lord MONBODDO justly says, "every kind of *relation* is a pure idea of *intellect*, which can never be apprehended by *sense*:" and when Mr. HORNE TOOKE denies this proposition, he shows strange ignorance of the human mind. Sense, taking that term in its widest acceptation, can only apprehend an external object, it can apprehend the *thing*, which is before another, or the *thing*, before which another is; but the *relation* of place, time, order, causation, or the like, which we express by the word *before*, is discerned not by a simple operation of sense, but by means of an exercise of our comparing and judging faculties. It is most extraordinary that Tooke, who asserts universally that "prepositions are the names of *real objects*," should say of the preposition, *for*, "I believe it to be no other than the Gothic substantive *fairina*, CAUSE." What *real object* is CAUSE? How is causation to be apprehended by sense? That we have a conception of cause is certain; but it is equally certain that we come at it by means of our intellect; and that it is in truth "a pure idea of intellect," which sense alone never did nor ever can give.

That the Gothic substantive *fairina* may have some etymological affinity to our preposition and conjunction, *for*, we do not mean to dispute; nor do we deny that *for* often expresses the relation of a final cause to its effect; but the reason of this is, that *the words* FOR *and* FORE *are the same.*

The identity of these words, both in their simple and compound states, may be shown in a variety of instances.

In "*Christ's descent into hell*," we have *fore* used as we now use *for*.—

> *Fore* Adame's sunne fol y wis,
> Ich haue tholed al this.

Our common words where*fore* and there*fore* are "*for* which," and "*for* this;" and the latter is often written *forthi* or *forthy*, in ancient authors; as the former is written *for why* by some of more modern date.

> *Forthi* myn wonges waxeth won.
> MSS. *Harl.* No. 2253. fol. 63.

> And *forthy* if it happe in any wise,
> That here be any louer in this place.
> CHAUCER'S *Troilus*.

> Solyman had three hundred field pieces, that a camel might well carry one of them, being taken from the carriage; *for why*, Solyman purposing to draw the emperor unto battle, had brought no greater pieces of battery with him.
> KNOLLES' *Hist. Turk.*

Forsaid is used as *foresaid*, or aforesaid, in a document of the year 1420. (*Rymer*, v. ix. p. 916.)

Forlok, for *forelook*, i. e. foresight, occurs in the romance of *Sir Amadas*.

> Ther Y had an bondorthe marke of rent,
> Y spentte hit all in lyghtte atent,
> Of suche *forlok* was Y.

In the same romance we have *fordryvon* for *foredriven*.

> Folke *fordryvon* in the schores
> Wrekkyd with the water lay.

So, *forward* for *foreword*; i. e. promise made *beforehand*.

> Thenke what *forward* that thou made,
> When thou full greyt myster hade.

In the romance of *Sir Tristem*, edited by Sir WALTER SCOTT, the preposition *before* is written *bi for*.

> The folk stode unfain
> *Bi for* that leuedi fre:
> Rouland mi lord is slain,
> He spekeih no more with me.

Mr. TOOKE has, with great parade of comment, in above twenty quarto pages, reviewed the seventeen significations of the word *for*, which are given by GREENWOOD, and the forty-six by JOHNSON; besides reprehending LOWTH and TYRWHIT, for their remarks on the same word. The result is, that in Mr. Tooke's opinion, *for* always signifies *cause*. Now, this is an error. *For* signifies merely *before*; and as the *final cause* is *before* the mind of the agent, *for* may, in some instances, be rendered *cause*; but there are other cases in which the notion of a final cause does not seem to be involved in the signification of the word *for*. When we say "Christ died *for* us," we mean that our salvation was *before* the contemplation of Christ as the *final cause* of his death. When we speak of "fighting *for* the public good," we mean that the public good is *before* the mind of the combatant, as the *final cause*

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of his fighting. But in the following instances, the notion of *cause* is very indistinctly, if at all alluded to. "He is big enough *for* his age;" i. e. having *before* your mind his age, considering his age, he is big enough. "He speaks one word *for* another," i. e. he speaks one word *before*, or in the place of another, he takes the wrong word first. "We sailed directly *for* Genoa;" i. e. Genoa was *before* us as the object toward which we sailed. In like manner, when *for* is used in the sense of equivalence, it is to be explained by the same word, *before*. "An eye *for* an eye;" i. e. having *before* you the consideration that an eye has been destroyed, another eye must be put out. "To translate line *for* line;" i. e. laying *before* you one line of the original, one line of the translation must follow. And this notion of equivalence is taken negatively in the common phrase,—*For all*. Thus, "*for all* his exact plot, down was be cast from his greatness;" i. e. the casting him down was effected *before*, in presence of, or, as we say, in the teeth of, all his plot.

For, when used as a conjunction, has manifestly the same force.

> Heav'n doth with us, as we with torches deal,
> Not light them for themselves; *for* if our virtues
> Did not go forth of us, 'twere all alike,
> As if we had them not. SHAKSPEARE.

Let us paraphrase this sentence:—"If our virtues did not proceed from us to others, we might as well not have them: and *therefore* I say, that Heaven uses us as we do our torches." The words *for* and *therefore* correspond; and the word "*therefore*," we know, is the same as "*that fore*," or "*that* being *before* me." Now that which is *before* the mind of the speaker, and which leads him to the conclusion that Heaven deals with us as we with torches, is the reflection—that if our virtues did not go forth of us, they would be useless; and this reflection is marked as being *before* the mind of the speaker, by the word *for* in the original sentence, as it is, in the paraphrase, by *therefore*. Again, in the 9th chapter of St. Mark's Gospel, verses 5 and 6:—

> Peter answered and said to Jesus, Master, it is good for us to be here; and let us make three tabernacles; one for thee, and one for Moses, and one for Elias. *For* he wist not what to say; *for* they were sore afraid.

Here we see, the construction is, "They were sore afraid; and *therefore* he wist not what to say; and *therefore* he talked (as Bishop TAYLOR says,) be knew not what, but nothing amiss, something prophetical." The fear and wonder of the apostles was *before* the ecstasy which deprived them of the power of connected reasoning; and the ecstasy was *before* the words which St. Peter uttered.

This double use of the conjunction, *for*, serves to explain the double use of the preposition *for* noticed by Tooke. "A writ was moved *for*, *for* Old Sarum." The representation of Old Sarum was *before* any writ, as an object in the mind of those who first devised such an instrument; and the desire of obtaining this particular writ was *before* the motion, as an object in the mind of the mover.

Nor let it be thought strange, that our wishes and intentions should be expressed in language which seems to indicate that they stand *before* the mind locally, as a tree or a house stands before the eye of a spectator; for this arises from the prevalent disposition to explain intellectual phenomena by material analogies, just as we call a certain faculty of the mind *imagination*, which word properly signifies the power of making visible or tangible representations of sensible objects. So we call another faculty *acuteness*, adopting our metaphor from the sharpness of a sword or knife. So we speak of *impressions* on the memory —of *reflections* on our acquired knowledge, &c. This disposition to speak of the faculties and operations of the mind in language originally applied to the powers and exercises of the body, may be said to have been at first and in the early stages of human society, a necessity; and if we confined ourselves to the etymological signification of words, it would be so still. In the rude ages of Gothic and Grecian barbarism, the action of *taking* was expressed by the radical *tak* or *dek*, whence our present verb, to *take*, and the Ionic δεκω. From its superior use in taking, the right hand was called *dextra*; he who was expert at any manual employment, was said by analogy to be *dextrous*; and by a further analogy, a superior readiness of contrivance, or quickness of expedient, (though a mental faculty,) was called *dexterity*. This sort of analogy must be resorted to, not merely by the untutored savage, in explaining the acts of his mind, but even by the most profound philosophers in investigating the same abstruse subject. Even the sublime speculations of PLATO are necessarily conveyed in metaphors derived from external and material nature; a circumstance which has occasioned some modern writers of eminence to form very erroneous notions of the doctrines of the Grecian sage. Plato never dreamt of "intelligible species," as actually distinct from the intellect itself, but merely as distinguishable for purposes of reasoning. He never thought that the νοουμενα were separate from the νου, as a picture is from the eye of the painter; but rather held that they were the very intellectual life and being itself. In like manner, when we say that motives, or objects of desire or aversion are *before* the mind, we do not suppose any local position*, either of the mind or of the mental conception; but we adopt an analogy on which are founded the various uses of the words *for*, *fore*, *before*, *therefore*, &c.

Prepositions.

In the use of the conjunction, *for*, already noticed, the words "I say," or some such phrase, must be supplied, to make up the full construction of the sentence: but there is another and now obsolete use of the same conjunction, in which the sense is perfect without such addition.

Thus, in the old satire on *Grooms and Stable-boys*, MSS. Harl. 2253, fol. 125.

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* *Τρίτον δὲ αὖ γένος ὂν τὸ τῆς χώρας, καὶ φθορὰν οὐ προσδεχόμενον, ἕδραν δὲ παρέχον ὅσα ἔχει γένεσιν πᾶσιν, αὐτὸ δὲ μετ' ἀναισθησίας ἁπτὸν, λογισμῷ τινὶ νόθῳ μόγις πιστον, πρὸς ὃ δὲ καὶ ὀνειροπολοῦμεν βλέποντες, καὶ φαμὲν ἀναγκαῖον εἶναί που, τὸ ὂν ἅπαν, ἔν τινι τόπῳ, καὶ κατέχον χώραν τινά. τὸ δὲ μήτε ἐν γῇ, μήτέ που κατ' οὐρανὸν, οὐδὲν εἶναι.*

The third kind of thing (having mentioned *intellect* and *sense*) is *space*, which is not subject to decay, but affords a place to all created things; this is touched without affording the sense of touch, and is obscurely understood to exist, by a spurious kind of reasoning. Looking on this, we dream, as it were, and say, it is somehow necessary that every existing thing should exist in some certain place, having some definite situation, and that what is neither on earth nor in heaven, is nothing.

PLATO, *in Timæo.*

Whil God wes on erthe and wandrede wyde,
What was the reson why he nolde ryde?
For he nolde no grome to go by ys syde.

And so in "*Christ's descent into Hell.*"

For y thyn heste hueldc noht,
Duere ich habbe hit her aboht.

The same use of the word *for* occurs occasionally in Shakspeare.

Heaven defend your good souls that you think
I will your serious and great bus'ness scant,
For she is with me. *Othello.*

In these passages, as well as in those before cited, *for* may, by transposition, be rendered there*fore*, as follows:—"He would not have a groom to go by his side, and *therefore* he would not ride;"—"I have not kept thy commandments, and *therefore* I have paid dearly for my conduct;"—"She is with me, but I will not *therefore* neglect your business." Or, to vary the phrases still more, with the same sense—"He would not have a groom to go by his side, and that determination being *before* his mind, he would not ride;"—"I have not kept thy commandments, and that misconduct having occurred *before* my present sufferings, has been their cause;"—"She is with me, but though her society be *before* my mind as a motive to idleness, it will not induce me to neglect your business."

We may sum up the different uses of the word, *for*, as follows. It is employed either as a preposition, as a conjunction, as an adverb, as an adjective, or as a component particle of a word. As a preposition, when properly used, and without ellipsis, it signifies a relation, 1st. of place; 2dly. of time; 3dly. of rank, or order; and 4thly. of cause, motive, or object. By an ellipsis, it may express the negative of its proper signification; and there are some uses of it in writers of repute, which are altogether improper. In the signification of rank, causation, &c. it expresses the future, co-existent, or previous cause of an action, the limitation of a quality, or the equivalence, substitution, similitude, or opposition of a substance. As a conjunction, adverb, adjective, or particle, its significations coincide with some of those which it has as a preposition. Upon the whole, it denotes that a person or thing is *before* another thing in place, time, or order; or that it is *before* the mind as a cause or object positively or relatively: and as similar relations are denoted by the terms *fore*, a*fore*, be*fore*, to*fore*, there*fore*, where*fore*, &c. the inference seems clear that *for* and *fore* were originally the same word.

When *for* is applied to place, it signifies that which is *before* us in intention, as "we sailed *for* Genoa." That which is before us, and becomes in fact the end of our journey, is expressed by *to*; as "we sailed to Malta."

When *for* is applied to time, it signifies, that the time in question is *before* the mind of the agent, as that which either continues, or is intended to continue, during the whole period of the action. "He is chosen *for* life;" i. e. he is chosen to serve *for* life, life being *before* the mind of the elector, as that which is to form the duration of the service. "He studied *for* a year; i. e. placing *before* your mind a year, that will be found to equal the time that he studied.

When *for* is applied to *causation*, or *motive*, the object is future in such sentences as the following;—"Chelsea Hospital was built *for* disabled soldiers;" i.e. the future accommodation of disabled soldiers was the object *before* the minds of those who directed the building. In like manner, when the poet exclaims—"O! *for* a muse of fire!" which is equivalent to "I wish for a muse of fire;" the muse is *before* his mind as the object of his wish.

The cause is co-existent in such sentences as these:—"Objects depend *for* their visibility, upon the light;" i. e. visibility being *before* the mind, when we consider objects, we find that in this respect they depend upon the light. "He does all things *for* the love of virtue;" i. e. in every action of his life, the love of virtue is *before* his mind as a motive.

The object or cause is past, in such phrases as these;—"to punish a man *for* his crimes;"—"to reward him *for* his valour." Here the crimes and the valorous deeds respectively, though they may have long gone by, are still *before* the mind of the person punishing and rewarding.

We find in Robert De Brunne, *ther-for*, employed to denote a cause, precedent. In speaking of the murder of Sir John Comyn, because he refused to rebel against King Edward, he says—

Sir Jon wild not so, *ther for* was he dede.

where, according to modern usage, we should say, "there*fore* was he killed."

For, used after an adjective or adverb, serves to limit and restrain the quality by reference to some certain object; as, "big *for* his age;" i. e. having *before* your mind his age, speaking with reference to that, you may call him big. "Situated commodiously *for* trade;" i. e. trade being *before* the mind when we speak of the situation, we may call it commodious.

When *for* is used after a substantive, it is generally with reference to some verb, expressed or understood, and then its use is similar to what we have already observed in speaking of verbs: e. gr. "an eye *for* an an eye;" "he takes Richard *for* Robert;" "he shot Peter *for* a deserter." Here an eye is *before* the mind as being *equivalent* to an eye: Robert is *before* the mind as being the person for whom Richard is *substituted*. The character of a deserter is *before* the mind as that to which the character of the person shot bore a real or supposed *similitude*; and the context will show whether it is meant to suggest *identity* or *diversity*; whether the individual was really a deserter, or whether his being alleged to be so, was merely a pretence to justify the execution.

Among the uses of the preposition, *for*, which may be regarded as improper, or at least have become obsolete, we may reckon the following, in which nevertheless, *for* always retains the sense of *before*.

1. Mr. Tyrwhitt, in his *Glossary*, says,—"For, *prep.* Sax. sometimes signifies *against*," and among other instances cites—

Some shall sow the sacke,
For shedding of the wheat. Chaucer.

Mr. Tooke says, that "this construction is aukward and faulty;" but that "the meaning of *for* is equally conspicuous;" "the *cause* of sowing the sack being that the wheat may *not* be shed." The shedding of the wheat is *before* the mind, but it is not before the mind as the proper object of the sowing; that is to say, as an end to be attained by sowing the sack; but

on the contrary, as an end to be prevented; and as this distinction may not immediately appear from the context, an obscurity is introduced into the sense, which renders the construction faulty, and has justly brought it into disuse.

2. The redundant use of *for*, preceding *to*, with an infinitive, is very ancient in English. It occurs frequently in Robert De Brunne.

> The yere next on hand yede the kyng of France
> To the holy land, with his purveiance,
> Upon Gode's ennys *forto* tak vengeance.

So in the song, on the *Battle of Lewes*, A. D. 1264.

> The kyng of Alemaigne, bi mi leaute,
> Thritti thousent pound askede he,
> *Forte* make the pees in the countre.

It was probably adopted in imitation of the French idiom, "*pour* prendre," "*pour* faire," &c.; inasmuch as *pour* and *for* equally indicate objects *before* the mind as causes of an action past or future; but the cases differ, because in French the termination *er* alone does not sufficiently denote motive, or cause; whereas, the preposition *to*, in English, has that force; and consequently it renders *for* redundant. This idiom therefore is at present confined to the vulgar.

3. The following use of *for* is elliptical.

> *For* tusks, with Indian elephants he strove.

Tusks were not *before* the monster as the object which he strove to attain; but he strove to attain celebrity, and tusks are *before* the mind of the narrator in speaking of that celebrity. The full construction therefore would be, "he strove with Indian elephants to attain celebrity for tusks;" but as the ellipsis introduces an obscurity into the sentence, this construction is also properly reprobated.

4. Dr. Lowth censures Swift for saying "he accused the ministers *for* betraying the Dutch;" and Dryden, for saying, "you accuse Ovid *for* luxuriancy of verse;" both which expressions Mr. Tooke defends. This, however, is a matter of idiom, and it turns on the force given in English to the verb *accuse*. We say, to accuse *of* a crime or fault, but not to accuse *for* a crime or fault; because the crime or fault is not regarded as the motive directly *before* the mind in such an act as accusation. We may reproach a minister *for* betraying an ally; or we may censure a poet *for* the luxuriancy of his verses; because it is the nature of censure and reproach to assume the fact as certain; whereas, in accusation, properly speaking, the fact remains in doubt. However this may be, it is certain that the passages above quoted from Swift and Dryden are not consistent with modern idiom; and they probably were the result of haste in their composition.

5. A somewhat similar observation may be made on the expressions "sick *for* disgust," and "sick *for* love," which also come recommended by the approbation of Mr. Tooke. The *Lady*, in Wycherley's play, says she is "sick *for* her gallant;" and *Falstaff*, in the 2d part of *King Henry IV.* says, "I know the young king is sick *for* me." There may be an object *before* the mind, occasioning sickness; as in these cases: but the feeling which constitutes the sickness, be it disgust, love, or any other, is not in modern use separated from it, and made a distinct object. Shakspeare indeed makes the *Duchess of York*, when interceding for the life of her son *Aumerle*, say—

> Yet am I sick *for* fear.

But here, it would seem, is meant an actual bodily sickness occasioned by fear: and even in this sense, the construction, however allowable in poetry, would appear harsh in common composition, or discourse.

The *conjunctional* use of the word *for* has already been noticed, at some length. The *adverbial* use is colloquial, and is generally considered inelegant in composition. Thus, instead of saying, "a writ was moved *for*," where *for* performs the function of an adverb, it would be advisable to say "a motion was made *for* a writ;" but on either construction, *for* implies that the writ was the object *before* the motion, as its cause, in the mind of the mover.

For is used *adjectivally* in such sentences as the following:—"It is *for* the general good of human society;"—"It were not *for* your quiet;"—"Moral considerations could not move us, were it not *for* the will." Here the general good of human life, and our own quiet, are laid *before* us as proper motives to action; and the will is stated to be *before* our capability of being moved by moral considerations, as the cause of such capability. In the colloquial phrase of vulgar combatants, "I am *for* you;" the meaning is, "I am *before* you, in opposition."

Lastly, *for*, when used as a *component* particle, agrees with *fore* when used in the same manner. Thus we have *forbear, forbid, forget, forlorn, forsake, forswear*, and *foreclose, forego, foreslack, forespent, forestall, forewaste*. Some words, too, are written indifferently either way,—as *forward* and *foreward, forfend* and *forefend*. Dr. Johnson says, "*for* has, in composition, the power of privation, as *forbear*; or depravation, as *forswear*; and other powers not easily explained." The explanation is easy enough, when we consider the various analogies of that which is *before*; inasmuch as it signifies going *forth*, going *out* of the ordinary limits, being *opposed* to, and the like.

To the same original, *fore*, we may trace many other English words, as *forth, further, first*, &c.

The word *forth* occurs in a charter of King Edward the Confessor, preserved in the very valuable work of Hickes, (*Thes. Ling. Sept.* v. i. 161.) It there appears to signify "freely" or "readily;" and is spelt *vorth*, as *for* is spelt in the same instrument *vor*; which is the more remarkable, because the charter relates to the county of Somerset, where that pronunciation is still preferved:—

> "Ich quethe eou that ich wille that Gyse Bissop beo thisses bissopriches;—swo uol, & swo *vorth* swo hit eni bissop him to uoren formest haueth on ealle thing."—"Significamus vobis nos velle quod episcopus Giso episcopatum possideat—adeò plenè et *liberè* per omnia sicut ullus episcoporum prædecessorum suorum unquam habebat."

In Robert De Brunne we find *forthely* used for "readily;" e. gr. "als *forthely* as he"—as readily as he.

Further (sometimes erroneously spelt *farther*,) was anciently in English *forther*; and in High German *forder*: e. gr. "Das volk zog nicht *forder* bis Mirjam aufgenommen wird;"—"The people journeyed not (went no further) till Miriam was brought in again." (*Numb.* c. xii. v. 15.) Ottfrid, in the *Frankish Gospels*, instead of this word, uses *furder*; in Anglo-

Saxon it is written *forthor*: in Low Saxon, *vorder*, *vurder*, *vudder*; in modern German, *vorder* is the foremost part, as "*vorderseite* des gebaudes," "the front of a house;—"*vordertheil* des schiffs," the prow of a ship. In old German, this word is written *furter* and *furder*. Adelung says *förder* is the comparative of *fort*; which, in some modern dialects, is pronounced *furt* and *furd*.

First, in English, was originally *fore-est*, i. e. *foremost*; and of the same origin is the German *fürst*, which properly signifies, according to Adelung, "the *first* and most eminent person of his nation, province, or state." It is commonly rendered "prince." In the German Bible, Abraham and Job are called *fürsten*, princes. *Furst* is written by Willeramus, *vorst*; by Ottfrid, *furista*; in Low Saxon, *förste* and *forste*; in Swedish, *forste*; in Danish, *fyrste*; and is the superlative (says Adelung) of *für*. "*Fur* and *vor* (pronounced *fore*) are sometimes distinguished, (says Wachter,) as if *vor* applied only to time, and not to place, or to cause; *fur* to place and cause, and not to time; but this distinction is not steadily observed among us, nor is there any trace of it in the ancient writers; for the Goths say *faur*, *foura*: the Anglo-Saxons, *for*, *fore*, *fyr*, *fyre*; the Franks and Alamans, *fora*, *furi*; the Belgians, *vor*; the English, *for*; the Swedes, *för*, &c." This author adds, that the Greek *προ*, and the Latin *pro*, differ not from *für* and *vor*, except in a slight change of the labial articulation, and in transposing the canine letter, *r*.

The simple Greek prepositinn *προ* signifies *before*, both in place and time; and the compounds in which it has that meaning are innumerable. The adverb *πρωι* denotes the early morning, the *fore*most part of the day. The adjective *πρωτος*, first, is evidently the superlative of *προ*, as our first is of *fore*. Πρωρα is the prow, the *fore*part of the ship.

In Latin, the prepositions *pro* and *præ* are both connected with the Greek *προ*. The ancients also used *pri* for *præ*; whence *prior* and *primus*, as also *pridem*, *pridie*, *princeps*, *priscus*, *pristinus*; all relating to that which is *before*, in time, or order. *Præ* signifies *before*, in place; e. gr. "I *præ*; sequar;"—"Go *before*, I will follow." "*Præfert* manus;"—"He stretches out his hands *before* him; he feels his way, like a person walking in the dark." "*Præcalvus*;"—"bald *before*, bald on the fore part of the head;" or *before*, in time; e. gr. "*præcanus*,"—"greyheaded *before* his time." "*Præcocia* poma;"—"apples, which grow ripe *before* the usual time;" or *before* as a cause, e. gr. "misera *præ* amore;" "wretched *for* love," love being that which was *before* her wretchedness, as its cause; or *before*, as denoting superiority or excess, as "*præaltus*," excessively high, *before* all others in height. In like manner, *pro* refers to place; e. gr. "hasta posita *pro* æde Jovis Statoris;"—"a spear placed *before* the temple of Jupiter Stator;" or to time, as "*proavus*," "a great grandfather; one who lived *before* the grandfather;" or to cause, e. gr. "pœnam *promerui*;"—"I have deserved punishment *for* my offences;" my evil deserts are *before* my punishment, as its cause.

Nor is the Latin language without closer traces of the Teutonic *for*, in *foris*, *foras*, *forum*, *forceps*; for these words signify respectively, *foris*, "the door;" which is in the *forepart* of the house; *foros*, "out of doors," abroad, *forth*, "from the house;" *forum*, "the market-place," or scene of public debates and trials, which were anciently carried on in an open space *before* the houses of the citizens; *forceps*, "the tongs;" the instrument with which a smith drew *forth* hot iron from the fire.

Again, in the base Latin of a subsequent age, we find such words as *foroneus*, *forensis*, *forasticus*, *foresto*, *forgeldum*, *forisfacere*, *forisbannitus*, &c. which appear to be of a similar origin. *Foraneus*, *forensis*, and *forasticus*, signify that which is *forth* of the house, or country; a thing or person that is external, strange, or foreign. Hence the Italian "grazie *foronee*," "external advantages;" "*forese* schiatta," "a rustic race," (che sta *fuor* della città, as it is explained in the *Vocabolario Della Crusca*.) So "*forasticæ* pugnæ," are foreign wars, (*Epist.* 3. S. Bonifac. Archiepisc. Mogunt.) and "*forastici* homines," are strangers, foreigners; (*Tabul.* S. Remigii. Rhemensis.) *Foresta* did not originally signify "a wild, uncultivated tract of ground with wood, as Dr. Johnson defines our word, *forest*; but rather as Giovanni Villani defines the Italian word *foresta*; "luogo di *fuora*, separato dalla congregazione e coabitazione degli uomini;" "a place that is *forth* from cities, and separated from the congregation and co-habitation of men." Whether these places did or did not abound in *trees* was accidental; but as it generally happened that they did so, the word *forest* came to be considered as indicating a woody tract of country. It is remarkable, however, that our word *wood*, itself, does not seem to have originally had a necessary connection with the notion of a tree, or its substance; but to have been of the same meaning as *wild*, *weald*, *wald*, *wold*, *wod*, *wud*, &c. denoting any thing uncultivated, savage, fierce, or mad. Hence, the *weald* of Kent was the wild, uncultivated part of that county. "St. Swithin footed thrice the *wold*;" i. e. the desert. Ottfrid, in the *Frankish Gospels*, translates "the voice of one crying in the wilderness,"—"Stimma ruafentes in wastinnu *waldes*." Tatian translates μελι ἄγριον (wild honey) "*wildi* honug;" but the Anglo-Saxon version renders it "*wudu* hunig." This word *wud* often occurs in Anglo-Saxon, signifying *wild*,—as "*wudu bucca*, a wild goat; "*wud*-culfer," a wild pigeon; "*wudu*-coc," a wild cock; which two last we still call a *wood*-pigeon, and a *wood*cock. *Wods*, in Gothic, is used for a demoniac madman; e. gr. "saei was *wods*," "he that had been possessed of the devil." (*St. Mark*, c. v. v. 18.) In Anglo-Saxon, *wod* is used for mad; hence "*wode-thistle*," i. e. mad-thistle, was the name of hellebore, a remedy against madness. In Frankish, *wotnissa* was madness. In Dutch, *woede* is fury; in Scottish, *wud* is mad. The English *wood*, in the same sense, has become obsolete; but is found in Spenser.

————Coal black steeds yborn of hellish brood,
That on their rusty bits did champ as they were *wood*.

To return to the derivatives of *for* and *forth*. *Forgeldum* was an impost probably on foreign goods:—

Omnibus geldis, tengeldis, horngeldis, *forgeldis*, penigeldis, &c.
Monast. Anglican. vol. i. p. 372.

Forisfacere, is explained by Ducange, "offendere, nocere, q. facere *foris*, i. e. extra rationem." Here the Latin *foris* is unnecessarily substituted for the Teutonic *for*. *Forfare* was the Italian word of which *forisfacere* was the barbarous Latin translation: and

Grammar. *for* in *forfare*, was employed exactly as *for* is in the English *forlorn*; and *ver* (pronounced *fer*) in the German *verloren*. In a secondary sense, *forfare* signified to *forfeit* lands or goods for one's misdeeds. So *forban* was one who acted against the *ban* or commandment of the law; (for Ottfrid translates "my commandment," "*ban* minnan,") and in a secondary sense, one who was banished, or exiled by command, *forth* from the state. In the former signification, the French still use *forban* for "a pirate:" and in the latter, Matthew, of Paris, uses *forisbannitus*, in his history, (ad ano. 1245.)

Expulsus a Scotiâ, *forisbannitus* ab Angliâ.

Forda is our word *ford*, which is manifestly from the Gothic *faran*.

Non liceat alicui facere dammas, aut *fordas*, aut alia impedimenta in watergangiis.
Ordinatio Marisci Ramesiensis.

Fordale appears to be of the same origin.

Tendit usque ad magnam aquam de Ayr, et *fordales* ejusdem prati. *Monast. Anglic.* vol. i. p. 657.

It is scarcely necessary to trace minutely the connection of *for* and *fore* with the German *für*, *ver*, and *vor*; the Dutch *voor*, the French *pour*, &c. One or two instances, however may be noted. The German *vorbey* is the old English *forbi*, and Scottish *forbye*; but with some variation in the use. *Vorbey* sometimes denotes the passing along *before* a place; e. gr. "Die flotte segelte die insel *vorbey*;" the fleet sailed along *before* the island. Sometimes it denotes a time that is past, and consequently a time *before* the present; e. gr. "Das jahr ist *vorbey*;"—the year is at an end. *Forbi* is used by Robert De Brunne in the following senses;—"before," "notwithstanding," "away," "therefrom;" "*forbi* euer ilkone," before every one. Burns uses *forbye* for "besides," "over and above." The Dutch *voor* is used in the senses of *before* and *for*, as "*voor* de deur," "*before* the door;"—"'s danges te *voore*," "the day *before*;"—"*voor* alle dingen," "*before* all things;"—"dat is *voor* hem," "that is *for* him;"—"jets *voor* verlooren houden," "to give up a thing *for* lost. *Voorburg* is a fenced suburb, built *before* a city. In old French, this was *forsbourg*, since corrupted into *fauxbourg*, *faubourg*, and *fauboar*.

Et pour ladite requeste, le sergent, en la ville et *forsbourgs*, n'aura que cinq sols. *Const. de Tourraine.*

The French *hors* was anciently written *fors*; and was probably derived, as Menage suggests, from the Latin *foris*.

Brantome uses *fors* in the sense of "except."

Ne furent à l'offrande, *fors* Monsieur D'Angoulesme.

And so La Fontaine—

Toute la troupe étoit lors endormie,
Fors le galant.

In like manner, *hormis* has been formed from *foris*, *missus*; and *dehors* from *de foris*.

There cannot be any doubt, but that the French *pour* is the Teutonic *for* or *fur*. In English compound words adopted from the French, it is spelt and pronounced *pur*; as *purchase*, *parport*, &c. *Purfle*, which Johnson defines "a border of embroidery," is simply *fore*worked, or *fore*-edged, *pour-filé*.

I saw his sleues *purfiled* at the hand
With gris, and that the finest in the land.
Chaucer.

Prepositions.

Gris is a better sort of fur. (v. Ducange, ad voc. *Griseum*.)

Thus have we seen that our words *for* and *fore* are alike connected with words of analogous sound and sense, both of Gothic and Grecian origin: and it seems not improbable, that they also agree with the verb, to *fare*, which is the Anglo-Saxon and Gothic *faran*, to go, or move *forward*. From *fare* doubtless comes the adverb *far*: and we find in old English, that the past tense of *fore* was *fore*.

Thorghe mountayn & more, the Bascles ge ther weie,
Our nesche and hard thei *fore*, & did the Walsch men deie.
Robert De Brunne.

Bot he mot quitely go, in world where he *fore*,
And frely passe him fro, fro whom that he to suore.
Idem.

As *before* is compounded of *be* and *fore*; so *but* is compounded of *be* and *out*. "*But*," says Skinner, "ab A. S. *bute*, *butan*, præter, nisi," &c.—"*bute* autem and *buton* tandem deflecti possint a præp. *be*, circa, vel *beon* esse, and *ute*, vel *utan*, foris." Mr. Tooke, however, has observed, that this word has in English two derivations; viz. that just quoted from Skinner, which is indisputably right; and another suggested by Tooke himself, which will require some observation hereafter. But.

1. We proceed, however, first, with *but* in the obvious sense of *be-out*; and for the present we assume, that the meaning of the word *out* is sufficiently understood, as denoting the opposite to *in*. By old English and Scottish writers we find it often written *bot*, or *bote*, possibly from some confusion with respect to its derivation: however, as there is no regularity in this respect, the orthography may merely have varied according to the accidental habits of the different writers.

But, answering to *without* is applied to *place* in the Scottish dialect, and opposed to *ben*, i. e. *within*; e. gr. "blithe was she *but* and *ben*," i. e. she was sprightly both *within* the house and *without*. We find *binnan* in Anglo-Saxon for *bi-innan*, or *be-innan*, in the same sense as the Scottish *ben*. The Dutch also use *buyten* and *binnen*, with these significations,—as "*buyten* deur," *without* doors; "*binnens* huys," *within* doors. In the old ballad of the *Goberlunzie Man*, ascribed to King James I. of Scotland, in the 15th century, we find both expressions.

Gae *butt* the house lass, and waken my bairn,
And bid her come quickly *ben*.

But, answering to *without* in the same sense of privation, is of very ancient use, both in the English and Scottish dialects.

Alsua that all the landis of the kinrik be taxt eftar as thai ar of vale now, and that *but* fraude or gile.
Scot. Act. Parl. 1424.

The sowking wolf furth strekyng breist and udyr,
About his palpis *but* fere, as thare modyr
The twa twynnyis. Gawin Douglas.

But mete or drinke, she dressed her to lie
In a darke corner of the hous alone. Chaucer.

But, in the sense of privation, answering to *except*, occurs in our common expression "all *but* one;" i.e. all *be-out* one, all, if one *be-out*. In this sense also it occurs frequently in old English and Scottish.

What is ther in paradis,
Bot grasse and flure, and greneris?
Descr. of Cokaygne.

Quhich has my hert for ever set abufe,
In perfyte joye that never may remufe
Bot onely deth. *King's Quair.*

In this sense it is sometimes preceded by a negative, as in the *Description of Cokaygne.*

Ther *nis* met *bote* frute,
Beth ther *no* men *bot* two.

So in the Anglo-Saxon Gospels, (*Luke* c. viii. v. 51.) "*ne* let he *nanne* mid hym ingan *buton* Petrum et Johannem et Jacobum;" "he suffered *no* man to go in *save* Peter and James and John." And again, (*Luke* c. ix. v. 13.) "We *nabbath buton* fif hlafas and twegen fixas, buton we gan, and us mete bicyon;" "we have *no* more *but* five loaves and two fishes, except we should go and buy meat."

In Chaucer we find (according to the idiom of that day,) no less than three negatives preceding *but.*

Ne neuer y nas but of my body trewe.

That is, "I never was otherwise than true."

In the present day, we omit the negative; which, as Mr. Tooke observes, often forms a very blameable and corrupt abbreviation of construction. Thus we say, "I saw *but* two plants;" which, in old English, would have been "I *ne* saw *but* two plants;" I saw no plants *be-out* two. So Chillingworth says, "If *but* wise men have the ordering of the building;" i. e. if *none* have the ordering of the building *but* wise men

Hence arises the conjunctional force of *but*, *bote*, or *bot*, answering to *unless.*

Thus, in the ballad of the *Mon in the Mone.*—

Nis no wyht in the world, that wot when he syt,
Ne, *bote* hit bue the hegge, whet wedes he wereth.

So Robert De Brunne,—

For slayn is Kyng Harald, & in lond may non be,
Bot of William he hald for homage & feaute.

So in *Kyng Alisaunder,*—

Al that we havith wonne and wrought,
Y no holde hit for nought,
Bote we mowe heom wynne.

So in *Richard Coer de Lion.*—

They tolde hym the hard caas,
Off the Sawdoun's hoost hou it was,
And *but* he come to hem anon,
They wer forlorne everilkon.

So Gawin Douglas.—

Blyn not, blyn not, thou grete Troian Enee,
Of thy bedis nor prayeris quod sche;
For *bot* thou do, thir grete durris, but dred,
And grislie yettis sall neuer warp on bred.

So Chaucer.—

But he wil hym repente.

But, or *bot*, in this sense, was often followed by *give* or *if.*

Thus, in the Scottish Act of 1424, before quoted,—

Thai salbe chalangit be the kyng as fautours of sik rebellyng, *bot gif* thai haif for thame resonable excusacion.

So in the romance of *Sir Tristrem.*—

The maiden of heighe kinne
She cald hir maisters thre;
Bot give it be thurch ginne,
A selly man is he.

So in *Richard Coer de Lion.*—

Wher thorwgh they myghten not withstonde,
But yiff Saladyn the Sawdan,
Come to help with many a man.

The last sense of this word, *but*, which we shall notice, is that of our common conjunction, answering to the French *mais* and the Anglo-Saxon, *ac.*

Robert De Brunne commonly spells it *bot.*

——————— Roberd thouht no gile,
Bot come on gode manere till his brother Henry.
* * * * * *
Roberd bi his letter his brother gan diffie,
Bot gode Anselme, that kept of Canterbirie the see,
Before the barons' lept, kried, pes per charitie.

Gawin Douglas sometimes spells it *bot*, and sometimes, though in this sense more rarely, *but.*

Sic wourdis vane & unsemelie of sound,
Furth warpis wyde this liger fulichelie;
Bot the Troiane baroun anabasitlie,
Na wourdis preisis to render him agane.
Booke x. p. 338.

Quhare sone forgadderit all the Troyane army,
And thyck about him flok and cam, but baid,
But nowthir scheild oor wappinis doun they laid.
Booke xii. p. 430.

So King James I. of Scotland, in his poem of *The King's Quair*, uses *bot.*

Bot for alsmoche as sum micht think or seyne,
Quhat nedis me apoun so lytill evyn
To writt all this? I ansuere thus ageyne.

In the poem of *Christis Kirk of the Grene*, by the same royal author, however, it is sometimes written *but.*

Twa, that wer herdsmen of the herd,
Ran upon udderis, lyk rammis;
But quhair thair gobbis wer ungeird,
Thay gat upon the gammis.

In the schedule of accusation against King Henry VI. presented in Parliament, A. D. 1461, it is written *but.*

Not oonlie in the north parties, *but* also oute of Scotland.

So in the English statute of 1483, before referred to.

That such exaccions, called benevolences, afore this tyme takyn be take for no example, to make suche or any lyke charge hereafter, *but* it be dampned and annulled for ever.

Dunbar, in his *Goldin Terge*, uses *but.*

All thir bure genyies to do me grivans;
But resoun bure the terge.
* * * * *
Thick was the schot of grundin arrows kene;
But Resoun, with the goldin schield sae schene,
Weirly deffendit quhosoeir assayit.

And so Montgomery, in the "*Cherrie and the Slae.*"

My agony was sae extreme,
I swelt and swound for feir;
But or I walkynt of my dreme,
He spulyied me of my geir.

2. We have seen that in the different uses of *bote*, *bot* or *but*, these words appear to be used almost indifferently: and perhaps they may all be referred to the same derivation, *be-out*; for that which is *out*, is *excepted* from that which is *in*; and it is likewise *over and above* that which is in. In this last acceptation, therefore, it may well answer to the French *mais*, which is a corruption of the Latin *magis*, more; and to the Anglo-Saxon *ac*, or *eac*, which is from *eacan*, to add to; as in Gothic, there are the conjunction *auk*, and the verb *aukan*, with the same significations, so

Grammar. in Greek, ἂν and αὔξω; in Latin, *aut* and *augeo*; in Alamannic, *auh* and *auhhon*; in Danish, *og* and *öge*; in Dutch, *ook*, and the old verb *oecken*; and in English, *eke*, and to *eke* out.

Mr. Tooke, however, thinks that in this signification of *over and above*, the word *bot* was the imperative of the Anglo-Saxon and Gothic verb *botan*, which (he says) "means to *boot*, i. e. to superadd, to supply, to substitute, to atone for, to compensate with, to remedy with, to make amends with, to add something *more*, in order to make up a deficiency in something else." We do not mean positively to reject one of the very *few* original etymologies in the first volume of the *Diversions of Purley*; but we must observe, that *botan* rather means to add something *better* than something *more*. The Gothic *botan* is our verb, to boot; and is explained by JUNIUS, proficere, prodesse, juvare. There can be little doubt but that it was of the same origin with the Anglo-Saxon, *bet*, *betera*, *best*, which two last we retain in our *better* and *best*. In Anglo-Saxon, *betan* was emendare, and *bot*, emendatio. Hence, perhaps, when the conjunction *but* implies preference, its original meaning was "*better*." Thus, "I will not do this, *but* I will do that," means "I will not do this, *better* I will do that;" i. e. I can do it better in fact, or better to my own satisfaction and pleasure. BEDA says, (i. 26.) "hi lefnysse onfengon cyrican to timbrianne, and to *betanne*;"—"They received permission to repair and amend the churches." LUPUS says, (*Serm.* i. 3.) "to myclan bryce sceal micel *bot* nyde;"—To a great breach shall need great amends.

Hence CHAUCER says,—

> God send every gode man *bote* of his bale!

Hence, also "nets to *bete*," in old English, was to *mend* nets.

Bot was used in a secondary sense for repentance, which was supposed to *amend* men; as "*bot* theawas awent;"—repentance changes manners; and also for compensation, as we say, to make *amends* for any thing; hence in the Anglo-Saxon *feohbote*, was a pecuniary mulct; and in the old English *theftbote*, was a fine for theft. Hence also *fire-bote*, *foldbote*, and *ploughbote*, were three rights anciently reserved to tenants of taking what *boots*, (i. e. profits, or is requisite) for fuel, for the fold, and for the plough. Straw and hay being among some of these *botes*: and the peasantry making covering for the legs of such materials, those coverings came to be called *boots*; and what is now called a *bottle* of hay, was the *botal*, or quantity, usually led home from the field for *bote*. The man in the moon is described in the verses often quoted above, as bearing his burthen on a *bot-fork*; that is, an instrument used to bring home thorns and other materials used for *bote*.

> Mon in the mone stont and stryt,
> On is *bot-forke* is burthen he bereth.

From this source evidently are our noun, *booty*; the Italian, *bottino*; French, *butin*; Spanish, *botin*; Dutch, *buyt*; and Danish *bytte*. Our verb, to *boot*, also is the Dutch *baeten*. Our *better* is the Dutch, *beter*; German, *besser*; Gothic, *batizo*; Frankish, *bezzer*; Alamannic, *pezzira*; Danish, *bedre*; Swedish, *baettre*; Islandic, *bettri*. The oldest form of the positive of these words, (says ADELUNG,) was in German, *bas*, and in Lower Saxon, *bat*; which brings us back again to the Anglo-Saxon *bet* and *bot*.

Prepositions. Thus have we seen the two connecting links; viz. *be-out* and *bet*, one or both of which connect what Mr. LOCKE calls "the several views, postures, stands, turns, limitations, and exceptions, and several other thoughts of the mind intimated by this particle *but*."

Out. In. The meaning of out we have hitherto explained only by its opposition to *in*; but what are these words? To these questions we have little more to answer, than that inasmuch as they name distinct conceptions of relation, they must have been originally nouns. Mr. Tooke observes, "that in the Gothic and Anglo-Saxon, *inna* means *uterus, viscera, venter, interior pars corporis*;" and that "there are some etymological reasons, which make it not improbable that *out* derives from a word, originally meaning *skin*." If these facts could be well established, they would prove but little. They would only prove that man, in the early progress of thought, applied (as it was most natural he should do) his conceptions of the relations of place to his own body, and distinguished the *inside* of his frame from the *outside*. "*Inna*, *inne* is also used in a secondary sense," says Mr. Tooke, "for cave, cell, cavern;" that is to say, it is used for the place *in* which a man or other animal dwelt.

Thus, in the song on *The Battle of Lewes*, (A. D. 1264.) we have *yn* for a place of abode.

> Sire Simond de Mounfort hath suore bi ys chyn,
> Hevede he nou here the Erl of Waryn,
> Shulde he neuer more com to is *yn*.

Hence our common noun, an *inn*, now used for a house of entertainment for travellers; the place where, after having been *out* all day on their journey, they are *in* at night. That this word was anciently applied to a more private and permanent residence, however, is evident, both from the passage just quoted, and also from a similar one in the ballad on *The Battle of Bruges*, (A. D. 1301.)

> Sir Jakes ascapede, by a coynte gyn.
> Out at one posterne, ther men solde wyn,
> Out of the fyhte, hom to ys *yn*.

The Anglo-Saxon verb *innan* is our verb, to *inn*, as in BUTLER.—

> I'm certain 'tis not in the scrowl
> Of all those beasts, and fish, and fowl,
> With which, like Indian plantations
> The learned stock the constellations;
> Nor those that drawn for signs have been
> To th' houses where the planets *inn*.

From the signification of *place*, the transition to signify *time*, is natural and easy.

> Danger before, and *in*, and after th' act,
> You needs must own is great.
>
> DANIEL, *Civil War*.

The signification of *circumstance* is still more comprehensive.

> *In* all things approving ourselves as the ministers of God; *in* much patience, *in* afflictions, *in* necessities, *in* distresses, *in* stripes, *in* imprisonments, *in* tumults, *in* labours, *in* watchings, *in* fastings.
>
> ST. PAUL, 2 *Cor.* c. vi. v. 4, 5.

Now, if we suppose any given space, or time, or circumstance, to be represented by a circle, whoever or whatever is between the periphery and the centre, bears to the thing given the relation, which we express by the word *in*, and whoever or whatever is farther from the centre than the periphery is, bears to the whole the relation which we express by the word *out*;

and this may be considered, either *simply*, or with reference to some other thing or person. Thus, a person may be said simply to be *out* of doors, or to play *out* of time, or *out* of tune, or to be *out* of his senses; or with reference to others, he may be said to *outdo* them, or to *outshine* them, or the like. In modern times, *out* is not used alone, as a preposition; but we find it so used in CHAUCER.—

Thou shuld neuer *out* this grove pace.

And the correspondent *aus* and *ausser*, in German, have the same force, as "*aus* dem hause gehen," to go out of the house; "*ausser* landes," out of the country. Most of the Teutonic dialects have this word,—as the Gothic *us*, *uzuh*, *ut*. *uta*; the Anglo-Saxon, *ut*; the Alamannic and Frankish, *uz*; the Dutch, *uyt*. "Even the Persian *ez*, and Latin *ex*," says AOBLONO, "belong to this root;" and if so, we may, of course, add the Greek ἐξ and ἐκ, and the Latin *e*.

We have noticed *within* and *without*: but instead of these, many old writers use *inwith* and *outwith*.

Thus, in the *Seuyn Sages* occurs *inwith*.

I sal him teche, with hert fre,
So that, *inwith* yeres thre,
Sal he be so wise of lare,
That ye sal thank me euermare.

BARBOUR has *outwith*.—

As he auised non bane thay done,
And to them *outwith* sent he soon,
And bad thame barbre thame that night,
And on the morn cum to the fight.

This word occurs in a curious passage of the Scottish Statute of A. D. 1427.

Item, at na lipirous folk sit to thig, nothir in kirk, nor in kirk yarde, na in nane vthir place *within* the borowis, bot at thare awin hospitale, ande at the porte of the toune, ant vthir places *outwith* the borowis.

We find in BARBOUR the words *withouten* and *foroutten*.—

For he would in his chalmer be,
A wel grete while in priuite
With hym a clerk *withoutten* mo.
* * * * * *
I ask you respite for to see
This letter, and therewith aduised be
Till to morn that ye be set,
And then, *foroutten* longer let,
This letter sal I enter here.

As *out* signifies privation in *without*, *foroutten*, and the like, so it has a like force in the word *outlaw*; which is, in Anglo-Saxon, *utlaga*. In the charter of Edward the Confessor, before quoted, we find *unlage*.—"And gif what sy mid *unlage* out of than hissopriche geydon;"—and if any thing be taken from that bishnprick with *un-law*; i. e. with injustice. Our word *idle* is derived from *ut*, or out. The German word *eitel* had for its first signification, *empty*; in Frankish, *itol*. "Sinan stual liaz er *italan*;"—he left his seat *empty*. "Thaz *itala* grab;"—the *empty* sepulchre (from which Christ had risen.) "Inti otage forliaz *itale*;"—the rich he sent away *empty*. "Origo vocis," says WACHTER, "est a particula privativa *ut*, ex;" and we have already seen that *ut* is our word *out*. This etymology may cast some light on Shakspeare's well known passage—

Of antres vast and desarts *idle*.

Eitel at present signifies in German, vain. "Significatos," adds WACHTER, "ex priori desumptus, quod vano nihil sit inertius nec magis vacuom." Hence, in the Alamannic, "*ital*-ruam," is vain-glory: and in the Anglo-Saxon, "*ydel*-yylp," is vain-boasting. In this sense HOOKER uses idle, "They are not in our estimation *idle* reproofs."

In is a word of still more general use among the European nations than *out*. We find it in the Greek ἐν, the Gothic, Italian, and Latin, *in*; the French and Spanish, *en*; the Swedish and Islandic, *inai*; the Frankish and Alamannic, *inna*; the Anglo-Saxon, *innan*; and many compound forms,—as the Gothic *innathro*, within; and *inngoggan*, to enter; the Latin *intra*, *infra*, &c.; the Italian and Spanish *dentro*, the French *dans*, and *dedans*, &c.

The Anglo-Saxon *innan* sometimes signifies *into*, as "heo beseah *innon* tha byrgenne;" she looked *into* the sepulchre: sometimes *within*, as "*innan* huse," *within* the house. We find it also further compounded, as in *oninnan* and *beinnan*, e. gr. "*oninnan* me selfum," within myself; "*beinnan* tham carcerne," *in* the prison.

In like manner we find that the Latin *in* signifies not only *within*, but *into*, *toward*, and consequently *against*; agreeing in this respect with *out*, which we have seen not only signifies *without*, but *beyond*, and not only privation, but excess. So, "*in* domo," signifies *within* the house; "Piso *in* ædem Vestæ pervasit," Piso came *into* the temple of Vesta; "*in* meridiem spectat," it looks *toward* the south; "hæc cum audio *in* te diei, excrucinr," when I hear these things said *against* thee, I am afflicted. From this last sense it would seem that the privative force which the Latin *in* has in composition is derived; as *infelix*, *inops*; and so in our English words *infamous*, *inactive*, *improbable*, &c. MILTON, however, seems to have somewhat exceeded the limits of grammatical analogy, when he invented the word *inabstinence*.

——————That thou may'st know
What misery th' *inabstinence* of Eve
Shall bring on man.

Of. Mr. Tooke says, "I imagine that *of*, in the Gothic and Anglo-Saxon *af*, is a fragment of the Gothic and Anglo-Saxon *afara* posteritas, *afora* proles, &c.; that it is a noun substantive, and means always *consequence*, *offspring*, *successor*, *follower*." That *of* or *af* was a noun, that is, the name of a conception, is not to be doubted; but to say that it is a fragment of *afara*, is probably as correct as to say, that the word *iron* is a fragment of the ancient noun substantive, *ironmonger*. If it be a fragment of any thing, it is more probably of *aft*, which we shall consider under the word *after*. However, the nouns, which by long use, for many centuries, and in various dialects, have come to serve as the most common prepositions, are in general so far removed from their source, that we cannot trace them back to it with certainty, as we can the more recently adopted prepositions, *touching*, *concerning*, *during*, and others already mentioned. It is very possible that the term *of*, *af*, or *op*, may, in certain early dialects have signified a *son*; and indeed some traces of this seem observable in the Sclavonic *of*, as *Peterhof*, the Welsh *ap*, as *ap-Rice*, and the Irish *o*, as *O'Hoalon*; but this fact, if it could be established, would be very far from proving, that the term might not have been Off.

Grammar.

so applied with reference to a more general idea, such as that of *proceeding from*, *depending on*, or *belonging to*, the parents.

The preposition *of*, and the preposition and adverb *off*, were anciently the same word, and the subsequeot variation of orthography was merely accidental.

> I schall you telle *of* a kynge,
> A doughty man withoute lesynge,
> *Off* body he was styffe and stronge.
> *Lyfe of Ipomydon.*

> Godwyn, an Erle of Kent, met with Alfred,
> Him and alle his feres vntille prison tham led :
> *Of* som smote *of* ther hedes, *of* som put out ther iyene.
> ROBERT DE BRUNNE.

> And at the last, with gret payne,
> Kyng Richard wan the Erl *off* Champagne ;
> The Erl *off* Leycetre, Sere Robard,
> The Erl *off* Rychemond and Kyng Richard.
> *Richard Coer de Lion.*

> And in the castle *off* Tyntagill.
> *Legend of King Arthur.*

> Quhare sodeynly a turture quhite as calk,
> So evinly vpon my hand gan lyt
> And vnto me sche turnyt hir full ryt
> *Off* quham the chere, in hir birdis assort,
> Gave me in hert kalendis of confort.
> KING JAMES. *King's Quair.*

Off signifies dissociation, or distance of place ; and this both adverbially and prepositionally.

> ———————— See
> The lurking gold upon the fatal tree ;
> Then read it *off*. DRYDEN.

> About thirty paces *off* were placed barquebusiers.
> KNOLLES.

> Cicero's Tusculum was at a place called Grotto ferrate, about two miles *off* this town. ADDISON.

"*Proceeding from*" may probably have been the original sense of the words *of* and *off*, both which in Dutch are written *af*; as "Ik weet'er niet *af*," I know nothing *of* it. "Zyn hoed is *af*," his hat is *off*. This word *af* was used in old French ; as "hostel *af* brebez," a sheepfold ; *hôtel aux brebis*. It is the Gothic *af*, as "wairp *af* thus." cast *from* thee ; "*af* missilhin taoya niwaiht," I do oothing *of* myself. It is the Lower Saxon, and Swedish *af*. It is without doubt the modern German *ab*, as "die farbe geht *ab*," the colour goes *off*, or fades ; "das feuer geht *ab*," the fire goes *out* ; "*abhangen*," to depend *on* ; "*ablassen*," to leave *off*. And it is probably connected with the Latin *ab*, and Greek ἀπο. "*Af* pro *ab* scribere antiqui solebant," says PRISCIAN : and we fiod on an ancient brazen tablet, "*af* vobīis" for "a vobis." GELLIUS, speaking of the verbs *aufugio* and *aufero*, says, "illod inspiel quærique dignum est, versane sit et mutata AB præpositio in AV syllabam propter levitatem vocis ; an *potius* AV particula suâ sit propria origine, et proinde, ut pleræque aliæ prepositiones à Græcis, ita hæc quoque inde accepta sit ; sicuti est, in illo versu Homeri :"—

> Αὖ ἔρυσαν μὲν πρῶτα, καὶ ἔσφαξαν, καὶ ἔδειραν.

If the word *af* was part of *aft*, it may possibly have signified "the *back*," and, consequently, "that which we leave *behind* ;" that "*before* which we are placed ;" or, that "*from* which we proceed." Hence *of* and *fore* may be regarded as expressing different stages of removal from an object ; and thus we may see how to be fond *of* an object, to wish *for* it, and to long *after* it, come to be nearly synooymous.

Prepositions.

In many old writers we find *of* employed as we now use *out of*, or *from*.

> I sal the brynge *of* hel pyne.
> M.S. *Harl.* No. 2253, fol. 55.

> Mote ye neuer *of* world wend.
> *Idem.* No. 913.

> Chargit he lous *of* this ilk mannis handis.
> GAWIN DOUGLAS, book ii. p. 43.

> Quhilk that he sayis *of* Frensche he did translait.
> IDEM. *Preface.*

There are several other uses of this preposition now obsolete, among which we may notice the following :

> Even like some empty creek, that long hath lain
> Left and neglected *of* the river by.
> DANIEL'S *Musophilus.*

> How many thousands never heard the name
> Of Sydney, or of Spenser, or their bookes,
> And yet brave fellowes, and presume *of* fame.
> *Ibid.*

> Lucifer *of* the brightest and most glorious angel, is become the blackest and the foulest fiend.
> *Homily against Disobedience, &c.*

> Bot yif I may with my brother go,
> Mine hert it breketh *of* thre.
> *Amis and Amiloun.*

> Then I, whiche had not slept *of* the hole nyght,
> By Morpheus sodaynly had lost my sight.
> GOODWYN'S *Mayden's Dreme.*

> Sir, said Reguawde, I thank you much *of* your good will.
> *Foure Sonnes of Aimon.*

> Holi Chirche was foundid *of* the apostlis on Crist the stoon.
> WICLIF.

> Soche an other for to ymake,
> That might *of* beaute be his make. CHAUCER.

The adverb *off* "is generally opposed," says JOHNSON, "to *on* ; as "to lay *on*, to take *off*." *On* would seem to apply adhesion to, as *off* does separation from ; as to stand *on* a table, to fall *off* a table ; to be fastened *on*, to be cut *off* ; to flow *on*, as a river, with continuity, to fly *off* as a bird, with separation. But in the signification of belonging to, *on* was anciently used where we now use *of* ; as in the Letter of HENRY III., A. O. 1258.—"Henr. thurg Godes fultume, King *on* Engleneloande, Lhloaverd *on* Yrloand, Duk *on* Norm. *on* Aquitain, Eorl *on* Aniou." In the old English it was also used for *in* ; as, in the same letter,—"to alle hise halde ilærde ilewed *on* Huntendon schir." In the Anglo-Saxon, besides this latter sense it had many others, as "tha comon fram eastdæle to yehiddenne hi *on* Ierusalem," then came they from the east parts *to* Jerusalem to pray : "sum feoll *on* tha thornas," some fell *among* thorns : "sceo fordælde *on* læcas eall that heo ahte," she had spent all her living *upon* physicians : "*on* thone heofen beseah," he looked *up to* heaven : "eode *on* anne munt," he went *up into* a mouotaio : "thære halyan rode, the ure drihten *on* throwode," the holy cross that our Lord suffered *on* : "Hwl ferde ye *on* westenne geseon ?" What went ye out *into* the wilderness to see ? "For *on*," says Hickes, "sometimes occurs *an*, from the Gothic *ana*." In Gothic, the preposition *ana* is used separately for *on* or *in*, as "*ana* staina," *on* a rock : "*ana* mesa," *in* a charger. The Goths, Franks, and Alamans, used also *an* and *ana* in many compounds ; as, the Gothic *anaaukan*, to add, or join *on* On.

Grammar.

to; the Frankish *anbeten*, to pray *to*, or, as we say, "to call *upon* the name of the Lord." The Alamannic "*angeuangen*," to lay hands *on*, or claim. This is also the Dutch *aan*, and German *an*, of which Wachter, in the 5th section of his *Prolegomena*, gives many significations; e. gr. denoting connection, as *anbinden*, to bind *on to*; denoting the direction of an act toward a particular object, as *anblicken*, to look *upon*, or toward; denoting continuity of time, as *anstehen*, to stop, to stand as it were *on* the same point of time; thus we say, a ship stands *on*, in the same course, using the word *on* for a continuous adherence, as in the other case it is used for a stationary adherence. Adelung considers the German *an* to be connected with the French *en*, the Latin *an* in composition, and the Greek ἀνὰ, as ἀνὰ μέσον, *in* the middle; ἀνὰ χθόνα, *on* the earth. It is plain that our *on*, though in modern use most frequently applied to that which is higher in place, did not, in its origin, necessarily imply such a position; for though it was added to *up*, in the word *upon*, it was also added to *nether* or *neder* in the word *on-neder*, under. It is difficult to assign with certainty any substantival form of this word. It has, however, been observed, that both in the Breton and Turkish languages *ana* signifies mother; and from this circumstance, the learned Pezron derives *Diana*, the mother of light, from *di*, day, or light, and *ana*, mother. The scriptural word "*abba*," father, is well known; and perhaps from *abba* and *ana*, some etymologists may be inclined to derive our prepositions *of* and *on*.

Up, upon, above, over.

We proceed to the word *upon* just noticed, and with this are connected *above* and *over*. The radix *up* implying superior elevation is most commonly employed, in modern English, as an adverb. As a preposition, we now use it, only to denote that an action is directed from a lower to a higher part, as

In going *up* a hill the knees will be most weary. Bacon.

But by old English writers it was used (as we now use *upon*,) to signify the being actually placed *above* and resting *on* an object.

Gyfre he rood all be hynde
Up Blaunchard whyt as flour.
MS. *Calig*. A. ii. fol. 36.

A wel vayre compayneye,
Vppe vayre wyte stedes, & in vayre armure also.
R. Gloucester.

And in Robert De Brunne we find "*up* that" used for "*upon* that," thereupon, upon that account.

Op, the corresponding word in the Dutch language, is used in the same manner; e. gr. "*op* een paerd ryden," to ride *on* horseback. So "*op* den tafel," is "*upon* the table;" "*op* de vloer," *on* the floor. And in the sense of completion, the Dutch *op* and our *up* also agree; as *opeeten*, to eat *up*; *opdrinken*, to drink *up*; *opbouwen*, to build *up*; *opgeschikt*, drest *up*.

Mr. Tooke, in his usual manner, raises a dispute about that, which properly understood, can admit of no dispute at all; namely, whether *up* was originally an adjective, a substantive, or a verb. "Upon, up, over, above," he justly says, "have all one common origin;" and he is clearly right in connecting them with the Anglo-Saxon *ufan*, high. He adds not an irrational conjecture, that *ufa*, or *up* may have anciently meant the same as *top*, or head; but when he goes on to infer from this, and other *conjectures* of a like kind, "that the names of all abstract relations (as it is called,) are taken either from the adjectived common names of objects, or from the participles of common verbs," he either means to advance an historical fact, or to lay down a necessary principle in the constitution of the human mind. If he means to speak historically, he asserts what it is utterly impossible either to prove or disprove: if he means to speak philosophically, his philosophy is destitute of common sense. We need only examine our own minds with a very slight degree of attention, to be satisfied that our conceptions of quality, positive or relative, are just as essential to human reason, as our conceptions of substance or of action. "The relations of place," says Tooke, "are more commonly from the names of some parts of our body; such as *head, toe, breast, side, back, womb, skin*, &c." It would have been equally correct, or rather equally incorrect, in a philosophical point of view, to have said, "the names of various parts of our body; as *head, toe, breast, side, back, womb, skin*, &c. are from the relations of place." As matter of history, both assertions are equally arbitrary. Mr. Tooke is very positive that the etymologists who derive *head* from the Scythian *ha*, German *hoch*, Dutch *hoog*, Alamannic *houch*, Gothic *hauh*, and Anglo-Saxon *heah*, high, are all wrong; and that it is the participle of the Anglo-Saxon verb, *heafan*, to heave. The fact, no doubt, is, that the same conception, and the same radical expression, was the origin of them all, as well as of the Islandic *haud*, and German *höhe*, height; the Anglo-Saxon *heafod*, Gothic *haubith*, Alamannic *haubit*, Islandic *hoffud*, Dutch *hoofd*, the head; the Anglo-Saxon *heofon*, heaven; the Alamannic *hebig*, and Anglo-Saxon *hæfig*, heavy, difficult to heave; the Alamannic *erhafan*, to ferment, to raise dough; the Anglo-Saxon, *hæf*, fermenting; the Anglo-Saxon *heap*, Alamannic *houph*, Dutch *hoop*, a heap; and numerous other cognate words in many languages.

Prepositions.

As the Anglo-Saxon *ufa* is our *up*, the Dutch *op*, and the German *auf*; so the Anglo-Saxon *ufera*, the comparative of *ufa*, and *ofer*, the preposition, are our *upper* and *over*, the Dutch *opper* and *over*, the German *über*, Alamannic *ubar*, Frankish *upar*, Gothic *ufar*, &c. The Anglo-Saxons also used *uppan* or *uppon*: and as they had *binnon* for *be-innon*, and *bæftan* for *be-æftan*, so they had *bufan* for *be-ufan*; as "*bufan* tham wætere," upon the water. This *bufan* is, no doubt, the origin of the Dutch *boven*, above: and our word *above*, written in old Scottish *abufe*, is *on-be-ufa*; as the Scottish *abune*, is *on-be-ufan*. In Danish, we find *over, ober, over, overste*; in Swedish, *uppe, up, öfwer, öfwerste, ofre, ypperst*. Wachter considers *über* to be connected with the Hebrew *eber*, Persian *avar*, Greek ὑπὲρ, and Latin *super*; and he traces its significations from that which is *above*, in place, to *above*, in power; *above*, in eminence; *above*, in the sense of prevailing over; *above*, in excellence; *over and above*, in abundance; *over*, in excess: and, again, from that which is *beyond* in place, to that which is *beyond* in quantity; hence, to *overlook*, is to look beyond, and therefore not to notice; while, on the other hand, to *look over*, is to examine carefully, by looking from point to point. After noticing these and many other meanings of this word, he concludes—"*Uber* plures habet significatus quorum racemationem aliis relinquo, qui hisce inves-

tigandis et in ordinem digerendis ad tædium usque defatigatus sum."

The adjectival use of *over* and *upperest* is common in CHAUCER.

Her *over* lyp wyped she so clene,
That in her cup was no ferthynge sene.
Prol. Cant. Tales.

By whiche degres men myght climben from the neytherest letter to the *upperest*. *Boecius*, book i.

So, in *Kyng Alisaunder*,—

Theose seresys as y fynde,
Uppurest folk buth of ynde.

The adverbial use of *over*, answering to our adverb *too*, is curiously marked in a passage of ALEXANDER MONTGOMERY's *Cherrie and Slae*.

All *owres* ar repute to be vyce;
Owre hich, *owre* law, *owre* rasch, *owre* nyce
Owre het, or yit *owre* cauld.

Up, in the sense of completion, occurs in our word *upshot*.

I cannot pursue this business with any safety to the *upshot*.
SHAKSPEARE.

JOHNSON derives *upshot* from *up* and *shot*; it would be more proper to derive it from *up* and *shut*; the *shutting up* of a business being formerly used for its close.

Altho' he was patiently heard, as he delivered his embassage, yet in the *shutting up* of all, he received no more but an insolent answer. KNOLLES.

The Dutch *boven* corresponds exactly with the old English *aboven*, which occurs in the ballad of the *Battle of Lewes*.

By God that is *aboven* ous, he dude muche synne.
That lette passen oversee the Erl of Warynne.

Among the combinations of *up* and *over*, we may notice *over that*, *over against*, out *owre*, and *uptak*.

Over that was formerly used, as we now use *moreover* to signify, " in addition to.—

That the same fyne be openly and solemply rad proclaymed in the same court; and *over that*, a transcript of the same fyne be sent by the seid justices unto the justices of assize.
Stat. I. Ric. III. c. 7. MS.

In the same sense, the Anglo-Saxon writers use "*ofer thæt*," and the Germans *über das*. Our compound preposition, *over against*, is transposed, in the German *gegenüber*.

Over against this church stands a large hospital, erected by a shoemaker. ADDISON, *on Italy*.

This is rendered in the Anglo-Saxon gospel *foran-ongean*. "Tha reowon hiy to Gerasenorum rice, that is *foran-ongean* Galileam;"—"And they arrived at the country of the Gadarenes, which is *over against* Galilee." *Luke*, c. viii. v. 26.

In the Scottish dialect, we find the compound preposition *out-owre*, which is used in two senses by BURNS.

The rising moon began to glow'r,
The distant Cumnock hills *out-ow'r*.
Death and Doctor Hornbook.

He by his shouther ga'e a keek,
An' tumbled, wi' a wintle,
Out-owre, that night.

The word "*uptak*" is also used colloquially in Scotland, to signify the power of *taking up*, or readily comprehending any notion; as in the phrase "dull i'th' *uptak*," which signifies slow in comprehending an idea; the mental faculty being in this instance, as in so many others, expressed by reference to a bodily action.

At.

Our preposition, *at*, is the Gothic *at*, and Anglo-Saxon *æt*. It may probably have been connected with the Latin *ad*, which Tooke awkwardly derives from *actum*. *Ad* was more probably the root of the verb *addo*; though it may not now be easy to trace it in a substantival form. For *ad* we sometimes meet with *ar*, *as*, and *at*. FULVIUS URSINUS quotes, from the *Laws of the Twelve Tables*, *arvorsom*, for *adversum*; as "*arvorsom* hostem æterna auctoritas estod;" that is, "*against* an alien, the right of property is never barred by prescription;" whereas, against a Roman it was so barred. VELIUS LONGUS says, that the old Romans not only used *arvorsum* for *adversum*, but *arvorsarius* for *adversarius*: and VOSSIUS observes, that in many ancient books and inscriptions, *ad* is written *at*.

The use of the preposition *at*, in Anglo-Saxon and old English, was much more loose and comprehensive than in our modern dialect. We find it used where we should now use *to*, *from*, *about*, *of*, *by*, *in with*, &c.

In the romance of the *Seuyn Sages*, "*at* lere" occurs for "*to* learn," or to be taught.

The sext maister rase vp onane.
Sir, he said, if thi will were,
Tak thi son to me, *at* lere.

In the Anglo-Saxon, "*æt* him" is used for "*from* him;" e. gr. "animath that pund æt him," take the talent *from* him.

Bishop LATIMER uses *at* for *about*, in the following passage,

What ado was there made in London *at* a certain man, because he said, Burgesses! nay, butterflies!

CHAUCER uses the phrase "to take leave *at*," for *of*.

She toke her leaue *at* hem ful thriftely.

This line is very similar to one in the romance of *Octouian Imperator*.

At all the cytè she tok her leue.

So, in the *Lyfe of Ipomydon*,—

He toke hys leve *at* Jason there,
And went forthe ellys where.

In *Richard Coer de Lion*, we have "to ask *at*."

He askyd *at* all the route,
Gyff ony durste com, and prove
A cours, for hys lemannes love.

"To ask *at* a person," is considered, in the present day, as a Scoticism.

Similar to this is Bishop LATIMER's phrase, "to learn *at*."—

He must study, and he must pray: and how shall he do both these? He maye learn *at* Salomon.

ROBERT DE BRUNNE uses *at* for *by*.

Sen thou has don amisse, *at* thin vnconyng,
We may not faile at this, to help the in alle thing.

At is also used for *by* in an old document of the year 1415. (9 Rymer, 301.)

Besechyng yow, *at* the reverence of God.

In the romance of *The Lyfe of Ipomydon*, *at* is used for *in*.

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> He wold wend into strange contrè,
> So that ye take it not *at* greffe.

ROBERT DE BRUNNE also uses *at*, where we should now use *with*; as in the form of Baliol's homage to King Edward.—

> I Jon Baliol, the Scottis kyng,
> I bicom thi man for Scotlond thing;
> The whilk I hold, & salle thorgh right,
> Clayme to hold, *at* all my myght.

This lax mode of using the preposition *at* is observable in our phrase "*at* all," which Johnson explains "*in* any manner, *in* any degree;" and which corresponds to the Scottish *ava*; i. e. *af* all, or *of* all.

> An' low'd his ill-tongu'd, wicked scawl,
> Was warst *ava'*. BURNS.

At is sometimes, though awkwardly, cumulated with other prepositions, as "*at about* six o'clock;" and so in a statute of the year 1495, "*at after* none."

Divers artificers and laborers reteyned to werke, and serve waste moch part of the day, and deserve not their wagis; sume tyme in late comyng unto their werke, erly departing therfro, longe sitting at ther brekfast, at ther dyner and noncmete, and long tyme of sleping *at after* none. *Stat.* 2. HEN. VII. c. xxii. MS.

Thus also in BARBOUR we find the expression "*at to* morn."

> That they may this nicht, if they will
> Gang harbry them, and sleep, and rest;
> And *at to* morn, but longer lest
> You shall ish forth to the battail.

To, too.

The origin of the word *to*, like that of the word *at*, can at best be but matter of conjecture. It may, however, be reasonably conjectured, that *at* and *to* are from the same root, "*per anastrophen*," as WACHTER expresses it; that is to say, that the vowel was sounded before the radical consonant in the one instance, and after it in the other. The primary conception, common to both words, seems to have been that of *touch*, either in consequence of moving the bodily organs *to*, or in consequence of their being *at* a specified place. Hence, the Latin *ad* coincides with both our *to* and *at*; e. gr. "Verres *ad* Messanam venit;" Verres came *to* Messina. "Mihi quoque etiam est *ad* portum negotium;" I too have business also *at* the harbour. And so in French, "il reste *à* la maison,"—"il est allé *à* la campagne." In the Devonshire dialect, *to* is used for *at*; as "he lives *to* Exmouth:" and we have seen above, that "*at* lere" was used for "*to* learn," *ad* discendum.

Mr. Tooke says, "the preposition *to*, in Dutch, written *toe* and *tot*, a little nearer to the original, is the Gothic substantive, *taui* or *tauhts*, i. e. *act*, *effect*, *result*, *consummation*; which Gothic substantive is indeed itself no other than the past participle, *tauid*, or *tauids*, of the verb *tauyan*, agere. In the Teutonic this verb is written *tuan* or *tuon*; whence the modern German *thun*, and its preposition varying like its verb, *tu*. In the Anglo-Saxon, the verb is *teogan*, and preposition *to*."

In all this, we see nothing of the "*real object*" which, according to Mr. Tooke's general theory, every preposition should signify; and it is a very circuitous mode of getting at a short monosyllabic preposition, to suppose that there first existed a dissyllabic verb, from which was formed a dissyllabic participle, and that this participle, a little differently articulated, became a dissyllabic substantive, which was shortened, we know not how, or wherefore, into the monosyllable in question.

Prepositions.

The German *zu* (not *tu*, as Mr. Tooke supposes,) answers, like the Latin *ad*, to our *to* and *at*; e. gr. "komm *zu* mir," come *to* me; "*zu* Windsor," *at* Windsor. WACHTER mentions, as connected with it, the Gothic *at* and *du*; Anglo-Saxon, *æt* and *to*; Frankish and Alamannic *az*, *zuo*, *zua*, *za*, *ze*, *zi*; Dutch, *toe* and *tot*; English, *to* and *at*. "Omnia," adds this learned author, "affinia Latino *ad*; nam *ad* et *to* se mutuò producunt per anastrophen."

The various uses of the German *zu*, the English, *to*, *too*, and *at*; the French *à*, the Latin *ad*, &c. will illustrate each other: and we may consider them as indicating approach *to*, or arrival *at*, a place, time, or circumstance; and thence, as having an objective force before a verb or substantive; moreover, since that to which a person or thing has attained, or which has come to it, is an addition to it, therefore *to* denotes addition; and thence excess; and thence, in composition, it has an intensive force; and, lastly, where the intensive force is very slight, the use of *to* seems almost superfluous.

In relation to place, we find *zu* used emphatically in German, "die thur ist *zu*," exactly corresponding with our English colloquial phrase—the door is shut *to*. So *zugang* is the Latin *aditus*, from *ad* and *ire*. *Zukunft* is the Latin *adventus*, from *ad* and *venire*, the coming of Christ *to* the world. The Frankish *zuochumft*, from *zuo* and *chommen*, is the Latin *aggressio*, from *ad* and *gradior*. In English the preposition *to* is not commonly prefixed, as in German to verbs in composition, but follows them, as "to fall *to*," "to bring a ship *to*;" and, so in the interjectional phrase, "Go *to*!" The Germans say "reit *zu*," for, ride on; "geh *zu*," go on, &c.

In relation to time, we find the German *zu nocht*, answering to the English "*at night*." *Zukunft*, in a secondary sense, signifies the time *to* come, the French *l'avenir*.

In relation to circumstance, the German *zufall* is the Latin *accidens*, from *ad* and *cado*, whatever befalls, or falls *to* a person; *zubringen*, to bring *to* an end; *zusagen*, to promise *to* a person. *Zu pferde* is the French *à* cheval, on horseback.

The objective force of *zu*, before a verb, is well explained by Dr. NOEHDEN, in his excellent *Grammar of the German Language*, (3d edit. p. 388, *et seq.*) whence it appears that the action may be either future, as "lust *zu* spielen," an inclination *to* play; or present, as "Das vergnügen sie *zu* sehen," answering to the French "j'ai grand plaisir *à* vous voir," I have great pleasure *in* seeing you; or past, as "müde *zu* stehen," tired *of* standing.

The English *to* had formerly a similar objective force before a substantive; but this construction is now obsolete.

> They have gruel *to* potage,
> And lekes kynde to companage. TREVISA.

> Tho thai were fiften winter old,
> He dubbed bothe tho bernes bold,
> *To* knightes, in that tide.
> *Amis and Amiloun.*

The English *too*, also, denoting addition, is the same word as *to*; and in the Anglo-Saxon and old English, is written *to*.

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The arriving to such a disposition of mind, as shall make a man take pleasure in other men's sins, is evident from the text, and from experience *too*. SOUTH.

The German *zu* in composition, possesses this same force; e. gr. *zunome*, a name *in addition to* another name, the Latin *agnomen*, from *ad* and *nomen*.

The word *toname* occurs in ROBERT DE BRUNNE, with the same meaning in speaking of *Statin*, whose nose had been cut off by King Isaac Comnenus.

> For Isaac did him schame, his lord suld be,
> Thei called him this *toname*, Satin the Nasee.

The German word *zugemus* (in Frankish, *zuomuse*,) signifies, in like manner, vegetables, or garnish of any kind *added to* the meat; from the German *mus*, Frankish *muos*, Alamannic *muas*, Gothic *mats*, French *mets*, Anglo-Saxon *mete*, and English *meat*. So the German verb *zugeben* is to give something *in addition to* the stipulated price.

The secondary sense of our *too* is excess, as "*too* great;" that is something added *to* the proper degree of greatness; and in this sense, *zu* is used in the German compounds *zuhoch*, too high, or overhigh; *zulang*, too long, or overlong; *zuwarm*, too warm, or overwarm.

Zu is used with an intensive force in such words as *zubereiten*, to make quite ready; *zulassen*, to grant *to*, &c.; and this may probably have some analogy to the Greek ζα, which has an intensive force; as in ζάπλουτος, very rich; ζάθεος, very divine; ζάκοτον, very furious, &c. The old English *to* before a verb or participle, appears to have had nearly this force.

> He schal therfore ben islawe,
> And afterward al *to*-drawe. *Seuyn Sages.*

> Th' emperour saide, I fond hire *to* rent,
> Hire her and hire face ishent. *Ibid.*

So, in the translation of the Bible, in the time of King HENRY VIII. "confregit cerebrum ejus," is rendered "all *to* brake hys brayne panne." (*Judges* c. ix. v. 53.) In the modern editions this is improperly printed "to break."

The intensive force of *zu* is scarcely, if at all perceivable in such words, as *zuvor*, before; *zuwider*, against; *zusamen*, along with. In English we still use *to*, thus in *together*, and in *heretofore*, as we formerly did in *tofore* and *toforne*.

There entered into the place, there I was lodged, a ladie, the moste semelich & moste goodly to my sight, that euer *toforne* appeared to any creature. CHAUCER. *Test. of Love.*

> *Tofore* the kyng com on harpour,
> And made a lay of gret favour.
> *Kyng Alisaunder.*

To appears to be superfluously used by BARBOUR in the preposition *into*.

> That he would travel owre the sea
> And a while *into* Paris be.

On the other hand, in the preposition *unto*, the syllable *un*, which seems to have been originally *on*, augments the force of *to*, and gives it the force of the Latin *usque*, *ad*, and French *jusqu'à*.

We have seen that *for* is unnecessarily prefixed to *to* before an infinitive; as "*for to* go," which is now reckoned a vulgarism. "*From to*" seems still more alien to the general idiom of our language: yet it occurs in poetry.—

> For not to have been dipp'd in Lethe's lake,
> Could keep the son of Thetis *from to* die.

Prepositions.

And there is something analogous to this in the German *ohne*, *zu*, e.gr. "*ohne zu* wissen," which we construe, with the participle, "without knowing," and the French, with the infinitive, *sans savoir*.

As the origin of the word *to* is matter of conjecture, of course we could only indicate conjecturally those words with which it may very anciently have been connected in sound and signification: and among these, it may be sufficient to notice the numeral *two*. This is in Gothic *two* or *twa*, in Anglo-Saxon *tu* or *twa*, in Greek δυω, in Latin *duo*, in Welsh *dau*, *dwy*, in Breton *dou*, in Tartarian *tua*, in Danish *tu*, in Frankish and Alamannic *zwei*, *zwo*, in German *zwey*, in Dutch *twee*, in Scottish *twae* and *two*. STAHENIUS endeavours to show that it is a word compounded of the Gothic *du*, to, and *a*, or *o*, one; so that it properly signifies "one *added to* one."

Till.

Till is used prepositionally and conjunctionally; but always, in modern English, with reference to time alone; e. gr.

> Unhappy *till* the last, the kind, releasing knell.
> COWLEY.

Meditate so long, *till* you make some act of prayer to God, or glorification of him. J. TAYLOR.

Dr. JOHNSON is mistaken in explaining the latter of these examples, as not signifying "to the time that," but "to the degree that;" for it palpably refers to the continuance of the meditation, which must occupy time. Mr. TOOKE, on the other hand, is right in saying (with reference to modern usage,) that "we apply *to* indifferently either to place or time; but *till* to time only and never to place. Thus we may say,

> From morn *to* night th' eternal larum rang;
> Or, from morn *till* night, &c.

But we cannot say, "From Turkey *till* England." He is, however, entirely mistaken in supposing "that *till* is a word compounded of *to* and *while*, i. e. time;" and that "the coalescence of these two words *tohwile* took place in the language long before the present wanton and superfluous use of the article *the*, which by the prevailing custom of modern speech is now interposed." For, on the contrary, the custom of confining the signification of the preposition and conjunction *till* to time, is comparatively of very modern date, and is confined solely to the English dialect.

"*Til*," says HICKES, "is a *Cimbric* word, signifying *ad*, *usque*, and it often occurs (in *Anglo-Saxon*,) as "yearwian *til* etanne," to make ready to eat; "ewæth *til* him Hælend," the Saviour said to him. "*Til*, in the old *Norwegian* and modern *Islandic* languages, governed the genitive." So we find in the Islandic *History of Hialmar*, "*til* borgarinar," *ad* propugnaculum.

In a marginal note to the letter of Henry III. (A. D. 1258,) we find this word written *tel*.

And al on tho ilche worden is isend in to aurihce othre sheire, ouer al thare kuneriche on Engleneloand, & ek in *tel* Irelonde.

ROBERT DE BRUNNE writes it *tille*.—

> A knyght was tham among, Sir Richard Seward,
> *Tille* our faith was he long, & with Kyng Edward,
> *Tille* our men he com tite.

CHAUCER uses *til*.—

> A doly season *til* a carefull dite
> Should correspond. *Test. of Cresside.*

Grammar.

GAWIN DOUGLAS, *tyll.*

> Ane young bullock of cullour quhite as snaw
> With hede equale *tyll* his moder on hicht.

In *Octouian Imperator* it is written *tylle.*

> Her pauyloun whan they com *tylle,*
> Ther that sche was,
> Her maydenys gonne to crye schylle,
> Treson, alas!

In like manner we have *until,* and *thertil,* for *unto* and *thereto.*

> Then strake the dagger *untill* his heart.
> *L. Thomas and Fair Annet.*

> His owin lady he toke by lius,
> And gaf the knyght *until* his wins.
> *Seuyn Sages.*

> *Vntil* his toure thus wendes he right,
> For to speke with his lady bright. *Ibid.*

> They found the gates shut them *untyll.*
> *Adam Bell, &c.*

(That is, shut *against* them.)

> I bicom thi man for Scotland thing
> With alle the purtenance *thertille.*
> ROB. DE BRUNNE.

> Thoffe thei haue not als tyte her wyll,
> Yette shall they cum sumtyme *thertyll.*
> *Sir Amadas.*

> And the knight and his lady
> Went tham forth with grete solas
> To the ship whare his godes in was.
> The Erl went with tham *thartill.* *Seuyn Sages.*

The word *while* is used in several of our provincial dialects, and by many old writers for *till.*

Thus in the Scottish Statute of 1430.

It is statute and ordanit, that the act of the fisching of Salmonde, maid be the King that now is and the thre estatis, be fermly kepit ay furth *quhil* it be reuokit be the King and the thre estatis.

So in an historian quoted by Mr. TOOKE, vol. I. p. 363.

"He commaunded her to be bounden to a wylde horse tayle, by the here of her hedde, and so to be drawen *whyle* she were dede."

In like manner the word *to* is used for *till* in the romance of *Sir Amadas.*

> And owtte of cuntre wille y wende,
> *To* y haue gold and syluer to spende.

But we have never met with the compound *to-hwile,* or *to-while* in any English or Saxon writer. The German *zuweilen,* which is the only compound resembling it, signifies "*sometimes,*" "now and then," and nearly answers to the Scottish adverb *whiles,* as in BURNS's inimitable description of *Tam O'Shanter.*

> *Whiles* haddin' fast his gude blue bonnet,
> *Whiles* crooning o'er some auld Scots sonnet,
> *Whiles* glow'ring round, wi' prudent cares,
> Lest bogles catch him unawares.

SKINNER says that *til* was used, in his time, in Lincolnshire for *to.* GROSE includes it in his *Provincial Glossary* as signifying *to* in the north of England: and it is to this day very generally so used in the country parts of Scotland.

> Gae farer up the burn *til* Habbie's how.
> ALLAN RAMSAY.

The substantival form of *til* is to be found in the word *Zil,* which WACHTER thus explains:

1. "Finis, limes, terminus temporis et loci. Anglosax: *till* apud BENSON. Græcis τέλος, a τελεῖν finire, terminare."
2. "Meta jaculantis, scopus agentis, terminus oculi et mentis. Cum scopus sit terminus agentis, quem Latini *finem,* et Scholæ *terminum ad quem,* vocant; hinc manifestum, sensum vocis a termino terminante ad terminum intentionalem translatum esse."

Prepositions. From.

The preposition *from* is the word *fro* which we still use adverbially.

> As when a heap of gathered thorns is cast
> Now *to,* now *fro,* before th' autumnal blast. POPE.

In the Anglo-Saxon, and old Scottish dialect it is often written *fra:* and there can be little doubt but that *fro* and *fra* are, in fact, the same word with the English adjective *free,* the Gothic *friya,* Anglo-Saxon *friy, freo,* Frankish *frio,* German *frey,* Swedish *fry,* and Dutch *vry,* all of the same meaning. These words too, were no doubt, connected with the German *freude* joy, and *froh* joyful, free *from* care, which last is in Frankish *fro,* and in Dutch *vro;* and also with the German *fremde,* and Anglo-Saxon *fremd,* a stranger, one who dwells far *from* us.

"*From,*" says Mr. Tooke, "means merely *beginning,* and nothing else. It is simply the Anglo-Saxon and Gothic noun *frum,* beginning, origin, source, foundation, author." But beginning is not "*a real object,*" and, therefore, this etymology, if it prove any thing, proves that Mr. Tooke's theory of prepositions is false. The word *frum,* was no doubt the same as *from,* and may have been used to signify that *from* which any thing proceeded; but this was probably with reference to a still more general conception involved in all the terms that we have above mentioned.

In the Gothic Gospel of St. John, (c. xv. v. 27,) we have *fram* and *frum* in immediate connection. "*Fram* fruma mith mis siyuth." From the beginning ye are with me.

The Anglo-Saxons used both *fram* and *fra.*

The old Scottish writers use *fra,* and frequently in the sense of "*from* the time of." Thus GAWIN DOUGLAS, (b. ii. p. 63,) "*fra* she was loist," i.e. from the time that she was lost.

So BARBOUR,

> And *fra* he wist what charge they had
> He busked him, but mare abad.

ROBERT DE BRUNNE uses *fro.*

> Andrew is wroth, the wax him loth, for ther pride
> He is tham *fro,* now salle thei go, schame to betide.

Through.

Mr. TOOKE says, that the preposition *through* is the name of a *real object,* namely, *door.* This notion he probably took from the following passage in VERSTEGAN's *Restitution of Decayed Intelligence.* "*Dure* or *dark,* now *a door;* it is as much to say as *through;* and not improper; because it is a *dark-fare* or thorow passage." Verstegan certainly reasons more correctly in deriving *door* from *through,* than Tooke in deriving *through* from *door;* the more general idea must have preceded the more particular; men must have passed *through* many places before *doors* were invented. Nevertheless there may have been a connection between the words *through* and *door,* as there probably was between the words *per* and *porta.*

Through is the Gothic *thairh;* the Anglo-Saxon *thurh;* the old English *thurg, thourh, thorgh, thorth, thorou,* &c.; the Alamannic *duruh, durich, dhurah;* the Frankish *tharuh, thuruhe, thurah, durh,* the German *durch,* the Dutch *door,* &c.

The following are old English and Scottish examples:

Henr: *thurg* Godes Fultume King on Englene loande.
Letter *Hen. III.* 1258.

Grammar.

For alle this thraldam, that now on Inglond es,
Throgh Normanz it cam, bondage and destres.
ROBERT DE BRUNNE.

The appel, where *thorou* al the wold was forlore."
MS. Homily, temp. RICHARD II.

Sixtene hundred of horsemen hede ther her fyn
Thourh huere oune prude.
Ballad on Battle of Bruges.

The lady rod *thorth* Cardeule. *Syr. Launfal.*

In like manner are the compounds, *therthrough*, *quhairthroughe*, *thorghout*, and *out through*.

But whatsoe'er made the debate
Therthrough he died, well I wat. BARBOUR.

Sik as has sufficiently of thar swin, *quhar throughe* thai mai be punyst gif thai trespass. *Scot. Stat.* A. D. 1424.

Diuers ar yit absent, *quhairthrow* large tyme is spent and nathing as yit done. *Scot. Stat.* A. D. 1567.

The kyng *thorghout* the lond, he did crie his pes,
And with the law tham bond, als skille wild he ches,
ROBERT DE BRUNNE.

————— an aizle brunt
Her braw new worset apron
Out through, that night.
BURNS, *Hallow E'en.*

It is probable that one of the most ancient substantival forms of the word *thraugh* is to be found in the Anglo-Saxon *throt*, or English *throat*.

ANALUNO considers that *durch*, &c. are connected with the Greek *τερεω*, Latin *tero*, and Swedish *taera*, to pierce through.

To these we may add the Anglo-Saxon *thirlian*, which is our verb to *drill* a hole, whence *nasethyrl*, was the *nostril*.

As that which has been gone *through* with, or which is *thoroughly* effected, is complete, so *duruh*, *durch*, *door*, &c. in composition signify completeness, or excellence; as in the Frankish *duruhtuan* "to accomplish," or do *thoroughly*; the German *durchlauchtig*, and Dutch *doorluchtig*, "most illustrious," or *thoroughly* illustrious.

After. In this sense we may explain the force of the termination *thra* in the Gothic *uzathro*, extra, completely, or *thoroughly* out of. And perhaps to this source is to be traced the Latin *tra*, in the prepositions *intra*, *extra*, *ultra*, *citra*, &c.

Mr. TOOKE is undoubtedly right in saying that this word is merely the comparative of *aft*; and he has acted with more prudence than usual, in not pretending to specify any particular object of which *aft* was originally the name. It may probably have been a term applied to the *back*; and, as we have before suggested, the radic of *aft*, may have been *af*; but these are all mere conjectures. It is certain, however, that our English words *aft* and *after* are related to the Gothic *aftara*, Anglo-Saxon *æfter*, Danish and Swedish *efter*, Dutch and Swedish *achter*, all of the same signification. In German *after* is not found in its separate state, but enters into many compounds, all with analogous significations, e. gr. *afterdarm*, the intestinum rectum; *after-geburt*, the after-birth; *afterkind*, a posthumous child, &c. What we express by "*fore* and *aft*," the Danes express by "*for og bag*;" and the Danish *bag* is no doubt our word *back*. They have also *bagdeel*, the breech, the stern of a ship; and *tilbage*, behind, analogous in construction to our old word *to-fore*.

Prepositions.

The nautical expression *abaft* is from the Anglo-Saxon *be-æftan*, or *bæftan*, as "gang *bæftan* me Satanas."—Get thee behind me Satan.

After is poetically used as an adjective in the beautiful ballad of *Gil Morice*.

To me nae *after* days, nor nichts
Will eir be saft or kind.

It is probable that the Greek *αὐτὰρ* may have been the Gothic *after*, with little, if any change in the pronunciation. Indeed a modern Greek would pronounce *αυταρ*, *avtar*.

From the signification of that which is *behind*, in place, naturally follows the signification of that which is subsequent in time, as "the *afternoon*." Hence our modern adverb *afterwards*, and the obsolete adverb *eftsoons*, signifying shortly afterwards. In this sense of *aft* it may have given rise to the Greek *αὖθις*.

As the effect comes *after* the cause, in order, and the copy *after* the model, we have the expressions "*after* our unrighteousness," "*after* Rembrandt," &c. which are expressed according to a similar analogy in Latin, by the word *secundum*. In this manner the Franks used the word after, as "*after* kewrahti," *after* what we have wrought. A singular instance of this use of the word *after* occurs in *Kyng Alisaunder*, where the poet is describing certain "bestes ferlich," called "Deutyrauns—"

More by ben than Olyfaunz;
Blake heueded *after* a palfray;
Ac in the forehede, parmafay,
Hy have thre hornes.

Obsolete and foreign.

Having thus examined at length the chief English prepositions now in use, it may not be necessary to consider so minutely the obsolete prepositions of our own language, or those which are only to be found in other languages or dialects. Some of these, however, we will briefly notice.

Mid, used in Anglo-Saxon and old English for *with*, is the Gothic *mith*, Frankish, Alamannic, and German *mit*, Dutch *met*, Danish *möd*, and probably the Greek *μετὰ*. It is evidently connected with the verb *meet*.

Emb, of which we retain a trace in the modern word *embassy*, was an Anglo-Saxon preposition signifying *about*. It seems to have had some analogy to the Anglo-Saxon substantive *wamb*, the belly; in the Scottish dialect *wame*: and was no doubt connected with the German *um*, and the old Latin *am*. "The particle *um*," says Dr. NOEHDEN, "is frequently joined with *zu*, which expresses the design still more distinctly. Liebet die Tugend *um* glücklich zu seyn, love virtue (in order) to be happy." Festus says, "*Am* præpositio loquelaris significat *circum*:" and R. STEPHANUS says "Verisimile est Latinos *ambi* suum, unde contractè *am*, Græcorum *ἀμφι*, debere." The old English *whilom* seems to be compounded of *while* and *om*, or *em*.

The Scottish participles *anent* and *forenent* are of doubtful origin; they may probably be derived from *ent*, for *end*. ROBERT DE BRUNNE uses *ent* for ended.

Be that the werre was *ent*, wynter was ther yare.
To Dounfermelyn he went, for rest wild he thare,

The German *ohne*, without, seems to have some affinity with our negative prefix *un*, where that particle is derived (as it seems to be in some instances) from *wan* or *want*. We have in Burns's poems, *wanchancie*,

wanrestfu,' &c. The Frankish preposition answering to *ohne* is *ano*, *as* "*ano* xwifal," without doubt. In the Swabian dialect this is *aun*; in Alamannic *anoh*, which nearly approaches the Greek *ανευ*.

The German preposition *wegen*, concerning, touching, &c. is evidently from *weg* motus, which is our verb *wag*, and substantive *way*.

The German preposition *sonder*, and Dutch *zonder*, without, or separated from, are doubtless connected with our words *sundry* and *asunder*, and these perhaps with *sand*.

The French preposition *chez* is correctly referred by Mr. Tooke to the Italian *casa*, so that "*chez moi*" is literally "*house me*," i. e. at my house.

The Dutch preposition *van*, of, or from, is retained in English as a substantive; but it does not, as Mr. Tooke seems to suppose, indicate a real *object*, but the *relation* which that object bears to some other; for when we speak of the *van* of an army, we do not mean merely to indicate a certain number of soldiers, but to signify that those soldiers are placed in a certain relation to the rest of their comrades.

Thus have we considered two of the three methods by which the relation of a substantive to a verb or to another substantive, may be expressed in language. The remaining mode of expressing such relation is by those changes or inflections of the word itself which are called *cases*. Of these we have considered the general use in treating of nouns and their incidents. The particular means employed to form such inflections will be most conveniently considered when we come to treat of the *particles* which enter into the composition of the great majority of words.

III. Having stated first the necessary *complexity* of every sentence in which a preposition is employed, and secondly the *origin and use* of many known prepositions, in expressing the relations of substantives, we have only, in the third place, to subjoin a few remarks on the *relations* ordinarily so expressed.

Now *relation*, which is the fourth of the logical predicaments, supposes three things, the *subject*, or thing related, the *object* or correlative, and the relation itself, or circumstance existing in the subject by means of which it is related to the object, and which logicians call the *foundation*. When we say "John is *before* Peter," "John" is the subject, "Peter" is the correlative, and "*before*" is the foundation, or, as we have been accustomed to speak, the conception of relation, expressed prepositionally.

It is manifest, that the circumstance, whatever it be, that forms the *foundation* of a logical relation, or (which is the same thing) that constitutes (when expressed in language together with its subject and object) a *preposition*, may either be common to the two *terms* (as they are called) of the relation, or it may belong to one of them exclusively. If I say "John is *with* Peter," the relation expressed by the preposition *with* belongs equally to Peter and to John; but if I say John is *before* Peter, the relation expressed by the preposition *before* belongs exclusively to John. In the first case it is perfectly indifferent whether I say "John is with Peter," or "Peter is with John;" it is perfectly indifferent which I make the subject and which the object of the relation; but in the other case, if I were to say "Peter is *before* John," I should not only vary the assertion, but I should directly contradict it.

Still the *foundation* of the relation would be the same: and we may illustrate this with the trivial comparison of two children playing at see-saw. If John and Peter be equally balanced at the opposite ends of a plank, John is level *with* Peter, and Peter is level *with* John, and the plank is the measure or standard of the *level*; but if John be lighter than Peter, John at once rises *above* Peter, and Peter sinks *below* John, and the same plank measures the *elevation* of one and the *depression* of the other. What the supposed plank is to the boys, the preposition is to the substantives related; and hence we may easily explain not only certain diversities in the idioms of different languages, but some apparent contradictions in the same idiom. Thus Mr. Tooke makes the following just observation on the Dutch preposition *van*: "The Dutch," says he, "are supposed to use VAN in two meanings, because it supplies indifferently the places both of our *of* and *from*. Notwithstanding which, VAN has always one and the same single meaning. And its use, both for *of* and *from*, is to be explained by its different *apposition*. When it supplies the place of *from*, VAN is put in *apposition* to the same term to which *from* is put in *apposition*. But when it supplies the place of OF, it is not put in *apposition* to the same term to which *of* is put in *apposition*, but to its correlative." The difference of idiom between the Dutch and English languages might have been still more strongly stated; for "*Von* Amsterdam gekomen" signifies "come *from* Amsterdam;" whereas "*Van* Amsterdam geboortig," is "born *at* Amsterdam:" and our prepositions *at* and *from* are commonly used in senses very opposite to each other.

But it is not only the different use of prepositions in different languages, but the apparent contradictions in the same language, which are thus to be explained. The prepositions *for* and *after* are of directly contrary origin and signification, being (as has been fully shown) the same as the words *fore* and *aft*. Nevertheless we say, "to seek *for* that which is lost," and "to seek *after* that which is lost." The thing sought is considered as *before* the mind of the seeker; and consequently the seeker is considered as *after*, or behind the thing sought; when, therefore, we use the word *before*, we specify the relation of which the thing sought is the subject; but when we use the word *after*, we specify a relation of which the subject is the seeker: or to use Mr. Tooke's phraseology, we put *before* in *apposition* with the thing sought; and *after* in *apposition* with the seeker.

From this statement it appears that the *subject* of the *relation* specified may or may not be the logical *subject* of the *proposition* enunciated in the sentence. In the sentences, "John seeks *for* Peter," and "John seeks *after* Peter," John is the logical subject; but the former sentence involves the expression of a relation of which Peter is the subject; the latter of one the subject of which is John. The relation of *foreness* exists in Peter; the relation of *afterness* exists in John.

How a particular preposition may be employed, in this respect, is mere matter of idiom, and depends solely on custom—

Quem penes arbitrium est, et jus, et norma loquendi.

But it will generally be found that the prepositions of most frequent use are employed with the greatest latitude, in the earlier stages of a language, and so continue, until their equivocal signification gives rise

to inconveniences which are only to be remedied by confining them to certain forms of construction.

Various prepositions may sometimes be used indifferently in a sentence; and sometimes a particular preposition is absolutely essential to the sense. This circumstance depends on the nature of the relation intended to be expressed. In general, the external and physical relations of objects must be expressed by their own proper and peculiar words. Thus we cannot substitute *in* for *out*, or *after* for *before*, in speaking of visible objects and bodily actions; but the case is different when we come to speak of the mind; for as the analogy of its states and operations to those of the material world are very loose and general, so we may adopt almost any external relation of things as a symbol whereby to explain mental relations. Thus we may say that a person did a certain act *in* envy, or *out of* envy, or *through* envy, or *from* envy, or *for* envy, or *with* envy; but we cannot say of the same man, under the same circumstances, that he was *in* his house and *out of* his house, passing *through* the town, and distant *from* the town, walking *with* another person, or a mile *before* him. Still there are limits, fixed by custom, to the use of each preposition; but these limits vary much in different languages; and hence a translation, correct in substance, often appears literally inaccurate. Thus the French "*sous* peine," answers to our "*on* pain," and to the old English "*up* peine."

No more *up* peine of lesing of your hed. CHAUCER.

Custom also varies in the course of time, as we have seen in many of the examples already cited, and which have now become obsolete, as "to learn *at*," "to accuse *for*," &c. But it must not always be supposed that the force of a preposition is varied, because the application is different; for the difference may arise from the other words in the sentence; thus the French *ôter à* and *donner à*, are our "take *from*," and "give *to*;" but in both cases *à* retains its primary force, and the apparent opposition depends on the contrariety between *ôter* and *donner*.

To suppose that the prepositions necessary to any language could be enumerated *à priori* would certainly be absurd. TOOKE has ridiculed the grammarians who have attempted to enumerate them, as matter of fact and history. It has been said, that the Greeks had eighteen prepositions, the Latins, forty-nine, and the French, (according to different authors,) thirty-two, forty-eight, and seventy-five. It is certainly a possible, but a very useless labour, to ascertain what words have been used as prepositions in a dead language. In a living language it is quite impracticable, for every day may enhance their number, by new combinations of thought and expression. A preposition is not like a piece of money stamped to pass for a certain value, and which cannot change its denomination or value. It is a word to which a transient function is assigned, and which, as soon as it has discharged that office, becomes available again for its former purposes, as a noun, verb, or other part of speech.

But although it be not possible to enumerate prepositions, yet they may be subjected to a general classification, according to the great distinctions of relation in human conceptions. M. COUR DE GEBELIN has attempted something of this kind, and Bishop WILKINS has also given an arrangement of thirty-six prepositions, "which," he says, "may, with much less equivocalness than is found in instituted languages, suffice to express those various respects, which are to be signified by this kind of particle." It may be doubted whether either of these schemes be sufficiently comprehensive, or perfectly philosophical. Prepositions must be classed, if at all, by their signification only, as expressing relations of parity or of disparity, of place, time, motion, order, causation, &c.; and in forming such an arrangement, the same word will frequently occur, with different powers, according as its force is primary, or figurative.

Although the proper function of a preposition be to modify a substantive, yet in several of the instances already quoted, we have seen prepositions accumulated on each other, either as separate words, or as compounds, and, of course, modifying each other.

In the earlier and less cultivated periods of a language, such cumulations of words may be expected to be more common; but as grammatical accuracy and elegance of style prevail, the prepositions (considered as distinct words,) are usually confined more strictly to their separate use. We find even in MILTON, the combination *at under*, as "some trifles composed *at under* tweoty;" but in the present day, such a construction would hardly be tolerated by the critics. In more ancient times this sort of construction was still more prevalent; and we find numberless such expressions as "of beyond," "for against," and the like.

Artifycers and other straungiers, from the parties *of beyonde* the see. *Stat.* 1. RIC. III. c. ix.

The shiref of the seid countie of Northumbreland, or wardeyn of the est and middell marchees *for ayenst* Scotlond. *Stat.* 11. HEN. VII. c. ix.

Where the combination has been such as to present to the mind the ready conception of a new relation, it has generally been received in language as a new preposition, as *throughout, into, overthwart*; and so perhaps the Latin *intra, extra*, &c. Custom too has sometimes given a distinct force to compounds, which appear originally to have had no signification different from that of the simple preposition which formed their basis. Thus we have in English distinguished *within* from *in*, *without* from *out*; and more slightly *unto* from *to*, *untill* from *till*, &c. So in French we find *en* and *dans*, *avant* and *devant*, *vers* and *devers*, *près* and *auprès*, with more or less of distinction in their modern use and application; and, in like manner, the Italians, from the Latin *ante*, have formed *innanzi*, formerly *inanti*, and *dianzi*; as from *pressus* they have formed *appresso* and *d'appresso*.

L'alma Ciprigna *inanti* i primi albori
Ridendo empia d'amor la terra e'l mare.
ANNIBAL CARO.

Torna amore a l'aratro, e i sette colli,
Ou 'era *dianzi* il seggio tuo maggiore.
F. M. MOLZA.

Io pur doveua il mio bel sole, io stesso
Seguir col piè, come segu'hor col core;
E le fredde Alpi, e'l Rhen, ch'aspro rigore,
Mai sempre agghiaccia rimir *d'appresso*.
IDEM.

Where the prepositions, as they are called, have entered into composition with nouns and verbs, they are in fact no more than adjectival and adverbial *particles*, and remain to be considered as such, in a future part of this essay. It is, however, to be observed, that when such a composition takes place, the adding of

Grammar. the same preposition to the sentence, in a separate form, is a redundancy, to be justified only by the energy of feeling which sanctions the repetition of words.

Dr. JOHNSON, citing the exquisite lines of *Hamlet*—

O! that this *too*, *too* solid flesh would melt,
Thaw, and resolve itself into a dew!

has frigidly observed, that *too* "is doubled to increase its emphasis;" but that "this reduplication seems *harsh!*" It is clear, that to repeat and dwell upon a conception often gives energy and weight to discourse. In the *Andria* of TERENCE we find—"Quid tibi videtur? *adeon' ad* eum?" So CICERO says "Nihil non consideratum *ex*ibat *ex* ore." So VIRGIL—"*Retro* sublapsa *re*ferri;" in all which instances it is impossible not to see that the repetition of the preposition is a great beauty. Nor is this observation to be confined to the repetition of the *same* preposition; for it applies substantially to all prepositions, and even adverbs, of *similar* meaning; as in TERENCE—"Nonne oportuit *præ*scisse me *ante*?"—"Multa *con*currunt *simul*." Grammarians of repute, it must be allowed, have censured these redundancies of expression, which, doubtless, are to be regarded as exceptions from a general rule, and ought not to enter into the ordinary construction of a sentence. But the censure, when directed against such passages as we have cited, rather shows an acquaintance with technicalities, than a nice feeling of the higher powers of language.

In like manner, the omission of prepositions, though sometimes owing to a defective construction, has been in other instances unnecessarily blamed. The omission of the preposition *of* is undoubtedly awkward in the following instances:—

That every person cõmyng to suche feires shulde have lawefull remed of *all maner contractes*. *Stat.* 1. Ric. III. c vi. MS.

But God that is of maist poustè
Reserued to his majestie;
For to knaw in his prescience,
Of *all kind time* the first movence. BARBOUR.

The kyng Robert wist he was there
And *what kind chiftains* with him were. IDEM.

Then should they full enforcedly
Right *in mids the kirk* assail
The Englishmen. IDEM.

So, in old French, the preposition *de* is often awkwardly omitted.

Wrepoch ab Edenauet, &c. overe tot le orgoyl de Gales—descendirent a *la terre nostre seigneurs* le rei.
Let. P. De Mounfort, A. D. 1256.

Qui *la maison son voisin* ardoir voit,
De la sienne douter se doit.

Faut noter—*la maison son voisin* estre dict à la façon ancienne; au lieu de dire "la maison *de* son voisin."
H. ESTIENNE.

So, also in Italian, the authors of the *Vocabolario della Crusca* observe, on the word *casa*: "Nome, dopo di cui vien lasciato talvolta dagli autori, per proprietà di linguaggio, l'articolo, o il segnacaso."

E sì sen' andaron di concordia a *casa i prestatori*.
BOCCACIO.

Cominciano a chiedere il Gonfalone che stava *in casa Germanico*.—"Vexillum *in domo Germanici* situm flagitare occipiunt."
DAVANZATI, *Tacit. Ann.*

In the construction of the Latin language, some grammarians contend, that where a noun is commonly said to be *governed* by another noun, or by a verb, it is proper to consider that a preposition has been suppressed; as, "Cicero fuit eloquentior (præ) fratre." But this seems an unnecessary refinement in grammar; for the particle *or* in eloquentior, and the termination *e* in fratre, sufficiently show the *relation* between *eloquence* and *frater*, which is all the effect that a preposition could produce.

Prepositions.

The same observation may be made on the expressions *ire rus, domum, Romam, Hierosolymam*, where Vossius seems to suppose an omission of *ad* or *in*; but he adds, "Latinis tam usitata est hæc ellipsis, in exemplis allatis, ut vulgo naturalis sermo existimetur."

It may, however, be doubted, whether such constructions as *alias res improbus, cætera lætus*, and the like, are not to be ranked among the negligences of composition, though sanctioned by names of high repute in Roman literature.

———— Ille eam rem adeò sobriè et frugaliter
Accuravit, ut *alias res* est impensè *improbus*.
PLAUT. *Epid.* iv. 1.

Excepto quòd non simul esses, *cætera lætus*.
HORAT. *Ep.* i. 10.

Similar observations may be made on the Greek writers, who are often censured for the omission of prepositions; and the remark is sometimes just, though in general the *relation* is sufficiently expressed, and the preposition would therefore be superfluous. The learned LAMBERTUS BOS says, "Præpositionum ellipsin tantopere amant scriptores Græci ut interdum duæ præpositiones in unâ orationis parte omittantur. Aristoph. *Nub.* v. 1083. Ἢν τοῦτο νικηθῇς ἐμοῦ: *Si* (in) *hoc* (a) *me victus fueris*. Plene: ἢν εἰς τοῦτο νικηθῇς ὑπ' ἐμοῦ." In this instance it would perhaps have been better, had the rhythm allowed it, to express the first of the two prepositions; but the relation of ἐμοῦ to νικηθῇς is sufficiently denoted by their respective terminations.

From all that has here been said of prepositions, the necessity, and even beauty, of such a part of speech is sufficiently manifest. "Most, if not all prepositions," says HARRIS, "seem originally formed to denote the *relations* of PLACE." "Omne corpus," says SCALIGER, "aut movetur aut quiescit: quare opus fuit aliquâ notâ, quæ τὸ ποῦ significaret, sive esset inter duo extrema, inter quæ motus fit, sive esset in altero extremorum, in quibus fit quies. Hinc efficiemus præpositionis essentialem definitionem." "But though the original use of prepositions," continues Harris, "was to denote the *relations of place*, they could not be confined to this office only. They, by degrees, extended themselves to subjects *incorporeal*, and came to denote relations, as well *intellectual* as *local*." "But how," says COUR DE GEBELIN, "can such words introduce into the pictures of speech so much harmony and clearness, and become so necessary, that without them, language would present but an imperfect delineation? How can these words produce such powerful effects, and diffuse throughout discourse so much warmth and delicacy?" The reason, he adds, is simple: "There is no object which does not suppose the existence of some other object to which it is bound, with which it is connected, to which it in some way or other bears relation. A valley supposes the existence of a mountain, a mountain that of less elevated lands: smoke implies fire, and there is "no

rose without a thorn." It is of necessity, then, that different objects should be bound together in speech as they are in nature; and that we should have words to express the *relations* which exist among things."

After this, it may be unnecessary to remark on Mr. Tooke's sweeping censure of the philosophers, that "though they have pretended to teach others, they have none of them known themselves *what* the nature of a preposition is."

§ 8. *Of conjunctions.*

We have seen that a perfect sentence is formed by a noun and a verb, as, "John walks;" that it is complicated by the addition of an adverb, which modifies the verb, as, "John walks *foremost*;" and that it is rendered still more complex by a preposition which shows the relation of the noun or verb to another noun, as, "John walks *before* Peter;" but it may be requisite to connect one sentence either simple or complex, with another; as "John walks, *and* Peter rides." Now the word which thus conjoins sentences is called a *conjunction*.

In the very commencement of our inquiry into this class of words, we are met by the broad, unqualified assertion of Mr. Tooke, "I deny them to be a separate sort of words, or part of speech by themselves." Such are the bold, but absurd or unmeaning propositions which have obtained for this etymologist the reputation not merely of a grammarian, but of an absolute inventor of the science of grammar! He himself tells us, "he means to discard all *mystery*." Why, what greater mystery can there possibly be, what greater confusion in the mind of a student of grammar than to be told that there is no order, no classification, among words,—that *if* is derived from *give*, and therefore *if* and *give* are words of the same sort, nay identically the same in all their uses—that they do not indicate by their use, any different "turns, stands, postures, &c. of the mind." The mystery here discarded is the mystery of learning. The student is stopped on the very threshold of his studies, by being assured that there is nothing for him to learn. And the sage who gives him this precious information, sets up for the great illuminator of mankind. "I believe I differ from *all* the accounts which have hitherto been given of language," says Mr. Tooke. Very true: and every patient in Bedlam differs from all other persons who give any account of his state of mind. It is somewhat strange, that in support of his title to absolute originality and exclusive knowledge of grammar, this writer should quote the following (among other) expressions of Lord Bacon:—"*Quæ in naturâ fundata sunt, crescunt et augentur; quæ autem in opinione* Variantur, *non augentur.*" The science of grammar, which is founded in nature, was taught, as we have shown above, by Plato and Aristotle. Since their time it has grown and been increased by the labours of grammarians in all ages, and in a great variety of languages down to the present time; and now we see it illustrated by application to languages dead and living, polished and barbarous, to the Sanskrit, Hebrew, Latin, and Gothic, as well as to the English and French, the Soosoo, and the Chinese: and we find the same principles running throughout them all, because language is the expression of thought, and human thought runs in the same channels, among all mankind. But when at the close of the eighteenth century of the Christian era, an individual professes to set aside every trace and vestige of the knowledge which preceded him, his doctrine is not an *augmentation*, but a *variation*, and we may be well assured that it is founded in the mere *opinion* of its pretended inventor. Now what is opinion? Mr. Tooke presumes to ridicule Lord Monboddo's account of it, derived from the Platonic philosophy, simply because Mr. Tooke could not or would not understand that philosophy. Plato says that the subject of opinion is neither το ὀν nor το μη ὀν. But this, however paradoxical it may appear to any person who will not take the trouble to reflect upon it, will be found extremely clear, with the help of a very slight degree of attention. By το ὀν he means that which *is*, in the absolute sense of the word—that which *is*, always, and certainly, and without any variation. By το μη ὀν he means that which *is not* at any time, or in any manner, and cannot be conceived to be. Thus it is always and certainly true that in our *idea* of a circle all the radii are equal; and it is not at any time or in any manner true that we can form an idea of a circle with unequal radii. But there is a third case which is continually occurring to us, namely, that an *object* is presented to our observation which may correspond more or less accurately with a given idea. We may see for instance a coach-wheel, or the dome of St. Paul's church, but we can only form an *opinion* how nearly either of these approaches to our idea of a perfect circle; for the life of man would not suffice to prove such coincidence beyond the possibility of a doubt. Now, Plato distinguished this class of objects by the expression to γιγνομενον, which he opposed to το ὀν, as in the following celebrated passage of the *Timæus*—Εστιν οὖν δὴ κατ' ἐμην δόξαν πρῶτον διαιρετέον τάδε· τί το 'ΟΝ μὲν 'αεί, γένεσιν δὲ 'ουχ' ἔχον· καὶ τί τὸ ΓΙΓΝΟ'ΜΕΝΟΝ μὲν, ὂν δὲ 'ουδέποτε· τὸ μὲν δὴ ΝΟΗ'ΣΕΙ, μέτα λόγο περιληπτὸν, 'αεὶ κάτα ταυτὰ ὂν. τὸ δ'αὖ ΔΟ'ΞΗ, μετ' αισθήσεως ἀλόγυ, δοξαστὸν, γιγνόμενον καὶ ἀπολλύμενον ὄντως δὲ 'ουδέποτε ὂν—which passage Cicero has thus freely rendered:—"Quid est, quod semper *sit*, neque ullum habet ortum? et quod *gignatur*, nec unquam sit? Quorum alterum *intelligentiâ* et ratione comprehenditur, quod unum semper atque idem est: alterum quod affert *opinionem* per sensus rationis expertes, quod totum opinabile est; id gignitur et interit, nec unquam esse vere potest."—And the general sense of both these great writers is, that *science* is founded on that which *is*; *opinion* on that which *seems*: science relates to that which is distinctly apprehended, because it is permanent, immutable, and consonant to the necessary laws of human existence; opinion to that which is vague and indistinct, arising from sensible impressions, and the casual accidents of time and place. What Mr. Tooke calls his "general doctrine," is of this latter kind: it is an *opinion* derived from comparing the *sound* of words, not only without regarding, but often in direct opposition to their *sense*. Should any one for a moment conceive that we are speaking without due respect to the literary reputation of Mr. Tooke, we beg to remind him that we speak of a passage in which Mr. Tooke himself has treated the profound wisdom of a Plato and a Cicero with the most sovereign contempt, and has even represented Lord Monboddo as an idiot, for quoting their very words. As to Lord Monboddo himself,

Mr. Tooke elsewhere says that his Lordship was "incapable of writing a sentence of common English;" but this is nothing to his abuse of one of his critics, the late Mr. Windham, an accomplished scholar, and as honourable a man as ever existed, hut whom Mr. Tooke calls in his chapter on conjunctions, a "cannibal," and "a cowardly assassin."

Defined from use. We call a word which conjoins sentences a conjunction. But to this also Mr. Tooke objects. "Conjunctions," says he, "it seems, are to have their denomination and definition from the use to which they are applied: *per accidens, essentiam.*" This leads us to ask what Mr. Tooke understands by the *essence* of a word? Its sense, or its sound? Evidently the latter, which is in truth, an *accident*. The words "*και*," "*et*," "*and*," are all essentially the same. The Greek, the Roman, the Englishman, who may have used each respectively, must have meant and intended the same thing; but by the thousand accidents which led to the formation of each separate language, the expression became varied in sound. Besides, this objection involves Mr. Tooke in a gross inconsistency. He admits that a noun differs from a verb; hut how does it differ, if not in use? How does the noun *love* differ from the verb *love*, or the noun *whip* from the verb *whip*, but in use? And if a noun differs from a verb in its use alone, why should not a conjunction differ from both, in the same manner? This is an essential difference; because the essence of a word is the *thought* which it conveys; but there is no more reason for calling the sound of a word its essence, than for giving that appellation to the colour of the ink with which it is printed.

"There is not such a thing," says Mr. Tooke, "as a conjunction in any language, which may not, by a skilful herald, be traced home to its own family and origin." This may, or may not, be the case; but it has nothing to do with the *science* of grammar. Mr. Tooke has accurately "traced home" some conjunctions: in regard to others, he has been mistaken; but whether right or wrong in the particular instances, his general doctrine can derive no benefit from them. To prove that a word performs one function at one time, does not disprove its performing another function at another time. In fact, most of Mr. Tooke's derivations in this part of his work are borrowed from former writers; hut those writers never conceived any thing so ahsurd, as that derivation was the whole of grammar.

Definition. The early grammarians included what we call *conjunctions* and *prepositions*, under the name of the *connective* Συνδεσμος: and the definition given of the Συνδεσμος by Aristotle, though commonly cited as that of a conjunction, is, in fact, equally applicable to a preposition. It is in part doubtful, owing to the diversity of the manuscripts, but, upon the whole, the following may he regarded as tolcrably correct: *Σύνδεσμος ἐστι φωνὴ ἄσημος, ἐκ πλειόνων μὲν φωνῶν μιᾶς, σημαντικῶν δὲ, ποιεῖν πεφυκυῖα μίαν σημαντικὴν φωνήν.*—"A connective is a non-significant word, formed to make one significant expression, out of words (or expressions) more than one, but (separately) significant." According to this definition, *of* is a connective in the phrase "the Son *of* man;" for both son and man are separately significant; but by the connective they are so united as to produce a third significant expression. According to the same definition, *but* is a connective in the expression "John danced, but Peter sang;" for "John danced" is one significant expression; and "Peter sang" is another significant expression; and they are both united together, so as to form one continued sense, by the word "*but*."

Subsequent writers, however, perceived that it would be useful to separate these two classes of connectives; they therefore gave to that which showed the relation of word to word the name of preposition; and to that which showed the relation of sentence to sentence, the name of conjunction. Hence Scaliger says "Conjunctio est quæ conjungit *orationes* plures;" and Sanctius, more briefly, "Conjunctio *orationes* inter se conjungit." Harris says, "The conjunction connects not words but sentences;" and he gives the definition of a conjunction fully, thus:—"A part of speech, void of signification itself, but so formed as to help signification, by making two or more significant *sentences* to be one significant *sentence*." Vossius says "Conjunctio est quæ *sententiam* sententiæ conjungit;" and he more formally defines it, "Dictio invariabilis, quæ conjungit verba, et *sententias*, actu vel potestate." We should be inclined to prefer the following definition—"A conjunction is a word used to show the relation of sentence to sentence." We designedly omit stating it as a characteristic of the conjunction to be "void of signification," or to be "invariable." Possibly these expressions may be understood in such senses, as to agree with the proper idea of a conjunction; but they may also serve to give a false idea of it. and, at all events, they are not essential to the character of a conjunction. Neither do we think it necessary to say, that the conjunction unites "*verbs* and sentences;" for, according to the definition which we have heretofore given of a sentence, it is clear that the uniting of verbs must he the uniting of sentences. Thus "he danced and sang" combine in reality the two sentences "he danced" and "he sang." Lastly, it seems scarcely necessary to add, as Vossius does, the words "actually or potentially;" for this seems merely to have relation to those cases which are to be explained by the figure, ellipsis, so common in all the constructions of speech.

The main point, however, is, that the conjunction receives its distinguishing characteristic from showing the relation of *sentences*, and not simply of words. Mons. Court de Gebelin expresses this in his figurative way, by saying "une conjunction est un mot, qui, de plusieurs *tableaux* de la parole fait un tout;" for, by *tableaux* he does not mean a single object, a single assertion, or a single sensation, but such a combination of these as we have called a sentence.

Mr. Tooke, however, objects that there are cases in which the words, commonly called conjunctions, do not connect sentences, or show any relation between them. "*You, and I, and Peter, rode to London*, is one sentence made up of three. Well! So far, matters seem to go on very smoothly. It is, *You rode, I rode, Peter rode.* But let us now change the instance, and try some others, which are full as common, though not altogether so convenient. *Two* and *two make four*; AB and BC and CA *form a triangle*; *John and Jane are a handsome couple.* Does AB form a triangle, BC form a triangle? &c. Is John a couple? Is Jane a couple? Are two, four?" To all this we answer, that if it could be shown that *and*, or any other

Grammar. word generally used as a conjunction, was occasionally used with a different force and effect, that circumstance would not make it less a conjunction, when used conjonctionally. In the instances cited, the word *and* serves merely to distribute the whole into its parts, all which bear relation to the verb: and it is observable, that though the verb be not twice expressed, yet it is expressed differently from what it would have been, had there been only a single nominative. We say, "John *is* handsome,"—"Jane *is* handsome;" but we say John and Jane *are* a handsome couple. In this particular, the use of the conjunction differs from that of the preposition: it varies the assertion, and thus does *potestate*, if not *actu*, (to use the phrase of Vossius,) combine different sentences; for though AB does not form a triangle, yet AB forms one part of a triangle, and BC forms another part, and CA the remaining part; and these three parts are the whole. So, when Perizonius says "Emi librum x drachmis *et* iv obolis," although the buying was not wholly effected by the ten drachmas, nor by the four oboli; yet the purchaser did employ ten drachmas in buying, and he did also employ four oboli in buying. The meaning, therefore, if fully developed, would exhibit two *sentences* connected by the conjunction *and*. Nevertheless, if any one contend that the word *and*, in the above sentences, does simply and solely connect together the nouns, then we say it must in such instances be called a preposition; but this will in no degree alter its property or character as a conjunction, when it is really employed to connect sentences.

In pursuance of the view exhibited by our definition of this part of speech, we proceed to consider the three following subjects: first, the sentences connected; secondly, the different relations between them, intimated by different conjunctions, or conjunctional forms; and thirdly, the words or phrases which are used to imply these relations.

Sentences connected. We have, in a former part of this treatise, distinguished sentences into *enunciative* and *passionate*: and in each, the *verb*, or the *interjection*, which stands in the place of a verb, is to be taken as the hinge on which all the rest of the sentence turns. By means of this we form an unity of thought, a distinct perception of some fact, or a feeling of some sentiment, connected with a distinct object. But thoughts and sentiments do not always succeed each other in the mind as detached, and perfectly separate things, but more commonly with associations of similarity or contrast, with relations of cause and effect, and with a thousand other modifications and mutual dependencies. Hence these first and elementary unities become parts of larger unities; the simple sentence forms only a phrase or paragraph in a more comprehensive sentence: and the longest sentence is more or less closely connected with what precedes or follows it, in a long discourse or poem.

Connection of nouns. When this compression (so to speak) of thoughts is the closest, it unites mere words, in the manner we have already described; thus, in the expressions, "I paid six shillings *and* twopence"—"I gave six shillings *want* twopence"—"Il est dix heures *moins* un quart"—"XY *plus* Z"—"AB *minus* C"—the words *and*, *want*, *moins*, *plus*, *minus*, all serve to connect words, and may be called *prepositions* if we regard only what is expressed in their respective sentences; but if we consider the sentences themselves to be formed on an elliptical construction, and resolve the assertion applying to all the objects as a whole, into separate assertions applying to the separate objects, as parts of that whole, then these same words may be properly called *conjunctions*. So, when Hamlet, addressing the ghost of his father, says, Conjunctions.

> If thou hast any sound, *or* use of voice,
> Speak to me!—

The word "*or*," if considered as merely pointing out a relation between the nouns, "sound," and "use," may be called a *connective preposition*; but if the sentence be supposed equivalent (as we think it is) to this, "if thou hast any sound, *or* if thou hast any use of voice," then *or* is certainly to be called a conjunction.

Whatever difficulty there may be when the verb is suppressed, there can be none when it is expressed—e. gr. Connection of verbs.

> ——————— Fairy elves,
> Whose midnight revels, by a forest side,
> Or fountain, some belated peasant sees,
> *Or* dreams he sees.

Here the sense is clearly, "the peasant sees revels, *or* the peasant dreams that he sees revels," and the latter *or* is therefore clearly a conjunction uniting those two short sentences, into one longer sentence.

How far these connections may go on, that is to say, how many conjunctions may be admitted into one comprehensive sentence, is a matter not to be determined by any grammatical rule, but must depend on the taste and judgment of the writer; and great writers, more particularly great poets and orators often seem to indulge in a more than common degree of continuity. Thus Milton— Length of passages.

> *Now*, Morn, her rosy steps in th' eastern clime
> Advancing, sow'd the earth with orient pearl,
> *When* Adam wak'd, so custom'd; *for* his sleep
> Was airy, light, from pure digestion bred,
> *And* temp'rate vapours bland, which th' only sound
> Of leaves and fuming rills, Aurora's fan,
> Lightly dispers'd, *and* the shrill, matin song
> Of birds on ev'ry bough.

Thus, too, Cicero—

> Potestne tibi hujus vitæ lux, Catilina, *aut* hujus cœli spiritus esse jucundus, *cum* scias, horum esse neminem qui nesciat, te pridie Kalendas Januarias, Lepido et Tullo Consulibus, stetisse in comitio cum telo; manum consulum *et* principum Civitatis interficiendorum causâ paravisse; sceleri *ac* furori tuo non mentem aliquam aut timorem tuum, *sed* fortunam Populi Romani obstitisse?

And it is to be observed, that in both these instances, the following sentence begins with a distinct expression of relation to that which preceded it. Milton, having described Adam's sleep as light, goes on to say, "so much the *more* his wonder was" to find that the rest of Eve had been unquiet: and Cicero having briefly alluded to the former atrocities of Catiline, proceeds, "*ac* jam illa omitto." Indeed there are some writers whose sentences, for whole pages together, are connected, and it is difficult to detach a short passage so as to show its whole force and effect, without referring to the previous and subsequent parts of the discourse. For instances of this continuous style, we may particularly refer to the *Sermons on the Creed* by the celebrated Dr. Isaac Barrow; who, it must be confessed, carries this method to an excess; for even in a continued argument the mind seems to require some short pauses,

and resting places, as it were, to enable it to pursue its steps with regularity and firmness.

Different relations of sentences.

A very slight degree of reflection must teach any one, that the relations of sentences to each other must be very various, and consequently that the modes of marking these different relations ought to be classed under several different heads. Those persons, however, whose vanity or ignorance prompts them to overturn the whole fabric of that wisdom which has preceded them, uniformly begin by decrying it as mere rubbish. Thus Mr. Tooke, speaking of conjunctions, says,—"At the same time we shall get rid of that farrago of useless distinctions into *conjunctive, adjunctive, disjunctive, subdisjunctive, copulative, negative-copulative, continuative, subcontinuative, positive, suppositive, causal, collective, effective, approbative, discretive, ablative, presumptive, abnegative, completive, augmentative, alternative, hypothetical, extensive, periodical, motival, conclusive, explicative, transitive, interrogative, comparative, diminutive, preventive, adequate-preventive, adversative, conditional, suspensive, illative, conductive, declarative*, &c. &c. which explain nothing; and (as most other technical terms are abused) serve only to throw a veil over the ignorance of those who employ them." As this mode of treating a scientific subject is extremely flattering to the indolence of mankind in general, the above passage may not improbably have produced an injurious effect, in deterring the grammatical student from investigations which it falsely describes as unprofitable: and we therefore think it proper to examine a declamation, which in any other point of view would be totally beneath notice.

In the first place, there is a manifest want of good faith in heaping together a number of words, "*conjunctive, adjunctive*," &c. &c. &c. which are not to be found in any one grammatical writer, and presenting the whole as a "farrago" common to such writers. This is a mere trick, and a trick extremely unworthy of any man with the least pretension to literary reputation. The thirty-nine terms above cited are indeed a "farrago;" they have no meaning as they stand, they are placed in no order, and they have no relation to each other; but whose fault is that? Undoubtedly Mr. Tooke's, for he was the sole author and inventor of the "farrago" which he pretends to ridicule.

"Most other technical terms," says he, "serve only to throw a veil over the ignorance of those who employ them." A profound remark! So, the geometrician must not tell us of a *parallelogram*, or of a *rhomboid*; a surgeon must not speak of the *metacarpal* bone, or of the *arterial* tube; nor an engineer of a *counterscarp*, or a *ravelin*, because these are all technical terms; and technical terms are a mere veil for ignorance! Mr. Tooke, however is not original, in applying this sort of reasoning to grammar. That philosophic statesman, Jack Cade, thus reproaches his prisoner Lord Say, "It will be proved to thy face, that thou hast men about thee, that usually talk of a *noun* and a *verb*, and such *abominable words*, as no Christian ear can endure to hear." Admitting however that some technical terms may be properly employed, Mr. Tooke asserts that the terms applied to classify conjunctions form only a "farrago of *useless* distinctions." Now, this it would have been better for him to prove than to assert: only assertion was the easier process of the two, and presented the shorter road to celebrity as a grammatical reformer! If Mr. Tooke had submitted to the labour of attempting this proof, he would have found that some, at least, of the terms which he has specified, serve to mark *useful* distinctions; and that that utility has been very well marked out by Mr. Harris, an author whom Mr. Tooke affects to hold in so much, but such very undeserved, contempt; for whatever may have been the errors of Harris, they are not a thousandth part so gross, or so injurious to the science of grammar, as those into which Tooke himself has fallen.

Harris's scheme.

Mr. Harris exhibits the following scheme of the different species, into which conjunctions may be divided. "Conjunctions while they connect sentences, either connect also their meanings or not." The first division of them therefore is into *connexive* and *disjunctive*. "Aut sensum conjungunt ac verba," says Scaliger, "aut verba tantum conjungunt, sensum vero disjungunt." So says Vossius, "Aliæ sunt *copulativæ*, ut, *et, que, ac*; aliæ sunt disjunctivæ ut *vel, aut*." The former of these terms adds he, is used in a strict sense, "nam omnis quidem conjunctio copulat; sed hæ simpliciter id præstant citra disjunctionem sententiæ, aut caussalitatem, vel ratiocinationem." On the other hand he defends the expression of *disjunctive conjunctions* because by them "conjunguntur voces materialiter, disjunguntur formaliter." And Boethius gives the same reason in different words, where he says, "conjonctionem ea quæ conjungit inter se, disjungere in tertio." We do not cite these expressions of Vossius and Boethius as most happily chosen to illustrate the distinction in question; yet that distinction is no less obvious than fundamental. Every one must perceive at first sight, the marked difference between these two passages, "Cæsar was ambitious *and* Rome was enslaved"—"Cæsar was ambitious, *or* Rome was enslaved." It is clear that the words *and* and *or* alike join the same sentences; but it is equally clear that they join them differently. In the one case, they intimate, that the propositions stand on the same basis, and are both meant to be asserted with the same degree of confidence: in the other, that the ground, on which the one assertion is made, excludes the other; and that if not contradictory they are at least meant to be contradistinguished. Both *and* and *or* are conjunctions; both mark that a *relation* exists between the sentences; but the particular relations, which they mark, are different: one serves to cumulate, the other to separate.

Gellius uses the word *connexiva* for that sort of conjunction, which Vossius calls *copulativa*; and the former term is better suited than the latter to the scheme adopted by Harris; for he divides "the conjunctions, which conjoin both sentences and their meanings," i. e. those which may be called *connexives*, into *copulatives* and *continuatives*. The copulative (which perhaps might be called the *cumulative* conjunction) "does no more" according to him, "than barely couple sentences; and is therefore applicable to all subjects whose natures are not incompatible. *Continuatives* on the contrary, by a more intimate connection, consolidate sentences into one continuous whole; and are therefore applicable only to subjects which have an essential coincidence. To explain by examples,—'Tis no way improper to say *Lysippus was a statuary*, and *Priscian was a grammarian*—*The sun shineth*, and *the sky is clear*. But 'twould be absurd to say *Lysippus was a statuary* because *Priscian was a grammarian*—though not to say *the sun shineth* because

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the sky is clear. The reason is, with respect to the first, the coincidence is merely accidental; with respect to the last, 'tis essential and founded in nature." The Greek name for the copulative (in this sense) was Σύνδεσμος συμπλεκτικός; for the continuative συναπτικός, or παρασυναπτικός.

The continuatives are subdivided by Harris into *suppositive* and *positive*. The suppositives are such as *if*; the positives, such as *because, therefore, as*, &c. The former denote (necessary) connection, but do not assert existence; the latter imply both the one and the other. The Greek term συναπτικὸς and the Latin *continuativa* was applied to the suppositive conjunctions, which extend not only to possible but even to impossible suppositions, as, "*if* the sky fall, we shall catch larks;" the positives were called παρασυναπτικοὶ or *subcontinuative*, and assumed the actual existence of the primary fact; and this either where the connection is strictly and logically necessary or where it is mere matter of analogy, the former case being expressed by *because*, &c. the latter by *as*, &c. Of the suppositives, GAZA says, ὕπαρξιν μὲν οὔ, ἀκολουθίαν δέ τινα, καὶ τάξιν δηλοῦσιν: PRISCIAN says they signify to us "qualis est ordinatio et natura rerum, cum *dubitatione* aliquâ essentiæ rerum." And SCALIGER says they conjoin "sine subsistentiâ *necessariâ*; potest enim subsistere, et non subsistere, utrumque enim admittunt."

The positives are either *causal* or *collective*. The causals are such as *because, since*, &c. which subjoin causes to effects; e. gr. *the sun is in eclipse*, BECAUSE *the moon intervenes*. The collectives are such as subjoin effects to causes; e. gr. *the moon intervenes*, THEREFORE *the sun is in eclipse*. The causals were called in Greek Αἰτιολογικοὶ, and in Latin *causales* or *causativæ*; the collectives were called in Greek Συλλογιστικοὶ, and in Latin collectivæ or illativæ.

The *disjunctive* conjunctions are in like manner divisible into various classes. Their first distinction is into *simple* and *adversative*. A simple disjunctive conjunction, disjoins and opposes indefinitely as *either it is day*, OR *it is night*. An adversative disjoins with a positive and definite opposition, asserting the one alternative and denying the other; as *it is not day*, BUT *it is night*.

The adversatives admit of two distinctions, first as they are either *absolute* or *comparative*, and secondly as they are either *adequate* or *inadequate*. The absolute adversative is where there is a simple opposition of the same attribute in different subjects, or of different attributes in the same subject, or of different attributes in different subjects; as 1. *Achilles was brave*, BUT *Thersites was not*; 2. *Gorgias was a sophist* BUT *not a philosopher*; 3. *Plato was a philosopher* BUT *Hippias was a sophist*. The comparative adversative marks the *equality* or *excess* of the same attribute in different subjects, as *Nireus was more beautiful* THAN *Achilles*—*Virgil was* AS *great a poet*, AS *Cicero was an orator*. These relate to substances and their qualities, but the other sort of adversatives relate to events, and their causes or consequences. Mr. Harris applies to these latter the terms *adequate* and *inadequate*; he however confesses that this is a distinction referring only to common opinion, and the form of language consonant thereto; for in strict metaphysical truth no cause that is not adequate is any cause at all. With this explanation the terms may be admitted into use. Thus we may say, *Troy will be taken* UNLESS *the Palladium be preserved*; where the word unless implies that the preservation of the Palladium will be an adequate preventive of the capture of Troy. On the other hand, when we say, *Troy will be taken* ALTHOUGH *Hector defend it*, we intimate that Hector's defending it, though employed to prevent the capture, will be an inadequate preventive. Conjunctions.

The following, then, is a comprehensive view of Mr. Harris's scheme for an arrangement of the conjunctions.

- 1. connexive
  - 1. copulative
  - 2. continuative
    - 1. suppositive
    - 2. positive
      - 1. causal
      - 2. collective
- 2. disjunctive
  - 1. simple
  - 2. adversative
    - 1. absolute, or comparative
    - 2. adequate, or inadequate

Priscian distinguishes the *subdisjunctive* from the *disjunctive*: and he gives the former appellation to the Latin *sive*, as *Alexander* SIVE Paris; where *sive* has nearly a similar force with the Greek εἴτ' οὖν. In English we use the conjunction *or* indifferently as a disjunctive or subdisjunctive; that is, we say, "Alexander *or* Paris," whether Alexander and Paris be two different persons, or only two different names for the same person. SCALIGER and VOSSIUS both approve of the distinction between the disjunctive and the subdisjunctive: and though, in our own language, we employ the same word for both purposes, yet it may not be amiss to distinguish its two functions by appropriate designations.

It remains to be seen what are the conjunctional forms in language. Now it is manifest that one sentence may, and generally speaking, in a long discourse, the majority of sentences must serve to lead the mind from what precedes to what follows. It would, however, be endless to attempt to point out the means by which this is effected; nor would such an explanation, if practicable, properly fall within the scope of grammar. The remark nevertheless is important; for a sentence is in this respect only the developement of an operation more briefly effected by a *word* or a *phrase*. In treating of prepositions, we first considered prepositional phrases, and then showed how those phrases were gradually compressed into words constituting that class to which the name of preposition is usually assigned. It may not be necessary to follow exactly the same order of discussion in this part of our treatise; but we will begin with some of the more common conjunctions, and afterwards advert to phrases, and to certain other modes by which a connection of thought is kept up between sentence and sentence. Conjunctional forms.

"The principal copulative," says HARRIS, "is *and*," And.
which answers to the Greek καὶ and the Latin *et*, and is found we apprehend substantially in all cultivated languages. VOSSIUS considers the Latin *et* to be derived *per apocopen* from the Greek ἔτι, *præterea, insuper*, or more properly speaking to be the very word ἔτι only pronounced more briefly by the Latins. It is remarkable that in the most ancient remains that we have of the Latin language, the fragments of the laws of the *Twelve Tables*, *et* rarely if ever occurs, but its place is supplied by the enclitic *que*, which is probably of the same origin as the Greek καὶ. The force and effect of all these words, as simply coupling toge-

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ther sentences, will be fully understood from what has been already said of the copulative conjunctions. Mr. Tooke derives our common word *and* from *An-ad*, which he says in Anglo-Saxon signifies *dare congeriem*. This etymology is altogether obscure. It has even been doubted whether *Anan* which he expounds *dare*, to give or grant, had any such meaning; and what to make of the syllable *ad* which he translates congeriem we do not know. However, with his usual confidence in his own judgment, he elsewhere says, "I have already given the derivation which I believe will alone stand examination." Skinner more modestly, but with quite as much plausibility, says, "And—nescio an a Lat. *addere*, q. d. *add*, interjectâ per epenthesin *n*, ut in *render*, a reddendo." A word of this very ancient use can only be guessed at with much doubt, and may probably be itself one of the original roots of language. We find terms of some analogy to it in the early Gothic dialects. In the Frankish and Alamannic it is written *indi, inti, enti, ante, unde*; in the modern German *and*; in Icelandic *end*, in Lower Saxon *an*. Adelung considering (like Skinner) that the letter *n* is often inserted in one dialect, while it is omitted in another, is of opinion that the Latin *et*, and Greek ἔτι are identical in origin with the Teutonic *enti, ante*, &c. It is possible too, that our word *and* may have a connection with the Mœso-Gothic *and*, which is used as a preposition answering to the Greek ἐν, εἰς, ἐπὶ, κατά; or with the word *andar*, which in the same language means "other." Upon the whole, Skinner's suggestion is probably not remote from the truth; for the meaning of *and* is clearly *add*; nay, in separate sentences we may always substitute the imperative *add* for the conjunction *and*, with little if any difference in the force or intelligibility of the sentence. Thus, "I rode, *add* Peter walked, *add* James sailed," will not only convey the same notions, but will connect them nearly in the same manner, as if it had been more elegantly written, "I rode, *and* Peter walked, *and* James sailed."

Ac, eke.

The Latin *ac*, which seems to be identical with our *eke*, is a copulative of nearly the same force as our *and*. The Latin language does not afford any obvious etymology for the conjunction *ac*; but of the etymology of *eke* there can be no doubt; and Tooke wisely adopts that of Junius. *Eke* as a conjunction, has become nearly obsolete in modern English, with the exception of a few colloquial phrases in which it is still employed; but it is clearly the same as the verb to *eke* out; and they are both from the Anglo-Saxon *eac*, also, again, and *eacan* to add to, or augment. In the Gothic, Frankish, and Alamannic we find it written *auk, auh, ouh*. The Gothic verb *aukan* is manifestly identical with the Greek αὔξειν and the Latin *augere*. In Alamannic and Frankish the verb is written *auchon, auhhon, ouhhon*, in Danish *oge*, in Islandic *auka*. Adelung says that some of the most ancient German writers use *auch* for *und* (our conjunction, *and*). Of similar origin too are the Lower Saxon *ook*, the Dutch *oock*, Swedish *ok*, Danish and Icelandic *og*: and it is observable that in old Frankish *ioh* was similarly used for a conjunction. Tooke reprehends Skinner for deriving *eacan* from *eac*, rather than *eac* from *eacan*. There is no doubt that *eac* is the root, and *eacan* the derivative; and so far Skinner is doubtless right; but that *eac* itself was used as a verb before it was used as a conjunction is not to be doubted, inasmuch as the former use depends on a more simple operation of thought than the latter. *Eac* might be a verb in a single and simple sentence: it could not be a conjunction except in a complex sentence, that is, in the union of several sentences. Mr. Tooke has made an observation which holds true in several instances, but which like all philosophy that is founded on mere observation would be calculated to mislead, if adopted as an universal truth. He remarks that "in each language where this imperative is used conjunctively, the conjunction varies just as the verb does."—Thus, says he,

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"In Danish the conjunction is *og* and the verb *oger*.
"In Swedish the conjunction is *och* and the verb *oka*.
"In Dutch the conjunction is *ook* from the verb *oecken*.
"In German the conjunction is *auch* from the verb *auchon*.
"In Gothic the conjunction is *auk* and the verb *aukan*.
"As in English the conjunction is *eke* or *eak* from the verb *eacan*."

So far he is right; but on the other hand, the Latin conjunction *ac* varies from the verb *augeo*: the Greek αυ wants the characteristic ξ of αὔξειν, and the Icelandic *og* differs from the verb *auka*.

Also.

As *eke* varies in a slight degree from the simple copulative, *and*, so *also* is a copulative with a still more specific meaning; inasmuch as it implies something of similitude with what went before. We have already seen that *so* when used as a pronoun, was originally equivalent to "*this*," and when used as an adverb, to "*thus*." Also, therefore, though by long use it has become a conjunction, may properly be regarded as an elliptical phrase, meaning "wholly thus," or "in like manner."

If.

We come now to the *continuative* conjunctions, that is to say, those which not only connect sentences and their meanings by coupling them together, but mark a dependence of one on the other; and this, first as suppositives—IF is called by Mr. Harris a *suppositive* conjunction: some other grammarians term it a *conditional*; but however it may be designated, the general force and effect of such a conjunction is obvious in most languages. It serves to mark the certain dependence of one event on another, without asserting the absolute existence of either. We therefore intimate, that if the one *be* the other must be its necessary result, that *when* we are sure of the one, then we may reckon upon the other also; or that the former being *given* as a datum, the latter follows by the power of reasoning. Hence the Greek εἰ, and the Latin *si* merely expressed *being*; for εἰ is part of the verb ἐω or εἰμι, and *si* is part of *siet* or *sit*. The power of the conjunction εἰ is thus elegantly illustrated by Plutarch, according to the free translation of the old English folio: "In logike this conjunction EI (that is to say *if*, which is so apt to continue a speech and proposition) hath a great force, as being that which giveth forme unto that proposition, which is most agreeable to discourse of reason and argumentation. And who can deny it? considering that the very brute beasts themselves have in some sort a certeine knowledge, and true intelligence of the subsistence of things; but nature hath given to man alone the notice of *consequence*, and the judgement for to know how to discerne that which followeth upon every thing. For that it *is* day, and that it *is* light, the very woolves, dogs, and cocks

perceive; but that *if* it be day, of necessitie it *must* make the aire light, there is no creature, save onely man that knoweth." The Greek or Latin construction therefore is "*be it* that there is day there must be light." Again, the German conjunction answering to our *if* is *wenn*, which also signifies *when*. Hence the expression, "*Wenn* man dich fragt, so antworte," which signifies "*if* any one asks you, answer thus," may be rendered with little difference of meaning, "*when* any one asks you, answer thus." Lastly, the English *if* is plainly in signification *give*; and hence Skinner's etymology of it has never been disputed. He says, "If (in agro Linc. *gif*) ab A. S. *gif*, si. Hoc a verbo *gifan*, dare, q. d. *dato*." Tooke justly adds that *gif* for *if* is to be found not only in Lincolnshire, but in all our old writers. It must be observed that the same letter was variously pronounced *g* and *y* in different dialects, as *gate* and *yate*, *give* and *yeve*. It is also to be observed that the participle *given* (approaching still more nearly to Skinner's *dato*) was used as well as the imperative give; and from these two sources we have for the conjunction *geve*, *gef*, *giffe*, *gif*, *yive*, *yef*, *yif*, *yf*, *if*, and *gin*: which may be still further illustrated by tracing the verb, participle, and substantives, *gyffe*, *yive*, *yeve*, *yave*, *gaff*, *yer*, *yth*, *yeft*, *yifte*, *gytys*, *yever*, *yevour*, *yeven*, *yevyn*, &c; as in the following examples:

Hartely myght thei warry me,
That of ther gud had ben so fre,
To *gyffe* me and to sende.

Sir Amadas.

Sir Amis answerd tho
Sir, therof *yive* Y nought a slo
Do al that thou may.

Amis and Amiloun.

Not Avarice the foule caytyfe
Was halfe to grype so ententyfe,
As Largesse is to *yeve* & spende.

CHAUCER.

And with hys hevy mase of stele
There he *gaff* the kyng hys dele.

Richard Coer de Lion.

And truely in the blustring of her looke, shee *yave* gladnes & comforte sodainly to all my wittes. CHAUCER. *Test. Lov.*

The remedy by the seid estatutes is not verray perfite nor *yevyth* certeyn ne hasty remedy.

Stat. 11. *Hen. VII.* c. 22. *MS.*

He *gaf gyftys* largelyche
Gold & syluer & clodes ryche.

Launfal Miles.

For gret *yeftys* that she gan bede,
To loode the schypmen goone her lede.

Octouian Imperator.

Every astate, feoffement, *yeft*, release, graunte, lesis and confirmacions of landys. *Stat.* 1. *Rich. III.* *e.* 1. *MS.*

Provided that this acte—extend not—to any graunte or grauntes, *yeft* or *yiftis* had or made by the kinges letres patentes to the same Anthony. *Stat.* 11. *Hen. VII.* c. 31. *MS.*

Ayenst the sellere, feffours, *yevours* or grauntours and his or their heires. *Stat.* 1. *Rich. III.* c. 1. *MS.*

That no artificer ne laborer herafter named take no more ne gretter wagis then in this estatute is lymytted, upon the payne assessed as well unto the taker as to the *yever*.

Stat. 11. *Hen. VII.* c. 22. *MS.*

Which lawe by negligence ys disused, and therby grete boldnes ys *goven* to sleers and murdrers. *Stat.* 3. *Hen. VII.* c. 2. *MS.*

Yeoven under our signet. *Q. Elizabeth, Let. to Sir W. Cecil.*

If the seid lessee or lesses within viii daies warnyng to theym *yeven* by any of the seid justices of the peas.

Stat. 11. *Hen. VII.* c. 9. *MS.*

Or yit *gewe* Virgil stude well before. GAWIN DOUGLAS.

Eorthliche knyght, or eorthliche kyng
Nis so swete in no thyng;
Gef he is God, he is mylde.

Kyng Alisaunder.

He askyd at all the route,
Gyff ony durste com and prove
A cours for hys lemannes love.

Richard Coer de Lion.

For *giff* he be of so grete excellence,
That he of every wight hath core & charge,
Quhat have I gilt to him, or doon offense?

K. JAMES I. *The King's Quair.*

The domes and law pronouncis sche to thaym then,
The feis of thare laubouris equalye
Gart distribute. *Gif* d not fallis thareby
Be cut or cavill that plede sone partid was.

GAWIN DOUGLAS.

Ich am comen hider to day,
For to sauen hem, *yive* Y may.

Amis and Amiloun.

Yef thou me louest ase mon says,
Lemmon as y wene;
Ant *yef* hit thi wille be
Thou loke that hit be sene.

MS. Harl. No. 2253, fol. 80.

Wurthe we never for men telde,
Sith he hath don us thys despyte,
Yiffe he agayn passe quyte.

Richard Coer de Lion.

He thought *yif* ich com hir to,
More than ichaue ydo,
The abbesse wil souchy gile.

Lay Le Freine.

The lawe of the land ys that *yf* eny man be slayne in the day, and the felon not taken, the townshipp wher the deth or murder is done shal be amerced. *Stat.* 3. *Hen. VII.* c. 2. *MS.*

Gin living worth cou'd win my heart,
You wou'd na speak in vain.

Scots Song.

These words *geve*, *gef*, *gyff*, *giff*, *gif*, *yive*, *yef*, *yiffe*, *yiff*, *yif*, *yf*, which in the last eleven examples are conjunctions, are doubtless the same in origin with the preceding verbs *geve*, *yeve*, *gyffe*, *gaff*, *yave*, *yevyth*, and the nouns *giftys*, *yeftys*, *yeft*, *yiftis*, *yevours*, *yevers*; and in like manner the conjunction *gi'n* is clearly nothing more than a new application of the participle *goven*, *yeoven*, or *yeven*, which is the modern *given*. But this new application causes the words *if*, *gif*, *gi'n*, &c. to express a new "posture, stand, turn, or thought of the mind," (as Mr. Locke speaks) and thus to perform a different function in language, or become a different "part of speech," namely, a conjunction. Mr. Tooke therefore is right so far as he follows SKINNER, who first showed the connection between *if* and *give*: but he is wrong, when, trusting to his own theory, he says "our corrupted *if* has always the signification of the English imperative *give*, and *no other*." In short he is right where he is not original, and original only where he is not right. Nor is his "additional proof" of much relevancy. "As an additional proof," says he, "we may observe, that whenever the *datum* upon which any conclusion depends, is a sentence, the article *that* if not expressed is always understood, and may be inserted after *if*; as in the instance,—

'——— My largesse
Hath lotted her to be your brother's mistresse,
Gif shee can be reclam'd; *gif* not, his prey.'

Sad Shepherd. act. 2. sc. 1.

the poet might have said,

'*Gif that* she can be reclaimed, &c.'

Grammar. But the article *that* is not noderstood and cannot be inserted after *if*, where the *datum* is not a sentence but some noun governed hy the verb *if* or *give*. Exam. 'How will the weather dispose of you to-morrow? *If* fair, it will send me abroad, &c.'"

Now the whole of this observation turns on the peculiar idiom of the English language, which admits one form of ellipsis and not another; for all these constructions are elliptical; and the word *that*, which is a conjunction as well as *if*, has not the least pretension in such sentences to he called an articie. We shall have occasioo hereafter to notice some other uses of this conjunction, when we speak of the phrases *O! si—O! gi'n, an if, as if*, &c.

An. The conjunction *an*, is not mentiooed by SKINNER, JUNIUS, LYE, or any writer of note, before Dr. JOHNSON, whose account of it is perfectly unintelligible. He says it is "sometimes a eontractioo of *and if*;" sometimes a contraction of "*and* before *if*;" sometimes a contraction of "*as if*;" and to complete this jumble of inconsisteocies, he elsewhere says, "*and* sometimes sigoifies *though*, and seems a cootractioo of *and if*."—Aod again, "in *and if*, the *and* is redundant."

TOOKE, who bas justly reprebended the errors of Johnson, thus speaks of the word *an* himself "We have in English another word, which, though now rather obsolete, is used frequently to supply the place of *if*; as, "*an* you bad any eye bebind you, you might see more detraction at your heels, than fortune before you. *Twelfth Night*, act ii. sc. 8." Again, "*An* is also a verb, and may very well supply the place of *if*: it being oothiog else but the imperative of the Anglo-Saxon verb *anan*, which likewise means to *give* or *grant*."

This eonjectural etymology of Mr. Tooke's is plausible, though not perfectly satisfactory. The verb *anan*, to grant, Is of duhious authority. The supposed instances of its oceurrence are rare, and may possihly be accouoted for from casual errors in manuscripts. Few words are brought into use as secoodary parts of speech, which have not also a very gcoeral use as primary parts, and that in different dialects; but we have in vain sought to trace this verb *anan* as a verb or nnun in any dialect ancieot or modern, beyond the two or three doubtful instaoces cited by Mr. Tooke. We do not positively reject bis etymology, but we must own it appears to us quite as probable that '*an* is only a further corruption than *gi'n* from *given* or *yeven*; and this is the more probahle because '*an* seems never to have been used but in the colloquial dialect of homely life, or of distant provinces.

Thus, in *Much Ado about Nothing*, Beatrice, who affects a homely and somewhat coarse kind of wit, replies to the messenger as follows:—

MESS. I see, lady, the gentleman is not in your books.
BEAT. No; '*an* he were, I would burn my study.

So we find, in an old Scotch song—

'*An* thou wert mine ain thing,
O! I wou'd lo'e thee, I wou'd lo'e thee!

But no serious and polished writer at any period of our literature uses *an* for if; and at present it is oot only "rather obsolete," but has long been obsolete altogether.

The circumstance wbich tends to give the most plausibility to Mr. Tooke's etymology, is, that this word is often speit by old writers *and*, which may seem to be a contraction of *anned*, i. e. granted, if there be such a verb as to *an*. Conjunctions.

Hereafter, litel in a stounde,
Comen vp, out of the grounde,
Amonge the folk sodeynlich,
Grete foxes, and griselich—
Her bytt envenymed was,
Man ne beest non there nas,
And he were of hem ybite,
That he nas ded, God it wyte.
Kyng Alisaunder.

So in an old MS. In the puhlic library at Cambridge—

Ther is Levthe, Reythe, and Meythe:
Meythe ouerset Reythe for the defawte of Leythe;
Bot *and* Reythe methe com to Leythe,
Scholden neuer Meythe ouerset Reythe.

In *Gammer Gurton's Needle*, Diecon says,

It is a marrion crafty drab and froward to be pleased,
And ye take not the better way, your nedle yet ye lese it.

Lord BACON, also, thus writes—

It is the nature of extreme self-lovers, as they will set an house on fire, *and* it were but to roast their eggs.

Still, in the very unsettled state of our ancient orthography, much stress cannot be laid on this circumstance: and it seems bardly sufficient to outweigh the presumption agaiost the derivation from *anan*, arisiog from the want of correspondent nouns and verbs io all the Teutonic dialects.

Whatever be the true etymology of *an*, its grammatical force and effect are exactly the same as those of *if*.

Because, since, as. *Because, since*, and *as* are enumerated by HARRIS as *causal* conjunctions. We have already noticed the word *because* as n preposition. It was originally a phrase or combination of tbe words *by* and *cause*, and we sometimes find *by cause that* used in old writers; e. gr.

On me no fatnesse wilbe seene,
By cause that pasture I fynde none.
Ballad of Chichevache, MS. Harl. 2251.

In modern use it commonly signifies a cause preeedent; but formerly it appears to have been applied to denote the *final* cause, or object of an action.

The word *since* will afford scope for more particular observation. Dr. JOHNSON, though he calls *since* an adverb, bas given the following iostances of its use evidently as a conjunction.

1. "From the time that"—

He is the most improved mind, *since* you saw him, that ever was, without shifting into a new body. POPE.

2. "Because that"—

Since the clearest discoveries we have of other spirits, besides God and our own souls, are imparted by revelation; the information of them should be taken from thence. LOCKE.

Mr. TOOKE says "*since* is a very corrupt abbreviation confoooding together differeot words, and differeot combinations of words;" and he afterwards classes the different uses of this word under four heads, viz.—

1. (As a preposition) for *siththan, sithence*: or *seen* and *thenceforward*.

2. (As a prepositioo) for *seand*, seeing as, or *seeing* that.

3. (As a conjunction) for *seand*, seeing, seeing as, or seeing that.

4. (As a conjunetion) for *siththe, sith*, seen as, or seen that.

And he adds in a note, "it is likewise used adverbially; as when we say—it is a year *since*; i. e. a year *seen*." In short, Mr. Tooke contends that it "is the participle of *seon*, to see."

We conceive that a little investigation will show this etymology to be entirely erroneous. There are in English two causal conjunctions, which, as such, have nearly the same force and effect, viz. *since* and *seeing*; the latter speaks for itself; the former requires to be traced to its source.

We say then, that *since* is a contraction of *sith thence*, or *sithens*, the root of which latter is the word *sith* or *sithe*: and we have before shown that *sithe* is identical with *tide*, which in German is pronounced *zeit*, in Frankish *zit*, and *citi*, and was probably the origin of the Latin *cito*, and in all these words the common idea expressed is *time*.

Now, as the noun *while*, which also signified *time*, came to be used *adverbially* in the forms of *while*, *whiles*, *whilst*, to signify the time during which an action continued, so the noun *sith*, time, in the forms of *sith*, *sithen*, *sythyn*, *seththen*, was used adverbially to signify the time from which an event was to be reckoned.

This adverb, like most others of a similar construction, came next to be employed prepositionally and conjunctionally, with the same reference to time past.

Finally, as the effect commonly succeeds the cause in time, *sith* came to be used as a causal conjunction, either distinctly referring to time, or without such distinct reference.

The different stages in this progress we shall proceed to illustrate, by adducing examples of the use of *sith* and its derivatives.

1. As a *noun*, signifying *time*.

> When he him segh'th, than was he blithe,
> And kest him wel *mani a sithe*. — *Seuyn Sages.*
>
> And such he was iproued *ofte sithes*. — CHAUCER.
>
> For thi was Tristrem oft
> To court cleped *fele sithe*. — *Sir Tristrem.*
>
> For nede now wo is me,
> Said Tristrem *that sithe*. — *Ibid.*
>
> Thai underfengen him with cher blithe,
> And thonged him *a thousand sithe*. — *Seuyn Sages.*

2. As an adverb, signifying *afterwards*, i. e. at a *time* subsequent.

> And is *sith* sogne dele changed. — TREVISA.
>
> The letter told him all the deed,
> And he unto his men gart read;
> And *sithin* said them sickerly,
> I hope Thomas his prophecy
> Of Ersiltoun verifyd be. — BARBOUR.
>
> Ae Alisaunder, his owen honde,
> Biheueded the prince of the londe,
> And *sithen*, withouten any pyté,
> Sette on fyre that cyté. — *Kyng Alisaunder.*
>
> He tok that blod that was so bright,
> And alied that gentil knight,
> That euer was hende in hale,
> And *seththen* in a bed him dight. — *Amis and Amiloun.*
>
> He gaffe ther ryche gyftes,
> Both to sqwyars and to knyghtes.
> Stedes, hawkes, and bowndes:
> And *sythyn* apon a day
> He buskyd hym on hys jornay. — *Sir Amadas.*

3. As a conjunction, simply signifying "from the *time* that."

> To his ostage she went right,
> There she nyver come byfore,
> *Sithe* his stedis harborowed thore. — *Lyfe of Ipomydon.*
>
> *Sethe* Normans came first into Engelonde. — TREVISA.
>
> Nas non so holy prophete
> *Seththe* Adam and Eue the appel ete. — *Christ's Descent to Hell.*
>
> *Siththe* that I was born to man,
> Swylke sorwe hadde I never nan. — *Richard Coer de Lion.*

4. As a conjunction, signifying "from the time that," with the farther idea of causation.

> *Sith* so is that sinne was first cause of thraldome, *than* (i. e. then) is it thus; that at the *time* that all this world was in sinne, *than* was all this world was in thraldome. — CHAUCER, P. T.
>
> For *sith* the daie is come that I shal die,
> I make plainly my confession. — CHAUCER, Kn. T.

5. As a conjunction, in relation to cause only.

> The wise eke saieth wo him that is alone,
> For and he fall he hath none help to rise,
> And *sith* thou hast a felow, tell thy mone. — CHAUCER.
>
> *Sithe* in thi support myn hope abidith al. — LIDGATE.
>
> And therefore madame, if your wil be,
> *Sithe* we have so grete plenté,
> Sende hym somme, while we may. — *Lyfe of Ipomydon.*

In the Scottish dialect, we find *sithin*, *syne*, and *sen*. *Sithin* we have already cited from Barbour. *Syne* appears to be a contraction from *sithin*, or *sithen*, used adverbially, and in contradistinction to a *time* preceding.

> He busked him, bot mair abade,
> And left purpois that he had tane,
> And to England again is gane,
> And *syne* to Scotland word sent he. — BARBOUR.
>
> By processe and by menys favourable,
> First of the blisful goddis purveyance,
> And *syne* throu long and trewe contynuance
> Of veray faith. — *The King's Quair.*
>
> Till first ae caper, *syne* anither,
> Tam tint his reason a' thegither. — BURNS.

Lang syne, long since, a time long past, is an expression well known from the admirable song of *Auld lang syne*. *Sen* may possibly have been the past participle *seen*, used as a causal conjunction, in the same manner as we employ the active participle *seeing*.

> Giff ye be warldly wight that dooth me sike,
> Quhy lest God mak you so, my derest hert,
> To do a sely prisoner thus smert,
> That lufis you all, and wote of nought but wo,
> And therefore merci suete! *sen* it is so. — *The King's Quair.*

Sensyne, a compound of *sen* and *syne*, is used adverbially, as in the Scottish translation of the *Romance of Alexander*, A. D. 1438.

> *Sensyne* is past ane thousand yeir,
> Four hundred and threttie thairto neir,
> And auch, and some dele mair I wis.

So in the Act of the Scottish Parliament, A. D. 1540—

> All his gudis movable and vnmovable pertening to him, the tyme of the committing of the said cryme, and *sensyne*, to be decernit to pertene to His Grace.

We now come to the word *as*, which HARRIS reckons among the *causal conjunctions*, ex. gr.

Grammar.

As when the moon hath comforted the night,
And set the world in silver of her light—
So, when the glories of our lives, &c.
CHAPMAN.

Here we see that *as* marks an analogical connection between one set of incidents and another. The first set are assumed to be well known and certain, the latter to be equally true but less obvious. Whether the term *causal* be strictly applicable to this sort of analogical connection may perhaps be doubted; but inasmuch as the certainty in both instances is first stated, *because* and *as* may properly enough be distinguished by a common appellation from *therefore* and *so*, which mark the less obvious or certain of the two facts.

Mr. TOOKE however seems to deny that *as* is a conjunction. His words are, "the truth is that *as* is also an article; and (however and whenever used in English) means the same as *it*, or *that*, or *which*." Why he calls it an article we know not; for in another part he says, "I should be sorry if any of my readers were—to believe—that articles and pronouns are neither nouns nor verbs—for I hope *hereafter* to satisfy the reader that they are nothing else, and can be nothing else." He afterwards published another volume on grammar; but though it contains a long chapter on "the Rights of Man," it has none on either article or pronoun. We are therefore left in the dark, as to Mr. Tooke's opinion of the word *as*; and know not whether he thought it a noun or a verb; why, being either, he called it an article; and why, if it could at once be either a noun or a verb and an article, it could not also be a conjunction.

In its etymology indeed Mr. Tooke is certainly right; *as* is the German *es*, it; and as we have elsewhere had occasion to observe, the same word which signified identity, by an easy transition came to signify likeness; and hence we often find in our ancient style the word *like*, either prefixed pleonastically to *as*, or else used with a corresponding force. Of the former we have an instance in *Psalm* ciii. 13.

Like as a father pitieth his children; so the Lord pitieth them that fear him."

The poet S. DANIEL furnishes an example of the latter kind.—

O! thou and I have heard, and read, and known,
Of *like*-proud states, *as* woefully incumbered,
And framed by them examples for our own,
Which now among examples must be numbered.

We use *so* as a relative to the antecedent *as*, or as an antecedent to the relative *that*; and *so* (as Mr. Tooke justly observes) is the Gothic *sa*, or *sa*, it or that; but *so* by some of our old writers was used where we now use *as*.

Bulsiful neied so loude
That it schrillith into the cloude—
Ac Alisaundre leop on his rugge,
So a goldfynch doth on the hegge;
Hit monteth, and he let him gon,
So of bowe doth the flon. *Kyng Alisaunder.*

In the German translation of the Bible, *so* is sometimes used as the relative pronoun *who*, in the same manner as we employ the pronoun *that*.

Alle Juden *so* in Ægyptenland wohneten.
All the Jews which dwell in the land of Egypt.
JEREMIAH, c. 44. v. 1.

Als is also used in the Anglo-Saxon and old English for *as*; and this word likewise is correctly explained by Mr. Tooke, as "a contraction of *al* and *es*, or *as*." —"This *al*," adds he, "which in comparisons used to be very properly employed before the first *es* or *as*, but was not employed before the second, we now in modern English suppress." It would not be quite correct to say that *als* was *never* employed before the second *es* or *as*; for examples of it sometimes occur.

Conjunctions.

Vnto the toure he takes the way
Als hastily *als* euer he may. *Seuyn Sages.*
Vntil the kirk than went he sone
And herd his mes, *als* he was wone. *Ibid.*

From *as* we naturally pass to the word *that*, which is also a pronoun conjunctionally used. It is rather singular that any difficulty should ever have occurred, respecting either this word, or the corresponding Latin words *quod* and *ut* or *uti*. Mr. Tooke says, "*that* is the article or pronoun *that*;" in which he seems to have copied VOSSIUS, who says, "*quod* pronomen est, etiam cùm dico, *gaudeo* QUOD *veneris*; vel illo Horatii, lib. 1. sat. 4." That.

—— Incolumis lætor *quod* vivit in urbe.

Nam integrè sit, gaudeo *eo* nomine, vel lætor ob *id*, sive propter id negotium, *quod est* te venisse.

That *quod* may be used as a pronoun is no reason why it should not also be used as a conjunction: and its use is what determines its grammatical character. *Ut* seems to have been an abbreviation of the later Romans from *uti*, and is manifestly the Greek conjunction ὅτι, which HOOGEVEN justly remarks is formed by uniting the pronouns ὅ and τι.

Mr. HARRIS calls *therefore* a *collective* conjunction, meaning that it subjoins an effect to a cause, e. gr. "The moon intervenes; *therefore* the sun is in eclipse." And he observes, "we use *causals* (such as *because*) in those instances where the effect being conspicuous we seek its cause; and *collectives* in demonstrations, and science properly so called, where the cause being known first, by its help we discern consequences. Our English word *therefore* is manifestly a phrase, or combination of words reduced by custom into one; like the Latin *propterea*, which for this reason Vossius excludes from the class of conjunctions.—"*Quamobrem, quasobres, propterea, quare*, et similia," says he, "non videntur hujus esse classis; quia non tam vox unica sunt, eaque composita, quàm plures: cui rei argumento nobis est, quòd structura, quæ in simplici voce locum non habet, in earum singulis observatur. Et vix caussa apparet cur *quamobrem* magis sit vox unica, quàm *eam ob rem*; vel *quare* quàm *ed re*." The latter part of this reasoning does not strictly apply to the English *therefore*, and even admitting it to be correct we may still call that word a conjunction. Its meaning, as we have elsewhere had occasion to show, is simply *for this* (subanditur *cause* or *reason*:) and it has two conjunctional meanings; first when we state the effect as a matter of fact; and secondly when we state it as a matter of reasoning. Therefore, wherefore, then.

1. This is the latest parley we will admit,
Therefore to our best mercy give yourselves.
SHAKSPEARE.

2. He blushes, *therefore* he is guilty.
Spectator.

The blush is not the cause of the guilt in fact; but it is the cause of our asserting the person to be guilty.

The statement would be the very reverse, if the fact alone were considered; for we should then say, "he is guilty, therefore he blushes;" but the full construction in the other sense is, "he blushes, *therefore* I conclude that he is guilty."

Wherefore is so similar in construction and effect to *therefore*, that it needs no further explanation.

Then, used as an adverb, signifies *at that time*, but used as a conjunction it not only has that meaning, but in a secondary sense it means "in consequence."

> 1. ——————— My brother's servants
> Were *then* my fellows, *now* they are my men.
> SHAKSPEARE.
>
> 2. *If* all this be so, *then* man has a natural freedom.
> LOCKE.

Either, neither, or, nor, else.

We call *either* and *neither*, *or* and *nor* simple disjunctives, in conformity with the scheme of HARRIS above particularised; but they might perhaps be more appropriately styled alternatives; *either* and *or* being set in opposition to each other affirmatively; *neither* and *nor* negatively. *Either* is clearly in origin a pronoun; and *or* is a contraction of *other*, which is also a pronoun. In old English *other* frequently occurs at length, in the sense of the modern *or*.

> ——— Ful feole and fille
> Beoth yfounde, in heorte and wille
> That hadde levere a ribaudye
> Than to here of God, *other* of seynte Marie.
> *Kyng Alisaunder.*

In a charter of king EDWARD the Confessor we have *oth* for *or*.

> Swo ful and swo forth, swo Duduc Bissop *oth* suy Bissop hit firmest him toforen havede.

The conjunction *or* is frequently followed by *else*. As *nor* is by *yet*. The word *else*, Mr. Tooke says, is "the imperative *ales* of the (Anglo-Saxon) verb *alesan* to dismiss." The learned HICKES, however, thinks it is contracted from the Latin *alias*: and of this opinion, which appears to us the more probable, are SKINNER and MINSHEW. It occurs both in the Scottish and English idioms, and is written *els*, *elles*, *ellis*, *ellys*, &c.

> To take where a man hath leue
> Good is: and *elles* he mote leue. GOWER.
>
> What man that in special
> Hath not him selfe he hath not *els*
> No more the perles than the shels. IDEM.
>
> Withouten noyse or clatteryng of belles
> Te Deum was our songe, and nothyng *elles*.
> CHAUCER.
>
> Him behoueth serve himselfe that has no swayn,
> Or *els* he is a fole, as clerkes sayn. IDEM.
>
> Traist not all talis that wanton wowaris tellis
> You to defloure purposyng, and not *ellis*.
> GAWIN DOUGLAS.
>
> Frehold withyn the same shires to the yerely value of xxs at the leste, or *ellys* londes and ten^ts holdyn by custume of manere.
> *Stat.* 1. *Ric. III.* c. 4. *MS.*
>
> As though they lacked wysedome and learnyng to be able for such offices, or *elles* were no men of conscience, or *els* were not meete to be trusted. LATIMER'S *Sermons*, Ed. 1562.
>
> Than may ye haue baith quaiffis and kellis
> Hich candle ruffes and barlet bellis
> All for your weiring and not *ellis*.
> *Philotus*, Edinburg ed.

Mr. Tooke very angrily accuses his critics of "ignorance and idleness," because they venture to suggest that *el* or *al* (signifying *other*) is the radix of the English *else* and the Latin *alius*; but certainly WACHTER was neither idle nor ignorant: and yet he has traced this radix, with a similar signification, through a great variety of languages. The passage is a very curious one, and well deserves attention.

"EL, *ell*, alius, alienus, peregrinus. HENISCHIUS in *Thes. L. Germ. el*, alius, *jemand el*, alter quispiam, alius quispiam, *niemand el*, nemo alius. Vox Celtica et primitiva, quæ Græcis effertur ἄλλος. Lat. *alius*. Inde composita et derivata in omnibus dialectis, et præcipuè."

"CAMBRICA, *aliwn*, alienus, *alon* alieni, inimici, *alltud* alienigena, advena, *alltudo* in exilium pellere, *altudaeth* exilium, *allwlad* alienigena, *arallu* alterare, *ellmyn* Alamanni, et usurpatur pro peregrino quovis. BOXHORN in *Leg. Ant. Brit.*"

"GOTHICA, *aljath* alio, aliorsum, peregre, *aljathro* aliunde, *aljakunja* alienigena, apud JUNIUM in *Gloss. Goth.* p. 49"

"ANGLO-SAXONICA, *elles* alias, alioquin, *elles—hwær* aliorsum, *eltheodig*, *ælitheodig*, exterus, extraneus, peregrinus, *elreordig*, barbarus, apud SOMN. et BENSON. Quibus addi potest *eltheodiscemen* peregrini, ex *Matth.* xxvii. 7."

"FRANCICA, *allasuuara*, alio, in Gloss. Pez. *eliporo*, alienigena, in *Gloss.* Boxh. *elirarter* barbarus, in *Gloss.* R. Mauri."

"ALAMANNICA, *allesuuanan*, aliunde, in *Gloss.* Keron."

"ISLANDICA, *ella*, alias, apud VEREL. in *Ind.*"

"ANGLICA, *else*, alius, alias, aliter, alioqui, *elsewhere*, alibi."

"GERMANICA hodierna, *alfanz* aliena loquens, *elgötze*, Idolum peregrinum, *elend* terra aliena, *buffel* bos peregrinus, &c."

Wachter goes on to cite the proper names derived from this root, as *Allobroges*, *Alamanni* and *Aliso*.

Neither and *nor* are merely *either* and *or* with a negative particle prefixed to them. To these two distinct words the Latin *nec*, or *neque*, answers when repeated. Vossius speaking of the passage "*neque* natare *neque* literas novit," says, "*neque* magis negandi adverbium est quàm conjunctio." In this position we cannot acquiesce, and indeed his subsequent argument shows that he had some doubt on the point himself. "Certè," says he, "in eâ particulâ duo sunt, τὸ *ne*, quod negandi adverbium est, et τὸ *que* quod copulativa est conjunctio. Utrumque munus præstat *neque*: ac quatenus negat, adverbium est: quia verò et disjunctas connectit sententias, quodammodo conjunctio est." We cannot but think that a little reflection would have shown this very acute and judicious grammarian, that, under such circumstances as he describes, a word becomes not merely *quodammodo*, but plainly and altogether a conjunction.

Both.

To the simple disjunctives *either*, *or*, and *neither*, *nor*, are opposed the simple connectives *both*, *and*. It is sufficient to observe that as *either* and *or* are pronouns used conjunctionally, so is *both* a pronoun employed in the same manner, and consequently converted into a real conjunction.

But, &c.

Of the etymology of *but* we have already spoken at length. It belongs to that class of conjunctions which Harris calls the *adversative absolute*. In these a positive and a negative are both asserted. We have a remarkable instance of this in MILTON, who reduplicates the conjunction *but* in application to

two different kinds of opposition in the same sentence.

> Virtue may be assailed *but* never hurt;
> *But* evil on itself shall back recoil,
> And mix no more with goodness.

Whether this reduplicated construction be a beauty, or a blemish, in style, we shall not here inquire: we only cite the passage to show the effect of the conjunction *but*, which in both cases is as above stated.—1. It is positively asserted that virtue may be assailed, and negatively asserted that virtue cannot be hurt. 2. It is negatively asserted that virtue cannot be hurt, and positively asserted that evil shall recoil on itself; i. e. shall be hurt. In the one case, the subject remains the same, but the predicates vary; in the other, the subjects are opposed to each other, but the predicates are, if not identical, at least equivalent.

Ac, which was probably identical with *eac*, *eke*, and originally signified *also*, is found in old English writers, for *but*; e. gr.

> With wraththe to Alisaundre he saide,
> "Quik tak thy wed for thy deth."
> Alisaundre, "Nay" onswerith
> Wed no schalt thou have of me,
> *Ac* Y wol have wed of the.
>
> *Kyng Alisaunder.*

Nor is this surprising, since the French *mais* and Italian *ma*, but, are merely the Latin *magis*, more; and therefore originally signified mere addition, without opposition.

Than. *Than* and *as* (which latter we have already considered as a causal) are reckoned by Harris among adversatives of comparison, the former implying superiority, the latter equality, as, "Nireus was fairer *than* Achilles."—"Virgil was not so great *as* Cicero." It is clear, therefore, that these words having a relative force, must be preceded either by some separate word, as an antecedent, or by some inflection which has the force of an antecedent. In the first of the examples just quoted, the comparative termination *er* renders the word *fairer* the antecedent of the relative *than*: in the second examples *so* acts as an antecedent to the relative *as*. The antecedents, when consisting of separate words, are commonly called adverbs, and properly so, inasmuch as they modify an adjective or another adverb. The relatives are also called adverbs by many grammarians, but improperly since they obviously connect sentences. It is of course matter of mere idiom whether the comparison be effected by an inflection in the antecedent, in the relative, or in both; or whether it be effected by a separate word, but in the latter case we call the relative a conjunction.

It is also matter of idiom whether the same conjunction answer one or several purposes. Thus the Latin *ac* and *atque*, which in their first sense are mere copulatives, become adversatives of comparison in such phrases as *æquè ac*, *æquè atque*, *aliter ac*, *aliter atque*.

> Somnia dormienti, non *æquè ac* vigilanti probantur. CICERO.
>
> Quæ beneficia *æquè* magna non sunt habenda *atque* ea quæ judicio, consideratè, constanterque delata sunt. IDEM.
>
> Ego isti nihilo sum *aliter ac* fui. TERENTIUS.
>
> Nunquam te *aliter atque* es in animum induxi meum. IDEM.

So we use *as*, with the force of a causal conjunction, or of a relative conjunction, or of the antecedent to such relative; as in the sentence, "Cæsar was *as* brave, *as* Alexander."

So in Greek, "the simple disjunctive ἢ, or *vel*," as HARRIS observes, "is mostly used indefinitely, so as to leave an alternative. But when it is used *definitely*, so as to leave no alternative, it is then a perfect disjunctive of the subsequent from the previous; and has the same force with καὶ οὐ, or *et non*. It is thus GAZA explains that verse of *Homer*"—

> Βούλομ' ἐγὼ λαὸν σόον ἔμμεναι, ἢ ἀπολέσθαι

"That is to say, *I desire the people should be saved* AND NOT be destroyed; the conjunction ἢ being ἀναιρετικὸν or *sublative*. It must however be confest, that this verse is otherwise explained by an ellipsis, either of μᾶλλον, or ἀντὶ, concerning which, see the commentators."

The grammarians seem to have doubted to what class they should assign most of these words. Thus PRISCIAN in one place calls *quàm* an *elective conjunction*, but in another an *adverb of comparison*. PLINY, according to Charisius and Diomedes designated these words generally as *conjunctiones relativæ ad aliquid*. VOSSIUS says "multa esse fatemur, quibus et in adverbiis et hic (sc. in conjunctionibus) rectè tribuatur locus—*Ac* et *atque* conjunctiones sunt, cùm dico *Brutus* AC vel ATQUE *Cassius*: adverbia sunt in isto *aliter facit* AC *tu*, vel ATQUE *tu*; nam idem valent ac adverbium comparandi *quàm*."

It is remarkable that all these words, *than*, *as*, *quàm*, ἢ were originally pronouns; for *than* and *then*, as has been observed, are the same word.—

> *Than* badde the douke ich vnderstond
> A chef steward of alle his lond.
>
> *Amis and Amiloun.*
>
> Hire swyre is whittore *then* the swon.
>
> *Ballad on Alisoun*, MS. Harl. No. 2253.

The French *que* is used both for our *than* and *as*, e. gr. *plus que*, "more than," and *autant que*, "as much as." In some provincial dialects of England they say "greater *as*" for "greater *than*;" and in the old Scottish dialect *na* or *nor* is used for *than*.

> Item it is statut that na man, of quhat estate degre or condicioun he be of, rydande or gangande in the cuntre, leide nor haif ma personis with him *na* may suffice him, or till his estate.
>
> *Scot. Act. Parl.* A. D. 1424.

> I levir haif evir
> A foul in hand or tway
> *Nor* sieand ten flieand
> About me all the day.
>
> *The Cherrie and the Slae.*

Unless. SKINNER has given two etymologies of the conjunction un*less*. He says, "*unless*, nisi, præter, præterquam, q. d. *one less*, uno dempto seu excepto: vel potiùs ab *onlesan* dimittere, liberare, q. d. *hoc dimisso*." TOOKE adopts the latter etymology, only suggesting that it is from the imperative *onles* dimitte. It does not appear to us that there is any reason to believe that the Anglo-Saxon verb *onlesan* was ever used in this conjunctional manner: and we rather incline to think that the present word was originally *on less that*, a phrase adopted as a literal translation of the French phrase *à moins que*.

> But alway sister remembre that Charitie is not perfect *onles that* it be burninge. *Treatise of Charitie.*

The *Dictionnaire de l'Academie* says—

> A MOINS QUE, sorte de conjonction qui regit le subjonctif, et qui signifie, si ce n'est que. *Il n'en fera rien* A MOINS QUE *vous ne luy parliez.*

Grammar.

This explanation is confirmed by observing, that in the old Scottish dialect the phrase *les than* was used instead of our modern *unless*.

That na notaris maid nor to be maid be the imperouris autorite haue faith in contractis ciuile within the realme, *les than* he be examinyt be the ordinare and approuit be the kingis hienes.
Scot. Act. Parl. A. D. 1469.

I shall distroye hyr landis alle
Hyr men sle bothe grete and smalle
Hyr castelle breke and hyr toure
With strenghe take hyr in hyr boure
Lesse than she may fynde a knyght
That for hyr loue with me darre fight.
The Lyfe of Ipomydon.

Mr. Tooke however is not only very positive in the etymology which he has borrowed from Skinner, but is extremely angry at the critics who presume to question it. What he says further of this word and of *less*, *lest*, and *least*, we shall have occasion to consider hereafter.

Unless is called by Harris an adversative adequate, with reference to the prevention of an event. Mr. Tooke says this is "a gross mistake;" but as Mr. Harris had explained the terms *adequate* and *inadequate preventive* by analogy to *adequate* and *inadequate* cause, and had expressly added that "this distinction has reference to common opinion and the form of language consonant thereto," there was little ground for Mr. Tooke's objection. When we say, "Troy will be taken *unless* the Palladium be preserved," we mean to express an *opinion* that if the Palladium be preserved Troy will not be taken. That opinion however we do not assert as a fact: and the fact may eventually happen to differ from it, without any great impeachment of our judgment in calling *unless* an adversative adequate.

Except, which is manifestly the imperative mood of a verb used conjunctionally, agrees in effect with *unless*. Thus we might say, "Troy will be taken, *except* the Palladium be preserved."

But if is a conjunctional phrase used formerly in a like sense—

That noon of thoo merchaunts of Venice convey into this said realme any merchandises, *but yf* the same merchaunt and merchaunts bryng with every butte of Malvesy x bowstaves.
Stat. 1. *Ric. III.* c. 11. *MS.*

Without is also conjunctionally used in the same sense.

This realme is like to lacke bothe stuff of artillary, and of artificers of the same, *without* a provision of due remedy in this behalf be the more spedely founde. *Ibid.*

We find in another statute of the same date, (A. D. 1483) the phrase *but if the rather* employed, with the same signification,—

Wheruppon *but if the rather* a remedy be purveid by youre most noble grace, of werry likelyhode consequently shall ensue the destruccion of drapery of all this your seid realme.
Stat. 1. *Ric. III.* c. 8. *MS.*

Though.

Harris calls *though*, or *although*, an *inadequate* adversative, that is to say a conjunction uniting two sentences, one of which states an event or circumstance, and the other states another event or circumstance as inadequate to prevent the former; e. gr. "*Troy will be taken* ALTHOUGH Hector defend it," where the conjunction *although* serves to shew that Hector defends Troy with a view to prevent its being taken; but that this preventive is inadequate to produce the intended effect. We may, however, observe that the same conjunction is used, and by a just analogy to mark an apparent incongruity of qualities, where the possession of the one does not in fact preclude the existence of the other, as, "*though* brave, yet pious," "*though* learned, yet polite."

Conjunctions.

Mr. Tooke says "*tho'* or *though* is the imperative *thaf* or *thafig*, from the verb *thafian* or *thafigan* to allow." This is one of the few original etymologies of Mr. Tooke; and we must confess we think it more ingenious than sound. In a charter of William the Conqueror we find, "ic nelle *gethafian* thæt ænig man this ahrecan," which in the ancient Latin version is thus rendered, "ego nolo *consentire* ut aliquis istud frangat:" and the same clause occurs in two other charters, one of Henry I. the other of Henry II. in the latter of which the verb is spelt *gethavian*, i. e. *gethavian*. These examples seem to show that in the Anglo-Saxon language *thafian* or *thavian* signified to consent or permit, neither of which ideas has much in common with the meaning of the conjunction *though*. If this however had been the origin of the conjunction, we might expect to find it in Anglo-Saxon *thaf* or *thav*; but it is *theah*. We might also expect to find the *f* or *v* in the numerous other Teutonic dialects in which a similar conjunction or adverb occurs; but there is no such thing. Adelung, under the German word *doch*, says, "In Low Saxon this particle is sounded *doch* and *dog*, by Ulphila *thau*, by Ottfried *thoh*, by Willeram *doh*, in Anglo-Saxon *theah*, in Dutch *doch*, in English *though*, in Danish *dog*, in Swedish *dock*."

In old English and Scottish we find it written very variously, *thah*, *thau*, *thaugh*, *thoffe*, *thof*, *thocht*, and *thought*.

Richard *thah* thou be euer trichard
Tricchen shalt thou neuer more.
Song on battle of Lewes.

Ant for ir feirnesse, *thau* ho be comen of threlle,
Hire wedlac ne scal ho nout lesen all.
Vita Sanctæ Margaretæ.

Thaugh me slowe feole of heom,
They slowe mo of the kyngis men.
Kyng Alisaunder.

Thoffe Y owe syche too. *Sir Amadas.*

Thof men wolde alle the londe seche.
MS. Harl. 7333. fol. 125.

Bot *thocht* I failyeit of rhyming,
Forgif me for my will was gude.
Scottish Rom. of Alexander.

Thocht be na reson persane I mycht but fale
Quhat than the force of armis coud auale.
Gawin Douglas.

Thocht be remission
Haif for prodission,
Schame and suspission
Ay with him dwells. Dunbar.

The king—woll—that suche possession—reste and be—holy in the other persone—in like wise as *thought* he had never be enfeoffed. *Stat.* 1. *Ric. III.* c. 5. *MS.*

It is to be observed that Gawin Douglas and other Scottish writers spell *thocht*, the past tense of the verb, to *think*, exactly as they do this conjunction.

So that we *thocht* maist semelye, in ane feild,
To de fechtand ennarmed vnder schield.
Gawin Douglas.

But mid they soold sound thair retreit
Because they *thocht* them nae ways meit
Conducters unto me.
Alex. Montgomery.

Grammar.

Add to this that the Anglo-Saxon *athoht*, the Dutch *gedocht*, and the German *gedacht* all answer to our substantive *thought*; and upon the whole it may not be deemed improbable, that the words *though* and *thought* are of the same origin.

Thus the example, "Troy will be taken *though* Hector defend it," may be paraphrased, "even if it be *thought* or *supposed* that Hector defends Troy, even if this supposition be admitted, or be true in fact, still Troy will be taken."

In confirmation of this etymology we may observe, that the word *suppose* is often used in the Scotch dialect for *though*.

> Yone slae, *suppose* thou think it sour,
> May satisfie to slokkin
> Thy drouth now. ALEX. MONTGOMERY.

> Stories to rede ar delectabil
> *Suppois* that thay be nocht but fabil. BARBOUR.

For *though* were also used *albe*, *albeit*, *howbeit*, *all had*, *all should*, *all were*, *all gwe*, &c.

Yet, still.

Yet and *still* are conjunctions used in English as relatives to the antecedent adversatives *though*, *suppose*, &c.; e. gr

> *Though* Birnam wood be come to Dunsinane,
> *Yet* will I try the last. SHAKSPEARE.

> —— *Though* I do contemn report myself,
> As a mere sound, I *still* will be so tender
> Of what concerns you in all points of honour,
> That the immaculate whiteness of your fame
> Shall ne'er be sullied. MASSINGER.

The use of *yet* as a conjunction is directly taken from its use as an adverb; e. gr.

> ———— Tarry, Jew;
> The law hath *yet* another hold of you.
> SHAKSPEARE.

Here *yet* means, "*at this time*, after you have come, as you suppose to the end of the legal proceedings against you, in addition to these there remains another." So, in the above example where it is employed as a conjunction, *yet* means—"*at this time*, after Birnam wood has come to Dunsinane, and when no hope seems to lie in resistance, I will nevertheless resist."

The etymology of this word has already been considered in our chapter on adverbs.

As *yet* refers primarily to time, so *still* refers primarily to place, but secondarily to time. *Stelle* in German is place, and it answers to our *stead*, as *an meiner stelle*, "in my stead." ADELUNG, speaking of this word, says, "By Notker it is written *stal*; in Swedish it is *ställe*; in the Anglo-Saxon *stealle*, *steale*, in Low Saxon *stede*, in the Swiss dialect *stahl*." Hence the Anglo-Saxon, English, Frankish, and German adjective *still*, means primarily remaining in the same place, motionless; and consequently quiet; secondarily it means that which remains unchanged by the lapse of time, or which moves on equally with it.

> 1. Thy stone, O Sysiphus, stands *still*. POPE.
> Such silence waits on Philomela's strain
> On some *still* evening. IDEM.
> 2. It hath been anciently reported and is *still* received.
> BACON
> A generation of *still*-breeding thoughts.
> SHAKSPEARE.

Still as a conjunction is manifestly the adverb so employed, and the adverb is taken from the adjective: it is not easy to conceive a direct transition from the Anglo-Saxon imperative *stell*, to our modern conjunction. The analysis of the above example is, "I contemn report as far as it merely affects myself; but *at the same time* (and indeed *at all times alike*) I will be tender of your reputation."

Conjunctions.

In treating of *yet*, Mr. Tooke has very erroneously explained *algate* as "meaning no other than *all get*." The very example which he adduces might have taught him a different origin of this word.

> "For *albeit* tarieng be noyful, *algate* it is not to be reproued in yeuynge of iugement, ne in vengeaunce takyng." CHAUCER.

French having long been the fashionable language, previously to the time of Chaucer, the construction of his sentences is generally to be explained by reference to French idioms; and *algate*, which is literally *all way*, was undoubtedly a translation of the French conjunction *toutefois*.

> Et se ele eschaoit à autre—*tote vois* qe ele eust este ainsi donée—il donroit la fie au Roi de Angleterre.
> *Treaty, England and France*, A. D. 1259.

> Si ne poons à vostre priere entendre quant à ores; *totevoies*, pur ceo qe nous ne voliens mie q'il fuissent en nostre terre surpris de leur corn ne de leur biens, si lor avoms suffert de marchander. *Letter K. Edw. I.* A. D. 1304.

> TOUTEFOIS. Conjonction adversat: *neanmoins*, *mais*, *pourtant*. Tous les hommes recherchent les richesses, and *toutesfois* on voit peu d'hommes riches heureux. *Dict. de l'Academie*.

> Kyng Alisaunder leoseth many men
> Ac *allegate* the kynges
> Losen ten ageyns on in werrynges.
> *Kyng Alisaunder*.

Gate, as has already been explained, is the same as *gait* from the verb *go* or *gae*.

HARRIS notices another class of conjunctions which he says "may be properly called *adverbial conjunctions*, because they participate the nature both of adverbs and conjunctions—of conjunctions as they conjoin sentences; of adverbs as they denote the attributes of time and place." Such are *when*, *where*, *whence*, *whither*, *whenever*, *wherever*, &c. Upon the principles which we have adopted, these are to be called conjunctions when they conjoin sentences; but the name adverbial, is not at all distinctive, because many other conjunctions have occasionally an adverbial use; and many prepositions when used conjunctionally serve to mark time or place. The scheme of arrangement which Harris has followed, is principally directed to the logical connection of sentences; but the connections of *time* and *place* are merely physical, and should therefore form a class apart. The term *ordinative* which VOSSIUS applies to *deinde*, *postea*, &c. may not improperly designate this class.

Thus, among *ordinatives of time* we should reckon *whiles*, *till*, *o that*, *or*, *be*.

> His Lord nold he neuer forsake
> *Whiles* he ware oliue.
> *Amis and Amiloun*

> Full ofte drinkes shee,
> *Till* ye may see
> The teares run down her cheeke.
> *Gammer Gurton's Needle*.

> Al the day and al the nyht
> *O that* sprong the day lyht.
> *Geste of Kyng Horn*.

> Sathanas Y bynde the her shalt thou lay,
> *O that* come Domesday.
> *Christ's Descent to Hell*.

> He it is my dedly foo;
> He schal abeyen it *or* he goo.
> *Richard Coer de Lion*.

Grammar.

Your madynis than sall haue your geir
Put in gude ordour and effeir
Ilk morning *or* yow ryse. *Philotus*

The supper done than vp ye ryse,
To gang ane quhyle as is the gyse;
Be ye haue rowmit ane alley thryse
It is ane myle almaist. *Ibid.*

So, *where* is an *ordinative of place* in the following passage.

——————— He rails
Even there, *where* marchants most do congregate.
SHAKSPEARE.

Conjunctional phrases.

We have seen that several of the conjunctions, now considered as single *words*, were formerly *phrases*; such are *because*, *therefore*, *wherefore*, *quamobrem*, and *toutefois*; but there are many other *conjunctional phrases*, which have been more or less generally appropriated to the connection of sentences, such are the following in old English, Scottish, and French, *Howe be it—for als moche—at least waye—not forcing whether—contrariwise—insafer as—pur ceo qe—cest asavoir—and over that—coment que—how often, so often—no the less—neuerthelas—not for thi—nought gaynstandand—forfered that—set in cais—put the cais—forseing that*, &c. &c.

Howe be it, the kynge held styll his siege. BERNES'S *Froissart.*

Bot *for als moche* as sum micht think or seyne
Quhat nedis me apoon so lytill evyn
To writt all this; I answere thus ageyne.
The King's Quair.

This geare lacketh wethering; *at least waye* it is not for me to plough. *Bishop* LATIMER.

These words goe generally to all the king's tenants—*not forcing whether* he haue the reuersion by dyscent.
Sir W. STAUNFORD, A. D. 1590.

Contrariwise, certain Laodiceans and lukewarm persons think they may accommodate points of religion by middle ways.
BACON. *Essays.*

And decernis the saidis actis and euery ane of thame to be abolishit and extinct for euer, *insafer as* ony of the saidis actis ar repugnant and contrarie to the confessioun and word of God foirsaidis. *Scot. Act. Parl.* A. D. 1567.

E *pur ceo qe* aucunes gentz de nre Roiaume se doutent qe les aides &c. pussent turner en servage a eus a s leurs heirs avoms graunte pur nous et pur nos heirs qe mes ticles aides &c. ne treroms a custume. *Stat.* 25. *Edw. I.* c. 1. A. D. 1297.

Meismes les chartres en toutz leur pointz en ples devaant eus e an jugementz les facent alower, *cest asavoir* la grand chartre des franchises come ley commune, e la chartre de la forest solom l'assise de la forest. *Ibid.*

That—the same fyne be openly and solemply rad and proclaymed in the same court—*And* in the same tyme that it is so redd and proclaymed all plees cease; *and over that* a transcript of the same fyne be sent by the seid justices unto the justices of assisez. *Stat.* 1. *Ric. III.* c. 7. *MS.*

Il de common droit poit distreiner pur le rent aderere, *coment que* tiel done fuit fait sauns fait. LITTLETON, Sect. 214.

How often his eye turned to his attractiue adamant, *so often* did an vnspeakeable horrour strike his noble heart.
SIR P. SIDNEY'S *Arcadia.*

What answere thei bare the sothe can I not say
Na the les of fele this was the comon sawe.
R. DE BRUNNE.

Youe knowe, Lordes Syracusans, that we haue hytherto done in thys warre, as men of honestie: *neuerthelas*, leste there be anny that vnderstandeth not fullye the affayre, I wolle well declare yt vnto hym. NICOLLS'S *Thucydides*, fo. 191.

Was mad another statute, that non erle no baroun
No other lorde stoute ne frankeleyn of toun
Tille holy kirke salle gyue tenement rent no lond.
Not for thi he wille that alle religioun
Haf and hold in skille that gyuen is at resoun.
R. DE BRUNNE.

Item it is ordanyt that all craftis &c be distroyit &c *nocht gaynstandand* ony priuilegis or fredome geifyn in the contrare.
Scot. Act. Parl. A. D. 1424.

Conjunctions.

He slogh him sone that ilk day
Forfered that he sold oght say.
The Seuyn Sages.

With stout curage agane him wend I will
Thocht he in proues pas the grete Achill,
Or *set in cais* sic armour he weris as he,
Wrocht be the handis of God Vulcanus sle.
GAWIN DOUGLAS.

And *put the cais* that I may not optene
From Latyne land thaim to expell all clene,
Yit at leist thare may fall stop or delay. IDEM.

It may be ordered that ii or iii of our owne shippes do see the sayde Frenche soldiers wafted to the coast of France; *forseing that* our sayd shippes entre no hauen there.
Q. ELIZABETH *to Sir W. Cecil.*

It is plain, that these phrases operate, with relation to the sentences between which they show a relation, exactly in the same manner as the words do which we call conjunctions. A phrase is first abbreviated into its principal words, and these are again contracted into one short word. Thus the French *c'est asavoir* above quoted was probably first translated into English, "it is to know," or "it is to wit," whence we now have in our legal documents the abbreviated phrase, "*to wit*," as from the Latin *videre licet* comes *videlicet*, which we have adopted into the English language. These abbreviations and contractions are very arbitrary in their use; thus our ancestors in the fifteenth century used to say *where*, for that conjunction which we now express by *whereas*, i. e. *where that.*

Wher in a statute made in the xvij yere of the reign of King Edward the iiijth hit was ordeigned &c &c Please it *therfore* youre highnesse &c to ordeign.
Stat. 1. *Ric. III.* c. 6. *MS.*

The *ordinals*, which we have included in the class of pronouns, such as *first*, *second*, &c. necessarily imply connection, and consequently the adverbs formed from them, are easily employed with a conjunctional force, as *primò*, *secundò*, *tertiò*, when placed at the beginnings of sentences. The same also is to be observed of the adverbs used as relatives to these antecedents, such as *deinde*, *item*, *puis*, *next*, *syne*, *lastly*, &c. "*Deinde*," says VOSSIUS, "cùm verbo jungitur, ad circumstantiam temporis indicandum, adverbium est: conjunctio autem, cùm tantùm ad orationis juncturam pertinet."

Accepit condiitionem; *dein* quæstum occipit. TERENTIUS.

Pergratum mihi feceris; spero *item* Scævolæ, &c. CICERO.

Ils font estat d'aller à Orleans, à Blois, *puis* à Tours.
Dict. de l'Academie.

First an caper, *syne* anither. BURNS.

The adverbs *where*, *when*, &c. which we have heretofore shown to be pronouns in origin, have often the same conjunctional force, and in such case are properly to be reckoned conjunctions.

It remains to be observed that some conjunctions are used singly, and others in a succession of two or more. Thus we may say, "John *and* William came," or, "*both* John *and* William came."—"It is ordained that proclamation be made, *and* that the judgment be recorded, *and furthermore* that the record be transmitted." Where two or more succeed each other, there is a certain order in the succession; ex. gr. "*as—so*;" "*so—that*:" "*when—then*," &c. On this subject VOSSIUS thus speaks—"Conjunctioni

Grammar.

etiam accidit ordo; secundum quem aliæ sunt *præpositivæ*, ut *et*, *nam*; aliæ *postpositivæ*, ut *quoque*, *autem*; aliæ communes, ut *equidem*, *itaque*. *Igitur* sæpiùs postponitur. *Enim* etiam est particula præpositiva, Terent. *Phor.* act v. sc. viii. *Enim nequeo solus.*" Ad postpositivas etiam pertinent encliticæ. Ex his, *que* interdum alteri verbo jungitur quàm nativus verborum ordo exigebat: ut apud HORAT. lib. ii. od. 19.

———— Ore pedes tetigitque crura.

Pro *eruraque tetigit*. These however are matters depending on the particular idiom of each language, and not governed by the philosophy of general grammar.

The case is different with the pleonasms and cumulations of conjunctions. These occur in all languages, and they therefore clearly arise out of principles common to the human mind in different countries. Hence VOSSIUS speaks of *expletive* conjunctions—"*Expletivæ* sunt, quæ nullâ necessitate sententiæ, sed explendi tantùm gratiâ usurpantur. Ut quæ metri vel ornatûs caussâ inseruntur. SALLUST in *Catil.* *Verùm enimverò is demum mihi vivere, et frui animâ videtur*; ubi *verùm* redundat." VIRGIL in xii.

"——*Equidem merui nec deprecor inquit.*

Plena fuerit sententia, licet *equidem* tollas." To this head are to be referred such expressions as "*an if.*"

———————— Well I know
The clerk will ne'er wear hair on's face that had it.
———He will *an' if* he live to be a man.

Where either *an* or *if* is redundant; for they both signify the same, and Johnson is wrong in supposing that an' in this instance is a contraction of *and*.

VOSSIUS refers these redundancies to the custom of ancient writers, "nempe is *veterum* mos fuit, ut interdum conjungerent voces idem significantes." But they are not peculiar to any age or nation: they are the result of hasty and inconsiderate habits of speech, which, it is true, are more common in the first formation of a language, than in more cultivated and civilized periods of history.

Cumulation, however, is not always redundancy. Thus when we find a sentence beginning thus—"*but nevertheless if*," the conjunction *but* connects it with what goes before, and *if* with some subsequent sentence, and the word *nevertheless* alone may be called redundant, and yet not strictly so, since it adds a great force and emphasis to the word *but*.

In the Greek language, this cumulation of conjunctions is frequent; and is sometimes explained by an ellipsis. Thus HOOGEVEN says—"hoc modo ἀλλὰ νῦνγε redditur *nunc maximè*, suppressâ per ellipsin voculâ ἐίποτε." Ita SOPHOCL in *Electr.* v. 413.—

Ὦ Θεοὶ πατρῷοι συγγένεσθέ γ' ἀλλὰ νῦν!

O Dii patrii, adeste nunc maximè, vel nunc saltem!

Plenior structura est Ὦ Θεοὶ πατρῷοι, ἐίποτε συγγένεσθέ μοι, ἀλλὰ νῦνγε συγγένεσθε!—*O Dii patrii, si unquam aliàs mihi adfuistis, at nunc adeste saltem!*

And so much for the conjunction, which may be considered as the completion of the parts of speech necessary in any language to discourse, so far as it consists merely of *enunciative* sentences!

§ 9. *Of interjections.*

"The brutish, inarticulate *interjection*," says Mr. HORNE TOOKE, "which has nothing to do with speech, and is only the miserable refuge of the speechless, has been permitted, because beautiful and gaudy, to usurp a place among words." This is what we learn from Mr. Tooke, on the subject of interjections: and surely this is sufficiently inconsistent with itself, and with common experience. How can a class of words be at once *beautiful*, *gaudy*, *brutish*, and *inarticulate*? And what is meant by saying that the interjection, which somehow or other has been enabled to occupy a place among *words*, has nothing to do with *speech*, and is only the miserable refuge of the *speechless*? If some grammarians have reckoned inarticulate sounds among interjections, it is certain that far the greater part of the sounds so designated are not only articulate, but like adverbs, conjunctions, &c. may generally be traced to a distinct connection with nouns and verbs. VOSSIUS, speaking of CHARISIUS, says—"Malè idem huc refert *trit* quæ murum vox est, apud Nævium Corollariâ. Par ratio erit Aristophani Βρεκεκέξ, quæ vox est ranarum. Idem censendum de rei inanimæ sono, vel humano quidem, sed nec ex instituto aliquid significante, nec animi affectum testante. Uti *bat* qui sonus est ex ore cornicinis lituum eximentis, quemadmodum, ex CÆSELLIO VINDICE, observat Charisius. (Utitur Plautus, Pseudo:) Item *bat tat ti* fluctus quidam, et sonus vocis effeminatior, ut esse in sacris anagmenorum, vocum veterum interpres scribit; et ex eo idem Charisius extremo, lib. ii." Upon this principle we may admit that sounds, whether articulate or inarticulate, which are merely intended as imitations of other sounds, not proceeding from the human mind nor expressing human passion or affection, are neither interjections nor parts of speech.

Interjections.

But excluding these, there are many sounds, more or less perfectly articulated, which occur constantly in human speech, but which yet are not to be reduced to any of the classes which we have hitherto discussed. These, generally speaking, we reckon among *interjections*: they do not form part of an enunciative sentence; but they are commonly *thrown in between* such sentences, or the parts of them, according as the impulse of a strong or sudden feeling dictates.

Now, as a botanist would but imperfectly teach his science, if he were to tell his scholars that certain large portions of the vegetable world were beneath their notice, as *weeds*; or as he would be a poor mineralogist who should disdain to cast an eye on *pebbles*; so he is a miserable grammarian who affects to disregard the numerous *interjections* and *interjectional phrases* which give such force, tenderness, variety, and truth to the works of the rhetorician and poet, and contribute so much toward rendering language an exact picture of the human mind.

Definition.

SANCTIUS, like Tooke, denied that the interjection was a part of speech; but he did this, with at least a show of argument: his conclusion was fairly derived from his premises: only those premises were built on too narrow and limited a view of his subject. "Interjectionem non esse partem orationis," says he, "sic ostendo: quod naturale est idem est apud omnes: sed gemitus et signa lætitiæ idem sunt apud omnes: sunt igitur naturales. Si vero naturales non sunt partes orationis. Nam eæ partes, secundum Aristotelem, ex instituto, non naturâ debent constare." The error here arises from giving too great a latitude to a proposition which within certain limits is true;

Grammar. viz. that words are significant *ex instituto*; for in truth this proposition applies only to *nouns* (i. e. names of distinct conception) and to words derived from them. But in the nature of the human mind, intellect is mixed up with feeling, the will is often confounded with the reason; and our desires, or fears, unconsciously modify our conceptions or assertions. We express in speech the transitions and mixt states of the mind, as well as its clear, fixt, and determinate distinctions; and hence the interjection rises from a scarcely articulate sound to a passionate, and almost to an enunciative sentence.

According to CHARISIUS, COMMINIANUS briefly defines the interjection thus, "pars orationis significans adfectum animi."—CAIUS JULIUS ROMANUS thus, "pars orationis motum animi significans;" and PALÆMON thus, "interjectiones sunt quæ nihil docibile habent, significant tamen adfectum animi." DIOMEDES gives the following definition—"pars orationis adfectum mentis adsignificans voce inconditâ." Vossius however observes that *apage! euge!* and many others, are not *voces inconditæ*; nor is the signifying an affection of the mind peculiar to the interjection, for even adverbs do this, as *iracundè, irridenter, timidè*, &c He also censures the following definition, *dictio invariabilis quæ interjicitur orationi ad declarandum animi affectum*; for says he, "interjections are not always *thrown in between* the parts of a sentence; since we may properly begin a sentence with an interjection." His own definition is, "vox affectum meatis significans, ac citra verbi opem sententiam complens." This definition agrees in the main with that which is to be gathered from the works of that excellent old grammarian, PRISCIAN; viz. "vox quæ alicujus passionis animi pulsu, per exclamationem, interjicitur:" and finally from all these authorities it is clear, that an interjection is *a word showing an actual emotion of the mind, without assertion*: which, therefore, we may adopt as the definition of this part of speech.

To illustrate this definition, it may be necessary to explain, *first*, what we here mean by a word; *secondly*, why we say the Interjection does not assert any thing, and, *thirdly*, what we understand by an emotion of the mind.

Interjectional forms. First then, we take the term *word* in a large and comprehensive sense, including not only what HARRIS calls "voices of art," but also what he terms "voices of nature, expressing those passions and natural emotions of the mind which spontaneously arise in the human soul, upon the view, or narrative of interesting events." Now, the expressions of mere passion or emotion, as such, are either effected with some degree of volition, or they are extorted by a physical necessity; but on the one hand, it may be doubted whether pure physical necessity can operate so as to produce speech properly so called, that is, with any the slightest degree of articulation. To take a striking instance, that of the *Philoctetes* of SOPHOCLES: we find him at one time exclaiming Ἆ ἆ, ἆ, ἆ, at another Αἶ, αἶ, αἶ, αἶ, and again Παπᾶ, παπᾶ, παπαῖ; but it is manifest that some power, beyond that of mere mechanical impulse, must intervene to give even the slightest of these articulations its difference from the rest. On the other hand, if we admit that some degree of thought enters into all those "voices," which express the emotions of the human mind, then it becomes difficult, if not impossible, for us to distinguish them grammatically into classes, having more or less distinctness of conception attached to them—to distinguish, for instance, in this respect, between O! ἰὼ! euge! evax! papæ! fie! harrow! pax! hush! hurrah! alas! bravo! &c. &c.; for such words may form an ascending gradation from that which is but just above mechanical impulse to that which is but just below the assertion of a proposition. Where indeed such an assertion takes place, that is (speaking as a grammarian) where a verb is connected with a noun, there is formed a sentence, which may be resolved grammatically into its separate parts of speech. But this is not all—the same difficulty which is found in the ascending scale of expression, occurs in the descending scale. A whole sentence is sometimes suddenly interposed in a discourse, by the mere effect of passion or strong feeling, without any direct connection with what goes before, or with what follows. Some such sentences become popular and common, they constitute *interjectional phrases*, expletive parts of the daily conversation of particular sects, parties, or classes of men; they become habitual; they are abbreviated, contracted, corrupted; they remain in language as *words*, sometimes with little more articulation, or distinct meaning than those other sounds which are ascribed to the effect of mere natural impulse. Here then is a wide field for *interjectional forms* in speech, comprehending the almost involuntary *exclamation*, the *word* more or less significant, and the *phrase* more or less imperfect and obscure. And thus we see, that the interjection, like the conjunction, preposition, or adverb, may often be traced home to its origin in the verb, or noun.

Do not assert. We have said that the interjection does not *assert*. It manifests the existence of an emotion, to the sympathies of mankind, but it does not declare that existence as a fact addressed to their judgment. In this respect therefore it differs from the verb. Again, we say it shows *actual* emotion. It does not merely name the conception of an emotion, but gives to that conception a vital energy as it were; it shows the speaker to be affected by its impulse, and is thus distinguished from a noun. It is true, that the limits between an interjection and a noun or verb are not always very easy to be observed. The *imperative* mood, and the *interrogative* form of a verb have so much of animation about them, that they easily pass into mere interjections, and the same may be said of the *vocative* case of nouns. In practice, we should be inclined to say, that so long as a noun or verb (distinguishable as such) enters into construction with other parts of a sentence, or admits of grammatical inflection, according to its particular application, it is to be considered as not having assumed the character of a mere interjection; whilst on the other hand, the simply articulated exclamation, or the noun or verb which has lost somewhat of its original form and signification, while it expresses emotion, is to be called an interjection.

Though the interjection itself does not assert, it may be coupled with an assertion, as one subordinate sentence is coupled with another in a larger sentence. This we have already exemplified in the passages—"O! that I had wings like a dove!"—"O! that this too solid flesh would melt!"—in both which, the verbs (had, and would melt) are put in the subjunctive mood, as dependent on the interjection, O!—just

Grammar. as they would have been had the place of O! been supplied by a verb, such as, "I wish," "I desire," or the like.

In an union of this kind the interjection precedes the sentence with which it is connected; for it has been observed by Vossius, that though the name interjection is given on account of its being *thrown in between* the parts of a sentence, yet this is not essential to the character of an interjection. It is so named, not because it is always, but because it is generally so placed. "Interjectiones dictæ sunt quia sæpe interserantur orationi, non quòd id perpetuum sit."—"Interjectio non semper interjicitur; quia ab eâ quoque rectè auspicamur."—"Nec tamen de *dota* ejus est, ut interjiciatur; cum per se compleat sententiam, nec rarò ab eâ incipiat oratio."

It is scarcely necessary to add, that the interjection may stand quite alone. The mind may be satisfied with giving utterance to its feelings, without entering into any train of reasoning whatever; or those feelings may be too intense and overpowering to admit of any exercise of the discursive faculty. In either case the interjection, to use the phrase of Vossius, "sententiam per se complet."

Emotions expressed. We come now to the most interesting part of this discussion, namely the consideration of the emotions expressed by interjections, or interjectional phrases. And it is to be observed, that we here use the term *emotions*, as we before used the term, *word*, in its most comprehensive sense, including not only the gentler movements of the mind which are sometimes so called, but all kinds and degrees of passion, feeling, or sentiment, which for the moment exclusively govern and direct expression in speech. In this view, so far is the interjection from being a "brutish" thing, that the nice and philosophical examination of it, as it has been practised in the different languages and ages of the world, would furnish matter for a better treatise than was ever yet written on the sensibilities and sympathies of the human mind. Mr. Tooke declares that "the dominion of speech is erected upon the downfal of interjections."—If so, the dominion of speech never was erected, nor ever will be, till the minds of all men are "a standing pool"—incapable of being moved or incited to action even by the naked calculations of a cold, exclusive, hateful selfishness. Mr. Tooke himself uses interjections, especially in those passages which relate to matters affecting his own personal feelings and interests. Yet he says, "where speech can be employed, they are totally useless; and are always insufficient for the purpose of communicating our thoughts." "And indeed," adds he, "where will you look for the interjection? Will you find it amongst laws, or in books of civil institutions, in history, or in any treatise of useful arts or sciences? No: you must seek for it in rhetoric and poetry, in novels, plays, and romances." Mr. Tooke has forgotten one book, in which interjections abound from the beginning to the end, and fill the mind with impressions of the highest sublimity and pathos—That book is the Bible. But if the interjection had *only* to do with "rhetoric and poetry," surely its sphere would not be narrow. If a knowledge of it *only* led us properly to appreciate the lofty mind of Demosthenes or Cicero, to read with true relish the immortal verses of Homer, Virgil, Tasso, Milton—if it were *only* to be met with in the "plays" of Sophocles, Plautus, Moliere, Shakspeare; or in the "romances and novels" of Sidney, Cervantes, Le Sage, Fielding, how lamentable must be the taste, how blind the philosophy, which would decline the examination of this interesting part of speech!

Interjections. The emotions expressed by interjections and interjectional forms of speech may be considered as of three kinds, each running into the others by scarcely distinguishable shades. The impulse of the mind, which leads to the expression, arises either from strong *passion*, from milder *affections*, (that is, emotions in the narrower sense of the word,) or else from certain feelings intimately connected with *particular objects* or events.

Let us first consider the interjectional expression of the stronger passions, such as *terror, fear, pain, sorrow, hatred, eager desire, warm affection*, and *enthusiastic joy*.

Chaucer uses *harrow!* as a common exclamation of the vulgar in situations of danger and terror. Harrow!

Or I will cry out *harrowe!* and alas!

And again,

That down he goeth, and crieth *harrowe!* I die.

So in the *Proces of the Seuyn Sages*.

With both honden here yaulew here
Out of the tresses sche hit tere:
And sche to-cragged hire visage
And gradde *harow*, with gret rage.

It is probable that this exclamation was brought by our Norman ancestors from France. In the old *coustumier* of Normandy *haro* or *harou* is the cry of the country, for pursuit of felons, or other demand of justice.

Duntalous in his *Rollo Normanicus* interprets it *ha! Raoul!* as a cry addressed to *Rollo* Duke of Normandy, whose name was formidable to all evil-doers.

This is what we now call the *hue and cry* from the French *huer*, to hiss or hoot; in the *Statute of Westminster, the First*, A. D. 1272, it is termed *crie de pays*, (see the ingenious remarks of the Hon. Daines Barrington on the statutes) and in the *Statute of Winchester*, A. D. 1285, *heu e cri*.

Other etymologists may perhaps prefer the derivation of this word from the adjective *horowe*, used in old English for filthy, odious, in Anglo-Saxon *horu*, *horwe*, from the Icelandic *hor*, mucor, probably not unconnected with the Latin *horreo*.

And thei wer naughtie, foule, and *horowe*. — Chaucer.

Sometime envious folke with tonges *horowe*. — Idem.

Be this as it may, the interjection *harrow*, although its origin is involved in some obscurity, was evidently used either to denote a strong feeling of horror, or a want of *help*, in which latter sense it would nearly resemble the invocations for help, common in old poetry.

God *help* Tristrem, the knight!
He faught for Ygland. — *Sir Tristrem.*

O empti saile! quhare is the wynd suld blowe
Me to the port quhare gyneth all my game?
Help Calyope! and wynd in Marye name.
The King's Quair

It is obvious, that the simpler any articulations are, the more easily they may be adapted by the flexibility Ah! Oh!

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of the voice to express different states of the mind: a slight degree of elevation or depression, of length or shortness, of weakness or force, serves to mark a very sensible difference in the emotion meant to be expressed. Hence CINONIO thus speaks of the Italian *ah* and *ohi*:—" I varj affetti cui serve questa interiezxione *ah* ed *ahi* sono più di venti; ma v'abbisogna d'un avvertimento; che nell' esprimerli sempre diversificano il suono, e vagliono quel tanto che, presso i Latini, *ah! proh! oh! væ! hei! papæ!* &c. Ma questa è parte spettante a chi pronunzia, che sappia dar loro l'accento di quell' affetto cui servooo; e sono—d'esclamazione—di dolersi—di svillaveggiare—di pregare—di gridare minacciando—di minacciare—di sospirare—di sgarare—di maravigliarsi—d'incitare—di dsegno—di desiderare—di reprendere—di vendicarsi—di raccomandazione—di commovimento per allegrezza—di lamentarsi—di beffare—ed altri varj." VOSSIUS observes of the Latin *oh*, that in ancieot books it is often written o without the aspiration; as *pro* is also written for *proh*; and indeed the Greeks write ἆ without the breathing. Thus the 739th, and the 746th lines of the *Philoctetes* are both written ᾯA, ἆ, ἆ, ἆ. PRISCIAN, too, says that o is the name of a letter, and a preposition, and also an interjection. We need scarcely observe that both *ah!* and *oh!* are used by English writers as interjections of pain and sorrow.

> In youth alone uohappy mortals live
> But *ah!* the mighty bliss is fugitive. DRYDEN.
>
> *Oh!* this will make my mother die with grief.
> SHAKSPEARE.

Dr. JOHNSON says "*ah*, interjection—a word noting sometimes dislike and censure—sometimes contempt and exultatioo—sometimes, and most frequently, compassion and complaint." He also says "*oh*, interjection—an exclamation denoting pain, sorrow, or surprise." The Greek Ἰὼ and Latin *Io*, varying but little in souod from O, were also sometimes used to denote pain or sorrow. Thus Phdoctetes in the agooy of his bodily torture cries ἰὼ, ἰὼ; and Polymestor in the *Hecuba* of Euripides uses the same exclamation. Thus TIBULLUS says,

> Uror, *io!* remove, sæva Puella faces!
> *Lib.* ii. *Eleg.* 4.

And in CLAUDIAN, *Io* seems to express the agony of grief.—

> Mater *io!* seu te Phrygiis in vallibus Ida
> Mygdonio buxus circomsonet horrida cantu;
> Seu te sanguineis ululantia Dindyma, Gallis
> Incolis *De rapt. Proserp.*

Alas! The wnrd *olas!* was manifestly adopted into the English language from the French *helas!* which is only a corruption of the Italian *ohi lasso*, "ah! weary!" It does uot appear to have been known in England mueh before the time of Chaucer, who frequently uses it.

> How shall I doen? whan shall she come againe?
> I note *alas!* why let I her go. *Troilus*, book v.

So in the early romances—

> Thurch the bodi him pight,
> With gile:
> To deth he him dight
> *Allas* that ich while! *Sir Tristrem.*
>
> *Allas* that he no hadde ywite,
> Er the forward were ysmite,
> That hye and his leman also
> Sostree were and tvinnes to. *Lay le Fraine.*

> Quhat sall I think? *Allace* quhat reverence
> Sall I mester to your excellence?
> *The King's Quair.*

Interjections.

> Evir *allace!* than said scho,
> Am I nocht cleirlie tynt?
> *Peblis to the play.*

The sensation of *weariness*, expressed in *ohi lasso*, is also to be found in the Scottish interjectional phrase "weary fa' you."

> *Weary fa' you* Duncan Gray! *Old Scottish Song.*

The latin *væ*, which is nsed only as an interjection, Væ! in that language, is no douht identical with the Anglo-Saxon *wa* and Scottish *wae*, which is our substantive *woe*: and it is to be observed that the Latin *v* was in all probability pronounced like our *w*.

> *Væ* misero mihi! TERENTIUS.

HICKES reckons *wa is me!* and *wam me!* among the Anglo-Saxon interjections of grief. In old English we find "wo the be!"—"woe worth!" &c.; and in Scottish "wae's me!" and "wae's my heart."—

> Wales *wo the be!* the fende the coofound!
> R. DE BRUNNE.
>
> Where er those worldlyngs now? *Wo worth* them, that euer they were about any kyng! LATIMER.
>
> Ah, *wae* be to yoo Gregory,
> An ill death may you dee!
> *Ballad of Lord Gregory.*
>
> *Wae's* my heart that we shou'd sunder!
> *Scottish Song.*

Welaway. From *woe* it is probable came the verb *woil*, and from *waile wae* came *waileway*, *welaway*, and corruptly *welladoy*.

HICKES expounds the Anglo-Saxon *wala wa!* hen! proh dolor! and he adds in a note, "hæc ioterjectio freqnenter tropicè ponitur pro dolore, præcipuè in scriptis Satyrographi, ut,

> "Wote no wyght what war is ther that peace reineth
> Ne what is witerly weale till *welaweye* him teache."

We find it written variously, *weylaway*, *waylewoy*, *waileway*, *wel owaie*.

> Betere hem were at home in huere londe,
> Thao forte seche Flemioysh by the see stronde,
> Whare routh moni Frensh wyf wryngeth hire houde.
> Aot singeth *weylaway*.
> *Battle of Bruges.*
>
> Sche seyd *waylеway*,
> When hye herd it was so:
> To her maistresse sche gan say,
> That hye was boun to go. *Sir Tristrem.*
>
> Biclept him in his armes twain,
> Aod oft allas he gan sain,
> His song was *waileway*.
> *Amis and Amiloun.*
>
> I set hem so a worke, by my faie,
> That many a night they songen *wel awaie*.
> CHAUCER.

Connected with *wae* and *wail* is the verb *waiment*, which Chaucer nses for *lament*.

> The swalow Proigne with a sorowful lay
> Whan morow come gun make her *waimenting*.
> *Troilus*, book ii.

Lastly, the Anglo-Saxon *wala* (in *wala wa*) seems to be still retained in the Scottish ioterjection *waly*.—

> O waly! waly! up the bank,
> And waly! waly! down the brae!
> *Scottish Song.*

Och hone! *Och hone!* or *O hone!* and *O hone-a-chree!* appear to be exclamations of grief used in the Gaelic language:

 and this word *hone* is evidently connected with the verb to *hone*, of which Lye gives this account.—"*Hone after a thing*, anxiè rem aliquam appetere, agi desiderio alicujus rei. Vox agro Devoniensi perfamiliaris; ab AS *honyian, hogian* quæ occurrunt apud *Boethium*, p. 31, 32. Unde hæc oisi a Goth. *hunyan?* —hwaiva agln ist thaim bunyandam afar faihu! quàm difficile est iis qui soliciti sunt de pecuniis!" To *hoe* for any thing, according to Grose, is in the Berkshire dialect to *long* for it

Pape! Dante commences his seventh canto of the *Inferno*, with the exclamation *pape!*—

> *Pape!* Satan, *pape!* Satan, alepe!
> Comincîò Pluto con la voce chioccia.

It is curious to trace this exclamation into the Italian from the Latin, and ioto the Latin from the Greek. In Latin it seems to have chiefly expressed wooder.

> Ecquid beo te? Mene? *Papæ!* — Terentius.

Donatus says "*papæ* ioterjectio mira subito accipientis:" and R. Stephanus says, "admirantis interjectio, habet enim in se affectum verhi miror." It is however admitted to be the Greek παπαῖ, which is manifestly used by Sophocles as an exclamation of pain.—

> Ἀπόλωλα, τέκνον· βρύχομαι, τέκνον· παπαῖ.
> Παπᾶ, παπᾶ· παπᾶ· παπᾶ· παπαῖ.
>
> *Philoct.* v. 752.

It is said to be synonymous with Βαβαί, called by Scapula "admirantis adverbium." Perhaps however it may have had some connection with πόποι, which is used by Homer, and generally rendered *O Dii!* or *papæ!*

> Ὦ πόποι, ἦ μέγα πένθος Ἀχαιΐδα γαῖαν ἱκάνει.
> *O Dii*, certè magnus luctus Achivam terram invadit.
>
> *Iliad.* α. v. 254.
>
> Ὦ πόποι, οἷον δή νυ θεοὺς βροτοὶ αἰτιόωνται
> *Papæ!* ut scilicet Deos mortales culpant!
>
> *Odyss.* α. v. 32.

In both which instances it is clear, that a strong feeling of dissatisfaction or reprohation is iotended to be expressed. Plutarch says that this word πόποι signified io the language of the Dryopans the same as δαίμονες; so that it was originally an invocation of the minor deities.

Fie Few words in any language more obviously deserve the title of interjection than *fie* does io English; yet Mr. Tooke ranks it among adverhs! It is certainly connected with the Gothic verh *fiyan*, Anglo-Saxon *feogan, fean, fiin*, Frankish and Alamanaic *fien, figen*, all which signify to hate; but that it is to be regarded as the imperative of any of these verbs, may be doubted. More probably the verb was formed from the exclamation, of which Wachter gives the following account. "*Fi*, interjectio aversantis apud Saxones Inferiores et Gallos hodiernos, sicut apud Latinos *fu*. Germanl superiores dicunt *phui* et *pfui*, Græci φεῦ. A flatu contra putidum." R. Stephanus explains the Latin *fue* "interjectio ructum exprimentis." (See Plautus, *Most.* 1. 37.) The Greek φεῦ sometimes expresses sorrow, and in this sense probably was the same as the Latin *heu!*—Thus Xenophon says, φεῦ ὦ ἀγαθὴ ψυχὴ—φεῦ τοῦ ἀνδρὸς—both relating to persons dead: and Sophocles says φεῦ τάλας, heu, me miserum! The same interjection is also used to express admiration; as by Aristophanes, φεῦ, φεῦ, ἦ μέγ' ἀναρῶ βούλευμ' ἐν ὀρνίθων γένει—where the scholiast observes, that φεῦ commonly expresses complaint or indignation, hut here admiration. So in Latin *phy* is an ioterjection of admiration, (see Terence, *Adelph.* 3. 3. 59.) With the verb *fian* are connected *feide* odium and *feind* hostis. *Feide* or *fede* is explained hy Wachter "inimicitia aperta, persecutio, viodicta. Anglo-Saxon *fehth*, Island. *fæd*, Latino Barbaris *faida* and *feida*, Belgis *veede*, Anglis *feud*." Thus in the *Lombard Laws* (lib. 1. tit. 7. art. 1 and 15.) we find "*faida*, id est inimicitia." From *faida* was formed the Barbarous Latin *diffidare*, which is the origin of the French *defier*, and of our verb to *defy*. The modern German *fehde*, the Low-Saxon *veide*, and the Danish *feide*, all sigaify enmity. *Feind*, hostis, an enemy, is properly, says Adelung, the participle of the old verb *fian* to hate. This word is written by Ulphila *fijand*, by Kero and Ottfried *fiant*, by Willeram *vient*, in Anglo-Saxon *feond, fynd*, in Lower-Saxon *fijnd*, io Daoish *fiende*, in Swedish *fiende*, in Icelandic *fiande*; and in many of those dialects it receives like the English *fiend*, the particular signification of an enemy to the soul, an evil spirit. So in old English.—

> The small *fendes* that bueth nout stronge
> He shulen among men gonge.
>
> *Christ's Descent to Hell.*

In the Scottish dialect the word *fient*, the Devil, is jocularly employed as a sort of adverb, answering to our colloquial use of such phrases as "the devil a hit," &c.

> When I looked to my dart,
> It was sae blaat,
> *Fient haet* o't wad hae pierced the heart
> Of a kail-runt. — Burns.
>
> They loiter, lounging, lank, and lazy,
> Tho' *de'il haet* ails them, yet uneasy. — Idem.

Fie is also related to the interjections *foh!* and *faugh!* and they all three express various modifications af dislike. Thus the French *fi, donc!* is a slight, and often a sportive reproof, while the English *foh!* is, as Dr. Johnson says, "ao interjection of *abhorrence*."

> *Foh!* one may smell, io such, a will most rank,
> Foul disproportions, thoughts unnatural.
>
> Shakspeare.

Both *foh!* and *faugh!* are connected with the Anglo-Saxon *fah*, and English *foe*, an enemy; but this circumstance has led Dr. Johnson into an error io grammatical reasoning. He says *foh!* is from the Saxon word *fah* an enemy, "as if one should at sight of any thing hated cry out a *foe!*" This supposes the conception of an enemy to be prior to the more general emotion of dislike, or at least to have received a name before the other had been expressed hy a sound. Now the contrary is so abviously probable, we had almost said so necessarily true, that it must be taken as a first principle in all rational etymology.

Most authors reckon such expressions as *ah me!* Ah me! &c. *Oἴμοι! hei mihi! me miserum*, &c. as real ioterjections. We may at least rank them among interjectiooal phrases.

> Cry but *ah me!* couple but love and dove.
>
> Shakspeare.

Vigar calls Oἴ an adverb of grieving: and he adds, "ex οἴ et dativo μοι conflatur oovum dolentis adverbium οἴμοι, unde factum est verhum οἰμώζειν, h. e. οἴμοι λέγειν." Vossius, however, contends that a word like *miserum* is not to be called an interjection.

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His expressions are these, "sunt alia, quæ etsi animi adfectum testentur, ad hanc tamen classem non pertinent. Ut *malum!* quod rectè interjectionum numero eximit Cælius Calcagninus, lib. ii. epist. 8. Sed est ἐπιφώνημα per interpositionem, ut Donatus quoque annotavit super Terentii non uno loco. Similis ratio in istis, *miserum! infandum! nefas!* atque aliis." How far an epiphonema per interpositionem, differs from an interjection, it may not be easy to say. We think, upon the whole, that when any word expressing emotion, be it noun, verb, or other part of speech, is prefixed or thrown into a sentence without connection, and does not enter into grammatical construction with the other members of the sentence, it may not improperly be called an interjection. Thus the furious clamor of the Jewish populace against our blessed Saviour—Ἆρον, Ἆρον—which is rendered in our translation by the interjectional phrase, "*Away* with him! *away* with him!" might perhaps be called an interjection, though it is in origin an imperative mood. The same may be said of the expression of Philoctetes, Ὄλωλα, and Ἀπόλωλα (γ. v. 749 and 752) which differ but little from the vulgar Irish exclamation, "I'm kil't."—Ἆρον, Ἆρον, may be compared, in point of grammatical form, to the expressions so common in popular meetings *off! off!—down! down!* &c. And "*away* with him" may in like manner be compared to the phrase "*out* upon it!"

> Shy. —— My own flesh and blood to rebel!
> Sal. *Out upon it*, old Carrion!
> Shakspeare.

Leeze me! &c.

From interjections expressing painful emotions we turn to those which express pleasurable emotions. In the Scottish dialect we find *leeze me!* signifying "it is dear to me."

> *Leeze me* on drink! it gie's us mair
> Than either school or college. Burns.

> *Leeze me* on your curly pow,
> Bonie Davie, daintie Davie!
> *Old Scottish Song.*

The explanation of this phrase is "*lief* is me;" and of the meaning of the word *lief*, carus, dear, we have already spoken under the head of adverbs.

The construction of the phrase is similar to the German *wohl uns!* and the English *well is him!*

> *Well is him* that dwelleth with a wife of understanding, and that hath not slipped with his tongue! *Ecclesiast.* xxv. 8.

The Latin *amabo*, the future tense of the verb *amo* I love, is often introduced interjectionally as an exclamation of fondness.—

> Vide, *amabo*, si non cum adspicias os impudens
> Videtur. Terentius.

Engraphius, an old commentator, on this passage, says that *amabo* is used by the poets without any meaning; but on this Vossius justly remarks—"si blandientis est, otiosum esse nequit, cum multùm blanditiæ et preces valeant."

O si!

Eager desire is shown by such expressions as *O si!*—*oh, gi'n!*—*O utinam!*—εἴθε, Εἰ γὰρ, &c.

> Quamquam *O si!* solitæ quicquam virtutis inesset.
> Virgil.

> *O!* gi'n my luve were yon red rose,
> That grows upon the castle wa'!
> *Old Scottish Song.*

> *O utinam* tunc, cùm Lacedæmona classe petebat,
> Obrutus insanis esset adulter aquis! Ovidius.

Interjections.

> Ὦ γέρον, εἴθ', ὡς θυμὸς ἐνὶ στήθεσσι φίλοισιν,
> Ὥς τοι γούναθ' ἕποιτο. Homerus.

> Εἰ γὰρ Αἰγίσθῳ γ' ὁμῶ. Sophocles.

Huzza! bravo! &c.

Our common *huzza!* and *hurrah!* seem to be old German shouts of exultation. It does not appear that they were ever used by the more southern nations. The Latin words *io! evoe!* and *evax!* express nearly similar emotions. We have adopted into the English language the Italian exclamation of applause *bravo!* but chiefly at our theatrical entertainments. Some persons affect to distinguish the persons applauded by the terminations *bravo, brava, bravi*; but this seems to be carrying the adoption of a foreign idiom further than is suited to the character of a mere interjection. A strong proof that interjections are not, as Sanctius contends, *naturales, et idem apud omnes*, is, that even in such apparently unpremeditated effusions of joy, people of different countries testify their feelings differently, and each in the mode of his own country. Thus where the Englishman exclaims *hurrah!* the Frenchman would cry *ha! bon!* and the Italian *bravo!*

Hech! &c.

Among milder emotions, we may reckon the various degrees of surprise, sometimes mixt with a certain dissatisfaction, which are indicated by the Scottish *hech!* the French *comment!* the English *good now!*—*good lack!*—*la, la, la!*—*dear me! sure!* &c. &c.

> *Hech* man! dear sirs! is that the gate
> They waste sae mony a braw estate! Burns.

> *Comment* donc! qu'est que c'est que ceci? On dit que vous voulez donner votre fille en mariage à un Carême-prenant! Moliere.

> *Good now! good now!* how your devotions jump with mine! Dryden.

> Flam. My lord, having great and instant occasion to use fifty talents, hath sent to your lordship to furnish him: nothing doubting your present assistance therein.
> Lucul. *La, la, la!*—nothing doubting, says he? A noble gentleman 'tis, if he would not keep so good a house.
> Shakspeare.

Yare! swith! &c.

Interjections expressing haste, slowness, and the like, are common in all languages, and are usually verbs or adjectives applied to this purpose. Such are the old English *yare!* the Scottish *swith!* and *hoolie!* the German *nur frisch!* the French *allons!* the Italian *piano!* the modern English *make haste! come, come! stay! stop! hold!* &c. &c.

Yare is from the Anglo-Saxon verb *yearwian*, to prepare, as *yearwian til etanne*, to prepare for eating.

> Heigh, my hearts! cheerly, cheerly, my hearts, *yare! yare!* Take in the top-sail. Shakspeare.

We have noticed *swithe* among the adverbs. It is used interjectionally by Burns.

> Then *swith* an get a wife to hug.

The same poet uses *hoolie* for gently.—

> But still the mair I'm that way bent,
> Something cries *hoolie!*
> I red you, honest man, tak tent
> Ye'll shaw your folly.

Prithee! pish! &c.

There are many interjections expressing slight contempt—some by way of expostulation, as *prithee!*—some indicating the trivial nature of the object, as *pish!*—some showing a degree of vexation in the speaker, as *pshaw!*—some denoting the absurdity of the thing in question, as the English *tut!* and *tush!* the Latin *vah!* the French *bah!* and the Scottish *hout!* whilst others mark in the speaker a certain

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feeling of disgust or weariness, as the English *humph!* the French *ouf!* &c.

Tooke ranks *prithee!* among adverbs. Johnson does not decide what part of speech it is, but merely calls it "a familiar corruption of *pray thee*." This corruption, however, becomes in use a real interjection. In the following instance the request is merely contemptuous.—

> Poh! *prithee!* ne'er trouble thy head with such fancies;
> But rely on the aid thou shalt have from St. Francis.
> *Old Song.*

In the next, the request is more serious, but still the abbreviation of the phrase marks a degree of familiarity.—

> Alas! why comest thou, at this dreadful moment,
> To shock the peace of my departing soul?
> *Away!* I *prithee* leave me! Rowe.

Of the interjection *pish!* Dr. Johnson thus speaks—"Pish! *interj.* a contemptuous exclamation. This is sometimes spoken and written *pshaw!* I know not their etymology, and imagine them formed by chance."

> She frowned, and cried *pish!* when I said a thing that I stole.
> *Spectator*, No. 268.

> A peevish fellow has some reason for being out of humour, or has a natural incapacity for delight, and therefore disturbs all with pishes and *pshaws*. *Ibid*, No. 438.

Pshaw would certainly be an odd way either of speaking or writing *pish*: and an interjection is no more formed by chance than a chronometer. *Pish* and *pshaw* both appear to be *natural* exclamations; but they express different shades of contempt, the latter showing more of ill humour and vexation than the former.

Dr. Johnson says of *tut!* "this seems to be the same with *tush!*" and of the latter he says—"Tush! *interj.*—of this word I can find no credible etymology—an expression of contempt."

> *Tut*, man! one fire burns out another's burning.
> Shakspeare.

> *Tush!* say they, how should God perceive it; is there knowledge in the Most High? *Psalm* lxxiii.

Among the *few* interjections, which, Wallis says, the English language possesses, he reckons "*tush* contemnentis." It is probably connected with the French verb *tousser*, to cough. Wallis renders it by the Latin *vah*, which sometimes has a similar force.—

> *Vah!* leno iniqua me non vult loqui. Terentius.

With this latter the French *bah*! is probably connected, and it may also have some relation to the French verb *bailler*, to yawn.

The interjection *hout!* is very common in the novels of the author of *Waverley*—

> Weel, but Troda kens this lad weel; and she has often spoke to me about him. They call his father the silent man of Sumburgh; and they say he's uncanny—*Hout! hout!* Nonsense! nonsense! they're aye at sic trash as that, said the brother.
> *The Pirate.*

Hout! seems to be an *onomatopoeia* of the same nature as the English verb, to *hoot*, or the Scottish, to *hoast*, to cough.

Humph! appears to be a mere imitation of the natural expression of contemptuous discontent in the following passage.

> Semp. Must he needs trouble me on t—*Humph!*
> 'Bove all others? Shakspeare.

Interjections.

Ouf! a similar expression of the pain arising from weariness, as in the *Bourgeois Gentilhomme* of Moliere.—

> Après que l'Invocation est finie les Derviches ôtent l'Alcoran de dessus le dos du Bourgeois, qui crie *ouf*, parce qu'il est las d'avoir été long temps en cette posture.

Euge! Gramercie, &c.

Among interjections of soothing and encouraging, of satisfaction, acquiescence, and the like, we may reckon *euge!* (well done); θάρσει (be of good cheer); *sodes*, (I pray you); *paramour* (for love's sake); *gramercy!* &c.

The Latin *euge!* was, in its origin, a compound of the Greek εὖ and γε.

The Greek imperative θάρσει is rendered by the interjectional phrase "be of good cheer!" in our translation of St. Matthew's *Gospel*, (c. ix. v. 2.)—θάρσει τέκνον· ἀφέωνταί σοι αἱ ἁμαρτίαι σου. In the Latin vulgate, the correspondent word is "*confide*." In the Anglo-Saxon translation, the passage runs thus—"*La!* bearn, *gelyfe!* the beoth thine synna forgifene." In the Gothic it is "*thrafstei* thur barnilo! afletanda thus frawaurhteis theinos;" where the verb *thrafstyan* appears to have some affinity both with the Greek θάρσειν, and the Teutonic *trost*, whence came the Barbarous-Latin *trustis* and *antrustio*, the Icelandic *traust*, the Dutch *troost*, the German *trost* and *getrost*, the Scottish *tryst*, and the English *trust* and *trusty*; all which have the analogous significations of *fiducia*, *solatium*, &c.

The French *courage!* answers not to the imperative θάρσει, but to the substantive θάρσειν. Of this word, *courage*, the *Dictionnaire de l'Academie* says, "il se dit quelquefois absolument, par maniere de particule exhortative, *courage* mes amis! *courage*, soldats!" Thus we use the words *courage! comfort! patience!* &c.

> Pand. *Courage!* and *comfort!* all shall yet go well.
> K. Phil. Patience, good lady! *Comfort* gentle Constance!
> Shakspeare.

The Latin *sodes!* is rendered by R. Stephanus δέομαι; and he calls it "blandientis vel exhortantis adverbium, seu mavis *interjectio*; quasi tu dicas *quæso*, *rogo*, *obsecro*." It is a contraction of the phrase *si audes*.

> Quintilio si quid recitares, corrige, *sodes!*
> Hoc aiebat, et hoc. Horatius.

Paramour! par charité! and such like words and phrases occur often in our old writers.

> He spak vnto the emperoure,
> Tak me thi sun, sir, *paramoure!*
> And I sal teche him ful trewly.
> *Seuyn Sages.*

> Ynough thai hadde of warldes wele,
> Togeder thai leved yeres fele,
> Thai ferd miri, and so mot we.
> Amen, amen, *par charité.*
> *How a Merchant*, &c.

Gramercy, or *gramercies!* which occurs often in our old writers as a mode sometimes serious, sometimes ironical, of returning thanks, is a contraction of the French *grand merci*, great thanks.

> When the king understood this word, he was right glad of it, and said to Regnawde, I right gladly grant this to you: and with the same ye shall have of me x thousand mark every year for to maintain your estate. Sir, said Regnawde, *gramercie!* and cast himself at his feet. *The Foure Sonnes of Aimon.*

> Gobbo. God bless your worship!
> Bass. *Gramercy!* Wouldst thou aught with me?
> Shakspeare.

FOOL. How do you, gentlemen?
SERV. *Gramercies!* good fool; how does your mistress?
SHAKSPEARE.

Grand merci façon de parler dont on se sert dans le style familier, pour dire, je vous rends graces. *Vous me donnez cela? Grand merci! monsieur.* *Dict. de l'Academie.*

Merci, peutestre de *misericorde*, par contraction.
MENAGE.

The signification of *thanks* is so different from that of *mercy*, as to render it probable that there were two derivations of this word in French, the one pointed out by Menage applying to the substantive *merci*, mercy; the other from *merces*, recompense; in which latter sense *grand merci* would be in Latin (cupio tibi) *grandem mercedem*. In the other sense, the French have an interjectional phrase, *merci de ma vie!* answering to our familiar exclamation, *mercy upon us!* expressing astonishment.

Eh! is used very ludicrously in the soothing expostulations of the *Bourgeois Gentilhomme*, with his dancing and fencing masters, when they quarrel.

M. D'ARMES. Comment! Petit impertinent!
JOURD. *Eh!* mon Mâitre d'armes!
M. A DANS. Comment! Grand cheval de carosse!
JOURD. *Eh!* mon Mâitre à danser! MOLIERE.

Hush! whisht! &c.

There are many words of admonition, such as *hush!* and *whisht!* to keep silence; *gare!* to beware, &c.

Hush! seems to be the Gothic imperative *hausei!* *hear!* from the verb *hausyan*, which occurs frequently in Ulfila's translation of the Gospels, e. gr. "*Hausei!* Israel, fan Goth unsar fan ains ist;" "*Hear*, O Israel! the Lord our God is one Lord." (*Mark*, c. xii. v. 29.) The verb *hausyan* is manifestly from *auso*, the ear. The denomination of this part of the body has a similarity in many dialects, which may be divided into two classes distinguished by the letters *s* and *r*. Of the former class are the Gothic *auso*, the old Latin *ausis*, and the Greek οὖς; of the latter are the more modern Latin *auris*, the Frankish and Alamannic *ora*, *ore*, *or*, the Low-Saxon and Dutch *oor*, the modern German *ohr*, the Danish *öre*, the Swedish *oera*, the Icelandic *eyra*, the Anglo-Saxon *eare*, and the English *ear*. The Italian *orecchio*, and Spanish *oreja*, are corruptions of the Latin diminutive *auriculus*, and from *orecchio* comes the French *oreille*.

Hark! is of the same family. From *ohr*, the ear, the Germans have formed *hören!* to hear, and *horchen!* to listen to; as the Latins, from *auris* or *ausis*, had *audire* and *auscultare;* and so the Anglo-Saxons had *hyran* and *heorchian*, which are our *hear* and *harken*, or *hark*, and of this last the imperative easily becomes an interjection.

The Scottish exclamation *whisht!* may not improbably be of the same origin as *hush!* We pronounce this word *whist!* and use it, as Johnson observes, 1st. as an interjection commanding silence, 2dly. as an adverb, 3dly. as a verb, and 4thly. as a noun, the name of a well known game, requiring silent attention. BURNS uses *whisht* as a noun implying silence.

A tight outlandish hizzie, braw,
Came full in sight.
Ye need na doubt I held my *whisht*.

Nearly similar to this is our word *hist!* of which JOHNSON thus speaks:—"HIST, *interj.* of this word I know not the original: probably, it may be a corruption of *hush, hush it, husht, hist.*"

Hist! Romeo, *hist!* O for a falc'ner's voice,
To lure this tassel-gentle back again!
SHAKSPEARE.

It is, however, to be observed, that the Romans used the imperfect articulation *'st* for the same purpose. R. STEPHANUS says "ST [r] vox est silentium indicentis. Ter. *Phorm.* v. 1. 16. Quid? Non is, obsecro, es, quem semper te esse dictitasti?—C. *'st*—S. Quid has metuis fores?" The Italians use the word *zitta!* and the French say *chut!* VARCHI, in his *Ercolana, or Dialoga sopra le lingue*, printed at Florence in 1570, says of this word, "Il quale *zitto*, credo che sia tolto da' Latini, i quali, quando volevano, che alcuno stesse cheto, usavano profferire verso quel tale, queste due consonanti *'st*, quasi come diciamo noi *zitto!*" It is used substantively for the slightest sound possible. Thus BOCCACCIO says "senza far motto, o *zitto* alcuno;" "without uttering a word, or sound, the slightest possible." It is also used adjectively, with the variation of gender and number, e. gr.

E i buon soldati, in campo, o in citadella,
Si stanno *zitti* in far la sentinella. ALLEGRI.

Of the French *chut!* the *Dictionnaire de l'Academie* merely says, "*Chut*, particule dont on se sert pour imposer silence."

Gare! is a French interjection, the imperative of the old verb *garer*. "On se sert," says the *Dictionnaire de l'Academie*, "pour avertir que l'on se range, que l'on se detourne pour laisser passer quelqu'un, ou quelque chose, *gare! gare! gare de la! gare l'eau!* il se dit aussi par maniere d'avertissement et de menace: ainsi on dit à un jeune escolier *gare le fouet!*" Hence the French *garenne*, a warren, or place for preserving rabbits: *guerir*, anciently *garir*, to cure, to preserve from disease. *Guerite*, anciently *garite*, a watch tower, or sentry box, which is the origin of our word *garret*. It is said to have been formerly a custom in the northern parts of Great Britain, in throwing water from a high window to cry to the passengers below *gardyloo!* a corruption of the above cited French phrase *gare l'eau!* or *gare! de l'eau!* The verb *garer* or *garir*, is only another pronunciation of the old Teutonic *waren*, which appears in so many forms and dialects. Hence WACHTER says 1. "*Waren* oculis usurpare, spectare, intueri. Hic primus verbi latissimè patentis significatus, quem MYLIUS quoque apud veteres animadvertit in *Archæolago Teutone*—Francis *uuara* sæpe est adspectus, et *uuara tuon* adspicere—ex eodem fonte haud dubiè est adjectivum *war* videns, in formulâ vetustissimâ *gewar werden* videre, videntem fieri visu cognoscere. 2. *Waren* ab oculo corporis transfertur ad oculum animi, et tunc significat, quantum potest, *considerare, curare, observare, servare, custodire, cavere.*" Of all these significations he gives examples, as follows:—

Considerare, hence the Frankish *uuara*, consideration, and *uuara tuon*, to consider; e. gr. "Ne *duont* thes niet *uuara* thaz ih so salo him;" "nolite attendere quòd tam fusca sim;" (Willeram. cap. i. 6.) hence also in modern German *warnehmen* is used for, to observe.

Curare, hence the Frankish *ungiuueri*, carelessness; *ungiuuarin*, inconsiderate: in modern German they sometimes say *warlos*, for careless.

Observare, hence the Frankish *uuara*, observation, used by WILLERAM, cap. ii. 15. "*Ir doctores ecclesiæ tuot uuara*," "The doctors of the church observe."

Servare, the Frankish *uuara* has also this sense, e. gr. *aim sin mihila uuara*, "take much care of him," "keep your eye on him."

Custodire, hence the Dutch *waarande*, Barharous-Latin *warenna*, French *garenne*, and English *warren*. The Germans also use *gewarsame* for *custodia*.

Cavere, as in the Frankish *geuueri* cautela. From the participle *warend* in the sense of caution, eame the Barbarous-Latin *warendia*, and *warandia*, bail; also *warendator*, *warandare*, and *warentizare*; the Freuch *garant*, *garantir*, *garantie*; the English *warrant*, *guarantee*, &c. From *waren* in this sense came *warnen*, to premonish, to provide against danger, to fortify. In the Icelandic, *varnaa* is *cautio*; in the German, *andern zu warnung* is, "as a warning to others." In the Frankish, we have *giuuarnot werdet*, "be prepared for defence," "arm yourselves." *Ingegin uuidaruuinon so sculun uuir unsih uuarnon*; "against the enemy so should we arm ourselves." (Ottfried, 1. ii. c. 3. 3.) *Warninga* is used hy Willeram for *munitio*. *Warnitus* signifies, in the Barbarous-Latin of the Frankish Capitulars, "armed." Hence the Italian *guarnire*, to fortify, and *guarnigione*, which is the French *garnison*, and English *garrison*.

Other analogous meanings might be added, as tho German *gewarten*, to expect or look for; and *gewartig einer sache seyn*, "to be aware that a thing will happen."

In the Anglo-Saxon we find *waer* cautus, *waere* cautin, *werian*, to defend, *warnian*, to heware, &c.

In Dutch *waarande*, a park for keeping deer or other animals; *waarborg*, a surety; *waaren*, to guarantee; *waarnemen*, to regard; *waarschauwen*, to premonish, &c.

From these sonrces lastly come our English words *aware*, *beware*, *wary*, and others already mentioned.

Ware! pronouneed hy the lower elasses of peopln *war!* is often used as an interjection of premonition, as in *war hawk!* a notice to smugglers to avoid thn excisemen. Hence Lye, in his edition of Junius's *Dictionary*, says—"War, cavere, prospicere. *War heads!* prospicite capitibus, ab A. S. *warian* ejusdem significationis.

> False witnesse is in word, and also in dede: in word, as for to bireue thy neighbour's good name.—*Ware!* ye questmongers and notaries! CHAUCER.

Among words of this kind we may reckon *mum! pax! paix! peace! silence!*

Johnson says, "Mum, *interject.* Of this word I know not the original: it may be observed, that when it is pronounced, it leaves the lips closed; a word denoting prohibition to speak, or resolution not to speak; silence; hush." It seems to be connected with the Latin *murmur*, the German *mummeln*, the Dutch *mompelen*, and the English to *mumble*.

Pax! is called hy R. Stephanus an interjectinn; as in Plautus, "*pax!* te trihus verhis volo." "Malè autem," says Vossius, "quidam interpretes posuêre *pax* inter *admirationis* interjectiones; nam, ut pluribus ostendit Jos. Scaliger in Ausonianis lectionihus est silentium sibi aut alteri imponentis." It is manifestly the nonn *pax*, used interjectionally, in the sense of peace, quietness, silence, as we say, "hold your peace!" for "be silent!" retain your peacefulness and quietness. So the French use the exclamation *paix!*

> Madame Jourd. Hélas! mon Mari est devenu fou.
> Mons. Jourd. *Paix!* insolente; portez respect à Monsieur le Mamamouchi! MOLIERE.

So Johnson says "Peace, *interjection*. A word commanding silence."

> ——————— Hark! *peace!*
> It was the owl that shriek'd, the fatal bellman
> Which gives the stern'st good night.
> SHAKSPEARE.

In pointing out an object, we say *lo!* in inviting attention *list!* in giving assurance *troth!* and there are many other such interjectiuns which occur in books and conversation, as *forsooth! indeed! well! why! hum! a'tweel! poz'!* &c. &c. Lo! Troth! &c.

Lo! is ranked by Mr. Tooke among adverhs: why, it would be in vain to ask, since the only thing he tells us of adverhs is, that they are not a separate part of speech. He is, however, right in his etymology. *Lo!* is the imperative of the verb *look*, used interjectionally. The old imperative, *loketh*, was used in the same manner.

> *Loketh!* Attyla the greate conquerour,
> Dyed in his slepe, with shame and dishonour.
> CHAUCER.

List! is in like manner the imperative of the verh to *list* or *listen*.

> ————*List! list!* Oh *list!*
> If thou didst ever thy dear father love.
> SHAKSPEARE.

Troth! is the old noun *troth*, truth, trust, fidelity.

> D. Ped. Now, Signor! where's the Count? Did you see him?
> Ben. *Troth!* my lord, I have played the part of lady Fame. I found him here, &c. SHAKSPEARE.

This exclamation is still common among the lower ranks of people in Scotland and Ireland: and it is the abhreviation of a sentence such as "I say the truth" —"the truth is;" or the like.

Forsooth is little different, in its original meaning, from troth; "the *sooth*" being "the truth."

> The pauyloun was wrouth, *for sothe ywis*,
> All of werk of sarsynys. *Syr Launfal.*

> The wer lasted so long,
> Till Morgan asked pes,
> Thurch pine!
> *For sothe*, withouten les,
> His liif he wende to tine. *Sir Tristrem.*

Indeed! This word, which Johnson notes as an adverb, and which is in fact made of the two words *in* and *deed*, serves interjectionally to denote surprise, with some degree of doubt.

Well! and *why!* are elliptical interjections commonly nsed at the beginning of a sentence. When any matter has been stated which is known to, or admitted hy the other party in conversatinn, the speaker introduces his next position with the interjection *well!*—or else the person addressed exclaims *well!* meaning to deny or dispute what follows. The meaning is "so far is spoken *well*;" but as to what follows a further consideration is necessary. When a certain degree of impatience is meant to be expressed hy the hearer, he exclaims *well, well!* meaning, so far it is well, but you must proceed.

Why! used in a similar manner, expresses a transient feeling of hesitation, or surprise.

> ——————— You have not been a-bed then?
> *Why!* no. The day had broke before we parted.
> SHAKSPEARE.

> Whence is this?—*Why!* From that essential suitableness, which obedience has to the relation which is between a rational creature and his Creator. SOUTH.

Ninus' tomb, man! — *Why!* you must not speak that yet.
That you answer to Pyramus. SHAKSPEARE.

In the two first of these instances, the speaker seems to have an indistinct intention of asking *why* the question is put: in the third *why* the fact happened. It is as if he had said, *why* do you ask whether I have been a-bed? the circumstance is trivial—*why* do you ask whence this happens? the reason is obvious—*why* do you speak of Ninus' tomb? you are not yet come to that part of the play. But in all these cases the emotion is transient, and satisfies itself, as it were, with a brief interjection, instead of proceeding to develope itself in a formal interrogatory.

Johnson calls *hum!* an interjection: and says it is "a sound implying doubt or deliberation."

> ——— See Sir Robert? *Hum!*
> And never laugh for all my life to come.
> POPE.

Pos' is a vulgar colloquial expression introduced into some of our comedies—a mere abbreviation of *positively*; to express that a thing is certain.

We shall not dwell on the interjections of compellation or inquiry—*hollo! eheu! heus tu! hark ye friend!* &c.—nor on those of vexation, *plague on't! peste! dear heart! O dear!* &c. &c.

The religious opinions and feelings of mankind have furnished a great variety of exclamations, imprecations, and asseverations, all constituting interjections or interjectional phrases. Hence the Greek Ναὶ μὰ Διὰ, the Latin *ædepol!* The old English and French *parde! perfay! parmafey! parbleu! morbleu! Christes rode! ouh for Saint Davy! for Seynt Martin! by Seint Eloy! with sorwe! Godamercy! God's face! gadso! egad! foregad! Gog's soul! by cock's bones! odsbodikins! zounds!* besides a number of whimsical and arbitrary exclamations of a similar nature, as *gemini! miga! cadedis! tete! ventre! by my pan! by my top! by bread and salt!* &c. &c.

EUSTATHIUS contends that in the phrase ναὶ μὰ Διὰ the particle ναὶ has the power of adjuration; but HOOGEVEEN more accurately says that ναὶ has only the power of confirming the adjuratory particles μὰ and πρὸς; as in Ναὶ μὰ τόδε σκῆπτρον (*Iliad*. α.) Ναὶ πρὸς τῶν θεῶν. (Aristoph. *Nub.*)

Ædepol! is commonly written with an *æ*, and is explained to be a contraction of *per ædem Pollucis*. VOSSIUS however contends that it should be written *edepol!* and that it is made up of three words *e* or *me*, a particle of adjuration, *deus*, and *Pollux*. But MEURSIUS suggests that it was originally *epol*, as we find *Ecastor! Equirine! Ejuno!* and that the *d* was inserted merely as it is in *medecum* for *mecum*. At all events this is a contracted exclamation relating to *Pollux*.

> *Ædepol!* mortalem parce parcum prædicas.
> PLAUTUS.

Parde! perfay! and *parmafey!* are the French *par Dieu, par foi*, and *par ma foi*.

> Ah! good sir host! I haue wedded be,
> These moneths two, and nat more, *parde!*
> CHAUCER.

> When Alexander the king was dead,
> That Scotland had to steer and lead,
> The land six years and more, *perfay!*
> Lay desolate aftir his day. BARBOUR.

> SATHAN. *Parmafey!* Ich holde myne
> Alle tho that bueth her yne.
> *Christ's Descent to Hell.*

Parbleu! is an exclamation of the same kind but not quite so intelligible. It seems to be connected with *morbleu!* or *mort bleu!* which was originally an imprecation of death with putrefaction, either on the speaker, if his words should not prove true, or on the person addressed. They have, however, both dwindled into mere ejaculations of surprise. *Ventrebleu!* and *tetebleu!* also occur in a similar sense.

> M. DE P. Que me voulez vous?
> MEDECIN. Vous guerir, selon l'ordre qui nous a été donné.
> M. DE P. *Parbleu!* je ne suis pas malade. MOLIERE.

> Comment, Marauts! vous avez la hardiesse de vous attaquer à moi! allons! *morbleu!* tue! point de quartier! IDEM.

> LE MARQ. ——— Vous, trêve de colere!
> Ou je me fâcherai.
> LE BAR. ——— Fâchez vous, *ventrebleu!*
> DESTOUCHES.

> LE COM. Moi, je ments! *tétebleu!* mon pere permettez.
> LE MARQ. Tout doux, il n'a pas tort, et c'est vous qui mentez.
> IDEM.

For Christes rode! for Seynt Martin! are solemn adjurations signifying *before the cross of Christ! before Saint Martin!* *By Saint Loy*, or *Saint Eloy*, is an asseveration of similar import.

> Scho crid merci anough,
> And seyd *for Cristes rode!*
> What have Y don wough?
> Whi wille ye spille mi blode?
> *Sir Tristrem.*

> Bi God, quoth Erl Florentin,
> Who may that be, *for Seynt Martin!*
> That Ich here in mi forest blow?
> *Guy of Warwick.*

> The Walsch without the toun euerilkon thei lay,
> Whan thei the trumpes herd; that he to bataile blew,
> And saw the yates sperd, than gamened tham no glewe.
> *Ouh! for Saynt Dawy!* the Flemmyng wille him gile.
> R. DE BRUNNE.

> There was also a nonne, a prioresse
> That of her smiling was simple and coy
> Her greatest oth was but *by Saint Loy!* CHAUCER.

The wife of Bath however swears much more roundly than the prioress.

> But now Sir, let me se, what shal I sain?
> Aha! *by God!* I haue my tale againe. IDEM.

And we find in old writers some singular adjurations of the divinity, as *be Goddis face! bi Godes ore!* &c.

> Evyn in the Peth was Erle Dawy,
> And til a gret stane that lay by,
> He sayd "*be Goddis face*, we twa
> The fleycht on us sall samyn ta."
> WINTON'S *Chronicle*.

> Brengwain the coupe bore
> Hem rewe that ferly fode
> He swore *bi Godes ore*
> In her hond fast it stode. *Sir Tristrem.*

Godamercy! is the more obvious exclamation (however improperly introduced into trivial discourse) *God have mercy!*

> CHAT. Go to then! what is your rede? say on your mind.
> Ye shall me rule herein.
> DIC. *Godamercy!* dame chat, in faith,
> Thou must the gere begin. *Gammer Gurton.*

This irreverent and irreligious use of the sacred name of God being felt to be very reprehensible, the vulgar resorted to various modes of avoiding its sinfulness, and yet giving way to their emotions in such exclamations as *Gog's soul! Gog's sides! cock's bones! gadso! foregad! egad! odsbodikins! bodikins! Godzounds! zounds! odd so! odd's life! slife!* &c. &c.

HODG. Daintrels, Diccon? *Gog's soul!* man, save
This pece of dry horsbred,
Chav byt no bit this live long day,
No crome come in my bed. *Gammer Gurton.*

HODG. *Gog's sides!* Diccon, me think ich hear him,
An tarry, chall mar all. *Ibid.*

See, how he nappeth. See, *for cock's bones!*
How he woll fall from his hors stones. CHAUCER.

Mr. Tooke has thought proper to call *gadso!* an adverb, and to explain it by *cazzo*, an obscene Italian word. He is wrong on both points. As *egad!* was an evasion of *by God*; and *fore gad!* of *before God*; so *gadso!* was an evasive contraction of *by God it is so*, or *by God, is it so?*

Od'sbodikins! is a diminutive of *by God's body*; and this is further corrupted to *bodikins!*—So *Godzounds!* and *zounds!* are *by God's wounds*—*odd's life!* and *'slife!* are by *God's life.*

SHAL. *Bodikins!* Master Page, though I now be old, and of the peace, if I see a sword out, my finger itches to be one.
SHAKSP. *Merry Wives.* &c.

He swore *by the wounds* in Jesu's side
He would proclaim it far and wide. COLERIDGE.

Zounds! sirrah, the lady shall be as ugly as I choose: she shall have a bump on each shoulder, &c.
SHERIDAN, *The Rivals.*

Odd's life! when I ran away with your mother, I would not have touched any thing old or ugly, to gain an empire.
Ibid.

O gemini! was probably an evasive imitation of *O Jesu!* What *ad's niggs* and *'snigs, odzooks!* and *zooks!* were meant to resemble, it would perhaps be difficult to ascertain. All these exclamations occur in ludicrous writings about the end of the seventeenth and beginning of the eighteenth century.

But the man of Clare Hall that proffer refuses:
'Snigs! he'll be beholden to none but the muses.
GEO. STEPNEY.

Ad's niggs, crys Sir Domine,
Gemini! gemini!
Shall a rogue stay?

Come strike hands I'll take your offer:
Farther on I may fare worse.
Zooks! I can no longer suffer. *Midas.*

So in French we find, among the vulgar, numerous exclamations of this kind, which it is not easy to explain. Such are *cadedis! perguenne! testigué! morguene! palsanguenne!*

Cadedis! is a Gascon expression, perhaps signifying originally *chef de Dieu!* "by the head of God!" for *cap* in the Gascon dialect was used for "head." Thus *cadet* the younger son of a family, anciently *capdet*, is derived from the Barbarous-Latin *capitetum*, or little chief, the elder being the great chief.

Vindrent deuant une place nommée Malaunoy, dedans laquelle estoit vn Capitaine Gascon, nommé le *Capdet* REMONNENT.
Chronique de Louys XI.

Perguenne! J'avons pris là, tous deux, une gueble de commission! *Medecin malgré lui.*

Testigué vela justement l'homme qu'il nous faut. *Ibid.*

Eh! *Morguene!* laissez nous faire. S'il ne tient qu'a battre, la vache est à nous. *Ibid.*

Palsanguenne! vela un medecin qui me plait. *Ibid.*

The custom of swearing by the head of the person making oath was very ancient; and is forbidden by our Saviour in the well known text, "Neither shalt thou swear by thy head; because thou canst not make one hair white or black." (*S. Matt.* c. vi. v. 36.) Nevertheless it appears to have long prevailed; for the words *pan* and *top* in the following examples both signify *head.*

Loue is a gretter lawe, *by my pan!*
Than may be yeuen to any erthly man.
CHAUCER.

Sire Simond de Mountfort hath swore *bi ys top*,
Hevede he nou here Sir Hue de Bigot
Al he shulde graunte hen twelfemonth scot
Shulde he neuer more with his fot pot
To help Wyndesore.
Battle of Lewes.

The military cries, *halt! or sus! az armes! God and St. George! Bourbon, nostre Dame! Montjoie, St. Denys!* &c. &c. are interjectional forms; as are the naval exclamations *yo, ho! avast! 'vast heaving!* &c.

Mr. Tooke reckons *halt!* among adverbs, and says it is the imperative of the Anglo-Saxon verb *healdan*, to hold. It was probably borrowed by us as a technical expression from the French, who use the exclamation *halte, la!* derived from the German *still halten*, to halt or stop.

Richard aros, and toke hys wede,
And lept on Favel hys gode stede,
And sayde, Lordynges *or sus! or sus!*
That hath us warned swete Jesus.
Richard Coer de Lion.

Az armes! he let crye there,
Ayenst the Sarazyns for to fare. *Ibid.*

God and St. George! Talbot and England's right!
Prosper our colours in this dangerous fight.
SHAKSPEARE.

Instead of the tumult and din of their anarchy, the human voice divine may yet be heard. The antient spirit may yet revive. The cry of *Bourbon, nostre Dame!* and *Montjoie St. Denys!* may again resound through France. WILDE. 1793.

We need not dwell on the modern popular cries, such as *England for ever! vive le roi! vivat! à bas! off! encore! bis!* But the old Scottish and English "*heve and how*," and "*rumbelow!*" is singular enough to be cited.

With *hey and how! rohumbelow!*
The young folk wer full bald.
Peblis to the Play.

They rowede hard, and sungge ther too
With *heuelow!* and *rumbeloo!*
Richard Coer de Lion.

Your maryners shall synge arowe,
Hey how! and *rumbylowe!*
Squyre of lowe degree.

Salutations and valedictions afford several interjections and interjectional forms, as *hail! allhail! welcome! benedicite! greeting! farewell, adieu!*

——— Farewell happy fields
Where joy for ever dwells. *Hail* horrors! *hail*
Infernal world. MILTON.

And while I stode, this derke and pale
Reason began to me her tale:
She saied *alhaile!* my swete frend.
CHAUCER.

Of *hail!* JUNIUS thus speaks, "hæc salutandi formula ex pervetustâ Gothorum, Anglo-Saxonumque, Francorumque, consuetudine." Hence in St. Mark's *Gospel*, (c. xv. v. 18.) the Greek Χαῖρε βασιλεῦ τῶν Ἰουδαίων; and Latin "*ave!* rex Judæorum," are rendered in Gothic "*hails* thiudan Judoie,"—in Anglo-Saxon "*hal* beo thu Jodea cyning," and in Frankish "*heil* coning Judeono." From *hail* or *heil* come the Alamannic *heilizen* salutare, *heilizung* salutation.

Grammar.

Wachter thinks that the root of *hal* was *all* "quod eleganti migratione ab *omni* pervenit ad *totum*, a toto ad *sanum* et *salvum*," and he might have added "a *salvo* ad *sanctum*."

1. In the sense of *totus*, we find the Greek ὅλος, the old English *hole*, and modern English *whole*.

2. In the sense of *sanus* are the Gothic *hailai* sani, *unhailans* ægroti, *unhaili* infirmitates, *hailyan* sanare—the Frankish and Alamannic *heilon* sani, *unanaheilem* ægroti, *heilas* sanitatis expers, *heil*, sanatus, *heilen*, sanare, *heillihoor* salubrius — the Anglo-Saxon *hæl* sanus, *unhæl* ægrotus, *unhælo* invaletudo, *hælan* sanare —in modern German *heil werden* to be cured, *heilen* to cure, *heilbar* curable, *heilkraut* a sanatory herb, *heilsam* salutary, *heilung* a cure, *unheilbar* incurable, &c. —in English *hale*, *heal*, *health*, *healthy*, *healthful*, &c.

3. In the sense of *salvus* we have the Anglo-Saxon *hal* salvus, e. gr. "hwa mæg *hal* beon?" "Who can be saved?" (*Mark* x. 26.) *hæl* salus, e. gr. "the *hæl* ys of Judeum," "salvation is of the Jews," (*John* iv. 22.) and *hælend* the Saviour—the Frankish and Alamannic *heil* and *heiler* salvus, e. gr. "ther gilouhit inti gitoufit uuirdit ther uuirdit *heil*," "he who shall believe and be baptized shall be saved:" *heili* salus, *heilen* salvare, *heilant* salvator—in modern German *heil* well being, *unheil* misfortune, *Heiland* the Saviour, &c.

4. In the sense of *sanctus*, are the Frankish *heilag* and *heilig*, the Dutch and German *heilig*, the Swedish *helig*, the Anglo-Saxon *halig*, and the English *holy*, sanctus. Hence the German verb *heiligen* and English to *hallow*, sanctificare; the old English *hallows*, saints, *Allhallows* all saints, &c.

The Saxons and old English used the expression *was hæl!* salvus sis! in drinking to each other: whence the *wassail* or *wassel-cup*, and *wasselling* for carousing. In Mr. Dibdin's *Typographical Antiquities* we find a collation of a MS. English Chronicle, with Caxton's printed *Cronicles of Englond*, ed. 1480. The MS. contains this passage,—

> The monke toke a cup, and fillede hit with gode ale, and broughte before the king and sette him on his knees, and saide Sir, *watsaille!* for nevere dayes of youre lyf ne dronke ye suche ale.

In the printed copy, the word is more accurately spelt *wassayl*, being derived from the Anglo-Saxon *wesan* to be, and *hæl*, well. *Wes hæl!* "be well," is therefore the same, in substantial meaning, as the modern English compliment, on a similar occasion, *your health!* the French *à votre santé!* the Italian *salute!* &c.

Welcome! is a literal translation of the two French words *bien* and *venu*, which when used together as a substantive, are thus explained in the *Dictionnaire de l'Academie*. "Bien-venue, s. f. L'heureuse arrivée de quelqu'un. Il ne se dit proprement que de la première fois qu'on arrive en quelque endroit, ou qu'on est reçu en quelque corps: et parceque la coustume est de payer quelque droit en y entrant, ou de faire quelque regale à ceux qui en sont, on dit *payer sa bienvenue*; donner un repas *pour sa bienvenue*."

Benedicite! This Latin verb was used by our ancestors, in its proper sense, as an interjection of salutation, and more loosely as a mere expression of surprise; as the common people still say *bless me! bless my soul!* &c.

> Rom. Good morrow, Father
> Friar. ———— *Benedicite!*
>
> Shakspeare.

> ———————— *A benedicite!*
> What aileth swiche an old man for to chide?
>
> Chaucer.

Interjections.

Greeting! is a word which has travelled very far from its origin. In Greek we find *κρίζω*, and *κράζω* clamo, *κραυγὴ* and *κριγή*, clamor. The Gothic *greitan*, Cimbric *grada*, Icelandic *graata*, Swedish *græta*, Danish *græde*, Spanish *gridar*, Italian *gridare*, French *crier*, Scottish to *greet*, Dutch *kryten*, old English *grede*, and modern English to *cry*, all signify to weep, cry, call aloud, &c. Wachter says the old German *kreide*, clamor, is from *kræhen*, clamare; and *kræhen* appears to be connected with our verb to *crow*, and to give name to *kræhe*, in Frankish *chraia*, Dutch *kraai*, Anglo-Saxon and Scottish *crawe*, English a *crow*, corvus. From *kreide* came the Barbarous-Latin *crida*, and Italian *grida*, a proclamation. *Gridare* in Italian is explained "mandar fuori la voce, con alto suono—manifestare, pubblicare—mostrare, far comprendere—garrire, riprendere."—*Grætig* is applied in Suabia to signify a *squalling* child. It seems that to *greet* was very early used in Anglo-Saxon and old English, for to salute or wish joy to a person: and *greeting* was consequently used as a noun, signifying salutation or well being. Thus a charter of King Edward the Confessor begins, "Eadweard Kyng *gret* Rodberd biscop." The letter of king Henry III. A. D. 1258, begins "Henr. thurg Godes fultume King (&c) *send igretinge* to alle hise halde," i. e. Henry by God's grace King of England, &c. sends salutation to all his subjects. Afterwards the verb "send" was omitted in English, as the correspondent verb had before been in Latin and French; for the French copy of the last mentioned letter has "Henri, par le grace Dieu, Rey de Engleterre (&c.) a tuz ses feaus, *saluz*:" and another letter of the same year begins, "Domino Papæ, Rex Angliæ *salutem*." Thus *greeting* having lost its use, in regular construction, as a noun; and its original signification as such, being almost forgotten, it remains in modern official documents merely as a sort of interjection.

Farewell! is absurdly called by Johnson an adverb. He says: "Farewell, *adv*. This word is originally the imperative of the verb *fare well*, or *fare you well*; *sis felix, abi in bonam rem*; or *bene sit tibi*; but in time, use familiarised it to an adverb; and it is used both by those who go, and by those who are left." So R. Stephanus says of the Latin *vale*: "imperativus, quo utimur quum recedimus, vel quum remanentes respondemus abeunti."

> The long day thus gan I prye and poure,
> Till Phebus endit had his bemes bryt,
> And bad go *farewele* every lef and floure.
>
> *The King's Quair.*

To *fare* well is in modern usage applied chiefly to the food and other enjoyments of life; and the noun *fare* has this among other significations; but they all come no doubt from the Gothic *faran*, Anglo-Saxon *færan*, Alamannic *faren*, Cimbric *fara*, Danish *fare*, and Dutch *vaeren*, to go; which are connected with our *for*, *fore*, *forth*, *further*, &c.

Well to fare! is used as an interjectional phrase in *Gammer Gurton's Needle*.

> Hail fellow Hodg! and *well to fare*,
> With thy meat, if thou have any!

The Italian and French valedictions *addio!* and *adieu!* are manifest interjections, being abbreviations of the phrase "I commit you *to God*."

Grammar.

O yes!

A more unmeaning exclamation cannot well be conceived, than that appears at first sight to be, which is used by our public criers to call attention in courts of justice, &c. viz. *O! yes!—O! yes!—O! yes!* It is however the imperative of the old French *oyer*, the modern *ouir*, a corruption of the Latin *audire*, to hear; so that it exactly coincides with the exclamation *hear! hear!* so much used in our senate. Both *O yes* and *hear!* may properly be styled interjections. The same we may say of many miscellaneous exclamations applied to incidental circumstances, as "*Anon! anon*, Sir!" used by the waiter, *Francis*, in *K. Henry IV.*—*coming!* the more modern exclamation of a waiter—*going!* that of an auctioneer—*lullaby!* and *hushaby!* those of a nurse lulling and hushing an infant, &c. &c.

Imitative sounds.

Finally, we may revert to the imitative sounds, of which we before spoke. Although considered as mere imitations they can hardly be called words, or reckoned among the parts of speech; yet it often happens that a certain degree of mental emotion is mixed with their utterance, and they may then perhaps be not improperly denominated interjections. Thus the lively Scottish poet BURNS gives great animation to his description by the sounds *click! jee! fuff!* &c.

> When *click!* the string the sneck did draw;
> And *jee!* the door gaed to the wa'. *The Vision.*

> He bleez'd owre her, and she owre him,
> As they wad never mair part;
> Till *fuff!* he started up the lum,
> And Jean had e'en a sair heart. *Hallow e'en.*

The German poet BURGER uses similar onomatopœias with equal effect,—

> Und jedes heer mit *sing* und *sang*,
> Mit paukenschlag, und *kling!* und *klang*.
> Geschmückt mit grünen reisen,
> Zog heim zu seinem häusen. *Lenore.*

The sounds *kling* and *klang* are connected with the German verb *klingen* to sound (like a bell) with our words *clink* and *clang*, the Greek *κλάγγω* and *κλαγγή*, and the Latin *clangor* and *clanga*. The German composite *wohlklang* signifies harmony.

Very similar to this is our *ding, dong bell!* used interjectionally.—

> Sea-nymphs hourly ring his bell.
> Hark! now I hear them—*ding, dong, bell!*
> SHAKSPEARE.

The same may be said of *tantara! tantara!* imitating the trumpet; *row-de-dow-dow!* the drum; *rat-a-tat-tat!* the knocking at a door, &c.

The German interjection *schnapp!* or *schnapps!* (quickly!) may perhaps be ranked among these imitative sounds. It is however connected with the German verb *schnappen*, the Swedish *snappa*, and our *snap*. The Dutch say *met een snop*, "in a trice." The French *habber* is to snap, and the Italian *chiappare* appears to be of the same family. Our word *slap!* is used like the German *schnapp!* in the following lines of a ludicrous poet,—

> Up comes a man, on a sudden, *slap! dash!*
> Snuffs the candles, and carries away all the cash.
> ANSTEY.

The French *glou! gloux!* is used to imitate the *gurgling* sound of liquor from a bottle, as by *Sganarelle* in the *Medécin malgré lui*.—

> Qu'ils sont doux, bouteille jolie!
> Qu'ils sont doux, vos petits *glou glous!*
> MOLIERE.

Interjections.

The songs and cries of birds are imitated by such sounds as *pepe!—jug, jug!—tirra-lirra!—too-hoo!—cuckoo*. &c.

> Now, swete bird, say ones to me *pepe!*
> *The King's Quair.*

> And murmurs musical, and swift *jug! jug!*
> COLERIDGE.

> Then nightly sings the staring owl,
> *Tu-whit! to-who!* a merry note!
> SHAKSPEARE.

> The lark that *tirra-lirra!* chaunts. IDEM.

> The cuckoo then on ev'ry tree,
> Mocks married men, for thus sings he—
> *Cuckoo! cuckoo!*—O word of fear! IDEM.

In like manner many loose syllables and imperfect articulations are used to imitate human laughter, coughing, whistling, singing, &c. such as *ha! ha! ha!—te! he!—ugh!—whew!—tol de rol lol!* &c. which require no particular notice.

We have not pretended to reduce the great variety of interjections to a complete and systematic arrangement. The only attempt of the kind which deserves attention is that of the very ingenious Bishop WILKINS; but it is a mere outline, and is meant to include only "rude, incondite sounds," the "natural signs of our mental motions or passions," and "several of which are common with us to brute creatures." It is as follows:—

1. *Solitary*, the result of a surprised
   1. *judgment*, denoting
      1. *admiration*, heigh!
      2. *doubt* or consideration, hem! hm! hy!
      3. *contempt*, pish! shy! tysh!
   2. *affection* moved by apprehension of good or evil
      1. *past* { *mirth*, ha! ha! he! / *sorrow*, hoi! oh! oh! ah!
      2. *present* { *love* and pity, ah! alack! alas! / *hate* and anger, vauh! hau!
      3. *future* { *desire*, O! O that! / *aversion*, phy!
2. *Social*
   1. *preceding* discourse
      1. *exclaiming*, oh! soho!
      2. *silencing*, 'st! hush!
   2. *beginning* discourse
      1. to dispose the *senses* of the hearer
         1. *bespeaking* attention, ho! oh!
         2. *expressing* attention, ha!
      2. to dispose the *affections* of the hearer
         1. by way of *insinuation*, eja! now!
         2. by way of *threatening*, væ! wo!

Even this short scheme shows the error of the learned WALLIS in supposing that there were but *few* interjections in the English language: and it furnishes ground for two or three other observations of some importance in grammatical science. The first is, that no precise line can be drawn between interjections consisting of "incondite sounds" the "*natural* signs" of mental emotion, and exclamations derived from a partial exercise of the reasoning faculty; for among the sounds enumerated by Wilkins we find *alas!* derived from the regular Latin adjective *lassus*—*alack!* from the English verb *to lack*, and Dutch *laecken*—*hush!* from the Gothic verb *hausyan*—and *væ!* identical with the English noun *woe*. That the noun and the mere incondite sound are used as equivalents, and with the same sort of grammatical construction, we see in the following lines of BUTLER.—

Intrust it under solemn vows
Of *mum!* and *silence*, and the *rose* — *Hudibras.*

We may next observe, that the same interjection expresses very different emotions. Thus we find Wilkins describing *oh!* as an expression of sorrow, as an exclamation preceding discourse, and as bespeaking attention in discourse. These variations then depend not on the articulation, but on the intonation; that is, not on the letters which go to form the word, but on the elevation or depression of voice in pronouncing it: but this is not peculiar to the interjection *oh!* or to the "incondite" interjections generally; for the same may be observed of any nouns or verbs used interjectionally. Thus we say impatiently, "*well!* and what of that?"—or with patient acquiescence, "*well!* never mind: it can't be helped." So there is great difference between the affected gravity of Falstaff's imprecation, *plague!* and the same imprecation seriously uttered against Apemantus.

FALST. A *plague* of sighing and grief! It blows a man up, like a bladder. — *First part Henry IV.*

CAPH. Stay, stay, here comes the fool, with Apemantus.
SERV. Hang him! He'll abuse us.
ISID. ——— A *plague* upon him! Dog! — *Timon.*

The scheme of Wilkins too, short as it is, helps to illustrate the connection which we have already pointed out between the *interjection* on the one hand, and the *vocative* case, *imperative* mood, and *interrogative* form of the verb on the other. *Wo!* which he properly ranks among interjections, is the vocative case of a noun, so used.—*Hush!* (like hark! lo! oyez! &c.) is the imperative mood of a verb. The interrogative is in some degree implied by *hem!* or *hm!* which he considers as interjections of doubt. It is more distinctly marked in French by the word *puis*, as explained in the *Dictionnaire de l'Academie*. "On dit, par ellipse, et par interrogation, *et puis!* pour dire, eh bien! qu'en arrivera-t-il? que s'ensuivra-t-il? que fera-t-on apres? Ou bien, qu'en arriva-t-il? que s'ensuivit-il?"

Thus have we shown the propriety of ranking the interjection as a separate part of speech, determinable as all the other parts of speech are, not by its sound or derivation, but by its use in the particular passage which may be under consideration. We have shown that it evinces actual emotion of the mind, but does not assert the existence of such emotion. Lastly, we have endeavoured to illustrate the nice shades and gradations by which as emotion passes into conception or assertion, in the human mind, and *vice versâ*; so the interjection rises to a noun, a verb, or a phrase, and the phrase, verb, or noun sinks into an interjection. And thus have we concluded our survey of *words* as distributed into those classes which grammarians call the *parts of speech.*

§ 10. *Of particles.*

Having treated of *sentences* and *words*, it only remains to inquire whether we cannot carry our grammatical analysis still further, and examine the *constituent parts* of words. Now, words may be resolved into syllables; and syllables may be resolved into the articulations, which are marked in writing by letters: and this part of grammar is called *orthography*; but as it relates wholly to the *sound*, and not at all to the *signification* of words, it has nothing to do with our present inquiry. It is part of the *art* of grammar; but no part of the *science*.

Nevertheless, though we have called words "the primary integers of significant language," and have denominated the classes into which they are divided the *parts of speech*; yet, even with reference to signification, there are certain fractions, if we may so speak, which go to make up these integers. Thus if we say, "*Johnson was learned*"—"*Friendship is delightful*;" each of these sentences, *as a sentence*, contains three, and only three, significant parts; viz. a subject, a predicate, and a copula; and each of these parts is a *word*. But if we take one of these words, and inquire how it comes to possess its actual signification, we may find that this is owing to the peculiar force and effect of its separate portions. Thus, in the word *Johnson*, there are two portions, *John* and *son*, which, taken separately, would be significant; and which, when put together, form a third signification relating to the two former. Again, in the word *friendship*, there are two portions, *friend* and *ship*, each significant, when taken separately; and the relation of the word *friend* to *friendship* is very obvious, but the relation of *ship* to *friendship* is not equally so, at first sight, though it may be discovered by study and reflection, as will hereafter be shown. Lastly, the word *learned*, may, in like manner, be divided into two portions, *learn* and *ed*, of which the former has a clear meaning of its own; but the latter, if it ever had a distinct and separate meaning, has long since lost it, and serves only to mark that *learned* is a participle of the verb to *learn*. The three words, *Johnson*, *friendship*, and *learned*, therefore, are manifest compounds, each consisting of a primary part, which is modified by a secondary part. *John* is modified by *son*, *friend* by *ship*, and *learn* by *ed*. The primary parts in such compounds are commonly *words*, that is, when used separately, they have a plain and distinct signification of their own. The secondary parts may or may not have such separate signification; and their signification, if any, may be more or less obvious. These secondary parts, we call *particles*, when so used in composition. Thus, we say, that in the word Johnson, *son* is a particle; in the word friendship, *ship* is a particle; and in the word learned, *ed* is a particle.

Particles modify words in three different ways, and with three different effects.

1. In the ordinary compounds, such as *Johnson*, *noonmete*, *overtake*, *forewarn*, *erewhile*, *elsewhere*, there is no alteration of the principal word, either by changing the grammatical class to which it belongs, or by varying the grammatical construction of the sentence in which it is used.

2. In such compounds as *friendship*, *bisyhed*, *procurour*, *gadelyng*, *avette*, *masterless*, *delightful*, *blaunchard*, *lovely*, *lolich*, *sweetly*, &c. the grammatical class of the word is more or less altered; thus, from the personal substantive, *friend*, we form the abstract substantive, *friendship*; from the common substantive *apis*, we form the diminutive substantive *avette*; from the common adjective *blanche*, we form the diminutive adjective *blaunchard*; from the adjective *busy*, we form the substantive *bisyhed*; from the substantive *master*, we form the adjective *masterless*; from the adjective *sweet*, we form the adverb *sweetly*, and so forth.

3. In such compounds as *growen*, *beon*, *makede*, *walked*, *monethes*, *children*, &c. the principal word is

Grammar. varied in its construction, by the particles *en*, *on*, *ede*, *ed*, *es*, &c.; and thus are formed those inflections, which grammarians call declensions and conjugations.

We shall trace the first sort of compounds, beginning with the more obvious, and proceeding to the more obscure.

Son, &c. The word *Johnson* was manifestly in its origin nothing more than *John's son*. Thus in all languages have been formed *patronymics*, the most ancient of all family names. The Greeks did this in several instances, whence such names as *Æacides*, *Pelides*, *Atrides*, &c.; but the Romans adopted it generally at a very early period of their history. "Remarquons sur les noms propres des familles Romaines, (says M. DE BROSSES,) qu'il n'y en a *pas un seul* chez eux, qui ne soit terminé en *ius*, desinence fort semblable à l' υἱος des Grecs, c'est-à-dire *filius*—par où on pourrait conjecturer que les noms des familles, du moins ceux des anciennes maisons, seraient du genre patronimique." Thus Cæcilius was *Cæculæ* υἱος, *Julius*, *Juli* υἱος, *Æmilius*, Æmili υἱος, &c. Mr. TOOKE says, "I think it not unworthy of remark, that whilst the old patronymical termination of our northern ancestors was *son*, the Sclavonic and Russian patronymic was *of*. Thus whom the English and Swedes named *Peterson*, the Russians called *Peterhof*. And as a polite foreign affectation afterwards lodged some of our ancestors to assume *Fitz* (i. e. *fils* or *filius*,) instead of *son*; so the Russian affectation, in more modern times changed *of* to *vitch*, (i. e. fitz, fils, or filios) and *Peterhof*, became *Petrovitch*, or *Petrowitz*." The Irish patronymic O' may possibly be of the same origin as the Russian *of*. The Welsh 'P is well known to be *ap*, an abbreviation of *mab*, a son, as *Price* for Ap Rhys, *Powell* for Ap Hoël, &c.: the Scottish highlanders used the cognate word *mac*, a son, for their patronymical prefix, as in *Mac Donald*, i. e. the son of Donald, *Mac Kenzie*, (i. e. the son of Kenneth,) &c.; while the lowland Scotch used still a different mode of expressing the same thing, by prefixing to the son's name the genitive case of the father's, as *Watt's Robin*, for Robert the son of Walter; *Sim's Will*, for William, the son of Simon, whence arose such family names, as *Watts*, *Sims*, and the like: and so much for the particles *son*, *ius*, *fitz*, *of*, *vich*, *mac*, *O'*, *'P*, and *'S*.

The proper name, *Johnson*, is no less obviously a compound, than *watchman*, *spearman*, *boat-hook*, and thousands of similar words in common use. There are also many that have fallen into disuse, though still perfectly intelligible; e. gr. *nonemete*, a meal formerly eaten by artificers at noon, but which seems to be distinguished from dinner.

> Divers artificers and laborers reteyned to werke and serve, waste moch part of the day, and deserve not their wagis, summe tyme in late commyng unto their werke, erly departing therfro, longe sitting at ther brekfast at ther dyner and *nonemete*, and long tyme of sleping at after none.
>
> *Stat.* 2 HEN. VII. c. xxii. MS.

And as we have the word *noonmeat*, so we have the words, *noontide*, *noonday*, *mid-day*, *mid-night*, *forenoon*, *afternoon*, &c. all nouns compounded on similar principles; for as *noon* modifies *meat*, so *mid* modifies *night*, and *fore* modifies *noon*: and thus *noon*, *mid*, and *fore* are equally to be considered in these three instances respectively, as particles. So, in the compound verb *overtake*, *over* is a particle modifying take; and this particle, *over*, is sometimes corrupted into *or*, as in the word *orlop*, which is a platform of planks laid over the beams in the hold of a ship of war; so named from the Dutch *overloopen*, to run over, and anciently written in English *overloppe*. Particles.

> Somuche as they shall put greater nomber of people in the castelles and *overloppe* of their shypps they shalbe the more oppressed. NICOLLS'S *Thucydides*, fol. 191. a.

In Danish, this same preposition *over*, written *ober*, is used as a particle in compound nouns, as *oberdommer* the chief-justice.

We have already noticed the particle *fore* which occurs in *forewarn*, and in many other compound verbs; e. gr.

> *Forwakit* and *forwalloit* thus musing
> Wery *forlyin*, I lestnyt sodaynlye.
>
> *The King's Quair.*

Erewhile and *elsewhere* are compound adverbs, of which we have already noticed the constituent parts *ere*, *else*, *while*, and *where*. In addition to what we have said of *else*, we may observe that the particle *el* occurs with a similar effect in the Danish *eller*, "or," and *ellers*, "else."

In proceeding to compounds, which, by course of time, and change of pronunciation, have become less obvious, we will begin as before, with some proper names. M. DE BROSSES says with great truth, "tous les mots formant les noms propres, ou appellatifs des personnes, ont, en quelque langage que ce soit, on origine certaine, une signification déterminée, une étymologie veritable." VERSTEGAN has preserved a rude distich not unworthy of notice, in this respect.

> In *foord*, in *ham*, in *ley*, in *tun* Ford, &c.
> The most of English surnames run.

Thus, says he, "the sirname of *Rainford*, now *Rainsford*, seemeth to have risen by reason that the first of this name had his dwelling at a passage or *foord* caused through *raine*."—"*Ham* originally signifieth a coverture or place of shelter, and is thence growo to signifie one's *home*, as now uncomposed we pronounce it—it is one of our greatest terminations of sirnames, as of *Denham*, for having his home or residence *downe* in a valley; of *Higham* for the situation of his *ham* or *home* upon *high* ground; and accordingly of many others."—"*Legh*, *ley*, or *lea*, howsoever wee do now distinguish these terminations, I take them to have been anciently all one, and to signifie ground that lieth unmanured and wildly overgrowne;"—hence *Berkley*, "of birch trees, anciently called *berk*," *Bromley*, "of the store of *broom*," and *Bramley*, "of *lee* or *legh* ground bearing bramhles." Of the name *Lesley*, he relates this story, "A combat being once fought in Scotland betweene a gentleman of the family of the Lesieyes and a knight of Hungary, wherein the Scottish gentleman was victor: in memory thereof, and of the place where it happened, these ensuing verses doe in Scotland yet remaine.

> 'Betweene the *lesse ley*, and the mare
> He slew the Knight, and left him there.'

"Though the name of hedge doe anciently appertaine to our language, yet we also used sometimes for the same thing the name of *tun*. In the Netherlands they yet call it a *tuyn*; and in some parts of England they will say" hedging and *tining*." Our ancestors, in time of war, to prevent themselves from being spoyled, would, in stead of a palizado as is now used, cast a ditch, and make a strong hedge about their houses,

Grammar.

and the houses, so environed about with *tunes* or *hedges*, gat the name of *tunes* annexed uoto them. As *Cote-tun*, oow *Cotton*, for that his *cote*, or house, was fenced or *tuned* about; *North-tun*, now *Narton* in regard of the opposite situation thereof from *South-tun* now *Sutton*. Moreover, when necessity, hy reasoo of warres and troubles, caused whole thorpes to be with such *tunes* eovironed about, those enclosed places did therehy take the name of *tunes* afterward pronounced *towns*." To the same effect Junius says "Town, villa, vicus, pagus, et in genere, quilibet locus conelusus et circumseptus. A. S. *tun*, Al. *zun*, B. *tuyn*, sunt ab A. S. *tynan*, *betynan*, claudere, circumsepire." And Lye says "*tine* the door, fores claudere, ab A. S. *tynan* claudere;" which expression "*tine* the door" is also noticed hy Grose in his *Provincial Glossary*. In Dutch, *tuyn* is in its first sense the hedge of a garden, and then the garden itself: it is also used for some other inclosed places, as *een hout-tuyn*, a wood-yard. So in Scotland, the *toon* means the inclosure round about a farm-house, and in Cornwall tha *town-place* means the farm-yard.

Worth.

The last mentioned particle *ton*, has much affinity in point of signification to the particle *worth*, also very common in English names of places and thence of persons. "Anciently," says Verstegan, "it was *wearth*, and *weard*, whereof yet the name of *werd* remaineth to divers places in Germany, as *Thonawerd*, (*Donawert*, *Danubii Insula*,) *Keyserwerd*, *Bomelswerd* and the like; and in England, to the same sense and signification, the names of *Tamwoorth*, *Kenelme-woorth*, and the like. A *wearth* or *werd* is a place situate betweene two rivers, or the nooke of land where two waters, passing hy the two sides thereof, doe enter ioto the other; such nooks of grouod having of old time beene chosen out for places of safety, where people might be *warded* or defended in." Verstegan has here described only ooe kind of *worth* or *wearth*; for this word, (which is the same with *garth* or *yard*,) signifies any inclosure whatever. Indeed its first signification is the act of *girding* or surrounding, then the thing which *girds* or surrounds, then the thing *girded* or surrounded, then the purpose for which it is surrounded, namely to *guard* it, then the thing *guarded*, the person *guarding* it, and so forth.

1. The act of *girding* or surrounding is expressed hy the Gothie verh *gaurdan*, the Anglo-Saxon *gyrdan*, the Frankish and Alamannic *gurten* and *curten*, the Danish *gyrde*, the Icelandic *gyrda*, the Swedish *giorda*, the Dutch *gorten*, the German *gurten*, and the English to *gird*: and all these have an evident affinity to the Greek κίρκος, and the Latin *circus*, *circulus*, *circum*, &c.

2. Various things used for *girding* or surrounding were hence named; e. gr.

A belt, which is tied *round* the body of a man, horse, &c.; in Gothic *gaird*, in modern German *gurt*, in English *girth* and *girdle*, in Anglo-Saxon aod Danish *gyrdel*, in Alamanoic *gurdel*, in Dutch *gortel*.

A curtain, which is drawn *round* a bed; io Dutch *gardyn*, in later Latin, Italian and Spanish *cortina*, in old French *courtine*, in English *curtain*.

The bark, which *surrounds* the body of a tree, in Latin *cortex*.

A hedge, which *surrounds* a garden or other ioclosed place; in Anglo-Saxon *geard*, in Swedish *girde*, in Danish *gierde*; and so the act of hedging *round* about a place, is in Cimbric *gertha*, in Swedish *gaerda*, and in Danish *at gierde*.

Particles.

Lastly any hoop or band which *surrounds* things, is called in the north of England a *garth*.

3. Among things *surrounded*, which derive their names from this sonrce, may he particularised the following. The old Latin *cors*, *cortis* signified a farm-yard, or inclosed space before a country house; whence the Barbarous-Latin *curtis*, Italian *corte*, old French *court* and English *court* often applied formerly as the name of a country house. The Gothie *gards*, Danish *gaard*, Icelandic *gard*, Cimbrie *garthur*, signified a house or farm; the Anglo-Saxon *geard*, or *yeard* an inclosed space, as *win-gaard* a vioeyard, *ortgeard*, an orehard or garden (in Gothic *aurtigards*, from the Gothic *waurts*, and Anglo-Saxon *wurt* or *ort*, a root) the Frankish and Alamannic *gardo* and *karto*, Welsh *gardd*, Danish *gaard*, Dutch *gaerde*, Italian *giardina*, Spanish *gardin*, French *jardin*, German *garten* and English *garden*, hortus. The modern English *yard*, the provincial English *garth*, and the old English *wearth* or *worth*, are only variatioos in pronunciation from the Anglo-Saxon *geard* or *yeard*. In the north of England *garth* is still used generally for a *yard* or inclosed place; so *churchgarth* is a chnrchyard, *stockgarth* a rickyard, &c. and in Scotland *ward* is used in the same sense.

> His braw *calf-ward*, whare gowans grew,
> Sae white and bonie,
> Nae doubt they'll rive it wi' the plew. Burns.

Hence origioated many English names of places, and consequently of persons; as *Kenilwarth*, i. e. *Kenelm's wearth*, or Kenelm's inclosure; *Wordsworth*, i. e. the *Wurts' wearth*, or garden of roots (as before explained under the word *orchard*;) *Holdsworth*, i. e. the *Holts' wearth*, or inclosure of trees; *Applegarth*, the inclosure of apple-trees; *Haygarth* the hay-yard; *Hoggart* the hog-yard (or sheep inclosnre, some sheep heing provincially called hogs,) *Garth*, the ioclosure, &c.

Moreover, as places were often inclosed for defence, *gard*, and its cogoate sounds came to signify a fortified place, or city. Heoce the Cimbric *gard* and *garthur*, a fortification; the Icelandic *gard*, a city; the Sclavonic terminations *grod* and *gradz*, as in *Novogrod* castrum novum, and *Belgrade* castrum album; the German termination *gard*, as in *Stutgard*, (from *stut* a horse,) civitas equaria: hence also the Freoch *boulevard*, corrupted from *burgward*, in Barbarous-Latin *burgwardium*, munitio oppidi.

4. The English verhs to *guard* and to *ward*, which are the same word differently pronounced, agree with the Gothic *wardyan*, Anglo-Saxon *weardian*, Alemannic *uuarten*, Icelandic *varda*, Italian *guardare*, Spanish *guardar*, and French *garder*, to protect, and keep. Hence the Anglo-Saxon *weard*, which is both custos and custodia. So in English we have *guard*, *guardian*, and *warden*, the persoo who defends, protects, or keeps; *ward* the act of safe custody, the place where prisoners or others are safely kept, and the person who is under the protection of a guardian. The Anglo-Saxon *weard* custos appears frequently in composition, as *dureweard* a porter, in old English a *gateward*—

> Wer ye nou this *gate ward*?
> Me thuncheth he is a coward.
> *Christ's Descent to Hell.*

Many other employments were designated hy this particle *ward*, which have since become proper names of families, whence *Howard*, *Hayward*, *Woodward*,

Grammar. *Stewart, Stoddart,* &c. *Howard,* says VERSTEGAN, "came of *Holdward,* which signifieth the governour or keeper of a castle, fort, or *hold* of warre." *Hayward* was the person who had the care of the hedges.

He hath hewe sumwher a burthen of brere,
Thare fore sum *hayward* hath taken ys wed.
Ballad of the Man in the Moon.

Woodward is explained by LYE, "sylvæ custos, saltuarius."—*Stewart* is from the Anglo-Saxon and old English *stiward* or *styward,* and modern English *steward.*

The kyng com in to halle,
Among his knyhtes alle,
Forth he clepeth Athebrus,
His *stiward,* and him seide thus;
Stiward tac thou here
My fundling forto lere
Geste of Kyng Horn.

The *styward* walkyd there withall
Among the lordes in the hall. *Sir Cleges.*

The *styward* tolde Rychard the Kyng
Sone anon off that tydng
Richard Coer de Lion.

Stywarde, as thou art me lefe,
Let no mon wytte of my myschefe.
Sir Amadas.

That every *styward, understyward,* baillif, commissarie or other mynystre holdyng and rulyng any of the seid courtes that doth the contrary of this ordinance shall forfeit an C s.
Stat. 1. RIC. III. c. 6. *MS.*

In the Icelandic, this word is *stivardur,* from *stia* opus and *vardur* custos: and the word *stia* seems to be connected with the Italian *stivare,* to *stow* goods or ballast in a ship.

Stoddart is from the old English *Stodward,* equorum custos. A family of this name was anciently settled near *Stodmarsh,* in Kent, and the name of the place as well as that of the person was derived from the Anglo-Saxon *stod* and *steda,* (in Swedish *stod,* in Alamannic *stud,* in Icelandic *stedda,*) a horse, whence come our modern *steed* and *stud.* In the Anglo-Saxon also are found *stod-hors,* a stallion, (connected with the Danish *stod-hest,*) *stod-myro,* a stud-mare, and *stod-fold* an inclosure where horses are kept. DUCANGE explains *stuot,* equus admissarius; and WACHTER gives the same explanation of the old German *stut.* The modern German *stute* is a brood mare. In old English we also find *stot,* used for a horse.

This Reue sate upon a right good *stot,*
That was all pomell gray, and hight Scot.
CHAUCER.

Hence are derived many other old English names of places and persons, as *Stodham, Stodinton, Stodelegh, Studlay, Stoteville, Stuteville, Stovel, Stotfald, Stoutesfeld, Stutfeld, Stottesden, Stottesdon, Stuton, Stotenghem, Stoteney, Stotesbrok, Stotholme,* &c.

Reverting to the particle *worth,* we must observe that it has sometimes a very different origin from that which we have above noticed; for the substantive *worth,* value, derived from *wirthan,* to be, is often used as a particle. Hence the substantive *worthship,* or *worship,* is estimation, and the verb to *worship,* to hold in esteem or reverence.

The proffit and the *worship* of the same roialme.
Treaty HEN. V. A. D. 1420.

Thaime, as our fadir and modir, we shall have and *worship*
Ibid.

Thowe haste onowryd all my fest,
And *worschepyd* me also. *Sir Cleges.* Particles.

In this sense, magistrates are called "Your *Worship;*" and designated "*Worshipful.*" We find in old English the adjective *derworth,* signifying precious.

Now Jesu for thi *derworth* blode.
MS. Harl. No. 913. fol. 29. b.

So, in Danish, *elskværdig* is "worthy of love or esteem," from *værer* to be, *værd,* worth.

Thus have we examined the particles, *ford, ham, ley, ton, worth, garth, word,* &c. which enter into the formation of so many proper names. Nor should the grammarians disregard this class of words; for in them are often preserved many traces of connection between different dialects, contributing much to the illustration of the whole. Thus the English name *Fairfax,* i. e. fair-haired, retains the Anglo-Saxon *feax* crinis. The Spanish *Ferdinand,* shows the connection of Spain with the Goths, being derived from *pferd dienend* equn serviens. The Scottish *Telfair,* anciently written *tailyfeir, tailliefeir,* and *tailefeir,* is the French *toille-fer,* cut-iron; as *Playfair* is *pli-fer,* or bend-iron.

Stead, &c. The particles *stead, rick,* and *dom,* are often applied, in modern use, to express locality. *Stead,* which we have before had occasion to notice, is the Anglo-Saxon noun *sted,* Gothic *stads,* Alamannic *stat,* Dutch *stad,* and old English *stede,* a place. This word is used as a particle in *gyrdylstede, hache-styd,* &c.

To ech a *stede* the Kyng hym sente
He wan the fyght. *Octovian Imperator.*
Some he hytte on the bacyn,
That he cleff him to the chyn;
And some to the *gyrdyl stede.*
Richard Coer de Lion.
Thei myghtt not passe the dure threscwold,
Nor lope ouer the *hache-styd.*
The Huntyng of the Hare.

And so in the modern words *bedstead, roadstead, homestead,* with which agrees the Danish *fyr-sted,* a fire-place.

Rick is the obsolete English noun *riche,* and modern German *reich,* a kingdom.

He that made heaven and erthe
And sun and mone for to shine
Bring ous into his *riche,*
And scheld ous fram helle pine.
Legend of Seynt Katerine.

It is used as a particle in the modern English word *bishoprick,* as in old English it was usual to say *kingriche.*

Over londes he gan fare,
With sorwe and rewful chere,
Seven *kingriche* and mare
Tristrem to finde there. *Sir Tristrem.*

Thar salhe rasyt a general gelde or ma gif it misteris throu the haile *kinryk.* *Scottish Act. Parl.* A. D. 1424.

Dom is a particle of obscure origin, but of very extensive use in the different northen dialects. In the Anglo-Saxon, *dom* is judgment, from *deman* to *judge,* whence our words *doom* and *deem,* and the proper name *demster,* a judge. In Frankish *duam* is power, which WACHTER and ADELUNG seem to consider as the primary signification, from whence the special power of jurisdiction was derived as a secondary signification was derived. Hence the Anglo-Saxon *cynedom,* Dutch *koningdom,* and English *kingdom,* first for the power, and then for the territories of a king. So, in Frankish *rihthuom* is empire, and *hertuom* government,

Grammar. and in modern German *kaiserthum* is the empire, *herzogthum* a dukedom, *bisthum* a bishoprick, &c. and in these senses *dom* is probably connected with the Latin *domo* and *dominus*. Where *dom* merely signifies a general state, it may perhaps be connected with the verb *do*, as the correspondent German particle *thum*, in the same sense, may with the verb *thun*. Thus we have *freedom* and *thraldom*, the Germans *olterthum*, &c.

Gate, gang, are. *Gate*, *gong*, and *fare*, all originally signify going, as in *Ludgate*, *gangway*, *thoroughfare*. Hence the old English *algate*, the old Scottish *howgate*, the Danish *mellemgang*, an intercession or *going* between; the Dutch *gangbaar*, passable; the Scottish *auldfarran*, &c.

Leman, &c. As we have seen the word *worth* corrupted into the particle *wor*, in *worship*, so *lief-man* was corrupted into *le-man*, *wyfman* into *wo-man*, *god-sib* into *gos-sip*—*sib* is a relation; whence Robert de Brunne uses the word *sibred* for *kindred*. In our modern word *harbour*, the particle *bour* has undergone such a change as not to be easily recognised. To *harbour*, was, in old English, *herbarewen*.

> Herkneth hideward horsmen,
> A tidyng ichou telle;
> That ye shulen hongen
> And *herbarewen* in helle.
>
> *Satire on Horsemen.*

It is the same verb as the modern German *herbergen*, and comes from the Alamannic *hereberga*, compounded of *her* an army and *berg* o fortification. *Hereberga* therefore first meant the safe quarters of an army; thence any place of safe resort, and thence a place of safe resort for travellers, or for ships. Hence the Dutch *herberg*, Italian *albergo*, Spanish *alvergue*, and French *auberge*. Hence also the old English *herbergeour*, a person sent before to announce the approaching arrival of an army at its quarters, or of a traveller at his inn; which word we have corrupted to *harbinger*, used generally as a forerunner or precursor.

> And now of love they treat, till th' ev'ning star,
> Love's *harbinger* appear'd. Milton.

The particle *bour*, in our word *neighbour*, is of a different origin. It seems to agree with the German *bar* in *nachbar*, which some derive from *nach* nigh and *bauer* an inhabitant. *Bar*, however, is a particle of extensive use in German, and may, in its various applications, come from *baeren* to bear, or *faren* to do; as in *lastbar*, *brauchbar*, *dienstbar*, &c. &c.

Certain particles are frequently confounded with words which they resemble only in sound. Thus the particle *gale*, in *nightingale*, has no relation to the noun

Nightingale, roundelay, &c. *gale*, a breeze; but like the German *nachtigal* is derived from the Icelandic *gala*, and Anglo-Saxon *galan*, to sing: and these seem to have some relation to *wail*, whence the name of the bird called the *wodewale*. To *gale* was metaphorically used, in old English, for "to jest."

> And whan the Sompnour herd the Frere *gale*.
> Chaucer.

So *round* in roundelay, has no relation to a circle; but is derived from the verb to *roun*, to sing, or hum over a song, whence a song was called a *roun*.

> Lenten ys come, with loue, to toune,
> With blosmen, and with briddes *roune*.
> *MS. Harl.* 2253. fol. 71. b.
> Geynest under gore, herkne to my *roun*.
> *Ibid.* fol. 63. b.

Particles. *Lace* did not anciently mean the elegant manufacture so termed in modern days, but any thing which served for the purpose of a girdle or strap: whence the *anlas* or *anlace*, was a kind of knife or dagger, so called because it hung *on* a *lace* or strap at the girdle, as described by Chaucer.

> A dagger hanging by a *las* had he.

The modern particle *lass* in *cutlass* seems to have been ignorantly taken from the old word *anlas*.

One, two &c. The numbers *one*, *two*, and *ten* are not at first sight obvious as particles, when entering into the compound words *eleven*, *twelve*, *forty*, &c.; but we easily see that the particle *on*, in the old English *onlevene*, is the numeral word *one*.

> *Onlevene* thousand off our meyné
> Ther were slayn withouten pyté.
> *Richard Coer de Lion.*

So the particles *twa* and *zuue* in the Gothic *twalif*, and Frankish *zuuelf*, twelve, are easily recognised as the numerals *twa* and *zuuei* two, in those languages respectively: hence the Gothic *twalif*, Swedish and Icelandic *tolf*, Dutch *twaalf*, Anglo-Saxon *twelf*, Frankish *zuuelif*, and German *zwölf*, all evidently mean *two left*, as *onlevene* means *one left*, over and above the perfect number *ten*.

In like manner the particle *tig*, which Junius supposes to have been the old Gothic numeral *ten*, is seen in *twaimtig*, *thrinstig*, *fidwortig*, which are twenty, thirty, and forty, in that language. And this same particle *tig* was also retained in old English; as in the letter of Henry III. before quoted.

> Witnesse usselven æt Lunden, thane egtetenthe day on the monthe of Octobr' in the two and *fowertigthe* yeare of ure cruninge.

There are numberless other compounds of the kind which we have hitherto considered; namely those, which merely unite two conceptions, without changing the grammatical class, to which the principal portion of the word belongs.

We now come to words in which, by a slight inflection, the class that the word belongs to is altered.

Ship, head, &c. *Friendship* is such a compound, and the word *friend*, which forms the primary part of it, is sufficiently obvious; but what is *ship?* In order to answer this we must look through the other dialects, in which it occurs. The Germans use the termination *schaft*, the Dutch *schap*, and the Swedes *skap*: and these are manifestly from the Gothic *skapan*, Anglo-Saxon *scaapan*, or *scyppan*, Frankish and Alamannic *scaffen*, Dutch *scheppen*, Icelandic *skapa* and *skipa*, Danish *skaber*, and old English to *shup*, i.e. to shape, make, or do.

> The *shuppere* that huem *shupte*
> To shome he huem shadde.
> *Satire on Horsemen.*
>
> Wymmen were the beste thing,
> That *shup* our heye heuene kyng.
> *MS. Harl.* 2253. fol. 71. b.

Friendship therefore is the action, the work, of a friend: Chaucer uses *gladshipe*.

> That *gladshipe* he hath al forsake.

In Danish we find *selkskab*, a fellowship; in Anglo-Saxon *ealdor-scipe*, *cynescipe*, *sib-scipe*, &c. In German *herrschaft*, *eigenschaft*, *gesellschaft*, &c. &c.

The particle *scape*, in *landscape*, is the same as *ship*; for we find in Anglo-Saxon *landscipe*, in Dutch *landschap*, and in German *landschafft*

Grammar

The particle *head* or *hood* has nothing to do with the common noun *head*, from which some ignorant grammarians have supposed it to be derived. It is the Saxon *had*, status, and is probably connected with the pronoun *hyt*, it. In Danish the particle is *hed*; in German *heit*, and *keit*. *Heit* is used in Frankish as a *word* signifying "person"—e.gr. der ander *heit* Gotes "the second *person* of the Divinity." We find in Frankish *magadheit* virginity, *uuipheit* womanhood; in Anglo-Saxon *cniht-hade* childhood, *preost-hade* priesthood; in German *freyheit* freedom, *menschheit* human nature, *einsamkeit* solitariness, *seligkeit* happiness. In old English the particle *head* occurs in many compounds now disused; as *yunghead*, *wighthede*, *fairehed*, *brotherhede*, *boldehed*, *bisyhed*, &c.

The particle *ness* has been still more absurdly derived from the French *nez*, the nose. How any human being could ever have dreamt that greatness, in the abstract, was named from a *great nose*, redness from a *red nose*, or sweetness from a *sweet nose*, it is difficult to conceive. *Ness* appears to be nothing more than the French termination *esse*, preceded by the Saxon infinitive termination *en*. Thus from *great* would be formed the verb *greaten*, which would be converted into the abstract *greaten-esse*; so, *sweet*, *sweeten*, *sweeten-esse*; *red*, *redden*, *redden-esse*, &c. It must not however be omitted that the learned Hickes, with some doubt, suggests this termination to be taken from the Gothic *eins*.

Ess or *esse* is a particle common to the Anglo-Saxons, and the French; and it is probably a mere corruption of the Latin termination *etia* or *itia*. Gower uses *tristesse*. Some old English words ending in *ess* have, by modern corruption, been used as plurals; such are *riches* and *alms*, anciently *richesse*, and *almesse* or *elmesse*.

> Dame *richesse* on her honde gan lede
> A yong man full of semelyhede. CHAUCER.

> Sende god biforen him man,
> The while he mai, to heuene;
> For betere is on *elmesse* biforen,
> Thanne ben after seuene.
> *Digby MS.* (circa 1066.)

Besides these terminations of abstract nouns, we have, from the Latin and French, *ance*, *ence*, *dge*, *ery*, *our*, *ty*, as in *finance*, *credence*, *courage*, *drapery*, *honour*, *piety*, all which are manifestly framed, in the Latin original, by combining pronouns, and participial terminations with the radical word.

Other terminations of abstract nouns we have from Teutonic sources; such are *ledge*, *red*, *er*, *th*, *t*, &c. as in *knowledge*, *kindred*, *sibred*, *hunger*, *murder*, *death*, *sloth*, *drift*, *thrift*. *Ledge* seems to be formed from *lagen*, and *red* from *ræden*. Thus *rædesmen*, in King Henry the Third's letter, are counsellors, and in Scotland they still say "I read you not to do such a thing," for "I advise you not to do it." *Er*, *th*, and *t* are probably remains of Teutonic pronouns; the two latter are still used in the conjugation of our verbs. Hence *death* is that which *dieth* or maketh to die; *drift* is that which hath driven; *thrift* that which hath thriven, &c.

Ling is a diminutive, which Wachter thinks to be derived from *langen*, in the sense of *tangere* or of *pertinere*; thus the Anglo-Saxons used the word *earth-ling* for a husbandman, as we use *worldling* for a man of this world.

Particles.

The diminutive *et* is from the French *ette* and Italian *etto*. Thus we find the word *baronette* long prior to its institution as a separate dignity by King James I.

> But he wer prelat, other *baronette*.
> *MS. Cotton, Calig.* A.2. fol. 33.

Mr. Tyrwhit thinks that *doucet* was the name of a particular kind of musical instrument; it was probably no more than our adjective *dulcet*.

> Ther were trumpes and trumpetes
> Lowde shalmys and *doucetes*. LIDGATE.

i. e. "there were large and *small* instruments of the trump kind; and there were loud and *soft* instruments of the shawm kind.'

Full, *less*, and *some* are particles which give an adjectival force to a compound. The particles *full* and *some* are obviously identical with the words *full* and *some*. The particle *less*, in such words as *hopeless*, *restless*, *deathless*, *motionless*, &c. Mr. Tooke explains to be the imperative *les!* which (he says) is *dismiss*. It does not appear that *les* means *dismiss*: and if it did, how are we to explain by *dismiss*, the *word* (less) the comparative of little. It is well known that many adjectives are used as comparatives which have little or no affinity with the positives. Thus ἀμείνων is used as the comparative of ἀγαθὸς, *melior* of *bonus*, and *better* of *good*. So *less* seems to have been an adjective originally implying *want*. When compared with *little*, therefore, it would signify that quality in a stronger degree: but when compounded with such words as those above quoted, it might denote a total want or privation of the ideas they express. In the following instance, it appears to be used in the sense of wanting honour, evil, worthless, as we now say a *loose*, bad man.

> Bysshopes ant barouns come to the kynges pes,
> Ase men that weren fals, fykel, ant *les*.
> *MS. Harl.* 2253. fol. 59. b.

Ish is a particle of very ancient and general use, as in *reddish*, *Turkish*, &c. It signifies "of the nature or substance of a thing;" and seems to have an affinity to the Greek verb εἴκω and termination ικος. It is undoubtedly the German *ische*, the Dutch *sche*, the Frankish *isc*, the Italian *esco*, and the French *esque*. In the *Edda* of SÆMUND, the first man (or perhaps the first substance) is called *ask*.

> *Unst thriar comu ur thai lide.*
> Until three came out of that company.
> *Auffigir og astgier aser ad huse.*
> Powerful and lovely *asans* to the house.
> *Fundu a lande lyte meigande.*
> They found on the land powerless.
> ASK *og emblo oerloeg lausa.*
> *Ask* and *emblo* strengthless.

"Ab hoc *asko* vel *æsco*, primo condito homine," says HICKES, "venit proprium nomen *æsc* apud Anglo-Saxones;" and he cites various instances in which *æsc* appears to have the general signification of *man*.

Ard is an adjectival particle somewhat similar in effect to *ish*, but appearing to have been derived into English immediately from the French. We find in old English *lyard*, *bayard*, *blaunchard*, *trichard*, *caynard*, &c. now obsolete; but we still retain *drunkard*, *coward*, *braggart*, and some others. It seems to exist, as a word, in the Scottish *airt*, a quarter of the heavens or portion of the earth.

Grammar.

The adjectival particle *wise* we have already shown to be the same as the word *guise*. *Rightwise* has been corrupted into *righteous*.

It is scarcely necessary to dwell on such particles as *mis* from the verb to *miss*—*wan* (as in the Scottish *wanchancy*) from the noun *want*—*fold* as in *twofold*, from the verb to *fold*—the Latin *plex*, as in *duplex* from the verb *plico*. We have specified enough to show, that the generality of particles which serve the purpose of changing the grammatical class to which a word belongs, originally existed in a separate shape, as significant words.

Declension and conjugation.

It is certainly not so easy to prove that the particles used in the declension of nouns and conjugation of verbs were originally significant words; yet we cannot but agree with Mr. Tooke that there is good reason to believe that they were.

One forcible reason for this opinion is, that what is done in some languages by terminations, is done in other languages by separate words, by prepositions, by adverbs, by auxiliary verbs, &c.; but we have already shown, that not only the auxiliary verbs, but the adverbs and prepositions, were significant; and hence it is reasonable to infer that what stands in their place is significant also.

The noun substantive, for instance, in some languages may be varied in gender, number and case, by its terminations. Thus the Latins expressed the children of the two sexes by the words puer and puella. *Puer* signifies what we mean by a *man-child*. We have therefore reason to believe, that as *man* is a word significant of a male of the human kind, so *er* when standing alone had a similar signification: and in fact we find that *er* is to this day the German masculine pronoun *he*. *Puella* signifies a girl: if we call *pu-er* a *he-child*, we may call *pu-ella* a *she-child*: and in fact *illa* is the Latin feminine pronoun *she*. In like manner our feminine particle *ess*, as in *shepherdess*, is found in the Italian pronoun *essa*, she.

The terminations of number and case, are not very clearly to be traced to their origin; but they seem in general to be pronouns. Thus the nominative case *lapis* a stone, is evidently made up of two parts, *lap*, which conveys the conception of stone through all its inflections, and *is*, which distinguishes this particular case. Now *is* is a Latin pronoun. So we think the final *o* in *homo*, is the Greek article ὁ, and the final *a* in *musa* the Greek article ἡ.

In nouns adjective, we have already said that the termination *ly* is the Gothic substantive *leik*, body; and if the *ly* in *greatly* have a separate meaning, it is probable that the *er* and *est* in *greater* and *greatest*, have also separate meanings.

Various explanations have been attempted of the terminations of the Latin and Greek verbs: and though they may none of them be perfectly satisfactory throughout, yet it can hardly fail to be admitted, that some of the particles have a connection with the words to which they have been traced. Thus in *capiam*, *capies*, *capiet*, the termination *am* has certainly some analogy to the Latin *me*, or the Greek εμε; the termination *es* to συ, and the termination *et* to τιν. M. de Brosses, after following the radical sound *cap* through all its developements in the verb *capio*, concludes with a just observation. "Toute cette composition est l'ouvrage non d'une combinaison reflechie, ni d'une philosophie raisonnée mais d'une metaphysique d'instinct." Now instinct could never have led men to form a complicated and beautiful system out of sounds altogether unmeaning; but it might easily lead to the gradual combination of known elements, until they formed at length the complete structure of a language.

Particles.

As the effect of a particle in declension or conjugation is sometimes supplied by a *word*; so on the other hand it is sometimes produced by a mere change in articulation: and this seems to be natural to mankind, because we find it in different languages, and in very various languages, and in very various ways, as *τυπτω*, *τυπω*—*capio*, *cepi*—*sing*, *sang*—*man*, *men*, &c.

Lastly we must observe, that there are numerous causes of anomaly in language, which render it more particularly difficult to systematise and explain the minor portions of speech, such as the prepositions, auxiliary verbs, and particles. One of these causes is a mistaken notion of analogies between particular words, where no such analogy exists. Thus our word *further*, which was the comparative of *forth*, has been supposed by many persons to be the comparative of *far*, and has therefore been erroneously written *farther*. A still more striking instance is that of the word *coud*, which we always pronounce properly, but spell *could*, inserting the *l*, without any reason whatever, but that there is an *l* in *would* and *should*. The two latter words are from the Anglo-Saxon *wille* and *sceal*, the former is from the Anglo-Saxon *cwethan*; and was always written in old English *couthe*, *cowthe*, or *coude*.

> That though he had me bete on every bone,
> He *couthe* winne agen my love anone. CHAUCER.
>
> He thowght to taste if he *cowthe*,
> And on he put in his mowth. *Sir Cleges.*
>
> Sir, quod this knyght myld of speche,
> Wold God I *cowthe* your sonne teche!
> *Lyfe of Ipomydon.*
>
> Ac he no *couthe* neuer mo
> Chese the better of hem to. *Amis and Amiloun.*
>
> Whiche was right displeasant to the kyng, but he *coude* nat amende it. BERNERS'S *Froissart*, fol. 43.

Another and a more effective cause of anomaly is the love of euphony, or easy pronunciation, which leads the ignorant especially to corrupt words by abbreviations and changes, as *Godild!* for God yelde, i. e. reward him. *Gossip* for god-sib, &c.

Allowing for the obscurities which these and other causes spread over the minor portions of speech, it may fairly be said, that in regard to *particles*, as well as to *words*, we have established the great principle of *transition*, by which significant sounds pass from one class and description of signs into another. The noun or verb becoming a particle, and the particle coalescing with another verb or noun, serve to modify their signification, and determine their grammatical use. And, finally, we may conclude, that language is, throughout, a combination of significant sounds, fitted to express thoughts and emotions, as they exist interchangeably in the human mind.

LOGIC.

INTRODUCTORY SECTION.

Logic in the most extensive sense which it can with propriety be made to bear, may be considered as the Science and also as the Art of Reasoning. It investigates the principles on which argumentation is conducted, and furnishes rules to secure the mind from error in its deductions. Its most appropriate office, however, is that of instituting an analysis of the process of the mind in Reasoning: and in this point of view it is, as has been stated, strictly a Science: while considered in reference to the practical rules above mentioned, it may be called the Art of Reasoning. This distinction, as will hereafter appear, has been overlooked, or not clearly pointed out by most writers on the subject, Logic having been in general regarded as merely an Art; and its claim to hold a place among the Sciences having been expressly denied.

Considering how early Logic attracted the attention of philosophers, it may appear surprising that so little progress should have been made, as is confessedly the case, in developing its principles, and perfecting the detail of the system: and this circumstance has been brought forward as a proof of the barrenness and futility of the study. But a similar argument might have been urged with no less plausibility, in past ages, against the study of Natural Philosophy, and very recently against that of Chemistry. No Science can be expected to make any considerable progress, which is not cultivated on right principles. Whatever may be the inherent vigour of the plant, it will neither be flourishing nor fruitful till it meet with a suitable soil and culture: and in no case is the remark more applicable than in the present; the greatest mistakes having always prevailed respecting the nature of Logic, and its province having in consequence been extended by many writers to subjects with which it has no proper connection. Indeed, with the exception of Aristotle, (who is himself not entirely exempt from the errors in question,) hardly a writer on Logic can be mentioned who has clearly perceived, and steadily kept in view throughout, its real nature and object. Before his time, no distinction was drawn between the Science of which we are speaking, and that which is now usually called Metaphysics: a circumstance which alone shews how small was the progress made in earlier times. Indeed those who first turned their attention to the subject, hardly thought of inquiring into the process of Reason itself, but confined themselves almost entirely to certain preliminary points, the discussion of which is (if logically considered) subordinate to that of the main inquiry.

Zeno the Eleatic, whom most accounts represent as the earliest systematic writer on the subject of Logic, or as it was then called, Dialectics, divided his work into three parts; the first of which (upon Consequences) is censured by Socrates [Plato, *Parmen.*] for obscurity and confusion. In his second part, however, he furnished that interrogatory method of disputation [ἐρώτησις] which Socrates adopted, and which has since borne his name. The third part of his work was devoted to what may not improperly be termed the art of wrangling, [ἐριστικὴ] which supplied the disputant with a collection of sophistical questions, so contrived that the concession of some point which seemed unavoidable, immediately involved some glaring absurdity. This, if it is to be esteemed as at all falling within the province of Logic, is certainly not to be regarded (as some have ignorantly or heedlessly represented it) as its principal or proper business. The Greek philosophers generally have unfortunately devoted too much attention to it: but we must beware of falling into the vulgar error of supposing the ancients to have regarded as a serious and intrinsically important study, that which in fact they considered as an ingenious recreation. The disputants diverted themselves in their leisure hours by making trial of their own and their adversary's acuteness, in the endeavour mutually to perplex each other with subtle fallacies; much in the same way as men amuse themselves with propounding and guessing riddles, or with the game of chess; to each of which diversions the sportive disputations of the ancients bore much resemblance. They were closely analogous to the wrestling and other exercises of the gymnasium, these last being reckoned conducive to bodily vigour and activity, as the former were to habits of intellectual acuteness; but the *immediate* object in each was a sportive, not a serious contest; though doubtless fashion and emulation often occasioned an undue importance to be attached to success in each.

Zeno then is hardly to be regarded as any further a logician than as to what respects his erotetic method of disputation; a course of argument constructed on this principle being properly an hypothetical sorites, which may easily be reduced into a series of syllogisms.

To Zeno succeeded Euclid of Megara, and Antisthenes, both pupils of Socrates. The former of these prosecuted the subject of the third part of his predecessor's treatise, and is said to have been the author of many of the fallacies attributed to the Stoical school.

Of the writings of the latter nothing certain is known: if, however, we suppose the above mentioned sect to be his disciples in this study, and to have retained his principles, he certainly took a more correct view of the subject than Euclid. The Stoics divided all λεκτὰ, every thing that could be said, into three classes: 1st, the simple term; 2d, the proposition; 3d, the syllogism; viz. the *hypothetical*: for they seem to have had little notion of a more rigorous analysis of argument than into that familiar form.

We must not here omit to notice the merits of Archytus, to whom we are indebted for the doctrine of the categories. He, however, (as well as the other writers on the subject,) appears to have had no distinct view of the proper object and just limits of the science of Logic; but to have blended with it Metaphysical discussions not strictly connected with it, and to have dwelt on the investigation of the nature of terms and propositions, without maintaining a constant reference to the principles of Reasoning, to which all the rest should be made subservient.

The state then in which Aristotle found the Science, (if indeed it can properly be said to have existed at all before his time,) appears to have been nearly this: the division into simple terms, propositions and syllogisms, had been slightly sketched out; the doctrine of the categories, and perhaps that of the opposition of propositions, had been laid down; and, as some believe, the analysis of species into genus and differentia, had been introduced by Socrates. These, at best, were rather the materials of the system than the system itself; the foundation of which indeed he distinctly claims the merit of having laid; and which remains fundamentally the same as he left it.

It has been remarked, that the Logical system is one of those few theories which have been begun and perfected by the same individual. The history of its discovery, as far as the main principles of the science are concerned, properly commences and ends with Aristotle. And this may perhaps in part account for the subsequent perversions of it. The brevity and simplicity of its fundamental truths, (to which indeed all real science is perpetually tending,) has probably led many to suppose that something much more complex, abstruse, and mysterious, remained to be discovered. The vanity by which all men are prompted unduly to magnify their own pursuits, has led unphilosophical minds, not in this case alone, but in many others, to extend the boundaries of their respective Sciences, not by the patient developement and just application of the principles of those Sciences, but by wandering into irrelevant subjects. The mystical employment of numbers by Pythagoras, in matters utterly foreign to Arithmetic, is perhaps the earliest instance of the kind. A more curious and important one is the degeneracy of Astronomy into judicial Astrology; but none is more striking than the misapplication of Logic, by those who have treated of it as "the art of rightly employing the rational faculties," or who have intruded it into the province of Natural Philosophy, and regarded the syllogism as an engine for the investigation of nature: overlooking the boundless field that was before them within the legitimate limits of the Science; and not perceiving the importance and difficulty of the task of completing and properly filling up the masterly sketch before them.

The writings of Aristotle were not only absolutely lost to the world for about two centuries, but seem to have been but little studied for a long time after their recovery. An Art, however, of Logic, derived from the principles traditionally preserved by his disciples, seems to have been generally known, and was employed by Cicero in his philosophical works; but the pursuit of the science seems to have been abandoned for a long time. Early in the Christian era, the Peripatetic doctrines experienced a considerable revival; and we meet with the names of Galen and Porphyry as Logicians: but it is not till the fifth century that Aristotle's Logical works were translated into Latin by the celebrated Boethius. Not one of these seems to have made any considerable advances in developing the Theory of Reasoning. Of Galen's labours little is known; and Porphyry's principal work is merely on the *predicables*. We have little of the Science till the revival of learning among the Arabians, by whom Aristotle's treatises on this as well as on other subjects were eagerly studied.

Passing by the names of some Byzantine writers of no great importance, we come to the times of the Schoolmen, whose waste of ingenuity and frivolous subtilty of disputation need not be enlarged upon. It may be sufficient to observe, that their fault did not lie in their diligent study of Logic, and the high value they set upon it, but in their utterly mistaking the true nature and object of the science; and by attempting to employ it for the purpose of physical discoveries, involving every subject in a mist of words, to the exclusion of sound philosophical investigation. Their errors may serve to account for the strong terms in which Bacon sometimes appears to censure Logical pursuits; but that this censure was intended to bear against the extravagant perversions, not the legitimate cultivation of the Science, may be proved from his own observations on the subject, in his *Advancement of Learning*.

His moderation, however, was not imitated in other quarters. Even Locke confounds in one sweeping censure the Aristotelic theory, with the absurd misapplications and perversions of it in later years. His objection to the Science, as unserviceable in the *discovery* of truth, (which has of late been often repeated) while it holds good in reference to many (misnamed) Logicians, indicates that with regard to the true nature of the Science itself he had no clearer notions than they have, of the proper province of Logic, viz. Reasoning; and of the distinct character of that operation from the observations and experiments which are essential to the study of nature.

An error apparently different, but substantially the same, pervades the treatises of Watts and other modern writers on the subject. Perceiving the inadequacy of the syllogistic theory to the vast purposes to which others had attempted to apply it, he still craved after the attainment of some equally comprehensive and all-powerful system; which he accordingly attempted to construct, under the title of *The Right Use of Reason*; which was to be a method of invigorating and properly directing all the powers of the mind: a most magnificent object indeed, but one which not only does not fall under the province of Logic, but cannot be accomplished by any *one* Science or system that can even be conceived to exist. The attempt to comprehend so wide a field is no extension of Science, but a mere verbal generalization, which

leads only to vague and barren declamation. In every pursuit, the more precise and definite our object, the more likely we are to attain some valuable result; if, like the Platonists, who sought after the αὐτάγαθον, the abstract idea of good, we pursue some specious but ill-defined scheme of universal knowledge, we shall lose the substance while grasping at a shadow, and bewilder ourselves in empty generalities.

It is not perhaps much to be wondered at, that in still later times several ingenious writers, forming their notions of the Science itself from professed masters in it, such as have just been alluded to, and judging of its value from their failures, should have treated the Aristotelic system with so much reprobation and scorn. Too much prejudiced to bestow on it the requisite attention for enabling them clearly to understand its real character and object, or even to judge correctly from the little they did understand, they have assailed the study with a host of objections, so totally irrelevant, and consequently impotent, that, considering the talents and general information of those from whom they proceed, they might excite astonishment in any one who did not fully estimate the force of very early prejudice.

Logic has usually been considered by these objectors as professing to furnish a peculiar method of Reasoning, instead of a method of analyzing that mental process which must *invariably* take place in all correct Reasoning; and accordingly they have contrasted the ordinary mode of reasoning with the syllogistic; and have brought forward with an air of triumph the argumentative skill of many who never learned the system: a mistake no less gross than if any one should regard Grammar as a peculiar language, and contend against its utility on the ground that many speak correctly who never studied the principles of Grammar; whereas Logic, which is, as it were, the Grammar of Reasoning, does not bring forward the regular syllogism as a distinct mode of argumentation, designed to be substituted for any other mode; but as the form to which all correct Reasoning may be ultimately reduced, and which consequently serves the purpose (when we are employing Logic as an Art) of a test to try the validity of any argument, in the same manner as by chemical analysis we develope and submit to a distinct examination the elements of which any compound body is composed, and are thus enabled to detect any latent sophistication and impurity.

Complaints have also been made that Logic leaves untouched the greatest difficulties, and those which are the sources of the chief errors in Reasoning; viz. the ambiguity or indistinctness of terms, and the doubts respecting the degrees of evidence in various propositions: an objection which is not to be removed by any such attempt as that of Watts to lay down "rules for forming clear ideas, and for guiding the judgment;" but by replying that no Art is to be censured for not teaching more than falls within its province, and indeed more than can be taught by any conceivable art. Such a system of universal knowledge as should instruct us in the full meaning of every term, and the truth or falsity, certainty or uncertainty, of every proposition, thus superseding all other studies, it is most unphilosophical to expect or even to imagine. And to find fault with Logic for not performing this is as if one should object to the Science of Optics for not giving sight to the blind; or as if (like the man of whom Warburton tells a story in his *Div. Leg.*) one should complain of a reading glass for being of no service to a person who had never learned to read.

In fact, the difficulties and errors above alluded to are *not* in the process of Reasoning itself, (which alone is the appropriate province of Logic,) but in the subject matter about which it is employed. This process will have been correctly conducted if it have conformed to the Logical rules which preclude the possibility of any error creeping in between the principles from which we are arguing, and the conclusions we deduce from them. But still that conclusion may be false, if the principles we start from are so. In like manner, no Arithmetical skill will secure a correct result to a calculation, unless the data are correct from which we calculate; nor does any one on that account undervalue Arithmetic; and yet the objection against Logic rests on no better foundation.

There is in fact a striking analogy in this respect between the two Sciences. All numbers (which are the subject of Arithmetic) must be numbers *of some things*, whether coins, persons, measures, or any thing else; but to introduce into the Science any notice of the *things* respecting which calculations are made, would be evidently irrelevant, and would destroy its scientific characters: we proceed therefore with arbitrary signs representing numbers in the abstract. So also does Logic pronounce on the validity of a regularly-constructed argument equally well, though arbitrary symbols may have been substituted for the terms, and consequently without any regard to the things signified by those terms. And the probability of doing this (though the employment of such arbitrary symbols has been absurdly objected to, even by writers who understood not only Arithmetic but Algebra) is a proof of the strictly scientific character of the system. But many professed Logical writers, not attending to the circumstances which have been just mentioned, have wandered into disquisitions on various branches of knowledge; disquisitions which most evidently be as boundless as human knowledge itself, since there is no subject on which Reasoning is not employed, and to which consequently Logic may not be *applied*. The error lies in regarding every thing as the *proper province* of Logic, to which it is *applicable*. A similar error is complained of by Aristotle, as having taken place with respect to Rhetoric; of which indeed we find specimens in the arguments of several of the interlocutors in *Cic. de Oratore*.

From what has been said, it will be evident that there is hardly any subject to which it is so difficult to introduce the student in a clear and satisfactory manner, as the one we are now engaged in. In any other branch of knowledge, the reader, if he have any previous acquaintance with the subject, will usually be so far the better prepared for comprehending the exposition of the principles; or if he be entirely a stranger to it, will at least come to the study with a mind unbiassed, and free from prejudices and misconceptions; whereas in the present case it cannot but happen that many who have given some attention to Logical pursuits, (or what are usually considered as such) will frequently have rather been bewildered by fundamentally erroneous views, than prepared by the acquisition of just principles for ulterior progress; and that not a few who pretend not to any acquaintance whatever with the Science, will yet

have imbibed either such prejudices against it, or such false notions respecting its nature, as cannot but prove obstacles in their study of it.

There is, however, a difficulty which exists more or less in all abstract pursuits, though it is perhaps more felt in this, and often occasions it to be rejected by beginners as dry and tedious; viz. the difficulty of perceiving to what ultimate end,—to what practical or interesting application the abstract principles lead which are first laid before the student; so that he will often have to work his way patiently through the most laborious part of the system before he can gain any clear idea of the drift and intention of it.

This complaint has often been made by chemical students, who are wearied with descriptions of oxygen, hydrogen, and other invisible elements, before they have any knowledge respecting such bodies as commonly present themselves to the senses. And accordingly some teachers of Chemistry obviate in a great degree this objection, by adopting the analytical instead of the synthetical mode of procedure, when they are first introducing the subject to beginners; i. e. instead of synthetically enumerating the elementary substances, proceeding next to the simplest combinations of these, and concluding with those more complex substances which are of the most common occurrence, they begin by *analyzing* these last, and resolving them step by step into their simple elements; thus presenting the subject at once in an interesting point of view, and clearly setting forth the object of it. The synthetical form of teaching is indeed sufficiently interesting to one who has made considerable progress in any study; and being more concise, regular, and systematic, is the form in which our knowledge naturally arranges itself in the mind, and is retained by the memory: but the analytical is the more interesting, easy, and natural kind of introduction, as being the form in which the first invention or discovery of any kind of system must originally have taken place.

It may be advisable, therefore, to begin by giving a slight sketch, in this form, of the Logical system, before we enter regularly upon the details of it. The reader will thus be presented with a kind of imaginary history of the course of inquiry by which the Logical system may be conceived to have occurred to a philosophical mind.

In every instance in which we *reason*, in the strict sense of the word, i. e. make use of arguments, whether for the sake of refuting an adversary, or of conveying instruction, or of satisfying our own minds on any point, whatever may be the subject we are engaged on, a certain process takes place in the mind, which is one and the same in all cases, provided it be correctly conducted.

Of course it cannot be supposed that every one is even conscious of this process in his own mind, much less is competent to explain the principles on which it proceeds; which indeed is, and cannot but be, the case with every other process respecting which any system has been formed; the practice not only may exist independently of the theory, but must have preceded the theory; there must have been language before a system of Grammar could be devised; and musical compositions previous to the science of Music. This by the way will serve to expose the futility of the popular objection against Logic, that men may reason very well who know nothing of it. The parallel instance adduced, shews that such an objection might be applied in many other cases, where its absurdity would be obvious; and that there is no reason for deciding thence, either that the system has no tendency to improve practice, or that even if it had not, it might not still be a dignified and interesting pursuit.

One of the chief impediments to the attainment of a just view of the nature and object of Logic, is the not fully understanding, or not sufficiently keeping in mind, the SAMENESS of the Reasoning process in all cases; if, as the ordinary mode of speaking would seem to indicate, Mathematical Reasoning, and Theological, and Metaphysical, and Political, &c. were essentially different from each other, i. e. different *kinds of reasoning*, it would follow, that supposing there could be at all any such Science as we have described Logic, there must be so many different species, or at least different branches of Logic. And such is perhaps the most prevailing notion. Nor is this much to be wondered at; since it is evident to all that some men converse and write in an argumentative way, very justly on one subject, and very erroneously on another, in which again others excel, who fail in the former. This error may be at once illustrated and removed, by considering the parallel instance of Arithmetic, in which every one is aware that the process of a calculation is not affected by the nature of the objects whose numbers are before us: but that (e. g.) the multiplication of a number is the very same operation, whether it be a number of men, of miles, or of pounds; though nevertheless men may perhaps be found who are accurate in calculations relative to Natural Philosophy, and incorrect in those of Political Economy, from their different degrees of skill in the subjects of these two Sciences; not surely because there are different arts of Arithmetic applicable to each of these respectively.

Others again, who are aware that the simple system of Logic may be applied to all subjects whatever, are yet disposed to view it as a peculiar method of Reasoning, and not as it is, a method of unfolding and analyzing our Reasoning: whence many have been led (e. g. the author of the Philosophy of Rhetoric) to talk of comparing syllogistic Reasoning with moral Reasoning, and to take it for granted that it is possible to reason correctly without reasoning Logically; which is in fact as great a blunder as if any one were to mistake Grammar for a peculiar language, and to suppose it possible to speak correctly without speaking Grammatically. They have in short considered Logic as *an* Art of Reasoning; whereas, so far as it is an Art, it is *the* Art of Reasoning: the Logician's object being, not to lay down principles by which one *may* reason, but by which all *must* reason, even though they are not distinctly aware of them: to lay down rules, not which may be followed with advantage, but which cannot possibly be departed from in sound reasoning. These misapprehensions and objections being such as lie on the very threshold of the subject, it would have been hardly possible, without noticing them, to convey any just notion of the nature and design of the Logical system.

Supposing it then to have been perceived that the operation of Reasoning is in all cases the same, the analysis of that operation could not fail to strike the

mind as an interesting matter of inquiry: and moreover, since (apparent) arguments which are unsound and inconclusive, are so often employed either *from* error or *from* design; and even those who are not misled by these fallacies, are so often at a loss to detect and expose them in a manner satisfactory to others, or even to themselves, it could not but appear desirable to lay down some general rules of Reasoning, applicable to all cases, by which a person might be enabled the more readily and clearly to state the grounds of his own conviction, or of his objection to the arguments of an opponent, instead of arguing at random without any fixed and acknowledged principles to guide his procedure. Such rules would be analogous to those of Arithmetic, which obviate the tediousness and uncertainty of calculations in the head, wherein, after much labour, different persons might arrive at different results, without any of them being able distinctly to point out the error of the rest. A system of such rules, it is obvious, must, instead of deserving to be called the Art of wrangling, be more justly characterised as "the Art of cutting short wrangling," by bringing the parties to issue at once, if not to agreement, and thus saving a waste of ingenuity.

In pursuing the supposed investigation, it will be found that every conclusion is deduced, in reality, from two other propositions, (thence called premises;) for though one of these may be, and commonly is, suppressed, it must nevertheless be understood as admitted; as may easily be made evident by supposing the DENIAL of the suppressed premiss, which will at once invalidate the argument: e. g. if any one from perceiving that the world exhibits marks of design, infers that "it must have had an intelligent author," though he may not be aware in his own mind of the existence of any other premiss, he will readily understand, if it be denied that "whatever exhibits marks of design must have had an intelligent author," that the affirmative of that proposition is necessary to the validity of the argument. An argument thus stated regularly and at full length is called a Syllogism; which therefore is evidently not a peculiar *kind of argument*, but only a peculiar *form* of expression, in which every argument may be stated. When one of the premises is suppressed, (which for brevity's sake it usually is) the argument is called an Enthymeme. And it may be worth while to remark, that when the argument is in this state, the objections of an opponent are (or rather appear to be) of two kinds; viz. either objections to the *assertion* itself, or objections to its *force* as an argument; e. g. in the above instance, an atheist may be conceived either denying that the world does exhibit marks of design, or denying that it *follows* from thence that it had an intelligent author. The only difference in the two cases is, that in the one the *expressed* premiss is denied, in the other the *suppressed*; for the *force as an argument* of either premiss depends on the other premiss: if both be admitted, the conclusion legitimately connected with them cannot be denied.

It is evidently immaterial to the argument whether the conclusion be placed first or last; but it may be proper to remark, that a premiss placed *after* its conclusion is called the *reason* of it, and is introduced by one of those conjunctions which are called causal; viz. "since," "because," &c. which may indeed be employed to designate a premiss, whether it came first or last; the illative conjunctions, "therefore," &c. designate the conclusion. It is a circumstance which often occasions error and perplexity, that both these classes of conjunctions have also another signification, being employed to denote, respectively, cause and effect, as well as premiss and conclusion: e. g. if I say, (to use an instance employed by Aristotle) "yonder is a fixed star, *because* it twinkles," or, "it twinkles, and therefore is a fixed star," I employ these conjunctions to denote the connection of premiss and conclusion; for it is plain that the twinkling of the star is not the cause of its being fixed, but only the cause of *my knowing* that it is so: but if I say, "it twinkles because it is a fixed star," or it is a fixed star, and therefore twinkles," I am using the same conjunctions to denote the connection of cause and effect; for in this case the twinkling of the star, being evident to the eye, would hardly need to be *proved*, but might need to be accounted for. There are, however, many cases in which the cause is employed to *prove* the existence of its effect; especially in arguments relating to *future* events: the *cause* and the reason, in that case, coincide; and this contributes to their being so often confounded together in other cases. In an argument, such as the example above given, it is, as has been said, impossible for any one, who admits both premises, to avoid admitting the conclusion; but there will be frequently an apparent connection of premises with a conclusion which does not in reality follow from them, though to the inattentive or unskilful the argument may appear to be valid. and there are many other cases in which a doubt may exist whether the argument be valid or not; i. e. whether it be possible or not to admit the premises, and yet deny the conclusion. It is of the highest importance, therefore, to lay down some regular form to which every valid argument may be reduced, and to devise a rule which shall prove the validity of every argument in that form, and consequently the unsoundness of any apparent argument which cannot be reduced to it:—e. g. if such an argument as this be proposed, "every rational agent is accountable; brutes are not rational agents; therefore they are not accountable:" or again, "all wise legislators suit their laws to the genius of their nation; Solon did this; therefore he was a wise legislator:" there are some, perhaps, who would not perceive any fallacy in such arguments, especially if enveloped in a cloud of words; and still more when the conclusion is true, or, which comes to the same point, if they are disposed to believe it; and others might perceive indeed, but might be at a loss to explain the fallacy. Now these (apparent) arguments exactly correspond respectively with the following, the absurdity of the conclusions from which is manifest: "every horse is an animal; sheep are not horses; therefore they are not animals:" and, "all vegetables grow; an animal grows; therefore it is a vegetable." These last examples, it has been said, correspond exactly (considered as arguments) with the former; the question respecting the validity of an argument being, not whether the conclusion be *true*, but whether it follows from the premises adduced. This mode of exposing a fallacy, by bringing forward a similar one whose conclusion is obviously absurd, is often, and very advantageously, resorted to in addressing

those who are ignorant of Logical rules; but to lay down such rules, and employ them as a test, is evidently a safer and more compendious, as well as a more philosophical mode of proceeding. To attain these, it would plainly be necessary to analyze some clear and valid arguments, and to observe in what their conclusiveness consists. Let us suppose, then, such an examination to be made of the syllogism above mentioned: "whatever exhibits marks of design had an intelligent author."

The world exhibits marks of design; therefore the world had an intelligent author. In the first of these premises we find it assumed universally of the *class* of "things which exhibit marks of design," that they had an intelligent author; and in the other premiss, "the world" is referred to that class as comprehended in it: now it is evident, that whatever is said of the whole of a class, may be said of any thing comprehended in that class; so that we are thus authorized to say of the world, that it had an intelligent author. Again, if we examine a syllogism with a negative conclusion, as, e. g. "nothing which exhibits marks of design could have been produced by chance: the world exhibits, &c.; therefore the world could not have been produced by chance." The process of Reasoning will be found to be the same; since it is evident, that whatever is *denied* universally of any class, may be denied of any thing that is comprehended in that class.

On further examination it will be found, that all valid arguments whatever may be easily reduced to such a form as that of the foregoing syllogisms; and that consequently the principle on which they are constructed is the universal principle of Reasoning. So elliptical indeed is the ordinary mode of expression, even of those who are considered as prolix writers, i. e. so much is implied and left to be understood in the course of argument, in comparison of what is actually stated, (most men being impatient, even to excess, of any appearance of unnecessary and tedious formality of statement,) that a single sentence will often be found, though perhaps considered as a single argument, to contain, compressed into a short compass, a chain of several distinct arguments; but if each of these be fully developed, and the whole of what the author intended to imply be stated expressly, it will be found that all the steps even of the longest and most complex train of Reasoning, may be reduced into the above form.

It is a mistake (which might appear scarcely worthy of notice had not so many, even esteemed writers, fallen into it) to imagine that Aristotle and other Logicians meant to propose that this prolix form of unfolding arguments should universally supersede, in argumentative discourses, the common forms of expression; and that to reason Logically, means, to state all arguments at full length in the syllogistic form: and Aristotle has even been charged with inconsistency for not doing so; it has been said, that "in his *Treatises of Ethics, Politics*, &c. he argues like a rational creature, and never attempts to bring his own system into practice:" as well might a Chemist be charged with inconsistency for making use of any of the compound substances that are commonly employed, without previously analyzing and resolving them into their simple elements; as well might it be imagined that, to speak grammatically, means, to parse every sentence we utter. The Chemist (to pursue the illustration) keeps by him his tests and his method of analysis, to be employed when any substance is offered to his notice, the composition of which has not been ascertained, or in which adulteration is suspected. Now a fallacy may aptly be compared to some adulterated compound; it consists of an ingenious mixture of truth and falsehood, so entangled, so intimately blended, that the falsehood is (in the chemical phrase) *held in solution*: one drop of sound Logic is that test which immediately disunites them, makes the foreign substance visible, and precipitates it to the bottom.

But to resume the investigation of the principles of Reasoning: the maxim resulting from the examination of a syllogism in the foregoing form, and of the application of which every valid argument is in reality an instance, is, "that whatever is predicated (i. e. affirmed or denied) universally, of any class of things, may be predicated, in like manner, (viz. affirmed or denied) of any thing comprehended in that class." This is the principle, commonly called the *dictum de omni et nullo*, for the establishment of which we are indebted to Aristotle, and which is the keystone of his whole Logical system. It is not a little remarkable that some, otherwise judicious writers, should have been so carried away by their zeal against that philosopher, as to speak with scorn and ridicule of this principle, on account of its obviousness and simplicity; though they would probably perceive at once, in any other case, that it is the greatest triumph of philosophy to refer many, and seemingly very various, phenomena to one, or a very few, simple principles; and that the more simple and evident such a principle is, provided it be truly applicable to all the cases in question, the greater is its value and scientific beauty. If, indeed, any principle be regarded as *not* thus applicable, *that* is an objection to it of a different kind. Such an objection against Aristotle's dictum, no one has ever attempted to *establish* by any kind of proof; but it has often been *taken for granted*; it being (as has been stated) very commonly supposed, without examination, that the syllogism is a *distinct kind of argument*, and that the rules of it do not apply, nor were intended to apply, to *all* Reasoning whatever. Under this misapprehension, Campbell (*Philosophy of Rhetoric*) labours, with some ingenuity, and not without an air of plausibility, to shew that every syllogism must be futile and worthless, because the premises virtually assert the conclusion: little dreaming, of course, that his objections, however specious, lie against the *process of Reasoning itself* universally; and will therefore, of course, apply to those very arguments which he is himself adducing.

It is much more extraordinary to find another author (Dugald Stewart) adopting, expressly, the very same objections, and yet distinctly admitting within a few pages, the possibility of reducing every course of argument to a series of syllogisms.

The same writer brings an objection against the dictum of Aristotle; which it may be worth while to notice briefly, for the sake of setting in a clearer light the real character and object of that principle. Its application being, as has been seen, to a regular and conclusive syllogism, he supposes it intended to prove and *make evident* the conclusiveness of such a syllogism; and remarks how unphilosophical it is to

attempt giving a *demonstration of a demonstration*. And certainly the charge would be just, if we could imagine the Logician's object to be, to *increase the certainty* of a conclusion which we are supposed to have already arrived at by the clearest possible mode of proof. But it is very strange that such an idea should ever have occurred to one who had even the slightest tincture of Natural Philosophy: for it might as well be imagined that a Natural Philosopher or a Chemist's design to strengthen the testimony of our senses by *a priori* reasoning, and to convince us that a stone when thrown will fall to the ground, and that gunpowder will explode when fired, because they shew that according to their principles those phenomena must take place as they do. But it would be reckoned a mark of the grossest ignorance and stupidity, not to be aware that their object is not to *prove* the existence of an individual phenomenon, which our eyes have witnessed, but (as the phrase is) to account for it: i. e. to shew according to what *principle* it takes place;—to refer, in short, the *individual case* to a *general law* of nature. The object of Aristotle's dictum is precisely analogous: he had, doubtless, no thought of adding to the force of any individual syllogism; his design was to point out the *general principle* on which that process is conducted which takes place in each syllogism. And as the laws of nature (as they are called) are in reality merely *generalized facts*, of which all the phenomena coming under them are particular instances; so the proof drawn from Aristotle's dictum is not a distinct demonstration brought to confirm another demonstration, but is merely a generalized and abstract statement of *all* demonstration whatever; and is therefore in fact, *the very demonstration* which (*mutatis mutandis*) accommodated to the various subject matters, is actually employed in each particular case.

In order to trace more distinctly the different steps of the abstracting process, by which any particular argument may be brought into the most general form, we may first take a syllogism stated accurately and at full length, such as the example formerly given, "whatever exhibits marks of design, &c.," and then somewhat generalize the expression, by substituting (as in Algebra) arbitrary unmeaning symbols for the significant terms that were originally used; the syllogism will then stand thus; "every B is A; C is B; therefore C is A." The Reasoning is no less evidently valid when thus stated, whatever terms A, B, and C, respectively may be supposed to stand for: such terms may indeed be inserted as to make all, or any of, the assertions *false*; but it will still be no less impossible for any one who *admits* the truth of the *premises*, in an argument thus constructed, to deny the conclusion; and this it is that constitutes the conclusiveness of an argument.

Viewing then the syllogism thus expressed, it appears clearly, that "A stands for *any thing whatever* that is predicated of a whole class," (viz. of *every* B) "which comprehends or contains in it *something else*," viz. C, of which B is, in the second premiss affirmed; and that consequently the first term (A) is, in the conclusion, predicated of the third C.

Now to assert the validity of this process, now before us, is to state the very dictum we are treating of with hardly even a verbal alteration, viz.:

1. Any thing whatever, predicated of a whole class,

2. Under which class something else is contained,

3. May be predicated of that which is so contained.

The three members into which the maxim is here distributed, correspond to the three propositions of the syllogism to which they are intended respectively to apply.

The advantage of substituting for the terms, in a regular syllogism, arbitrary unmeaning symbols such as letters of the alphabet, is much the same as in Mathematics: the Reasoning itself is then considered, by itself, clearly, and without any risk of our being misled by the truth or falsity of the conclusion, which are, in fact, accidental and variable; the essential point, being, as far as the *argument* is concerned, the *connection between* the premises and the conclusions. We are thus enabled to embrace the general principle of all Reasoning, and to perceive its applicability to an indefinite number of individual cases. That Aristotle, therefore, should have been accused of making use of these symbols for the purpose of *darkening* his demonstrations, and that too, by persons not unacquainted with Geometry and Algebra, is truly astonishing. If a Geometer, instead of designating the four angles of a square, by four letters, were to call them *north, south, east*, and *west*, he would not render the demonstration of a theorem the easier; and the learner would be much more likely to be perplexed in the application of it.

It belongs then exclusively to a syllogism, properly so called (i. e. a valid argument, so stated that its conclusiveness is evident from the mere *form* of the expression) that if letters or any other unmeaning symbols be substituted for the several terms, the validity of the argument shall still be evident. Whenever this is not the case, the supposed argument is either unsound and sophistical, or else may be reduced, (without any alteration of its meaning) into the syllogistic form; in which form, the test just mentioned may be applied to it.

What is called an unsound or fallacious argument, i. e. an *apparent* argument which is, in reality, none, cannot, of course, be reduced into this form; but when stated in the form most nearly approaching to this that is possible, its fallaciousness becomes more evident, from its nonconformity to the foregoing rule: e. g. "whoever is capable of deliberate crime is responsible; an infant is not capable of deliberate crime; therefore, an infant is not responsible:" here, the term "responsible" is affirmed universally of "those capable of deliberate crime;" it might, therefore, according to Aristotle's dictum, have been affirmed of any thing contained under that class; but in the instance before us nothing is mentioned as contained under that class, only the term infant is *excluded* from that class; and though what is affirmed of a whole class may be affirmed of any thing that *is* contained under it, there is no ground for supposing that it may be *denied* of whatever is *not* so contained; for it is evidently possible that it may be applicable to a whole class and to something else besides: to say, e. g. that all trees are vegetables, does not imply that *nothing else* is a vegetable. It is evident, therefore, that such an apparent argument as the above does not comply with the rule laid down, and is consequently invalid.

Again, in this instance, "food is necessary to life;

corn is food; therefore corn is necessary to life:" the term "necessary to life" is affirmed of food, but *not universally*; for it is not said of *every kind of food*: the meaning of the assertion being manifestly that *some* food is necessary to life: here again therefore the rule has not been complied with, since that which is predicated, (i. e. affirmed or denied,) not of the *whole*, but of a *part* only of a certain class, cannot be predicated of any thing, whatever is contained under that class.

The fallacy in this last case is, what is usually described in Logical language as consisting in the "non-distribution of the middle term." In order to understand this phrase, it is necessary to observe, that a proposition being an expression in which one thing is affirmed or denied of another; e. g. "A is B," both that of which something is said, and that which is said of it, (i. e. both A and B,) are called "Terms," from their being (in their nature) the extremes or *boundaries* of the proposition; and there are, of course, two, and but two, terms in a proposition, (though it may so happen that either of them may consist either of one *word*, or of several;) and a term is said to be "distributed," when it is taken universally, so as to stand for every thing it is capable of being applied to; and consequently "undistributed," when it stands for a part only of the things signified by it; thus, "*all* food," or *every* kind of food, are expressions which imply the distribution of the term "food;" "*some* food" would imply its non-distribution: and it is also to be observed, that the term of which, in one premiss, something is affirmed or denied, and to which in the other premiss something else is referred as contained in it, is called the "middle" term in the syllogism, as standing *between* the other two, (viz. the two terms of the conclusion,) and being the medium of proof. Now it is plain, that if in each premiss a *part* only of this middle term is employed, i. e. if it be not at all distributed, no conclusion can be drawn. Hence, if in the example formerly adduced, it had been merely stated that "*something*" (not "*whatever*," or "*every thing*") "which exhibits marks of design, is the work of an intelligent author," it would not have followed, from the world's exhibiting marks of design, that that is the work of an intelligent author.

It is to be observed, also, that the words "all," and "every," which mark the distribution of a term, and "some," which marks its non-distribution, are not always introduced: they are frequently understood, and left to be supplied by the context; e. g. "food is necessary:" viz. "*some* food;" "man is mortal;" viz. "*every* man." Propositions thus expressed are called by Logicians "*indefinite*," because it is left undetermined by the form of the expression whether the "subject," (the term of which something is affirmed or denied being called the "subject" of the proposition, and that which is said of it, the "predicate") be distributed or not. Nevertheless it is plain that in every proposition the subject either is, or is not, distributed, though it be not declared whether it is or not; consequently every proposition, whether expressed indefinitely or not, must be either "universal" or "particular;" those being called universal, in which the predicate is said of the whole of the subject, (or in other words, where the subject is distributed;) and those, particular, in which it is said only of a part of the subject: e. g. "All men are sinful," is universal; "some men are sinful," particular: and this division of propositions is in Logical language said to be according to their "*quantity*."

But the distribution or non-distribution of the *predicate* is entirely independent of the *quality* of the proposition; nor are the signs "all" and "some" ever affixed to the predicate; because its distribution depends upon, and is indicated by the "*quality*" of the proposition; i. e. its being affirmative or negative; it being a universal rule, that the predicate of a negative proposition is distributed, and, of an affirmative, undistributed. The reason of this may easily be understood, by considering that a term which stands for a whole class may be applied to (i. e. *affirmed* of) any thing that is comprehended under that class, though the term of which it is thus affirmed may be of much narrower extent than that other, and may, therefore, be far from coinciding with the *whole* of it: thus it may be said with truth, that "the Negroes are uncivilized," though the term uncivilized be of much wider extent than "Negroes," comprehending, besides them, Hottentots, &c.: so that it would not be allowable to assert, that "*all* who are uncivilized are Negroes;" it is evident, therefore, that it is a *part* only of the term "uncivilized" that has been affirmed of "Negroes:" and the same reasoning applies to every affirmative proposition; for though it may so happen that the subject and predicate coincide, i. e. are of equal extent, as, e. g. "all men are rational animals," (it being equally true, that "all rational animals are men,) yet this is not *implied by the form of the expression*; since it would be no less true, that "all men are rational animals," even if there were other rational animals besides man.

It is plain, therefore, that if *any part* of the predicate is applicable to the subject, it may be affirmed, and, of course, cannot be denied of that subject; and consequently, when the predicate *is denied* of the subject, it is implied that *no part* of that predicate is applicable to that subject; i. e. that the *whole* of the predicate is denied of the subject: for to say, e. g. that "no beasts of prey ruminate," implies that beasts of prey are excluded from the *whole class* of ruminant animals, and consequently that "no ruminant animals are beasts of prey. And hence results the above mentioned rule, that the distribution of the predicate is implied in negative propositions, and its non-distribution in affirmatives.

It is to be remembered, therefore, that it is not sufficient for the middle term to *occur* in a universal proposition, since if that proposition be an affirmative, and the middle term be the *predicate* of it, it will not be distributed: e. g. if in the example formerly given it had been merely asserted, that "all the works of an intelligent author shew marks of design," and that "the universe shows marks of design," nothing could have been proved; since, though both these propositions are universal, the middle term is made the predicate in each, and both are affirmative; and accordingly the rule of Aristotle is not here complied with, since the term, "work of an intelligent author," which is to be proved applicable to "the universe," is not affirmed of the middle term, ("what shows marks of design,") under which "universe" is contained; but the middle term on the contrary is affirmed of it.

If, however, one of the premises be negative,

the middle term may then be made the predicate of it, and will thus, according to the above remark, be distributed: e. g. "no ruminant animals are predacious; the lion is predacious; therefore the lion is not ruminant:" this is a valid syllogism; and the middle term (predacious) is distributed by being made the predicate of a negative proposition. The form, indeed, of the syllogism, is not that prescribed by the dictum of Aristotle, but it may easily be reduced to that form, by stating the first proposition thus; no predacious animals are ruminant; which is manifestly implied (as was above remarked) in the assertion, that "no ruminant animals are predacious." The syllogism will thus appear in the form to which the dictum applies.

It is not every argument, indeed, that can be reduced to this form by so short and simple an alteration as in the case before us: a longer and more complex process will often be required; and rules will hereafter be laid down to facilitate this process in certain cases: but there is no sound argument but what *can* be reduced into this form, without at all departing from the real meaning and drift of it: and the form will be found (though more prolix than is needed for ordinary use) the most perspicuous in which an argument can be exhibited.

All reasoning whatever, then, rests on the one simple principle laid down by Aristotle; that, "what is predicated, either affirmatively or negatively, of a term distributed, may be predicated, in like manner, (i. e. affirmatively or negatively) of any thing contained under that term." So that when our object is to prove any proposition, i. e. to shew that one term may rightly be affirmed or denied of another, the process which really takes place in our minds is, that we refer that term (of which the other is to be thus predicated,) to some class, (i. e. middle term) of which that other may be affirmed, or denied, as the case may be. Whatever the subject matter of an argument may be, the Reasoning itself, considered by itself, is in every case the same process; and if the writers against Logic had kept this in mind, they would have been cautious of expressing their contempt of what they call "syllogistic Reasoning," which is in truth *all* Reasoning; and instead of ridiculing Aristotle's principle for its obviousness and simplicity, would have perceived that these are in fact its highest praise: the easiest, shortest, and most evident theory, provided it answer the purpose of explanation, being ever the best.

If we conceive an inquirer to have reached, in his investigation of the theory of Reasoning, the point to which we have now arrived, a question which would be likely next to engage his attention, is, that of predication; i. e. since in Reasoning we are to find a middle term, which may be predicated affirmatively of the subject in question, we are led to inquire what terms may be affirmed, and what denied, of what others.

It is evident that proper names, or any other terms, which denote each but a single individual, as "Cæsar," "the Thames," "the Conqueror of Pompey," "this river," (hence called in Logic, "singular terms") cannot be affirmed of any thing besides themselves, and are therefore to be denied of any thing else; we may say, "this river is the Thames," or "Cæsar was the conqueror of Pompey;" but we cannot say of any thing else that it is the Thames.

On the other hand, those terms which are called "common," as denoting any one individual of a whole class, as "river," "conqueror," may of course be affirmed of any, or all that belong to that class; as, "the Thames is a river;" "the Rhine and the Danube are rivers."

Common terms, therefore, are called "predicables," (viz. affirmatively predicable,) from their capability of being affirmed of others: a singular term on the contrary may be subject of a proposition, but never the predicate, unless it be of a negative proposition; (as, e. g. the first-born of Isaac was not Jacob;) or, unless the subject and predicate be only two expressions for the same individual object, as in some of the above instances.

The process by which the mind arrives at the notions expressed by these "common" (or in popular language, "general") terms, is properly called generalization; though it is usually (and truly) said to be the business of *abstraction*; for generalization is one of the purposes to which abstraction is applied: when we *draw off*, and *contemplate separately*, any part of an object presented to the mind, disregarding the rest of it, we are said to *abstract* that part. Thus, a person might, when a rose was before his eyes or mind, make the scent a distinct object of attention, laying aside all thought of the colour, form, &c.; and thus, though it were the only rose he had ever met with, be would be employing the faculty of abstraction; but if, in contemplating several objects, and finding that they agree in certain points, we abstract the circumstances of agreement, disregarding the differences, and give to all and each of these objects a name applicable to them in respect of this agreement, i. e. a common name, (as "rose,") we are then said to generalize. Abstraction, therefore, does not necessarily imply generalization, though generalization implies abstraction

Much needless difficulty has been raised respecting the results of this process; many having contended, and perhaps more having taken for granted, that there must be some really existing *thing*, corresponding to each of these general or common terms, and of which such term is the name, standing for and representing it: e. g. that as there is a really existing being corresponding to the proper name Ætna, and signifying it, so the common term "mountain," must have some one really existing thing corresponding to it, and of course *distinct* from each individual mountain, (since the term is not singular, but common,) yet existing in each, since the term is applicable to each of them. "When many different men," it is said, "are at the same time thinking or speaking about a mountain, i. e. not any particular one, but a mountain generally, their minds must be all employed on *something*; which must also be *one* thing, and not several, and yet cannot be any one individual:" and hence a vast train of mystical disquisitions about ideas, &c. has arisen, which are at best nugatory, and tend to obscure our view of the process which actually takes place in the mind.

The fact is, the notion expressed by a common term is merely an inadequate (or incomplete) notion of an individual; and from the very circumstance of its inadequacy, it will apply equally well to any one of several individuals: e. g. if I omit the mention and the consideration of every circumstance which

distinguishes Ætna from any other mountain, I then form a notion (expressed by the common term mountain) which inadequately designates Ætna, and is equally applicable to any one of several other individuals.

Generalization, it is plain, may be indefinitely extended by a further abstraction applied to common terms: e. g. as by abstraction from the term Socrates we obtain the common term philosopher; so from "philosopher," by a similar process, we arrive at the more general term "man;" from "man" to "animal," &c.

The employment of this faculty at pleasure has been regarded, and perhaps with good reason, as the characteristic distinction of the human mind from that of the brutes. We are thus enabled, not only to separate, and consider singly, one part of an object presented to the mind, but also to fix arbitrarily upon whatever part we please, according as may suit the purpose we happen to have in view: e. g. any individual person to whom we may direct our attention, may be considered either in a political point of view, and accordingly referred to the class of merchant, farmer, lawyer, &c. as the case may be; or physiologically, as negro, or white man; or theologically, as Pagan or Christian, Papist or Protestant; or geographically, as European, American, &c. &c. And so, in respect of anything else that may be the subject of our Reasoning: we arbitrarily fix upon and abstract that point which is essential to the purpose in hand; so that the same object may be referred to various different classes, according to the occasion. Not, of course, that we are allowed to refer anything to a class to which it does *not* really belong; which would be pretending to abstract from it something that was *no* part of it; but that we arbitrarily fix on *any part* of it which we choose to abstract from the rest. It is important to notice this, because men are often disposed to consider each object as really and properly belonging to some one class alone, from their having been accustomed, in the course of their own pursuits, to consider in one point of view only things which may with equal propriety be considered in other points of view also: i. e. referred to various classes, (or predicates.) And this is that which chiefly constitutes what is called narrowness of mind: e. g. a mere Botanist might be astonished at hearing such plants as clover and lucerne included, in the language of a farmer, under the term "grasses," which he has been accustomed to limit to a tribe of plants widely different in all Botanical characteristics; and the mere farmer might be no less surprised to find the troublesome "weed," (as he has been accustomed to call it,) known by the name of couch grass, and which he has been used to class with nettles and thistles, to which it has no *Botanical* affinity, ranked by the Botanist as a species of wheat, (*Triticum Repens*.) And yet neither of these classifications is in itself erroneous or irrational; though it would be absurd in a Botanical treatise to class plants according to their Agricultural use; or in an Agricultural treatise, according to the structure of their flowers.

The utility of these considerations, with a view to the present subject, will be readily estimated, by recurring to the account which has been already given of the process of Reasoning; the analysis of which shews, that it consists in referring the term we are speaking of to some class, viz. a middle term; which term again is referred to or excluded from (as the case may be) another class, viz. the term which we wish to affirm or deny of the subject of the conclusion. So that the quality of our Reasoning in any case must depend on our being able, correctly, clearly, and promptly, to *abstract* from the subject in question that which may furnish a middle term suitable to the occasion.

The imperfect and irregular sketch which has here been attempted, of the Logical System, may suffice (even though some parts of it should not be at once fully understood by those who are entirely strangers to the study) to point out the general drift and purpose of the Science, and to render the details of it both more interesting and more intelligible. The analytical form, which has here been adopted, is, generally speaking, the best suited for *introducing* any science in the plainest and most interesting form; though the synthetical, which will henceforth be employed, is the most regular and the most compendious form for storing it up in the memory.

CHAPTER I.

OF THE OPERATIONS OF THE MIND AND OF TERMS.

There are three operations of the mind which are concerned in argument: 1st. Simple Apprehension; 2d. Judgment; 3d. Discourse or Reasoning. 1st. Simple apprehension is the notion (or conception) of any object in the mind, analogous to the perception of the senses. It is either incomplex or complex: incomplex apprehension is of one object, or of several without any *relation* being perceived between them, as of "a man," "a horse," "cards:" complex is of several *with* such a relation, as of "a man on horseback," "a pack of cards."

2d. Judgment is the comparing together in the mind two of the notions, (or ideas) whether complex or incomplex, which are the objects of apprehension, and pronouncing that they *agree* or *disagree* with each other; (or that one of them *belongs* or does not belong to the other.) Judgment therefore is either *affirmative* or *negative*.

3d. Reasoning (or discourse) is the act of proceeding from one judgment, to another *founded* upon it, (or the result of it.)

§ 2. Language affords the *signs* by which these operations of the mind are expressed and communicated. An act of *Apprehension* expressed in language, is called a *Term*; an act of *Judgment*, a *Proposition*; an act of *Reasoning*, an *Argument* or Syllogism; as e. g.

"Every dispensation of Providence is beneficial;
Afflictions are dispensations of Providence,
Therefore they are beneficial:" is a Syllogism;

(the act of Reasoning being indicated by the word "*therefore*,") it consists of three *Propositions*, each of which has (necessarily) two *Terms*, as "beneficial," "dispensations of Providence," &c.

Language is employed for various purposes, e. g. the province of an historian is to *convey information*; of an orator, to *persuade*, &c. Logic is concerned with it only when employed for the purpose of *Reasoning*, (i. e. in order to *convince*;) and whereas, in reasoning, *Terms* are liable to be *indistinct*, (i. e. without any clear determinate meaning,) *Propositions*, to be *false*, and *Arguments*, *inconclusive*, Logic undertakes directly and completely to guard against *this last* defect, and incidentally and in a certain degree against the others, as far as can be done by the *proper use of language*: it is, therefore, (when regarded as an *art**) "the art of employing language properly for the purpose of Reasoning." Its importance no one can rightly estimate who has not long and attentively considered how much our thoughts are influenced by words, and how much error, perplexity, and labour, are occasioned by a faulty use of language.

A Syllogism being, as aforesaid, resolvable into three Propositions, and each Proposition containing two Terms; of these Terms, that which is spoken of, is called the *Subject*; that which is said of it, the *Predicate*; and these two together are called the *Terms*, (or extremes,) because, logically, the subject is placed *first*, and the predicate *last*: and, in the middle, the Copula, which indicates the act of Judgment, as by it, the Predicate is affirmed or denied of the Subject. It must be either IS or IS NOT; the substantive verb being the only *verb* recognised by Logic: all others are resolvable, by means of the verb, "to be," and a participle or adjective; e. g. "the Romans conquered:" the word "conquered" is both Copula and Predicate, being equivalent to "*were* (Cop.) *victorious*" (Pred.)†

§ 3. It is evident that a Term may consist either of one word or of several; and that it is not every word that is capable of being employed by itself as a Term; e. g. adverbs, prepositions, &c. and also nouns in any other case besides the nominative. A noun may be by itself a Term; a verb (all except the substantive verb used as the Copula,) is resolvable into the Copula and Predicate, to which it is equivalent, and indeed is often so resolved in the mere rendering out of one language into another; as "*ipse adest*," he is present. It is to be observed, however, that under "verb," we do not include the infinitive, which is properly a noun substantive, nor the participle, which is a noun adjective. They are *verbals*, being related to their respective verbs in respect of *the things they signify*: but not *verbs*, inasmuch as they differ entirely in their *mode of signification*. It is worth observing, that an infinitive (though it often comes last in the sentence) is *never the Predicate*, except when another infinitive is the Subject. It is to be observed, also, that in English there are two infinitives, one, in "*ing*," the same in sound and spelling as the participle present, from which, however, it should be carefully distinguished; e.g. "rising early is healthful," and "it is healthful to rise early," are equivalent.

An adjective (including participles) cannot, by itself, be made the Subject of a Proposition; but is often employed as a Predicate; as "Crassus was rich;" though some choose to consider some substantives as understood in every such case, (e. g. rich man) and consequently do not reckon adjectives among simple Terms; i. e. words which are capable, simply, of being employed as Terms. This, however, is a question of no practical consequence.

Of simple Terms, then, (which are what the first part of Logic treats of) there are many divisions;* of which, however, one will be sufficient for the present purpose; viz. into *singular* and *common*; because, though any Term whatever may be a *Subject*, none but a *common* Term can be affirmatively *predicated* of *several* others. A singular Term stands for *one individual*, as "Cæsar," "the Thames;" (these, it is plain, cannot be said [or predicated] *affirmatively*, of any thing but themselves.) A common Term stands for several individuals: i. e. can be *applied* to any of them, as comprehending them in its *single signification*; as "man," "river," "great." The notions expressed by these common Terms, we are enabled to form, by the faculty of *abstraction*: for by it, in contemplating any object (or objects,) we can attend exclusively to some particular circumstances belonging to it, [some certain parts of its nature as it were] and quite withhold our attention from the rest. When, therefore, we are thus contemplating several individuals which *resemble* each other in some *part* of their nature, we can (by attending to *that part alone*, and not to those points in which they differ) assign them *one common name*, which will express or stand for them merely as far as they all *agree*; and which of course will be applicable to all or any of them; (which process is called *generalization*,) and each of these names is called a *common* Term, from its belonging to them *all alike*; or a *Predicable*, because it may be predicated affirmatively of them, or of any one of them.

Generalization (as has been remarked) implies abstraction, but it is not the same thing; for there may be abstraction without generalization: when we are speaking of an individual, it is usually an abstract notion that we form; e. g. suppose we are speaking of the present King of France; he must actually be

* It is to be observed, however, that as a *science* is conversant about *knowledge only*, an *art* is the *application* of knowledge to *practice*; hence Logic (as well as any other system of knowledge) becomes, when applied to practice, an *art*; while confined to the *theory* of Reasoning, it is strictly a *science*: and it is as such that it occupies the higher place in point of dignity, since it professes to develope some of the most interesting and curious intellectual phenomena.

† It is proper to observe, that the Copula, as such, has no relation to *time*; but expresses merely the agreement or disagreement of two given terms: hence, if any other *tense* of the substantive verb, besides the present, is used, it is either to be understood as the same in sense, (the difference of tense being regarded as a matter of grammatical convenience only;) or else, if the circumstance of time really do *modify the sense* of the whole proposition, so as to make the use of that tense an essential, then this circumstance is to be regarded as a part of one of the terms: "*at that time*," or some such expression, being understood. Sometimes the substantive verb is both Copula and Predicate; i. e. where existence only is predicated: e. g. *Deus est*.

* The usual divisions of words into univocal, equivocal, and analogous, and into words of the first and second intention, however, are not, strictly speaking, divisions of *words*, but divisions of the *manner of employing* them. the *same* word may be employed either univocally, equivocally, or analogously; either in the first intention or in the second.

either at Paris or elsewhere; sitting, standing, or in some other posture; and in such and such a dress, &c. Yet many of these circumstances, (which are *separable* accidents, (vide § 7.) and consequently) which are regarded as *non-essential to the individual*, are quite disregarded by us; and we *abstract* from them what we consider as essential; thus forming an *abstract* notion of the individual. Yet there is here no generalization.

§ 4. Whatever Term can be affirmed of several things, must express either their *whole essence*, which is called the *Species*; or a *part* of their essence, (viz. either the material part, which is called the *Genus*, or the *formal* and *distinguishing part*, which is called *Differentia*,) or in common discourse, *characteristic*, or something *joined to the essence*, whether *necessarily*, which is called a *property*, or *contingently*, which is an *accident*.

Every Predicable expresses either

The *whole* essence of its subject: viz.: Species— | or part of its essence | or something *joined* to its essence

Genus—Difference

Property | Accident

universal but not peculiar | peculiar but not universal | universal and peculiar

inseparable—separable.

It is evident from what has been said, that the Genus and Difference put together make up the Species: e.g. "rational" and "animal" constitute "man;" so that, *in reality*, the Species contains the Genus (i.e. implies it;) and when the Genus is called a *whole*, and is said to *contain* the Species, this is only a *metaphorical* expression, signifying that it *comprehends* the Species, in its own more *extensive* signification: e.g. if I predicate of Cæsar that he is an *animal*, I say the truth indeed, but not the *whole* truth; for he is *not only* an animal, but a man; so that "man" is a more *full and complete* expression than "animal;" which for the same reason is more *extensive*, as it contains, (or rather comprehends) and may be predicated of, several other Species, i.e. "beast," "bird," &c. In the same manner the name of a *Species* is a more *extensive*, but less *full and complete* term than that of an *individual*, (viz. a singular term;) since the Species may be predicated of each of these. [Note, that Genus and Species are commonly said to be *predicated in quid*, (τί) (i.e. to answer to the question "what?" as, "what is Cæsar?" Answer, "a man:" "what is a man?" Answer, "an animal.") Difference, in "quale quid;" (ποῖόν τι) Property and Accident in *quale* (ποῖον).]

§ 5. A *Genus, which is also a Species*, is called a *subaltern* Genus or Species; as "bird," which is the Genus of "pigeon, (i.e. of which "pigeon" is a Species) is itself a *Species* of "animal." A Genus which is *not* considered as a *Species* of anything, is called *summum* (the highest) Genus; a Species which is *not* considered as a *Genus* of any thing, i.e. is regarded as containing under it only *individuals*, is called *infima* (the lowest) Species.

When I say of a magnet, that it is "a kind of *iron ore*," that is called its *proximum* Genus, because it is the closest (or lowest) Genus that can be predicated of it: "mineral" is its more *remote* Genus.

When I say that the *Differentia* of a magnet is its "*attracting iron*," and that its *Property* is "*polarity*," these are called respectively a specific Difference and Property; because magnet is an *infima Species*, (i.e. *only* a Species.)

When I say that the *Differentia* of iron ore is its "*containing iron*," and its Property "*being attracted by the magnet*," these are called respectively, a *generic* Difference and Property, because iron ore is a *subaltern* Species or Genus, being both the *Genus* of *magnet*, and a *Species* of *mineral*.

That is the most strictly called a Property, which belongs to the *whole* of a Species, and to that Species *alone*; as polarity to the magnet. [And such a *property*, it is often hard to distinguish from the *Differentia*; but whatever you *consider* as the most *essential to the nature* of a Species with respect to the matter you are engaged in, you must call the *Differentia*; as "rationality" to "man;" and whatever you consider as rather an *accompaniment* (or result) of that Difference, you must call the *Property*; as the "use of speech" seems to be a result of rationality.] But very many Properties which belong to the *whole* of a Species are not peculiar to it; as, "to breathe air" belongs to every *man*, but not to man alone; and it is, therefore, strictly speaking, not so much a Property of the Species "man," as of the higher, i.e. more comprehensive, Species, which is the Genus of that, viz. of "land animal." Other Properties, as some Logicians call them, are *peculiar* to a Species, but do not belong to the whole of it: e.g. man *alone* can be a poet, but it is not *every* man that is so. These, however, are more commonly and more properly reckoned as *Accidents*.

For that is most properly called an Accident, which may be absent or present, the essence of the Species continuing the same; as, for a man to be "*walking*," or a "*native of Paris*:" of these two examples, the former is what Logicians call a *separable* Accident, because it may be separated from the *individual*: (e.g. he may sit down;) the latter is an *inseparable* Accident, being not separable from the individual, (i.e. he who is an individual of Paris can never be otherwise;) "from the individual," I say, because *every Accident* must be separable from the *Species*, else it would be a *Property*.

Let it here be observed, that both the general name "Predicable," and each of the classes of Predicables, (viz. Genus, Species, &c.) are *relative*; i.e. we cannot say *what Predicable* any Term is, or whether it is any at all, unless it be specified *of what* it is to be predicated: e.g. the Term "red" would be considered a *Genus*, in relation to the Terms "pink," "scarlet," &c. it might be regarded as the *Differentia*, in relation to "red rose;"—as a property of "blood;"—as an Accident of "a house," &c.

And universally, it is to be steadily kept in mind, that no "common Terms" have, as the names of individuals have, any *real thing existing in nature* corresponding to them; (τόδε τι, as Aristotle expresses it, though he has been represented as the champion of

the opposite opinion: vide *Categ*. c. 3.) but is merely a name denoting a certain *inadequate notion* which our minds have formed of an individual, and which, consequently, not including any thing wherein that individual differs from certain others, is applicable equally well to all or any of them: thus "man" denotes no real thing (as the sect of the Realists maintained,) distinct from each individual, but merely, *any* man, viewed *inadequately*, i. e. so as to omit and abstract from all that is peculiar to each individual; by which means the Term becomes applicable alike to any one of several individuals, or (in the plural) to several together; and we arbitrarily fix on the circumstance which we thus choose to abstract and consider separately, disregarding all the rest; so that the same individual may thus be referred to any of several different Species, and the same Species to several Genera, as suits our purpose. Thus it suits the farmer's purpose to class his cattle with his ploughs, carts, and other possessions, under the name of "*stock*:" the naturalist, suitably to *his* purpose, classes them as "*quadrupeds*," which Term would include wolves, deer, &c., which to the farmer would be a most improper classification: the commissary, again, would class them with corn, cheese, fish, &c. as "*provision*." That which is most essential in one view, being subordinate in another.

§ 6. An *individual* is so called because it is incapable of *logical Division*; which is a metaphorical expression to signify "the distinct (i. e. separate) enumeration of several things signified by one common name." This operation is directly opposite to *generalization*, (which is performed by means of abstraction;) for as in that, you *lay aside* the *difference* by which several things are distinguished, so as to call them all by one *common name*, so, in Division, you *add on* the differences, so as to enumerate them by their *several particular* names. Thus, "mineral" is said to be divided into "stones, metals," &c.; and metals again into "gold, iron," &c. and these are called the parts (or members) of the Division.

The rules for Division are three: 1st. each of the parts, or any of them short of *all*, must *contain less* (i. e. have a narrower signification) than the thing divided. 2d. All the parts together must be exactly equal to the thing divided; (therefore we must be careful to ascertain that the *summum Genus* may be predicated of *every* Term placed under it, and of nothing else.) 3d. The parts or members must be *opposed*; i. e. must not be contained in one another: e. g. if you were to divide "book" into "poetical, historical, folio, quarto, French, Latin," &c. the members would be contained in each other; for a French book may be a quarto, and a quarto, French, &c. You must be careful, therefore, to keep in mind the *principle of Division* with which you set out: e. g. whether you begin dividing books according to their *matter*, their *language*, or their *size*, &c. these being also so many *cross Divisions*. And when any thing is capable (as in the above instance) of being divided in several different ways, we are not to reckon one of these as the true, or real, or right one, without specifying what the object is which we have in view: for one mode of dividing may be the most suitable for one purpose, and another, for another; as e. g. one of the above modes of dividing books would be the most suitable to a bookbinder; another in a philosophical, and the other in a philological view.

It must be carefully remembered, that the word "Division," as employed in Logic, is, as has been observed already, *metaphorical*; for to divide, means originally and properly to separate the component parts of any thing, each of which is of course absolutely less than the whole: e. g. a tree (i. e. *any individual tree*) might be divided "physically," as it is called, into root, trunk, branches, leaves, &c. Now it cannot be said that a root or a leaf is a tree: whereas in a logical Division each of the members is, in reality, *more* than the whole: e. g. if you divide tree (i. e. *the Genus, tree*) into oak, ash, elm, &c. we may say of the oak, or of any individual oak, that "it is a tree;" for by the very word "oak," we express not only the general notion of a tree, but *more*, viz. the peculiar characteristic (i. e. difference) of that kind of tree.

It is plain, then, that it is *logically* only, i. e. in our mode of speaking, that a Genus is said to contain (or rather, *comprehend*) its Species; while metaphysically, i. e. in our conceptions, a Species contains, i. e. *implies*, its Genus.

Care must be taken not to confound a *physical* Division with a *Logical*, against which a caution is given under R. 1.

§ 7. *Definition* is another metaphorical word, which literally signifies, "laying down a boundary;" and is used in Logic to signify an expression which explains any term, so as to *separate* it from every thing else, as a boundary separates fields. A nominal Definition (such as are those usually found in a *dictionary* of one's own language) explains only the *meaning of the term*, by giving some equivalent expression, which may happen to be better known. Thus you might define a "Term," that which forms one of the *extremes* or *boundaries* of a "Proposition;" and a "Predicable," that which may be predicated; "decalogue," "ten commandments;" "telescope," an instrument for viewing distant objects, &c. A real *Definition* is one which explains and unfolds the *nature of the thing*; and each of these kinds of Definition is either *accidental* or *essential*. An *essential* Definition assigns (or lays down) the *constituent parts of the essence*, (or nature.) An *accidental* Definition (which is commonly called a *Description*) assigns the circumstances *belonging* to the essence, viz. Properties and Accidents, (e. g. causes, effects, &c.) thus, "man" may be *described* as "an animal that uses fire to dress his food," &c. [And here note, that in *describing a Species*, you cannot mention any thing which is strictly an *Accident*, because if it does not belong to the *whole* of the Species, it cannot define it: in describing an *individual*, on the contrary, you enumerate the *Accidents*, because by them it is that one individual differs from another, and in this case you add *the Species*: e. g. "Philip was a *man* of Macedon, who subdued Greece," &c. Individuals, it is evident, can be defined in this way alone.]

Lastly, the essential Definition is divided into *physical* (i. e. natural) and *Logical* or *Metaphysical*: the *physical* Definition lays down the *real parts of the essence* which are actually separable; the *logical*, lays down the *ideal* parts of it, which cannot be separated except in *the mind*: thus, a plant would be defined *physically*, by enumerating the leaves, stalks, roots,

&c. of which it is composed: *logically*, it would be defined an organized being, destitute of sensation; the former of these expressions expressing the Genus, the latter, the Difference: for a *logical Definition* must always consist of the *Genus* and *Differentia*, which are the parts of which Logic considers every thing as consisting, and which evidently are separable in the *mind alone*. Thus "man" is defined "a rational animal," &c. So also a "Proposition" might be defined, physically, a Subject and Predicate combined by a Copula: the parts here enumerated being actually separable; but logically it would be defined "a sentence which affirms or denies;" and these two parts of the essence of a Proposition (which are the *Genus* and *Differentia* of it) can be separated in the mind only. And note, that the difference is not always *one* quality, but is frequently *compounded* of several together, no one of which would alone suffice.

Definitions are divided into nominal and real, according to the *object accomplished* by them; whether to explain, merely, the meaning of the word, or the nature of the thing: they were divided into accidental, physical, and logical, according to the *means employed* by each for accomplishing their respective objects, whether it be the enumeration of attributes, or of the physical or the metaphysical parts of the essence. These, therefore, are evidently two cross divisions. In this place we are concerned with nominal Definitions only, (except, indeed, of logical Terms,) because all that is requisite for the purposes of *Reasoning* (which is the proper province of Logic,) is, that a Term shall not be used in *different senses*: a real Definition of any thing belongs to the science or system which is employed about that thing. It is to be noted, that in Mathematics the nominal and real Definition exactly coincide; the *meaning of the word*, and the *nature of the thing*, being exactly *the same*. This holds good also with respect to logical Terms, most legal, and many ethical terms.

It is scarcely credible how much confusion has arisen from the ignorance of these distinctions which has prevailed among logical writers.

The principal rules for Definition are three; viz. 1st. The Definition must be *adequate*; i. e. neither too extensive nor too narrow for the thing defined: e. g. to define "fish," "an animal that lives in the water," would be too *extensive*, because many insects, &c. live in the water; to define it, "an animal that has an air-bladder," would be too *narrow*; because many fish are without any.

2d. The Definition must be in itself plainer than the thing defined, else it would not *explain it*: I say, "in itself," (i. e. generally) because, to some particular person, the term defined may happen to be even more familiar and better understood, than the terms of the definition.

3d. It must be couched in a *convenient number* of *appropriate* words, (if such can be found suitable for the purpose:) for *figurative* words (which are opposed to appropriate) are apt to produce ambiguity or indistinctness: too great *brevity* may occasion *obscurity*; and too great *prolixity*, *confusion*.

CHAPTER II.

OF PROPOSITIONS.

§ 1. The second part of Logic treats of the *Proposition*; which is, "*Judgment expressed in words.*"

A proposition is defined logically "*a sentence indicative*," i. e. affirming or denying; (this excludes *commands* and *questions*.) "Sentence" being the *Genus*, and "indicative" the *Difference*, this definition expresses the whole essence; and it relates entirely to the *words* of a Proposition. With regard to the *matter*, its Property is to be *true* or *false*, and therefore it must not be *ambiguous*, (for that which has more than one *meaning*, is in reality *several* Propositions;) nor *imperfect*, nor *ungrammatical*, for such an expression has no meaning at all.

Since the *Substance* (i. e. *Genus*, or material part) of a Proposition is, that it is a *sentence*; and since every *sentence* (whether it be a *Proposition* or not) may be expressed either *absolutely*, (as "Cæsar deserved death;" "did Cæsar deserve death?") or under an *hypothesis*, (as, "if Cæsar was a tyrant, what did he deserve?" "Was Cæsar a hero or a villain?" "If Cæsar was a tyrant, he deserved death;" "he was either a hero or a villain,") on this we found the division of Propositions according to their *substance*; viz. into *categorical* and *hypothetical*. And as Genus is said to be *predicated in quid* (what,) it is by the members of this division that we answer the question, *what* is this Proposition? *(quæ est propositio.)* Answer, categorical or hypothetical.

Categorical Propositions are subdivided into *pure*, which asserts *simply* or *purely*, that the Subject does or does not agree with the predicate, and modal, which expresses in what *mode* (or manner) it agrees; e. g. "an intemperate man will be sickly;" "Brutus killed Cæsar;" are *pure*. "An intemperate man will *probably* be sickly;" "Brutus killed Cæsar *justly*;" are *modal*. At present we speak only of *pure categorical* Propositions.

It being the *Differentia* of a *Proposition*, that it *affirms or denies*, and its *Property* to be *true or false*; and Differentia being predicated in *quale quid*; Property in *quale*, we hence form another division of Propositions, viz. according to their *quality*, into *affirmative*, and *negative*, (which is the *quality of the expression*, and therefore (in Logic) essential;) and into true and false, (which is the quality of the *matter*, and therefore *accidental*.) An affirmative Proposition is one whose *Copula* is affirmative, as "birds fly;" "not to advance is to go back;" a negative proposition is one whose *Copula* is negative, as "man is not perfect;" no "miser is happy."

Another division of Propositions is according to their *quantity*, (or extent;) if the Predicate is said of the *whole* of the Subject, the Proposition is *universal*; if of a *part* of it only, the Proposition is *particular*, (or partial;) e. g. "England is an island;" "all tyrants are miserable;" "no miser is rich;" are *universal* Propo-

sitions, and their Subjects are therefore said to be *distributed*, being understood to stand, each, for *the whole* of its significates: but, "*some* islands are fertile;" "all tyrants are not assassinated;" are *particular*, and their Subjects, consequently not *distributed*, being taken to stand for a part only of their significates.

As every Proposition must be either *affirmative* or *negative*, and must also be either *universal* or *particular*, we reckon in all, four kinds of pure categorical Propositions, (i. e. considered as to their quantity and quality *both*;) viz. universal affirmative, whose symbol (used for brevity,) is *A*; universal negative, *E*; particular affirmative, *I*; particular negative, O.

§ 2. When the subject of a Proposition is a common Term, the *universal signs* ("all, no, every,") are used to indicate that it is distributed, (and the Proposition consequently is universal;) the *particular signs*, ("some, &c.") the contrary; should there be *no sign* at all to the common Term, the quantity of the Proposition (which is called an *indefinite* Proposition) is ascertained by the *matter*; i. e. the nature of the connection between the extremes; which is either necessary, impossible, or contingent. In necessary and in impossible matter, an indefinite is *understood as a universal*: e. g. "birds have wings;" i. e. *all*: "birds are not quadrupeds;" i. e. *none*: in contingent matter, (i. e. where the terms partly (i. e. sometimes) agree, and partly not,) an indefinite is understood as a particular; e. g. "food is necessary to life;" "birds sing;" i. e. *some* do; "birds are not carnivorous;" i. e. "*some* are not," or, "all are not."

As for *singular* Propositions, (viz. those whose Subject is either a *proper name*, or a common Term with a *singular* sign,) they are reckoned as universals, (see ch. iv. § 2.) because in them we speak of the *whole* of the subject; e. g. when we say, "Brutus was a Roman," we mean, the *whole* of Brutus: this is the general rule; but some singular Propositions may fairly be reckoned *particular*; i. e. when some qualifying word is inserted, which indicates that you are not speaking of the *whole* of the subject; e. g. "Cæsar was not *wholly* a tyrant;" "this man is *occasionally* intemperate;" "*non omnis moriar*." It is evident that the *Subject* is *distributed* in every universal Proposition, and never in a *particular*; (that being the very difference between universal and particular Propositions;) but the distribution or non-distribution of the *Predicate*, depends (not on the *quantity*, but) on the *quality*, of the Proposition; for, if any *part of the Predicate* agrees with the Subject, it must be *affirmed* and not *denied* of the Subject; therefore, for an affirmative Proposition to be true, it is *sufficient* that some *part of the Predicate* agree with the Subject; and (for the same reason) for a negative to be true, it is necessary that the *whole* of the Predicate should *disagree* with the Subject: e. g. it is true that "learning is useful," though the whole of the Term "useful" does not agree with the Term "learning," (for many things are useful besides learning,) but "no vice is useful," would be false, if any part of the Term "useful" agreed with the Term "vice;" (i. e. if you could find any one useful thing which was a vice.) The two practical rules then to be observed respecting distribution, are,

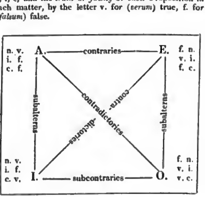
1st. All universal Propositions (and no particular) distribute the *Subject*.

2d. All *negative*, (and no affirmative) the Predicate.

It *may happen* indeed, that the whole of the Predicate in an affirmative may agree with the Subject; e. g. it is equally true, that "all men are rational animals;" and "all rational animals are men:" but this is merely *accidental*, and is not at all *implied* in the *form of expression*, which alone is regarded in Logic.

Of Opposition.

§ 3. Two Propositions are said to be opposed to each other, when having the same Subject and Predicate; they differ in *quantity*, or *quality*, or *both*. It is evident, that with any given Subject and Predicate, you may state four distinct Propositions, viz. A, E, I, and O; and any two of these are said to be *opposed*; hence there are four different kinds of opposition, viz. 1st. the two universals (A and E) are called *contraries* to each other; 2d. the two particular, (I and O,) *subcontraries*: 3d. A and I, or E and O, *subalterns*; 4th. A and O, or E and I, *contradictories*. As it is evident that the truth or falsity of any Proposition (its quantity and quality being known,) must depend on the *matter* of it, we must bear in mind that, "*in necessary matter all affirmatives are true and negatives false*; *in impossible matter, vice versâ*; *in contingent matter, all universals false, and particulars true*;" (e. g. "*all* islands, (or, *some* islands,) are surrounded by water," must be true, because the *matter is necessary*: to say, "*no* islands, or some — *not*, &c." would have been false; again, "*some* islands are fertile; "*some* are not fertile," are both true, because it is contingent matter: put "*all*" or "no" instead of "*some*," and the propositions will be false.) Hence it will be evident, that contraries will be *both false* in contingent matter, but never *both true*: subcontraries, *both true* in contingent matter, but never *both false*: contradictories, always *one true* and *the other false*, &c. with other observations, which will be immediately made on viewing the scheme? in which the four Propositions are denoted by their symbols; the different kinds of *matter*, by the initials n, i, c, and the *truth* or *falsity* of each Proposition in each matter, by the letter v. for (*verum*) true, f. for (*falsum*) false.



By a careful study of this scheme, bearing in mind, and applying the above rule concerning matter, the learner will easily elicit all the maxims relating

to Opposition; as that, in the subalterns, the truth of the particular (which is called the *subalternate*) follows from the truth of the universal *(subalternans)* and the falsity of the universal from the falsity of the particular: that subalterns differ in *quantity alone*; contraries, and also subcontraries in *quality alone*; contradictories, in both: and hence, that if any Proposition is known to be true, we infer that its contradictory is false; if false, its contradictory true, &c.

Of Conversion.

§ 4. A Proposition is said to be *converted* when its Terms are *transposed*: when nothing more is done, this is called *simple* Conversion. No Conversion is of any use, unless it be *illative*; i.e. *when the truth of the converse follows from* the truth of the exposita, (or proposition given;) e. g.

> "No virtuous man is a rebel, *therefore*
> No rebel is a virtuous man."
>
> "Some boasters are cowards, *therefore*
> Some cowards are boasters."

Conversion can then only be illative when *no Term is distributed in the converse, which was not distributed in the exposita*: (for if that be done, you will employ a Term *universally* in the converse, which was only used *partially* in the exposita.) Hence, as E distributes both Terms, and I neither, these Propositions may be illatively converted in the simple manner; (vid. Rule 2.) But as A does not distribute the Predicate, its simple Conversion would not be illative; (e. g. from "all birds are animals," you cannot infer that "all animals are birds,") as there would be a Term distributed in the converse, which was not before. We must therefore *limit its quantity* from universal to particular, and the Conversion will be illative. (e. g. "some animals are birds;") this might be fairly named Conversion by *limitation*; but is commonly called "*Conversion per accidens.*" E may thus be converted also. But in O, whether the quantity be changed or not, there will still be a Term (the Predicate of the converse) distributed, which was not before: you can therefore only convert it by changing the quality; i. e. considering the negative as attached to the *Predicate instead of to the Copula*, and thus regarding it as I. One of the Terms will then not be the same as before; but the Proposition will be æquipollent; (i. e. convey the same meaning,) e. g. "some members of the University are not learned:" you may consider "*not learned*" as the Predicate, instead of "*learned*;" the Proposition will then be I, and of course may be simply converted, "some who are not learned are members of the University." This may be named Conversion by negation; or as it is commonly called, by contra-position. A may also be fairly converted in this way, e. g.

> "Every poet is a man of genius; *therefore*
> He who is not a man of genius, is not a poet:"
> (or, "None but a man of genius can be a poet.")

For (since it is the same thing, to *affirm* some Attribute of the Subject, or to *deny* the *absence* of that Attribute,) the original Proposition is precisely æquipollent to this,

$$\text{“No } \overbrace{\text{poet}}^{\text{subj.}} \text{ is } \overbrace{\text{not a man of genius}}^{\text{pred.}}\text{;”}$$

which, being E, may of course be simply converted. Thus, in one of these three ways, every Proposition may be illatively converted: viz. "*E, I, simply; A, O, by negation*; A, E, limitation." Note, that as it was remarked, that in some affirmatives, the whole of the Predicate does actually agree with the Subject; so, when this is the case, A may be illatively converted, simply; but this is an accidental circumstance. In a *just definition*, this is always the case; for there the Terms being *exactly equivalent*, (or, as they are called, *convertible Terms)* it is no matter which is made the Subject, and which the Predicate, e. g. "a good government is that which has the happiness of the governed for its object;" if this be a *right definition*, it will follow that "a government which has the happiness of the governed for its object, is a good one." Most Propositions in Mathematics are of this description: e. g.

> "All equilateral triangles are equiangular;" and
> "All equiangular triangles are equilateral."

CHAPTER III.

OF ARGUMENTS.

§ 1. The third operation of the mind, viz. *Reasoning* (or discourse) expressed in words, is *Argument*; and an Argument stated *at full length*, and in its *regular form* is called a *Syllogism*: the third part of Logic therefore treats of the *Syllogism*. Every Argument consists of two parts; that which is *to be proved*; and that *by means of which* it is proved: the former is called *before* it is proved the *Question*; *when proved*, the *Conclusion*, (or *inference*;) that which is used to prove it, if stated *last*, (as is often done in *common discourse*,) is called the *Reason*, and is introduced by "*because*," or some other casual conjunction; (e. g. "Cæsar deserved death, *because* he was a tyrant, and all tyrants deserve death.") If the Conclusion be stated *last*, (which is the strict *logical form*, to which all Reasoning may be reduced,) then that which is employed to prove it is called the *Premises*; and the Conclusion is then introduced by some *illative* conjunction, as "therefore" e. g.

> "All tyrants deserve death;
> Cæsar was a tyrant;
> *therefore* he deserved death."

Since then an Argument is an expression in which "*from something laid down and granted as true, (i. e. the Premises) something else, (i. e. the Conclusion) beyond this, must be admitted to be true, as following necessarily, (or resulting) from the other*;" and since Logic is

wholly concerned in the use of language, it follows that a Syllogism (which is an Argument stated in a regular logical form,) must be "an Argument so expressed, that the conclusiveness of it is manifest from *the mere force of the expression*," i. e. without considering the *meaning of the Terms*: e. g. in this syllogism, "B is A, C is B, therefore C is A:" the Conclusion is inevitable, whatever Terms A, B, and C, respectively, are understood to stand for. And to this form, all legitimate Arguments may ultimately be brought.

§ 2. The rule or axiom, (commonly called "*dictum de omni et nullo*,") by which Aristotle proves the validity of this Argument is this: "*whatever is predicated of a Term distributed, whether affirmatively or negatively, may be predicated in like manner, of every thing contained under it.*" Thus, in the examples above, A is predicated of B distributed, and C is *contained under* B, (i. e. is its Subject;) therefore A is predicated of C: so "all tyrants, &c." (p. 208.) This rule may be *ultimately* applied to all Arguments; (and their validity ultimately rests on their conformity thereto;) but it cannot be *directly and immediately* applied to *all*, even of pure *categorical* Syllogisms; for the sake of brevity therefore some other axioms are commonly applied in practice, to avoid the occasional tediousness of reducing all Syllogisms to that form in which Aristotle's *dictum* is applicable.

We will speak first of pure categorical Syllogisms; and the axioms or canons by which their validity is to be proved: viz. first, *if two Terms agree with one and the same third, they agree with each other*: second, *if one Term agrees and another disagrees with one and the same third, these two disagree with each other.* On the former of these canons rests the validity of *affirmative* conclusions; on the latter, of *negative*: for no Syllogism can be faulty which does not violate these canons; none correct which does: hence on these two canons are built the rules or cautions which are to be observed with respect to Syllogisms, for the purpose of ascertaining whether those canons have been strictly observed or not.

1st. *Every Syllogism has three, and only three Terms*; viz. the two Terms (or *extremes*, as they are commonly called) of the Conclusion, (or question;) (whereof first, the *Subject* is called the *minor Term*; second, the *Predicate*, the *major*;) and third, the *middle Term*, with which each of them is separately compared, in order to judge of their agreement or disagreement with each other. If therefore there were *two* middle Terms, the *extremes*, (or *Terms of the Conclusion*) not being both compared to the *same*, could not be compared to each other.

2d. Every syllogism has three, and only three Propositions; viz. first, the *major Premiss*, (in which the major Term is compared with the *middle*;) second, the *minor Premiss*, (in which the minor Term is compared with the *middle*;) and third, the Conclusion, in which the minor Term is compared with the major.

3d. Note, that if the *middle Term is ambiguous, there are in reality two middle Terms, in sense*, though but one in *sound*. An ambiguous middle Term is either an equivocal Term, used in different senses in the two Premises; (e.g.

> "*Light* is contrary to darkness;
> Feathers are *light*; therefore
> Feathers are contrary to darkness.")

Or a Term *not distributed*; for as it is then used to stand for a *part* only of its signification, it may happen that one of the extremes may have been compared with *one part* of it, and the other, with another part of it; e. g.

> "White is a colour,
> Black is a colour; therefore
> Black is white."—Again,
> "Some animals are beasts,
> Some animals are birds; therefore
> Some birds are beasts."

The middle Term therefore must be distributed once, at least, in the Premises; (i. e. by being the Subject of an universal, or Predicate of a negative, Ch. II. § 2. p. 207.) and once is sufficient; since if one extreme has been compared to a *part* of the middle Term, and another to the *whole* of it, they must have been both compared to the *same*.

4th. *No Term must be distributed in the Conclusion which was not distributed in one of the Premises*; for that (it is called an *illicit* process, either of the major or the minor Term) would be to employ the *whole* of a Term in the Conclusion, when you had employed only a *part* of it in the Premiss; and thus, in reality, to introduce a fourth Term; e. g.

> "All quadrupeds are animals,
> A bird is not a quadruped; therefore
> It is not an animal."—Illicit process of the major.

5th. *From negative Premises you can infer nothing.* For in them the middle is pronounced *to disagree with both extremes*; not to *agree with both*; or to agree with one, and *disagree* with the other; therefore they cannot be compared together; e. g.

> "A fish is not a quadruped,"
> "A bird is not a quadruped," proves nothing.

6th. *If one Premiss be negative, the conclusion must be negative*; for in that Premiss the middle Term is pronounced to disagree with one of the extremes, and in the other Premiss, (which of course is affirmative, by the preceding rule) to agree with the other extreme; therefore the extremes disagreeing with each other, the conclusion is negative. In the same manner it may be shewn, that *to prove a negative conclusion one of the Premises must be a negative.*

By these six rules, all Syllogisms are to be tried; and from them it will be evident; first, that *nothing can be proved from two particular Premises*; (for you will then have *either the middle Term undistributed*, or an *illicit process*; e. g.

> "Some animals are sagacious;
> Some beasts are not sagacious;
> Some beasts are not animals.")

And for the same reason secondly, that if one of the Premises be particular, the Conclusion must be particular; e. g. from

> "All who fight bravely deserve reward;
> Some soldiers fight bravely;" you can only infer that some soldiers deserve reward.

For to infer a universal Conclusion, would be an illicit process of the minor. But from two universal

Premises you cannot always infer a universal Conclusion; e. g.

> "All gold is precious,
> All gold is a mineral; therefore
> *Some* mineral is precious."

And even when we *can* infer a universal, we are always *at liberty* to infer a particular; since what is predicated of *all* may *of course be* predicated of some.

Of Moods.

§ 3. When we designate the three Propositions of a Syllogism in their order, according to their respective quantity and quality, (i. e. their *symbols*) we are said to determine the *Mood* of the Syllogism; e. g. the example just above, "all gold, &c." is in the Mood A, A, I. As there are four kinds of Propositions, and three Propositions in each Syllogism, all the possible ways of combining these four, (A, E, I, O,) by threes, are sixty-four. For any one of these four may be the major Premiss; each of these four majors may have four different minors, and of these sixteen pairs of Premises, each may have four different Conclusions. 4×4 $(=16) \times 4 = 64$. This is a mere arithmetical calculation of the Moods, without any regard to the Logical rules: for many of these Moods are inadmissible in practice, from violating some of those rules; e. g. the Mood E, E, E, must be rejected, as having *negative Premises*; I, O, O, for *particular Premises*; and many others for the same faults. By examination then of all, it will be found that of the sixty-four, there remain but twelve Moods, which can be used in a legitimate Syllogism, viz. A, A, A, A, A, I, A, E, E, A, E, O, A, I, I, A, O, O, E, A, E, E, A, O, E, I, O, I, A, I, I, E, O, O, A, O.

Of Figure.

§ 4. The Figure of a Syllogism consists in the situation of the middle Term with respect to the extremes of the conclusion, (*i. e. the major and minor term.*) When the middle Term is made the *subject of the major Premiss, and the Predicate of the minor*, that is called the first Figure; (which is far the most natural and clear of all, as to this alone, Aristotle's dictum may be *at once* applied.) In the second Figure the middle Term is the Predicate of *both* Premises: in the third, *the Subject* of both: in the fourth, the Predicate of the major Premiss, and the Subject of the minor. (This is the most awkward and unnatural of all, being the very reverse of the first.) Note, that the *proper order* is to place the major Premiss *first*, and the minor *second*; but this does not *constitute* the major and minor Premises; for that Premiss (wherever placed) is the major which *contains the major Term*, and the minor, the minor, (v. R. 2. p. 209.) Each of the allowable Moods mentioned above, will not be allowable in every Figure; since it may violate some of the foregoing rules, in one Figure, though not in another: e. g. I, A, I, is an allowable Mood in the third Figure; but in the first, it would have an *undistributed middle*. So A, E, E, would in the first Figure have an *illicit process of the major*, but is allowable in the second; and A, A, A, which in the first Figure is allowable, would in the third have an *illicit process of the minor*: all which may be ascertained by trying the different Moods in each Figure, as per scheme.

Let A represent the major Term, C the minor, B the middle.

1st Fig.	2d Fig.	3d Fig.	4th Fig.
B, A,	A, B,	B, A,	A, B,
C, B,	C, B,	B, C,	B, C,
C, A,	C, A,	C, A,	C, A.

The Terms alone being here stated, the *quantity and quality* of each Proposition (and consequently the Mood of the whole Syllogism) is left to be filled up: (i. e. between B, and A, I may place either a negative or affirmative Copula; and I may prefix either a *universal* or *particular* sign to B.) By applying the Moods then to each Figure, it will be found that each Figure will admit six Moods only, as not violating the rules against *undistributed middle*, and against *illicit process*: and of the Moods so admitted, several (though valid) are *useless*, as having a particular Conclusion, when a *universal might have* been drawn; e. g. A, A, I, in the first Figure,

> "All human creatures are entitled to liberty;
> All slaves are human creatures; therefore
> *Some* slaves are entitled to liberty."

Of the twenty-four Moods then (six in each Figure) five are for this reason neglected: for the remaining nineteen, Logicians have devised names to distinguish both the Mood itself, and the Figure in which it is found; since when one Mood (i. e. *one in itself*, without regard to Figure) occurs in two different Figures, (as E, A, E, in the first and second) the mere letters denoting the Mood would not inform us concerning the Figure. In these names then, the *three vowels* denote the Propositions of which the Syllogism is composed; the consonants (besides their other uses, of which hereafter) serve to keep in mind the Figure of the Syllogism.

Fig. 1. bArbArA, cElArEnt, dArII, fErIOque prioris.
Fig. 2. cEsArE, cAmEstrEs, fEstInO, bArOkO, secundæ.
Fig. 3. { tertia, dArAptI, dIsAmIs, dAtIsI, fElAptOn, bOkArdO, fErIsO, habet: quarta insuper addit.
Fig. 4. brAmAntIp, cAmEnEs, dImArIs, fElApo, frEsIsOn.

By a careful study of these mnemonic lines (which must be committed to memory) you will perceive that A can only be proved in the first Figure, in which also every other Proposition may be proved; that the second proves only *negatives*; the third only *particulars*, &c.; with many other such observations, which will readily be made, (on trial of several Syllogisms, in different Moods) and the reasons for which will be found in the foregoing rules. E. G. to shew why the second Figure has only negative Conclusions, we have only to consider, that in it the middle Term being the *Predicate in both Premises*, would not be *distributed* unless one Premiss were *negative*; (v. R. 2. p. 205.) therefore the conclusion must be negative also, by R. 6. p. 209. One Mood in each Figure may suffice in this place by way of example; first, *Barbara*, viz. (bAr.)

Every B is A; (bA) every C is B; therefore (rA) every C is A, e. g. let the major Term (which is represented by A) be "one who possesses all virtue;"

the minor term (C) "every man who possesses one virtue;" and the middle term (B) "every one who possesses prudence;" and you will have the celebrated argument of Aristotle, *Eth.* sixth book, to prove that the virtues are inseparable; viz.

"He who possesses prudence, possesses all virtue;
He who possesses one virtue, must possess prudence; therefore
He who possesses one, possesses all."

Second, *Camestres*, (cAm) every A is B; (Es) no C is B; (trES) no C is A. Let the major term (A) be "true philosophers," the minor (C) "the Epicureans;" the middle (B) "reckoning virtue a good in itself;" and this will be part of the reasoning of Cicero, *Off.* book first and third, against the Epicureans. Third, *Darapti*, viz. (*dA*) every B is A; (*rAp*) every B is C; therefore (*tI.*) Some C is A. e. g.

"Prudence has for its object the benefit of individuals;
But prudence is a virtue; therefore
Some virtue has for its object the benefit of the individual," is part of Adam Smith's reasoning, (*Moral Sentiments*,) against Hutcheson and others, who placed all virtue in benevolence. Fourth, Camenes, viz. (*cAm*) every A is B; (*En*,) no B is C; therefore (Es,) no C is A, e. g.

"Whatever is expedient, is conformable to nature;
Whatever is conformable to nature, is not hurtful to society; therefore
What is hurtful to society is never expedient,"

is part of Cicero's argument in *Off.* third book: but it is an inverted and clumsy way of stating what would much more naturally fall into the first Figure; for if you examine the propositions of a Syllogism in the fourth Figure, *beginning* at the *Conclusion*, you will see that as the major Term is predicated of the minor, so is the minor of the middle, and that again of the major: so that the major appears to be merely *predicated of itself*. Hence the five Moods in this Figure are seldom or never used; some one of the fourteen (*Moods with names*) in the first three Figures, being the forms into which all Arguments may most readily be thrown; but of these, the four in the first Figure are the clearest and most natural; as to them, Aristotle's dictum will *immediately* apply. And as it is on this dictum that all Reasoning *ultimately* depends, so all Arguments may be somehow or other brought into some one of these four Moods; and a Syllogism is, in that case, said to be *reduced*: (i. e. *to the first Figure*.) These four are called the *perfect* Moods, and all the rest, *imperfect*.

Ostensive Reduction.

§ 5. In reducing a Syllogism, we are not of course allowed to introduce any new Term or Proposition, having nothing granted but the truth of the Premises; but these Premises are allowed to be *illatively converted*, (because the truth of any Proposition *implies* that of its illative converse) *or transposed:* by taking advantage of this liberty, where there is need, we deduce in Figure one, from the Premises originally given, either the *very same Conclusion* as the original one, or another from which the *original Conclusion* follows, by illative Conversion; e. g. *Darapti*

"All wits are dreaded;
All wits are admired;
Some who are admired are dreaded."

Into *Darii*, by converting by limitation (*per accidens*) the minor Premiss.

"All wits are dreaded;
Some who are admired are wits; therefore
Some who are admired are dreaded."

Camestres.

"All true philosophers account virtue a good in itself;
The advocates of pleasure do not account, &c.
Therefore they are not true philosophers."

Reduced to *Celarent*, by simply converting the minor, and then transposing the Premises.

"Those who account virtue a good in itself, are not advocates of pleasure;
All true philosophers account virtue, &c.; therefore
No true philosophers are advocates of pleasure."

This Conclusion may be *illatively converted* into the original one.

Baroko, e. g.

"Every true patriot is a friend to religion;
Some great statesmen are not friends to religion;
Some great statesmen are not true patriots."

To *Ferio*, by converting the major *by negation* (contraposition) vide Ch. II. § 4.

"He who is not a friend to religion, is not a true patriot;
Some great statesmen, &c."

and the rest of the Syllogism remains the same; only that the minor Premiss must be considered as affirmative, because you take "not a friend to religion" as the middle Term. In the same manner *Bokardo* to *Darii*; e. g.

"Some slaves are not discontented;
All slaves are wronged; therefore
Some who are wronged are not discontented."

Convert the major by negation (contraposition) and then transpose them; the Conclusion will be the *converse by negation of the original one*, which therefore may be inferred from it; e. g.

"All slaves are wronged;
Some who are not discontented are slaves;
Some who are not discontented are wronged."

In these ways (which are called *Ostensive Reduction*, because you prove in the first Figure, either the *very same* conclusion as before, or one *which implies it*) all the imperfect Moods may be reduced to the four perfect ones. But there is also another way, called *reductio ad impossibile*,

§ 6. By which we prove (in the first Figure) not directly that the original Conclusion is *true*, but that it *cannot be false*; i. e. that an absurdity would follow from the supposition of its being false; e. g.

"All true patriots are friends to religion;
Some great statesmen are not friends to religion;
Some great statesmen are not true patriots.

If this conclusion be not true, its contradictory must be true; viz.

Logic.

"All great statesmen are true patriots."

Let this then be assumed, in the place of the minor Premiss of the original Syllogism, and a false conclusion will be proved; e. g. bAr.

"All true patriots are friends to religion;
bA, All great statesmen are true patriots;
rA, *All great statesmen are friends to religion:*"

for as this Conclusion is the contradictory of the original minor Premiss, it must be false, since the Premises are always supposed to be granted; therefore one of the *Premises* (by which it has been correctly proved) must be false also; but the major Premiss (being one of those originally granted) is *true*; therefore the *falsity must be in the minor Premiss*; which is the *contradictory* of the original Conclusion; therefore the original Conclusion must be true. This is the *indirect* mode of Reasoning.

§ 7. This kind of Reduction is seldom employed but for *Baroko* and *Bokardo*, which are thus reduced by those who confine themselves to *simple* Conversion, and Conversion by limitation, (*per accidens*;) and they framed the names of their Moods with a view to point out the manner in which each is to be reduced; viz. B, C, D, F, which are the initial letters of all the Moods, indicate to which Mood of the first Figure, (*Barbara*, *Celarent*, *Darii*, and *Ferio*,) each of the others is to be reduced: *m*, indicates that the Premises are to be *transposed*; *s*, and *p*, that the Proposition denoted by the vowel immediately preceding, is to be *converted*; *s*, simply, *p*, *per accidens*, (by limitation:) thus, in *Camestres*, (see example, p. 211,) the *C*, indicates that it must be reduced to *Celarent*; the two *ss*, that the minor Premiss and Conclusion must be *converted simply*; the *m*, that the Premises must be *transposed*. *K*, (which indicates the reduction *ad impossibile*) is a sign that the Proposition denoted by the vowel immediately before it, must be left out, and the contradictory of the Conclusion substituted; viz. for the *minor* premiss in *Baroko*, and the *major* in *Bokardo*. But it has been already shewn, that the Conversion by contraposition, (by negation,) will enable us to reduce these two Moods, *ostensively*.

Chap. III.
Chap. IV.

CHAPTER IV

OF MODAL SYLLOGISMS, AND OF ALL ARGUMENTS BESIDES REGULAR AND PURE CATEGORICAL SYLLOGISMS.

Of Modals.

§ 1. Hitherto we have treated of *pure* categorical Propositions, and the Syllogisms composed of such: a *Modal* Proposition may be stated as a *pure* one, by *attaching the Mode to one of the Terms*: and the Proposition will in all respects fall under the foregoing rules; e. g. "John killed Thomas *wilfully and maliciously*;" here the Mode is to be regarded as part of the Predicate. "It is *probable* that all knowledge is useful;" "probably useful" is here the Predicate; but when the Mode is only used to express the necessary, contingent, or impossible connection of the Terms, it may as well be attached to the *Subject*: e. g. "man is *necessarily* mortal;" is the same as, "*all* men are mortal:" and "this man is *occasionally* intemperate," has the force of a *particular*: (vide Part II. § 2. p. 207.) It is thus that two singular Propositions may be contradictories; e. g. "this man is *never* intemperate," will be the *contradictory* of the foregoing. Indeed every sign (of universality or particularity) may be considered as a *Mode*. Since, however, in all Modal Propositions, you assert that the dictum (i. e. the *assertion itself*) and the *mode*, agree together, or disagree, so, in some ases, this may be the most convenient Way of stating a Modal, purely: e. g. "$\overbrace{\text{It}}^{\text{subj.}}$ $\overbrace{\textit{is}}^{\text{cop.}}$ $\overbrace{\textit{impossible}}^{\text{pred.}}$ $\overbrace{\text{that all men should be virtuous.}}^{\text{subject.}}$" Such is a proposition of St. Paul's: "$\overbrace{\text{This}}^{\text{subj.}}$ $\overbrace{\text{is}}^{\text{cop.}}$ $\overbrace{\text{a faithful saying, \&c.}}^{\text{pred.}}$ $\overbrace{\text{that Jesus Christ came into the world to save sinners.}}^{\text{subject.}}$" In these cases, one of your *Terms* (the Subject) is itself *an entire Proposition*. Thus much for Modal Propositions.

Of Hypotheticals.

§ 2. A hypothetical Proposition is defined to be, *two or more categoricals united by a Copula*, (or conjunction;) and the different kinds of hypothetical Propositions are named from their respective conjunctions; viz. conditional, disjunctive, causal, &c.

When a hypothetical Conclusion is inferred from a hypothetical Premiss, so that the force of the Reasoning does not turn on the hypothesis, then the hypothesis (as in Modals) must be considered *as part of one of the Terms*; so that the *Reasoning* will be, in effect, categorical: e. g.

"Every conqueror is $\overbrace{\text{either a hero or a villain:}}^{\text{predicate.}}$
Cæsar was a conqueror; therefore
He was $\overbrace{\text{either }\textit{a hero or a villain.}}^{\text{predicate.}}$"

"Whatever comes from God is entitled to reverence;
$\overbrace{\text{If the Scriptures are not wholly false,}}^{\text{subject.}}$ they must come from God;
If they are not wholly false, they are entitled to reverence."

But when the *Reasoning* itself *rests* on the hypothesis, (in which way a categorical Conclusion may be drawn from a hypothetical Premiss,) this is what is called a *hypothetical Syllogism*; and rules have been devised for ascertaining the validity of such Arguments, at once, without bringing them into the categorical form. (And note, that in these Syllogisms the *hypothetical Premiss* is called the *major*, and the *categorical one*, the *minor*.) They are of two kinds, *conditional* and *disjunctive*.

Of Conditionals.

§ 3. A *Conditional* Proposition has in it an *illative force*; i.e. it contains two, and only two categorical Propositions, whereof one *results* from the other, (or, follows from it,) e. g.

antecedent.

" If the Scriptures are not wholly false,

consequent.

they are entitled to respect."

That from which the other results, is called the *antecedent*; *that which results from it*, the *consequent*, (*consequens*;) and the *connection* between the two, (expressed by the word " if ") the *consequence*, (*consequentia*.) The natural order is, that the antecedent should come *before* the consequent; but this is frequently reversed: e. g. " the husbandman is well off if he knows his own advantages;" Virg. *Geor.* And note, that the truth or falsity of a conditional Proposition depends entirely on the *consequence*: e. g " if Logic is useless, *it* deserves to be neglected;" here both antecedent and consequent are *false*: yet the whole proposition is *true*: i.e. it it true that the consequent *follows* from the antecedent. " If Cromwell was an Englishman, he was an usurper," is just the reverse case: for though it is true that " Cromwell was an Englishman," and also that " he was an usurper," yet it is not true that the latter of these Propositions *depends on* the former; the whole Proposition, therefore, is false, though both antecedent and consequent are true. A Conditional Proposition, in short, may be considered as an assertion of the *validity* of a certain Argument; since to assert that an Argument is *valid*, is to assert that the Conclusion necessarily results from the Premises, whether those Premises be *true* or not. The meaning, then, of a Conditional Proposition is this; that, *the antecedent being granted, the consequent* is granted: which may be considered in two points of view: first, if the antecedent *be* true, the consequent *must* be true; hence the first rule; *the antecedent being granted, the consequent may be inferred*: secondly, if the antecedent *were* true, the consequent *would* be true; hence the second rule; *the consequent being denied, the antecedent may be denied*; for the antecedent must in that case be false; since if it were true, the consequent (which is granted to be false) would be true also: e. g. " if this man has a fever, he is sick;" here, if you *grant the antecedent*, the first rule applies, and you infer the truth of the consequent; " he has a fever, therefore he is sick:" if A is B, C is D; but A is B, therefore C is D, (and this is called a *constructive* Conditional Syllogism;) but if you *deny the consequent* (i. e. grant its *contradictory*,) the second rule applies, and you infer the *contradictory of the antecedent*: " he is not sick, therefore he has not a fever:" this is the *destructive* Conditional Syllogism: if A is B, C is D; C is not D, therefore A is not B. Again, " if the crops are not bad, corn must be cheap:" for a major; then, " but the crops are not bad, therefore corn must be cheap," is constructive. " Corn is not cheap, therefore the crops are bad," is destructive. " If every increase of population is desirable, some misery is desirable; but no misery is desirable, therefore, some increase of population is not desirable," is destructive. But if you *affirm* the *consequent*, or *deny* the *antecedent*, you can infer nothing; for the same consequent may *follow from other* antecedents: e. g. in the example above, a man may be sick *from other* disorders besides a fever; therefore it does not follow from his being sick, that he has a fever; nor (for the same reason) from his *not* having a fever, that he is *not* sick. There are, therefore, two, and only two kinds of Conditional Syllogisms; the *constructive*, founded on the first rule, and answering to *direct* Reasoning; and the *destructive* on the second, answering to *indirect*. And note, that a conditional Proposition may (like the categorical A,) be *converted by negation*; i. e. you may take the *contradictory of the consequent*, as an *antecedent*, and the *contradictory of the antecedent*, as a *consequent*: e. g. " if this man is not sick, he has not a fever." By this conversion of the major Premiss, a constructive Syllogism may be reduced to a destructive, and vice versa. (See § 6. Ch. IV. p. 214.)

Of Disjunctives.

§ 4. A disjunctive Proposition may consist of any number of categoricals; and, of these, *some one*, at least, must be *true*, or the whole Proposition will be false: if, therefore, one or more of these categoricals be denied, (i. e. granted to be false,) you may infer that the remaining one, or (if several) *some one* of the remaining ones is true: e. g. " either the earth is eternal, or the work of chance, or the work of an intelligent being; it is not eternal, nor the work of chance; therefore it is the work of an intelligent being." " It is either spring, summer, autumn, or winter; but it is neither spring nor summer, therefore it is either autumn or winter." Either A is B, or C is D: but A is not B, therefore C is D. Note, that in these two examples (as well as very many others,) it is implied not only that *one* of the members (the categorical Propositions) must be *true*, but that *only one* can be true; so that, in such *cases*, if one or more members be *affirmed*, the rest may be *denied*; [the members may then be called *exclusive*:] e.g. " it is summer, therefore it is neither spring, autumn, nor winter;" " either A is B, or C is D; but A is B, therefore C is not D." But this is by no means universally the case; e. g. " virtue tends to procure us either the esteem of mankind or the favour of God:" here both members are true, and consequently from one being affirmed, we are not authorized to deny the other. It is evident that a disjunctive Syllogism may easily be reduced to a *conditional*: e. g. if it is not spring or summer, it is either autumn or winter, &c.

The Dilemma.

§ 5. Is a complex kind of Conditional Syllogism.

1st. If you have in the *major* Premiss *several antecedents* all with the *same consequent*, then these antecedents, being (in the *minor*) *disjunctively granted*, (i. e. it being granted that *some one* of them is true,) the one *common consequent* may be inferred, (as in the case of a simple constructive syllogism:) e.g. If A is B, C is D; and if X is Y, C is D; but either A is B, or X is Y; therefore C is D. " If the blest in heaven have no desires, they will be perfectly content; so they will, if their desires are fully gratified; but

either they will have no desires, or have them fully gratified; therefore they will be perfectly content." Note, in this case, the two conditionals which make up the major Premiss may be united in one Proposition by means of the word "*whether*:" e.g. "whether the blest, &c. have no desires, or have their desires gratified, they will be content."

2d. But if the *several antecedents* have each a *different consequent*, then the antecedents, being as before, disjunctively granted, you can only *disjunctively* infer the consequents: e.g. if A is B, C is D; and if X is Y, E is F: but either A is B, or X is Y; therefore either C is D, or E is F. "If Æschines joined in the public rejoicings, he is inconsistent; if he did not, he is unpatriotic; but he either joined, or not, therefore he is either inconsistent or unpatriotic." (Demost. *For the Crown*) This case, as well as the foregoing, is evidently *constructive*. In the destructive form, whether you have one antecedent with several consequents, or several antecedents, either with one, or with several consequents; in all these cases, if you deny the *whole* of the consequent or consequents, you may in the conclusion, deny the *whole* of the antecedent or antecedents: e.g. "If this fact be true, it must be recorded either in Herodotus, Thucydides, or Xenophon: it is not recorded in any of the three, therefore it is not true." "If the world existed from eternity there would be records prior to the Mosaic; and if it were produced by chance, it would not bear marks of design: there are no records prior to the Mosaic; and the world does bear marks of design; therefore it neither existed from eternity, nor is the work of chance." These are commonly called Dilemmas, but hardly differ from *simple* conditional Syllogisms. Nor is the case different if you have *one* antecedent with several consequents, which consequents you *disjunctively* deny; for that comes to the same thing as *wholly* denying them; since if they be not *all* true, the *one antecedent* must equally fall to the ground; and the Syllogism will be equally simple: e.g. "if we are at peace with France by virtue of the treaty of Paris, we must acknowledge the sovereignty of Buonaparte; and also we must acknowledge that of Louis: but we cannot do both of these; therefore we are not at peace," &c.; which is evidently a plain destructive. The true dilemma is, "*a conditional Syllogism with several antecedents in the major, and a disjunctive minor*;" hence,

3d. That is most properly called a *destructive Dilemma*, which has (like the constructive ones) a *disjunctive minor Premiss*: i.e. when you have several antecedents with each a different consequent; which consequents, (instead of wholly denying them, as in the last case,) you *disjunctively* deny; and thence, in the Conclusion, deny disjunctively the antecedents: e.g. if A is B, C is D; and if X is Y, E is F: but either C is not D, or E is not F; therefore, either A is not B, or X is not Y. "If this man were wise, he would not speak irreverently of Scripture in jest; and if he were good he would not do so in earnest; but he does it, either in jest or in earnest; therefore he is either not wise or not good." Every Dilemma may be reduced into two or more simple Conditional Syllogisms: e.g. "if Æschines joined, &c. he is inconsistent; he *did* join, &c. therefore he is inconsistent: and again, if Æschines did not join, &c. he is unpatriotic; he *did not*, &c. therefore he is unpatriotic."

Now an opponent might deny *either* of the minor Premises in the above Syllogisms, but he could not deny *both*; and therefore he must admit one or the other of the Conclusions: for, when a Dilemma is employed, it is supposed that *some one* of the antecedents must be true, (or, in the destructive kind, *some one* of the consequents false,) but that we cannot tell *which* of them is so; and this is the reason why the argument is stated in the form of a Dilemma. From what has been said, it may easily be seen that all Dilemmas are in fact *conditional syllogisms*; and that disjunctive Syllogisms may also be reduced to the same form: but as it has been remarked, that all Reasoning whatever may *ultimately* be brought to the one test of Aristotle's "dictum," it remains to shew how a Conditional Syllogism may be thrown into such a form that that test will at once apply to it; and this is called the

Reduction of Hypotheticals.

§ 6. For this purpose we must consider every Conditional Proposition as a universal affirmative categorical Proposition, of which the Terms are entire Propositions, viz. the antecedent answering to the *Subject*, and the consequent to the *Predicate*; e.g. to say, "if Louis is a good king, France is likely to prosper;" is equivalent to saying, "the case of Louis being a good king, is a case of France being likely to prosper:" and if it be granted, as a minor Premiss to the Conditional Syllogism, that "Louis is a good king;" that is equivalent to saying, "the present case is the case of Louis being a good king:" from which you will draw a conclusion in *Barbara*, (viz. "the present case is a case of France being likely to prosper,") exactly equivalent to the original Conclusion of the Conditional Syllogism; viz. "France is likely to prosper." As the constructive condition may thus be reduced to *Barbara*, so may the destructive in like manner, to *Celarent*, e.g. "if the Stoics are right, pain is no evil: but pain is an evil; therefore, the Stoics are not right;" is equivalent to, "the case of the Stoics being right, is the case of pain being no evil; the present case is not the case of pain being no evil; therefore the present case is not the case of the Stoics being right." This is *Camestres*, which of course is easily reduced to *Celarent*. Or, if you will, all Conditional Syllogisms may be reduced to *Barbara*, by considering them all as constructive; which may be done, as mentioned above, by converting by negation the major Premiss, (see p. 212. § 3. Ch. IV.) The reduction of Hypotheticals may always be effected in the manner above stated; but as it produces a circuitous awkwardness of expression, a more convenient form may in some cases be substituted: e.g. in the example above, it may be convenient to take, "*true*," for one of the Terms: "that pain is no evil is not true; that pain is no evil is asserted by the Stoics; therefore something asserted by the Stoics is not true." Sometimes again it may be better to unfold the argument into two Syllogisms: e.g. in a former example; first, "Louis is a good king; the governor of France is Louis; therefore the governor of France is a good king." And then, second, "every country governed by a good king is likely to prosper," &c. [A Dilemma is generally to be reduced into two or more categorical Syllogisms.] And when the antecedent and consequent have each the *same* Subject, you may sometimes reduce the Conditional by merely

substituting a categorical major Premiss for the conditional one: e. g. instead of "if Cæsar was a tyrant, he deserved death; he was a tyrant, therefore he deserved death;" you may put for a major, "all tyrants deserve death," &c. But it is of no great consequence, whether Hypotheticals are reduced in the most *neat and concise* manner or not; since it is not intended that they should be reduced to categorical, in *ordinary practice*, as the *readiest way* of trying their validity, (their own rules being quite sufficient for that purpose;) but only *that we should be able*, if required, to subject any argument whatever to the test of Aristotle's dictum, in order to shew that all Reasoning turns upon one simple principle.

Of Enthymeme, Sorites, &c.

§ 7. There are various abridged forms of Argument which may be easily expanded into regular Syllogisms: such as, first, the Enthymeme, which is a Syllogism with one Premiss suppressed. As all the Terms will be found in the remaining Premiss and Conclusion, it will be easy to fill up the Syllogism by supplying the Premiss that is wanting, whether major or minor: e. g. "Cæsar was a tyrant; therefore he deserved death." "A free nation must be happy; therefore the English are happy."

This is the ordinary form of speaking and writing. It is evident that Enthymemes may be filled up hypothetically.

2d. When you have a string of Syllogisms, in which the Conclusion of each is made the Premiss of the next, till you arrive at the main and ultimate Conclusion of all, you may sometimes state these briefly, in a form called *Sorites;* in which the Predicate of the first proposition is made the Subject of the next; and so on, to any length, till finally the Predicate of the last of the Premises is predicated (in the Conclusion) of the Subject of the first: e. g. A is B, B is C, C is D, D is E; therefore A is E. "The English are a brave people; a brave people are free; a free people are happy; therefore the English are happy." A Sorites then has as many middle Terms as there are intermediate Propositions between the first and the last; and consequently it may be drawn out into as many separate Syllogisms; of which the first will have, for its *major Premiss*, the second; and for its *minor*, the *first* of the Propositions of the Sorites; as may be seen by the example. It is also evident, that in a Sorites you cannot have more than *one* negative Proposition, and one particular; for else, one of the Syllogisms would have its Premises both negative or both particular, (vid. p. 209.) A string of Conditional Syllogisms may in like manner be abridged into a Sorites; e. g. if A is B, C is D; if C is D, E is F; if E is F, G is H; but A is B, therefore G is H. "If the Scriptures are the word of God, it is important that they should be well explained; if it is important, &c. they deserve to be diligently studied; if they deserve, &c. an order of men should be set aside for that purpose: but the Scriptures are the word, &c.; therefore an order of men should be set aside for the purpose, &c." Hence, it is evident, how injudicious an arrangement has been adopted by former writers on Logic, who have treated of the Sorites and Enthymeme before they entered on the subject of Hypotheticals.

Those who have spoken of induction or of example, as a distinct kind of Argument in a Logical point of view, have fallen into the common error of confounding *Logical* with *Rhetorical* distinctions, and have wandered from their subject as much as a writer on the orders of Architecture would do, who should introduce the distinction between buildings of stone and of marble. Logic takes no cognizance of induction, for instance, or of *a priori* reasoning, &c. as distinct *Forms* of argument; for when thrown into the syllogistic form, and when letters of the alphabet are substituted for the Terms (and it is thus that Argument is properly to be brought under the cognizance of Logic,) there is no distinction between them; e. g. a Property which belongs to the ox, sheep, deer, goat, and antelope, belongs to all horned animals; rumination belongs to these; therefore, to all. This, which is an inductive argument, is evidently a Syllogism in *Barbara*. The essence of an inductive argument (and so of the other kinds which are distinguished for it,) consists, not in the *form of the Argument*, but in the relation which the *Subject matter* of the Premises bears to that of the Conclusion.

3d. There are various other abbreviations commonly used, which are so obvious as hardly to call for explanation: as, where one of the Premises of a Syllogism is itself the Conclusion of an Enthymeme which is expressed at the same time: e. g. "all useful studies deserve encouragement; Logic is such, (*since it helps us to reason accurately*,) therefore it deserves encouragement;" here, the minor Premiss is what is called an *Enthymematic sentence*. The *antecedent* in that minor Premiss, (i. e. that which *makes* it Enthymematic,) is called by Aristotle the *Prosyllogism*.

It is evident that you may for brevity substitute for any term an *equivalent*; as in the last example, "*it*" for "Logic;" "*such*" for "a useful study," &c.

4th. And many Syllogisms, which at first appear faulty, will often be found, on examination, to contain correct reasoning, and, consequently, to be reducible to a regular form; e. g. when you have, *apparently*, *negative Premises*, it may happen, that by considering one of them as *affirmative*, (see Ch. II. § 4. p. 208.) the Syllogism will be regular: e. g. "no man is happy who is not secure; no tyrant is secure; therefore no tyrant is happy," is a Syllogism in *Celarent*.* Sometimes there will appear to be too many terms; and yet there will be no fault in the Reasoning, only an irregularity in the expression: e. g. "no irrational agent could produce a work which manifests design; the universe is a work which manifests design; therefore no irrational agent could have produced the universe." Strictly speaking, this Syllogism has five Terms; but if you look to the meaning, you will see, that in the first Premiss (considering it *as a part of this Argument*,) it is not, properly, "an irrational agent" that you are speaking of, and of which you predicate that it could not produce a work manifesting design; but rather it is this "work," &c. of which you are speaking, and of which it is predicated that it could

* If this experiment be tried on a Syllogism which has *really* negative Premises, the only effect will be to change that fault into another: viz. an excess of Terms, or, (which is substantially the same) an undistributed middle; e. g. "an enslaved people is not happy; the English are not enslaved; therefore they are happy:" if "enslaved" be regarded as one of the Terms, and "not enslaved" as another, there will manifestly be four. Hence you may see how very little difference there is in reality between the different faults which are enumerated.

not be produced by an irrational agent; if then you state the Propositions in that form, the Syllogism will be perfectly regular.

Thus, such a Syllogism as this, "every true patriot is disinterested; few men are disinterested; therefore few men are true patriots;" might appear at first sight to be in the second Figure, and faulty; whereas it is *Barbara*, with the *Premises transposed*; for you do not really predicate of "few men," that they are "disinterested," but of "*disinterested persons*," that they are "few." Again, "none but candid men are good reasoners; few infidels are candid; few infidels are good reasoners." In this it will be most convenient to consider the major Premiss as being "all good reasoners are candid," (which of course is precisely æquipollent to its illative converse by negation;) and the minor Premiss and Conclusion may in like manner be fairly expressed thus —"most infidels are not candid; therefore most infidels are not good reasoners:" which is a regular Syllogism in *Camestres*. Or, if you would state it in the first Figure, thus—those who are not candid (or uncandid) are not good reasoners; most infidels are not candid; most infidels are not good reasoners.

§ 8. The foregoing rules enable us to develope the principles on which all Reasoning is conducted, whatever be the Subject matter of it, and to ascertain the validity or fallaciousness of any apparent argument, as far as the *form of expression* is concerned; that being alone the proper province of Logic.

But it is evident that we may nevertheless remain liable to be deceived or perplexed in Argument by the assumption of false or doubtful Premises, or by the employment of indistinct or ambiguous terms; and, accordingly, many Logical writers, wishing to make their systems appear as perfect as possible, have undertaken to give rules "for attaining clear ideas," and for "guiding the judgment;" and fancying or professing themselves successful in this, have consistently enough denominated Logic, the "Art of using the Reason;" which in truth it would be, and would supersede all other studies, if it could alone ascertain the meaning of every Term, and the truth or falsity of every Proposition, in the same manner as it actually can the validity of every Argument. And they have been led into this, partly by the consideration that Logic is concerned about the three operations of the mind—simple Apprehension, Judgment, and Reasoning; not observing that it is not equally concerned about all; the last operation being alone its appropriate province; and the rest being treated of only in reference to that.

The contempt justly due to such pretensions has most unjustly fallen on the Science itself, much in the same manner as Chemistry was brought into disrepute among the unthinking by the extravagant pretensions of the Alchemists. And those Logical writers have been censured, not (as they should have been) for *making* such professions, but for *not fulfilling* them. It has been objected, especially, that the rules of Logic leave us still at a loss as to the most important and difficult point in Reasoning; viz. the ascertaining the sense of the terms employed, and removing their ambiguity. A complaint resembling that made (according to a story told by Warburton in his *Div. Leg.*) by a man who found fault with all the reading-glasses presented to him by the shopkeeper; the fact being that he had never *learnt to read*. In the present case, the complaint is the more unreasonable, inasmuch as there neither is, nor ever *can possibly be*, any such system devised as will effect the proposed object of clearing up the ambiguity of Terms. It is, however, no small advantage, that the rules of Logic, though they cannot alone, ascertain and clear up ambiguity in any Term, point out in *which* Term of an Argument it is to be sought for, directing our attention to the middle Term, as the one on the ambiguity of which a fallacy is likely to be built.

It will be useful, however, to class and describe the different kinds of ambiguity which are to be met with; and also the various ways in which the insertion of false, or, at least, unduly assumed Premises, is most likely to elude observation. And though the remarks which will be offered on these points may not be considered as strictly forming a *part* of Logic, they cannot be thought out of place, when it is considered how essentially they are connected with the application of it.

CHAPTER V.

OF FALLACIES.

Introduction.

Logic. By a Fallacy is commonly understood, "any unsound mode of arguing, which appears to demand our conviction, and to be decisive of the question in hand, when in fairness it is not so." As we consider the ready detection and clear exposure of Fallacies to be both more extensively important, and also more difficult than many are aware of, we propose to take a Logical view of the subject; referring the different Fallacies to the most convenient heads, and giving a scientific analysis of the procedure which takes place in each.

After all, indeed, in the practical detection of each individual Fallacy, much must depend on natural and acquired acuteness; nor can any rules be given, the mere learning of which will enable us to apply them with mechanical certainty and readiness: but still we shall find that to take correct general views of the subject, and to be familiarized with scientific discussions of it, will tend, above all things, to *engender such a habit of mind* as will best fit us for practice.

Indeed the case is the same with respect to Logic in general; scarce any one would in ordinary practice, state to himself either his own or another's reasoning in Syllogisms in *Barbara* at full length; yet a familiarity with Logical principles, tends very much, (as all feel, who are really well acquainted with them,) to beget a habit of clear and sound Reasoning. The truth is, that in this, as in many other things, there are processes going on in the mind (when we are practising any thing quite familiar to us) with such rapidity as to leave no trace in the memory; and we often apply principles which did not, as far as we are conscious, even occur to us at the time.

It would be foreign, however, to the present purpose, to investigate fully the manner in which certain studies operate in remotely producing certain effects on the mind: it is sufficient to establish the *fact*, that habits of scientific analysis (besides the intrinsic beauty and dignity of such studies) lead to practical advantage. It is on Logical principles therefore that we propose to discuss the subject of Fallacies: and it might, indeed, seem to be unnecessary to make any apology for so doing, after what has been formerly said, generally, in defence of Logic: if the majority of Logical writers had not usually followed a very opposite plan. Whenever they have to treat of any thing that is beyond the mere elements of Logic, they totally lay aside all reference to the principles which they have been occupied in establishing and explaining, and have recourse to a loose, vague, and popular kind of language; such as would be the best suited indeed to an exoterical discourse, but seems strangely incongruous in a professed Logical treatise. What should we think of a Geometrical writer, who, after having gone through the Elements with strict definitions and demonstrations, should, on proceeding to Mechanics, totally lay aside all reference to scientific principles,—all use of technical terms,—and treat of the subject in undefined terms, and with probable and popular arguments? It would be thought strange, if even a Botanist, when addressing those whom he had been instructing in the principles and the terms of his system, should totally lay these aside when he came to describe plants, and should adopt the language of the vulgar. Surely it affords but too much plausibility to the cavils of those who scoff at Logic altogether, that the very writers who profess to teach it, should never themselves make any application of, or reference to its principles, on those very occasions, when, and *when only*, such application and reference are to be expected. If the principles of any system are well laid down,—if its technical language is well framed,—then, surely those principles and that language will afford, (for those who have once thoroughly learned them,) the best, the most clear, simple, and concise method of treating any subject connected with that system. Yet even the accurate Aldrich, in treating of the Dilemma and of the Fallacies, has very much forgotten the Logician, and assumed a loose and rhetorical style of writing, without making any application of the principles he had formerly laid down, but on the contrary, sometimes departing widely from them. Chap. V.

The most experienced teachers, when addressing those who are familiar with the elementary principles of Logic, think it requisite, not indeed to lead them, on each occasion, *through the whole detail* of those principles, when the process is quite obvious, but always to *put them on the road*, as it were, to those principles, that they may plainly see their own way to the end, and take a scientific view of the subject: in the same manner as Mathematical writers, avoid indeed the occasional tediousness of going all through a very simple demonstration which the learner, if he will, may easily supply; but yet always speak in strict Mathematical language, and with reference to Mathematical principles, though they do not always state them at full length. We would not profess, therefore, any more than they do, to write (on subjects connected with the science,) in a language intelligible to those who are ignorant of its first rudiments; to do so, indeed, would imply that we were not taking a scientific view of the subject, nor availing ourselves of the principles which had been established, and the accurate and concise technical language which had been framed.

§ 1. The division of Fallacies into those in the words, IN DICTIONE, and those in the matter EXTRA DICTIONEM, has not been, by any writers hitherto, grounded on any distinct principle; at least, not on any that they have themselves adhered to. The confounding together, however, of these two classes is highly detrimental to all clear notions concerning Logic; being obviously allied to the prevailing erroneous views which make Logic the *art of employing the intellectual faculties in general*, having the *discovery of truth* for its object, and all kinds of knowledge for its proper subject matter; with all that train of vague and groundless speculations which have led to such interminable confusion and mistakes, and afforded a pretext for such clamorous censures.

It is important, therefore, that rules should be given for a division of Fallacies into Logical, and Non-logical, on such a principle as shall keep clear of all this indistinctness and perplexity.

If any one should object that the division we adopt is in some degree arbitrary, placing under the one head Fallacies, which many might be disposed to place under the other, let him consider not only the indistinctness of all former divisions, but the utter impossibility of framing *any* that shall be completely secure from the objection urged, in a case where men have formed such various and vague notions, from the very want of some clear principle of division. Nay, from the elliptical form in which all Reasoning is usually expressed, and the peculiarly involved and oblique form in which Fallacy is for the most part conveyed, it must of course be often a matter of doubt, or rather, of arbitrary *choice*, not only to which genus each *kind* of Fallacy should be referred, but even to which kind to refer any *one individual* Fallacy: for since in any course of argument, one Premiss is usually suppressed, it frequently happens, in the case of a Fallacy, that the hearers are left to the alternative of supplying *either* a Premiss which is *not true*, or *else*, one which *does not prove* the conclusion; e. g. if a man expatiates on the distress of the country, and thence argues that the government is tyrannical, we must suppose him to assume *either* that "every distressed country is under a tyranny," which is a manifest falsehood, *or*, merely that "every country under a tyranny is distressed," which, however true, proves nothing, the middle term being undistributed. Now, in the former case, the Fallacy would be referred to the head of "*extra dictionem*;" in the latter, to that of "*in dictione*:" which are we to suppose the speaker meant us to understand? surely just whichever each of his hearers might happen to prefer: some might assent to the false Premiss; others, allow the unsound Syllogism: to the Sophist himself it is indifferent, as long as they can but be brought to admit the conclusion.

Without pretending then to conform to every one's mode of speaking on the subject, or to lay down rules which shall be, in themselves, (without any call for labour or skill in the person who employs them,) readily applicable to, and decisive on each individual case; we propose a division which is at least perfectly clear in its main principle, and coincides, perhaps, as nearly as possible with the established notions of Logicians on the subject.

§ 2. In every Fallacy, the conclusion either *does*, or *does not follow from the Premises*: where the conclusion does not follow from the Premises, it is manifest that the fault is in the *Reasoning*, and in that alone; these, therefore, we call Logical Fallacies,* as being properly violations of those rules of Reasoning which it is the province of Logic to lay down. Of these, however, one kind are more *purely Logical*, as exhibiting their fallaciousness by the bare *form* of the expression, without any regard to the meaning of the terms: to which class belong: 1st. undistributed middle; 2d. illicit process; 3d. negative Premises, or affirmative conclusion from a negative Premiss, and *vice versâ*: to which may be added, 4th. those which have palpably (i. e. *expressed*) more than three terms. The other kind may be most properly called semi-logical; viz. all the cases of ambiguous middle term except its non-distribution: for though in such cases the Conclusion does not follow, and though the rules of Logic shew that it does not, *as soon as the ambiguity of the middle term is ascertained*, yet the discovery and ascertainment of this ambiguity requires attention to the *sense of the term*, and knowledge of the subject matter; so that here, Logic "teaches us not *how to find* the Fallacy, but only *where to search* for it," and on what principles to condemn it. Accordingly it has been made a subject of bitter complaint against Logic, that it presupposes the most difficult point to be already accomplished, viz. the sense of the terms to be ascertained. A similar objection might be urged against every other art in existence; e. g. against Agriculture, that all the precepts for the cultivation of land presuppose the possession of a farm; or against Perspective, that its rules are useless to a blind man. The objection is indeed peculiarly absurd when urged against Logic, because the object which it is blamed for not accomplishing, *cannot possibly* be within the province of *any one* art whatever. Is it indeed possible or conceivable that there should be any method, science, or system, that should enable one to know the full and exact meaning of every term in existence? The utmost that can be done is to give some general rules that may assist us in this work; which is done in the two first parts of Logic.

The very author of the objection says, "this (the comprehension of the meaning of general terms) is a study which every individual must carry on for himself; and of which no rules of Logic (how useful soever they may be in directing our labours) can supersede the necessity." D. Stewart, *Phil.* vol. ii. ch. ii. s. 2.

Nothing perhaps tends more to conceal from men their imperfect conception of the meaning of a term, than the circumstance of their being able fully to comprehend a *process of Reasoning* in which it is involved, without attaching any distinct meaning, or perhaps any meaning at all to that term; as is evident when A B C, are used to stand for terms, in a regular Syllogism: thus a man may be *familiarized* with a term, and never find himself *at a loss* from not comprehending it; from which he will be very likely to infer that he *does* comprehend it, when perhaps he does not, but employs it vaguely and incorrectly, which leads to fallacious reasoning and confusion. It must be owned, however, that many Logical writers have, in great measure, brought on themselves the reproach in question, by calling Logic "the right use of Reason," laying down "rules for gaining clear ideas," and such-like ἀλαζονεία, as Aristotle calls it. *Rhet.* book i. ch. ii.

§ 3. The remaining class (viz. where the Conclusion does follow from the Premises) may be called the Material, or Non-logical Fallacies: of these there are two kinds; 1st. when the Premises are such as ought not to have been assumed; 2d. when the Conclusion is not the one required, but irrelevant; which Fallacy is called "*ignoratio elenchi*," because your argument is not the *elenchus*, (i. e. proof of the *contradictory*) of your opponent's assertion, which it should be; but proves, instead of that, some other proposition resembling it. Hence, since Logic defines what Contradiction is, some may choose rather to range this with the *Logical* Fallacies, as it seems, so far, to come under the jurisdiction of that art. nevertheless, it is perhaps better to adhere to the original

* Just as we call that a *criminal* Court in which crimes are judged.

division, both on account of its clearness, and also because few would be inclined to apply to the Fallacy in question the accusation of being *inconclusive*, and consequently illogical reasoning: besides which, it seems an artificial and circuitous way of speaking, to suppose in all cases an *opponent* and a *contradiction*; the simple statement of the matter being this,—I am required, by the circumstances of the case, (no matter why) to prove a certain Conclusion; I prove, not that, but one which is likely to be mistaken for it;—in this lies the Fallacy.

It might be desirable therefore to lay aside the name of "*ignoratio elenchi*," but that is so generally adopted as absolutely to require some mention to be made of it. The other kind of Fallacies in the matter will comprehend, (as far as the vague and obscure language of Logical writers will allow us to conjecture,) the Fallacy of "*non causa pro causâ*," and that of "*petitio principii*:" of these, the former is by them distinguished into "*a non verâ pro verâ*, and "*a non tali pro tali*;" this last would appear to be arguing from a case *not parallel* as if it *were* so; which, in Logical language, is, having the *suppressed* Premiss false; (for it is in *that* the *parallelism* is affirmed) and the "*a non verâ pro verâ*" will in like manner signify the *expressed* Premiss being false; so that this Fallacy will turn out to be, in plain terms, neither more nor less than falsity, (or unfair assumption) of a Premiss.

The remaining kind, "*petitio principii*," (begging the question) takes place when a Premiss, whether true or false, is either plainly equivalent to the Conclusion, or depends on it for its own reception. It is to be observed, however, that in all correct Reasoning the Premises must, virtually, imply the conclusion; so that it is not possible to mark precisely the distinction between the Fallacy in question and fair argument; since that may be correct and fair Reasoning to one person, which would be, to another, begging the question, since to one the Conclusion might be more evident than the Premiss, and to the other, the reverse. The most plausible form of this Fallacy is arguing in a circle; and the greater the circle, the harder to detect.

§ 4. There is no Fallacy that may not properly be included under some of the foregoing heads; those which in the Logical Treatises are separately enumerated, and contradistinguished from these, being in reality instances of them, and therefore more properly enumerated in the subdivision thereof; as in the scheme annexed.

Fallacies.

- Logical. (i. e. when the fault is, strictly, in the very process of Reasoning; the conclusion *not following* from the Premises.)
  - Purely-logical. (§ 7.) (i. e. where the fallaciousness is apparent from the mere *form of expression*.)
    - Undistributed middle.
    - Illicit process, &c.
  - Semi-logical. (the middle term being ambiguous *in sense*.)
    - in itself,
      - accidentally.
      - from some connection between the different senses,
        - resemblance.
        - analogy.
        - cause and effect, &c.
    - from the context,
    - (§ 10.) Fallacy of Division and Composition.
    - (§ 11.) *fallacia accidentis*, &c.
- Non-logical or Material. (i. e. when the conclusion *does follow* from the Premises.)
  - Premiss unduly assumed.
    - (§ 12.) (*Petitio principii.*) Premiss depending on the Conclusion,
      - circle.
      - assuming a proposition not the very same as the question, but unfairly implying it.
    - (§ 13.) Premiss false or unsupported.
  - Conclusion irrelevant. (*ignoratio elenchi.*)
    - (§ 16.) Fallacy of objections, &c.
    - (§ 15.) Fallacy of shifting ground,
      - to something wholly irrelevant.
      - from Premiss to Premiss alternately.
    - (§ 14.) Fallacy of using complex and general terms.
    - (§ 14.) Fallacy of appeals to the passions; *ad hominem*; *ad verecundiam*, &c.

§ 5. On each of the Fallacies which have been thus enumerated and distinguished, we propose to offer some more particular remarks: but before we proceed to this, it will be proper to premise two general observations, 1st. on the *importance*, and 2d. the *difficulty*, of detecting and describing Fallacies; both have been already slightly alluded to, but it is requisite that they should here be somewhat more fully and distinctly set forth.

1st. It seems by most persons to be taken for granted

that a Fallacy is to be dreaded merely as a weapon fashioned and wielded by a skilful Sophist: or if they allow that a man may with honest intentions slide into one, unconsciously, in the heat of *argument*, still they seem to suppose that where there is no *dispute*, there is no cause to dread Fallacy; whereas there is much danger, even in what may be called *solitary Reasoning*, of sliding unawares into some Fallacy, by which one may be so far deceived as even to *act upon* the Conclusion thus obtained. By solitary Reasoning is meant the case in which we are not seeking for *arguments* to prove a *given question*, but labouring to elicit from our previous stock of knowledge *some useful inference*. To select one from innumerable examples which might be cited, and of which some more will occur in the subsequent part of this Essay; it is not improbable that many indifferent sermons have been produced by the ambiguity of the word "*plain*:" a young divine perceives the truth of the maxim, that "for the lower orders one's language cannot be too *plain*;" (i. e. *clear* and *perspicuous*, so as to require no learning nor ingenuity to understand it,) and when he proceeds to practice, the word "*plain*" indistinctly flits before him, as it were, and often checks him in the use of *ornaments* of style, such as metaphor, epithet, antithesis, &c. which are opposed to "plainness" in a totally different sense of the word, being by no means necessarily adverse to *perspicuity*, but rather, in many cases, conducive to it; as may be seen in several of the clearest of our Lord's discourses, which are of all others the most richly adorned with figurative language. So far indeed is an ornamented style from being unfit for the vulgar, that they are pleased with it even in excess. Yet the desire to be "plain," combined with that dim and confused notion which the ambiguity of the word produces in such as do not separate in their minds, and set distinctly before themselves, the two meanings, often causes them to write in a dry and bald style, which has no advantage in point of perspicuity, and is least of all suited to the taste of the vulgar. The above instance is not drawn from mere conjecture, but from actual experience of the fact.

Another instance of the strong influence of words on our ideas may be adduced from a widely different subject: most persons feel a certain degree of *surprise* on first hearing of the result of some late experiments of the agricultural Chemists, by which they have ascertained that universally what are called *heavy* soils are specifically the lightest; and *vice versâ*. Whence this *surprise?* for no one ever *distinctly believed* the established names to be used in the literal and *primary* sense, in consequence of the respective soils having been *weighed* together; indeed it is obvious on a moment's reflection that *tenacious clay* soils (as well as muddy roads) are *figuratively* called heavy from the difficulty of ploughing or passing over them, which produces an *effect like* that of bearing or dragging a heavy weight; yet still the terms, "light" and "heavy," though used figuratively, have most undoubtedly introduced into men's minds something of the ideas expressed by them in their primitive sense. So true is the ingenious observation of Hobbes, that "words are the counters of wise men, and the money of fools."

More especially deserving of attention is the influence of analogical terms in leading men into erroneous notions in Theology; where the most important terms are analogical; and yet, they are continually employed in Reasoning without due attention (oftener through want of caution than by unfair design) to their analogical nature; and most of the errors into which Theologians have fallen may be traced, in part, to this cause.

Thus much, as to the extensive practical influence of Fallacies, and the consequent high importance of detecting and exposing them.

§ 6. 2dly. The second remark is, that while sound Reasoning is ever the more readily admitted, the more clearly it is perceived to be such, Fallacy, on the contrary, being rejected as soon as perceived, will, of course be the more likely to obtain reception, the more it is obscured and disguised by obliquity and complexity of expression: it is thus that it is the most likely either to slip accidentally from the careless reasoner, or to be brought forward deliberately by the Sophist. Not that he ever wishes that obscurity and complexity to be perceived; on the contrary it is for his purpose that the expression should *appear* as clear and simple as possible, while in reality it is the most tangled net he can contrive. Thus, whereas it is usual to express our Reasoning elliptically, so that a Premiss, (or even two or three entire steps in a course of argument) which may be readily supplied, as being perfectly obvious, shall be left to be understood, the Sophist in like manner suppresses what is *not* obvious, but is in reality the weakest part of the argument; and uses every other contrivance to withdraw our attention (his art closely resembling the juggler's) from the quarter where the Fallacy lies. Hence the uncertainty before mentioned, to which class any individual Fallacy is to be referred: and hence it is that the *difficulty* of detecting and exposing Fallacy, is so much greater than that of comprehending and developing a process of sound argument. It is like the detection and apprehension of a criminal in spite of all his arts of concealment and disguise; when this *is accomplished*, and he is *brought to trial* with all the evidence of his guilt produced, his conviction and punishment are easy; and this is precisely the case with those Fallacies which are given as examples in Logical Treatises; they are in fact already detected, by being stated in a plain and regular form, and are, as it were, only brought up to *receive sentence*. Or again, fallacious Reasoning may be compared to a perplexed and entangled mass of *accounts*, which it requires much sagacity and close attention to clear up, and display in a regular and intelligible form; though *when this is once accomplished*, the whole appears so perfectly *simple*, that the unthinking are apt to undervalue the skill and pains which have been employed upon it.

Moreover, it should be remembered that a very *long* discussion is one of the most effectual veils of Fallacy. Sophistry, like poison, is at once detected, and nauseated when presented to us in a concentrated form; but a Fallacy which when stated barely, in a few sentences, would not deceive a child, may deceive half the world if *diluted* in a quarto volume. To speak therefore of all the Fallacies that have ever been enumerated as too glaring and obvious to need even being mentioned, because the simple instances given in books, and there stated in the plainest and consequently most easily detected form, are such as would (in that form) deceive no one; this, surely, shews

either extreme weakness, or else unfairness. It may readily be allowed, indeed, that to detect *individual* Fallacies, and *bring them under* the general rules, is a harder task than to *lay down* those general rules; but this does not prove that the latter office is trifling or useless, or that it does not essentially conduce to the performance of the other: there may be more ingenuity shewn in detecting and arresting a malefactor, and convicting him of the fact, than in laying down a law for the trial and punishment of such a person; but the latter office, i. e. that of a legislator, is surely neither unnecessary nor trifling.

It should be added that a close observation and Logical analysis of fallacious arguments, as it tends (according to what has been already said) to form a habit of mind well suited for the practical detection of Fallacies; so, for that very reason, it will make us the more careful in making *allowance* for them; i. e. bearing in mind how much men in general are liable to be influenced by them: e. g. a *refuted argument ought* to *go for nothing*; but in *fact* it will generally prove *detrimental* to the cause, from the Fallacy which will be presently explained. No one is more likely to be practically aware of this, and to take precautions accordingly, than he who is most versed in the whole theory of Fallacies; for the best Logician is the least likely to calculate *on men in general being such*.

Of Fallacies in form.

§ 7. Enough has already been said in the preceding compendium; and it has been remarked above, that it is often left to our *choice* to refer an individual Fallacy to this head or to another.

To the present class we may the most conveniently refer those Fallacies, so common in practice, of supposing the Conclusion false, because the Premiss is false, or because the argument is unsound; and inferring the truth of the Premiss from that of the Conclusion; e. g. if any one argues for the existence of a God, from its being universally believed, a man might perhaps be able to refute the argument by producing an instance of some nation destitute of such belief; the argument *ought* then (as has been observed above) to *go for nothing*: but many would go further, and think that this refutation had *disproved* the existence of a God; in which they would be guilty of an *illicit process* of the major term; viz. "whatever is universally believed must be true; the existence of a God is not universally believed; therefore it is not true." Others again from being convinced of the truth of the Conclusion would infer that of the Premises; which would amount to the Fallacy of *undistributed middle*: viz. "what is universally believed, is true; the existence of a God is true; therefore it is universally believed." Or, these Fallacies might be stated in the hypothetical form; since the one evidently proceeds from the denial of the antecedent to the denial of the consequent; and the other from the establishing of the consequent to the inferring of the antecedent; which two Fallacies correspond respectively with those of *illicit process* of the major, and *undistributed middle*.

Fallacies of this class are very much kept out of sight, being seldom perceived even by those who employ them; but of their practical importance there can be no doubt, since it is notorious that a weak argument is always, in practice, *detrimental*; and that there is no absurdity so gross which men will not readily admit, if it appears to lead to a Conclusion of what they are already convinced. Even a candid and sensible writer is not unlikely to be, by this means, misled, when he is seeking for arguments to support a Conclusion which he has long been fully convinced of himself; i. e. he will often use such arguments as would never have *convinced* himself, and are not likely to convince others, but rather (by the operation of the converse Fallacy) to *confirm* in their dissent those who before disagreed with him.

It is best therefore to endeavour to put yourself in the place of an *opponent* to your own arguments, and consider whether you could not find some objection to them. The applause of *one's own party* is a very unsafe ground for judging of the real force of an argumentative work, and consequently of its real utility. To satisfy those who were doubting, and to convince those who were opposed, is the only sure test; but *these* are seldom very loud in their applause, or very forward in bearing their testimony.

Of Ambiguous middle.

§ 8. That case in which the middle is undistributed, belongs of course to the preceding head, the fault being perfectly manifest from the mere form of the expression: in that case the extremes are compared with *two parts of the same* term; but in the Fallacy which has been called semi-logical, (which we are now to speak of) the extremes are compared with *two different* terms, the middle being used in two different senses in the two Premises.

And here it may be remarked, that when the argument is brought into the form of a *regular Syllogism*, the contrast between these two senses will usually appear very striking, from the two Premises being *placed together*; and hence the scorn with which many have treated the very mention of the Fallacy of equivocation, deriving their only notion of it from the *exposure* of it in Logical Treatises; whereas, in practice it is common for the two Premises to be placed very far apart, and discussed in different parts of the discourse; by which means the inattentive hearer overlooks any ambiguity that may exist in the middle term. Hence the advantage of Logical habits, to fix our attention strongly and steadily on the *important* terms of an argument.

One case which may be regarded as coming under the head of Ambiguous middle, is, what is called "*Fallacia Figuræ Dictionis*," the Fallacy built on the grammatical structure of language, from men's usually taking for granted that *paronymous* words, (i. e. those belonging to each other, as the substantive, adjective, verb, &c. of the same root) have a precisely correspondent meaning: which is by no means universally the case. Such a Fallacy could not indeed be even exhibited in strict Logical form, which would preclude even the attempt at it, since it has two middle terms in sound as well as sense; but nothing is more common in practice than to vary continually the terms employed, with a view to grammatical convenience; nor is there any thing unfair in such a practice, as long as the *meaning* is preserved unaltered: e. g. "murder should be punished with death; this man is a murderer; therefore he deserves to die;" &c. &c. Here we proceed on the assumption (in this case just) that to commit murder and to be a murderer,—to deserve death and to be one who ought to

die, are, respectively, equivalent expressions; and it would frequently prove a heavy inconvenience to be debarred this kind of liberty; but the abuse of it gives rise to the Fallacy in question. e. g. *projectors* are unfit to be trusted; this man has formed a *project*, therefore he is unfit to be trusted:* here the Sophist proceeds on the hypothesis that he who forms a *project* must be a *projector*; whereas the bad sense that commonly attaches to the latter word, is not at all implied in the former.

This Fallacy may often be considered as lying not in the middle, but in one of the terms of the Conclusion; so that the Conclusion drawn shall not be, in reality, at all warranted by the Premises, though it will appear to be so, by means of the grammatical affinity of the words: e. g. "to be acquainted with the guilty is a *presumption* of guilt; this man is so acquainted; therefore we may *presume* that he is guilty:" this argument proceeds on the supposition of an exact correspondence between "*presume*" and "*presumption*," which however does not really exist; for "presumption" is commonly used to express a kind of *slight suspicion*; whereas "to presume" amounts to *absolute belief*.

The above remark will apply to some other cases of ambiguity of term; viz. the Conclusion will often contain a term, which (though not as here, different in *expression* from the corresponding one in the Premiss, yet) is *liable* to be understood in a sense different from that which it bears to the Premiss; though of course such a Fallacy is less common, because less likely to deceive, in *those* cases, than in *this*; where the term used in the Conclusion, though professing to correspond with one in the Premiss, is not the very *same in expression*, and therefore is more *certain* to convey a different sense; which is what the Sophist wishes.

There are innumerable instances of a non-correspondence in paronymous words, similar to that above instanced; as between *art* and *artful*, *design* and *designing*, *faith* and *faithful*, &c.; and the more slight the variation of meaning, the more likely is the Fallacy to be successful; for when the words have become so widely removed in sense as "pity" and "pitiful," every one would perceive such a Fallacy, nor could it be employed but in jest.

This Fallacy cannot in practice be refuted, by stating merely the impossibility of reducing such an argument to the strict Logical form; (unless indeed you are addressing regular Logicians,) you must find some way of pointing out the non-correspondence of the terms in question; e. g. with respect to the example above, it may be remarked, that we speak of *strong* or *faint* "presumption," but yet we use no such expression in conjunction with the verb "presume," because the word itself *implies strength*.

No Fallacy is more common in controversy than the present, since in this way the Sophist will often be able to misinterpret the propositions which his opponent admits or maintains, and so employ them against him: thus in the examples just given, it is natural to conceive one of the Sophist's Premises to have been borrowed from his opponent.

Perhaps a dictionary of such paronymous words as do not regularly correspond in meaning, would be nearly as useful as one of synonyms; i. e. properly speaking, of *pseudo-synonyms*. The present Fallacy is nearly allied to, or rather perhaps may be regarded as a branch of that founded on *Etymology*; viz. when a term is used, at one time, in its customary, and at another, in its Etymological sense. Perhaps no example of this can be found that is more extensively and mischievously employed than in the case of the word *representative*; assuming that its right meaning must correspond exactly with the strict and original sense of the verb represent, the Sophist persuades the multitude, that a member of the House of Commons is bound to be guided in all points by the opinion of his constituents; and, in short, to be merely their *spokesman*; whereas law and custom, which in this case may be considered as fixing the meaning of the term, require no such thing, but enjoin the representative to act according to the best of his own judgment, and on his own responsibility. H. Tooke has furnished a whole magazine of such weapons for any Sophist who may need them, and has furnished some specimens of the employment of them.

§ 9. It is to be observed, that to the head of *Ambiguous middle* should be referred what is called "*Fallacia plurium Interrogationum*," which may very properly be named, simply, "the Fallacy of Interrogation;" viz. the Fallacy of asking several questions which appear to be but one; so that whatever *one* answer is given, being of course applicable to one only of the implied questions, may be interpreted as applied to the other; the refutation is, of course, to reply *separately* to each question, i. e. to detect the ambiguity.

We have said several "questions which *appear to be but one*, for else there is no *Fallacy*; such an example therefore, as "*estne homo animal et lapis?*" which Aldrich gives, is foreign to the matter in hand; for there is nothing unfair in asking two distinct questions, or asserting two distinct propositions, *distinctly and avowedly*.

This Fallacy may be referred, as has been said, to the head of *Ambiguous middle*: in all Reasoning it is very common to state one of the Premises in form of a question, and when that is admitted, or supposed to be admitted, then to fill up the rest; if then one of the terms of that question be ambiguous, whichever sense the opponent replies to, the Sophist assumes the *other* sense of the term in the remaining Premiss. It is therefore very common to state an unequivocal argument, in form of a question so worded, that there shall be *little doubt which* reply will be given: but if there *be* such doubt, the Sophist must have *two* Fallacies of equivocation ready: e. g. the question "whether any thing vicious is expedient," discussed in Cic. *Off.*, book iii. (where, by the bye, he seems not a little perplexed with it himself,) is of the character in question, from the ambiguity of the word "*expedient*," which means sometimes, "conducive to temporal prosperity," sometimes, "conducive to the greatest good:" whichever answer therefore was given, the Sophist might have a Fallacy of equivocation founded on this term; viz. if the answer be in the negative, his argument Logically developed, will stand thus,—"what is vicious is not expedient; whatever conduces to wealth and aggrandizement is expedient, therefore it cannot be vicious:" if, in the affirmative, then thus, "whatever is expedient is desirable; something vicious is expedient, therefore desirable."

* *Wealth of Nations*, A. Smith: Usury.

This kind of Fallacy is frequently employed in such a manner, that the uncertainty shall be, not about the *meaning*, but the *extent* of a term, i. e. whether it is *distributed* or not: e. g. "did A B in this case act from such and such a motive?" which may imply either, "was it his *sole* motive?" or "was it one of his motives?" in the former case the term "that which actuated A B" is distributed; in the latter not: now if he acted from a mixture of motives, whichever answer you give, may be misrepresented and thus disproved.

§ 10. In some cases of *Ambiguous middle*, the term in question may be considered as having *in itself*, from its own equivocal nature, two significations; (which apparently constitutes the "*Fallacia equivocationis* of Logical writers;) others again have a middle term which is ambiguous from the context, i. e. from what is *understood in conjunction* with it: this division will be found useful, though it is impossible to draw the line accurately in it.

There are various ways in which words come to have two meanings; 1st. by *accident*; (i. e. when there is no perceptible connection between the two meanings) as "*light*" signifies both the contrary to "heavy," and the contrary to "dark." Thus, such proper names as John or Thomas, &c. which happen to belong to several different persons, are ambiguous, because they have a different signification in each case where they are applied. Words which fall under this first head are what are the most strictly called *equivocal*.

2dly. There are several terms in the use of which it is necessary to notice the distinction between *first* and *second intention*: the "first intention" of a term, (according to the usual acceptation of this phrase,) is a certain *vague* and *general* signification of it, as opposed to one more *precise* and *limited*, which it bears in some particular art, science, or system, and which is called its "second intention." Thus, among farmers in some parts, the word "beast" is applied particularly and especially to the ox kind; and "bird," in the language of many sportsmen is in like manner appropriated to the partridge: the common and general acceptation (which every one is well acquainted with) of each of those two words, is the first intention of each; the other, its second intention.

It is evident that a term may have several second intentions, according to the several systems into which it is introduced, and of which it is one of the technical terms: thus line signifies, in the Art Military, a certain form of drawing up ships or troops; in Geography, a certain division of the earth; to the fisherman, a string to catch fish, &c. &c.; all which are so many distinct second intentions, in each of which there is a certain signification of "extension in length" which constitutes the first intention, and which corresponds pretty nearly with the employment of the term in Mathematics.

It will sometimes happen, that a term shall be employed always in some one or other of its second intentions; and never, strictly, in the first, though that first intention is a *part of its signification* in each case. It is evident, that the utmost care is requisite to avoid confounding together, either the first and second intentions, or the different second intentions with each other.

3dly. When two or more things are connected by *resemblance* or *analogy*, they will frequently have the same name. Thus a "*blade* of grass," and the contrivance in building called a "*dove-tail*," are so called from their *resemblance* to the *blade** of a sword, and the *tail* of a real dove: but two things may be connected by *analogy*, though they have in *themselves* no *resemblance*: for analogy is the resemblance of *ratios*, (or relations) thus,—as a *sweet* taste *gratifies* the palate, so does a *sweet* sound *gratify* the ear; and hence the same word, "*sweet*," is applied to both, though no flavour can resemble a sound in itself: so, the *leg of a table*, does not resemble that of an animal; nor the foot of a mountain that of an animal: but *the leg answers the same purpose* to the table, as the leg of an animal to that animal; the foot of a mountain has the *same situation* relatively to the mountain, as the foot of an animal, to the animal; this analogy therefore may be expressed like a Mathematical analogy; (or proportion) leg: animal :: supporting stick : table.— In all these cases, (of this 3d head) one of the meanings of the word is called by Logicians *proper*, i. e. original or primary; the other *improper*, secondary or transferred: thus, *sweet*, is originally and properly applied to *tastes*; secondarily and *improperly* (i. e. by analogy,) to sounds: thus also, dove-tail is applied secondarily though not by analogy, but by direct resemblance to the contrivance in building so called. When the secondary meaning of a word is founded on some *fanciful* analogy, and especially when it is introduced for ornament sake, we call this a *metaphor*; as when we speak of "a ship's *ploughing* the deep." The turning up of the surface being essential indeed to the plough, but incidental only to the ship; but if the analogy be a more important and essential one, and especially if we have no other word to express our meaning but this transferred one, we then call it *merely* an *analogous* word, (though the *metaphor* is analogous also;) e. g. one would hardly call it *metaphorical* or *figurative* language to speak of the leg of a table, or mouth of a river.

4thly. Several things may be called by the same name, (though they have no connection of resemblance or analogy) from being *connected by vicinity of time or place*; under which head will come the connection of *cause and effect*, or of part and whole, &c. Thus a *door* signifies both an opening in the wall, (more strictly called the door-way,) and a board which closes it: which are things neither similar nor analogous. When I say, "the rose *smells* sweet" and "I smell the rose:" the word "smell" has two meanings: in the latter sentence, I am speaking of a certain *sensation* in my own mind; in the former, of a certain *quality* in the flower, which produces that sensation, but which of course cannot in the least resemble it: and here the word *smell*, is applied with equal propriety to both. Thus we speak of Homer, for "the works of Homer;" and this is a secondary or transferred meaning: and so it is when we say, "a good shot," for a good marksman: but the word "*shot*" has two other meanings, which are both equally proper; viz. *the thing put into a gun* in order to be discharged from it, and the *act* of discharging it.

* Unless indeed the primary application of the term be to the leaf of grass, and the secondary, to cutting instruments; which is perhaps more probable; but the question is unimportant in the present case.

Thus, "*learning*" signifies either the act of acquiring knowledge, or the knowledge itself; e. g. "he neglects his learning." "Johnson was a man of learning." *Possession* is ambiguous in the same manner; and a multitude of others. Much confusion often arises from ambiguity of this kind, when unperceived; nor is there any point in which the copiousness and consequent precision of the Greek language is more to be admired than in its distinct terms for expressing an act, and the result of that act; e. g. *πρᾶξις* "the doing of anything;" *πρᾶγμα*, "the thing done;" so, *δόσις* and *δῶρον*, *λῆψις* and *λῆμμα*, &c. It will very often happen, that two of the meanings of a word will have no connection with one another, but will each have some connection with a third. Thus "martyr," originally signified a *witness*, thence it was applied to those who *suffered* in bearing testimony to Christianity; and thence again it is often applied to sufferers in general: the first and third significations are not the least connected. Thus "*post*" signifies originally a pillar, (*póstum*, from *pono*;) then a distance marked out by posts; and then the carriages, messengers, &c. that travelled over this distance.

Innumerable other ambiguities might be brought under this fourth head, which indeed comprehends all the cases which do not fall under the three others.

The remedy for ambiguity is a definition of the term which is suspected of being used in two senses; viz. a verbal, not necessarily a real definition; as was remarked in the Compendium.

But here it may be proper to remark, that for the avoiding of Fallacy or of verbal controversy, it is only requisite that the term should be employed uniformly in the same sense *as far as the existing question is concerned*. Thus, two persons might, in discussing the question, whether Buonaparte was a GREAT man, have some difference in their acceptation of the epithet "great," which would be non-essential to that question; e. g. one of them might understand by it nothing more than eminent intellectual, and moral *qualities*; while the other might conceive it to imply the performance of splendid *actions*: this abstract difference of meaning would not produce any disagreement in the existing question, because both those circumstances are united in the case of Buonaparte; but if one of the parties understood the epithet "great" to imply GENEROSITY of character, &c. then there *would* be a disagreement. Definition, the specific for ambiguity, is to be employed and demanded with a view to this principle; it is sufficient on each occasion to define a term *as far as regards the question in hand*.

Of those cases in which the ambiguity arises *from the context*, there are many species; several of which Logicians have enumerated, but have neglected to refer them, in the first place, to one common class, (viz. the one under which they are here placed;) and have even arranged some under the head of Fallacies "*in dictione*," and others, "*extra dictionem*."

We may consider, as the first of these species, the Fallacy of "Division" and that of "Composition," taken together, since in each of these the middle term is used in one Premiss *collectively*, in the other, *distributively*: if the former of these is the major Premiss, and the latter the minor, this is called the "Fallacy of division;" the term which is first taken collectively being afterwards divided; and *vice versâ*. The ordinary examples are such as these; all the angles of a triangle are equal to two right angles: A B C, is an angle of a triangle; therefore A B C, is equal to two right angles. Five is one number; three and two are five; therefore three and two are one number; or, three and two are two numbers, five is three and two, therefore five is two numbers: it is manifest that the middle term, three and two, (in this last example) is ambiguous, signifying, in the major Premiss "taken distinctly," in the minor, "taken together:" and so of the rest.

To this head may be referred the Fallacy by which men have sometimes been led to admit, or pretend to admit, the doctrine of necessity; e. he who necessarily goes or stays (i. e. in reality, "who *necessarily goes*, or who *necessarily stays*") is not a free agent; you must necessarily go or stay; (i. e. "you must necessarily *take the alternative*,") therefore you are not a free agent. Such also is the Fallacy which probably operates on most adventurers in lotteries; e. g. the gaining of a high prize is no uncommon occurrence; and what is no uncommon occurrence may reasonably be expected; therefore the gaining of a high prize "may reasonably be expected:" the conclusion when applied to the individual, (as in practice it is) must be understood in the sense of "reasonably expected *by a certain individual*;" therefore for the major Premiss to be true the middle term must be understood to mean, "no uncommon occurrence to some one *particular* person;" whereas for the minor (which has been placed first) to be true, you must understand it of "no uncommon occurrence to *some one or other*;" and thus you will have the Fallacy of Composition.

There is no Fallacy more common, or more likely to deceive than the one now before us: the form in which it is most usually employed, is, to establish some truth, *separately*, concerning *each single* member of a certain class, and thence to infer the same of the *whole collectively*: thus some infidels have laboured to prove concerning *some one* of our Lord's miracles, that it might have been the result of an accidental conjuncture of natural circumstances; next, they endeavour to prove the same concerning *another*; and so on; and thence infer that *all* of them might have been so. They might argue in like manner, that because it is not very improbable one may throw sixes in any one out of a hundred throws, therefore it is no more improbable that one may throw sixes a hundred times running.

This Fallacy may often be considered as turning on the ambiguity of the word "all;" which may easily be dispelled by substituting for it the word "each" or "every," where that is its signification; e. g. "all these trees make a thick shade" is ambiguous, meaning, either "every one of them," or "all together."

This is a Fallacy with which men are extremely apt to deceive *themselves*: for when a multitude of particulars are presented to the mind, many are too weak or too indolent to take a comprehensive view of them; but confine their attention to each single point, by turns; and then decide, infer, and act, accordingly; e. g. the imprudent spendthrift, finding that he is able to afford this, *or* that, *or* the other expense, forgets that *all of them together* will ruin him.

To the same head may be reduced that fallacious reasoning by which men vindicate themselves to their own conscience and to others, for the neglect of those *undefined* duties, which though indispensable, and

therefore not left to our choice *whether* we will practise them or not, are left to our discretion as to the *mode*, and the particular occasions of practising them; e. g. "I am not bound to contribute to this charity in particular; nor to that; nor to the other:" the *practical* conclusion which they draw, is, that *all* charity may be dispensed with.

As men are apt to forget that any two circumstances (not naturally connected) are more rarely to be met with combined than separate, though they be not at all incompatible; so also they are apt to imagine from finding that they *are* rarely combined, that there is an incompatibility; e. g. if the chances are ten to one against a man's possessing strong reasoning powers, and ten to one against exquisite taste, the chances against the combination of the two (supposing them neither connected nor opposed) will be a hundred to one. Many therefore, from finding them so rarely united, will infer that they are in some measure incompatible; which Fallacy may easily be exposed in the form of Undistributed middle: "qualities unfriendly to each other are rarely combined; excellence in the reasoning powers and in taste are rarely combined; therefore they are qualities unfriendly to each other."

§ 11. The other kind of ambiguity arising from the context, and which is the last case of Ambiguous middle that we shall notice, is the "*fallacia accidentis*," together with its converse "*fallacia a dicto secundum quid ad dictum simpliciter*;" in each of which the middle is used in one Premiss to signify something considered simply, in itself, and as to its essence; and in the other Premiss, so as to imply that its accidents are taken into account with it: as in the well-known example, "what is bought in the market is eaten; raw meat is bought in the market; therefore raw meat is eaten." Here the middle has understood in conjunction with it, in the major Premiss "*as to its substance merely*;" in the minor, "*as to its condition and circumstances*."

To this head perhaps, as well as to any, may be referred the Fallacies which are frequently founded on the occasional, partial, and temporary variations in the acceptation of some term, arising from circumstances of person, time, and place, which will occasion something to be understood in conjunction with it beyond its strict literal signification; e. g. the phrase "Protestant ascendancy," having become a kind of watch-word or gathering-cry of a party, the expression of good wishes for it would commonly imply an adherence to certain measures not literally expressed by the words; to assume therefore that one is unfriendly to "Protestant ascendancy" in the literal sense, because he has declared himself unfriendly to it when implying and connected with such and such other sentiments, is a gross Fallacy; and such an one as perhaps the authors of the above would much object to, if it was assumed of *them* that they were adverse to "the cause of liberty throughout the world," and to "a fair representation of the people," from their objecting to join with the members of a factious party in the expression of such sentiments.

Such Fallacies may fairly be referred to the present head.

§ 12. Of the Non-logical (or material) Fallacies, and first of begging the question.

The indistinct and unphilosophical account which has been given by Logical writers of the Fallacy of "*non-causd*," and that of "*petitio principii*," makes it very difficult to ascertain wherein they conceived them to differ, and what, according to them, is the nature of each; without therefore professing to conform exactly to their meaning, and with a view to distinctness only, which is the main point, let us confine the name "*petitio principii*" to those cases in which the Premiss either appears manifestly to be the same as the Conclusion, or is actually proved from the Conclusion, or is such as would naturally and properly so be proved; (as if one should attempt to prove the being of a God from the authority of holy writ;) and to the other class be referred all other cases, in which the Premiss (whether the expressed or the suppressed one) is either proved false, or has no sufficient claim to be received as true. Let it however be observed, that in such cases (apparently) as this, we must not too hastily pronounce the argument fallacious; for it may be perfectly fair at the *commencement* of an argument to assume a Premiss that is not more evident than the Conclusion, or is even ever so paradoxical, provided you proceed to prove fairly that Premiss: and in like manner it is both usual and fair to *begin* by deducing your Conclusion from a Premiss exactly equivalent to it; which is merely throwing the proposition in question into the form in which it will be most conveniently proved. Arguing in a circle however must necessarily be unfair; though it frequently is practised undesignedly; e. g. some Mechanicians, attempt to prove, (what they ought to lay down as a probable but doubtful hypothesis,) that every particle of matter gravitates equally; "why?" because those bodies which contain more particles ever gravitate more strongly, i. e. are heavier: "but (it may be urged) those which are heaviest are not always more bulky;" "no, but still they contain more particles, though more closely condensed;" "how do you know that?" "because they are heavier;" "how does that prove it?" "because all particles of matter gravitating equally, that mass which is specifically the heavier, must needs have the more of them in the same space."

Obliquity and disguise being of course most important to the success of the *petitio principii*, as well as of other Fallacies, the Sophist will in general either have recourse to the circle, or else not venture to *state* distinctly his assumption of the point in question, but will rather assert some other proposition which *implies* it; thus keeping out of sight (as a dexterous thief does stolen goods) the point in question, at the very moment when he is taking it for granted: hence the frequent union of this Fallacy with "*ignoratio elenchi*:" vide § 14. The English language is perhaps the more suitable for the Fallacy of *petitio principii*, from its being formed from two distinct languages, and thus abounding in synonymous expressions which have no resemblance in sound, and no connection in etymology; so that a Sophist may bring forward a proposition expressed in words of Saxon origin, and give as a reason for it, the very same proposition stated in words of Norman origin; e. g. "to allow every man an unbounded freedom of speech, must always be, on the whole, advantageous to the State; for it is highly conducive to the interest of the community, that each individual should enjoy a liberty perfectly unlimited of expressing his sentiments."

§ 13. The next head is, the falsity, or at least, undue assumption of a Premiss when it is not equivalent to, or dependent on the Conclusion; which, as has

been before said, seems to correspond nearly with the meaning of Logicians, when they speak of "*non causa pro causâ*:" this name indeed would seem to apply a much narrower class, there being one species of arguments which are *from cause to effect*, in which of course two things are necessary; 1st. the *sufficiency* of the cause, 2d. its establishment; these are the two Premises; if therefore the *former* be unduly assumed, we are arguing from that which is *not* a sufficient cause as if it *were* so; e.g. as if one should contend from such a man's having been unjust or cruel, that he will certainly be visited with some heavy temporal judgment, and come to an untimely end. In this instance the Sophist, from having assumed in the Premiss, the (granted) existence of a pretended cause, infers in the conclusion the existence of the pretended effect, which we have supposed to be the Question: or *vice versâ*, the pretended effect may be employed to establish the cause; e. g. inferring sinfulness from temporal calamity: but when *both* the pretended cause, and effect are *granted*, i. e. granted to exist, then the Sophist will infer something from their pretended *connection*; i. e. he will assume as a Premiss, that "of these two admitted facts, the one is the cause of the other;" as the opponents of the Reformation assumed that it was the cause of the troubles which took place at that period, and thence inferred that it was an evil. Such an argument as either of these might strictly be called "*non causa pro causâ*;" but it is not probable, that the Logical writers intended any such limitation, (which indeed would be wholly unnecessary and impertinent,) but rather that they were confounding together *cause and reason*; the sequence of *Conclusion* from *Premises* being perpetually mistaken for that of *effect* from physical *cause*. It is indeed a very necessary caution in *philosophical investigation* not to assume too hastily that one thing is the *cause* of another, when perhaps it is only an *accidental concomitant*; (as was the case in the assumption of the Premises of the last mentioned examples:) but *investigation* is a perfectly distinct business from *argumentation*; and to mingle together the rules of the two, (as Logical writers have generally done, especially in the present case,) tends only to produce confusion in both. It may be better therefore to drop the name which tends to perpetuate this confusion, and simply state (when such is the case) that the Premiss is unduly assumed; i. e. without being either self-evident, or satisfactorily proved.

The contrivances by which men may deceive themselves or others, in assuming Premises unduly, *so that that undue assumption shall not be perceived*, (for it is in *this* the *Fallacy* consists) are of course infinite. Sometimes (as was before observed) the *doubtful Premiss* is *suppressed*, as if it were too evident to need being proved, or even stated, and as if the whole question turned on the establishment of the other Premiss.

Thus H. Tooke proves, by an immense induction, that all particles were originally nouns or verbs; and thence concludes, that in reality they are so still, and that the ordinary division of the parts of speech is absurd; keeping out of sight, as self-evident, the other Premiss, which is absolutely false; viz. that the meaning and force of a term, now and for ever, must be that, which it, or its root originally bore.

Sometimes men are shamed into admitting an unfounded assertion, by being assured, that it is so evident it would argue great weakness to doubt it. In general, however, the more skilful Sophist will avoid a *direct assertion* of what he means unduly to assume; since that might direct the reader's attention to the consideration of the question whether it be *true* or not, since that which is indisputable does not so often need to be asserted: it succeeds better, therefore, if you *allude* to the proposition as something *curious* and remarkable; just as the Royal Society were imposed on by being asked to *account for* the fact that a vessel of water received no addition to its weight by a live fish put into it; while they were seeking for the *cause*, they forgot to ascertain the *fact*, and thus admitted without suspicion a mere fiction. Thus an eminent Scotch writer, instead of *asserting* that "the advocates of Logic have been worsted and driven from the field in every controversy," (an assertion, which if made, would have been the more readily ascertained to be perfectly groundless,) merely *observes*, that "it is a circumstance not a little *remarkable*."

Frequently the Fallacy of *ignoratio elenchi* is called in to the aid of this; i. e. the Premiss is assumed on the ground of another proposition, *somewhat like it*, having been proved; thus in arguing by example, &c. the parallelism of two cases is often assumed from their being in *some respects* alike, though perhaps they differ in the very point which is *essential* to the argument; e. g. from the circumstance that some men of humble station, who have been well educated, are apt to think themselves above low drudgery, it is argued that universal education of the lower order, would beget general idleness: this argument rests of course on the assumption of *parallelism* in the two cases, viz. the past and the future; whereas there is a circumstance that is absolutely essential, in which they differ; for when education is *universal* it must cease to be a *distinction*; which is probably the very circumstance that renders men too proud for their work.

This very same Fallacy is often resorted to on the opposite side; an attempt is made to invalidate some argument from example, by pointing out a difference between the two cases, though they agree in every thing that is essential to the question. Lastly, it may be here remarked, conformably with what has been formerly said, that it will often be left to your choice whether to refer this or that fallacious argument to the present head, or that of Ambiguous middle; "*if* the middle term is here used in *this* sense, there is an *ambiguity*; *if* in *that* sense, the proposition is *false*."

§ 14. The last kind of Fallacy to be discussed is that of Irrelevant Conclusion, commonly called *ignoratio elenchi*. Various kinds of propositions are, according to the occasion, substituted for the one of which proof is required.

Sometimes the particular for the universal; sometimes a proposition with different terms: and various are the contrivances employed to effect and to conceal this substitution, and to make the Conclusion which the Sophist has drawn, answer, practically, the same purpose as the one be *ought* to have established. We say, "practically the same purpose," because it will very often happen that some *emotion* will be excited—some sentiment impressed on the mind—(by a dexterous employment of this Fallacy) such as shall bring men into the *disposition* requisite for your purpose, though they may not have assented to, or even stated distinctly in their own minds the *proposition* which it was your business to establish. Thus if a Sophist has to

defend one who has been guilty of some *serious* offence, which he wishes to extenuate, though he is unable distinctly to prove that it is not such, yet if he can succeed in *making the audience laugh* at some casual matter, he has gained practically the same point. So also if any one has pointed out the extenuating circumstances in some particular case of offence, so as to shew that it differs widely from the generality of the same class, the Sophist, if he find himself unable to disprove these circumstances, may do away the force of them, by simply *referring the action to that very class*, which no one can deny that it belongs to, and the very name of which will excite a feeling of disgust sufficient to counteract the extenuation; e. g. let it be a case of peculation, and that many mitigating circumstances have been brought forward which cannot be denied; the sophistical opponent will reply, "well, but after all, the man is a *rogue*, and there is an end of it;" now in reality this was (by hypothesis) never the question; and the mere assertion of what was never denied, *ought* not, in fairness, to be regarded as decisive; but, practically, the odiousness of the word, arising in great measure from the *association of those very circumstances* which belong to *most of the class*, but which we have supposed to be *absent* in *this particular* instance, excites precisely that *feeling of disgust*, which in effect destroys the force of the defence. In like manner we may refer to this head all cases of improper appeals to the passions, and every thing else which is mentioned by Aristotle as extraneous to the matter in hand, (*ἔξω τοῦ πράγματος*.)

In all these cases, as has been before observed, if the Fallacy we are now treating of be employed for the apparent establishment, not of the *ultimate* Conclusion, but (as it very commonly happens) of a *Premiss*, (i. e. if the Premiss required be assumed on the ground that some proposition resembling it has been proved,) then there will be a combination of this Fallacy with the last mentioned. A good instance of the employment and exposure of this Fallacy occurs in Thucydides, in the speeches of Cleon and Diodotus concerning the Mityleneans: the former (over and above his appeal to the angry passions of his audience,) urges the *justice* of putting the revolters to death; which, as the latter remarked, was nothing to the purpose, since the Athenians were not sitting in *judgment*, but in *deliberation*, of which the proper end is *expediency*.

It is evident that *ignoratio elenchi* may be employed as well for the apparent *refutation* of your opponent's proposition, as for the apparent establishment of your own; for it is substantially the same thing to *prove* what was not denied, or to *disprove* what was not asserted: the latter practice is not less common, and it is more offensive, because it frequently amounts to a personal affront, in attributing to a person opinions, &c. which he perhaps holds in abhorrence. Thus, when in a discussion one party vindicates, on the ground of general expediency, a particular instance of resistance to Government in a case of intolerable oppression, the opponent may gravely maintain that "we ought not to do evil that good may come:" a proposition which of course had never been denied, the point in dispute being "whether resistance in this particular case *were* doing evil or not." In this example it is to be remarked, (and the remark will apply very generally,) that the Fallacy of *petitio principii* is combined with that of *ignoratio elenchi*, which is a very common and successful practice; viz. the Sophist proves, or disproves, not the proposition which is really in question, but one which so implies it as to proceed on the supposition that it is already decided, and can admit of no doubt; by this means his "assumption of the point in question" is so indirect and oblique, that it may easily escape notice; and he thus establishes, practically, his Conclusion, at the very moment when he is withdrawing your attention from it to another question.

There are certain kinds of argument recounted and named by Logical writers, which we should by no means universally call Fallacies; but which *when unfairly* used, and *so far as they* are fallacious, may very well be referred to the present head; such as the "*argumentum ad hominem*," or personal argument, "*argumentum ad verecundiam*," "*argumentum ad populum*," &c. all of them regarded as contradistinguished from "*argumentum ad rem*," or according to others (meaning probably the very same thing) "*ad judicium*." These have all been described in the lax and popular language before alluded to, but not scientifically: the "*argumentum ad hominem*" they say, "is addressed to the peculiar circumstances, character, avowed opinions, or past conduct of the individual, and therefore has a reference to him only, and does not bear directly and absolutely on the real question, as the '*argumentum ad rem*' does:" in like manner the "*argumentum ad verecundiam*" is described as an appeal to our reverence for some respected authority, some venerable institution, &c. and the "*argumentum ad populum*," as an appeal to the prejudices, passions, &c. of the multitude, and so of the rest. Along with these is usually enumerated "*argumentum ad ignorantiam*," which is here omitted as being evidently nothing more than the employment of some kind of Fallacy, in the widest sense of that word, towards such as are likely to be deceived by it. It appears then, (to speak rather more technically,) that in the "*argumentum ad hominem*" the Conclusion which actually is established, is not the absolute and general one in question, but relative and particular; viz. not that "such and such is the fact," but that "*this man* is bound to admit it, in conformity to his principles of Reasoning, or in consistency with his own conduct, situation, &c." Such a Conclusion it is often both fair and necessary to establish, in order to silence those who will not yield to fair general argument; or to convince those whose weakness and prejudices would not allow them to assign to it its due weight: it is thus that our Lord on many occasions silences the cavils of the Jews; as in the vindication of healing on the Sabbath, which is paralleled by the authorized practice of drawing out a beast that has fallen into a pit. All this, as we have said, is perfectly fair, provided it be done plainly, *knowingly*, and *avowedly*; but if you attempt to *substitute* this partial and relative Conclusion for a more general one—if you triumph as having established your proposition absolutely and universally, from having established it, in reality, only as far as it relates to your opponent, then you are guilty of a Fallacy of the kind which we are now treating of: your Conclusion is not in reality that which was, by your own account, proposed to be proved: the fallaciousness depends upon the *deceit* or attempt to deceive. The same observations will *apply* to "*argumentum ad verecundiam*," and the rest.

It is very common to employ an ambiguous term for the purpose of introducing the Fallacy of Irrelevant Conclusion; i. e. when you cannot prove your proposition in the sense in which it was maintained, to prove it in some other sense; e. g. those who contend against the efficacy of *faith*, usually employ that word in their arguments in the sense of mere belief, unaccompanied with any moral or practical result, but considered as a mere intellectual process; and when they have thus proved their Conclusion, they oppose it to one in which the word is used in a widely different sense.

§ 15. The Fallacy of *ignoratio elenchi* is no where more common than in protracted controversy, when one of the parties, after having attempted in vain to maintain his position, *shifts his ground* as covertly as possible to another, instead of honestly giving up the point. An instance occurs in an attack made on the system pursued at one of our Universities. The objectors finding themselves unable to maintain their charge of the *present* neglect of Mathematics in that place, (to which neglect they had attributed the late *general decline* in those studies,) they shifted their ground, and contended that that University was *never* famous for Mathematicians; which not only does not establish, but absolutely overthrows their own original assertion; for if it *never* succeeded in those pursuits, it could not have caused their late *decline*.

A practice of this nature is common in oral controversy especially; viz. that of combating *both* your opponent's Premises *alternately*, and shifting the attack from the one to the other, without waiting to have either of them decided upon before you quit it.

It has been remarked above, that one class of the propositions that may be, in this Fallacy, substituted for the one required, is the *particular* for the *universal*: nearly akin to this is the very common case of proving something to be *possible* when it ought to have been proved highly *probable*; or *probable*, when it ought to have been proved *necessary*; or, which comes to the very same, proving it to be *not necessary*, when it should have been proved *not probable*; or *improbable*, when it should have been proved *impossible*. Aristotle, (*in Rhet.* book ii.) complains of this last branch of the Fallacy, as giving an undue advantage to the respondent: many a guilty person owes his acquittal to this; the jury considering that the evidence brought does not demonstrate the absolute impossibility of his being innocent, though perhaps the chances are innumerable against it.

§ 16. Similar to this case is that which may be called the *Fallacy of objections*; i. e. shewing that *there are* objections against some plan, theory or system, and thence inferring that it should be rejected; when that which ought to have been proved, is, that there are *more*, or *stronger* objections against the receiving than the rejecting of it. This is the main, and almost universal Fallacy of infidels, and is that of which men should be first and principally warned. This is also the stronghold of bigoted anti-innovators, who oppose all reforms and alterations indiscriminately; for there never was, nor will be, any plan executed or proposed, against which strong and even unanswerable objections may not be urged; so that unless the opposite objections be set in the balance on the other side, we can never advance a step. "There are objections," said Dr. Johnson, "against a *plenum*, and objections against a *vacuum*; but one of them must be true."

The very same Fallacy indeed is employed on the other side, by those who are for overthrowing whatever is established as soon as they can prove an objection against it, without considering whether more and weightier objections may not lie against their own schemes: but their opponents have this decided advantage over them, that they can urge with great plausibility, "we do not call upon you to *reject* at once whatever is objected to, but merely to *suspend your judgment* and not come to a decision as long as there are reasons on both sides:" now since there always *will be* reasons on both sides, this *non*-decision is practically the very same thing as a *decision in favour of the existing state* of things; the *delay* of trial becomes equivalent to an *acquittal*.*

§ 17. Another form of *ignoratio elenchi*, which is also rather the most serviceable on the side of the respondent, is, to prove or disprove *some part* of that which is required, and dwell on *that*, suppressing all the rest.

Thus, if a University is charged with cultivating *only* the mere elements of Mathematics, and in reply a list of the books studied there is produced, should even *any one* of those books be *not elementary*, the charge is in fairness refuted; but the Sophist may then earnestly contend that *some* of those books *are* elementary; and thus keep out of sight the real question, viz. whether they are *all* so.

Hence the danger of ever advancing more than can be well maintained; since the refutation of *that* will often quash the whole: a guilty person may often escape by having too much laid to his charge: so he may also by having too much evidence against him, i. e. some that is not in itself satisfactory: thus, a prisoner may sometimes obtain acquittal by shewing that one of the witnesses against him is an infamous informer and spy; though perhaps if that part of the evidence had been omitted, the rest would have been sufficient for conviction.

Cases of this nature might very well be referred also to the Fallacy formerly mentioned, of inferring the Falsity of the Conclusion from the Falsity of a Premiss, which indeed is very closely allied to the present Fallacy: the real question is "whether or not this Conclusion *ought to be admitted*;" the Sophist confines himself to the question, "whether or not it is *established by this particular argument*;" leaving it to be inferred by the audience, if he has carried his point as to the latter question, that the former is thereby decided.

§ 18. It will readily be perceived that nothing is less conducive to the success of the Fallacy in question than to state clearly, in the outset, either the proposition you are about to prove, or that which you ought to prove; it answers best to *begin* with the *Premises*, and to introduce a pretty long chain of argument before you arrive at the Conclusion. The careless hearer takes for granted, at the beginning, that this chain

* "Not to resolve, is to resolve." Bacon.
How happy it is for mankind that in the most momentous concerns of life their decision is generally formed *for* them by external circumstances; which thus saves them not only from the perplexity of doubt and the danger of delay, but also from the pain of regret, since we acquiesce much more cheerfully in that which is unavoidable.

will lead to the Conclusion required; and by the time you are come to the end, he is ready to take for granted that the Conclusion which you draw *is* the one required; his idea of the question having gradually become indistinct. This Fallacy is greatly aided by the common practice of suppressing the Conclusion and leaving it to be supplied by the hearer, who is of course less likely to perceive whether it be really that "which was to be proved," than if it were distinctly stated. The practice therefore is at best suspicious; and it is better in general to avoid it, and to give and require a distinct statement of the Conclusion intended.

§ 19. Before we dismiss the subject of Fallacies, it may not be improper to mention the just and ingenious remark, that Jests are Fallacies; i. e. Fallacies so palpable as not to be likely to deceive any one, but yet bearing just that resemblance of argument which is calculated to amuse by the contrast; in the same manner that a parody does, by the contrast of its levity with the serious production which it imitates. There is indeed something laughable even in Fallacies which are intended for serious conviction, when they are thoroughly exposed. There are several different kinds of joke and raillery, which will be found to correspond with the different kinds of Fallacy: the pun (to take the simplest and most obvious case) is evidently a mock argument founded on a palpable equivocation of the middle term: and the rest in like manner will be found to correspond to the respective Fallacies, and to be *imitations* of serious argument. It is probable indeed that all jests, sports, or games, (παιδιαι) properly so called, will be found, on examination, to be *imitative* of serious transactions: but to enter fully into this subject would be unsuitable to the present occasion.

We shall conclude the consideration of this subject with some general remarks on the legitimate province of Reasoning, and on its connection with Inductive philosophy, and with Rhetoric: on which points much misapprehension has prevailed, tending to throw obscurity over the design and use of the Science under consideration.

ESSAY
ON THE
PROVINCE OF REASONING.

Logic.

Logic being concerned with the theory of Reasoning it is evidently necessary, in order to take a correct view of this Science, that all misapprehensions should be removed, relative to the occasions on which the Reasoning process is employed, the purposes it has in view, and the limits within which it is confined.

Simple and obvious as such questions may appear to those who have not thought much on the subject, they will appear on further consideration to be involved in much perplexity and obscurity, from the vague and inaccurate language of many popular writers. To the confused and incorrect notions that prevail respecting the Reasoning process, may be traced most of the common mistakes respecting the Science of Logic, and much of the unsound and unphilosophical argumentation which is so often to be met with in the works of ingenious writers.

These errors have been incidentally adverted to in the foregoing part of this article; but it may be desirable, before we dismiss the subject, to offer on these points some further remarks, which could not have been there introduced without too great an interruption to the developement of the system. Little or nothing indeed remains to be said that is not *implied* in the principles which have been already laid down; but the results and applications of those principles are liable in many instances to be overlooked if not distinctly pointed out. These supplementary observations will neither require, nor admit of, so systematic an arrangement as has hitherto been arrived at, as they will be such as are suggested principally by the objections and mistakes of those who have misunderstood, partially, or entirely, the nature of the Logical system.

Of Induction.

§ 1. Much has been said by some writers of the superiority of the Inductive to the Syllogistic method of seeking truth, as if the two stood opposed to each other; and of the advantage of substituting the Organon of Bacon for that of Aristotle, &c. &c. which indicates a total misconception of the nature of both. There is, however, the more excuse for the confusion of thought which prevails on this subject, because eminent Logical writers have treated or at least have appeared to treat of Induction as a distinct kind of argument from the Syllogism; which if it were, it certainly might be contrasted with the Syllogism: or rather the whole Syllogistic theory would fall to the ground, since one of the very first principles it establishes, is that all Reasoning, on whatever subject, is one and the same process, which may be clearly exhibited in the form of Syllogisms. It is hardly to be supposed, therefore, that this was the meaning of those writers; though it must be admitted that they have countenanced the error in question, by their inaccurate expressions. This inaccuracy seems chiefly to have arisen from a vagueness in the use of the word Induction, which is sometimes employed to designate the process of investigation and of collecting facts; sometimes the deducing of an inference from those facts. The former of these processes (i. e. that of observation and experiment) is undoubtedly *distinct* from that which takes place in the Syllogism; but then it is not a process of *argument*; the latter again *is* an argumentative process; but then it is, like all other arguments, capable of being Syllogistically expressed. And hence Induction has come to be regarded as a distinct kind of argument from the Syllogism. This Fallacy cannot be more concisely or clearly stated, than in the technical form with which we may now presume our readers to be familiar.

Essay on the Province of Reasoning.

Induction is distinct from Syllogism:
Induction is a process of Reasoning; therefore
There is a process of Reasoning distinct from Syllogism.

Here, "Induction" which is the middle term, is used in different senses in the two Premises.

In the process of Reasoning by which we deduce, from our observation of certain known cases, an inference with respect to unknown ones, we are employing a Syllogism in *Barbara* with the major* Premiss suppressed; that being always substantially the same, as it asserts that "what belongs to the individual or individuals we have examined, belongs to the whole class under which they come:" e. g. from an examination of the history of several tyrannies, and finding that each of them was of short duration, we conclude that "the same is likely to be the case with all tyrannies;" the suppressed major Premiss being easily supplied by the hearer; viz. "that what belongs to the tyrannies in question is likely to belong to all."

Induction, therefore, so far forth as it is an *argument*, may of course be stated Syllogistically; but so far forth as it is a *process of inquiry* with a view to obtain the Premises of that argument, it is of course out of the province of Logic. Whether the Induction (in this last sense) has been sufficiently ample, i. e. takes in a sufficient number of individual cases,—whether the character of those cases has been correctly ascertained—and how far the individuals we have examined are *likely to resemble*, in this or that circumstance, *the rest* of the class, &c. &c. are points that require indeed great judgment and caution; but this judgment and caution are not to be aided by Logic, because they are, in reality, employed in deciding whether or not it is fair and allowable to *lay down your Premises*; i. e. whether you are authorized or not, to assert that "what is true of the individuals you

* Not the minor, as Aldrich represents it.

have examined, is true of the whole class:" and that this or that *is* true of those individuals. Now the rules of Logic have nothing to do with the truth or falsity of the Premises, but merely teach us to decide (not whether the Premises are *fairly laid down*, but) whether the Conclusion *follows fairly from* the Premises or not.

Whether the Premises may fairly be assumed, or not, is a point which cannot be decided without a competent knowledge of the *nature of the subject*, e. g. in Natural Philosophy, in which the circumstances which in any case affect the result, are usually far more clearly ascertained, a *single instance* is often accounted a sufficient Induction: e. g. having once ascertained that an individual magnet will attract iron, we are authorized to conclude that this property is universal: in the affairs of human life, a much fuller Induction is required; as in the former example. In short the *degree of evidence* for any proposition we originally assume as a Premiss, (whether the expressed, or the suppressed one) is not to be learned from Logic, nor indeed from *any one distinct* Science; but is the province of whatever Science furnishes the subject matter of your argument. None but a Politician can judge rightly of the degree of evidence of a proposition in Politics; a Naturalist, in Natural History, &c. &c. e. g. from examination of many horned animals, as sheep, cows, &c. a Naturalist finds that they have cloven feet; now his *skill as a Naturalist* is to be shewn in judging whether these animals are likely to resemble in the form of their feet all other horned animals; and it is the exercise of this judgment, together with the examination of individuals, that constitutes what is usually meant by the *Inductive process*: which is that by which we gain *new truths*, and which is not connected with Logic; being not what is strictly called *Reasoning*, but *Investigation*. But when this major Premiss is *granted* him, and is combined with the minor, viz. that the animals he has examined have cloven feet, then he *draws the conclusion Logically*: viz. that "the feet of all horned animals are cloven." Again, if from several times meeting with ill-luck on a Friday, any one concluded that Friday, universally, is an unlucky day, one would object to his *Induction*; and yet it would not be, as an *argument, illogical*; since the conclusion *follows* fairly, *if* you *grant* his implied Premiss, that the events which happened on those particular Fridays are such as must happen on all Fridays;" but we should object to his *laying down this* Premiss; and therefore should justly say that his *Induction* was faulty, though his *argument* was correct.

And here it may be remarked that the ordinary rule for fair argument, viz. that in an Enthymeme the suppressed Premiss should be always the one of whose truth least doubt can exist, is not observed in Induction; for the Premiss which is usually the more doubtful of the two, is, in that, the major; it being in few cases quite certain that the individuals respecting which some point has been ascertained are to be fairly regarded as a sample of the whole class; the major Premiss nevertheless is seldom expressed, for the reason just given, that it is easily understood, as being *mutatis mutandis*, the same in every Induction.

What has been said of Induction will equally apply to Example, which differs from it only in having a singular instead of a general conclusion: e. g. in the instance above, if the conclusion had been drawn, not respecting tyrannies in general, but respecting this or that tyranny, that it was not likely to be lasting, each of the cases adduced to prove this, would have been called an Example.

On the Discovery of Truth.

§ 2. Whether it is by a process of Reasoning that New Truths are brought to light, is a question which seems to be decided in the negative by what has been already said, though many eminent writers seem to have taken for granted the affirmative. It is perhaps, in a great measure, a dispute concerning the use of words; but it is not for that reason either uninteresting or unimportant, since an inaccurate use of language may often, in matters of Science, lead to confusion of thought, and to erroneous conclusions. And in the present instance much of the undeserved contempt which has been bestowed on the Logical system may be traced to this source; for when any one has laid down that "Reasoning is important in the discovery of Truth," and that "Logic is of no service in the discovery of Truth," each of which propositions is true in a certain sense of the terms employed, but not in the *same* sense; he is naturally led to conclude that there are processes of Reasoning to which the Syllogistic theory does not apply, and of course to misconceive altogether the nature of the Science.

In maintaining the negative side of the above question, three things are to be premised: first, that it is not contended that Discoveries of any kind of Truth can be made (or at least are usually made) *without* Reasoning; only that Reasoning is not the *whole* of the process, nor the whole of that which is important therein: secondly, that Reasoning shall be taken in the sense, not of every exercise of the Reason, but of *Argumentation*, in which we have all along used it, and in which it has been defined by all the Logical writers, viz. "from certain granted propositions to infer another proposition as the consequence of them:" thirdly, that by a "New Truth," be understood something neither expressly nor virtually asserted before,—not implied and involved in any thing already known.

To prove then this point demonstratively becomes in this manner perfectly easy; for since all Reasoning (in the sense above defined) may be resolved into Syllogisms; and since even the objectors to Logic make it a subject of complaint, that in a Syllogism the Premises do virtually assert the Conclusion, it follows at once that no New Truth (as above defined) can be elicited by any process of Reasoning.

It is on this ground indeed, that the justly celebrated author of the *Philosophy of Rhetoric* objects to the Syllogism altogether, as necessarily involving a *petitio principii*; an objection which, of course, he would not have been disposed to bring forward, had he perceived that, whether well or ill founded, it lies against all arguments whatever.

Had he been aware that a Syllogism is no distinct kind of argument otherwise than in form, but is, in fact, any argument whatever stated regularly and at full length, he would have obtained a more correct view of the object of all Reasoning, which is merely to expand and unfold the assertions wrapt up, as it were, and implied in those with which we set out, and to

bring a person to perceive and acknowledge the full force of that which he has admitted,—to contemplate it in various points of view,—to admit in one shape what he has already admitted in another, and to give up and disallow whatever is inconsistent with it.

Nor is it always a very easy task even to bring before the mind the several bearings,—the various applications,—of any one proposition. A common term comprehends several, often numberless individuals, and these often, in some respects, widely differing from each other; and no one can be, on each occasion of his employing such a term, attending to and fixing his mind on each of the individuals, or even of the species so comprehended. It is to be remembered too, that both Division and Generalization are in a great degree arbitrary; i. e. that we may both divide the same genus on several different principles, and may refer the same species to several different classes, according to the nature of the discourse and drift of the argument; each of which classes will furnish a distinct middle term for an argument, according to the question: e. g. if we wished to prove that "a horse feels," (to adopt an ill-chosen example from the above writer,) we might refer it to the genus "animal;" to prove that "it has only a single stomach," to the genus of "non-ruminants;" to prove that it is "likely to degenerate in a very cold climate," we should class it with "original productions of a hot climate, &c. &c." Now each of these, and numberless others to which the same thing might be referred, are implied by the very term "horse;" yet it cannot be expected that they all be at once present to the mind whenever that term is uttered. Much less, when instead of such a term as that, we are employing terms of a very abstract, and perhaps complex signification,* as "government, justice, &c."

The ten Categories† or Predicaments which Aristotle and other Logical writers have treated of, being certain general heads or *summa-genera*, to one or more of which every term may be referred, serve the purpose of marking out certain tracks, as it were, which are to be pursued in searching for middle terms in each argument respectively; it being essential that we should generalize on a right principle, with a view to the question before us; or, in other words, that we should abstract that portion of any object presented to the mind, which is important to the argument in hand. There are expressions in common use which have a reference to this caution; such as "this is a question, not as to the nature of the object, but the magnitude of it;" "this is a question of time, or of place, &c." i. e. "the subject must be referred to this or to that Category."

With respect to the meaning of the terms in question, "Discovery," and "New Truth;" it matters not whether we confine ourselves to the narrowest sense, or admit the widest, provided we do but *distinguish*; there certainly *are* two kinds of "New Truth, and of "Discovery," if we take those words in the widest sense in which they are ever used. First, such Truths as were, before they were discovered, *absolutely* unknown, being not implied by any thing we previously knew, though we might perhaps suspect them as probable; such are all *matters of fact* strictly so called, when first made known to one who had not any such previous knowledge, as would enable him to ascertain them *a priori*: i. e. by Reasoning; as if we inform a man that we have a colony at Botany Bay; or that the earth is at such a distance from the sun; or that platina is heavier than gold. The communication of this kind of knowledge is most usually and most strictly called *information*: we gain it from *observation*, and from *testimony*; no *mere internal workings* of our own minds, (except when the mind itself is the very object to be observed,) or *mere* discussions in words, will make these known to us; though there is great room for sagacity *in judging what testimony to admit*, and forming *conjectures* that may *lead to profitable observation*, and to experiments with a view to it. The other class of Discoveries is of a very different nature; that which may be elicited by Reasoning, and consequently is implied in that which we already know, we assent to on that ground, and not from observation or testimony: to take a Geometrical truth upon *trust*, or to attempt to ascertain it by *observation*, would betray a total ignorance of the nature of the Science. In the longest demonstration the Mathematical teacher seems only to lead us to make use of our own stores, and point out to us how much we had already admitted; and in the case of many Ethical propositions, we assent at first hearing, though perhaps we had never heard or thought of the proposition before; so also do we readily *assent* to the *testimony* of a respectable man who tells us that our troops have gained a victory; but how different is the nature of the assent in the two cases. In the latter, we are ready to thank the person for his information, as being such as no wisdom or learning would have enabled us to ascertain; in the former we usually exclaim "*very true!*" "that is a valuable and just remark; that never *struck* me before!" implying at once our practical ignorance of it, and also our consciousness that we possess, in what we already know, the means to ascertain the truth of it.

To all practical purposes, indeed, a Truth of this description may be as completely unknown to a man as the other; but as soon as it is set before him, and the argument by which it is connected with his previous notions is made clear to him, he *recognises* it as something conformable to, and contained in his former belief.

It is not improbable that Plato's doctrine of Reminiscence arose from a hasty extension of what he had observed in this class, to all acquisition of knowledge whatever.

His Theory of ideas served to confound together *matters of fact* respecting the nature of things, (which may be perfectly new to us,) with propositions *relating to our own notions*, and modes of thought; (or to speak perhaps more correctly, our own arbitrary signs) which propositions must be contained and implied in those very complex notions themselves; and whose truth is a conformity, not to the nature of things, but to

* On this point there are some valuable remarks in the *Philosophy of Rhetoric* itself, book iv. ch. vii.

† The Categories enumerated by Aristotle, are *οὐσία, ποσόν, ποιόν, πρός τι, ποῦ, πότε, κεῖσθαι, ἔχειν, ποιεῖν, πάσχειν*; which are usually rendered, as adequately as perhaps they can be in our language, Substance, Quantity, Quality, Relation, Place, Time, Situation, Possession, Action, Suffering. The catalogue has been by some writers enlarged, as it is evident may easily be done by subdividing some of the heads; and by others curtailed, as it is no less evident that all may ultimately be referred to the two heads of Substance and Attribute, or in the language of some Logicians, Accident.

our own hypothesis. Such are all propositions in pure Mathematics, and many in Ethics, viz. those which involve no assertion as to real matters of fact. It has been rightly remarked, that Mathematical propositions are not properly true or false in the same sense as any proposition respecting real fact is so called; and hence the truth (such as it is) of such propositions is necessary and eternal; since it amounts only to this, that any complex notion which you have arbitrarily framed, must be exactly conformable to itself. The proposition that "the belief in a future state, combined with a complete devotion to the present life, is not consistent with the character of prudence," would be not at all the less true if a future state were a chimera, and prudence a quality which was nowhere met with; nor would the truth of the Mathematician's conclusion be shaken, that "circles are to each other as the squares of their diameters," should it be found that there never had been a circle or a square, conformable to the definition, in *rerum naturâ*.

The Ethical proposition just instanced, is one of those which Locke calls "trifling," because the Predicate is merely a part of the complex idea implied by the subject; and he is right, if by "trifling" he means that it gives not, strictly speaking, any *information*; but he should consider that to *remind* a man of what he had not, and what he would have thought of, may be, practically, as valuable as giving him information; and that most propositions in the best sermons, and all in pure Mathematics, are of the description which he censures.

It is indeed rather remarkable that he should speak so often of building Morals into a demonstrative Science, and yet speak so slightingly of those very propositions to which we must absolutely *confine* ourselves, in order to give to Ethics even the appearance of such a Science; for the instant you come to an assertion respecting a *matter of fact*, as that "men (i. e. *actually* existing men) are bound to practise virtue," or "are liable to many temptations," you have stepped off the ground of strict demonstration, just as when you proceed to practical Geometry.

But to return: it is of the utmost importance to distinguish these two kinds of Discovery of Truth; to the former, as we have said, the word "*information*" is most strictly applied; the communication of the latter is more properly called "*instruction*." We speak of the *usual* practice; for it would be going too far to pretend that writers are uniform and consistent in the use of these, or of any other term. We say that the Historian gives us *information* respecting past times; the Traveller, respecting foreign countries: on the other hand, the Mathematician gives *instruction* in the principles of his Science; the Moralist *instructs* us in our duties; and we generally use the expressions "a well-informed man," and "a well-instructed man," in a sense conformable to that which has been here laid down. However, let the words be used as they may, the things are evidently different, and ought to be distinguished. It is a question comparatively unimportant, whether the term "Discovery" shall or shall not be extended to the eliciting of those Truths, which, being implied in our previous knowledge, may be established by mere strict Reasoning. Similar verbal questions indeed might be raised respecting many other cases; e. g. one has forgotten (i. e. cannot *recollect*) the name of some person or place; perhaps we even try to think of it, but in vain; at last some one reminds us, and we instantly *recognise* it as the one we wanted to recollect; it may be asked, was this in our mind or not? The answer is, that in one sense it was, and in another sense, it was not. Or, again, suppose there is a vein of metal on a man's estate which he does not know of; is it part of his possessions or not? and when he finds it out and works it, does he then *acquire* a *new* possession or not? Certainly not, in the same sense as if he has a fresh estate bequeathed to him, which he had formerly no *right* to; but to all practical purposes, it *is* a new possession. This case indeed may serve as an illustration of the one we have been considering; and in all these cases, if the real distinction be understood, the verbal question will not be of much consequence. To use one more illustration; Reasoning has been aptly compared to the piling together blocks of stone; on each of which, as on a pedestal, a man can raise himself a small, and but a small, height above the plain; but which, when skilfully built up, will form a flight of steps, which will raise him to a great elevation. Now (to pursue this analogy) when the materials are all ready to the builder's hand, the blocks ready dug and brought, his work resembles one of the two kinds of Discovery just mentioned, viz. that to which we have assigned the name of *instruction*: but if his materials are to be entirely, or in part, provided by himself,—if he himself is forced to dig fresh blocks from the quarry,—this corresponds to the other kind of Discovery.

We have hitherto spoken of the employment of argument in the establishment of those hypothetical Truths (as they may be called) which relate only to our own abstract notions; it is not, however, meant to be insinuated that there is no room for Reasoning in the establishment of a matter of fact; but the other class of Truths have first been treated of, because in discussing subjects of that kind the process of Reasoning is always the *principal*, and often the *only* thing to be attended to; if we are but certain and clear as to the meaning of the terms; whereas, when assertions respecting real existence are introduced, we have the additional and more important business of ascertaining and keeping in mind the *degree* of *evidence* for those facts, since, otherwise, our Conclusions could not be relied on, however accurate our Reasoning; but, undoubtedly, we may by Reasoning arrive at matters of fact, *if* we have *matters of fact to set out with as data*; only that it will very often happen that "from certain facts," as Campbell remarks, "we draw only probable Conclusions;" because the other Premiss introduced (which he overlooked) is only probable. He observed that in such an instance, for example, as the one lately given, we infer from the *certainty* that such and such tyrannies have been short-lived, the *probability* that others will be so; and he did not consider that there is an understood Premiss which is essential to the argument; (viz. that all tyrannies will resemble those we have already observed) which being only of a probable character, must attach the same degree of uncertainty to the Conclusion. An *individual* fact is not unfrequently elicited by skilfully combining, and Reasoning from, those already known; of which many curious cases occur in the detection of criminals by officers of justice and Barristers, who acquire by practice such dexterity in that particular depart-

ment, as sometimes to draw the right Conclusion from data, which might be in the possession of others, without being applied to the same use. In all cases of the establishment of a *general* fact from Induction, that *general* fact (as has been formerly remarked) is *ultimately* established by Reasoning; e. g. Bakewell, the celebrated cattle-breeder, observed, in a great number of individual beasts, a tendency to fatten readily, and in a great number of others the absence of this constitution; in every individual of the former description, he *observed* a certain peculiar *make*, though they differed widely in size, colour, &c. Those of the latter description differed no less in various points, but agreed in being of a different make from the others: these facts were his data; from which, combining them with the general principle that Nature is steady and uniform in her proceedings, he *Logically* drew the conclusion that beasts of the specified make have universally a peculiar tendency to fattening: but then his principal merit consisted in *making* the observations, and in so *combining* them as to *abstract* from each a multitude of cases, differing widely in many respects, the circumstances in which they all agreed; and also in conjecturing skilfully how far those circumstances were likely to be found in the whole class; the making such observations, and still more the combination, abstraction, and judgment employed, are what men commonly mean (as was above observed) when they speak of *Induction*; and these operations are certainly distinct from Reasoning. The same observations will apply to numberless other cases, as, for instance, to the Discovery of the law of "*vis inertiæ*," and the other principles of Natural Philosophy.

But to what class, it may be asked, should be referred the Discoveries thus made? All would agree in calling them, when first ascertained, "New Truths," in the strictest sense of the word; which would seem to imply their belonging to the class which may be called, by way of distinction, "*Physical Discoveries*:" and yet their being ultimately established by Reasoning, would seem, according to the foregoing rule, to refer them to the other class, viz. what may be called "*Logical* Discoveries;" since whatever is established by Reasoning, must have been contained and virtually asserted in the Premises. In answer to this, it is to be observed, that they certainly do belong to the latter class, *relatively*, to a person who is *in possession of the data*: but to him who is *not*, they are New Truths of the other class; for it is to be remembered, that the words "Discovery" and "New Truths" are necessarily *relative*: there may be a proposition which is to one person absolutely *known*; to another, (viz. one to whom it has never *occurred*, though he is in possession of all the data from which it may be *proved*) it will be, when he comes to perceive it, by a process of *instruction*, what we have called a *Logical Discovery*; to a third, (viz. one who is ignorant of these data) it will be absolutely *unknown*, and will have been, when made known to him, a perfectly and properly New Truth,—a piece of information,—a *Physical Discovery* as we have called it. To the Philosopher, therefore, who arrives at the Discovery by Reasoning from his observations, and from established principles combined with them, the Discovery is of the former class; to the multitude, probably of the latter, as *they* will have been most likely not possessed of all his data. It follows from what has been said, that in Mathematics, and in such Ethical propositions as we were lately speaking of, we do not allow the possibility of *any* but a Logical Discovery; i. e. no proposition, of that class, can be true, which was not implied in the definitions we set out with, which are the first principles: for since these propositions do not profess to state any matter of fact, the only Truth they *can* possess, *consists* in conformity to the original principles; to one, therefore, who *knows* these principles, such propositions are Truths already implied, since they may be developed to him by Reasoning, if he is not defective in the discursive faculty; to one who does *not* understand those principles, (i. e. is not master of the definitions) such propositions are absolutely unmeaning. On the other hand, propositions relating to matters of fact, *may be*, indeed, implied in what he already knew; (as he who knows the climate of the Alps, the Andes, &c. &c. has virtually admitted the general fact, that "the tops of mountains are comparatively cold;") but as these possess an absolute and physical Truth, they *may* also be absolutely "new," their Truth not being *implied by the mere terms of the propositions*. The truth or falsity of any proposition concerning a triangle, is implied by the meaning of that and of the other Geometrical terms; whereas, though one may *understand* (in the ordinary sense of that word) the full meaning of the terms, "moon" and "inhabited," and of all the other terms in the language, he cannot thence be certain that the moon is, or is not, inhabited.

It has probably been the source of much perplexity that the term "*true*" has been applied *indiscriminately* to two such different classes of propositions. The term *definition* is used with the same laxity; and much confusion has thence resulted.

Such Definitions as the Mathematical, must imply every attribute that belongs to the thing defined; because that *thing* is merely *our meaning*, which meaning the Definition lays down; whereas, real substances, having an independent existence, may possess innumerable qualities (as Locke observes) not implied by the meaning we attach to their names, or, as Locke expresses it, by our ideas of them. "Their *nominal essence* (to use his language) is not the same as their *real essence*:" whereas the nominal essence, and the real essence, of a circle, &c. are the same. A Mathematical Definition, therefore, cannot properly be called *true*, since it is not properly a *proposition*, (any more than an article in a Dictionary,) but merely an explanation of the meaning of a term. Perhaps in Definitions of this class, it might be better to substitute (as Aristotle usually does) the *imperative* mood for the indicative; thus bringing them into the form of postulates; for the Definitions and the postulates in Mathematics differ in little or nothing but the form of expression: e. g. "let a four-sided figure, of equal sides and right angles, be called a square," would clearly imply that such a figure is conceivable, and that the writer intended to employ that term to signify such a figure; which is precisely all that is intended to be asserted. If, indeed, a Mathematical writer mean to assert that the *ordinary* meaning of the term is that which he has given, *that*, certainly, *is* a proposition, which must be either true or false; but in defining a *new* term, the term indeed may be ill-chosen and improper, or the Definition may be self-contradictory, and consequently unintelligible;

but the words, "true," and "false," do not apply. The same may be said of what are called *nominal* Definitions of other things, i. e. those which merely explain the meaning of the word; viz. they can be true or false only when they profess (and so far as they profess) to give the ordinary and established meaning of the term. But those which are called *real* Definitions, viz. which unfold the nature of the thing, (which they may do in *various degrees*,) to these the epithet "true" may be applied; and to make out such a Definition will often be the very *end* (not as in Mathematics the *beginning*) of our study.

In Mathematics there is no such distinction between nominal and real Definition; *the meaning of the term*, and the *nature of the thing*, being one and the same: so that *no correct Definition whatever* of any Mathematical term can be devised, which shall not imply every thing which belongs to the term.

When it is asked, then, whether such great Discoveries, as have been made in Natural Philosophy, were accomplished, or can be accomplished by *Reasoning?* the inquirer should be reminded, that the question is ambiguous; it may be answered in the affirmative, if by "Reasoning" is meant to be included the assumption of Premises; to the right performance of *that* work, is requisite, not only in many cases, the ascertainment of facts, and of the degree of evidence for doubtful propositions, (in which observation and experiment will often be indispensable,) but also a skilful *selection* and *combination* of known facts and principles; such as implies, amongst other things, the exercise of that powerful *abstraction* which seizes the common circumstances—the point of agreement—in a number of, otherwise dissimilar, individuals: it is in this that the greatest genius is shewn. But if "Reasoning" be understood in the limited sense in which it is usually defined, then we must answer in the negative; and reply that such Discoveries are made by means of Reasoning *combined* with other operations.

In the process we have been speaking of, there is much Reasoning throughout; and thence the whole has been carelessly called a "Process of Reasoning."

It is not, indeed, any just ground of complaint that the word Reasoning is used in two senses; but that the two senses are perpetually confounded together: and hence it is that some Logical writers fancied that Reasoning (viz. that which Logic treats of) was the method of discovering Truth; and that so many other writers have accordingly complained of Logic for not accomplishing that end, urging that "Syllogism (i. e. Reasoning; though they overlooked the coincidence) never established any thing that is, strictly speaking, unknown to him who has granted the Premises: and proposing the introduction of a certain "rational Logic" to accomplish this purpose; i. e. to direct the mind in the progress of investigation. Supposing that some such system could be devised—that it could even be brought into a Scientific form, (which he must be more sanguine than Scientific who expects,) that it were of the greatest conceivable utility, and that it should be allowed to bear the name of "Logic," since it would not be worth while to contend about a word, still it would not, as these writers seem to suppose, have the same object proposed with the Aristotelian Logic; nor be in any respect a rival to that system. A *plough* may be a much more ingenious and valuable instrument than a *flail*, but it never can be substituted for it.

Those Discoveries of general laws of Nature, &c. of which we have been speaking, being of that character which we have described by the name of "Logical Discoveries," *to him who is in possession of all the Premises from which they are deduced*; but being, *to the multitude* (who are unacquainted with many of those Premises) strictly "New Truths;" hence it is, that men in general give to the *general* facts, and to them, most peculiarly, the name of *Discoveries*; for to *themselves* they *are* such, in the strictest sense; the Premises from which they were inferred being not only originally unknown to them, but frequently *remaining unknown to the very last*: e. g. the general conclusion concerning cattle, which Bakewell made known, is what most Agriculturists (and many others also) are acquainted with; but the Premises he set out with, viz. the facts respecting this, that, and the other, individual ox, (the ascertainment of which facts was his first *Discovery*) these are what few know, or care to know, with any exact particularity.

And it may be added, that these discoveries of particular facts, which are the *immediate* result of *observation*, are, in themselves, uninteresting and insignificant, *till* they are combined so as to lead to a grand general result; those who on each occasion watched the motions, and registered the date of a comet, little thought, perhaps, themselves, what magnificent results they were preparing the way for. So that there is an additional cause which has confined the term Discovery to these grand general conclusions; and, as was just observed, they are, to the generality of men, perfectly New Truths in the strictest sense of the word, not being implied in any previous knowledge *they* possessed. Very often it will happen, indeed, that the conclusion thus drawn will amount only to a *probable conjecture*; which conjecture will dictate to the inquirer such an *experiment*, or course of experiments, as will fully establish the fact; thus Sir H. Davy, from finding that the flame of hydrogen gas was not communicated through a long slender tube, conjectured that a shorter, but still slenderer tube, would answer the same purpose; this led him to try the experiments, in which, by continually shortening the tube, and at the same time lessening its bore, he arrived at last at the wire-gauze of his safety-lamp.

It is to be observed also, that whatever *credit* is conveyed by the word "Discovery," to him who is regarded as the author of it, is well deserved by those who skilfully select and combine known Truths, (*especially* such as have been *long* and *generally known*,) so as to elicit important, and hitherto unthought-of, conclusions; theirs is the master mind; *αρχιτεκτονικὴ φρόνησις·* whereas men of very inferior powers may sometimes, by immediate observation, discover perfectly new facts, empirically, and thus be of service in furnishing materials to the others; to whom they stand in the same relation (to recur to a former illustration) as the brickmaker or stonequarrier, to the architect. It is peculiarly creditable to A. Smith, and to Mr. Malthus, that the data from which they drew such important Conclusions had been in every ones hands for centuries.

As for Mathematical Discoveries, they (as we have before said) must *always* be of the description to which

we have given the name of "Logical Discoveries;" since to him who properly comprehends the meaning of the Mathematical terms, (and to no other are the Truths themselves, properly speaking, intelligible,) those results are implied in his previous knowledge, since they are Logically deducible therefrom. It is not, however, meant to be implied that *Mathematical* Discoveries are effected by pure Reasoning, and by that *singly*. For though there is not here, as in Physics, any exercise of judgment as to the degree of evidence of the Premises, nor any experiments and observations, yet there is the same call for skill in the *selection* and *combination* of the Premises in such a manner as shall be best calculated to lead to a new, that is, *unperceived* and *unthought-of* Conclusion.

In *following*, indeed, and *taking in* a demonstration, nothing is called for but pure Reasoning; but *the assumption of Premises is not a part* of Reasoning, in the strict and technical sense of that term. Accordingly, there are many who can *follow* a demonstration, or any other train of argument, who would not succeed well in *framing* one of their own.*

For both kinds of Discovery then, the Logical, as well as the Physical, certain operations are requisite, beyond those which can fairly be comprehended under the strict sense of the word "Reasoning;" in the Logical, is required a skilful *selection* and *combination of known Truths*; in the Physical we must employ, in *addition* (generally speaking) to that process, *observation* and *experiment*. It will generally happen, that in the study of Nature, and, universally, in all that relates to matters of fact, *both* kinds of investigation will be united; i. e. some of the facts or principles you *reason from* as Premises, must be ascertained by *observation*; or, as in the case of the safety-lamp, the ultimate *Conclusion* will need *confirmation* from experience; so that both Physical and Logical Discovery will take place in the course of the same process: we need not, therefore, wonder, that the two are so perpetually confounded. In Mathematics, on the other hand, and in great part of the discussions relating to Ethics and Jurisprudence, there being no room for any Physical Discovery whatever, we have only to make *a skilful* use of the propositions in our possession, to arrive at every attainable result.

The investigation, however, of the latter class of subjects differs in *other* points also from that of the former; for setting aside the circumstance of our having, in these, no question as to facts,—no room for observation,—there is also a considerable difference in what may be called the process of *Logical investigation*; the *Premises* on which we proceed being of so different a nature in the two cases.

To take the example of Mathematics, the definitions, which are the principles of our Reasoning, are very *few*, and the axioms still fewer; and both are, for the most part, *laid down*, and *placed before the student in the outset*; the introduction of a new definition or axiom, being of comparatively rare occurrence, at wide intervals, and with a *formal* statement; besides which, there is no room for *doubt* concerning either. On the other hand, in all Reasonings which regard matters of fact, we introduce, almost at *every step*, fresh and fresh propositions (to a very great number) which had not been elicited in the course of our Reasoning, but are taken for granted; viz. facts and laws of Nature which are here the principles of our Reasoning, and *maxims*, or "elements of belief," which answer to the axioms in Mathematics. If, at the opening of a Treatise, for example, on Chemistry, on Agriculture, on Political Economy, &c. the author should make, as in Mathematics, a formal statement of all the propositions he intended to assume, as granted throughout the whole work, both he and his readers would be astonished at the number: and, of these, many would be only probable, and there would be much room for doubt as to the *degree* of probability, and for judgment, in ascertaining that degree.

Moreover, Mathematical axioms are always employed precisely *in the same simple form*; e. g. the axiom that "things equal to the same, are equal to one another," is cited, whenever there is need, in those very words; whereas the maxims employed in the other class of subjects, admit of, and require, continual modifications in the application of them: e. g. "the stability of the laws of Nature," which is our constant assumption in inquiries relating to Natural Philosophy, assumes many different shapes, and in some of them, does not possess the same absolute certainty as in others: e. g. when from having always observed a certain sheep ruminating, we infer, that this individual sheep will continue to ruminate, we assume that "the property which has hitherto belonged to this sheep, will remain unchanged;" when we infer the same property of all sheep, we assume that "the property which belongs to this individual, belongs to the whole species:" if, on comparing sheep with some other kinds of horned animals, and finding that all agree in ruminating, we infer that, "all horned animals ruminate," we assume that "the whole of a genus or class are likely to agree in any point wherein many species of that genus agree;" or in other words, "that if one of two properties, &c. has *often* been found accompanied by another, and never without it, the former will be *universally* accompanied by the latter;" now all these are merely different forms of the maxim, that "nature is uniform in her operations;" which, it is evident, varies in expression in almost every different case where it is applied, and admits of every degree of evidence, from absolute moral certainty, to mere conjecture.

The same may be said of an infinite number of principles and maxims appropriated to, and employed in each particular branch of study. Hence, all such Reasonings are, in comparison of Mathematics, very complex; requiring so much *more* than that does, beyond the process of merely deducing the Conclusion Logically, from the Premises; so that it is no wonder that the longest Mathematical demonstration should be so much more easily constructed and understood, than a much shorter train of just Reasoning concerning real facts. The former has been aptly compared to a long and steep, but even and regular, flight of steps, which tries the breath, and the strength, and the perseverance, only; while the latter resembles a short, but rugged and uneven, ascent up a precipice, which requires a quick eye, agile limbs, and a firm step; and in which we have to tread now on this side, now on that; ever considering, as we proceed, whether this projection will afford room for our foot, or whether some loose stone may not slide from under us.

* Hence the Student must not confine himself to this passive kind of employment, if he would become truly a Mathematician.

As for those Ethical and Legal Reasonings which were lately mentioned, as in some respects resembling those of Mathematics, (viz. such as keep clear of all assertions respecting facts,) they have this difference; that not only men are not so completely *agreed* respecting the maxims and principles of Ethics and Law, but the meaning also of each term cannot be absolutely, and for ever, fixed by an arbitrary definition; on the contrary, a great part of our labour consists in distinguishing accurately the various senses in which men employ each term, ascertaining which is the most proper, and taking care to avoid confounding them together.

Of Inference and Proof.

§ 3. Since it appears, from what has been said, that universally a man must possess something else besides the Reasoning faculty, in order *to apply* that faculty properly to his own purpose, whatever that purpose may be; it may be inquired whether some theory could not be made out, respecting those "*other operations*," and "intellectual processes distinct from Reasoning, which it is necessary for us sometimes to employ in the investigation of truth;"* and whether rules could not be laid down for conducting them.

Something has, indeed, been done in this way by more than one writer; and more might probably be accomplished by one who should fully comprehend and carefully bear in mind the principles of Logic, properly so called; but it would hardly be possible to build up any thing like a regular *Science*, respecting these matters, such as Logic is, with respect to the theory of Reasoning. It may be useful, however, to observe, that these "*other operations*" of which we have been speaking, and which are preparatory to the exercise of Reasoning, are of *two kinds*, according to the nature of the end proposed; for Reasoning comprehends *Inferring* and *Proving*; which are not two different things, but the same thing regarded in *two different points of view*: (like the road from London to York, and the road from York to London,) he who infers,† proves; and he who proves, infers; but the word "infer" fixes the mind *first* on the Premiss, and then on the *Conclusion*; the word "prove," on the contrary, leads the mind *from* the Conclusion *to* the Premiss. Hence, the substantives derived from these words respectively, are often used to express that which, on each occasion, is *last* in the mind; *Inference* being often used to signify the *Conclusion*, (i. e. *Proposition inferred*) and *Proof*, the *Premiss*. We say also "How do you *prove* that?" and "What do you *infer from* that?" which sentences would not be so properly expressed if we were to transpose those verbs. One might, therefore, define *Proving*, "the assigning of a reason or argument for the support of a *given* proposition;" and "*Inferring*," the "deduction of a Conclusion from *given* Premises." In the one case our *Conclusion* is *given*, (i. e. set before us) and we have to *seek* for *arguments*; in the other, our Premises are *given*, and we have to *seek* for a Conclusion; i. e. to put together our own propositions, and try *what* will follow from them; or, to speak more Logically, in the one case, we seek to refer the subject of which we would predicate something, to a class to which that predicate will (affirmatively or negatively) apply; in the other we seek to find comprehended, in the subject of which we have predicated something, some other term to which that predicate had not been before applied. Each of these is a definition of Reasoning.

To *infer*, then, is the business of the *Philosopher*; to *prove*, of the *Advocate*; the former, from the great mass of known and admitted truths, wishes to elicit *any* valuable additional truth whatever, that has been hitherto unperceived; and, perhaps, without knowing, with certainty, what will be the terms of his Conclusion. Thus the Mathematician, e. g. seeks to ascertain *what* is the ratio of circles to each other, or *what* is the line whose square will be equal to a given circle: the Advocate, on the other hand, has a proposition put before him, which he is to maintain as well as he can; his business, therefore, is to *find middle terms*, (which is the *inventio* of Cicero;) the Philosopher's, to combine and select known facts, or principles, suitably for gaining from them conclusions which, though implied in the Premises, were before unperceived; in other words, for making "Logical Discoveries." Such are the respective preparatory processes in these two branches of study. They are widely different;—they arise from, and generate, very different habits of mind; and require a very different kind of training and precept. The Lawyer, or Controversialist, or, in short, the Rhetorician in general, who is, in his own province, the most skilful, may be but ill-fitted for Philosophical investigation, even where there is no *observation* wanted;—when the facts are all *ready ascertained* for him. And again, the ablest Philosopher may make an indifferent disputant; especially, since the arguments which have led *him* to the conclusion, and have, with him, the most weight, may not, perhaps, be the most powerful in controversy. The commonest fault, however, by far, is to forget the Philosopher or Theologian, and to assume the Advocate, improperly. It is therefore of great use to dwell on the distinction between these two branches: as for the bare process of Reasoning, *that* is the same in both cases; but the preparatory processes which are requisite *in order to employ* Reasoning profitably, these we see branch off into two distinct channels. In each of these undoubtedly, useful rules may be laid down; but they should not be confounded together. Bacon has chosen the department of Philosophy, giving rules in his *Organon*, (not only for the conduct of experiments to ascertain new facts, but also for the selection and combination of known facts and principles,) with a view of obtaining valuable *Inferences*; and it is probable that a system of such rules is what some writers mean (if they have any distinct meaning) by their proposed "Logic." In the other department, precepts have been given by Aristotle and other Rhetorical writers, as a part of their plan. How far these precepts are to be considered as belonging to the present system,—whether "method" is to be regarded as a *part of Logic*,—whether the matter of Logic is to be included in the system,—whether Bacon's is properly to be reckoned a kind of Logic; all these are merely verbal questions relating to the extension, not of the *Science*, but of the *name*. The bare process of Reasoning, i. e. deducing a Conclusion from Premises,

* D. Stewart.

† We mean, of course, when the word is understood to imply *correct* Inference.

must ever remain a distinct operation from the *assumption* of Premises, however useful the rules may be that have been given, or may be given, for conducting this latter process, and others connected with it; and however properly such rules may be subjoined to the precepts of that system to which the name of Logic is applied in the narrowest sense. Such rules as we now allude to may be of eminent service; but they must always be, as we have before observed, comparatively vague and general, and incapable of being built up into a regular demonstrative theory like that of the Syllogism; to which theory they bear much the same relation as the principles and rules of Poetical and Rhetorical criticism, to those of Grammar; or those of practical Mechanics, to strict Geometry. We find no fault with the extension of a term; but we would suggest a caution against confounding together, by means of a common name, things essentially different: and above all we deprecate the sophistry of striving to depreciate what is called "the school Logic," by perpetually *contrasting* it with systems with which it has nothing in common but the *name*; and whose object is essentially different.

It is not a little remarkable that writers whose expressions tend to confound together, by means of a common name, two branches of study which have nothing else in common, (as if they were two different plans for attaining *one and the same* object,) have themselves complained of one of the effects of this confusion, viz. the introduction, early in the career of Academical Education, of a course of *Logic*; under which name, they observe "men now universally comprehend the works of Locke, Bacon, &c." which, as is justly remarked, are unfit for beginners. Now this would not have happened, if men had always kept in mind the meaning or meanings of each name they used. And it may be added, that, however justly the word Logic may be thus extended, we have no ground for applying to the Aristotelian Logic, the remarks above quoted respecting the Baconian; which the ambiguity of the word, if not carefully kept in view, might lead us to do. Grant that Bacon's work is a part of Logic; it no more follows from the unfitness of *that* for *learners*, that the elements of the theory of Reasoning should be withheld from them, than it follows that the elements of Euclid, and common Arithmetic, are unfit for boys, because *Newton's Principia*, which also bears the title of *Mathematical*, is above their grasp. Of two branches of study which bear the same name, or even of two parts of the same branch, the one may be suitable to the commencement, the other to the close, of the Academical career.

At whatever period of that career it may be proper to introduce the study of such as are usually called Metaphysical writers, it may be safely asserted, that those who have had the most experience in the business of giving instruction in Logic, properly so called, together with other branches of knowledge, prefer and generally pursue the plan of letting their pupils *enter* on that study next in order, after the elements of Mathematics.

Of Verbal and Real Questions.

§ 4. The ingenious author of the *Philosophy of Rhetoric* having maintained, or rather assumed, that Logic is applicable to Verbal controversy alone, there may be an advantage, though it has been our aim throughout to shew the application of it to all Reasoning, in pointing out the difference between Verbal and Real Questions, and the probable origin of Campbell's mistake; for to trace any error to its source, will often throw more light on the subject in hand than can be obtained if we rest satisfied with merely detecting and refuting it.

Essay on the Province of Reasoning.

Every Question that can arise, is in fact a Question whether a certain Predicate is or is not applicable to a certain subject; and whatever other account may be given by any writer of the nature of any matter of doubt or debate, will be found, ultimately, to resolve itself into this. But sometimes the Question turns on the meaning and extent of the terms employed; sometimes on the things signified by them. If it be made to appear therefore, that the opposite sides of a certain Question may be held by persons not differing in their opinion of the matter in hand, then that Question may be pronounced Verbal, as depending on the different senses in which they respectively employ the terms. If on the contrary it appears that they employ the terms in the same sense, but still differ as to the application of one of them to the other, then it may be pronounced that the Question is Real,—that they differ as to the opinions they hold of the things in Question.

If, for instance, two persons contend whether Augustus deserved to be called a great man, then if it appeared that the one included under the term "great," disinterested *patriotism*, and *on that ground* excluded Augustus from the class, as wanting in that quality, and that the other also gave him no credit for that quality, but understood no more by the term "great," than high intellectual qualities, energy of character, and brilliant actions, it would follow that the parties did not differ in opinion except as to the use of a term, and that the Question was Verbal. If again it appeared that the one *did* give Augustus credit for such patriotism as the other denied him, both of them including that idea in the term great, then the Question would be Real. Either kind of Question, it is plain, is to be argued according to Logical principles; but the middle terms employed would be different; and for this reason among others it is important to distinguish Verbal from Real controversy. In the former case, e. g. it might be urged with truth, that the common use of the expression "great and good" proves that the idea of good is not implied in the ordinary sense of the word great; an argument which could have, of course, no place in deciding the other Question.

It is by no means to be supposed that all Verbal Questions are trifling and frivolous; it is often of the highest importance to settle correctly the meaning of a word, either according to ordinary use or according to the meaning of any particular writer, or class of men; but when Verbal Questions are *mistaken* for Real, much confusion of thought and unprofitable wrangling will be generally the result. Nor is it always so easy and simple a task, as might at first sight appear, to distinguish them from each other: for several objects to which one common name is applied will often have many points of difference, and yet that name may perhaps be applied to them all in the same sense, and may be fairly regarded as the genus they come under, if it appear that they all agree in what is designated by that name, and that the differences between them are in points not essential to the

character of the genus. A cow and a horse differ in many respects, but agree in all that is implied by the term "quadruped," which is therefore applicable to both in the same sense. So also the houses of the ancients differed in many respects from ours, and their ships, still more; yet no one would contend that the terms "house" and "ship," as applied to both, were ambiguous, or that οἶκος might not fairly be rendered *house*, and ναῦς, *ship*: because the essential characteristic of a house is, not its being of this or that form or materials, but its being a dwelling for men; these therefore would be called *two different kinds* of houses; and consequently the term "house" would be applied to each, without any equivocation, in the same sense: and so in the other instances. On the other hand, two or more things may bear the same name, and may also have a resemblance in many points, and may from that resemblance have come to bear the same name, and yet if the circumstance which is essential to each be wanting in the other, the term may be pronounced ambiguous: e. g. the word "Priest" is applied to the ministers of the Jewish and of the Pagan religions, and also to those of the Christian: and doubtless the term is so used in consequence of their being both *ministers*, (in some sort) of religion. Nor would every difference that might be found between the Priests of different religions constitute the term ambiguous, provided such differences were non-essential to the idea suggested by the word Priest; as e. g. the Jewish Priest served the true God, and the Pagan, false Gods: this is a most important difference, but does not constitute the term ambiguous, because neither of these circumstances is implied and suggested by the term Ἱερεὺς, which accordingly was applied both to Jewish and Pagan Priests. But the term Ἱερεὺς does seem to have implied the office of offering *sacrifice*, atoning for the sins of the people, and acting as mediator between man and the object of his worship; and accordingly that term is never applied to any one under the Christian system, except to the one great Mediator. The Christian ministers not having that office which was implied as essential in the term Ἱερεὺς, were never called by that name, but by that of πρεσβύτερος. It may be concluded therefore, that the term Priest is ambiguous, as corresponding to the terms Ἱερεὺς and πρεσβύτερος respectively, notwithstanding that there are points in which these two agree. These therefore should be reckoned, not *two different kinds* of Priests, but Priests in two different senses; since, (to adopt the phraseology of Aristotle,) the definition of them so far forth as they are Priests, would be different.

It is evidently of much importance to keep in mind the above distinctions, in order to avoid, on the one hand, stigmatizing as Verbal controversies, what in reality are not such, merely because the Question turns on the applicability of a certain Predicate to a certain subject; or on the other hand, falling into the opposite error of mistaking words for things, and judging of men's agreement or disagreement in opinion in every case, merely from their agreement or disagreement in the terms employed.

Of Realism.

§ 5. Nothing has a greater tendency to lead to the mistake just noticed, and thus to produce *undetected* Verbal Questions and fruitless Logomachy, than the prevalence of the notion of the Realists, that genus and species were some real Things, existing independently of our conceptions and expressions, and that, as in the case of singular terms, there is some real individual corresponding to each, so in common terms also there is something corresponding to each, which is the object of our thoughts when we employ any such term.* Few, if any indeed, in the present day avow and maintain this doctrine; but those who are not especially on their guard, are perpetually sliding into it unawares. Nothing so much conduces to this as the transferred and secondary use of the words "same," "one and the same," "identical, &c." when it is not clearly perceived and carefully borne in mind that they *are* employed in a secondary sense, and *that* more frequently even than in the primary. Suppose e. g. a thousand persons are thinking of the sun, it is evident it is one and the same individual object on which all these minds are employed; so far all is clear: but suppose all these persons are thinking of a triangle; not any individual triangle, but triangle in general; and considering perhaps the equality of its angles to two right angles; it would seem as if in this case also, their minds were all employed on "one and the same" object: and this object of their thoughts, it may be said, cannot be the *mere word* triangle, but that which is meant by it; nor again, can it be everything that the word will apply to, for they are not thinking of *triangles*, but of *one* thing: those who do not acknowledge that this "one thing" has an existence independent of the human mind, are in general content to tell us by way of explanation, that the object of their thoughts is the abstract "idea" of a triangle; an explanation which satisfies, or at least silences many, though it may be doubted whether they very clearly understand what sort of a thing an idea is, which may thus exist in a thousand different minds at once, and yet be "one and the same."

The fact is, that "unity" and "sameness" are in such cases employed, not in the primary sense, but to denote perfect *similarity*. When we say that ten thousand different persons have all "one and the same" idea in their minds, or are all of "one and the same" opinion, we mean no more than that they are all *thinking exactly alike*: when we say that they are all in the "same" posture, we mean that they are all *placed alike*: and so also they are said all to have the "same" disease when they are all diseased alike.

The origin of this secondary sense of the words, "same," "one," "identical," &c. (an attention to which would clear away an incalculable mass of confused Reasoning and Logomachy,) is easily to be traced to the use of language and of other signs, for the purpose of mutual communication. If any one utters the "one single" word "triangle," and gives "one single" definition of it; each person who hears him forms a certain notion in his own mind, not differing in any respect from that of each of the rest; they are said therefore to have all "one and the same" notion, because, resulting from, and corresponding with, that which is in the primary sense "one and the same" expression; and there is said to be "one single" idea of every triangle, (considered merely as a triangle,) because one single name or definition is equally applicable to each. In like manner all the coins struck by

* A doctrine commonly, but falsely, attributed to Aristotle, who expressly contradicts it. Categories, περὶ οὐσίας.

the same single die, are said to have "one and the same" impression, merely because the one description which suits one of these coins will equally suit any other that is exactly like it.

It is not intended to recommend the disuse of the words "same," "identical," &c. in this transferred sense; which, if it were desirable, would be utterly impracticable; but merely, a steady attention to the ambiguity thus introduced, and watchfulness against the errors thence arising. The difficulties and perplexities which have involved the questions respecting personal identity, among others, may be traced principally to the neglect of this caution. But the further consideration of that question would be unsuitable to the subject of this article.

RHETORIC.

INTRODUCTORY SECTION.

Of Rhetoric various definitions have been given by different writers; who, however, seem not so much to have disagreed in their conceptions of the nature of the same thing, as to have had different things in view while they employed the same terms. Not only the word Rhetoric itself, but also those used in defining it, have been taken in various senses; as may be observed with respect to the word "Art" in Cic. *de Orat.* where a discussion is introduced as to the applicability of that term to Rhetoric; manifestly turning on the different senses in which "Art" may be understood.

To enter into an examination of all the definitions that have been given, would lead to much uninteresting and uninstructive verbal controversy. It is sufficient to put the reader on his guard against the common error of supposing that a general term has some real object, properly corresponding to it, independent of our conceptions;—that, consequently, some one definition is to be found which will comprehend every thing that is rightly designated by that term;—and that all others must be erroneous; whereas in fact it will often happen, as in the present instance, that both the wider, and the more restricted sense of a term, will be alike sanctioned by use, (the only competent authority;) and that the consequence will be a corresponding variation in the definitions employed, none of which perhaps may be fairly chargeable with error, though none can be framed that will apply to every acceptation of the term.

It is evident that in its primary signification, Rhetoric had reference to public *Speaking* alone, as its etymology implies: but as most of the rules for speaking are of course applicable equally to writing, an extension of the term naturally took place; and we find even Aristotle, the earliest systematic writer on the subject whose works have come down to us, including in his Treatise such compositions as were not intended to be publicly recited.* And even as far as relates to Speeches, properly so called, he takes, in the same Treatise, at one time a wider, and at another a more restricted view of the subject; including under the term Rhetoric, in the opening of his work, nothing beyond the finding of topics of Persuasion, as far as regards the *matter* of what is spoken; and afterwards embracing the consideration of Style, Arrangement, and Delivery.

The invention of Printing, by extending the sphere of operation of the Writer, has of course contributed to the extension of those terms which in their primary signification had reference to Speaking alone. Many objects are now accomplished through the medium of the Press, which formerly came under the exclusive province of the Orator; and the qualifications requisite for success are so much the same in both cases, that we apply the term "Eloquent" as readily to a Writer as to a Speaker; though etymologically considered it could only belong to the latter. Indeed "Eloquence" is often attributed even to such compositions, e. g. Historical works, as have in view an object entirely different from any that could be proposed by an Orator; because *some part* of the rules to be observed in Oratory, or rules analogous to these, are applicable to such compositions. Conformably to this view therefore, some writers have spoken of Rhetoric as the Art of Composition, universally; or, with the exclusion of Poetry alone, as embracing all Prose composition.

A still wider extension of the province of Rhetoric has been contended for by some of the ancient writers; who thinking it necessary to include, as belonging to the Art, every thing that could conduce to the attainment of the object proposed, introduced into their systems Treatises on Law, Morals, Politics, &c. on the ground that a knowledge of these subjects was requisite to enable a man to speak well on them; and even insisted on Virtue,* as an essential qualification of a perfect Orator, because a good character, which can in no way be so surely established as by deserving it, has great weight with the audience.

These notions are combated by Aristotle; who attributes them either to the ill-cultivated understanding (ἀπαιδευσία) of those who maintained them, or to their arrogant and pretending disposition, ἀλαζονεία; i. e. a desire to extol and magnify the Art they professed. In the present day, the extravagance of such doctrines is so apparent to most readers, that it would not be worth while to take much pains in refuting them. It is worthy of remark however, that the very same erroneous view is, even now, often taken of Logic, (as was remarked under that article;) which

* Arist. *Rhet.* book iii.

* See Quinctilian.

has been considered by some as a kind of system of universal knowledge, on the ground that argument may be employed on all subjects, and that no one can argue well on a subject which he does not understand; and which has been complained of by others as not supplying any such universal instruction as its unskilful advocates have placed within its province; such as in fact no one Art or System can possibly afford.

The error is precisely the same in respect of Rhetoric and of Logic; both being *instrumental* arts; and, as such, applicable to various kinds of subject-matter, which do not properly come under them.

So judicious an author as Quinctilian would not have failed to perceive, had he not been carried away by an inordinate veneration for his own Art, that as the possession of building materials is no part of the art of Architecture, though it is impossible to build without materials, so, the knowledge of the subjects on which the Orator is to speak, constitutes no part of the art of Rhetoric, though it be essential to its successful employment; and that though virtue and the good reputation it procures, add materially to the Speaker's influence, they are no more to be, for that reason, considered as belonging to the Orator, as such, than wealth, rank, or a good person, which manifestly have a tendency to produce the same effect.

In the present day however, the province of Rhetoric, in the widest acceptation that would be reckoned admissible, comprehends all "Composition in Prose;" in the narrowest sense, it would be limited to "Persuasive Speaking."

We propose in the present article to adopt a middle course between these two extreme points; and to treat of *Argumentative Composition* generally, and exclusively; considering Rhetoric (in conformity with our original plan, and with the very just and philosophical view of Aristotle) as an off-shoot from Logic.

It was remarked in our article on that Science, that Reasoning may be considered as applicable to two purposes, which we ventured to designate respectively by the terms "Inferring, and Proving;" i. e. the ascertainment of the truth by investigation, and the establishment of it to the satisfaction of another: and it was there remarked, that Bacon, in his *Organon*, had laid down rules for the conduct of the former of these processes, and that the latter belonged to the province of Rhetoric: and it was added, that to *infer* is to be regarded as the proper office of the Philosopher;—to *prove*, of the Advocate. It is not however to be understood that Philosophical works are to be excluded from the class to which Rhetorical rules are applicable; for the Philosopher who undertakes, by writing or speaking, to convey his notions to others, assumes for the time being, the character of Advocate of the doctrines he maintains; the process of *investigation* must be supposed completed, and certain conclusions arrived at by that process, before he begins to impart his ideas to others in a treatise or lecture; the object of which must of course be to *prove* the justness of those conclusions. And in doing this, he will not always find it expedient to adhere to the same course of reasoning by which his own discoveries were originally made; other arguments may occur to him afterwards, more clear or more concise, or better adapted to the understanding of those he addresses. In explaining therefore, and establishing the truth, he may often have occasion for rules of a different kind from those employed in its discovery. Accordingly, when we remarked, in the article above alluded to, that it is a common fault, for those engaged in Philosophical and Theological inquiries, to forget their own peculiar office, and assume that of the Advocate, improperly, this caution is to be understood as applicable to the process of *forming their own opinions*; not, as excluding them from advocating by all fair arguments, the conclusions at which they have arrived by candid investigation. But if this candid investigation do not take place in the first instance, no pains that they may bestow in searching for arguments, will have any tendency to ensure their attainment of truth. If a man begins (as is too plainly a frequent mode of proceeding) by hastily adopting or strongly leaning to some opinion, which suits his inclination, or which is sanctioned by some authority that he blindly venerates, and then studies with the utmost diligence, not as an Investigator of Truth, but as an Advocate labouring to prove his point, his talents and his researches, whatever effect they may produce in making converts to his notions, will avail nothing in enlightening his own judgment and securing him from error.

Composition however, of the Argumentative kind, may be considered (as has been above stated) as coming under the province of Rhetoric. And this view of the subject is the less open to objection, inasmuch as it is not likely to lead to discussions that can be deemed superfluous, even by those who may choose to consider Rhetoric in the most restricted sense, as relating only to "Persuasive Speaking;" since it is evident that *Argument* must be, in most cases at least, the basis of Persuasion.

We propose then, to treat first, and principally, of the Discovery of Arguments, and of their Arrangement; secondly, to lay down some Rules respecting the excitement and management of the Passions, with a view to the attainment of any object proposed,—principally, Persuasion in the strict sense, i. e. the influencing of the Will; thirdly, to offer some remarks on Style; and fourthly, to treat of Elocution.

It may be expected that before we proceed to treat of the Art in question, we should present our readers with a sketch of its history. Little however is required to be said on this head, because the present is not one of those branches of study in which we can trace with interest a progressive improvement from age to age. It is one, on the contrary, to which more attention appears to have been paid, and in which greater proficiency is supposed to have been made, in the earliest days of Science and Literature, than at any subsequent period. Among the ancients, Aristotle, who was the earliest, may safely be pronounced to be also the best, of the systematic writers on Rhetoric. Cicero is hardly to be reckoned among the number; for he delighted so much more in the practice than in the theory of his art, that he is perpetually drawn off from the rigid Philosophical analysis of its principles, into discursive declamations, always eloquent indeed, and often highly interesting, but adverse to regularity of system, and frequently as unsatisfactory to the practical student as to the Philosopher. He abounds indeed with excellent practical remarks, though the best of them are scattered up and down his works with much irregularity; but his precepts, though of great weight, as being the result of experience, are

not often traced up by him to first principles; and we are frequently left to guess, not only on what basis his rules are grounded, but in what cases they are applicable. Of this latter defect a remarkable instance will be hereafter cited.

Quinctilian is indeed a systematic writer; but cannot be considered as having much extended the Philosophical views of his predecessors in this department. He possessed much good sense, but this was tinctured with pedantry;—with that ἀλαζονεία as Aristotle calls it, which extends to an extravagant degree the province of the Art which he professes. A great part of his work indeed is a Treatise on education generally, in the conduct of which he was no mean proficient; for such was the importance attached to public Speaking, even long after the downfall of the Republic had cut off the Orator from the hopes of attaining, through the means of this qualification, the highest political importance, that he who was nominally a Professor of Rhetoric, had in fact the most important branches of instruction intrusted to his care.

Many valuable maxims however are to be found in this author; but he wanted the profundity of thought, and power of analysis which Aristotle possessed.

The writers on Rhetoric among the ancients whose works are lost, seem to have been numerous; but most of them appear to have confined themselves to a very narrow view of the subject; and to have been occupied, as Aristotle complains, with the minor details of style and arrangement, and with the sophistical tricks and petty artifices of the Pleader, instead of giving a masterly and comprehensive sketch of the essentials.

Among the moderns, few writers of ability have turned their thoughts to the subject; and but little has been added, either in respect of matter, or of system, to what the ancients have left us. It were most unjust however to leave unnoticed Dr. Campbell's *Philosophy of Rhetoric:* a work which does not enjoy indeed so high a degree of popular favour as Dr. Blair's, but is incomparably superior to it, not only in depth of thought and ingenious original research, but also in practical utility to the student. The title of Dr. Campbell's work has perhaps deterred many readers, who had concluded it to be more abstruse and less popular in its character than it really is. Amidst much however that is readily understood by any moderately intelligent reader, there is much also that calls for some exertion of thought, which the indolence of most readers refuses to bestow. And it must be owned that he also in some instances perplexes his readers by being perplexed himself, and bewildered in the discussion of questions through which he does not clearly see his way. His great defect, which not only leads him into occasional errors, but leaves many of his best ideas but imperfectly developed, is his ignorance and utter misconception of the nature and object of Logic, on which some remarks were made in our article on that Science. Rhetoric being in truth an off-shoot of Logic, that Rhetorician must labour under great disadvantages who is not only ill-acquainted with that system, but also utterly unconscious of his deficiency.

From a general view of the history of Rhetoric, two questions naturally suggest themselves, which on examination will be found very closely connected together: 1st, what is the cause of the careful and extensive cultivation, among the ancients, of an Art which the moderns have comparatively neglected; and 2dly, whether the former or the latter are to be regarded as the wiser in this respect;—in other words, whether Rhetoric be worth any diligent cultivation.

With regard to the first of these questions, the answer generally given is that the nature of the Government in the ancient democratical States caused a demand for public speakers, and for such speakers as should be able to gain influence not only with educated persons in dispassionate deliberation, but with a promiscuous multitude; and accordingly it is remarked, that the extinction of liberty brought with it, or at least brought after it, the decline of Eloquence; as is justly remarked (though in a courtly form) by the author of the dialogue on Oratory, which passes under the name of Tacitus: "*Quid enim opus est longis in Senatu sententiis, cum optimi cito consentiant? quid, multis apud populum concionibus, cum de Republica non imperiti et multi deliberent, sed sapientissimus, et unus?*"

This account of the matter is undoubtedly correct as far as it goes; but the importance of public speaking is so great, in our own, and all other countries that are not under a despotic Government, that the apparent neglect of the study of Rhetoric seems to require some further explanation. Part of this explanation may be supplied by the consideration, that the difference in this respect between the ancients and ourselves, is not so great in reality as in appearance. When the only way of addressing the public was by orations, and when all political measures were debated in popular assemblies, the characters of Orator, Author, and Politician, almost entirely coincided; he who would communicate his ideas to the world, or would gain political power, and carry his legislative schemes into effect, was necessarily a Speaker; since as Pericles is made to remark by Thucydides, "one who forms a judgment on any point, but cannot explain himself clearly to the people, might as well have never thought at all on the subject."* The consequence was, that almost all who sought, and all who professed to give, instruction, in the principles of Government, and the conduct of judicial proceedings, combined these, in their minds and in their practice, with the study of Rhetoric, which was necessary to give effect to all such attainments; and in time the Rhetorical writers (of whom Aristotle makes that complaint) came to consider the Science of Legislation and of Politics in general, as a part of their own Art.

Much therefore of what was formerly studied under the name of Rhetoric is still, under other names, as generally and as diligently studied as ever.

It cannot be denied however that a great difference, though less, as we have said, than might at first sight appear, does exist between the ancients and the moderns in this point;—that what is strictly and properly called Rhetoric, is much less studied, at least less systematically studied, now, than formerly. Perhaps this also may be in some measure accounted for from the circumstances which have been just noticed. Such is the distrust excited by any suspicion of Rhetorical artifice, that every speaker or writer who is anxious to carry his point, endeavours to disown or to keep out of sight, any superiority of

* Thucydides, book ii.

skill; and wishes to be considered as relying rather on the strength of his cause, and the soundness of his views, than on his ingenuity and expertness as an advocate. Hence it is, that even those who have paid the greatest and the most successful attention to the study of Composition and of Elocution, are so far from encouraging others by example or recommendation to engage in the same pursuit, that they labour rather to conceal and disavow their own proficiency; and thus, theoretical rules are decried, even by those who owe the most to them. Whereas among the ancients, the same cause, did not, for the reasons lately mentioned, operate to the same extent; since, however careful any speaker might be to disown the artifices of Rhetoric properly so called, he would not be ashamed to acknowledge himself, generally, a student, or a proficient in an Art which was understood to include the elements of Political wisdom.

With regard to the other question proposed, viz. concerning the utility of Rhetoric, it is to be observed that it divides itself into two; 1st, whether Oratorical skill be, on the whole a public benefit, or evil; and 2ndly, whether any artificial System of Rules is conducive to the attainment of that skill. The former of these questions was eagerly debated among the ancients; on the latter but little doubt seems to have existed. With us, on the contrary, the state of these questions seems nearly reversed. It seems generally admitted that skill in Composition and in Speaking, liable as it evidently is, to abuse, is to be considered, on the whole, as advantageous to the public; because that liability to abuse is neither in this, nor in any other case, to be considered as conclusive against the utility of any kind of art, faculty, or profession;—because the evil effects of misdirected power, require that equal powers should be arrayed on the opposite side;—and because truth having an intrinsic superiority over falsehood, may be expected to prevail when the skill of the contending parties is equal; which will be the more likely to take place, the more widely such skill is diffused. But many, perhaps most persons, are inclined to the opinion that Eloquence either in writing or speaking, is either a natural gift, or at least, is to be acquired only by practice, and is not to be attained or improved by any system of rules. And this opinion is favoured not least by those (as has been just observed) whose own experience would enable them to decide very differently; and it certainly seems to be in a great degree practically adopted. Most persons, if not left entirely to the disposal of chance in respect of this branch of education, are at least left to acquire what they can by *practice*, such as school or college exercises afford, without much care being taken to initiate them systematically into the principles of the Art; and that, frequently, not so much from negligence in the conductors of education, as from their doubts of the utility of any such regular system.

It certainly must be admitted, that rules not constructed on broad Philosophical principles, are more likely to cramp, than to assist the operations of our faculties;—that a pedantic display of technical skill is more detrimental in this than in any other pursuit, since by exciting distrust, it counteracts the very purpose of it;—that a system of rules imperfectly comprehended, or not familiarized by practice, will, (while that continues to be the case,) prove rather an impediment than a help; as indeed will be found in all other Arts likewise;—and that no system can be expected to equalise men whose natural powers are different: but none of these concessions at all invalidate the positions of Aristotle; that some succeed better than others in explaining their opinions, and bringing over others to them; and that, not merely by superiority of natural gifts, but by acquired habit; and that consequently if we can discover the causes of this superior success,—the means by which the desired end is attained by all who *do* attain it,—we shall be in possession of rules capable of general application: ὅπερ ἐστὶ, says he, τεχνῆς ἔργον.* Experience so plainly evinces, what indeed we might naturally be led antecedently to conjecture, that a right judgment on any subject is not necessarily accompanied by skill in effecting conviction,—nor the ability to discover truth, by a facility in explaining it,—that it might be matter of wonder how any doubt should ever have existed as to the possibility of devising, and the utility of employing, a System of Rules for "Argumentative Composition," generally, distinct from any system conversant about the subject-matter of each composition. It is probable that the existing prejudices on this subject may be traced in great measure to the imperfect or incorrect notions of some writers, who have either confined their attention to trifling minutiæ of style, or at least have in some respect failed to take a sufficiently comprehensive view of the principles of the Art. One distinction especially is to be clearly laid down and carefully borne in mind by those who would form a correct idea of those principles; viz. the distinction already noticed under the article Logic, between *an* Art, and *the* Art. "*An* Art of Reasoning" would imply, "a System of Rules by the observance of which one may Reason correctly;" "*the* Art of Reasoning" would imply a System of Rules to which every one *does* conform, (whether knowingly, or not) who reasons correctly: and such is Logic, considered as an Art. In like manner "*an* Art of Composition" would imply "a System of Rules by which a good Composition may be produced;" "*the* Art of Composition,"—"such rules as *every* good Composition must conform to," whether the author of it had them in his mind or not. Of the former character appear to have been (among others) many of the Logical and Rhetorical Systems of Aristotle's predecessors in those departments: he himself evidently takes the other and more Philosophical view of both branches: as appears (in the case of Rhetoric) both from the plan he sets out with, that of investigating the causes of the success of all who do succeed in effecting conviction, and from several passages occurring in various parts of his Treatise, which indicate how sedulously he was on his guard to conform to that plan. Those who have not attended to the important distinction just alluded to, are often disposed to feel wonder, if not weariness, at his reiterated remarks, that "*all* men effect persuasion either in this way or in that;" "it is *impossible* to attain such and such an object in any other way;" &c. which doubtless were intended to remind his readers of the nature of his design; viz. not, to teach *an* Art of Rhetoric but *the* Art;—not to instruct them

* *Rhet.* book i. ch. i.

merely how conviction *might* be produced but how it *must*.

If this distinction were carefully kept in view by the teacher and by the learner of Rhetoric, we should no longer hear complaints of the natural powers being fettered by the formalities of a System; since no such complaint can lie against a System whose Rules are drawn from the invariable practice of all who succeed in attaining their proposed object. No one would expect that the study of Sir Joshua Reynolds's lectures, would cramp the genius of the painter. No one complains of the Rules of Grammar as fettering Language; because it is understood that correct use is not founded on Grammar, but Grammar upon correct use. A just system of Logic or of Rhetoric, is analogous, in this respect, to Grammar.

CHAPTER I.

OF THE INVENTION, ARRANGEMENT, AND INTRODUCTION OF ARGUMENTS.

It has been formerly remarked in our Treatise on Logic, that in the process of *Investigation* properly so called, viz. that by which we endeavour to discover Truth, it must of course be uncertain to him who is entering on that process, what the conclusion will be, to which his researches will lead; but that in the process of *conveying truth* to others by reasoning, (i. e. that which according to the view we have at present taken, may be termed the Rhetorical process,) the conclusion or conclusions which are to be established must be present to the mind of him who is conducting the Argument, and whose business is to find *Proofs* of a given proposition.

It is evident therefore, that the first step to be taken by him, is, to lay down distinctly in his own mind, the proposition or propositions to be proved. It might indeed at first sight appear superfluous even to mention so obvious a rule; but experience shows that it is by no means uncommon for a young or ill-instructed writer to content himself with such a vague and indistinct view of the point he is to aim at, that the whole train of his reasoning is in consequence affected with a corresponding perplexity, obscurity, and looseness. It may be worth while therefore to give some hints for the conduct of this preliminary process,—the choice of propositions. Not, of course that we are supposing the author to be in doubt what opinion he shall adopt: the process of Investigation (which does not fall within the province of Rhetoric) being supposed to be concluded; but still there will often be room for deliberation as to the form in which an opinion shall be stated, and, when several propositions are to be maintained, in what order they shall be placed.

On this head therefore we shall proceed to propose some rules; after having premised (in order to anticipate some objections or doubts which might arise) one remark relative to the object to be effected. This is of course, what may be called, in the widest sense of the word, Conviction; but under that term are comprehended 1st, what is strictly called *Instruction*; and 2ndly, *Conviction* in the narrower sense; i. e. the Conviction of those who are either of a *contrary opinion* to the one maintained, or who are *in doubt* whether to admit or deny it. By Instruction on the other hand, is commonly meant the Conviction of those who have neither formed an opinion on the subject, nor are deliberating whether to adopt or reject the proposition in question, but are merely desirous of ascertaining *what* is the truth in respect of the case before them. The former are supposed to have before their minds the terms of the proposition maintained, and are called upon to consider *whether that particular proposition* be true or false; the latter are not supposed to know the terms of the conclusion, but to be inquiring *what proposition* is to be received as true. It is evident that the speaker or writer is, relatively to these last, (though not to himself,) conducting a process of Investigation; as is plain from what has been said of that subject, in the article Logic.

The distinction between these two objects gives rise in some points to corresponding differences in the mode of procedure, which will be noticed hereafter; these differences however are not sufficient to require that Rhetoric should on that account be divided into two distinct branches, since, generally speaking, though not universally, the same rules will be serviceable for attaining each of these objects.

§ 1. The first step is, as we have observed, to lay down, (in the author's mind,) the proposition or propositions to be maintained, clearly, and in a suitable form. He who makes a point of observing this rule, and who is thus brought to view steadily the point he is aiming at, will be kept clear, in a great degree, of some common faults of young writers; viz. entering on too wide a field of discussion, and introducing many propositions not sufficiently connected; an error which destroys the unity of the composition. This last error those are apt to fall into, who place before themselves a *Term* instead of a *Proposition*; and imagine that because they are *treating of one thing*, they are *discussing one question*. In an Ethical work, for instance, one may be *treating of virtue*, while discussing all or any of these questions; "Wherein virtue consists?" "Whence our notions of it arise?" "Whence it derives its obligation?" &c., but if these questions were confusedly blended together, or if all of them were treated of within a short compass, the most just remarks and forcible arguments would lose their interest and their utility in so perplexed a composition.

Nearly akin to this fault, is the other just mentioned, that of entering on too wide a field for the length of the work; by which means the writer is confined to barren and uninteresting generalities; as e. g. in general exhortations to virtue, (conveyed, of course, in very general terms,) in the space of a discourse only of sufficient length to give a characteristic description of some one branch of duty, or

of some one particular motive to the practice of it. Unpractised composers are apt to fancy that they shall have the greater abundance of matter, the wider extent of subject they comprehend; but experience shows that the reverse is the fact: the more general and extensive view will often suggest nothing to the mind but vague and trite remarks, when upon narrowing the field of discussion, many interesting questions of detail present themselves. Now a writer who is accustomed to state to himself precisely, in the first instance, the conclusions to which he is tending, will be the less likely to content himself with such as consist of very general statements; and will often be led, even where an extensive view is at first proposed, to distribute it into several branches, and waiving the discussion of the rest, to limit himself to the full developement of one or two; and thus applying, as it were, a microscope to a small space, will present to the view much that a wider survey would not have exhibited.

It may be useful, for one who is about thus to lay down his propositions, to ask himself these three questions: 1st, What is the fact? 2ndly, Why (i. e. from what Cause) is it so; or, in other words, how is it accounted for? and 3rdly, What Consequence results from it?

The last two of these questions, though they will not in every case suggest such answers, as are strictly to be called the Cause and the Consequence of the principal truth to be maintained, may, at least, often furnish such propositions as bear a somewhat similar relation to it.

It is to be observed that in recommending the writer to begin by laying down in his own mind the propositions to be maintained, it is not meant to be implied that they are always to be *stated* first; that will depend upon the nature of the case, and rules will hereafter be given on that point.

It is to be observed also, that by the words "Proposition" or "Assertion," throughout this Treatise, is to be understood some *conclusion* to be established *for itself*; not with a view to an ulterior conclusion: those propositions which are intended to serve as premises, being called, in allowable conformity with popular usage, *Arguments*; it being customary to argue in the enthymematic form, and to call, for brevity's sake, the expressed premiss of an enthymeme, the argument by which the conclusion of it is proved.

Of Arguments.

§ 2. Arguments are divided according to several different principles; i. e. logically speaking, there are *several divisions* of them. And these *cross-divisions* have proved a source of endless perplexity to the Logical and Rhetorical student, because the writers on those subjects have not been aware of them. Hardly any thing perhaps has contributed so much to lessen the interest and the utility of systems of Rhetoric, as the indistinctness hence resulting. When in any subject the members of a division are not opposed, but are in fact members of different divisions crossing each other, it is manifestly impossible to obtain any clear notion of the species treated of; nor will any labour or ingenuity bestowed on the subject be of the least avail, till the original source of perplexity is removed;—till, in short, the cross-division is detected and explained.

Arguments then may be divided,

1st, Into Irregular, and Regular, i. e. Syllogisms; these last into Categorical and Hypothetical; and the former into Syllogisms in the first Figure, and in the other figures, &c. &c.

2ndly, They are frequently divided into "Moral," (or "Probable,") and "Demonstrative," (or "Necessary.")

3rdly, Into "Direct" and "Indirect," (or *reductio ad absurdum*,) the Deictic and Elenctic of Aristotle.

4thly, Into Arguments from "Example," from "Testimony," from "Cause to Effect," from "Analogy," &c. &c.

It will be perceived on attentive examination, that several of the different species just mentioned will occasionally *contain* each other; e. g. a probable Argument may be at the same time a Categorical Argument, a Direct Argument, and an Argument from Testimony, &c.; this being the consequence of Arguments having been divided on several different principles; a circumstance so obvious the moment it is distinctly stated, that we apprehend such of our readers as have not been conversant in these studies, will hardly be disposed to believe that it could have been (as is the fact) generally overlooked, and that eminent writers should in consequence have been involved in inextricable confusion. We need only remind them however of the anecdote of Columbus breaking the egg; that which is perfectly obvious to any man of common sense, as soon as it is mentioned, may nevertheless fail to occur, even to men of considerable ingenuity.

It will also be readily perceived, on examining the principles of these several divisions, that the last of them alone is properly and strictly a division of *Arguments as such*. The 1st is evidently a division of the *Forms of stating them*; for every one would allow that the *same* Argument may be either stated as an enthymeme, or brought into the strict syllogistic form; and that, either categorically or hypothetically, &c., e. g. "Whatever has a beginning has a cause; the earth had a beginning, therefore it had a cause;" or, "*If* the earth had a beginning it had a cause: it had a beginning," &c. every one would call the *same* Argument, differently stated. This, therefore, evidently is not a division of Arguments *as such*.

The 2nd is plainly a division of Arguments according to their *subject-matter*, whether Necessary or Probable, certain or uncertain. In Mathematics, e. g. every proposition that can be stated is either an immutable truth, or an absurdity and contradiction; while in human affairs the propositions which we assume are only true for the most part, and as general rules; and in Physics, though they must be true as long as the laws of nature remain undisturbed, the contradiction of them does not imply an absurdity; and the conclusions of course, in each case, have the same degree and kind of certainty with the premises. This, therefore, is properly a division, not of *Arguments* as such, but of the *Propositions* of which they consist.

The 3rd is a division of Arguments according to the purpose for which they are employed;—according to the *intention* of the reasoner; whether that be to establish "directly" (or "ostensively") the conclusion drawn, or ("indirectly") by means of an absurd conclusion to disprove one of the premises: (i. e. to prove its contradictory) since the alternative proposed in every valid Argument is, either to admit the conclusion, or to deny one of the premises. Now it may so happen

that in some cases, one person will choose the former, and another the latter, of these alternatives. It is probable, e. g. that many have been induced to admit the doctrine of Transubstantiation, from its clear connection with the infallibility of the Romish Church; and many others, by the very same Argument, have surrendered their belief in that infallibility. Again, Berkley and Reid seem to have alike admitted that the non-existence of matter was a necessary consequence of Locke's Theory of Ideas; but the former was hence led, *bonâ fide*, to admit and advocate that non-existence, while the latter was led by the very same Argument to reject the Ideal Theory. Thus, we see it is possible for the very same Argument to be Direct to one person, and Indirect to another; leading them to different results, according as they judge the original conclusion, or the contradictory of a premiss, to be the more probable. This, therefore, is not properly a division of Arguments as such, but a division of the purposes for which they are employed.

The 4th, which alone is properly a division of Arguments as such, and accordingly will be principally treated of, is a division according to the "relation of the subject-matter of the premises to that of the conclusion." We say, "of the subject-matter," because the *logical* connection between the premises and conclusion is independent of the meaning of the terms employed, and may be exhibited with letters of the alphabet substituted for the terms; but the relation we are now speaking of between the premises and conclusion, (and the varieties of which form the several species of Arguments,) is in respect of their subject-matter; as e. g. an "Argument from Cause to Effect" is so called and considered, in reference to the relation existing between the premiss, which is the Cause, and the conclusion, which is the Effect; and an "Argument from Example," in like manner, from the relation between a known and an unknown instance, both belonging to the same class. And it is plain that the present division, though it has a reference to the subject-matter of the premises, is yet not a division of *propositions* considered by themselves, (as in the case with the division into probable and demonstrative,) but of *Arguments* considered as such; for when we say, e. g. that the premiss is a Cause, and the conclusion the Effect, these expressions are evidently relative, and have no meaning, except in reference to each other; and so also when we say that the premiss and the conclusion are two *parallel* cases, that very expression denotes their relation to each other.

In distributing, then, the several kinds of Arguments, according to this division, it will be found convenient to lay down first two great classes, under one or other of which all can be brought; viz. 1st, such Arguments as might have been employed to *account for* the fact or principle maintained, supposing its truth granted; 2nd, such as could *not* be so employed. The former class (to which in this Treatise, the name of "*A priori*" Argument will be confined,) is manifestly Argument from *Cause* to Effect; since to *account* for any thing, signifies to assign the Cause of it. The other class, of course, comprehends all other Arguments, of which there are several kinds, which will be mentioned hereafter.

The two sorts of proof which have been just spoken of, Aristotle seems to have intended to designate by the titles of ὅτι for the latter, and διότι for the former; but be has not been so clear as could be wished, in observing the distinction between them. The only decisive test by which to distinguish the Arguments which belong to the one, and to the other of these classes is, to ask the question, "Supposing the proposition in question to be admitted, would this Argument serve to *account* for the truth, or not?" It will then be readily referred to the former or to the latter class, according as the answer is in the affirmative or the negative, as, e. g. if a murder were imputed to any one on the grounds of his "having a hatred to the deceased, and an interest in his death," the Argument would belong to the former class; because, *supposing* his guilt to be *admitted*, and an inquiry to be made how he came to commit the murder, the circumstances just mentioned would serve to *account* for it; but not so, with respect to such an Argument as his "having blood on his clothes;" which would therefore be referred to the other class.

And here let it be observed, once for all, that when we speak of arguing from Cause to Effect, it is not intended to maintain the real and proper efficacy of what are called Physical Causes to produce their respective Effects, nor to enter into any discussion of the controversies which have been raised on that point, which would be foreign from the present purpose. The word "Cause," therefore, is to be understood as employed in the popular sense; as well as the phrase of "accounting for" any fact.

As far, then, as any Cause, popularly speaking, has a tendency to produce a certain Effect, so far its existence is an Argument for that of the Effect. If the Cause be fully *sufficient*, and no *impediments* intervene, the Effect in question follows certainly; and the nearer we approach to this, the stronger the Argument.

This is the kind of Argument which produces, (when short of absolute certainty,) that species of the Probable which is usually called the Plausible. On this subject Dr. Campbell has some valuable remarks in his *Philosophy of Rhetoric* (book i. §5. ch. vii.) though he has been led into a good deal of perplexity, partly by not having logically analyzed the two species of probabilities he is treating of, and partly by departing, unnecessarily, from the ordinary use of terms, in treating of the Plausible as something *distinct from* the Probable, instead of regarding it as a *species* of Probability.

This is the only kind of Probability which poets, or other writers of fiction, aim at; and in such works it is often designated by the term "natural." Writers of this class, as they aim not at producing belief, are allowed to take their "Causes" for granted, (i. e. to assume any hypothesis they please,) provided they make the Effects follow naturally; representing, that is, the personages of the fiction as acting, and the events as resulting, in the same manner as might have been expected, supposing the assumed circumstances to have been real. And hence, the great Father of Criticism establishes his paradoxical maxim, that impossibilities which appear probable, are to be preferred to possibilities which appear improbable. For, as he justly observes, the impossibility of the hypothesis, as e. g. in Homer, the familiar intercourse of God with mortals, is no bar to the kind of Pro-

bability required, if those mortals are represented as acting in the manner men naturally would have done under those circumstances.

The Probability, then, which the writer of fiction aims at, has, for the reason just mentioned, no tendency to produce a *particular*, but only a *general* belief; i. e. not that these particular events actually took place, but that *such* are likely, generally, to take place under such circumstances.* In Argumentative Compositions however, as the object of course is to produce conviction as to the particular point in question, the Causes from which our Arguments are drawn, must be such as are either admitted, or may be proved, to be actually existing, or likely to exist.

On the appropriate use of this kind of Argument, (which is probably the εἰκὸς of Aristotle, though unfortunately he has not furnished any example of it,) some Rules will be laid down hereafter; our object at present having been merely to ascertain the nature of it. And here it may be worth while to remark, that though we have applied to this mode of Reasoning the title of "*a priori*," it is not meant to be maintained that all such Arguments as have been by other writers so designated, correspond precisely with what has been just described † The phrase, "*a priori*" Argument, is not, indeed, employed by all in the same sense; it would however generally be understood to extend to any argument drawn from an *antecedent* or *forerunner*, whether a *Cause* or not; e. g. "the mercury sinks, therefore it will rain." Now this Argument being drawn from a circumstance which though an antecedent, is in no sense a Cause, would fall not under the former, but the latter, of the classes laid down, since when rain comes, no one would *account* for the phenomenon by the falling of the mercury; and yet most, perhaps, would class this among "*a priori*" Arguments. In like manner the expression, "*a posteriori*" Arguments, would not in its ordinary use, coincide precisely, though it would, very nearly, with the second class of Arguments. The division, however, which has bere been adopted, appears to be both more Philosophical, and also more precise, and consequently more practically useful than any other; since there is so easy and decisive a test by which an Argument may be at once referred to the one or to the other of the classes described.

The second, then, of these classes, (viz. "Arguments which could not be used to account for the fact in question, supposing it granted,") may be sub-divided into two kinds; which will be designated by the terms "Sign" and "Example."

By "Sign," (so called from the Σημεῖον of Aristotle,) is meant a species of Argument of which the analysis is as follows: As far as any circumstance is, what may be called, a *Condition* of the existence of a certain effect or phenomenon, so far it may be inferred from the existence of that Effect: if it be a Condition *absolutely essential*, the Argument is, of course, demonstrative; and the Probability is the stronger in proportion as we approach to that case. Of this kind is the Argument in the instance lately given: a man is suspected as the perpetrator of the supposed murder, from the circumstance of his clothes being bloody; the murder being considered as in a certain degree a probable *condition* of that appearance; i. e. it is presumed that his clothes would *not otherwise* have been bloody. Again, from the appearance of ice, we infer, decidedly, the existence of a temperature below freezing point, that temperature being an essential Condition of the crystallization of water.

Among the circumstances which are conditional to any Effect, must evidently come the Cause or Causes; and if there be only one possible Cause, this being absolutely essential, may be demonstratively proved from the Effect: if the same Effect might result from other Causes, then the Argument is, at best, but probable. But it is to be observed, that there are also many circumstances which have no tendency to *produce* a certain Effect, though it cannot exist *without* them, and from which Effect, consequently, they may be inferred, as Conditions, though not Causes; e. g. a man's "being alive one day," is a circumstance necessary, as a Condition, to his "dying the next;" but has no tendency to produce it: his having been alive, therefore, on the former day, may be proved from his subsequent death, but not *vice versâ*.*

It is to be observed therefore, that though it is very common for the Cause to be proved from its Effect, it is never so proved, *so far forth as* [ᾗ] it is a *Cause*, but so far forth as it is a *condition*, or necessary circumstance.

A Cause, again, may be employed to prove an Effect, (this being the first class of Arguments already described,) so far as it *has a tendency* to produce the Effect, even though it be not at all *necessary* to it; (i. e. when other Causes may produce the same Effect,) and in this case, though the Effect may be inferred from the Cause, the Cause cannot be inferred from the Effect; e. g. from a mortal wound you may infer death, but not *vice versâ*.

Lastly, when a Cause is also a necessary or probable *condition*, i. e. when it is the *only* possible or likely Cause, then we may argue both ways; e. g. we may infer a General's success from his known skill, or, his skill, from his known success: these two

* On which ground Aristotle contends that the end of Fiction is more Philosophical than that of History, since it aims at general, instead of particular Truth.

† Some Rhetorical students accordingly, partly with a view to keep clear of any ambiguity that might hence arise, and partly for the sake of brevity, have found it useful to adopt, in drawing up an outline, or analysis of any composition, certain arbitrary symbols, to denote, respectively, each class of Arguments and of Propositions; viz. A, for the former of the two classes of Arguments just described, (to denote "*A priori*," or "Antecedent," probability,) and B, for the latter, which, as consisting of several different kinds, may be denominated "the *Body* of evidence." Again, they designate the proposition, which *accounts for* the principal and original assertion, by a small "a," or Greek α, to denote its identity *in substance* with the *Argument* bearing the symbol "A," though employed for a different purpose; viz. not to *establish* a fact that is doubtful, but to *account for* one that is admitted. The proposition, again, which results as a Consequence or Corollary from the principal one, they designate by the symbol C. There seems to be the same convenience in the use of these symbols as Logicians have found in the employment of A, E, I, O, to represent the four kinds of Propositions according to quantity and quality.

* It is however very common, in the carelessness of common language, to mention, as the Causes of phenomena, circumstances which every one would allow, on consideration, to be not Causes, but only Conditions, of the Effects in question; e. g. it would be said of a tender plant, that it was destroyed in consequence of not being covered with a mat; though every one would mean to imply that the *frost* destroyed it; this being a Cause too well known to need being mentioned; and that which is spoken of as the Cause, viz. the absence of a covering, being only the Condition, without which the real Cause could not have operated.

Rhetoric. **Arguments belonging, respectively, to the two classes** originally laid down. And it is to be observed that, in such Arguments from Sign as this last, the conclusion which *follows, logically*, from the premiss, being the Cause from which the premiss *follows, physically*, i. e. as a natural Effect, there are in this case two different kinds of *Sequence* opposed to each other. In Arguments of the first class, on the contrary, these two kinds of Sequence are combined; i. e. the Conclusion which follows logically from the premiss, is also the Effect following physically from it as a Cause; a General's skill, e. g. being both the Cause and the Proof of his being likely to succeed.

It is most important to keep in mind the distinction between these two kinds of Sequence, which are, in Argument, sometimes *combined*, and sometimes *opposed*. There is no more fruitful source of confusion of thought than that ambiguity of language employed on these subjects, which tends to confound together these two things, so entirely distinct in their nature. There is hardly any argumentative writer on subjects involving a discussion of the Causes or Effects of any thing, who has clearly perceived and steadily kept in view the distinction we have been speaking of, or who has escaped the errors and perplexities thence resulting. The wide extent accordingly, and the importance of the mistakes and difficulties arising out of the ambiguity complained of, is incalculable. To dilate upon this point as fully as might be done with advantage, would lead us beyond our present limits; but it will not be foreign to the purpose of this article to offer some remarks on the origin of the ambiguity complained of, and on the cautions to be used in guarding against being misled by it.

The premiss by which any thing is proved, is not necessarily the Cause of the fact's *being such* as it is; but it is the Cause of our *knowing* and being convinced that it is so; e. g. the wetness of the earth is not the Cause of rain, but it is the Cause of our knowing that it has rained. These two things, the premiss which produces *our conviction*, and the Cause which produces that *of which* we are convinced, are the more likely to be confounded together, in the looseness of colloquial language, from the circumstance that (as has been above remarked) they frequently coincide; as, e. g. when we *infer* that the ground will be wet, from the fall of rain which *produces* that wetness. And hence it is that the same words have come to be applied, in common, to each kind of Sequence; e. g. an Effect is said to "follow" from a Cause, and a Conclusion to "follow" from the premises; the words "Cause" and "Reason," are each applied indifferently, both to a Cause, properly so called, and to the premiss of an Argument; though "Reason," in strictness of speaking, should be confined to the latter. "Therefore," "hence," "consequently," &c., and also, "since," "because," and "why," have likewise a corresponding ambiguity. The multitude of the words which bear this double meaning, (and that, in all languages,) greatly increases our liability to be misled by it; since thus the very means men resort to for ascertaining the sense of any expression, are infected with the very same ambiguity; e. g. if we inquire what is meant by a "Cause," we shall be told that it is that from which something "follows;" or, which is denoted by the words "therefore," "consequently," &c., all which expressions are as equivocal and uncertain in their signification as the original one. It is in vain to attempt ascertaining by the balance the true amount of any commodity, if false weights are placed in the opposite scale. Hence it is that so many writers, in investigating the Cause to which any fact or phenomenon is to be attributed, have assigned that which is not a *Cause*, but only a *Proof* that the fact is so; and have thus been led into an endless train of errors and perplexities.

Several, however, of the words in question, though employed indiscriminately in both significations, seem (as was observed in the case of the word "Reason,") in their primary and strict sense, to be confined to one, "δή," in Greek, and "ergo,"* or "itaque," in Latin, seem originally and properly to denote the Sequence of Effect from Cause; "ἄρα,"† and "igitur," that of conclusion from premises. The English word "accordingly," will generally be found to correspond with the Latin "itaque."

The interrogative "why," is employed to inquire, either, 1st, the "Reason," (or "Proof;") 2ndly, the "Cause;" or 3rdly, the "object proposed," or final Cause: e. g. 1st, Why are the angles of a triangle equal to two right angles? 2nd, Why are the days shorter in winter than in summer? 3rd, Why are the works of a watch constructed as they are? If any one were to ask "Why the Gospel-revelation is to be received?" he might intend by this question any one of these three inquiries; which would of course require very different answers.

It is to be observed that the discovery of *Causes* belongs properly to the province of the Philosopher; that of "Reasons," strictly so called, (i. e. Arguments) to that of the Rhetorician; and that, though each will have frequent occasion to assume the character of the other, it is most important that these two objects should not be confounded together.

Of Signs then one kind are such as from a certain Effect or phenomenon, infer the "Cause" of it; and the other, such as, in like manner, infer some "Condition" which is not the Cause. Of these last, one species is the Argument from Testimony; the premiss being the existence of the Testimony, the Conclusion, the truth of what is attested; which is considered as a "Condition" of the Testimony having been given; since it is evident that so far only as this is allowed, (i. e. so far only as it is allowed that the Testimony would not have been given, had it not been true,) can this Argument have any force.

Testimony is of various kinds; but the distinction between them is so obvious, as well as the various circumstances which add to, or diminish the weight of any Testimony, that it is not necessary to enter into any detailed discussion of the subject. It may be worth remarking, however, that one of the most important distinctions is between Testimony to matters of *Fact*, and to *Doctrines* or Opinions: in estimating the weight of the former, we look chiefly to the *honesty* of the witness, and his means of obtaining information: in the latter, his *ability* to judge is equally to be taken

* Most Logical writers seem not to be aware of this, as they generally, in Latin Treatises, employ "ergo" in the other sense; it is from the Greek ἔργῳ, i. e. "in fact."

† Ἄρα having a signification of *fitness or coincidence*; whence ἄρω.

into consideration. With respect however to the credibility of witnesses, it is evident that when many coincide in their testimony, (where no previous concert can have taken place,) the probability resulting from this concurrence does not rest on the supposed veracity of each considered separately, but on the improbability of such an agreement taking place by chance. For though in such a case each of the witnesses should be considered as unworthy of credit, and even much more likely to speak falsehood than truth, still the chances might be infinite against their all agreeing in the *same* falsehood. This remark is applied by Dr. Campbell to the Argument from Testimony; but he might have extended it to other Arguments also, in which a similar calculation of chances will enable us to draw a Conclusion, sometimes even amounting to moral certainty, from a combination of data which singly would have had little or no weight; e. g. if any one out of a hundred men throw a stone which strikes a certain object, there is but a slight probability, from that fact alone, that he aimed at that object; but if all the hundred threw stones which struck the same object, no one would doubt that they aimed at it. It is from such a combination of Argument that we infer the existence of an intelligent Creator from the marks of contrivance visible in the Universe, though many of these are such as, taken singly, might well be conceived undesigned and accidental; but that they should *all* be such, is morally impossible. Great care is requisite in setting forth clearly, especially in any popular discourse, Arguments of this nature; the generality of men being better qualified for understanding, (to use Lord Bacon's words,) "particulars, one by one," than for taking a comprehensive view of a whole; and therefore in a *Galaxy* of Evidence, as it may be called, in which the brilliancy of no single star can be pointed out, the lustre of the combination is often lost on them. Hence it is, as was remarked in the Treatise on Fallacies, that the sophism of "Composition," as it is called, so frequently misleads men: it is not improbable, (in the above example,) that *each* of the stones, considered *separately*, may have been thrown at random; and therefore the same is concluded of *all*, considered in *conjunction*. Not that in such an instance as the above, any one would reason so weakly; but that a still greater absurdity of the very same kind is involved in the rejection of the evidences of our religion, will be plain to any one who considers, not merely the individual force, but the *number* and *variety* of those evidences.

And here it may be observed, that though the easiest and most popular way of practically refuting the Fallacy just mentioned, (or indeed any Fallacy,) is, by bringing forward a parallel case, where it leads to a manifest absurdity, a metaphysical objection may still be urged against many cases in which we thus reason from calculation of chances; an objection not likely indeed practically to influence any one, but which may afford the Sophist a triumph over those who are unable to find a solution. If it were answered then to those who maintain that the universe, which exhibits so many marks of design, might be the work of non-intelligent causes, that no one would believe it possible for such a work as the Iliad, e. g. to be produced by a fortuitous shaking together of the letters of the alphabet, the Sophist might challenge us to explain why even this last supposition should be regarded as less probable than any other; since the letters of which the Iliad is composed, if shaken together at random, must fall in *some* form or other; and though the chances are millions of millions to one against that, or any other *determinate* order, *there are precisely as many chances against one, as against another*: and in like manner, astonished as we should be, and convinced of the intervention of artifice, if we saw any one draw out all the cards in a pack in regular sequences, it is demonstrable that the chances are not more against that order, than against any one determinate order we might choose to fix upon. The multitude of the chances, therefore, he would say, against any series of events, does not constitute it improbable; since the like happens to every one every day; e. g. a man walking through London streets on his business, meets accidentally hundreds of others passing to and fro on theirs: and he would not say at the close of the day that any thing *improbable* had occurred to him; yet it would almost baffle calculation to compute the Chances against his meeting precisely those very persons, in the order, and at the times and places of his meeting each. The paradox thus seemingly established, though few might be practically misled by it, many would be at a loss to solve. The truth is, that any supposition is justly called improbable, not from the number of chances against it, considered *independently*, but from the number of chances against it *compared* with those which lie against some *other* supposition: we call the drawing of a prize in the lottery improbable, though there be but five to one against it, because there are *more* chances of a blank; on the other hand, if any one was cast on a desert island under circumstances which warranted his believing that the chances were a hundred to one against any one's having been there before him, yet if he found on the sand pebbles so arranged as to form the letters of a man's name, he would not only conclude it probable, but absolutely certain that some human being had been there; because there would be millions of chances against those forms having been produced by the fortuitous action of the waves. So also, in the instance above given, any *unmeaning* form into which a number of letters might fall, would not be called improbable, countless as the chances are against that particular order, because there are just as many against each one of all other unmeaning forms; but if the letters formed a coherent poem, it would then be called incalculably improbable that this form should have been fortuitous, though the chances against it remain the very same; because there must be much *fewer* chances against the supposition of its having been the work of *design*. The probability in short, of any supposition, is estimated from a *comparison* with each of its alternatives.

The foregoing observations however, as was above remarked, are not confined to Arguments from Testimony, but apply to all cases in which the degree of probability is estimated from a calculation of chances.

Before we dismiss the consideration of Signs, it may be worth while to notice another case of combined Argument different from the one lately mentioned, yet in some degree resembling it. The combination just spoken of is where several Testimonies or other Signs, singly perhaps of little weight, produce jointly, and by their coincidence, a degree of probability far exceeding the *sum* of their several forces, taken separately; in the case we are now about to notice, the

combined force of the series of Arguments resulting from the order in which they are considered, and from their *progressive* tendency to establish a certain conclusion. E. g. one part of the law of oatnre called the "vis inertiæ," is established by the Argument we allude to; viz. that a body set in motion will eternally continue in motion with uniform velocity in a right line, so far as it is not acted upon by any causes which retard or stop, accelerate or divert its course. Now, as in every case which can come under our observation, some such causes do intervene, the assumed supposition is practically impossible, and we have no opportunity of verifying the law by direct experiment; but we may *gradually approach* indefinitely near to the case supposed: and on the result of such experiments our conclusion is founded. We find that when a body is projected along a rough surface, its motion is speedily retarded and soon stopped; if along a smoother surface, it continues longer in motion; if upon ice, longer still, and the like with regard to wheels, &c. in proportion as we gradually lessen the friction of the machinery; if we remove the resistance of the air, by setting a wheel or pendulum in motion under an air-pump, the motion is still longer continued. Finding then that the effect of the original impulse is more and more protracted, in proportion as we more and more remove the impediments to motion from friction and resistance of the air, we reasonably conclude that if this could be *completely* done, (which is out of our power,) the motion would never cease, since what appear to be the only causes of its cessation, would be absent.

Again, in arguing for the existence and moral attributes of the Deity from the authority of men's opinions, great use may be made of a like progressive course of Argument, though it has been often overlooked. Some have argued for the being of a God from the universal or at least general consent of mankind; and some have appealed to the opinions of the wisest and most cultivated portion, respecting both the existence and the moral excellence of the Deity. It cannot be denied that there is a presumptive force in each of these Arguments; but it may be answered that it is conceivable an opinion common to almost all the species, may possibly be an error resulting from a constitutional infirmity of the human intellect;—that if we are to acquiesce in the belief of the majority, we shall be led to Polytheism; such being the creed of the greater part: and that though more weight may reasonably be attached to the opinions of the wisest and best-instructed, still, as we know that such men are not exempt from error, we cannot be perfectly safe in adopting the belief they hold, unless we are convinced that they hold it *in consequence* of their being the wisest and best instructed;—*so far forth* as they are such. Now this is precisely the point which may be established by the above-mentioned progressive Argument. Nations of Atheists, if there are any such, are confessedly among the rudest and most ignorant savages: those who represent their God or Gods as malevolent, capricious, or subject to human passions and vices, are invariably to be found, (in the present day at least,) among those who are brutal and uncivilized; and among the most civilized nations of the ancients, who professed a similar creed, the more enlightened members of society seem either to have rejected altogether, or to have explained away, the popular belief. The Mahometan nations, again, of the present day, who are certainly more advanced in civilisation than their Pagan neighbours, maintain the unity and the moral excellence of the Deity; but the nations of Christendom, whose notions of the divine goodness are more exalted, are undeniably the most civilized part of the world, and possess, generally speaking, the most cultivated and improved intellectual powers. Now if we would ascertain, and appeal to, the sentiments of man *as* a rational being, we must surely look to those which not only prevail most among the *most* rational and cultivated, but towards which also a *progressive* tendency is found in men in proportion to their degrees of rationality and cultivation. It would be most extravagant to suppose that man's advance towards a more improved and exalted state of existence should tend to obliterate true and instil false notions. On the contrary we are authorized to conclude, that those notions would be the most correct, which men would entertain, whose knowledge, intelligence, and intellectual cultivation should have reached the highest pitch of perfection; and that those consequently will approach the nearest to the truth which are entertained, more or less, by various nations, in proportion as they have advanced towards this civilized state.

Many other instances might be adduced, in which truths of the highest importance may be elicited by this process of Argumentation, which will enable us to decide with sufficient probability what consequence would follow from an hypothesis which we have never experienced; it might, not improperly, be termed the Argument from Progressive Approach.

The third kind of Arguments to be considered being the other branch of the second of the two classes originally laid down, may be treated of under the general name of Example, taking that term in its widest acceptation, so as to comprehend the Arguments designated by the various names of Induction, Experience, Analogy, Parity of Reasoning, &c. all of which are essentially the same, as far as regards the fundamental principles we are here treating of; for in all the Arguments designated by these names it will be found, that we consider one or more, known, individual objects or instances, of a certain class, as fair *specimens*, in respect of some point or other, of that class; and consequently draw an inference from them respecting either the whole class, or other, less known, individuals of it. In Arguments of this kind then it will be found, that universally we assume as a major premiss that what is true, (in regard to the point in question,) of the individual or individuals which we bring forward and appeal to, is true of the whole class to which they belong; the minor premiss next asserts something of that individual; and the same is then inferred respecting the whole class: whether we stop at that general conclusion, or descend from thence to another, unknown, individual; in which last case, which is the most usually called the Argument from Example, we generally omit, for the sake of brevity, the intermediate step, and pass at once in the expression of the Argument from the known, to the unknown, individual. This ellipsis however does not, as some seem to suppose, make any essential difference in the mode of Reasoning; the reference to a common class being always, in such a case, understood, though not expressed; for it is evident that there can be no

reasoning from one individual to another, onless they come under some commoo genus, and are considered in that point of view; e.g.

Astronomy was decried at its first introduction, as adverse to religion:	Geology is likely to be decried, &c.
therefore	*therefore*

every Science is likely to be decried at its first introdnction, as adverse to religion.

This kind of example, therefore, appears to be a compound Argument, consisting of two enthymemes; and when (as often happens) we infer from a known Effect a certain Cause, and again, from that Cause, another unknown Effect, we then unite in this example, the argument from Effect to Cause, and that from Cause to Effect, e. g. we may from the marks of Divine benevolence in this world, argue, that "the like will be shown in the next;" through the intermediate conclusion, that "God is benevolent." This is not indeed always the case; but there seems to be in every example, a *reference* to some Cause, though that Cause may frequently be unknown; e.g. we suppose, in the instance above given, that there is some Cause, though we may be at a lost to assign it, which leads men generally to decry a new Science.

The term "Indnction" is commonly applied to such Arguments as stop short at the *general* conclusion; and is thus contradistinguished, in common use, from Example. There is also this additional differeoce, that when we draw a general conclusion from *several* individnal cases, we use the word Induction in the singular number, while each one of these cases, if the application were made to another individual, woold be called a distinct Example. This difference, however, is not essential, since whether the inference be made from one instance or from several, it is equally called an Induction, if a general conclusion be legitimately drawn, and this is to be determined by the nature of the subject-matter; in the investigation of the laws of nature, a single experiment, fairly and carefully made, is usually allowed to be conclusive, because we can then pretty nearly ascertain all the circumstances operating: a Chemist who has ascertained, in a single specimen of gold, its capability of combining with mercury, would not think it necessary to try the same experiment with several other specimens, but would draw the conclusion concerning those metals universally, and with certainty; in human affairs on the contrary, our uncertainty respecting many of the circumstances that may affect the result, obliges us to collect many coinciding instances to warrant even a probable conclusioo. From one instance, e.g. of the assassination of an Usurper, it would not be allowable to infer the certainty, or even the probability, of a like fate attending all Usurpers.*

Experience, in its original and proper sense, is applicable to the *premises* from which we argue, not to the *inference* we draw. Strictly speaking, we know by Experience only the *past*, and what has passed under our own observation; thus, we know by *Experience* that the tides *have* daily ebbed and flowed, during such a time; and from the Testimony of others as to their owo experience, that they have formerly done so; and *from* this experience, we conclude, *by* Induction, that the same phenomenon will cootinue.

The word Analogy again is generally employed in the case of Arguments in which the instance adduced is somewhat more remote from that to which it is applied; e. g. a physician would be said to know by experience the noxious effects of a certain drug on the human constitution if he had frequently seen men poisoned by it; but if he thence coojectured that it would be noxious to some other species of animal, he would be said to reason from *Analogy*; the only difference being that the resemblance is less, betweeo a man and a brute, than between one man and another; and accordingly it is found that many brutes are not acted upon by some drugs which are pernicious to man. But more strictly speaking, Analogy ought to be distinguished from *direct* resemblance, with which it is often confounded in the language even of eminent writers (especially on Chemistry and Natural History) in the present day. Analogy being a "resemblance of ratios,"* that should strictly be called an Argument from Analogy, in which the two cases (viz. the ooe *from* which, and the one *to* which we argue) are not themselves alike, but stand in a similar *relation* to something else; or in other words that the common genus which they both fall under, consists in a relation. Thus an egg and a seed are not in themselves alike, but bear a like relation to the parent bird and to her future nestling, on the one hand, and to the old and young plant on the other, respectively; this *relation* being the genus which both fall under: and many Argumeots might be drawn from this Analogy. Again the fact that from birth differeot persons have different bodily constitutions, in respect of complection, stature, strength, shape, liability to particular disorders, &c. which constitutions, however, are capable of being, to a certain degree, modified by regimeo, medicine, &c. affords an Analogy by which we may form a presumption, that the like takes place in respect of mental qualities also; though it is plain that there can be no direct resemblance either between body and mind, or their respective attributes.

In this kind of Argument one error, which is very common, and which is to be sedulously avoided, is that of concluding the *things* in question to be *alike*, because they are *Analogous*;—to resemble each other in themselves, because there is a resemblance in the relation they bear to certain other things; which is manifestly a groundless inference. Another caution is applicable to the whole class of Arguments from Example; viz. oot to consider the resemblance or Analogy to extend further (i. e. to more particulars) than it does. The resemblance of a picture to the object it represeots, is direct; but it extends no further than the one sense of seeing is concerned. In the parable of the unjust steward an Argument is drawn from Analogy, to recommend prudence and foresight to Christians in spiritual cooceros; but it woold be absurd to cooclude that fraud was recommended to our imitation; and yet mistakes very similar to such a perversion of that Argument are by no means rare.†

* See article Logic, "On the Province of Reasoning," (p. 230.)

* Λόγων ὁμοιότης, Aristotle.

† "Thus, because a just Analogy has been discerned between the metropolis of a country, and the heart of the animal body, it has been sometimes contended that its increased size is a disease,—that it may impede some of its most important functions, or even be the cause of its dissolution." Copleston's *Inquiry into the*

The Argument from *Contraries*, (ἐξ ἐναντίων) noticed by Aristotle, falls under the class we are now treating of; as it is plain that Contraries must have something in common; and it is so far forth only as they *agree*, that they are thus employed in Argument. Two things are called "Contrary," which, coming under the same class, are the *most dissimilar* in that class. Thus, virtue and vice are called Contraries, as being, *both*, "*moral habits*," and the most *dissimilar* of moral habits. mere dissimilarity, it is evident, would not constitute Contrariety; for no one would say that virtue was contrary to a mathematical problem, the two things having nothing in common. In this then, as in other Arguments of the same class, we may infer that the two Contrary terms have a similar relation to the same third, or respectively to two corresponding, (i. e. in this case, Contrary) terms: we may conjecture e. g. that since virtue may be acquired by education, so may vice; or again, that since virtue leads to happiness, so does vice to misery.

The phrase "Parity of Reasoning," is commonly employed to denote Analogical Reasoning.

Aristotle, in his *Rhetoric*, has divided Examples into *Real* and *Invented*: the one being drawn from actual matter of fact; the other, from a supposed case. And he remarks, that though the latter is more easily adduced, the former is more convincing. If however due care be taken, that the fictitious instance,—the supposed case, adduced, be not wanting in *probability*, it will often be no less convincing than the other. For it may so happen, that one, or even several historical facts may be appealed to, which being nevertheless exceptions to a general rule, will not prove the probability of the conclusion. Thus, from several known instances of ferocity in black tribes, we are not authorized to conclude, that blacks are universally, or generally ferocious; and in fact, many instances may be brought forward on the other side. Whereas in the supposed case, (instanced by Aristotle, as employed by Socrates,) of mariners choosing their steersman by lot, though we have no reason to suppose such a case ever occurred, we see so plainly the *probability*, that if it did occur, the lot might fall on an unskilful person, to the loss of the ship, that the argument has considerable weight against the practice, so common in the ancient republics, of appointing magistrates by lot. There is, however, this important difference; that a fictitious case which has not this intrinsic probability, has absolutely no weight whatever; so that of course such arguments might be multiplied to any amount without the smallest effect: whereas any matter of fact which is well established, however *unaccountable* it may seem, has some degree of weight in reference to a parallel case; and a sufficient number of such arguments may fairly establish a general rule, even though we may be unable, after all, to account for the alleged fact in any of the instances; e. g. no satisfactory reason has yet been assigned for a connection between the absence of upper cutting teeth, or of the presence of horns and rumination; but the instances, are so numerous and constant of this connection, that no Naturalist would hesitate, if on examination of a new species be found those teeth absent, and the head horned, to pronounce the animal a ruminant. Whereas on the other hand, the fable of the countryman, who obtained from Jupiter the regulation of the weather, and in consequence found his crops fail, does not go one step towards proving the intended conclusion; because that consequence is a mere gratuitous assumption without any probability to support it. There is an instance of a like error in a tale of Cumberland's, intended to prove the advantage of a public over a private education; he represents two brothers educated, on the two plans respectively; the former turning out very well, and the latter very ill: and had the whole been matter of fact, a sufficient number of such instances would have had weight as an Argument; but as it is a fiction, and no reason is shown why the result should be such as represented, except the supposed superiority of a public education, the Argument involves a manifest *petitio principii*; and resembles the appeal made in the well-known fable, to the picture of a man conquering a lion; a result which might just as easily have been reversed, and which would have been so, had lions been painters. It is necessary, in short, to be able to maintain, either that such and such an event *did* actually take place, or that, under a certain hypothesis, it would be likely to take place.

Under the head of Invented Example, a distinction is drawn by Aristotle, between παραβολὴ et λόγοι: from the instances he gives, it is plain that the former corresponds (not to Parable, in the sense in which we use the word, derived from that of παραβολὴ in the Sacred Writers, but) to Illustration; the latter to Fable or Tale. In the former, an *allusion* only is made to a case easily supposable; in the latter, a fictitious story is narrated. Thus, in his instance above cited, of Illustration, if any one, instead of a mere allusion, should relate a tale, of mariners choosing a steersman by lot, and being wrecked in consequence, Aristotle would evidently have placed that under the head of Logos. The other method is of course preferable, from its brevity, whenever the allusion can be readily understood: and accordingly it is common, in the case of well-known fables, to allude to, instead of narrating, them. That, e. g. of the horse and the stag, which he gives, would, in the present day, be rather alluded to than told, if we wished to dissuade a people from calling in a too powerful auxiliary. It is evident that a like distinction might have been made in respect of historical examples; those cases which are well known, being often merely alluded to, and not recited. The word "Fable" is at present generally limited to those fictions in which the resemblance to the matter in question is not direct, but analogical; the other class being called Novels, Tales, &c. Those resemblances are, (as Dr. A. Smith has observed) the most *striking*, in which the things compared are of the most *dissimilar* nature; as is the case in what we call Fables; and such accordingly are generally preferred for Argumentative purposes, both from that circumstance itself, and also on account of the greater *brevity* which is, for that reason not only allowed but required in them.* For a Fable spun out to a great length becomes an Allegory, which generally satiates and disgusts; on the other hand, a fictitious Tale, having a more direct,

Doctrines of Necessity and Predestination, note to Disc. iii. q. v. for a very able dissertation on the subject of Analogy, in the course of an analysis of Dr. King's *Discourse on Predestination*.

* A Novel or Tale may be compared to a Picture; a Fable to a Device.

and therefore less striking, resemblance to reality, requires that an interest in the events and persons should be created by a longer detail, without which it would be insipid. The Fable of the Old Man and the Bundle of Sticks, compared with the Iliad, may serve to exemplify what has been said; the moral conveyed by each being the same, viz. the strength acquired by union, and the weakness resulting from division; the latter fiction would be perfectly insipid if conveyed in a few lines; the former, in twenty-four books, insupportable.

Of the various uses, and of the real or apparent refutation, of Examples, (as well as of other Arguments), we shall treat hereafter; but it may be worth while here to observe, that we have been speaking of Example as a kind of *Argument*, and with a view therefore to that purpose alone; it often happens, that a resemblance, either direct, or analogical, is introduced for other purposes; viz. not to *prove* anything, but either to illustrate and *explain* one's meaning, (which is the strict etymological use of the word Illustration,) or to amuse the fancy by ornament of language. It is of course most important to distinguish, both in our own compositions and those of others, between these different purposes.

Of the various use and order of the several kinds of Proposition and of Argument, in different cases.

§ 3. The *first rule* to be observed is, that it should be considered, whether the principal object of the discourse be, to give *satisfaction to a candid mind*, and convey *instruction* to those who are ready to receive it, or to *compel* the assent, or silence the objections, of an opponent. The former of these purposes is, in general, principally to be accomplished by the former of those two great classes into which arguments were divided; (viz. by those from Cause to Effect,) the other, by the latter

To whatever class, however, the Arguments we resort to may belong, the general tenour of the reasoning will, in many respects, be affected by the present consideration. The distinction in question is nevertheless in general little attended to. It is usual to call an Argument, simply, *strong* or *weak*, without reference to the purpose for which it is designed; whereas the Arguments which afford the most *satisfaction* to a candid mind, are often such as would have less weight in *controversy* than many others, which again would be less suitable for the former purpose.* E. g. the internal evidence of Christianity in general, proves the most satisfactory to a believer's mind, but is not that which makes the most show in the refutation of infidels; the Arguments from Analogy on the other hand, which are the most *unanswerable*, are not so pleasing and consolatory.

Rule second. Matters of Opinion, (as they are called; i. e. where we are not said properly to *know*, but to *judge*,) are established chiefly by Antecedent-probability; (Arguments of the first class, viz. from Cause to Effect,) though the testimony of wise men is also admissible; past Facts, chiefly by Signs, of various kinds; (that term, it must be remembered, including Testimony,) and future events by Antecedent-probabilities and Examples.

Example, however, is not excluded from the proof of matters of opinion; since a man's judgment in one case, may be aided or corrected by an appeal to his judgment in another similar case. It is in this way that we are directed, by the highest authority, to guide our judgment in those questions, in which we are most liable to deceive ourselves; viz. what, on each occasion, ought to be our conduct towards another; we are directed to frame for ourselves a similar supposed case, by imagining ourselves to change places with our neighbour, and then considering how, in that case, we should wish to be treated.

It happens more frequently, however, that, when in the discussion of matters of opinion, an Example is introduced, it is designed, not for Argument, but, strictly speaking, for *Illustration*;—not to *prove* the proposition in question, but to make it more *clearly understood*; e. g. the Proposition maintained by Cicero, (*de Off.* book iii.) is what may be accounted a matter of opinion; viz. that "nothing is expedient which is dishonourable;" when then he adduces the Example of the supposed design of Themistocles to burn the allied fleet, which he maintains, in contradiction to Aristides, would not have been expedient, because it would have been unjust, it is manifest, that we must understand the instance brought forward as no more than an Illustration of the general principle he intends to establish; since it would be a plain begging of the question to *argue* from a particular assertion, which could only

* Our meaning cannot be better illustrated than by an instance referred to in that incomparable specimen of Reasoning, Dr. Paley's *Horæ Paulinæ*. "When we take into our hands the letters," (viz. St. Paul's Epistles,) "which the suffrage and consent of antiquity hath thus transmitted to us, the first thing that strikes our attention is the air of reality and business, as well as of seriousness and conviction, which pervades the whole. Let the sceptic read them. If he be not sensible of these qualities in them, the argument can have no weight with him. If he be; if he perceive in almost every page the language of a mind actuated by real occasions, and operating upon real circumstances, I would wish it to be observed, that the proof which arises from this perception is not to be deemed occult or imaginary, because it is incapable of being drawn out in words, or of being conveyed to the apprehension of the reader in any other way, than by sending him to the books themselves." p. 403.

There is also a passage in Dr. A. Smith's *Theory of Moral Sentiments*, which illustrates very happily one of the applications of the principle in question. "Sometimes we have occasion to defend the propriety of observing the general rules of justice by the consideration of their necessity to the support of society. We frequently hear the young and the licentious ridiculing the most sacred rules of morality, and professing, sometimes from the corruption, but more frequently from the vanity of their hearts, the most abominable maxims of conduct. Our indignation rouses, and we are eager to refute and expose such detestable principles. But though it is their intrinsic hatefulness and detestableness which originally inflames us against them, we are unwilling to assign this as the sole reason why we condemn them, or to pretend that it is merely because we ourselves hate and detest them. The reason, we think, would not appear to be conclusive. Yet, why should it not; if we hate and detest them because they are the natural and proper objects of hatred and detestation? But when we are asked why we should not act in such or such a manner, the very question seems to suppose that, to those who ask it, this manner of acting does not appear to be for its own sake the natural and proper object of those sentiments. We must show them, therefore, that it ought to be so for the sake of something else. Upon this account we generally cast about for other arguments, and the consideration which first occurs to us, is the disorder and confusion of society which would result from the universal prevalence of such practices. We seldom fail, therefore, to insist upon this topic." (Part ii. sec. ii. p. 151, 152, vol. i. ed. 1812.)

be admitted by those who assented to the general principle.

It is important to distinguish between these two uses of Example; that on the one hand we may not be led to mistake for an Argument such an one as the foregoing; and that on the other hand, we may not too hastily charge with sophistry him who adduces such an one simply with a view to explanation.

It is also of the greatest consequence to distinguish between Examples (of the invented kind,) properly so called, i. e. which have the force of Arguments, and Comparisons introduced for the ornament of style, in the form, either of simile, as it is called, or a Metaphor. Not only is an ingenious comparison often mistaken for a proof, though it be such as, when tried by the rules laid down in the present Article, and under the head of Logic, affords no proof at all; but also on the other hand, a real and valid argument is not unfrequently considered merely as an ornament of style, if it happen to be such as to produce that effect; though there is evidently no reason why that should not be fair Analogical Reasoning, in which the new idea introduced by the Analogy chances to be a sublime or a pleasing one. E. g. "The efficacy of penitence, and piety, and prayer, in rendering the Deity propitious, is not irreconcileable with the immutability of his nature, and the steadiness of his purposes. It is not in man's power to alter the course of the sun; but it is often in his power to cause the sun to shine or not to shine upon him; if he withdraws from its beams, or spreads a curtain before him, the sun no longer shines on him; if he quits the shade, or removes the curtain, the light is restored to him; and though no change is in the mean time effected in the heavenly luminary, but only in himself, the result is the same as if it were. Nor is the immutability of God any reason why the returning sinner, who tears away the veil of prejudice or of indifference, should not again be blessed with the sunshine of divine favour." The image here introduced is ornamental, but the Argument is not the less perfect; since the case adduced fairly establishes the general principle required, that "a change effected in one of two objects having a certain relation to each other, may have the same practical result as if it had taken place in the other."

The mistake in question, is still more likely to occur when such an Argument is conveyed in a single term employed metaphorically; as is generally the case where the allusion is common and obvious; e. g. "we do not receive as the genuine doctrines of the primitive Church what have passed down the *polluted stream* of Romish tradition." The Argument here is not the less valid for being conveyed in the form of a Metaphor.

The employment, in questions relating to the future, both of the Argument from Example, and of that from Cause to Effect, may be explained from what has been already said concerning the connection between them; some cause, whether known or not, being always *supposed*, whenever an Example is adduced.

Rule third. When Arguments of each of the two formerly-mentioned classes are employed, those from Cause to Effect (Antecedent-probability) have usually the precedence.

Men are apt to listen with prejudice to the Arguments adduced to prove any thing which appears *abstractedly* improbable; i. e. according to what has been above laid down, *unnatural*, or (if such an expression might be allowed) *unplausible*; and this prejudice is to be removed by the Argument from Cause to Effect, which thus prepares the way for the reception of the other Arguments; e. g. if a man who bore a good character, were accused of corruption, the strongest evidence against him might avail little; but if he were proved to be of a covetous disposition, this, though it would not alone be allowed to substantiate the crime, would have great weight in inducing his judges to lend an ear to the evidence. And thus, in what relates to the future also, the *a priori* Argument and Example support each other, when thus used in conjunction and in the order prescribed; a sufficient cause being established, leaves us still at liberty to suppose that there may be circumstances which will prevent the effect from taking place; but Examples subjoined show that these circumstances do not, at least always, prevent that effect; and on the other hand, Examples introduced at the first, may be suspected of being exceptions to the general rule, (unless they are very numerous,) instead of being instances of it; which an adequate cause previously assigned, will show them to be; e. g. if any one had argued, from the temptations and opportunities occurring to a military commander, that Buonaparte was likely to establish a despotism on the ruins of the French Republic, this Argument, by itself, would have left men at liberty to suppose that such a result would be prevented by a jealous attachment to liberty in the citizens, and a fellow feeling of the soldiery with them; then, the Examples of Cæsar and of O. Cromwell would have proved, that such preventives are not to be trusted.

Aristotle accordingly has remarked on the expediency of not placing Examples in the foremost rank of Arguments; in which case, he says, a considerable number would be requisite; whereas, in *confirmation*, even one will have much weight. This observation, however, he omits to extend, as he might have done, to Testimony and every other kind of Sign, to which it is no less applicable.

Another reason for adhering to the order here prescribed is, that if the Argument from Cause to Effect, were placed after the others, a doubt might often exist, whether we were engaged in *proving* the point in question, or (assuming it as already proved) in seeking only to *account* for it; that Argument being, by the very nature of it, such as *would* account for the truth contended for, supposing it were granted. Constant care, therefore, is requisite to guard against any confusion or indistinctness as to the object in each case proposed; whether that be, when a proposition is admitted, to assign a cause which *does* account for it, (which is one of the classes of *Propositions* formerly noticed) or, when it is *not* admitted, to *prove* it by an *Argument* of that kind which *would* account for it, if it were granted.

With a view to the Arrangement of Arguments, no rule is of more importance than the one now under consideration; and Arrangement is a more important point than is generally supposed; indeed it is not perhaps of less consequence in Rhetoric than in the Military Art; in which it is well known, that with an equality of forces, in numbers, courage, and every other point, the manner in which they are drawn up, so as either to afford mutual support, or on the other hand, even to impede and annoy each other, may make the difference of victory or defeat.

E. g. in the statement of the Evidences of our Religion, so as to give them their just weight, much depends on the Order in which they are placed. The Antecedent-probability that a Revelation should be given to man, and that it should be established by miracles, all would allow to be, considered by itself, in the absence of strong direct testimony, utterly insufficient to establish the Conclusion. On the other hand, miracles considered abstractedly, as represented to have occurred without any occasion or reason for them being assigned, carry with them such a strong intrinsic improbability as could not be wholly surmounted even by such evidence as would fully establish any other matters of fact. But the evidences of the former class, however inefficient alone towards the establishment of the Conclusion, have very great weight in preparing the mind for receiving the other Arguments; which again, though they would be listened to with prejudice if not so supported, will then be allowed their just weight. The writers in defence of Christianity have not always attended to this principle; and their opponents have often availed themselves of the knowledge of it, by combating in detail Arguments the combined force of which would have been irresistible. They argue respecting the credibility of the Christian miracles, abstractedly, as if they were insulated occurrences, without any known or conceivable purpose; as e. g. "what testimony is sufficient to establish the belief that a dead man was restored to life?" and then they proceed to show that the probability of a Revelation, abstractedly considered, is not such at least as to establish the fact that one has been given. Whereas, if it were *first* proved (as may easily be done) merely that there is no such abstract improbability of a Revelation as to exclude the evidence in favour of it, and that if one were given, it might be expected to be supported by miraculous evidence, then, just enough reason would be assigned for the occurrence of miracles, not indeed to establish them, but to allow a fair hearing for the Arguments by which they are proved.

The importance attached to the Arrangement of Arguments by the two great rival orators of Athens, may serve to illustrate and enforce what has been said. Æschines strongly urged the judges (in the celebrated contest concerning the crown) to confine his adversary to the same order in his reply to the charges brought, which he himself had observed in bringing them forward. Demosthenes however was far too skilful to be thus entrapped; and so much importance does he attach to this point, that he opens his speech with a most solemn appeal to the Judges for an impartial hearing; which implies, he says, not only a rejection of prejudice, but no less also a permission for each speaker to adopt whatever *Arrangement* he should think fit. And accordingly he proceeds to adopt one very different from that which his antagonist had laid down; for he was no less sensible than his rival that the same Arrangement which is the most favourable to one side, is likely to be the least favourable to the other.

It is to be remembered however, that the rules which have been given respecting the Order in which different kinds of Argument should be arranged, relate only to the different kinds of Arguments adduced in support of each separate Proposition; since of course the refutation of an opposed assertion, effected by means of signs, may be followed by an *a priori* Argument in favour of our own Conclusion; and the like in many other such cases.

Rule fourth. A Proposition that is *well known* (whether easy to be established or not) should in general be stated at once, and the Proofs subjoined; but if it be not familiar to the hearers, and especially if it be likely to be unacceptable, it is usually better to state the Arguments first, or at least some of them, and then introduce the Conclusion.

There is no question relating to Arrangement more important than the present; and it is therefore the more unfortunate that Cicero, who possessed so much practical skill, should have laid down no rule on this point, (though it is one which evidently had engaged his attention,) but should content himself with saying that sometimes he adopted the one mode and sometimes the other,* (which doubtless he did not do at random,) without distinguishing the cases in which each is to be preferred, and laying down principles to guide our decision. Aristotle also, when he lays down the two great heads into which speech is divisible, the Proposition and the Proof,† is equally silent as to the order in which they should be placed; though he leaves it to be understood, from his manner of speaking, that the Conclusion (or Question) is to be first stated, and then the Premises, as in Mathematics. This indeed is the usual and natural way of speaking or writing; viz. to begin by declaring your Opinion, and then to subjoin the Reasons for it. But there are many occasions on which it will be of the highest consequence to reverse this plan. It will sometimes give an offensively dogmatical air to a Composition to begin by advancing some new and unexpected assertion; though sometimes again this may be advisable, when the Arguments are such as can be well relied on, and the principal object is to excite attention, and awaken curiosity. And accordingly, with this view, it is not unusual to present some doctrine, by no means really novel, in a new and paradoxical shape. But when the Conclusion to be established is one likely to hurt the feelings and offend the prejudices of the hearers, it is essential to keep out of sight, as much as possible, the point to which we are tending, till the principles from which it is to be deduced shall have been clearly established; because men listen with prejudice, if at all, to Arguments which are avowedly leading to a Conclusion which they are indisposed to admit; whereas if we thus, as it were, mask the battery, they will not be able to shelter themselves from the discharge. The observance accordingly, or neglect, of this rule, will often make the difference of success or failure.

And it will often be advisable to advance very gradually to the full statement of the Proposition required, and to prove it, if one may so speak, by instalments, establishing separately, and in order, each part of the truth in question. It is thus that Aristotle establishes many of his doctrines, and among others his definition of happiness, in the beginning of the *Nicomachean Ethics*; he first proves in what it does *not* consist, and then establishes, one by one, the several points which together constitute his notion.

Rule fifth. If the Argument *a priori* has been introduced in the proof of the main Proposition in

* *De Orat.* † *Rhet.* book iii.

question, there will generally be no need of afterwards adducing Causes to account for the truth established; (since that will have been already done in the course of the Argument,) on the other hand, it will often be advisable to do this when Arguments of the other class have alone been employed.

For it is in every case agreeable and satisfactory, and may often be of great utility, to explain, where it can be done, the Causes which produce an Effect that is itself already admitted to exist. But it must be remembered that it is of great importance to make it clearly appear *which* object is, in each case, proposed; whether to establish the fact, or to account for it; since otherwise we may often be supposed to be employing a feeble Argument; for that which is a satisfactory explanation of an admitted fact, will frequently be such as would be very insufficient to prove it, supposing it were doubted.

Rule sixth. Refutation of Objections should generally be placed in the midst of the other Arguments, but nearer the beginning than the end.

If indeed very strong Objections have obtained much currency, or have been just stated by an opponent, so that what is asserted is likely to be regarded as paradoxical, it may be advisable to begin with a Refutation; but when this is not the case, the mention of Objections in the opening will be likely to give a paradoxical air to our assertion, by implying a consciousness that much may be said against it. If again all mention of Objections be deferred till the last, the other Arguments will often be listened to with prejudice by those who may suppose us to be overlooking what may be urged on the other side.

Sometimes indeed it will be difficult to give a satisfactory Refutation of the opposed Opinions till we have gone through the Arguments in support of our own: even in that case however it will be better to take some brief notice of them early in the Composition, with a promise of afterwards considering them more fully, and refuting them. This is Aristotle's usual mode of procedure.

A Sophistical use is often made of this last rule, when the Objections are such as cannot really be satisfactorily answered. The skilful Sophist will often, by the promise of a triumphant Refutation hereafter, gain attention to his own statement, which, if it be made plausible, will so draw off the hearer's attention from the Objections, that a very inadequate fulfilment of that promise will pass unnoticed, and due weight will not be allowed to the Objections.

It may be worth remarking, that Refutation will often occasion the introduction of fresh Propositions; i. e. we may have to disprove Propositions, which, though incompatible with the principal one to be maintained, will not be directly contradictory to it; e. g. Burke, in order to the establishment of his theory of beauty, refutes the other theories which have been advanced by those who place it in "fitness" for a certain end—in "proportion"—in "perfection," &c.: and Dr. A. Smith, in his *Theory of Moral Sentiments*, combats the opinion of those who make expediency the test of virtue—of the advocates of a "Moral sense," &c. which doctrines respectively are at variance with those of these authors, and *imply*, though they do not express, a contradiction of them.

Though we are at present treating principally of the proper collocation of Refutation, some remarks on the conduct of it will not be unsuitable in this place. In the first place, it is to be observed that there is (as Aristotle remarks, *Rhet.* book ii., apparently in opposition to some former writers) no distinct class of refutatory Arguments, since they become such merely by the circumstances under which they are employed.

There are two ways in which any Proposition may be refuted;* first, by proving the contradictory of it; second, by overthrowing the Arguments by which it has been supported. The former of these is less strictly and properly called Refutation, being only accidentally such, since it might have been employed equally well had the opposite Argument never existed; and in fact it will often happen that a Proposition maintained by one author may be in this way refuted by another, who had never heard of his Arguments. Thus Pericles is represented by Thucydides as proving, in a speech to the Athenians, the probability of their success against the Peloponnesians, and thus, virtually, refuting the speech of the Corinthian ambassador at Sparta, who had laboured to show the probability of their speedy downfal.† In fact, every one who argues in favour of any Conclusion is virtually refuting, in this way, the opposite Conclusion.

But the character of Refutation more strictly belongs to the other mode of procedure; viz. in which a reference is made, and an answer given, to some specific Arguments in favour of the opposite Conclusion. This may consist either in the denial of one of the *Premises*, or an Objection against the *conclusiveness* of the Reasoning. And here it is to be observed that the Objection is often supposed, from the mode in which it is expressed, to belong to this last class, when in truth it does not, but consists in the contradiction of a Premiss; for it is very common to say, "I admit your principle, but deny that it leads to such a consequence;" "the assertion is true, but it has no force as an Argument to prove that Conclusion;" this sounds like an objection to the Reasoning itself, but it will often be found to amount only to a denial of the *suppressed* Premiss of an Enthymeme; the assertion which is admitted being only the expressed Premiss, whose force as an Argument must of course depend on the other Premiss which is understood. Thus Warburton admits that in the Law of Moses the doctrine of a future state was not revealed; but contends that this, so far from disproving, as the Deists pretend, his Divine mission, does, on the contrary, establish it. But the Objection is not to the Deist's Argument properly so called, but to the other Premiss, which they so hastily took for granted, and which he disproves, viz. "that a divinely-commissioned Lawgiver would have been sure to reveal that doctrine." The Objection is then only properly said to lie against the Reasoning itself, when it is shown that granting all that is assumed on the other side, whether expressed or understood, still the Conclusion contended for would not follow from the Premises, either on account of some ambiguity in the Middle Term, or

* Ἀντισυλλογισμὸς and ἔνστασις of Aristotle, book ii.

† The speeches indeed are avowedly the composition of the historian; but he professes to give the substance of what was either actually said, or *likely* to be said, on each occasion; and the Arguments urged in the speeches now in question are undoubtedly such as the respective speakers would be likely to employ.

some other fault of that class. (See Logic, chapter *on Fallacies*.)

It may be proper in this place to remark, that "Indirect Reasoning" is sometimes confounded with "Refutation," or supposed to be peculiarly connected with it; which is not the case; either Direct or Indirect Reasoning being employed indifferently for Refutation as well as for any other purpose. The application of the term "elenctic," (from ἐλέγχειν to refute or disprove,) to Indirect Arguments, has probably contributed to this confusion; which, however, principally arises from the very circumstance that occasioned such a use of that term; viz. that in the Indirect method the absurdity or falsity of a Proposition (opposed to our own) is proved; and hence is suggested the idea of an *adversary* maintaining that Proposition, and of the Refutation of that adversary being necessarily accomplished in this way. But it should be remembered that Euclid and other mathematicians, though they can have no opponent to refute, often employ the Indirect Demonstration; and that on the other hand, if the contradictory of an opponent's Premiss can be satisfactorily proved in the Direct Method, the Refutation is sufficient. It is true however that while in science the Direct Method is considered preferable, in controversy the Indirect is often adopted by choice, as it affords an opportunity for holding up an opponent to scorn and ridicule, by deducing some very absurd Conclusion from the principles he maintains, or according to the mode of arguing he employs. Nor indeed can a fallacy be so clearly exposed to the unlearned reader in any other way. For it is no easy matter to explain, to one ignorant of Logic, the grounds on which you object to an inconclusive Argument, though he will be able to perceive its correspondence with another brought forward to illustrate it, in which an absurd Conclusion may be introduced, as drawn from true Premises.

It is evident that either the Premiss of an opponent or his Conclusion may be disproved, either in the Direct or in the Indirect Method; i. e. either by proving the truth of the Contradictory, or by showing that an absurd Conclusion may fairly be deduced from the Proposition in question: when this latter mode of Refutation is adopted with respect to the Premiss, the phrase by which this procedure is usually designated, is, that the "Argument proves too much;" i. e. that it proves, besides the Conclusion drawn, another, which is manifestly inadmissible; e. g. the Argument by which Dr. Campbell labours to prove that every correct Syllogism must be nugatory, as involving a "*petitio principii*," proves, if admitted at all, more than he intended, since it may easily be shown to be equally applicable to all Reasoning whatever.

It is worth remarking, that that which is in substance an Indirect Argument, may easily be altered in form so as to be stated in the Direct Mode. For, strictly speaking, that is Indirect Reasoning in which we assume as true the Proposition whose Contradictory it is our object to prove; and deducing regularly from it an absurd Conclusion, infer thence that the Premiss in question is false; the alternative proposed in all correct Reasoning being either to admit the Conclusion, or to deny one of the Premises; but by adopting the form of a Destructive Conditional,* the same Argument as this in substance may be stated directly; e. g. we may say, let it be admitted that no testimony can satisfactorily establish such a fact as is not agreeable to our experience; thence it will follow, that the Eastern Prince judged wisely and rightly in at once rejecting, as a manifest falsehood, the account given him of the phenomenon of ice; but he was evidently mistaken in so doing; therefore the Principle assumed is unsound. Now the substance of this Argument remaining the same, the form of it may be so altered as to make the Argument Direct; viz. "*if* it be true that no testimony, &c. that Eastern Prince must have judged wisely, &c. but he did not; therefore that Principle is not true."

Universally indeed a Conditional Proposition may be regarded as an assertion of the validity of a certain Argument; the Antecedent corresponding to the Premises, and the Consequent to the Conclusion; and neither of them being asserted as true, only the dependence of the one on the other; the alternative then is, to admit the Consequent, (which forms the Constructive Syllogism,) or to deny the Antecedent, which forms the Destructive; and the former accordingly corresponds to Direct Reasoning, the latter to Indirect, being, as has been said, a mode of stating it in the Direct form, as is evident from the examples adduced.

The difference between these two modes of stating such an Argument is considerable, when there is a long chain of Reasoning; for when we employ the Categorical form, and assume as true the Premises we design to disprove, it is evident we must be speaking *ironically*, and in the character, assumed for the moment, of an adversary; when, on the contrary, we use the hypothetical form, there is no irony. Butler's *Analogy* is an instance of the latter procedure; he contends that *if* such and such objections are admissible against Religion, they must be applied equally to the constitution and course of nature. Had he, on the other hand, assumed, for the Argument's sake, that such objections against Religion *are* valid, and had thence proved the condition of the natural world to be totally different from what we see it to be, his Arguments, which would have been the same in substance, would have assumed an ironical form. This form has been adopted by Burke in his celebrated *Defence of Natural Society, by a late noble Lord*;* in which, assuming the person of Bolingbroke, he proves, according to the principles of that author, that the Arguments he brought against ecclesiastical, would equally lie against civil institutions.

It is in some respects a recommendation of this latter method, and in others an objection to it, that the sophistry of an adversary will often be exposed by it in a *ludicrous* point of view; and this, even where no such effect is designed; the very essence of jest being its *mimic sophistry*.† This will often give additional force to the Argument, by the vivid impression which ludicrous images produce: but again, it will not unfrequently have this disadvantage, that weak men, perceiving the wit, are apt to conclude that nothing but wit is designed, and lose sight perhaps of

* See Logic.

* This is an Argument from *Analogy*, as well as Bishop Butler's, though not relating to the same point, Butler's being a defence of the *Doctrines* of Religion.

† See Logic, Chapter *on Fallacies*, at the conclusion.

a solid and convincing Argument, which they regard as no more than a good joke. Having been warned that "ridicule is not the test of truth," and that "wisdom and wit" are not the same thing, they distrust every thing that can possibly be regarded as witty; not having judgment to perceive the combination, when it occurs, of wit with sound Reasoning. The Ivy-wreath completely conceals from their view the point of the Thyrsus: and moreover if such a mode of Argument be employed on serious subjects, the "weak brethren" are sometimes scandalized by what appears to them a profanation; not having discernment to perceive when it is that the ridicule does, and when it does not, affect the solemn subject itself. But for the respect paid to Holy Writ, the taunt of Elijah against the prophets of Baal would probably appear to such persons irreverent. And the caution now implied will appear the more important when it is considered how large a majority they are, who, in this point, come under the description of "weak brethren:" he that can laugh at what is ludicrous, and at the same time preserve a clear discernment of sound and unsound Reasoning, is no ordinary man.

It may be observed generally that too much stress is often laid, especially by unpractised Reasoners, on Refutation; (in the strictest and narrowest sense, i. e. of Objections to the Premises, or to the Reasoning,) they are apt both to expect a Refutation where none can fairly be expected, and to attribute to it, when satisfactorily made out, more than it really accomplishes.

For first, not only specious, but real and solid Arguments, such as it would be difficult or impossible to refute, may be urged against a Proposition which is nevertheless true, and may be satisfactorily established by a *preponderance* of probability. It is in strictly scientific Reasoning alone that all the Arguments which lead to a false Conclusion must be fallacious: in what is called moral or probable Reasoning, there may be sound Arguments and valid objections on both sides;* e. g. it may be shown that each of two contending parties has some reason to hope for success; and this, by irrefragable Arguments on both sides, leading to Conclusions which are not contradictory to each other; for though only one party can obtain the victory, it may be true that each has some reason to expect it. The real question in such cases is, which event is the *more* probable;—on which side the evidence preponderates. Now it often happens that the inexperienced Reasoner, thinking it necessary that every Objection should be satisfactorily answered, will have his attention drawn off from the Arguments of the opposite side, and will be occupied perhaps in making a weak defence, while victory was in his hands. The Objection perhaps may be unanswerable, and yet may safely be allowed, if it can be shown that more and weightier Objections lie against every other supposition. This is a most important caution for those who are studying the Evidences of Religion.

Secondly, the force of a Refutation is often overrated: an Argument which is satisfactorily answered ought to *go for nothing*; but it is possible that the Conclusion drawn may nevertheless be true: yet men are apt to take for granted that the Conclusion itself is disproved, when the Arguments brought forward to establish it have been satisfactorily refuted; assuming, when perhaps there is no ground for the assumption, that these are *all* the Arguments that could be urged. This may be considered as the fallacy of denying the Consequent of a Conditional Proposition, from the Antecedent having been denied: "if such and such an Argument be admitted, the Assertion in question is true; but that Argument is inadmissible; *therefore the Assertion is not true.*" Hence the injury done to any cause by a weak advocate; the cause itself appearing to the vulgar to be overthrown when the Arguments brought forward are answered.

On the same principle is founded a most important maxim, that it is not only the fairest, but also the wisest plan, to *state Objections in their full force*; at least, wherever there does exist a satisfactory answer to them; otherwise, those who hear them stated more strongly than by the uncandid advocate who had undertaken to repel them, will naturally enough conclude that they are unanswerable. It is but a momentary and ineffective triumph that can be obtained by manœuvres like those of Turnus's charioteer, who furiously chased the feeble stragglers of the army, and evaded the main front of the battle.

Rule seventh. The Arguments which should be placed first in order are, *cæteris paribus*, the most *Obvious*, and such as naturally first occur.

This is evidently the natural order; and the adherence to it gives an easy, natural air to the Composition. It is seldom therefore worth while to depart from it for the sake of beginning with the most powerful Arguments, (when they happen not to be also the most Obvious) or on the other hand, for the sake of reserving these to the last, and beginning with the weaker: or, again, of imitating, as some recommend, Nestor's plan of drawing up troops, placing the best first and last, and the weakest in the middle. It will be advisable however (and by this means you may secure this last advantage) when the strongest Arguments naturally occupy the foremost place, to *recapitulate in a reverse order*; which will destroy the appearance of anti-climax, and is also in itself the most easy and natural mode of recapitulation. Let, e. g. the Arguments be A, B, C, D, E, &c. each less weighty than the preceding; then in recapitulating proceed from E to D, C, B, concluding with A.

Of Introduction.

§ 4. A Proem, Exordium, or Introduction, is, as Aristotle has justly remarked, not to be accounted one of the essential parts of a Composition, since it is not in every case necessary. In most, however, except such as are extremely short, it is found advisable to premise something before we enter on the main Argument, to avoid an appearance of abruptness, and to facilitate, in some way or other, the Object proposed. In larger works this assumes the appellation of Preface or Advertisement; and not unfrequently two are employed, one under the name of Preface, and another, more closely connected with the main work, under that of Introduction.

The rules which have been laid down already will

* "There are objections against a *Plenum*, and objections against a *Vacuum*; but one of them must be true." Johnson.

apply equally to that preliminary course of Argument of which Introductions often consist.

The writers before Aristotle, are censured by him for inaccuracy, in placing under the head of Introductions, as properly belonging to them, many things which are not more appropriate in the beginning than elsewhere; as, e. g. the contrivances for exciting the hearer's *attention*; which, as he observes, is an improper Arrangement; since, though such an Introduction may sometimes be required, it is, generally speaking, anywhere else rather than in the *beginning*, that the attention is likely to flag.

The rule laid down by Cicero, (*De Orat.*) not to compose the Introduction first, but to consider first the main Argument, and let that suggest the Exordium, is just and valuable; for otherwise, as he observes, seldom any thing will suggest itself but vague generalities; "*common*" topics, as he calls them, i. e. what would equally well suit several different Compositions; whereas the Introduction, which is composed last, will naturally spring out of the main subject, and appear appropriate to it.

1. One of the Objects most frequently proposed in an Introduction, is, to show that the subject in question is *important*, *curious*, or otherwise *interesting*, and worthy of attention. This may be called an "Introduction inquisitive."

2. It will frequently happen also, when the point to be proved or explained is one which may be very fully established, or on which there is little or no doubt, that it may nevertheless be *strange*, and different from what might have been expected; in which case it will often have a good effect in rousing the attention, to set forth as strongly as possible this *paradoxical* character, and dwell on the seeming improbability of that which must, after all, be admitted. This may be called an "Introduction paradoxical."*

* "If you should see a flock of pigeons in a field of corn: and if (instead of each picking where and what it liked, taking just as much as it wanted, and no more) you should see ninety-nine of them gathering all they got, into a heap; reserving nothing for themselves, but the chaff and the refuse; keeping this heap for one, and that the weakest, perhaps worst, pigeon of the flock; sitting round, and looking on, all the winter, whilst this one was devouring, throwing about, and wasting it; and if a pigeon, more hardy or hungry than the rest, touched a grain of the hoard, all the others instantly flying upon it, and tearing it to pieces; if you should see this, you would see nothing more than what is every day practised and established among men. Among men, you see the ninety and nine toiling and scraping together a heap of superfluities for one, (and this one too, oftentimes the feeblest and worst of the whole set, a child, a woman, a madman, or a fool;) getting nothing for themselves all the while, but a little of the coarsest of the provision, which their own industry produces; looking quietly on, while they see the fruits of all their labour spent or spoiled; and if one of the number take or touch a particle of the hoard, the others joining against him, and hanging him for the theft.

"There must be some very important advantages to account for an institution, which, in the view of it above given, is so paradoxical and unnatural.

"The principal of these advantages are the following:" &c.—Paley's *Moral Philosophy*, book iii. part i. c. 1 and 2.

3. What may be called an "Introduction corrective," is also in frequent use; viz. to show that the subject has been *neglected*, *misunderstood*, or *misrepresented* by others. This will, in many cases, remove a most formidable obstacle in the hearer's mind, the anticipation of triteness, if the subject be, or may supposed to be, a hacknied one; and it may also serve to remove or loosen such prejudices as might be adverse to the favourable reception of our Arguments.

4. It will often happen also, that there may be need to explain some *peculiarity* in the mode of Reasoning to be adopted; to guard against some possible *mistake* as to the Object proposed; or to apologise for some *deficiency*: this may be called the "Introduction preparatory."

5. And lastly, in many cases there will be occasion for what may be called a "Narrative Introduction," to put the reader or hearer in possession of the outline of some transaction, or the description of some state of things, to which references and allusions are to be made in the course of the Composition. Thus, in Preaching, it is generally found advisable to detail, or at least briefly to sum up, a portion of Scripture history, or a parable, when either of these is made the subject of a Sermon.

Two or more of the Introductions that have been mentioned are often combined, especially in the Preface to a work of any length.

And very often the Introduction will contain appeals to various passions and feelings in the hearers; especially a feeling of approbation towards the Speaker, or of prejudice against an opponent who has preceded him; but this is, as Aristotle has remarked, by no means confined to Introductions.*

* It has not been thought necessary to treat of Conclusion, Peroration, or Epilogue, as a distinct head: the general rules, that a Conclusion should be neither sudden and abrupt, (so as to induce the hearer to say, "I did not know he was going to *leave off*,") nor, again, so long as to excite the hearer's impatience after he has been led to expect an end, being so obvious as hardly to need being mentioned. The matter of which the concluding part of a Composition consists, will, of course, vary according to the subject and the occasion; but that which is most appropriate, and consequently most frequent (in Compositions of any considerable length,) is a Recapitulation, either of a part or the whole of the Arguments that have been used; respecting which a remark has been made at the end of Section 3.

Any thing relative to the Feelings and the Will, that may be especially appropriate to the Conclusion, will be mentioned in its proper place.

CHAPTER II.

OF PERSUASION.

Persuasion, properly so called, i. e. the Art of influencing the *Will*, is the next point to be considered. And Rhetoric is often regarded (as was formerly remarked) in a more limited sense, as conversant about this head alone. But even, according to that view, the rules above laid down will be found not the less relevant; since the *Conviction* of the understanding (of which we have hitherto been treating) is an essential part of Persuasion, and will generally need to be effected by the Arguments of the Writer or Speaker. For in order that the Will may be influenced, two things are requisite; viz. that the proposed *Object* should appear desirable; and that the *Means* suggested should be proved to be conducive to the attainment of that Object; and this last, evidently, must depend on a process of Reasoning. In order, e. g. to induce the Greeks to unite their efforts against the Persian invader, it was necessary to prove that cooperation could alone render their resistance effectual, and also to awaken such feelings of patriotism, and abhorrence of a foreign yoke, as might prompt them to make these combined efforts. For it is evident, that however ardent their love of liberty, they would make no exertions if they apprehended no danger; or if they thought themselves able, separately, to defend themselves, would be backward to join the confederacy; and on the other hand, that if they were willing to submit to the Persian yoke, or valued their independence less than their present ease, the fullest conviction that the Means recommended would secure their independence, would have no practical effect.

Persuasion, therefore, depends on 1st, *Argument*, (to prove the expediency of the Means proposed) and 2ndly, What is usually called *Exhortation*, i. e. the incitement of men to adopt those Means, by representing the End as sufficiently desirable. It will happen indeed, not unfrequently, that the one or the other of these Objects will have been already, either wholly or in part, accomplished, so that the other shall be the only one that it is requisite to insist on; viz. sometimes the hearers will be sufficiently intent on the pursuit of the End, and will be in doubt only as to the Means of attaining it; and sometimes, again, they will have no doubt on that point, but will be indifferent, or not sufficiently ardent, with respect to the proposed End, and will need to be stimulated by Exhortations. Not sufficiently ardent, we have said, because it will not so often happen that the Object in question will be one to which they are *totally* indifferent, as that they will, practically at least, not reckon it, or not feel it, to be worth the requisite pains. No one is absolutely indifferent about the attainment of a happy immortality; and yet a great part of the Preacher's business consists of Exhortation, i. e. endeavouring to induce men to use those exertions which they themselves know to be necessary for the attainment of it.

Aristotle, and many other writers, have spoken of Appeals to the Passions as an unfair mode of influencing the hearers; in answer to which Dr. Campbell has remarked, that there can be no Persuasion without an address to the Passions:* and it is evident, from what has been just said, that he is right, if under the term Passion is included every active principle of our nature. This however is a greater latitude of meaning than belongs even to the Greek word Πάθη, though the signification of that is wider than, according to ordinary use, that of our term "Passions." But Aristotle by no means overlooked the necessity for Persuasion, properly so termed, calling into action some motive that may influence the Will; it is plain that whenever he speaks with reprobation of an appeal to the Passions, his meaning is, the excitement of such feelings as *ought not to influence* the decision of the question in hand. A desire to do justice may be called, in Dr. Campbell's wide acceptation of the term, a Passion: this is what ought to influence a Judge; and no one would ever censure a Pleader for striving to excite and heighten this desire; but if the decision be influenced by an appeal to Anger, Pity, &c. the feelings thus excited being such as ought not to have operated, the Judge must be allowed to have been unduly biassed; and that this is Aristotle's meaning is evident from his characterising the introduction of such topics, as ἔξω τοῦ πράγματος, "foreign to the matter in hand." And it is evident that as the motives

* "To say, that it is possible to persuade without speaking to the passions, is but at best a kind of specious nonsense. The coolest reasoner always in persuading, addresseth himself to the passions some way or other. This he cannot avoid doing, if he speak to the purpose. To make me believe, it is enough to show me that things are so; to make me act, it is necessary to show that the action will answer some End. That can never be an End to me which gratifies no passion or affection in my nature. You assure me, 'It is for my honour.' Now you solicit my pride, without which I had never been able to understand the word. You say, 'It is for my interest.' Now you bespeak my self-love. 'It is for the public good.' Now you rouse my patriotism. 'It will relieve the miserable.' Now you touch my pity. So far therefore it is from being an unfair method of persuasion to move the passions, that there is no persuasion without moving them.

"But if so much depend on passion, where is the scope for argument? Before I answer this question, let it be observed, that, in order to persuade, there are two things which must be carefully studied by the orator. The first is, to excite some desire or passion in the hearers; the second is, to satisfy their judgment that there is a connection between the action to which he would persuade them, and the gratification of the desire or passion which he excites. This is the analysis of persuasion. The former is effected by communicating lively and glowing ideas of the object; the latter, unless so evident of itself as to supersede the necessity, by presenting the best and most forcible arguments which the nature of the subject admits. In the one lies the pathetic, in the other the argumentative. These incorporated together constitute that vehemence of contention to which the greatest exploits of Eloquence ought doubtless to be ascribed."—Campbell's *Philosophy of Rhetoric*, book i. c. vii. sec. 4.

which ought to operate will be different in different cases, the same may be objectionable and not fairly admissible in one case, which in another would be perfectly allowable.* An instance occurs in Thucydides, in which this is very judiciously and neatly pointed out: in the debate respecting the Mitylenæans, who had been subdued after a revolt, Cleon is introduced contending for the *justice* of inflicting on them capital punishment; to which Diodatus is made to reply, that the Athenians are not sitting in *judgment* on the offenders, but in *deliberation* as to their own *interest*; and ought therefore to consider, not the *right* they may have to put the revolters to death, but the *expediency* or inexpediency of such a procedure.

In judicial cases, on the contrary, any appeal to the personal interests of the Judge, or even to public expediency, would be irrelevant. In *framing* laws indeed, and (which comes to the same thing) giving those decisions which are to operate as precedents, the public good is the Object to be pursued; but in the mere *administering* of the established laws, it is inadmissible.

There are many feelings, again, which it is evident should in no case be allowed to operate, as Envy, thirst for Revenge, &c. &c. the excitement of which by the Orator is to be reprobated as an unfair artifice; but it is not the less necessary to be well acquainted with them, in order to allay them when previously existing in the hearers, or to counteract the efforts of an adversary in producing or influencing them. It is evident, indeed, that all the weaknesses, as well as the powers of the human mind, and all the arts by which the Sophist takes advantage of these weaknesses, must be familiarly known by a perfect Orator; who, though he may be of such a character as to disdain employing such arts, must not want the ability to do so, or be would not be prepared to counteract them. An acquaintance with the nature of poisons is necessary to him who would administer antidotes.

The active principles of our nature may be classed in various ways; the arrangement adopted by Mr. Dugald Stewart† is, perhaps, the most correct and convenient; the heads he enumerates are *Appetites*, (which have their origin in the body,) *Desires*, and *Affections*; these last being such as imply some kind of disposition relative to another *Person*; to which must be added, Self-love, or the desire of Happiness as such, and the Moral faculty, called by some writers Conscience, by others the Moral sense, and by Dr. A. Smith, the sense of Propriety.

Under the head of Affections may be included the sentiments of Esteem, Regard, Admiration, &c. which it is so important that the audience should feel towards the Speaker. Aristotle has considered this as a distinct head, separating the consideration of the speaker's Character (Ἦθος τοῦ λέγοντος) from that of the disposition of the hearers; under which, however, it might, according to his own views, have been included; it being plain from his manner of treating of the speaker's Character, that he means, not his real character, (according to the fanciful notion of Quinctilian,) but the impression produced on the minds of the hearers, by the speaker, respecting himself. He remarks, justly, that the Character to be established

* See the Treatise on FALLACIES, sec. 14.

† *Outlines of Moral Philosophy.*

is that of, 1st, Good Principle, 2ndly, Good Sense, and 3rdly, Goodwill and friendly disposition towards the audience addressed;* and that if the Orator can completely succeed in this, he will persuade more powerfully than by the strongest Arguments. He might have added, (as indeed he does slightly hint at the conclusion of his Treatise,) that, where there is an opponent, a like result is produced by exciting the contrary feelings respecting him; viz. holding him up to contempt, or representing him as an object of reprobation or suspicion.

To treat fully of all the different emotions and springs of action which an Orator may at any time find it necessary to call into play, or to contend against, would be to enter on an almost boundless field of Metaphysical inquiry, which does not properly fall within the limits of the subject now before us: and on the other hand, a *brief* definition of each passion, &c. &c. a few general remarks on it, could hardly fail to be trite and uninteresting. A few miscellaneous Rules therefore may suffice, relative to the conduct, generally, of those parts of any Composition which are designed to influence the Will.

§ 1. The first and most important point to be observed in every address to any Passion, Sentiment, Feeling, &c. is, that it should not be introduced as such, and plainly avowed; otherwise the effect will be, in great measure, if not entirely lost. This circumstance forms a remarkable distinction between the head now under consideration, and that of Argumentation. When engaged in Reasoning, properly so called, our purpose not only need not be concealed, but may, without prejudice to the effect, be distinctly declared: on the other hand, even when the feelings we wish to excite are such as ought to operate, so that there is no reason to be ashamed of the endeavours thus to influence the hearer, still, our purpose and drift should be, if not absolutely concealed, yet not openly declared, and made prominent. Whether the motives which the Orator is endeavouring to call into action, be suitable or unsuitable to the occasion, such as it is right, or wrong for the hearer to act upon, the same rule will hold good. In the latter case it is plain, that the speaker who is seeking to bias unfairly the minds of the audience will be the more likely to succeed by going to work clandestinely, in order that his hearers may not be on their guard, and prepare and fortify their minds against the impressions he wishes to produce; in the other case, where the motives dwelt on are such as ought to be present and strongly to operate, men are not likely to be pleased with the idea that they *need* to have these motives urged upon them, and that they are not already sufficiently under the influence of such sentiments as the occasion calls for. A man may indeed be convinced that he is in such a predicament, and may ultimately feel obliged to the Orator for exciting or strengthening such sentiments; but while he confesses this, he cannot but feel a degree of mortification in making the confession, and a kind of jealousy of the apparent assumption of superiority in a speaker, who seems to say, "now I will exhort you to feel as you ought on this occasion;" "I will endeavour to inspire you with such noble and generous, and amiable sentiments as you ought to entertain;" which is, in effect,

* Ἀρετὴ, Φρόνησις, Εὔνοια, book ii. c. i.

the tone of him who avows the purpose of Exhortation. The mind is sure to revolt from the humiliation of being thus moulded and fashioned, in respect to its feelings, at the pleasure of another; and is apt, perversely, to resist the influence of such a discipline.

Whereas there is no such implied superiority in avowing the intention of convincing the understanding: men know, and (what is more to the purpose) feel, that be who presents to their minds a new and cogent train of Argument, does not necessarily possess or assume any offensive superiority, but may, by merely having devoted a particular attention to the point in question, succeed in setting before them Arguments and Explanations which had not occurred to themselves; and even if the Arguments adduced, and the Conclusions drawn, should be opposite to those with which they had formerly been satisfied, still there is nothing in this so humiliating, as in that which seems to amount to the imputation of a moral defect.

It is true that Sermons not unfrequently prove popular, which consist avowedly and almost exclusively of Exhortation, strictly so called,—in which the design of influencing the sentiments and feelings is not only apparent, but prominent throughout; but it is to be feared, that those who are the most pleased with such discourses are more apt to apply these Exhortations to their neighbours than to themselves; and that each bestows his commendation rather from the consideration that such admonitions are much needed, and must be generally useful, than from finding them thus useful to himself.

When indeed the speaker has made some progress in exciting the feelings required, and has in great measure gained possession of his audience, a direct and distinct Exhortation to adopt the conduct recommended will often prove very effectual; but never can it be needful or advisable to *tell* them (as some do) that you are *going* to *exhort* them.

It will, indeed, sometimes happen that the excitement of a certain feeling will depend, in some measure, on a process of Reasoning; e. g. it may be requisite to prove, where there is a doubt on the subject, that the person recommended to the Pity, Gratitude, &c. of the hearers, is really an object deserving of these sentiments: but even then, it will almost always be the case, that the chief point to be accomplished shall be to raise those feelings to the requisite height, after the understanding is convinced that the occasion calls for them. And this is to be effected not by Argument, properly so called, but by presenting the circumstances in such a point of view, and so fixing and detaining the attention upon them, that corresponding sentiments and emotions shall gradually, and as it were spontaneously, arise.

§ 2. Hence arises another Rule, closely connected with the foregoing, though it also so far relates to Style that it might with sufficient propriety have been placed under that head; viz. that in order effectually to excite feelings of any kind, it is necessary to employ some copiousness of detail, and to dwell somewhat at large on the several circumstances of the case in hand; in which respect there is a wide distinction between strict Argumentation, with a view to the conviction of the understanding alone, and the attempt to influence the will by the excitement of any emotion. With respect to Argument itself indeed, different occasions will call for different degrees of Copiousness, Repetition, and Expansion; the chain of Reasoning employed, may, in itself, consist of more or fewer links; abstruse and complex Arguments must be unfolded at greater length than such as are more simple; and the more uncultivated the audience, the more full must be the explanation and illustration, and the more frequent the repetition of the Arguments presented to them; but still the same general principle prevails in all these cases; viz. to aim merely at letting the Arguments be fully *understood and admitted*; this will indeed occupy a shorter or longer space, according to the nature of the case and the character of the hearers; but all Expansion and Repetition *beyond* what is necessary to accomplish conviction, is in every instance tedious and disgusting. On the contrary, in a description of anything that is likely to act on the feelings, this effect will by no means be produced as soon as the understanding is sufficiently informed; detail and expansion are here not only admissible, but absolutely necessary, in order that the mind may have leisure and opportunity to form vivid and distinct ideas. For as Quinctilian well observes, he who tells us that a city was sacked, although that one word implies all that occurred, will produce little, if any, impression on the feelings, in comparison of one who sets before us a lively description of the various lamentable circumstances; to tell the *whole*, he adds, is by no means the same as to tell *every thing*.

§ 3. It is not however, always advisable to enter into a *direct* detail of circumstances, which would often have the effect of wearying the hearer beforehand, with the expectation of a long description of something in which he probably does not as yet feel much interest; and would also be likely to prepare him too much, and forewarn him as it were of the object proposed,—the design laid against his feelings. It will often, therefore, have a better effect to describe obliquely, (if we may so speak,) by introducing circumstances connected with the main Object or event, and affected by it, but not absolutely forming a part of it. And circumstances of this kind may not unfrequently be selected, so as to produce a more striking impression of anything that is in itself great and remarkable, than could be produced by a minute and direct description; because in this way the general and collective result of a whole, and the effects produced by it on other objects, may be vividly impressed on the hearer's mind; the circumstantial detail of *collateral* circumstances not drawing off the mind from the contemplation of the principal matter as one and complete. Thus the woman's application to the King of Samaria, to compel her neighbour to fulfil her agreement of sharing with her the infant's flesh, gives a more frightful impression of the horrors of the famine than any more direct description could have done; since it presents to us the picture of that hardening of the heart to every kind of horror, and that destruction of the ordinary state of human sentiment, which is the *result* of long-continued and extreme misery. Nor could any detail of the particular vexations suffered by the exiled Jews for their disobedience, convey so lively an idea of them as that description of their *result* contained in the denunciation of Moses; "in the evening thou shalt say, would

God it were morning, and in the morning thou shalt say, would God it were evening."

In the poem of Rukeby, a striking exemplification occurs of what has been said: Bertram is describing the prowess he had displayed as a Buccaneer, does not particularize any of his exploits, but alludes to the terrible impression they had left:

> "Panama's maids shall long look pale,
> When Risingham inspires the tale;
> Chili's dark matrons long shall *tame*,
> *The froward child* with Bertram's name."

The first of Dramatists, who might have been perhaps the first of Orators, has afforded some excellent exemplifications of this rule, especially in the speech of Antony over Cæsar's body.

§ 4. Comparison is one powerful means of exciting or heightening any emotion; viz. by presenting a parallel between the case in hand and some other that is calculated to call forth such emotions: taking care, of course, to represent the present case as stronger than the one it is compared with, and such as ought to affect us more powerfully.

When several successive steps of this kind are employed to raise the feelings gradually to the highest pitch, (which is the principal employment of what Rhetoricians call the Climax,*) a far stronger effect is produced than by the mere presentation of the most striking object at once. It is observed by all travellers who have visited the Alps, or other stupendous mountains, that they form a very inadequate notion of the vastness of the greater ones, till they ascend some of the less elevated, (which yet are huge mountains,) and thence view the others still towering above them. And the mind, no less than the eye, cannot so well take in and do justice to any vast object, at a single glance, as by several successive approaches and repeated comparisons. Thus in the well-known Climax of Cicero in the Oration against Verres, shocked as the Romans were likely to be at the bare mention of the crucifixion of one of their citizens, the successive steps by which he brings them to the contemplation of such an event, were calculated to work up their feelings to a much higher pitch: "It is an outrage to bind a Roman citizen; to scourge him is an atrocious crime; to put him to death is almost parricide; but to *crucify* him—what shall I call it?"

It is observed, accordingly, by Aristotle, in speaking of Panegyric, that the person whom we would hold up to admiration, should always be compared, and advantageously compared, if possible, with those that are already illustrious, but if not, at least with some person whom he excells: to *excel*, being in itself, he says, a ground of admiration. The same rule will apply, as has been said, to all other feelings as well as to Admiration: Anger, or Pity, for instance, are more effectually excited if we produce cases such as would call forth those passions, and which though similar to those before us, are not so strong; and so with respect to the rest.

When it is said, however, that the Object which we compare with another, should be one which ought to excite the feeling in question in a higher degree than that other, it is not meant that this must actually be, already, the impression of the hearers: the reverse will more commonly be the case; that the instances adduced will be such as actually affect their feelings more strongly than that to which we are endeavouring to turn them, till the flame spreads, as it were, from the one to the other. This will especially hold good in every case where *self* is concerned; e. g. men feel naturally more indignant at a slight affront offered to themselves, or those closely connected with them, than at the most grievous wrong done to a stranger; if therefore you would excite their utmost indignation in such a case, it must be by comparing it with a parallel case that concerns themselves; i. e. by leading them to consider how they would feel were such and such an injury done to themselves. And, on the other hand, if you would lead them to a just sense of their own faults, it must be by leading them to contemplate like faults in others; of which the celebrated parable of Nathan, addressed to David, affords an admirable instance.

§ 5. Another Rule, (which also is connected in some degree with Style) relates to the tone of feeling to be manifested by the writer or speaker himself, in order to excite the most effectually the desired emotions in the minds of the hearers. And this is to be accomplished by two opposite methods: the one, which is most obvious, is to *express* openly the feeling in question; the other, to seem labouring to *suppress* it: in the former method, the most forcible remarks are introduced,—the most direct as well as impassioned kind of description is employed,—and something of exaggeration introduced, in order to carry the hearers as far as possible in the same direction in which the Orator seems to be himself hurried, and to infect them to a certain degree with the emotions and sentiments which he thus manifests: the other method, which is often no less successful, is to abstain from all remarks, or from all such as come up to the expression of feeling which the occasion seems to authorize,—to use a gentler mode of expression than the case might fairly warrant,—to deliver "an unvarnished tale," leaving the hearers to make their own comments,—and to appear to stifle and studiously to keep within bounds such emotions as may seem natural. This produces a kind of reaction in the hearers' minds; and being struck with the inadequacy of the expressions, and the laboured calmness of the speaker's manner of stating things, compared with what he may naturally be disposed to feel, they will often rush into the opposite extreme, and become the more strongly affected by that which is set before them in so simple and modest a form. And though this method is in reality more artificial than the other, the artifice is the more likely (perhaps for that very reason) to escape detection; men being less on their guard against a speaker who does not seem so much labouring to work up *their* feelings, as to repress or moderate his own; provided that this calmness and coolness of manner be not carried to such an extreme as to bear the appearance of affectation; which caution is also to be attended to in the other mode of procedure no less; an excessive hyperbolical exaggeration being likely to defeat its own object. Aristotle mentions, (*Rhet.* book Ix.) though very briefly,

* An analogous Arrangement of *Arguments* in order to set forth the full force of the one we mean to dwell upon, would also receive the same appellation, and in fact is very often combined and blended with that which is here spoken of.

 these two modes of rousing the feelings, the latter under the name of *Eironeia*, which in his time was commonly employed to signify, not according to the modern use of "Irony, saying the *contrary* to what is meant," but, what later writers usually express by *Litotes*, i. e. "saying *less* than is meant."

The two methods may often be both used on the same occasion, beginning with the calm, and proceeding to the impassioned, afterwards, when the feelings of the hearers are already wrought up to a certain pitch: *ὅταν ἔχῃ ἤδη τοὺς ἀκροατὰς, καὶ ποιήσῃ ἐνθουσιάσαι.** Universally indeed it is a fault carefully to be avoided, to express feeling more vehemently than that the audience can go along with the speaker; who would, in that case, as Cicero observes, seem like one raving among the sane, or intoxicated in the midst of the sober. And accordingly, except where from extraneous causes the audience are already in an excited state, we must carry them forward gradually, and allow time for the fire to kindle. The blast which would heighten a strong flame, would, if applied too soon, extinguish the first faint spark. The speech of Antony over Cæsar's corpse, which has been already mentioned, affords an admirable example of that combination of the two methods which has been just spoken of.

Generally however, it will be found that the same Orators do not excel equally in both modes of exciting the Passions; and it should be recommended to each to employ principally that in which he succeeds best, since either, if judiciously managed, will generally prove effectual for its object. The well-known tale of Inkle and Yarico, which is an instance of the *extenuating* method, (as it may be called,) could not, perhaps, have been rendered more affecting, if equally so, by the most impassioned vehemence and rhetorical heightening.

§ 6. When the occasion or Object in question is not such as calls for, or as is likely to excite in those particular readers or hearers, the emotions required, it is a common Rhetorical artifice to turn their attention to some Object which *will* call forth these feelings; and when they are too much excited to be capable of judging calmly, it will not be difficult to turn their Passions, once roused, in the direction required, and to make them view the case before them in a very different light. When the metal is heated, it may easily be moulded into the desired form. Thus vehement indignation against some crime, may be directed against a person who has not been proved guilty of it; and vague declamations against corruption, oppression, &c. or against the mischiefs of anarchy; with high-flown panegyrics on liberty, rights of man, &c. or on social order, justice, the constitution, law, religion, &c. will gradually lead the hearers to take for granted, without proof, that the measure proposed will lead to these evils or these advantages; and it will in consequence become the object of groundless abhorrence or admiration. For the very utterance of such words as have a multitude of what may be called *stimulating* ideas associated with them, will operate like a charm on the minds, especially of the ignorant and unthinking, and raise such a tumult of feeling, as will effectually blind their judgment; so that a string of vague abuse or panegyric, will often have the effect of a train of sound Argument. This artifice falls under the head of "Irrelevant Conclusion," or *ignoratio elenchi*, mentioned in the Treatise on *Fallacies*. (Art. Logic, ch. v. sec. 14.) Mr. Bentham has treated of the employment of these "passion-kindling appellatives," as he calls them, under the head of "Fallacies of Confusion," in his work entitled the *Book of Fallacies*. Many other observations, also occurring in that Treatise, will be found very nearly to coincide with that which has been said in the fifth chapter of the Article on Logic just referred to; though not to be so strictly tried by Logical rules. Of many popular Sophisms he has given (though in a singular manner,) an able exposure; and of many others, unfortunately, the most striking exemplifications may be found in his own reasonings; in which *petitio principii* in particular, occurs perpetually; as well as the one now before us, the employment of vituperative, or as he calls it "Dyslogistic" language. that also which we there described as the "Fallacy of Objections," (which might be called by a lover of new-coined epithets, in language similar to that often employed by Mr. Bentham, a reverse-of-wrong-for-right-mistaking Fallacy) is skilfully described, and but too often employed; as if, because existing abuses are maintained by those who have an interest to *keeping* what they have, no apprehension were to be entertained of new evils being introduced through the interested conduct of those who wish to *acquire* what they have not;* and as if, because many are misled by a blind veneration for "Autbority" and the "Wisdom of our Ancestors," there did not exist also, as antagonist muscles, as it were, to these, an equally blind craving after novelty for its own sake, and a veneration for the ingenuity of one's own inventions.

It is matter of regret that the powers of such a mind as that of Mr. Bentham, should be to so great a degree wasted. Such, however, must always be the case, when a Scientific work is composed (with whatever sincerity) for party purposes, or with any object foreign to the precise End of the Science in question. Many Arguments accordingly are, in the work alluded to, stigmatised as Fallacies, which may be, either sophistical, or sound and fair, according to the circumstances in which they are employed; such as that a certain proposed reformation ought to be effected "gradually;" that we must "wait awhile, the present not being the time for such and such a measure;" or that "this or that proposal comes from a suspicious quarter," &c. which are topics that may be fairly or unfairly urged. And it is but too plain that the line is drawn not with a view to the *mode* in which, but to the *object* for which, and the *party* by which, each Argument is urged. It is only when certain classes

* Aristotle, *Rhet.* book iii. ch. vii.

* Those who will not profit by the lessons of the French Revolution, are not, perhaps, likely to learn wisdom from the great historian of Greece, who has so well described the workings of human passion as manifested in the civil commotions of Corcyra and other States. But to such as are willing to receive instruction, that which he affords can never cease to be valuable.—Ἐν δ' οὖν τῇ Κερκύρᾳ τὰ πολλὰ αὐτῶν προυτολμήθη, καὶ ὁπόσα ὕβρει μὲν ἀρχόμενοι τὸ πλέον ἢ σωφροσύνῃ, ὑπὸ τῶν, τὴν τιμωρίαν παρασχόντων, οἱ ἀνταμυνόμενοι δράσειαν· ΠΕΝΙΑΣ ΔΕ ΤΗΣ ΕΙΩΘΥΙΑΣ ΑΠΑΛΛΑΞΕΙΟΝΤΕΣ τινὲς, μάλιστα δ' ἂν διὰ πάθους ἐπιθυμοῦντες τὰ τῶν πέλας ἔχειν, παρὰ δίκην γιγνώσκοιεν· * * * * * ξυνταραχθέντος τε τοῦ βίου, ἐς τὸν καιρὸν τοῦτον, τῇ πόλει, καὶ τῶν νόμων κρατήσασα ἡ ἀνθρωπεία φύσις, εἰωθυῖα καὶ παρὰ τοὺς νόμους ἀδικεῖν, ἀσμένη ἐδήλωσεν, ἀκρατὴς μὲν ὀργῆς οὖσα, κρείσσων δὲ τοῦ δικαίου, ΠΟΛΕΜΙΑ ΔΕ ΤΟΥ ΠΡΟΥΧΟΝΤΟΣ. Thucyd. book iii. sec. 84.

of Propositions are distinctively pointed out as absolutely false, inadmissible, or irrelevant, or certain deductions from true ones shown to be unfair, that any useful warning can be supplied. The hopes, therefore, which the author entertains (p. 410,) that by the general study and adoption of his Principles, debates may be cleared and shortened, (each Fallacy being detected, exposed by name, and exploded, as soon as uttered) seem more sanguine than well-founded. If the general adoption, by the great majority of the audience, of the same system, means, their being of the same party, no doubt they would readily and easily silence by clamour every opposite Argument; but if they are merely to agree in adopting Logical principles as ill-defined as those we are speaking of, the proposed plan for the ready exposure of each fallacious Argument, resembles that by which children are deluded, of catching a bird by laying salt on its tail; the existing doubts and difficulties of debate being no greater than, on the proposed system, would be found in determining *what* Arguments were, and what not, to be classed with the Fallacies in question.

The work, however may be read safely, and, perhaps, not without advantage, by those who have sufficient interest in the subject to encounter the obscurity of the style, and sufficient patience in investigation, and power of discrimination, to separate the particles of gold-dust from the mass of sand and weed with which they are blended. It has been thought advisable therefore to make this reference to an author who is, perhaps, too generally regarded, except by the very small number of disciples who idolize him, with that unmixed contempt which is due to a *portion* only (though certainly no inconsiderable portion) of his tenets. Among posterity, the opinions entertained of him may probably be less violently contrasted, and, on the whole, more favourable; at least it usually happens that those who have manifested any considerable original powers, and have elicited valuable truths, however contaminated by the most extravagant errors, are remembered, even more favourably than is strictly their due; their absurdities are gradually forgotten, like the inscription on plaster on the light-house of Pharos, which mouldered away by the action of the weather; while the value of their discoveries is durably recorded, and becomes more and more conspicuous, like the inscription engraved on the marble beneath.

§ 7. In raising a favourable impression of the speaker, or an unfavourable one of his opponent, a peculiar tact will of course be necessary; especially in the former, since direct self-commendation will usually be disgusting to a greater degree, even than a direct personal attack on another; though, if the Orator is pleading his own cause, or one in which he is personally concerned, (as was the case in the speech of Demosthenes concerning the Crown,) a greater allowance will be made for him on this point; especially if he be a very eminent person, and one who may safely appeal to public actions performed by him. Thus Pericles is represented by Thucydides as claiming directly, when speaking in his own vindication, exactly the qualities (good Sense, good Principle, and Good-will,) which Aristotle lays down as constituting the character which we must seek to appear in. But then it is to be observed, that the historian represents him as accustomed to address the people with more authority than others for the most part ventured to assume. It is by the expression of wise, amiable, and generous *Sentiments*, that Aristotle recommends the speaker to manifest his own character; but even this must generally be done in an *oblique** and seemingly incidental manner, lest the hearers be disgusted with a pompous and studied display of fine sentiments; and care must also be taken not to affront them by seeming to inculcate as something likely to be new to them, maxims which they regard as almost truisms. Of course the application of this last caution must vary according to the character of the persons addressed; that might excite admiration and gratitude in one audience, which another would receive with indignation and ridicule. Most men, however, are disposed rather to overrate than to extenuate their own moral judgment; or at least to be jealous of any one's appearing to underrate it.

Universally indeed, in the Arguments used, as well as in the appeals made to the Feelings, a consideration must be had of the hearers, whether they are learned or ignorant,—of this or that profession,—nation,—character, &c. and the address must be adapted to each; so that there can be no excellence of writing or speaking in the abstract; nor can we any more pronounce on the Eloquence of any Composition, than upon the wholesomeness of a medicine, without knowing for whom it is intended. The less enlightened the hearers, the harder, of course, it is to make them comprehend a long and complex train of Reasoning; so that sometimes the Arguments, in themselves the most cogent, cannot be employed at all with effect; and the rest will need an expansion and copious illustration which would be needless, and therefore tiresome, (as has been above remarked,) before a different kind of audience: on the other hand, their feelings may be excited by much bolder and coarser expedients; such as those are the most ready to employ, and the most likely to succeed in, who are themselves but a little removed above the vulgar; as may be seen in the effects produced by fanatical preachers. But there are none whose feelings do not occasionally need and admit of excitement by the powers of Eloquence; only there is a more exquisite skill required in thus affecting the educated classes than the populace.†

* E. g. "It would be needless to impress upon you the maxim," &c. "You cannot be ignorant," &c. &c. "I am not advancing any high pretensions in expressing the sentiments which such an occasion must call forth in every honest heart," &c.

† "The less improved in knowledge and discernment the hearers are, the easier it is for the speaker to work upon their passions, and by working on their passions, to obtain his end. This, it must be owned, appears on the other hand, to give a considerable advantage to the preacher, as in no congregation can the bulk of the people be regarded as on a footing, in point of improvement, with either House of Parliament, or with the Judges in a Court of Judicature. It is certain, that the more gross the hearers are, the more avowedly may you address yourself to their passions, and the less occasion there is for argument; whereas, the more intelligent they are, the more covertly must you operate on their passions, and the more attentive must you be in regard to the justness, or at least the speciousness of your reasoning. Hence some have strangely concluded, that the only scope for eloquence is in haranguing the multitude; that in gaining over to your purpose men of knowledge and breeding, the exertion of Oratorical talents hath no influence. This is precisely as if one should argue, because a mob is much more easily subdued than regular troops, there is no occasion for the

Rhetoric.

Chap. II.

In no point more than in that now under consideration, viz. the Conciliation (to adopt the term of the Latin writers) of the hearers, is it requisite to consider who and what the hearers are; for when it is said that good Sense, good Principle, and Good-will, constitute the character which the speaker ought to establish of himself, it is to be remembered that every one of these is to be considered in reference to the opinions and habits of the audience. To think very differently from his hearers, may often be a sign of the Orator's wisdom and worth; but *they* are not likely to consider it so. A witty Satirist,* has observed, that "it is a short way to obtain the reputation of a wise and reasonable man, whenever any one tells you his opinion, to agree with him." Without going the full length of completely acting on this maxim, it is absolutely necessary to remember, that in proportion as the speaker manifests his dissent from the opinions and principles of his audience, so far he runs the risk at least, of impairing their estimation of his judgment. But this it is often necessary to do when any serious object is proposed; because it will commonly happen that the very End aimed at shall be one which implies a change of sentiments, or even of principles and character, in the hearers. Those indeed who aim only at popularity, are right in conforming their sentiments to those of the hearers, rather than the contrary; but it is plain that though in this way they obtain the greatest reputation for Eloquence, they deserve it the less; it being much easier, according to the tale related of Mahomet, to go to the mountain, than to bring the mountain to us.† There is but little Eloquence in convincing men that they are in the right, or inducing them to approve a character which coincides with their own.

The Christian preacher therefore is in this respect placed in a difficult dilemma, since he may be sure that the less he complies with the depraved judgments of man's corrupt nature, the less acceptable is he likely to be to that depraved judgment.

But he who would claim the highest rank as an Orator, (to omit all higher considerations) must be the one who is the most successful, not in gaining popular applause, but in *carrying his point*, whatever it be. The preacher, however, who is intent on this object, should use all such precautions as are not inconsistent with it, to avoid raising unfavourable impressions in his hearers. Much will depend on a gentle and conciliatory manner; nor is it necessary that he should, at once, in an abrupt and offensive form, set forth *all* the differences of sentiment between himself and his congregation, but win them over by degrees; and in whatever point, and to whatever extent, he may suppose them to agree with him, it is allowable, and for that reason advisable, to dwell on that agreement; as the Apostles began every address to the Jews by an appeal to the Prophets, whose authority they admitted; and as St. Paul opens his discourse to the Athenians (though unfortunately the words of our translation are likely to convey an opposite idea,*) by a commendation of their respect for religion. And above all, where censure is called for, the speaker should avoid, on Christian, as well as on Rhetorical principles, all appearance of exultation in his own superiority,—of contempt,—or of uncharitable triumph in the detection of faults; "in *meekness, instructing* them that oppose themselves."

Of intellectual qualifications, there is one which it is evident, should not only not be blazoned forth, but should in a great measure be concealed, or kept out of sight; viz. Rhetorical skill; since whatever is attributed to the Eloquence of the speaker is so much deducted from the strength of his cause. Hence, Pericles is represented by Thucydides as artfully claiming, in his vindication of himself, the power of *explaining* the measures he proposes, not, Eloquence in *persuading* their adoption. And accordingly a skilful Orator seldom fails to notice and extol the Eloquence of his opponent, and to warn the hearers against being misled by it. It is a peculiarity therefore in the Rhetorical art, that in it, more than in any other, vanity has a direct and immediate tendency to interfere with the proposed object. Excessive vanity may indeed, in various ways, prove an impediment to success in other pursuits; but in the endeavour to Persuade, all wish to appear excellent in that art, operates as a hindrance. A Poet, a Statesman, or a General, &c. though extreme covetousness of applause may mislead them, will, however, attain their respective Ends, certainly not the less for being admired as excellent in Poetry, Politics, or War; but the Orator attains his End the better the less he is regarded as an Orator; if he can make the hearers believe that he is not only a stranger to all unfair artifice, but even destitute of all Persuasive skill whatever, he will Persuade them the more effectually; and if there ever could be an absolutely perfect Orator,

art of war, nor is there a proper field for the exertion of military skill, unless when you are quelling an undisciplined rabble. Every body sees in this case, not only how absurd such a way of arguing would be, but that the very reverse ought to be the conclusion. The reason why people do not so quickly perceive the absurdity in the other case, is, that they affix no distinct meaning to the word *eloquence*, often denoting no more by that term than simply the power of moving the passions. But even in this improper acceptation, their notion is far from being just; for wherever there are men, learned or ignorant, civilized or barbarous, there are passions; and the greater the difficulty is in affecting these, the more art is requisite." Campbell's *Rhetoric*, book i. ch. x. sec. 2. p. 224, 225.

* Swift.

† "Little force is necessary to push down heavy bodies placed on the verge of a declivity, but much force is requisite, to stop them in their progress, and push them up. If a man should say, that because the first is more frequently effected than the last, it is the best trial of strength, and the only suitable use to which it can be applied, we should at least not think him remarkable for distinctness in his ideas. Popularity alone, therefore, is no test at all of the eloquence of the speaker, no more than velocity alone would be, of the force of the external impulse originally given to the body moving. As in this the direction of the body, and other circumstances, must be taken into the account; so in that, you must consider the tendency of the teaching, whether it favours or opposes the vices of the hearers. To head a sect, to infuse party-spirit, to make men arrogant, uncharitable, and malevolent, is the easiest task imaginable, and to which almost any blockhead is fully equal. But to produce the contrary effect, to subdue the spirit of faction, and that monster spiritual pride, with which it is invariably accompanied, to inspire equity, moderation, and charity into men's sentiments and conduct with regard to others, is the genuine test of eloquence." Campbell's *Rhetoric*, book i. ch. x. sec. 5. p. 239.

* Δεισιδαιμονεστέρους, not "too superstitious," but (as almost all commentators are now agreed) "very much disposed to the worship of Divine beings."

no one would, at the time at least, discover that he was so. And this consideration may serve to account for the fact which Cicero remarks upon (*De Oratore*, book i.) as so inexplicable; viz. the small number of persons who, down to his time, had obtained high reputation as Orators, compared with those who had attained excellence in other pursuits. Few men are destitute of the desire of admiration; and most are especially ambitious of it in the pursuit to which they have chiefly devoted themselves; the Orator therefore is continually tempted to sacrifice the substance to the shadow, by aiming rather at the admiration of the hearers, than their conviction; and thus to fall of that excellence in his art which he might otherwise be well qualified to attain, through the desire of a reputation for it. And on the other hand, some may have been really Persuasive speakers, who yet may not have ranked high in men's opinion, and may not have been known to possess that art of which they gave proof by their skilful concealment of it. There is no point, in short, in which report is so little to be trusted.

Of the three points which Aristotle directs the Orator to claim credit for, it might seem at first sight that one, viz. "Good-will," is unnecessary to be mentioned; since Ability and Integrity would appear to comprehend, in most cases at least, all that is needed; a virtuous man, it may be said, must wish well to his countrymen, or to any persons whatever, whom he may be addressing. But on a more attentive consideration, it will be manifest that Aristotle had good reason for mentioning this head; if the speaker were believed to wish well to his *Country*, and to every individual of it, yet if he were suspected of being unfriendly to the political or other *Party* to which his hearers belonged, they would listen to him with prejudice. The abilities and the conscientiousness of Phocion seem not to have been doubted by any; but they were so far from gaining him a favourable hearing among the Democratical party at Athens, (who knew him to be no friend to Democracy,) that they probably distrusted him the more; as one whose public spirit would induce him, and whose talents would enable him, to subvert the existing Constitution.

One of the most powerful engines, accordingly, of the Orator, is this kind of appeal to party-spirit. Party-spirit may, indeed, be considered in another point of view, as one of the Passions which may be *directly* appealed to, when it can be brought to operate in the direction required; i. e. when the conduct the writer or speaker is recommending appears likely to gratify party-spirit; but it is the *indirect* appeal to it which is now under consideration; viz. the favour, credit, and weight which the speaker will derive from appearing to be of the same party with the hearers, or at least not opposed to it. And this is a sort of credit which he may claim more openly and avowedly than any other; and likewise may throw discredit on his opponent in a less offensive, but not less effectual manner. A man cannot say in direct terms, "I am a wise and worthy man, and my adversary the reverse;" but he is allowed to say, "I adhere to Whig or Tory principles," (as the case may be,) and "my opponent the reverse;" which is not regarded as an offence against modesty, and yet amounts virtually to as strong a self-commendation, and as decided vituperation, in the eyes of those imbued with party-spirit, as if every kind of merit and of demerit had been enumerated: for to zealous party men, zeal for their party will very often either imply, or stand as a substitute for, every other kind of worth.

Hard, indeed, therefore is the task of him whose object is to counteract party-spirit and to soften the violence of those prejudices which spring from it.* His only resource must be to take care that he give no ground for being supposed imbued with the violent and unjust prejudices of the opposite party,—that he give his audience credit (since it rarely happens but that each party has some tenets that are reasonable,) for whatever there may be that deserves praise,—that he proceed gradually and cautiously in removing the errors with which they are infected,—and above all, that he studiously disclaim and avoid the appearance of any thing like a feeling of *personal* hostility, or personal contempt.

If the Orator's character can be sufficiently established in respect of Ability, and also of Good-will towards the hearers, it might at first sight appear as if this would be sufficient; since the former of these would imply the Power, and the latter, the Inclination, to give the best advice, whatever might be his Moral character; but Aristotle (in his *Politics*) justly remarks that this last is also requisite to be insisted on, in order to produce entire confidence; for, says he, though a man cannot be suspected of wanting Good-will towards *himself*, yet many very able men act most absurdly, even in their own affairs, for want of Moral virtue, being either blinded or overcome by their Passions, so as to sacrifice their own most important interests to their present gratification; and much more, therefore, may they be expected to be thus seduced by personal temptations, in the advice they give to others. Pericles, accordingly, in the speech which has been already referred to, is represented by Thucydides as insisting not only on his political ability and his patriotism, but also on his unimpeached integrity, as a qualification absolutely necessary to entitle him to their confidence: for "the man," says he, "who possesses every other requisite, but is overcome by the temptation of a bribe, will be ready to sell every thing for the gratification of his avarice."

From what has been said of the speaker's recommendation of himself to the audience, and establishment of his authority with them, sufficient Rules may readily be deduced for the analogous process, the depreciation of an opponent. Both of these, and especially the latter, under the offensive title of *personality*, are by many indiscriminately decried as unfair Rhetorical artifices; and, doubtless they are, in the majority of cases, sophistically employed; and by none more effectually than by those who are perpetually declaiming against such Fallacies; the unthinking hearers not being prepared to expect them from

* "Of all the prepossessions in the minds of the hearers, which tend to impede or counteract the design of the speaker, party-spirit, where it happens to prevail, is the most pernicious, being at once the most inflexible, and the most unjust. * * * * Violent party men not only lose all sympathy with those of the opposite side, but even contract an antipathy to them. This, on some occasions, even the divinest eloquence will not surmount." Campbell's *Rhetoric*.

 those who represent themselves as holding them in such abhorrence. But surely it is not in itself an unfair topic of Argument, in cases not admitting of decisive and unquestionable proof, to urge that the one party deserves the hearers' confidence, or that the other is justly an object of their distrust. "If the measure is a good one," says Mr. Bentham, "will it become bad because it is supported by a bad man? if it is bad, will it become good, because supported by a good man? If the measure be really inexpedient, why not at once show that it is so? Your producing these irrelevant and inconclusive Arguments, in lieu of direct ones, though not sufficient to prove that the measure you thus oppose is a good one, contributes to prove that you yourself regard it as a good one." Now there is no doubt that the generality of men are too much disposed to consider more, *who* proposes a measure, than *what* it is that is proposed; and probably would continue to do so, even under a system of annual Parliaments and universal suffrage; and if a warning be given against an excessive tendency to this way of judging, it is reasonable, and may be useful; nor should any one escape censure who *confines* himself to these topics, or dwells principally on them, in cases where "direct" Arguments are to be expected; but they are not to be condemned *in toto* as "irrelevant and inconclusive," because they are only probable, and not in themselves decisive; it is only in matters of strict science, and that too, in arguing to scientific men, that the character of the advisers (as well as all other *probable* Arguments,) should be wholly put out of the question. And it is remarkable that the necessity of allowing some weight to this consideration, in political matters, increases in proportion as any country enjoys a free government; if all the power be in the hands of a few of the higher orders, who have the opportunity at least, of obtaining education, it is conceivable, whether probable or not, that they may be brought to try each proposed measure exclusively on its intrinsic merits, by abstract Arguments; but can any man, in his senses, really believe that the great mass of the people, or even any considerable portion of them, can ever possess so much political knowledge, patience in investigation, and sound Logic, (to say nothing of candour,) as to be able and willing to judge, and to judge correctly, of every proposed political measure, in the abstract, without any regard to their opinion of the person who proposes it? And it is evident that in every case, in which the hearers are not completely competent judges, they not only will, but must, take into consideration the characters of those who propose, support, or dissuade any measure;—the persons they are connected with,—the designs they may be supposed to entertain, &c.; though, undoubtedly, an *excessive* and *exclusive* regard to Persons rather than Arguments, is one of the chief Fallacies against which men ought to be cautioned.

In no way, perhaps, are men, not bigoted to party, more likely to be misled by their favourable or unfavourable judgment of their advisers, than in what relates to the authority derived from *Experience*; not that Experience ought not to be allowed to have great weight; but that men are apt not to consider with sufficient attention, what it is that constitutes Experience in each point; so that frequently one man shall have credit for much Experience, in what relates to the matter in hand, and another, who, perhaps, possesses as much, or more, shall be underrated as wanting it. The vulgar, of all ranks, need to be warned, 1st, that *time* alone does not constitute Experience; so that many years may have passed over a man's head, without his even having had the same opportunities of acquiring it, as another, much younger: 2nd, that the longest practice in conducting any business in *one* way, does not necessarily confer any Experience in conducting it in a different way; e. g. an experienced Husbandman, or a Minister of State, in Persia, would be much at a loss in Europe; and if they had some things less to learn than an entire novice, on the other hand they would have much to unlearn: and, 3rd, that merely being conversant about a certain class of *subjects*, does not confer Experience in a case where the *Operations*, and the *End* proposed, are different. It is said that there was an Amsterdam merchant, who had dealt largely in corn all his life, who had never seen a field of wheat growing; this man had doubtless acquired, by Experience, an accurate judgment of the qualities of each description of corn,—of the best methods of storing it,—of the arts of buying and selling it at proper times, &c.; but he would have been greatly at a loss in its cultivation; though he had been, in a certain way, long *conversant* about corn. Nearly similar is the Experience of a practised Lawyer, (supposing him to be nothing more) in a case of *Legislation*; because be has been long *conversant about Law*, the unreflecting attribute great weight to his judgment; whereas his constant habits of fixing his thoughts on what the law *is*, and withdrawing it from the irrelevant question of what the law *ought* to be;—his careful observance of a multitude of Rules, (which afford the more scope for the display of his skill, in proportion as they are arbitrary, unreasonable, and unaccountable,) with a studied indifference as to that which is foreign from his business, the *convenience* or *inconvenience* of those Rules, may be expected to operate unfavourably on his judgment in questions of Legislation; and are likely to counterbalance the advantages of his superior knowledge, even in such points as do bear on the question. The consideration then of the character of the speaker, and of his opponent, being of so much importance, both as a legitimate source of Persuasion, in many instances, and also as a topic of Fallacies, it is evidently incumbent on the Orator to be well versed in this branch of the art, with a view both to the justifiable advancement of his own Cause, and to the detection and exposure of unfair artifice in an opponent. It is neither possible, nor can it, in justice be expected, that this mode of Persuasion should be totally renounced and exploded, great as are the abuses to which it is liable; but the speaker is bound, in conscience, to abstain from those abuses himself, and, in prudence, to be on his guard against them in others.

It only remains to observe, on this head, that, as Aristotle teaches, the *place* for the disparagement of an opponent is, for the first speaker, near the close of his discourse, to weaken the force of what may be said in reply; and, for the opponent, near the opening, to lessen the influence of what has been already said.

§ 8. Either a personal prejudice, such as has been just mentioned, or some other passion unfavourable to the speaker's Object, may already exist in the

minds of the hearer, which it must be his business to allay.

It is obvious that this will the most effectually be done, not by endeavouring to produce a state of perfect calmness and apathy, but by exciting some contrary emotion. And here it is to be observed that some passions may be, Rhetorically speaking, opposite to each other, though in strictness they are not so; viz. whenever they are incompatible with each other: e. g. the opposite, strictly speaking, to Anger, would be a feeling of Good-will and approbation towards the person in question; but it is not by the excitement of this, alone, that Anger may be allayed; for Fear is, practically, contrary to it also; as is remarked by Aristotle; who Philosophically accounts for this, on the principle that Anger implying a desire to inflict *punishment*, must imply also a supposition that it is *possible* to do so; and accordingly men do not, he says, feel Anger towards one who is so much superior as to be manifestly out of their reach; and the Object of their Anger ceases to be so, as soon as he becomes an Object of Apprehension. Of course the converse also of this holds good; Anger, when it prevails, in like manner subduing Fear. Compassion, likewise, may be counteracted either by Disapprobation, by Jealousy, by Fear, or by Disgust and Horror; and Envy, either by Good-will, or by Contempt.

This is the more necessary to be attended to, in order that the Orator may be on his guard against inadvertently defeating his own Object, by exciting feelings at variance with those he is endeavouring to produce, though not strictly contrary to them. Aristotle accordingly notices, with this view, the difference between the "Pitiable," (ἐλεεινὸν) and the "Horrible or Shocking," (δεινὸν;) which, as he observes, excite different feelings, destructive of each other; so that the Orator must be warned, if the former is his Object, to keep clear of any thing that may excite the latter.

It will often happen that it will be easier to give a *new direction* to the unfavourable passion, than to subdue it; e. g. to *turn* the indignation or the laughter of the hearers against a different object. Indeed, whenever the case will admit of this, it will generally prove the more successful expedient, because it does not imply the accomplishment of so great a change in the minds of the hearers

CHAPTER III

OF STYLE.

Though the consideration of Style has been laid down as holding a place in a Treatise of Rhetoric, it would be neither necessary nor pertinent, to enter fully into a *general* discussion of the subject, which would evidently embrace much that by no means peculiarly belongs to our present inquiry. It is requisite for an Orator, e. g. to observe the rules of Grammar; but the same may be said of the Poet and the Historian, &c. nor is there any *peculiar* kind of grammatical propriety belonging to Persuasive or Argumentative compositions; so that it would be a departure from our subject to treat at large, under the head of Rhetoric, of such rules as equally concern every other of the purposes for which Language is employed.

Conformably to this view we shall, under the present head, notice but slightly such principles of composition as do not exclusively or peculiarly belong to the present subject; confining our attention chiefly to such observations on Style as have an especial reference to Argumentative and Persuasive works.

§ 1. It is sufficiently evident (though the maxim is often practically disregarded) that the first requisite of Style not only in Rhetorical, but in all compositions, is Perspicuity; since, as Aristotle observes, language which is not intelligible, or not clearly and readily intelligible, fails, in the same proportion, of the purpose for which language is employed. And it is equally self-evident (though this truth is still more frequently overlooked) that Perspicuity is a *relative* quality, and consequently cannot properly be predicated of any work, without a tacit reference to the class of readers or hearers for whom it is designed. Nor is it enough that the Style be such as they are *capable* of understanding, *if* they bestow their utmost attention t· the degree and the kind of attention, which they have been accustomed, or are likely to bestow, will be among the circumstances that are to be taken into the account, and provided for. The *kind*, as well as the degree, of attention, is mentioned, because some hearers and readers will be found slow of apprehension indeed, but capable of taking in what is very copiously and gradually explained to them, while others on the contrary, who are much quicker at catching the sense of what is expressed in a short compass, are incapable of *long* attention, and are not only wearied, but absolutely bewildered, by a diffuse Style.

When a numerous and very mixed audience is to be addressed, much skill will be required in adapting the Style, (both in *this*, and in other respects,) and indeed the Arguments also, and the whole structure of the discourse, to the various minds which it is designed to impress; nor can the utmost art and diligence prove after all more than partially successful in such a case; especially when the diversities are so many and so great, as exist in the congregations to which most Sermons are addressed, and in the readers for whom popular works of an argumentative, instructive, and hortatory character, are intended. It is possible, however, to approach indefinitely to an object which cannot be completely attained, and to adopt such a Style and such a mode of Reasoning, as shall be level to the comprehension of the greater part, at least, even of a promiscuous audience, without being distasteful to any.

It is obvious, and sufficiently well known, that extreme conciseness is ill suited to hearers or readers, whose intellectual powers and cultivation are but small: the usual expedient, however, of employing a

prolix Style by way of accommodation to such minds, is seldom successful: most of those who could have comprehended the meaning, if more briefly expressed, and many of those who could not do so, are likely to be bewildered by tedious expansion; and being unable to maintain a steady attention to what is said, they forget part of what they have heard before the whole is completed. Add to which, that the feebleness produced by excessive dilution, (if such an expression may be allowed,) will occasion the attention to languish; and what is imperfectly attended to, however clear in itself, will usually be but imperfectly understood. Let not an author, therefore, satisfy himself by finding that he has expressed his meaning so that, if attended to, he cannot fail to be understood; he must consider also (as was before remarked) what attention is likely to be paid to it: if on the one hand much matter is expressed in very few words, to an unreflecting audience, or if, on the other hand, there is a wearisome prolixity, the requisite attention may very probably not be bestowed.*

The best general rule for avoiding the disadvantages both of conciseness and of prolixity, is to employ *Repetition*: to repeat, that is, the same sentiment and Argument in many different forms of expression; each in itself brief, but all, together, affording such an expansion of the sense to be conveyed, and so detaining the mind upon it, as the case may require. Cicero among the ancients, and Burke among the modern writers, afford, perhaps, the most abundant practical exemplifications of this rule. The latter sometimes shows a deficiency in correct taste, and lies open to Horace's censure of an author, "*Qui variare cupit rem prodigaliter unam*;" but it must be admitted that he seldom fails to make himself thoroughly understood, and does not often weary the attention, even when he offends the taste of his readers.

Care must of course be taken that the repetition may not be too glaringly apparent; the variation must not consist in the mere use of other, synonymous, words; but what has been expressed in appropriated terms may be repeated in metaphorical; the antecedent and consequent of an Argument, or the parts of an antithesis may be transposed; or several different points that have been enumerated, presented in a varied order, &c.

It is not necessary to dwell on that obvious rule laid down by Aristotle, to avoid uncommon, as they are vulgarly called, *hard* words, i. e. those which are such to the persons addressed; but it may be worth remarking, that to those who wish to be understood by the lower orders, one of the best principles of selection is to prefer terms of *Saxon* origin, which will generally be more familiar to them, than those derived from the Latin, (either directly or through the medium of the French,) even when the latter are more in use among persons of education. Our language being (with very trifling exceptions) made up of these elements, it is very easy for any one, though unacquainted with Saxon, to observe this precept, if he has but a knowledge of French or of Latin; and there is a remarkable scope for such a choice as we are speaking of, from the multitude of synonymes derived, respectively, from those two sources. The compilers of our Liturgy being anxious to reach the understandings of all classes, at a time when our language was in a less settled state than at present, availed themselves of this circumstance in employing many synonymous, or nearly synonymous expressions, most of which are of the description just alluded to. Take as an instance, the Exhortation: "acknowledge" and "confess;" "dissemble" and "cloak;" "humble" and "lowly;" "goodness" and "mercy;" "assemble" and "meet together:" and here it may be observed that, as in this last instance, a word of French origin will very often not have a *single word* of Saxon derivation corresponding to it, but may find an exact equivalent in a *phrase* of two or more words: e. g. "constitute," "go to make up;" "arrange," "put in order;" "substitute," "put in the stead," &c. &c.

It is worthy of notice that a Style composed chiefly of the words of French origin, while it is less intelligible to the lowest classes, is characteristic of those who in cultivation of taste are below the highest. As in dress, furniture, deportment, &c. so also in language, the dread of vulgarity constantly besetting those who are half conscious that they are in danger of it, drives them into the extreme of affected finery. So that the precept which has been given with a view to perspicuity, may, to a certain degree, be observed with an advantage in point of elegance also.

In adapting the Style to the comprehension of the illiterate, a caution is to be observed against the ambiguity of the word "*Plain*;" which is opposed sometimes to *Obscurity*, and sometimes to *Ornament*; the vulgar require a perspicuous, but by no means, a dry and unadorned Style; on the contrary, they have a taste rather for the over-florid, tawdry, and bombastic: nor are the ornaments of style by any means necessarily inconsistent with perspicuity; Metaphor, which is among the principal of them, is indeed, in many cases, the clearest mode of expression that can be adopted; it being usually much easier for uncultivated minds to comprehend a similitude or analogy, than an abstract term. And hence the language of savages, as has often been remarked, is highly metaphorical; and such appears to have been the case with all languages in their earlier, and consequently ruder and more savage state; many terms relating to the mind and its operations, being, as appears from their etymology, originally metaphorical, though by long use they have ceased to be so: e. g. the words "ponder," "deliberate," "reflect," and many other such, are evidently drawn by analogy from external sensible bodily actions.

In respect to the Construction of sentences, it is an obvious caution to abstain from such as are too long; but it is a mistake to suppose that the obscurity of many long sentences depends on their length

* It is remarked by Anatomists that the nutritive quality is not the only requisite in food;—that a certain degree of *distention* of the stomach is required, to enable it to act with its full powers;—and that it is for this reason hay and straw must be given to horses, as well as corn, in order to supply the necessary bulk. Something analogous to this takes place with respect to the generality of minds, which are incapable of thoroughly digesting and assimilating what is presented to them, however clearly, in a very small compass. Many a one is capable of deriving that instruction from a moderate sized volume, which he could not receive from a very small pamphlet, even more perspicuously written, and containing every thing that is to the purpose. It is necessary that the attention should be detained for a certain time on the subject: and persons of unphilosophical mind, though they can attend to what they read or hear, are unapt to dwell upon it in the way of subsequent meditation.

alone; a well constructed sentence of very considerable length may be more readily understood, than a shorter one which is more awkwardly framed. If a sentence be so constructed that the meaning of each part can be taken in as we proceed, (though it be evident that the sense is not brought to a close) its length will be little or no impediment to perspicuity; but if the former part of the sentence convey no distinct meaning till we arrive nearly at the end, however plain it may then appear, it will be on the whole deficient in perspicuity; for it will need to be read over, or *thought over*, a second time, in order to be fully comprehended; which is what few readers or hearers are willing to be burthened with. Take as an instance such a sentence as this: "It is not without a degree of patient attention and persevering diligence, greater than the generality are willing to bestow, though not greater than the object deserves, that the habit can be acquired of examining and judging of our own conduct with the same accuracy and impartiality as that of another:" this labours under the defect we are speaking of, which may be remedied by some such alteration as the following: "the habit of examining our own conduct as accurately as that of another, and judging of it with the same impartiality, cannot be acquired without a degree of patient attention and persevering diligence, not greater indeed than the object deserves, but greater than the generality are willing to bestow." The two sentences are nearly the same in length, and in the words employed; but the alteration of the arrangement allows the latter to be understood clause by clause, as it proceeds. The caution just given is the more necessary to be insisted on, because an author is apt to be misled by reading over a sentence to himself, and being satisfied on finding it perfectly intelligible, forgetting that he himself has the advantage, which a hearer has not, of knowing at the beginning of the sentence what is coming in the close.

Universally, indeed, an unpractised writer is liable to be misled by his own knowledge of his own meaning, into supposing those expressions clearly intelligible, which are so to himself; but which may not be so to the reader, whose thoughts are not in the same train. And hence it is that some do not write or speak with so much perspicuity on a subject which has long been very familiar to them, as on one which they understand indeed, but with which they are less intimately acquainted, and in which their knowledge has been more recently acquired. In the former case it is a matter of some difficulty to keep in mind the necessity of carefully and copiously explaining principles which by long habit have come to assume in our minds the appearance of self-evident truths. So far is Blair's notion from being correct, that obscurity of Style necessarily springs from indistinctness of Conception.

The foregoing rules have all, it is evident, proceeded on the supposition that it is the writer's *intention* to be understood; and this cannot but be the case in every legitimate exercise of the Rhetorical art; and generally speaking, even where the design is Sophistical. For, as Dr. Campbell has justly remarked, the Sophist may employ for his purpose what are in themselves real and valid Arguments, since probabilities may lie on opposite sides, though truth can be but on one; his fallacious artifice consisting only in keeping out of sight the stronger probabilities which may be urged against him, and in attributing an undue weight to those which he has to allege. Or again he may, either directly or indirectly, assume as self-evident a premiss which there is no sufficient ground for admitting; or he may draw off the attention of the hearers to the proof of some irrelevant point, &c. according to the various modes described in the Treatise on FALLACIES; but in all this there is no call for any departure from perspicuity of Style, properly so called; not even when he avails himself of an ambiguous term. "For though," as Dr. Campbell says, "a Sophism can be mistaken for an Argument only where it is not rightly understood," it is the aim of him who employs it, rather that the matter should be *misunderstood* than *not* understood;—that his language should be deceitful rather than obscure or unintelligible. The hearer must not indeed form a *correct*, but he must form some, and if possible, a distinct, though erroneous idea of the Arguments employed, in order to be misled by them. The obscurity in short, if it is to be so called, must not be obscurity of *Style*; that must be, not like a mist which dims the appearance of objects, but like a coloured glass which disguises them.

There are, however, certain spurious kinds, as they may be called, of writing or speaking, (distinct from what is strictly termed Sophistry,) in which obscurity of Style may be apposite. The object which has all along been supposed, is that of convincing or persuading; but there are some kinds of Oratory, if they are to be so named, in which different ends are proposed. One of these ends is, (when the cause is such that it cannot be sufficiently supported even by specious Fallacies,) to *appear to say something*, when there is in fact nothing to be said; so as at least to avoid the ignominy of being silenced. To this end, the more confused and unintelligible the language, the better, provided it carry with it the appearance of profound wisdom, and of being something to the purpose.

"Now though nothing (says Dr. Campbell) would seem to be easier than this kind of Style where an Author falls into it naturally; that is, when he deceives himself as well as his reader, nothing is more difficult when attempted of design. It is beside requisite, if this manner must be continued for any time, that it be artfully blended with some glimpses of meaning; else, to persons of discernment, the charm will at length be dissolved, and the nothingness of what has been spoken will be detected; nay even the attention of the unsuspecting multitude, when not relieved by any thing that is level to their comprehension, will infallibly flag. The Invocation in the Dunciad admirably suits the Orator who is unhappily reduced to the necessity of taking shelter in the unintelligible:

> "Of darkness visible so much be lent,
> As half to show, half veil the deep intent."
>
> Chap. viii. sec. 1. p. 119.

This artifice is distinguished from Sophistry, properly so called, (with which Dr. Campbell seems to confound it,) by the circumstance that its tendency is not, as in Sophistry, to convince, but to have the appearance of arguing, when in fact, nothing is urged; for in order for men to be convinced, on however insufficient grounds, they must (as was remarked above) understand *something* from what is said, though, if it be fallacious, they must not understand it *rightly*; but if this cannot be accomplished, the Sophist's next

 resort is the unintelligible, which indeed is very often intermixed with the Sophistical, when the latter is of itself too scanty or too weak. Nor does the adoption of this Style serve merely to save his credit as an Orator or Author; it frequently does more: ignorant and unreflecting persons, though they cannot be, strictly speaking, convinced, by what they do not understand, yet will very often suppose, each, that the rest understand it; and each is ashamed to acknowledge, even to himself, his own darkness and perplexity; so that if the speaker with a confident air announces his conclusion as established, they will often, according to the maxim "*omne ignotum pro mirifico*," take for granted that he has advanced valid Arguments, and will be loth to seem behind hand in comprehending them. It usually requires that a man should have some confidence in his own understanding, to venture to say, "what has been spoken is unintelligible to me."

Another purpose sometimes answered by a discourse of this kind, is that it serves to furnish an excuse, flimsy indeed, but not unfrequently sufficient, for men to vote or act according to their own inclinations; which they would perhaps have been ashamed to do, if strong Arguments had been urged on the other side, and had remained confessedly unanswered; but they satisfy themselves if something has been said in favour of the course they wish to adopt, though that something be only fair-sounding sentences that convey no distinct meaning. They are content that *an answer* has been made, without troubling themselves to consider what it is.

Another end, which in speaking, is sometimes proposed, and which is, if possible, still more remote from the legitimate province of Rhetoric, is to *occupy time*. When an unfavourable decision is apprehended, and the protraction of the debate may afford time for fresh voters to be summoned, or may lead to an adjournment, which will afford scope for some other manœuvre;—when there is a chance of so wearying out the attention of the hearers, that they will listen with languor and impatience to what shall be urged on the other side;—when an advocate is called upon to plead a cause in the absence of those whose opinion it is of the utmost importance to influence, and wishes to reserve all his Arguments till they arrive, but till then, must apparently proceed in his pleading; in these and many similar cases, which it is needless to particularize, it is a valuable talent to be able to pour forth with fluency an unlimited quantity of well-sounding language which has little or no meaning;—which shall not strike the hearers as unintelligible or nonsensical, though it convey to their minds no distinct idea. Perspicuity of Style, real, not apparent perspicuity, is in this case never necessary, and sometimes, studiously avoided. If any distinct meaning were conveyed, and that which was said were irrelevant, it would be perceived to be so, and would produce impatience in the hearers, or afford an advantage to the opponents; if, on the other hand, the speech were relevant, and there were no Arguments of any force to be urged, except such as either had been already dwelt on, or were required to be reserved (as in the case last alluded to) for a fuller audience, the speaker would not further his cause by bringing them forward. So that the usual resource on these occasions, of such Orators as thoroughly understand the tricks of their art, and do not disdain to employ them, is to amuse their audience with specious emptiness.

Another kind of spurious Oratory, and the last that will be noticed, is that which has for its object the hearer's admiration of the Eloquence displayed. This, indeed, constitutes one of the three kinds of Oratory enumerated by Aristotle, and is regularly treated of by him along with the deliberative and judicial branches; though it hardly deserves the place he has bestowed on it.

When this is the end pursued, perspicuity is not indeed to be avoided, but it may often without detriment be disregarded.* Men frequently admire as eloquent,

* In Dr. Campbell's ingenious dissertation, (*Rhetoric*, book ii. c. vii.) "on the causes that nonsense often escapes being detected, both by the writer and the reader," he remarks, (sec. 2,) that "there are particularly three sorts of writing wherein we are liable to be imposed upon by words without meaning."

"The first is, where there is an exuberance of metaphor. Nothing is more certain than that this trope, when temperately and appositely used, serves to add light to the expression, and energy to the sentiment. On the contrary, when vaguely and intemperately used, nothing can serve more effectually to cloud the sense, where there is sense, and by consequence to conceal the defect, where there is no sense to show. And this is the case, not only where there is in the same sentence a mixture of discordant metaphors, but also where the metaphoric Style is too long continued, and too far pursued. [*Ut modicus autem atque opportunus translationis usus illustrat orationem: ita frequens et obscurat et tædio complet; continuus vero in allegoriam et ænigmata exit.* Quint. lib. viii. c. vi.] The reason is obvious. In common speech the words are the immediate signs of the thought. But it is not so here; for when a person, instead of adopting metaphors that come naturally and opportunely in his way, rummages the whole world in quest of them, and piles them one upon another, when he cannot so properly be said to use metaphor, as to talk in metaphor, or rather when from metaphor he runs into allegory, and thence into enigma, his words are not the immediate signs of his thought; they are at best but the signs of the signs of his thought. His writing may then be called, what Spenser not unjustly styled his Fairy Queen, *a perpetual allegory or dark conceit*. Most readers will account it much to bestow a transient glance on the literal sense, which lies nearest; but will never think of that meaning more remote, which the figures themselves are intended to signify. It is no wonder then that this sense, for the discovery of which it is necessary to see through a double veil, should, where it is, more readily escape our observation, and that where it is wanting we should not so quickly miss it.

"There is, in respect of the two meanings, considerable variety to be found in the tropical Style. In just allegory and similitude there is always a propriety, or, if you choose to call it, congruity, in the literal sense, as well as a distinct meaning or sentiment suggested, which is called the figurative sense. Examples of this are unnecessary. Again, where the figurative sense is unexceptionable, there is sometimes an incongruity in the expression of the literal sense. This is always the case in mixed metaphor, a thing not unfrequent even in good writers. Thus, when Addison remarks that 'there is not a single view of human nature, which is not sufficient to *extinguish* the *seeds* of pride,' he expresses a a true sentiment somewhat incongruously; for the terms *extinguish* and *seeds* here metaphorically used, do not suit each other. In like manner, there is something incongruous in the mixture of tropes employed in the following passage from Lord Bolingbroke: 'Nothing less than the *hearts* of his people will content a patriot Prince, nor will he think his *throne* established, till it is established *there*.' Yet the thought is excellent. But in neither of these examples does the incongruity of the expression hurt the perspicuity of the sentence. Sometimes, indeed, the literal meaning involves a direct absurdity. When this is the case, as in the quotation from *The Principles of Painting* given in the preceding chapter, it is natural for the reader to suppose that there must be something under it; for it is not easy to say how absurdly even just sentiments will sometimes be expressed. But when no such hidden sense can be discovered, what, in the first view conveyed to our minds a glaring *absurdity*, is rightly on reflection denominated *nonsense*. We are satisfied that De Piles neither thought, nor wanted his readers to think, that Rubens was really the original performer, and God the copier. This

and sometimes admire the most, what they do not at all, or do not fully comprehend, if elevated and high sounding words be arranged in graceful and sonorous periods. Those of uncultivated minds especially, are apt to think meanly of any thing that is brought down perfectly to the low level of their capacity; though to do this with respect to valuable Truths which are not trite, is one of the most admirable feats of genius; they admire the profundity of one who is mystical and obscure; mistaking the muddiness of the water for depth; and magnifying in their imaginations what is viewed through a fog; and they conclude that brilliant language must represent some brilliant ideas, without troubling themselves to inquire what those ideas are.

Many an enthusiastic admirer of a "fine discourse," or a piece of "fine writing," would be found on examination to retain only a few sonorous, but empty phrases; and not only to have no notion of the general drift of the Argument, but not even to have ever considered whether the Author had any such drift or not.

It is not meant to be insinuated that in every such case the composition is in itself unmeaning, or that the Author had no other object than the credit of Eloquence: he may have had a higher end in view; and he may have expressed himself very clearly to *some* hearers, though not to all: but it is most important to be fully aware of the fact, that it is possible to obtain the highest applause from those who not only receive no edification from what they hear, but absolutely do not understand it. So far is popularity from being a safe criterion of the usefulness of a Preacher.

§ 2. The next quality of Style to be noticed is what may be called Energy; the term being used in a wider sense than the Ἐνέργεια of Aristotle, and nearly corresponding with what Dr. Campbell calls Vivacity; so as to comprehend every thing that may conduce to stimulate attention,—to impress strongly on the mind the Arguments adduced,—to excite the Imagination, and to arouse the Feelings.

This Energy then, or Vivacity of Style, must depend (as is likewise the case in respect of Perspicuity,) on three things; 1st, the *Choice* of words, 2d, their *Number*, and 3d, their *Arrangement*.

With respect to the Choice of words, it will be most convenient to consider them under those two classes which Aristotle has described under the titles of

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then was not his meaning. But what he actually thought and wanted them to think, it is impossible to elicit from his words. His words then may justly be attributed *bold*, in respect of their literal import, but *unmeaning* in respect of the author's intention.

"It may be proper here to observe, that some are apt to confound the terms *absurdity* and *nonsense* as synonymous, which they manifestly are not. An absurdity, in the strict acceptation, is a proposition either intuitively or demonstratively false. Of this kind are these: 'Three and two make seven.' 'All the angles of a triangle are greater than two right angles.' That the former is false we know by intuition; that the latter is so, we are able to demonstrate. But the term is further extended to denote a notorious falsehood. If one should affirm, that 'at the vernal equinox the sun rises in the north and sets in the south,' we should not hesitate to say, that he advances an absurdity; but still what he affirms has a meaning; insomuch, that on hearing the sentence we pronounce its falsity. Now *nonsense* is that whereof we cannot say either that it is true, or that it is false. Thus, when the Teutonic Theosopher enounces, that 'all the voices of the celestial joyfulness, qualify, commix, and harmonize in the fire which was from eternity in the good quality,' I should think it equally impertinent to aver the falsity as the truth of this enunciation. For, though the words grammatically form a sentence, they exhibit to the understanding no judgment, and consequently admit neither assent nor dissent. In the former instances I say the meaning, or what they affirm, is absurd; in the last instance I say there is no meaning, and therefore properly nothing is affirmed. In popular language, I own, the terms absurdity and nonsense are not so accurately distinguished. Absurd positions are sometimes called nonsensical. It is not common, on the other hand, to say of downright nonsense, that it comprises an absurdity.

"Further, in the literal sense there may be nothing unsuitable, and yet the reader may be at a loss to find a figurative meaning, to which his expressions can with justice be applied. Writers immoderately attached to the florid, or highly figured diction, are often misled by a desire of flourishing on the several attributes of a metaphor, which they have pompously ushered into the discourse, without taking the trouble to examine whether there be any qualities in the subject, to which these attributes can with justice and perspicuity be applied." This immoderate use of metaphor, Dr. Campbell observes, "is the principal source of all the nonsense of Orators and Poets.

"The second species of writing wherein we are liable to be imposed on by words without meaning, is that wherein the terms most frequently occurring, denote things which are of a complicated nature, and to which the mind is not sufficiently familiarised. Many of those notions which are called by Philosophers mixed modes, come under this denomination. Of these the instances are numerous in every tongue; such as *government*, *church*, *state*, *constitution*, *polity*, *power*, *commerce*, *legislature*, *jurisdiction*, *proportion*, *symmetry*, *elegance*. It will considerably increase the danger of our being deceived by an unmeaning use of such terms, if they are besides (as very often they are) of so indeterminate, and consequently equivocal significations, that a writer, unobserved either by himself or by his reader, may slide from one sense of the term to another, till by degrees he fall into such applications of it as will make no sense at all. It deserves our notice also, that we are in much greater danger of terminating in this, if the different meanings of the same word have some affinity to one another, than if they have none. In the latter case, when there is no affinity, the transition from one meaning to another, is taking a very wide step, and what few writers are in any danger of; it is, besides, what will not so readily escape the observation of the reader. So much for the second cause of deception, which is the chief source of all the nonsense of writers on politics and criticism.

"The third and last, and I may add, the principal species of composition, wherein we are exposed to this illusion by the abuse of words, is that in which the terms employed are very abstract, and consequently of very extensive signification. It is an observation that plainly ariseth from the nature and structure of language, and may be deduced as a corollary from what hath been said of the use of artificial signs, that the more general any name is, as it comprehends the more individuals under it, and consequently requires the more extensive knowledge in the mind that would rightly apprehend it, the more it must have of indistinctness and obscurity. Thus the word *lion* is more distinctly apprehended by the mind than the word *beast*, beast than *animal*, animal than *being*. But there is, in what are called abstract subjects, a still greater fund of obscurity, than that arising from the frequent mention of the most general terms. Names must be assigned to those qualities as considered abstractly, which never subsist independently, or by themselves, but which constitute the generic characters and the specific differences of things. And this leads to a manner which is in many instances remote from the common use of speech, and therefore must be of more difficult conception." (Book ii. sec. 2. p. 102, 103.)

It is truly to be regretted that an author who has written so justly on this subject, should within a few pages so strikingly exemplify the errors he has been treating of, by indulging in a declamation against Logic which could not even to himself have conveyed any distinct meaning. When he says that a man who had learned Logic was "qualified without any other kind of knowledge, to defend any position whatever, however contradictory to common sense; and that that art observed the most absolute indifference to truth and error," he cannot mean that a false conclusion could be logically proved from true premises; since, ignorant as he was of the subject, he was aware, and has in another place distinctly acknowledged, that this is not the case; nor could he mean merely that a false conclusion could be proved from a false premiss, since that would evidently be a nugatory and ridiculous objection. He seems to have had, in truth, no meaning at all; though like the authors he had been so ably criticising, he was perfectly unaware of the emptiness of what he was saying.

Κύρια and Ξένα, for which our language does not afford precisely corresponding names: "Proper," "Appropriate," or "Ordinary," terms, will the most nearly designate the former; the latter class including all others;—all that are in any way removed from common use;—whether uncommon terms, or ordinary terms, either transferred to a different meaning from that which strictly belongs to them, or employed in a different manner from that of common discourse. All the Tropes and Figures, enumerated by Grammatical and Rhetorical Writers, will of course fall under this head.

With respect then to "Proper" terms, the principal rule for guiding our Choice with a view to Energy, is to prefer, ever, those words which are the least *abstract* and *general*. Individuals alone having a real existence,* the terms denoting them (called by Logicians "Singular terms,") will of course make the most vivid impression on the mind, and exercise most the power of Conception; and the less remote any term is from these, i. e. the more *specific*, the more Energy it will possess, in comparison of such as are more general. The impression produced on the mind by a Singular term, may be compared to the distinct view taken in by the eye of any object (suppose a man) near at hand, in a clear light, which enables us to distinguish the features of the Individual; in a fainter light, or rather farther off, we merely perceive that the object is *a man*; this corresponds with the Idea conveyed by the name of the Species; yet further off, or in a still feebler light, we can distinguish merely some *living object*, and at length, merely *some object*; these views corresponding respectively with the terms denoting the genera, less or more remote: and as each of these views conveys, as far as it goes, an equally *correct* impression to the mind, (for we are equally certain that the object at a distance is *something*, as that the one close to us is such and such an individual,) though each, successively, is less *vivid*; so, in language, a General term may be as clearly *understood*, as a Specific or Singular term, but will convey a much less *forcible* impression to the hearer's mind. "The more General the terms are," (as Dr. Campbell justly remarks,) "the picture is the fainter; the more Special they are, the brighter. The same sentiment may be expressed with equal justness, and even equal perspicuity, in the former way, as in the latter; but as the colouring will in that case be more languid, it cannot give equal pleasure to the fancy, and by consequence will not contribute so much either to fix the attention, or to impress the memory."

It might be supposed at first sight, that an Author has little or no Choice on this point, but must employ either more or less General terms according to the objects he is speaking of. There is, however, in almost every case, great room for such a Choice as we are speaking of; for, in the first place, it depends on our Choice whether or not we will employ terms *more* General than the subject requires; which may almost always be done consistently with Truth and Propriety, though not with Energy: if it be true that a man has committed *murder*, it may be correctly asserted, that he has committed a *crime*; if the Jews were "exterminated," and "Jerusalem demolished" by "Vespasian's army," it may be said, with truth, that they were "subdued" by "an Enemy," and their "Capital" taken. This substitution then of the General for the Specific, or of the Specific for the Singular, is always within our reach; and many, especially unpractised Writers, fall into a feeble Style by resorting to it unnecessarily; either because they imagine there is more appearance of refinement or of profundity, in the employment of such terms as are in less common use among the vulgar, or, in some cases, with a view to give greater comprehensiveness to their Reasonings, and to increase the utility of what they say by enlarging the field of its application. Inexperienced Preachers frequently err in this way, by dwelling on Virtue and Vice, Piety and Irreligion, in the abstract, without particularizing; forgetting that while they *include* much, they *impress* little or nothing.

The only Appropriate occasion for this Generic language, (as it may be called,) is when we wish to *avoid* giving a vivid impression,—when our Object is to soften what is offensive, disgusting, or shocking; as when we speak of an "execution," for the infliction of the sentence of death on a criminal: of which kind of expressions, common discourse furnishes numberless instances. On the other hand, in Antony's speech over Cæsar's body, his object being to *excite* horror, Shakspeare puts into his mouth the most *particular* expressions: "those honourable men (not, who *killed* Cæsar, but) whose *daggers* have *stabbed* Cæsar."

But in the second place, not only does a regard for Energy require that we should not use terms *more* General than are exactly adequate to the objects spoken of, but we are also allowed, in many cases, to employ *less* General terms than are exactly Appropriate. In which case we are employing words not "Appropriate," but belonging to the second of the two classes just mentioned. The use of this Trope,* (enumerated by Aristotle among the Metaphors, but since more commonly called Synecdoche) is very frequent, as it conduces much to the Energy of the expression, without occasioning, in general, any risk of its meaning being mistaken. The passage cited by Dr. Campbell,† from one of our Lord's discourses, (which are in general of this character,) together with the remarks made upon it, will serve to illustrate what has been just said: "'Consider,' says our Lord, 'the lilies how they grow: they toil not, they spin not; and yet I say

* Thence called by Aristotle, (*Categ.* sec. 3,) "primary substances," (πρῶται οὐσίαι,) Genus and Species, being denominated "secondary," as not properly denoting a "really-existing-thing," (τόδε τι,) but rather an attribute. He has, indeed, been considered as the great advocate of the opposite doctrine; *i. e.* of the system of "Realism;" which was certainly embraced by many of his professed followers; but his own language is sufficiently explicit. Πᾶσα δὲ οὐσία δοκεῖ τόδε τι σημαίνειν. Ἐπὶ μὲν οὖν τῶν πρώτων οὐσιῶν ἀναμφισβήτητον καὶ ἀληθές ἐστιν, ὅτι τόδε τι σημαίνει· ἄτομον γὰρ, καὶ ἓν ἀριθμῷ τὸ δηλούμενόν ἐστιν. Ἐπὶ δὲ τῶν δευτέρων οὐσιῶν, ΦΑΙΝΕΤΑΙ μὲν ὁμοίως τῷ σχήματι τῆς προσηγορίας τόδε τι σημαίνειν, ὅταν εἴπῃ, ἄνθρωπος, ἢ ζῶον· ΟΥ ΜΗΝ ΓΕ ΑΛΗΘΕΣ· ἀλλὰ μᾶλλον ποιόν τι σημαίνει κ. τ. λ. Aristotle, *Categ.* sec. 3.

* From τρέπω; any word *turned* from its primary signification.

† The ingenious Author cites this in the Section treating of "*Proper* terms," which is a trifling oversight; as it is plain that "lily" is used for the Genus "flower,"—"Solomon," for the Species "King," &c.

unto you, that Solomon in all his glory, was not arrayed like one of these. If then God so clothe the grass which to-day is in the field, and to-morrow is cast into the oven, how much more will he clothe you?'* Let us here adopt a little of the tasteless manner of modern paraphrasts, by the substitution of more General terms, one of their many expedients of infrigidating, and let us observe the effect produced by this change. 'Consider the flowers, how they gradually increase in their size, they do no manner of work, and yet I declare to you, that no king whatever, in his most splendid habit, is dressed up like them. If then God in his providence doth so adorn the vegetable productions, which continue but a little time on the land, and are afterwards devoted to the meanest uses, how much more will he provide clothing for you?' How spiritless is the same sentiment rendered by these small variations? The very particularizing of *to-day* and *to-morrow*, is infinitely more expressive of transitoriness, than any description wherein the terms are General, that can be substituted in its room." It is a remarkable circumstance that this characteristic of Style is perfectly retained in *translation*, in which every other excellence of expression is liable to be lost; so that the prevalence of this kind of language in the Sacred writers, may be regarded as something providential. It may be said with truth, that the book which it is the most necessary to translate into every language, is chiefly characterised by that kind of excellence in diction which is least impaired by translation.

But to proceed with the consideration of Tropes; the most employed and most important of all those kinds of expressions which depart from the plain and strictly Appropriate Style,—all that are called by Aristotle, Ξένα,—is the Metaphor, in the usual and limited sense; viz. a word substituted for another, on account of the Resemblance or Analogy between their significations. The Simile or Comparison may be considered as differing in form only from a Metaphor; the Resemblance being in that case *stated*, which in the Metaphor is implied. Each may be founded either on Resemblance, strictly so called, i. e. *direct* Resemblance between the objects themselves in question, (as when we speak of "*table*-land," or compare great waves to *mountains*,) or on Analogy, which is the Resemblance of ratios,—a similarity of the relations they bear to certain other objects; as when we speak of the "*light* of reason," or of "revelation;" or compare a wounded and captive warrior to a stranded ship.† The Analogical Metaphors and Comparisons are both the most frequent and the most striking. They are the most frequent, because almost every object has such a multitude of relations, of different kinds, to many other objects; and they are the most striking, because (as Dr. A. Smith has well remarked,) the more remote and unlike in themselves any two objects are, the more is the mind impressed and gratified by the perception of some point in which they agree.

It has been already observed, under the head of Example, (chap. 1,) that we are carefully to distinguish between an *Illustration*, i. e. an *Argument* from Analogy or Resemblance, and what is properly called a Simile or Comparison, introduced merely to give force or beauty to the *expression*. The aptness and beauty of an Illustration sometimes leads men to overrate, and sometimes to underrate, its force as an Argument. (Vol. I. p. 255.)

With respect to the choice between the Metaphorical form and that of Comparison, it may be laid down as a general rule, that the former is always to be preferred,* wherever it is sufficiently simple and plain to be immediately comprehended; but that which as a Metaphor would sound obscure and enigmatical, may be well received if expressed as a Comparison. We may say, e. g. with propriety, that "Cromwell *trampled* on the laws:" it would sound flat to say that "he treated the laws with the same contempt as a man does any thing which he tramples under his feet." On the other hand it would be harsh and obscure to say, "the stranded vessel lay shaken by the waves," meaning the wounded chief tossing on the bed of sickness; it is therefore necessary in such a case to *state* the Resemblance. But this is never to be done more fully than is necessary to perspicuity, because all men are more gratified at catching the Resemblance for themselves, than at having it pointed out to them.† And accordingly the greatest masters of this kind of Style, when the case will not admit of pure Metaphor, generally prefer a mixture of Metaphor with Simile; first pointing out the similitude, and afterwards employing metaphorical terms which imply it; or, *vice versâ*, explaining a Metaphor by a statement of the Comparison. To take examples from an Author who particularly excels in this point; (speaking of a morbid Fancy,)

> "———— like the bat of Indian brakes,
> Her pinions fan the wound she makes,
> And soothing thus the dreamer's pain,
> She drinks the life-blood from the vein."‡

The word "*like*" makes this a Comparison; but the three succeeding lines are Metaphorical. Again, to take an instance of the other kind,

> "They *melted* from the field, *as* snow,
> When streams are swoln, and south winds blow,
> Dissolves in silent dew:"§

Of the words here put in italics, the former is a Metaphor, the latter, introduces a Comparison. Though the instances here adduced are taken from a Poet, the judicious management of Comparison which they exemplify, is even more essential to a Prose writer, to whom less licence is allowed in the employment of them. It is a remark of Aristotle, (*Rhet.* book iii. c. 4,) that the Simile is more suitable in Poetry, and that Metaphor is the only ornament of language in which the Orator may freely indulge. He should therefore be the more careful to bring a Simile as near as possible to the Metaphorical form. The following is an example of the same kind of expression: "These metaphysic rights entering into common life, like rays of light which pierce into a dense medium, are, by the laws of nature, refracted

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* *Luke*, ch. xii. ver. 27, 28.

† Roderic Dhu, in the *Lady of the Lake*.

* Ἔστιν ἡ εἰκὼν μεταφορὰ, διαφέρουσα προθέσει· διὸ ἧττον ἡδὺ, ὅτι μακροτέρως· κ. τ. λ. Aristotle, *Rhet.* book iii. c. 10.

† Τὸ μανθάνειν ῥᾳδίως ἡδὺ φύσει. Aristotle, *Rhet.* book iii. c. 5.

‡ *Rokeby.*

§ *Marmion.*

from their straight line. Indeed, in the gross and complicated mass of human passions and concerns, the primitive rights of man undergo such a variety of refractions and reflections, that it becomes absurd to talk of them as if they continued in the simplicity of their original direction."*

Metaphors may be employed, as Aristotle observes, either to elevate or to degrade the subject, according to the design of the Speaker; being drawn from similar or corresponding objects of a higher or lower character. Thus a loud and vehement Speaker may be described either as *bellowing*, or as *thundering*. And in both cases, if the Metaphor is apt and suitable to the purpose designed, it is alike conducive to Energy. He remarks that the same holds good with respect to Epithets also, which may be drawn either from the highest or the lowest attributes of the thing spoken of.† Metonymy likewise (in which a part is put for a whole, a cause for an effect, &c.) admits of a similar variety in its applications.

Any *Trope* (as is remarked by Dr. Campbell,) adds force to the expression, when it tends to fix the mind on that part, or *circumstance*, in the object spoken of, which is most essential to the purpose in hand. Thus, there is an Energy in Abraham's Periphrasis for "God," when he is speaking of the allotment of Divine punishment: "shall not the *Judge of all the earth* do right?" If again we were alluding to His *omniscience*, it would be more suitable to say, "this is known only to the *Searcher of hearts*:" if, to his *power*, we should speak of Him as "the *Almighty*," &c.

Of Metaphors, those generally conduce most to that Energy or Vivacity of Style we are speaking of, which illustrate an *intellectual* by a *sensible* object; the latter being always the most early familiar to the mind, and generally giving the most distinct impression to it. Thus we speak of "*unbridled* rage," "*deep-rooted* prejudice," "*glowing* eloquence," a "*stony* heart," &c. And a similar use may be made of Metonymy also; as when we speak of the "*Throne*," or the "*Crown*" for "Royalty,"—the "*sword*" for "military violence," &c.

But the highest degree of Energy (and to which Aristotle chiefly restricts the term) is produced by such Metaphors as attribute *life* and *action* to things inanimate; and that, even when by this means the last mentioned rule is violated, i. e. when sensible objects are illustrated by intellectual. For the disadvantage is overbalanced by the vivid impression produced by the idea of *personality* or *activity*;‡ as when we speak of the *rage* of a torrent, a *furious* storm, a river *disdaining* to endure its bridge, &c.§ Many such expressions, indeed, are in such common use as to have lost all their Metaphorical force, since they cease to suggest the idea belonging to their primary signification, and thus are become, practically, Proper terms. But a new, or at least, unhackneyed, Metaphor of this kind, if it be not far-fetched and obscure, adds greatly to the force of the expression. This was a favourite figure with Homer, from whom Aristotle has cited several examples of it; as "the *raging* arrow," "the darts *eager* to taste of flesh,"* "the *shameless*" (or as it might be rendered with more exactness, though with less dignity, "the *provoking*) stone" (λᾶαν ἀναιδῆ) which mocks the efforts of Sisyphus, &c. Our language possesses one remarkable advantage, with a view to this kind of Energy, in the constitution of its *genders*. All nouns in English, which express objects that are really neuter, are considered as strictly of the neuter gender; the Greek and Latin, though possessing the advantage, which is wanting in the languages derived from them, of having a neuter gender, yet lose the benefit of it by fixing the masculine or feminine genders upon many nouns denoting things inanimate; whereas in English, when we speak of any such object in the masculine or feminine gender, that form of expression at once confers *personality* upon it. When "Virtue," e. g. or our "Country," are spoken of as females, or "Ocean" as a male, &c. they are, by that very circumstance, *personified*; and a stimulus is thus given to the imagination, from the very circumstance that in calm discussion or description, all of these would be neuter; whereas in Greek or Latin, as in French or Italian, no such distinction could be made. The employment of "*Virtus*," and "Ἀρετή," in the feminine gender, can contribute, accordingly, no animation to the Style, when they could not, without a Solecism, be employed otherwise.

There is, however, very little, comparatively, of Energy produced by any Metaphor or Simile that is in common use, and already familiar to the hearer; indeed, what were originally the boldest Metaphors, are become, by long use, virtually, Proper terms; as is the case with the words "source," "reflection," &c. in their transferred senses; and frequently are even nearly obsolete in the literal sense, as in the words "ardour," "acuteness," "ruminate," &c. If, again, a Metaphor or Simile that is not so hackneyed as to be considered common property, be taken from any known Author, it strikes every one, as no less a plagiarism than if an entire argument or description had been thus transferred. And hence it is, that, as Aristotle remarks, the skilful employment of these, more than of any other, ornaments of language, may be regarded as a mark of genius; (εὐφυΐας σημεῖον,) not that he means to say, as some interpreters suppose, that this power is entirely a gift of nature, and in no degree to be learnt; on the contrary, he expressly affirms, that the "perception of Resemblances,"† on which it depends, is the fruit of "Philosophy;"‡ but he means that Metaphors are not to be, like other words and phrases, selected from

* Burke, *On the French Revolution.*

† A happier example cannot be found than the one which Aristotle cites from Simonides, who, when offered a small price for an Ode to celebrate a victory in a *mule*-race, expressed his contempt for *half-asses*, (ἡμίονοι) as they were commonly called; but when a larger sum was offered, addressed them in an Ode as "Daughters of Steeds swift-as-the-storm." ἀελλοπόδων θύγατρες ἵππων.

‡ The figure called by Rhetoricians Prosopopœia (literally, Personification) is, in fact, no other than a Metaphor of this kind: thus, in Demosthenes, *Greece* is represented as *addressing* the Athenians. So also in the book of Genesis, (chap. iv. ver. 10,) "the *voice* of thy brother's blood *crieth* unto me from the ground."

§ *Pontem indignatus.*

* There is a peculiar aptitude in some of these expressions which the modern student is very likely to overlook; an arrow or dart, from its flying with a *spinning* motion, *quivers* violently when it is fixed; thus suggesting the idea of a person *trembling with eagerness.*

† Τὸ ὅμοιον ὁρᾶν. Aristotle, *Rhet.* book ii.

‡ Ῥᾷον ἐκ φιλοσοφίας, Ibid. book ii. and iii.

 comon use, and transferred from one composition to another,* but must be formed for the occasion. Some care is accordingly requisite, in order that they may be readily comprehended, and may not have the appearance of being far-fetched and extravagant; for this purpose it is usual to combine with the Metaphor a Proper term which explains it; viz. either attributing to the term in its *transferred* sense, something which does not belong to it in its *literal* sense; or, *vice versâ*, denying of it in its transferred sense, something which *does* belong to it in its literal sense. To call the Sea the "*watery* bulwark" of our island, would be an instance of the former kind; an example of the latter is the expression of a writer who speaks of the dispersion of some hostile fleet by the winds and waves, "those ancient and *unsubsidized* allies of England."

It is hardly necessary to mention the obvious and hackneyed cautions against *mixture* of Metaphors;† and against any that are complex and far-pursued, so as to approach to Allegory. In this last case, the more apt and striking is the Analogy suggested, the more will it have of an artificial appearance; and will draw off the reader's attention from the subject, to admire the ingenuity displayed in the Style. Young writers, of genius, ought especially to be admonished to ask themselves frequently, not whether this or that is a *striking expression*, but whether it makes the *meaning* more striking than another phrase would,—whether it impresses more forcibly the *sentiment* to be conveyed.

It is a common practice with some writers to endeavour to add force to their expressions by accumulating high-sounding Epithets,‡ denoting the greatness, beauty, or other admirable qualities of the things spoken of; but the effect is generally the reverse of what is intended. Most readers, except those of a very vulgar or puerile taste, are disgusted at studied efforts to point out and force upon their attention whatever is remarkable; and this, even when the ideas conveyed are themselves striking. But when an attempt is made to cover poverty of thought with mock sublimity of language, and to set off trite sentiments and feeble arguments by tawdry magnificence, the only result is, that a kind of indignation is superadded to contempt; as when (to use Quinctilian's comparison) an attempt is made to supply, by paint, the natural glow of a youthful and healthy complexion.§

We expect, indeed, and excuse in ancient writers, as a part of the unrefined simplicity of a ruder language, such a redundant use of Epithets as would not be tolerated in a modern, even in a translation of their works; the "white milk," and "dark gore," &c. of Homer, must not be retained, at least, not so frequently as they occur in the original. Aristotle, indeed, gives us to understand that in his time this liberty was still allowed to Poets; but later taste is more fastidious. He censures, however, the adoption by prose writers of this, and of every other kind of ornament that might seem to border on the *poetical*; and he bestows on such a Style, the appellation of "*frigid*," (ψυχρὸν,) which, at first sight may appear somewhat remarkable, (though the same expression, "frigid," might very properly be so applied by us,) because "*warm*," "*glowing*," and such like Metaphors, seem naturally applicable to poetry. This very circumstance, however, does not in reality account for the use of the other expression. We are, in poetical prose, *reminded* of, and for that reason disposed to *miss*, the "warmth and glow" of poetry: it is on the same principle that we are disposed to speak of *coldness* in the rays of the *moon*, because they *remind* us of sunshine, but want its warmth; and that (to use an humbler and more familiar instance) an empty fire-place is apt to suggest an idea of cold.

The use of Epithets however, in prose composition, is not to be proscribed; as the judicious employment of them is undoubtedly conducive to Energy. It is extremely difficult to lay down any precise rules on such a point. The only safe guide in practice must be a taste formed from a familiarity with the best Authors, and from the remarks of a skilful Critic, on one's own composition. It may, however, be laid down as a general caution, more particularly needful for young writers, that an excessive luxuriance of Style, and especially a redundancy of Epithets, is the worse of the two extremes; as it is a positive fault, and a very offensive one; while the opposite is but the absence of an excellence. It is also an important rule that the boldest and most striking, and almost poetical, turns of expression, should be reserved (as Aristotle has remarked, book iii. c. 7,) for the most impassioned parts of a discourse; and that an Author should guard against the vain ambition of expressing *every thing* in an *equally* high-wrought, brilliant, and forcible Style. The neglect of this caution often occasions the imitation of the best models to prove detrimental. When the admiration of some fine and animated passages leads a young writer to take those passages for his *general* model, and to endeavour to make every sentence he composes equally fine, he will, on the contrary, give a flatness to the whole, and destroy the effect of those portions which would have been forcible if they had been allowed to stand *prominent*. To brighten the dark parts of a picture, produces much the same result as if one had darkened the bright parts; in either case there is a want of *relief* and *contrast*; and Composition, as well as Painting, has its lights and shades, which must be distributed

* 'Ουκ ἔστι γαρ' ἄλλου λαβεῖν. Aristotle, *Rhet.* book iii.

† Dr. Johnson justly censures Addison for speaking of "*bridling* in his muse, who longs to *launch* into a nobler strain;" "which," says the Critic, "is an act that was never restrained by a bridle." Some, however, are too fastidious on this point. Words, which by long use in a transferred sense, have lost nearly all their metaphorical force, may fairly be combined in a manner which, taking them literally, would be incongruous. It would savour of hypercriticism to object to such an expression as "fertile source."

‡ Epithets, in the Rhetorical sense, denote, not every adjective, but those only which do not add to the sense, but signify something already implied in the noun itself; as, if one says, "the *glorious* sun;" on the other hand, to speak of the "*rising*" or "*meridian* sun," would not be considered as, in this sense, employing an Epithet.

§ "A principal device in the fabrication of this Style," (the mock-eloquent,) "is to multiply epithets,—dry epithets, laid on the outside, and into which none of the vitality of the sentiment is found to circulate. You may take a great number of the words out of each page, and find that the sense is neither more nor less for your having cleared the composition of these Epithets of chalk of various colours, with which the tame thoughts had submitted to be rubbed over, in order to be made fine." Foster, *Essay* iv.

with no less skill, if we would produce the desired effect.*

In no place, however, will it be advisable to introduce any Epithet which does not fulfil one of these two purposes; 1st, to *Explain a Metaphor*; a use which has been noticed under that head, and which will justify, and even require, the introduction of an Epithet, which, if it had been joined to the Proper term, would have been glaringly superfluous; thus, Æschylus,† speaks of the "*winged hound* of Jove," meaning the Eagle: to have said the "*winged eagle*," would have had a very different effect: 2dly, when the Epithet, expresses something which, though *implied* in the subject, would not have been likely to occur at once spontaneously to the hearer's mind, and yet is important to be noticed with a view to the purpose in hand. Indeed it will generally happen, that the Epithets employed by a skilful Orator, will be found to be, in fact, so many *abridged arguments*, the force of which is sufficiently conveyed by a mere hint; e. g. if any one says, "we ought to take warning from the *bloody* revolution of France," the Epithet suggests one of the reasons for our being warned; and that, not less clearly, and more forcibly, than if the Argument had been stated at length.

With respect to the use of Antiquated, Foreign, New-coined or New-compounded words,‡ or words applied in an unusual sense, it may be sufficient to observe, that all writers, and prose writers most, should be very cautious and sparing in the use of them; not only because in excess they produce a barbarous dialect, but because they are so likely to suggest the idea of *artifice*; the perception of which is most especially adverse to Energy. The occasional apt introduction of such a term, will sometimes produce a powerful effect; but whatever may seem to savour of affectation, or even of great solicitude and study in the Choice of terms, will effectually destroy the true effect of Eloquence. The language which betrays art, and carries not an air of simplicity and sincerity, may, indeed, by some hearers, be thought not only very fine, but even very Energetic; this very circumstance, however, may be taken for a proof that it is not so; for if it *had* been, *they would not have thought about it*, but would have been occupied, exclusively, with the *subject*. An unstudied and natural air, therefore, is an excellence to which the true Orator, i. e. he who is aiming to *carry his point*, will be ready to sacrifice any other that may interfere with it.

The principle here laid down will especially apply to the Choice of words, with a view to their Imitative, or otherwise, Appropriate *sound* The attempt to make the sound an echo to the sense, is indeed more frequently to be met with in poets than in prose writers; but it may be worth remarking, that an evident *effort* after this kind of excellence, as it is offensive in any kind of Composition, would in prose appear peculiarly

disgusting. Critics treating on this subject have gone into opposite extremes; some fancifully attributing to words, or combinations of words, an Imitative power far beyond what they can really possess,* and representing this kind of Imitation as deserving to be studiously aimed at; and others, on the contrary, considering nearly the whole of this kind of excellence as no better than imaginary, and regarding the examples which do occur, and have been cited, of a congruity between the sound and the sense as purely accidental. The truth probably lies between these two extremes. In the first place, that words denoting *sounds*, or employed in describing them, may be Imitative of those sounds, must be admitted by all; indeed this kind of Imitation is, to a certain degree, almost unavoidable, in our language at least, which abounds perhaps more than any other, in these, as they may be called, naturally expressive terms; such as "hiss," "rattle," "clatter," "splash," and many others. In the next place, it is also allowed by most, that quick or slow *motion* may, to a certain degree at least, be imitated or represented by words; many short syllables (unincumbered by a clash either of vowels, or of consonants coming together,) being pronounced in the same time with a smaller number of long syllables, abounding with these incumbrances, the former seems to have a natural correspondence to a quick, and the latter to a slow motion, since in the one a greater, and in the other a less space, seem to be passed over in the same time. In the ancient Poets, their hexameter verses being always considered as of the same *length*, i. e. in respect of the *time* taken to pronounce them, whatever proportion of dactyls or spondees they contained, this kind of Imitation of quick or slow motion, is the more apparent; and after making all allowances for fancy, it seems impossible to doubt that in many instances it does exist; as, e. g. in the often-cited line which expresses the rolling of Sisyphus's stone down the hill:

Αὖθις ἔπειτα πέδονδε κυλίνδετο λᾶας ἀναιδής.

The following passage from the *Æneid* can hardly be denied to exhibit a correspondence with the slow and quick *motions* at least, which it describes; that of the Trojans laboriously hewing the foundations of a tower on the top of Priam's palace, and that of its sudden and violent fall:

> † "*Āggrēssī fērrō circūm, quā sūmma labantes,*
> *Jūnctūrās tabulata dabat, divellimus altis*
> *Sedĭbŭs, impŭlĭmusque, ĕă lapsă rĕpentĕ rŭinam*
> *Cum sŏnĭtu dĕdit, et Dănăum sŭpĕr agmĭnă late*
> *Concidit.*"————

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* Omnia *vult belle Matho dicere; dic aliquando*
Et bene; dic neutrum, dic aliquando male.

† *Prometheus.*

‡ It is a curious instance of whimsical inconsistency, that many who, with justice, censure as *pedantic*, the frequent introduction of *Greek* and *Latin* words, neither object to, nor refrain from, a similar pedantry with respect to *French* and *Italian*.

This kind of affectation is one of the "dangers" of "a *little* learning:" those who are really good linguists are seldom so anxious to display their knowledge.

* Pope has accordingly been justly censured for his inconsistency in making the Alexandrine represent both a quick and a slow motion:

1. "Flies o'er the unbending corn, and skims along the main."
2. "Which, like a wounded snake, drags its slow length along."

In the first instance, he forgot that an *Alexandrine* is long, from containing more *feet* than a common verse; whereas a long *hexameter* has but the same number of feet as a short one, and therefore being pronounced in the same *time*, seems to move more rapidly.

† The slow movement of this line would be much more perceptible, if we pronounced (as doubtless the Latins did,) the *doubled consonants*; "*ag-gres-si fer-ro——sum-ma*:" but in English, and consequently in the English way of reading Latin or Greek, the doubling of a consonant only serves to fix the place of the accent; the latter of the two being never pronounced, except in a very few compound words; as "innate," "connatural," "poor-rate," "hop-pole."

But, lastly, it seems not to require any excessive exercise of fancy to perceive, if not, properly speaking, an *Imitation*, by words, of other things besides sound and motion, at least, an Analogical aptitude. That there is at least an apparent Analogy between things sensible, and things intelligible, is implied by numberless Metaphors; as when we speak of "*rough*, or *harsh*, *soft*, or *smooth* manners," "*turbulent* passions," the "*stroke*, or the *storms* of adversity," &c. Now if there are any words, or combinations of words, which have in their sound a congruity with certain sensible objects, there is no reason why they should not have the same congruity with those *emotions*, *actions*, &c. to which these sensible objects are analogous. Especially, as it is universally allowed that certain musical combinations are, respectively, appropriate to the expression of grief, anger, agitation, &c.

On the whole, the most probable conclusion seems to be, that many at least of the celebrated passages that are cited as Imitative in sound, were, on the one hand, not the result of *accident*, nor yet, on the other hand, of *study*; but that the idea in the author's mind spontaneously suggested appropriate sounds: thus, when Milton's mind was occupied with the idea of the opening of the infernal gates, it seems natural that his expression—

"And on their hinges grate harsh thunder,"

should have occurred to him without any distinct intention of imitating sounds.

It will be the safest rule, therefore, for a prose writer at least, never to make any distinct effort after this kind of Energy of expression, but to trust to the spontaneous occurrence of suitable sounds on every occasion where the introduction of them is likely to have a good effect.

It is hardly necessary to give any warning, generally, against the unnecessary introduction of *Technical* language of any kind, when the meaning can be adequately, or even tolerably, expressed in common, i. e. unscientific words; the terms and phrases of Art have an air of pedantic affectation, for which they do not compensate, by even the smallest appearance of increased Energy. But there is an apparent exception to this rule, in the case of what may be called the "Theological Style;" a peculiar phraseology, adopted more or less by a large proportion of writers of Sermons and other religious works; consisting partly of peculiar terms, but chiefly of common words used in a peculiar sense or combination, so as to form altogether a kind of diction widely differing from the classical standard of the language. This phraseology having been formed partly from the Style of some of the most eminent Divines, partly, and to a much greater degree, from that of the Scriptures, i. e. of our Version, has been supposed to carry with it an air of appropriate dignity and sanctity, which greatly adds to the force of what is said. And this may, perhaps, be the case when what is said is of little or no intrinsic weight, and is only such meagre commonplace as many religious works consist of; the associations which such language will excite in the minds of those accustomed to it, supplying, in some degree, the deficiencies of the matter. But this diction, though it may serve as a veil for poverty of thought, will be found to produce no less the effect of obscuring the lustre of what is truly valuable: if it adds an appearance of strength to what is weak, it adds weakness to what is strong; and if pleasing to those of narrow and ill cultivated mind, it is in a still higher degree repulsive to persons of taste. It may be said, indeed, with truth, that the improvement of the majority is a higher object than the gratification of a refined taste in a few; but it may be doubted whether any real Energy, even with respect to any class of hearers, is gained by the use of such a diction as that of which we are speaking. For it will often be found that what is received with great approbation, is yet, even if, strictly speaking, understood, but very little attended to or impressed upon the minds of the hearers. Terms and phrases which have been long familiar to them, and have certain vague and indistinct notions associated with them, men often suppose themselves to understand much more fully than they do; and still oftener give a sort of indolent assent to what is said, without making any effort of thought. It is justly observed by Mr. Foster, (*Essay* iv.) when treating on this subject, that "with regard to a considerable proportion of Christian readers and hearers, a reformed language would be excessively strange to them;" but that "its *being* so strange to them, would be a proof of the necessity of adopting it, at least, in part, and by degrees. For the manner in which some of them would receive this altered diction, would prove that the customary phraseology had scarcely given them any clear ideas. It would be found that the peculiar phrases had been not so much the vehicles of ideas, as the substitutes for them. These readers and hearers have been accustomed to chime to the sound, without apprehending the sense; insomuch, that if they hear the very ideas which these phrases signify, expressed ever so simply in other language, they do not recognise them." He observes also, with much truth, that the studied incorporation and imitation of the language of the Scriptures in the texture of any Discourse, neither indicates reverence for the Divine composition, nor adds to the dignity of that which is human; but rather diminishes that of such passages as might be introduced from the sacred writings in pure and distinct quotation, standing contrasted with the general Style of the work.

Of the Technical terms, as they may be called, of Theology, there are many the place of which might easily be supplied by corresponding expressions in common use; there are others, doubtless, which, denoting ideas exclusively belonging to the subject, could not be avoided without a tedious circumlocution; these, therefore, may be admitted as allowable peculiarities of diction; and the others, perhaps, need not be entirely disused: but it is highly desirable that *both* should be very frequently exchanged for words or phrases entirely free from any Technical peculiarity, even at the expense of some circumlocution. Not that this should be done so constantly as to render the terms in question obsolete; but by introducing frequently both the term and a sentence explanatory of the same idea, the evil just mentioned,—the habit of not thinking, or not thinking attentively, on the meaning of what is said, will be, in great measure, guarded against,—the Technical words themselves will make a more forcible impression,—and the danger of sliding into unmeaning *cant* will be materially lessened. Such repetitions, therefore, will more than compensate for, or rather will be exempt from, any appearance of

 tediousness, by the addition both of Perspicuity and Energy.* It may be asserted, with but too much truth, that a very considerable proportion of Christians have a habit of laying aside, in a great degree, their *common sense*, and letting it, as it were, lie dormant, when points of Religion come before them;—as if Reason were utterly at variance with Religion, and the ordinary principles of sound judgment were to be completely superseded on that subject; and accordingly it will be found, that there are many errors which are adopted, many truths which are overlooked, or not clearly understood, and many difficulties which stagger and perplex them, for want, properly speaking, of the exercise of their common sense; i. e. in cases precisely *analogous* to such as daily occur in the ordinary affairs of life, in which those very same persons would form a correct, clear, prompt, and decisive judgment. It is well worthy of consideration, how far the tendency to this habit might be diminished by the use of a diction conformable to the suggestions which have been here thrown out.

With respect to the *Number* of words employed, "it is certain," as Dr. Campbell observes, "that of whatever kind the sentiment be, witty, humorous, grave, animated, or sublime, the more *briefly* it is expressed, the Energy is the greater."—"As when the rays of the sun are collected into the focus of a burning-glass, the smaller the spot is which receives them, compared with the surface of the glass, the greater is the splendour, so, in exhibiting our sentiments by speech, the narrower the compass of words is, wherein the thought is comprised, the more energetic is the expression. Accordingly, we find that the very same sentiment expressed diffusely, will be admitted barely to be just;—expressed concisely, will be admired as spirited." He afterwards remarks, that though a languid redundancy of words is in all cases to be avoided, the energetic brevity which is the most contrary to it, is not adapted alike to every subject and occasion. "The kinds of writing which are less susceptible of this ornament, are, the Descriptive, the Pathetic, the Declamatory,† especially the last. It is, besides, much more suitable in writing than in speaking. A reader has the command of his time; he may read fast or slow, as he finds convenient; he can peruse a sentence a second time when necessary, or lay down the book and think. But if, in haranguing the people, you comprise a great deal in few words, the hearer must have uncommon quickness of apprehension to catch the meaning, before you have put it out of his power, by engaging his attention to something else." The mode in which this inconvenience should be obviated, and in which the requisite expansion may be given to any thing which the persons addressed cannot comprehend in a very small compass, is, as we have already remarked, not so much by increasing the number of words in which the sentiment is conveyed in each sentence, (though in this some variation must of course be admitted,) as by *repeating* it in various forms. The uncultivated and the dull will require greater expansion, and more copious illustration of the same thought, than the educated and the acute; but they are even still more liable to be wearied or bewildered by prolixity. If the material is too stubborn to be speedily cleft, we must patiently continue our efforts for a longer time, in order to accomplish it: but this is to be done, not by making each blow fall *more slowly*, which would only enfeeble them, but by *often-repeated* blows.

It is needful to insist the more on the energetic effect of Conciseness, because so many, especially young writers and speakers, are apt to fall into a style of pompous verbosity, not from negligence, but from an idea that they are adding both Perspicuity and Force to what is said, when they are only incumbering the sense with a needless load of words. And they are the more likely to commit this mistake, because such a style will often appear not only to the author, but to the vulgar (i. e. the vulgar in *intellect*,) among his hearers, to be very majestic and impressive. It is not uncommon to hear a speaker or writer of this class, mentioned as having a "very fine command of language," when, perhaps, it might be said with more correctness, that "his language has a command of him;" i. e. that be follows a train of words rather than of thought, and strings together all the striking expressions that occur to him on the subject, instead of first forming a clear notion of the sense he wishes to convey, and then seeking for the most appropriate vehicle in which to convey it. If, indeed, any class of men are found to be the most effectually *convinced*, *persuaded*, or *instructed*, by a turgid amplification, it is the Orator's business, true to his object, not to criticise or seek to improve their taste, but to accommodate himself to it. But it will be found that this is not near so often the case as many suppose. The Orator may often by this kind of style gain great *admiration*, without being the nearer to his proper end, which is to *carry his point*. It will frequently happen that not only the approbation, but the whole attention of the hearers will have been confined to the Style, which will have drawn their minds, not *to* the subject, but *from* it. In those spurious kinds of Oratory, indeed, which have been above mentioned, (p. 272, 273,) in which the inculcation of the Subject-matter is not the principal object proposed, a redundancy of words may often be very suitable; but in all that comes within the

* "It must indeed be acknowledged, that in many cases innovations have been introduced, partly by the *ceasing* to employ the words designating those doctrines which were designed to be set aside: but it is probable they may have been still more frequently and successfully introduced under the advantage of *retaining the terms*, while the principles were gradually subverted. And therefore, since the peculiar words can be kept to one invariable signification only by keeping that signification clearly in sight, by means of something separate from these words themselves, it might be wise in Christian authors and speakers sometimes to express the ideas in common words, either in connexion with the peculiar terms, or, occasionally, instead of them. Common words might less frequently be applied, as affected denominations of things, which have their own direct and common denominations, and be less frequently combined into uncouth phrases. Many peculiar and antique words might be exchanged for other single words of equivalent signification, and in common use. And the small number of peculiar terms acknowledged and established, as of permanent use and necessity, might, even separately from the consideration of modifying the diction, be, occasionally, with advantage to the explicit declaration and clear comprehension of Christian truth, made to give place to a fuller expression, in a number of common words, of those ideas of which they are the single signs." Foster, *Essay* iv. p. 304.

† This remark is made, and the principle of it (which Dr. Campbell has omitted) subjoined, in chap. ii. sec. 2, of this Article, p. 262.

legitimate province of Rhetoric, there is no fault to be more carefully avoided.*

It will therefore be advisable for a tiro in composition to look over what he has written, and to strike out every word and clause which he finds will leave the passage neither less perspicuous nor less forcible than it was before; "*quamvis invita recedant*;" remembering that, as has been aptly observed, "nobody knows what good things you leave out:" if the general effect is improved, that advantage is enjoyed by the reader unalloyed by the regret which the author may feel at the omission of any thing which he may think in itself excellent. But this is not enough; he must study contraction, as well as omission. There are many sentences which would not bear the *omission* of a single word consistently with perspicuity, which yet may be much more concisely expressed, with equal clearness, by the employment of different words, and by *recasting* a great part of the expression. Take for example such a sentence as the following: "A severe and tyrannical exercise of power must become a matter of necessary policy with Kings, when their subjects are imbued with such principles as justify and authorize rebellion;" this sentence could not be advantageously, nor to any considerable degree, abridged, by the mere *omission* of any of the words; but it may be expressed in a much shorter compass, with equal clearness and far greater energy; thus, "Kings will be tyrants from policy, when subjects are rebels from principle."† The hints we have thrown out on this point coincide pretty nearly with Dr. Campbell's remark on "*Verbosity*," as contra-distinguished from "*Tautology*,"‡ and from "*Pleonasm*." "The third and last fault I shall mention against vivid Conciseness is *Verbosity*. This it may be thought coincides with the Pleonasm already discussed. One difference however is this; in the Pleonasm there are words which add nothing to the sense; in the Verbose manner, not only single words, but whole clauses, may have a meaning, and yet it were better to omit them, because what they mean is unimportant. Instead, therefore, of enlivening the expression, they make it languish. Another difference is, that in a proper Pleonasm, a complete correction is always made by razing. This will not always answer in the Verbose style; it is often necessary to alter as well as blot."*

It is of course impossible to lay down precise rules as to the degree of Conciseness which is, on each occasion that may arise, allowable and desirable; but to an author who is, in his expression of any sentiment, wavering between the demands of Perspicuity and of Energy, (of which the former of course requires the first care, lest he should fail of both,) and doubting whether the phrase which has the most forcible brevity will be readily taken in, it may be recommended to use *both* expressions;—first to expand the sense, sufficiently to be clearly understood, and then to contract it into the most compendious and striking form. This expedient might seem at first sight the most decidedly adverse to the brevity recommended; but it will be found in practice that the addition of a compressed and pithy expression of the sentiment, which has been already stated at greater length, will produce the *effect* of brevity. For it is to be remembered that it is not on account of the actual *Number of words* that diffuseness is to be condemned, (unless one were limited to a certain space, or time,) but to avoid the flatness and tediousness resulting from it; so that if this appearance can be obviated by the insertion of such an abridged repetition as is here recommended, which adds poignancy and spirit to the whole, Conciseness will be, practically, promoted by the addition. The hearers will be struck by the forcibleness of the sentence which they will have been prepared to comprehend; they will *understand* the longer expression, and *remember* the shorter. But the force will, in general, be totally destroyed, or much enfeebled, if the order be reversed;—if the brief expression be put first, and afterwards expanded and explained; for it loses much of its force if it be not clearly understood the moment it is uttered; and if it be, there is no need of the subsequent expansion. The sentence recently quoted from Burke, as an instance of Energetic brevity, is in this manner brought in at the close of a more expanded exhibition of the sentiment, as a condensed conclusion of the whole. "Power, of some kind or other, will survive the shock in which manners and opinions perish; and it will find other and worse means for its support. The usurpation which, in order to subvert ancient institutions, has destroyed ancient principles, will hold power by arts similar to those by which it has acquired it. When the old feudal and chivalrous spirit of *fealty*, which, by freeing kings from fear, freed both kings and subjects from the precaution of tyranny, shall be extinct in the minds of men, plots and assassinations will be anticipated by preventive murder and preventive confiscation, and that long roll of grim and bloody maxims, which form the political code of all Power, not standing on its own honour, and the honour of those who are to obey it. Kings will be tyrants from policy when subjects are rebels from principle." Burke, *Reflections on the Revolution in France*, *Works*, vol. v. p. 153.

The same writer, in another passage of the same work, has a paragraph in like manner closed and summed up by a striking metaphor, (which will often

* "By a multiplicity of words, the sentiment is not set off and accommodated, but like David, in Saul's armour, it is incumbered and oppressed.

"Yet this is not the only, or perhaps the worst, consequence resulting from this manner of treating Sacred writ," [*paraphrasing*] "we are told of the torpedo, that it has the wonderful quality of numbing every thing it touches; a paraphrase is a torpedo. By its influence the most vivid sentiments become lifeless, the most sublime are flattened, the most fervid chilled, the most vigorous enervated. In the very best compositions of this kind that can be expected, the Gospel may be compared to a rich wine of a high flavour, diluted in such a quantity of water as renders it extremely vapid." Campbell, *Rhetoric*, book iii. ch. ii. sec. 2.

† Burke.

‡ Tautology, which he describes as "either a repetition of the same sense in different words, or a representation of any thing as the cause, condition, or consequence, of itself," is, in most instances, (of the latter kind at least,) accounted no offence rather against *correctness* than *brevity*: the example he gives from Bolingbroke, "how many are there by whom these *tidings* of good *news* were never heard," would usually be reckoned a *blunder* rather than an instance of *prolixity*: like the expression of "*Sinecure* places which have no duty annexed to them." "The Pleonasm," he observes, "implies merely superfluity. Though the words do not, as in the Tautology, repeat the sense, they add nothing to it; e. g. They returned [back again] to the [same] city [from] whence they came [forth.]" Book iii. ch. ii. sec. 2.

* Campbell, *Rhetoric*, book iii. ch. ii. sec. 2. part iii.

prove the most *concise*, as well as in other respects, striking, form of expression,) such as would not have been so readily taken in if placed at the beginning. "To avoid therefore the evils of inconstancy and versatility, ten thousand times worse than those of obstinacy and the blindest prejudice, we have consecrated the State, that no man should approach to look into its defects or corruptions but with due caution; that he should never dream of beginning its reformation by its subversion; that he should approach to the faults of the State as to the wounds of a father, with pious awe and trembling solicitude. By this wise prejudice we are taught to look with horror on those children of their country who are prompt rashly to hack that aged parent in pieces, and put him into the kettle of magicians, in hopes that by their poisonous weeds, and wild incantations, they may regenerate the paternal constitution, and renovate their father's life."* Burke, *Reflections on the Revolution in France, Works*, vol. v. p. 183.

So great, indeed, is the effect of a skilful interspersion of short, pointed, forcible sentences, that even a considerable violation of some of the foregoing rules may be by this means, in a great degree, concealed; and vigour may thus be communicated (if vigour of thought be not wanting) to a Style chargeable even with Tautology. This is the case with much of the language of Dr. Johnson, who is certainly, on the whole, an Energetic writer, though he would have been much more so, had not an over attention to the roundness and majestic sound of his sentences, and a delight in balancing one clause against another, led him so frequently into a faulty redundancy. Take, as an instance, a passage in his life of Prior, which may be considered as a favourable specimen of his style: "Solomon is the work to which he intrusted the protection of his name, and which he expected succeeding ages to regard with veneration. His affection was natural; it had undoubtedly been written with great labour; and who is willing to think that he has been labouring in vain? He had infused into it much knowledge, and much thought; had often *polished* it to *elegance*, often *dignified* it with *splendour*, and sometimes *heightened* it to *sublimity*; he perceived in it many excellences, and did not discover that it wanted that without which all others are of small avail, the power of *engaging attention and alluring curiosity*. Tediousness is the most fatal of all faults; negligences or errors are single and local; but tediousness pervades the whole; other faults are censured and forgotten, but the power of tediousness propagates itself. He that is weary the first hour, is more weary the second; as bodies forced into motion contrary to their tendency, pass more and more slowly through every successive interval of space. Unhappily this pernicious failure is that which an author is least able to discover. We are seldom tiresome to ourselves; and the act of composition fills and delights the mind with change of language and succession of images; every couplet when produced is new, and novelty is the great source of pleasure. Perhaps no man ever thought a line superfluous when he first wrote it, or contracted his work till his ebullitions of invention had subsided." It would not have been just to the author, nor even so suitable to the present purpose, to cite less than the whole of this passage, which exhibits the characteristic merits, even more strikingly than the defects, of the writer. Few could be found in the works of Johnson, and still fewer in those of any other writer, more happily and forcibly expressed; yet it can hardly be denied that the parts here distinguished by italics are chargeable, more or less, with Tautology.

It happens, unfortunately, that Johnson's Style is particularly easy of imitation, even by writers utterly destitute of his vigour of thought; and such imitators are intolerable. They bear the same resemblance to their model, that the armour of the Chinese, as described by travellers, consisting of thick quilted cotton covered with stiff glazed paper, does to that of the ancient knights; equally glittering, bulky, and cumbersome, but destitute of the temper and firmness which was its sole advantage. At first sight, indeed, this kind of Style appears far from easy of attainment; on account of its being remote from the colloquial, and having an elaborately artificial appearance; but in reality, there is none less difficult to acquire. To string together *substantives*, connected by conjunctions, which is the characteristic of Johnson's Style, is, in fact, the rudest and clumsiest mode of expressing our thoughts: we have only to find *names* for our ideas, and then put them together by connectives, instead of interweaving, or rather *felting* them together, by a due admixture of verbs, participles, prepositions, &c. So that this way of writing, as contrasted with the other, may be likened to the primitive rude carpentry, in which the materials were united by coarse external implements, pins, nails, and cramps, when compared with that art in its most improved state, after the invention of dovetail joints, grooves, and mortices, when the junctions are effected by forming properly the extremities of the pieces to be joined, so as at once to consolidate and conceal the juncture.

If any one will be at the pains to compare a few pages, taken from almost any part of Johnson's works, with the same quantity from any other of our admired writers, noting down the number of *substantives* in each, he will be struck with the disproportion. This would be still greater, if he were to examine with the same view an equal portion of Cicero; but it must be acknowledged that the genius of the Latin language allows and requires a much smaller proportion of substantives than are necessary in our own.

In aiming at a Concise Style, however, care must of course be taken that it be not *crowded*; the frequent recurrence of considerable ellipses, even when obscurity does not result from them, will produce an appearance of affected and laborious compression, which is offensive. The author who is studious of Energetic brevity, should aim at what may be called a *Suggestive* Style; such, that is, as, without making a distinct, though brief, mention of a multitude of particulars, shall put the hearer's mind into the same *train of thought* as the speaker's, and suggest to him

* This, however, being an instance of what may be called the *classical* Metaphor, no preparation or explanation, even though sufficient to make it *intelligible*, could render it very *striking* to those not thoroughly and early familiar with the ancient fables of Medea.

The Preacher has a considerable resource, of an analogous kind, in similar allusions to the history, description, parables, &c. of Scripture, which will often furnish useful illustrations and forcible metaphors, in an address to those well acquainted with the Bible; though these would be frequently unintelligible, and always comparatively feeble, to persons not familiar with Scripture.

more than is actually expressed.* Aristotle's Style, which is frequently so elliptical as to be dry and obscure, is yet often, at the very same time, unnecessarily diffuse, from his enumerating much that the reader would easily have supplied, if the rest had been fully and forcibly stated. He seems to have regarded his readers as capable of going *along with him* readily, in the deepest discussions, but not, of going *beyond him*, in the most simple; i. e. of filling up his meaning, and inferring what he does not actually express; so that in many passages a free translator might convey his sense in a shorter compass, and yet in a less cramped and elliptical diction. A particular statement, of which the general application is obvious, will often save a long abstract rule, which needs much explanation and limitation; and will thus suggest much that is not actually said; thus answering the purpose of a mathematical diagram, which though itself an individual, serves as a representative of a class. Slight *hints* also respecting the subordinate branches of any subject, and notices of the principles that will apply to them, &c. may often be substituted for digressive discussions, which, though laboriously compressed, would yet occupy a much greater space. Judicious divisions likewise and classifications, save much tedious enumeration; and, as has been formerly remarked, a well-chosen epithet may often suggest, and therefore supply the place of, an entire argument. It would not be possible, within a moderate compass, to lay down precise rules for the Suggestive kind of writing we are speaking of; but if the slight hints here given are sufficient to convey an idea of the object to be aimed at, practice will enable a writer gradually to form the habit recommended. It may be worth while, however, to add, that those accustomed to rational *conversation*, will find in that a very useful exercise, with a view to this point, (as well as to almost every other connected with Rhetoric;) since, in conversation, a man naturally tries first one and then another mode of expressing his thoughts, and stops as soon as he perceives that his companion fully comprehends his sentiments, and is sufficiently impressed with them.

We have dwelt the more earnestly on the head of Conciseness, because it is a quality in which young writers (who are the most likely to seek for practical benefit in a Treatise of this kind,) are usually most deficient; and because it is commonly said that, in them, exuberance is a promising sign; without sufficient care being taken to qualify this remark, by adding, that this over-luxuriance must be checked by judicious pruning. If an early proneness to redondancy be an indication of natural genius, those who possess this genius should be the more sedulously on their guard against it; and those who do not, should be admonished that the want of a *natural* gift cannot be supplied by copying its attendant defects. The praises which have been bestowed on *Copiousness* of diction, have probably tended to mislead authors into a cumbrous verbosity. It should be remembered, that there is no real Copiousness in a multitude of synonymes and circumlocutions. A house would not be the better furnished for being stored with ten times as many of some kinds of articles as were needed, while it was perhaps destitute of those required for other purposes; nor was Lucullus's wardrobe which, according to Horace, boasted five thousand mantles, necessarily well stocked, if other articles of dress were wanting. The completeness of a library does not consist in the number of volumes, especially if many of them are *duplicates*; but in its containing copies of all the most valuable works. And in like manner, true Copiousness of language consists in having at command, as far as possible, a suitable expression for each *different* modification of thought. This, consequently, will often *save* much circumlocution; so that the greater our command of language, the more concisely we shall be enabled to write. To an author who is attentive to these principles, diffuseness may be accounted no dangerous fault of Style, because practice will gradually correct it: but it is otherwise with one who *pleases himself* in stringing together well-sounding words into an easy, flowing, and (falsely-called) Copious Style, destitute of nerve; and who is *satisfied* with a small portion of matter; seeking to increase, as it were, the appearance of his wealth by hammering out his metal thin. *This* is far from a curable fault. When the Style is fully formed in other respects, pregnant fulness of meaning is seldom superadded; but when there is a basis of Energetic condensation of thought, the faults of harshness, baldness, or even obscurity, are much more likely to be remedied. Solid gold may be new-moulded and polished; but what can give solidity to gilding?

* Such a Style may be compared to a good map, which marks distinctly the great outlines, setting down the principal rivers, towns, mountains, &c. and leaving the imagination to supply the villages, hillocks, and streamlets; which if they were all inserted in their due proportions would crowd the map, though, after all, they could not be discerned without a microscope.

Lastly, the *Arrangement* of words may be made highly conducive to Energy. The importance of an attention to this point, with a view to Perspicuity, has been already noticed: but of two sentences equally perspicuous, and consisting of the very same words, the one may be a feeble and languid, the other a striking and Energetic expression, merely from the difference of Arrangement.

Some, among the moderns, are accustomed to speak of the *Natural* order of the words in a sentence, and to consider, each, the established Arrangement of his own language as the nearest to such a natural order; regarding that which prevails in Latin and in Greek as a sort of deranged and irregular structure. We are apt to consider that as most natural and intrinsically proper, which is the most familiar to ourselves; but there seems no good ground for asserting, that the customary structure of sentences in the ancient languages is less natural, or less suitable for the purposes for which language is employed, than in the modern. Supposing the established order in English or in French, for instance, to be more closely conformed to the grammatical or logical analysis of a sentence, than that of Latin or Greek, because we place the Subject first, the Copula next, and the Predicate last, &c. it does not follow that such an Arrangement is necessarily the best fitted in every case to excite the attention,—to direct it to the most essential points,—to gratify the imagination,—or to affect the feelings: it is, surely, the natural object of language to express as strongly as possible the speaker's sentiments, and to convey the same to the hearers; and that Arrangement of words may fairly be accounted the most natural by

 which all men are naturally led, as far as the rules of their respective languages allow them, to accomplish this object. The rules of many of the modern languages do indeed frequently confine an author to an order which he would otherwise never have chosen; but what translator of any taste would ever *voluntarily* alter the Arrangement of the words in such a sentence, as Μεγάλη ἡ Ἄρτεμις Ἐφεσίων, which our language allows us to render exactly, "*Great* is Diana of the Ephesians!" How feeble in comparison is the translation of Le Clerc, "*La Diane des Ephesiens est une grande Déesse!*" How imperfect that of Beausobre, "*La grande Diane des Ephesiens!*" How undignified that of Saci, "*Vive la grande Diane des Ephesiens!*"

Our language indeed is, though to a less degree, very much hampered by the same restrictions; it being in general necessary, for the expression of the sense, to adhere to an order which may not be in other respects the most eligible: "Cicero praised Cæsar," and "Cæsar praised Cicero," would be two very different propositions; the situation of the words being all that indicates, (from our want of *Cases*,) *which* is to be taken as the nominative, and which as the accusative; but such a restriction is far from being an advantage. The transposition of words which the ancient languages admit of, conduces, not merely to variety, but to Energy, and even to Precision. If, for instance, a Roman had been directing the attention of his hearers to the circumstance that *Cæsar* had been the object of Cicero's praise, he would, most likely, have put "*Cæsarem*" first; but he would have put "Cicero" first, if he had been remarking that not only others, but even *he*, had praised Cæsar.

It is for want of this liberty of Arrangement that we are often compelled to mark the *emphatic* words of our sentences by the voice, in speaking, and by italics, in writing; which would, in Greek or in Latin, be plainly indicated, in most instances, by the collocation alone. The sentence which has been often brought forward as an example of the varieties of expression which may be given to the same words, "Will you ride to London to-morrow?" and which may be pronounced and understood in, at least, five different ways, according as the first, second, &c. of the words is printed in italics, would be, by a Latin or Greek writer, arranged in as many different orders, to answer these several intentions. The advantage thus gained must be evident to any one who considers how important the object is which is thus accomplished, and for the sake of which we are often compelled to resort to such clumsy expedients; it is like the proper distribution of the *lights* in a picture; which is hardly of less consequence than the correct and lively representation of the objects.

It must be the aim then of an author, who would write with Energy, to avail himself of all the liberty which our language does allow, so to arrange his words that there shall be the least possible occasion for under-scoring and italics; and this, of course, must be more carefully attended to by the *writer* than by the *speaker*, who may, by his mode of utterance, conceal, in great measure, a defect in this point. It may be worth observing, however, that some writers, having been taught that it is a fault of Style to require many of the words to be in italics, fancy they avoid the fault, by omitting those indications where they are really needed; which is no less absurd than to attempt remedying the intricacies of a road by removing the direction-posts.* The proper remedy is, to endeavour so to construct the Style, that the collocation of the words may, as far as is possible, direct the attention to those which are emphatic. And the general maxim that should chiefly guide us, is, as Dr. Campbell observes, the homely saying, "Nearest the heart, nearest the mouth;" the idea, which is the most forcibly impressed on the author's mind, will naturally claim the first utterance, as nearly as the rules of the language will permit. And it will be found that, in a majority of instances, the most Emphatic word will be the *Predicate*; contrary to the rule which the nature of our language compels us, in most instances, to observe. It will often happen, however, that we do place the Predicate first, and obtain a great increase of Energy by this Arrangement. Of this licence our translators of the Bible have, in many instances, very happily availed themselves; as, e. g. in the sentence lately cited, "Great is Diana of the Ephesians;" so also, "Blessed is he that cometh in the name of the Lord:" it is evident how much this would be enfeebled by altering the Arrangement into "He that cometh in the name of the Lord is blessed." And, again, "To Him give all the prophets witness:" here, indeed, it may be said that that is properly the Subject which comes first; since that of which we are speaking is He, of whom we assert, that all the prophets bear Him witness; but still, the placing of the oblique case first, is a departure from the most common, and, what many call, the Grammatical order of our language. And, again, "Silver and Gold have I none; but what I have, that give I unto thee."† Another passage, in which they might advantageously have adhered to the order of the original, is, "Ἔπεσεν, ἔπεσε Βαβυλὼν, ἡ μεγάλη,"‡ which would certainly have been rendered as correctly, and more forcibly, as well as more closely, "Fallen, fallen is Babylon, that great city," than, "Babylon is fallen, is fallen."

The word "IT" is frequently very serviceable in enabling us to alter the Arrangement: thus, the sentence, "Cicero praised Cæsar," which admits of at least two modifications of sense, may be altered so as to express either of them, by thus varying the order: "It was Cicero that praised Cæsar," or, "It was Cæsar that Cicero praised." "IT" is, in this mode of using it, the *representative* of the Subject,§ which it thus enables us to place, if we will, after the Predicate.

With respect to *Periods*, it would be neither prac-

* The censure of frequent and long Parentheses also leads some writers into the like preposterous expedient of leaving out the marks ( ) by which they are indicated, and substituting commas; instead of so framing each sentence that they shall not be needed. It is no cure to a lame man, to take away his crutches.

† *Acts*, ch. v. ver. 6.

‡ *Rev.* ch. xviii. ver. 2.

§ Of whatever *gender* or *number* the subject referred to may be, "IT" may, with equal propriety, be employed to represent it. Our translators of the Bible have not scrupled to make "IT" refer to a *masculine* noun: "It is I, be not afraid;" but they seem to have thought it not allowable, as perhaps it was not, at the time when they wrote, to make such a reference to a *plural* noun. "Search the Scriptures——*they are they* which testify of Me:" we should now say, without any impropriety, "*IT is they, &c.*"

tically useful, nor even suitable to the present object, to enter into an examination of the different senses in which various authors have employed the word. A technical term may allowably be employed, in a scientific work, in any sense not very remote from common usage (especially when common usage is not uniform, and invariable, in the meaning affixed to it,) provided it be clearly defined, and the definition strictly adhered to. By a Period, then, is to be understood in this place, any sentence, whether simple or complex, which is so framed that the Grammatical construction will not admit of a close, before the end of it; in which, in short, the meaning remains suspended, as it were, till the whole is finished. A *loose* sentence, on the contrary, is, any that is not a Period;—any, whose construction will allow of a stop, so as to form a perfect sentence, at one or more places, before we arrive at the end. E. g. "We came to our journey's end—at last—with no small difficulty—after much fatigue—through deep roads—and bad weather." This is an instance of a *very* loose sentence; (for it is evident that this kind of structure admits of degrees,) there being no less than five places, marked by dashes, at any one of which the sentence might have terminated, so as to be grammatically perfect. The same words may be formed into a Period, thus: "At last, after much fatigue, through deep roads, and bad weather, we came, with no small difficulty, to our journey's end." Here, no stop can be made at any part, so that the preceding words shall form a sentence before the final close. These are both of them *simple* sentences; i. e. not consisting of several clauses, but having only a single verb; so that it is plain we ought not, according to this view, to confine the name of Period to *complex* sentences; as Dr. Campbell has done, notwithstanding his having adopted the same definition as has been here laid down.

Periods, or sentences nearly approaching to Periods, have certainly, when other things are equal, the advantage in point of Energy. An unexpected continuation of a sentence which the reader had supposed to be concluded, especially if in reading aloud, he had, under that supposition, dropped his voice, is apt to produce a sensation in the mind of being disagreeably balked; analogous to the unpleasant jar which is felt, when in ascending or descending stairs, we meet with a step more than we expected: and if this be often repeated, as in a *very* loose sentence, a kind of weary impatience results from the uncertainty when the sentence is to close. This, however, must have been much more the case in the ancient languages, than in the modern; because the variety of Arrangement which they permitted, and, in particular, the liberty of reserving the *verb*, on which the whole sense depends, to the end, made that structure natural and easy, in many instances in which, in our language, it would appear forced, unnatural, and affected. But the agreeableness of a certain degree, at least, of Periodic structure, in all languages, is apparent from this; that they all contain words which may be said to have no other use or signification but to *suspend the sense*, and lead the hearer of the first part of the sentence to expect the remainder. He who says, "the world is not eternal, nor the work of chance," expresses the same sense as if he said, "The world is *neither* eternal, nor the work of chance;" yet the latter would be generally preferred. So also, "The vines afforded *both* a refreshing shade, and a delicious fruit;" the word "both," would be missed, though it adds nothing to the sense. Again, "While all the Pagan nations consider Religion as one part of Virtue, the Jews, on the contrary, regard Virtue as a part of Religion;"* the omission of the first word would not alter the sense, but would destroy the Period; to produce which is its only use. The ΜΕΝ, ΔΕ, and ΤΕ of the Greek are, in many places, subservient to this use alone.

The modern languages do not indeed admit, as was observed above, of so Periodic a Style as the ancient do: but an author, who does but clearly understand what a Period is, and who applies the test we have laid down, will find it very easy, after a little practice, to compose in Periods, even to a greater degree than, in an English writer, good taste will warrant. His skill and care will be chiefly called for in avoiding all appearance of stiffness and affectation in the construction of them,—in not departing, for the sake of a Period, too far from colloquial usage,—and in observing such moderation in the employment of this Style, as shall prevent any betrayal of artifice,—any thing savouring of elaborate stateliness, which is always to be regarded as a worse fault than the slovenliness and languor which accompany a very loose Style.

It should be observed, however, that, as a sentence which is not *strictly* a Period, according to the foregoing definition, may yet approach indefinitely near to it, so as to produce nearly the same effect, so on the other hand, Periods may be so constructed as to produce much of the same feeling of weariness and impatience which results from an excess of loose sentences. If the clauses be very long, and contain an enumeration of many circumstances, though the sentence be so framed, that we are still kept in expectation of the conclusion, yet it will be an impatient expectation; and the reader will feel the same kind of uneasy uncertainty *when* the *clause* is to be finished, as would be felt respecting the *sentence*, if it were loose. And this will especially be the case, if the rule formerly given with a view to Perspicuity be not observed,† of taking care that each part of the sentence be understood, as it proceeds. Each clause, if it consist of several parts, should be continued with the same attention to their mutual connection, so as to suspend the sense, as is employed in the whole sentence; that it may be, as it were a *Periodic* clause; and if one clause be long and another short, the *shorter* should, if possible, be put *last*. Universally indeed a sentence will often be, practically, too long, i. e. will have a tedious, dragging effect, merely from its concluding with a much longer clause than it began with; so that a composition which most would censure as abounding too much in long sentences, may often have its defect, in great measure, remedied without shortening any of them; merely by reversing the order of each. This of course holds good with respect to all complex sentences of any considerable length, whether Periods or not. An instance of the difference of effect produced by this means, may be seen in such a sentence as the following: "The State was made, under the pretence of serving it, in reality, the prize of their contention, to each of those opposite parties,

* Josephus. † P. 272.

who professed in specious terms, the one, a preference for moderate Aristocracy, the other, a desire of admitting the people at large, to an equality of civil privileges." This may be regarded as a complete Period; and yet, for the reason just mentioned, has a tedious and cumbrous effect. Many critics might recommend, and perhaps with reason, to break it into two or three; but it is to our present purpose to remark that it might be, in some degree at least, decidedly improved, by merely reversing the clauses; as thus: "The two opposite parties, who professed in specious terms, the one, a preference for moderate Aristocracy, the other, a desire of admitting the people at large to an equality of civil privileges, made the State, which they pretended to serve, in reality the prize of their contention."* Another instance may be cited from a work, in which any occasional awkwardness of expression is the more conspicuous, on account of its general excellence, the Church Liturgy; the style of which is so justly admired for its remarkable union of energy with simplicity, smoothness, and elegance: the following passage from the Exhortation is one of the very few, which, from the fault just noticed, it is difficult for a good reader to deliver with spirit: "And although we ought at all times humbly to acknowledge our sins before God,|| yet ought we most chiefly so to do,|| when we assemble and meet together—to render thanks for the great benefits that we have received at his hands,—to set forth his most worthy praise, to hear his most holy word, and to ask those things which are requisite and necessary,—as well for the body as the soul." This is evidently a very loose sentence, as it might be supposed to conclude at any one of the three places which are marked by dashes (—); this disadvantage, however, may easily be obviated by the suspension of voice, by which a good reader, acquainted with the passage, would indicate that the sentence was not concluded; but the great fault is the length of the *last* of the three principal clauses, in comparison of the former two; (the conclusions of which we have marked ||) by which a dragging and heavy effect is produced, and the sentence is made to appear longer than it really is. This would be more manifest to any one not familiar, as most are, with the passage; but a good reader of the Liturgy will find hardly any sentence in it so difficult to deliver to his own satisfaction. It is perhaps the more profitable to notice a blemish occurring in a composition so well known, and so deservedly valued for the excellence, not only of its sentiments, but of its language.

It is a useful admonition to young writers, with a view to what has lately been said, that they should always attempt to *recast* a sentence which does not please; altering the Arrangement and entire construction of it, instead of merely seeking to change one word for another. This will give a great advantage in point of *Copiousness* also: for there may be, suppose, a *substantive*, which, either because it does not fully express our meaning, or for some other reason, we wish to remove, but can find no other to supply its place; but the object may perhaps be easily accomplished by means of a *verb*, adverb, or some other part of speech, the substitution of which implies an alteration of the construction. It is an exercise accordingly which may be recommended as highly conducive to the improvement of Style, to practise casting a sentence into a variety of different forms.

It is evident, from what has been said, that in compositions intended to be delivered, the Periodic Style is much less necessary, and therefore much less suitable, than in those designed for the closet. The speaker may, in most instances, by the skilful suspension of his voice, give to a loose sentence the effect of a Period: and though, in both species of composition the display of art is to be guarded against, a more unstudied air is looked for in such as are spoken.

The study of the best Greek and Latin writers may be of great advantage towards the improvement of the Style in the point concerning which we have now been treating, (for the reason lately mentioned,) as well as in most others: and there is this additional advantage, (which, at first sight, might appear a disadvantage,) that the Style of a foreign writer cannot be so *closely* imitated as that of one in our own language: for this reason there will be the less danger of falling into an *obvious* and *servile* imitation. Bolingbroke may be noted as one of the most Periodic of English writers; Swift and Addison, (though in other respects very different,) are among the most loose.

Antithesis has been sometimes reckoned as one form of the Period; but it is evident that, according to the view here taken, it has no necessary connection with it. One clause may be *opposed* to another, by means of some *contrast* between corresponding words in each, whether or not the clauses be so connected that the former could not, by itself, be a complete sentence. Tacitus, who is one of the most Antithetical, is at the same time one of the least Periodic, of all the Latin writers.

There can be no doubt that this figure is calculated to add greatly to Energy. Every thing is rendered more striking by contrast; and almost every kind of subject-matter affords *materials* for contrasted expressions. Truth is *opposed* to error; wise conduct to foolish; different causes often produce opposite effects; different circumstances dictate to prudence opposite conduct; opposite impressions may be made by the same object, on different minds; and every extreme is opposed both to the Mean, and to the other extreme. If, therefore, the language be so constructed as to contrast together these opposites, they throw light on each other by a kind of mutual reflexion, and the view thus presented will be the more striking. By this means also we may obtain, consistently with Perspicuity, a much greater degree of Conciseness; which in itself is so conducive to Energy; e. g. "When Reason is against a man, he will be against Reason;"* it would be hardly possible to express this sentiment, not Antithetically, so as to be clearly intelligible, except in a much longer sentence. Again, "Words are the Counters of wise men, and the Money of fools;" here we have an instance of the combined effect of Antithesis and Metaphor in producing increased Energy, both directly, and at the same time, (by the Conciseness resulting from them,) indirectly; and accordingly, in such pointed and pithy expressions, we obtain the gratification which, as Aristotle remarks, results from "the act of *learning* quickly and easily." It is a remark of the same au-

* Thucydides, on the Corcyrean sedition.

* Hobbes.

 thor, that, in Antithesis, either "contraries are joined to contraries," in the two clauses respectively, or "the *same* thing is joined to contraries;" of this last, the former of the two examples, just cited, is an instance;—the condemnation pronounced on any one's principles or conduct, by Reason, being contrasted with his dislike and defiance of it: the other example is an instance of the former kind; "Counters" being opposed to "Money," and "wise men" to "fools." Of the same nature is the Antithetical expression, "Party is the madness of many, for the gain of a few;" which affords, likewise, an instance of this construction in a sentence which does not contain two distinct clauses. Frequently the same words, placed in different relations with each other, will stand in contrast to themselves; as in the expression, "A fool with judges; among fools, a judge;"* and in that given by Quinctilian, "*non ut edam vivo, sed ut vivam edo;*" "I do not live to eat, but eat to live;" both of these are instances also of perfect Antithesis, without Period; for each of these sentences might, grammatically, be concluded in the middle. Of the same kind is an expression in a Speech of Mr. Wyndham's, "Some contend that I disapprove of this plan, because it is not my own; it would be more correct to say, that it is not my own, because I disapprove it."

The use of Antithesis has been censured by some, as if it were a paltry and affected decoration, unsuitable to a chaste, natural, and masculine Style. Pope, accordingly, himself one of the most Antithetical of our writers, speaks of it in the *Dunciad* with contempt:

> "I see a Chief who leads my chosen sons,
> All arm'd with Points, Antitheses, and Puns."

The excess, indeed, of this Style, by betraying artifice, effectually destroys Energy; and draws off the attention, even of those who are *pleased* with effeminate glitter, from the matter to the Style. But, as Dr. Campbell observes, "the excess itself into which some writers have fallen, is an evidence of its value—of the lustre and emphasis which Antithesis is calculated to give to the expression. There is no risk of intemperance in using a liquor which has neither spirit nor flavour."

It is, of course, impossible to lay down precise rules for determining, *what* will amount to excess, in the use of this, or of any other figure: the great safeguard will be the formation of a pure taste, by the study of the most chaste writers, and unsparing self-correction. But one rule always to be observed in respect to the antithetical construction, is to remember that in a true Antithesis the opposition is always in the *ideas* expressed. Some writers abound with a kind of mock-antithesis, in which the same, or nearly the same sentiment which is expressed by the first clause, is repeated in a second; or at least, in which there is but little of real contrast between the clauses which are expressed in a contrasted form. This kind of style not only produces disgust instead of pleasure, when once the artifice is detected, which it soon must be, but also, instead of the brevity and vigour resulting from true Antithesis, labours under the fault of prolixity and heaviness. Sentences which might have been expressed simply, are expanded into complex ones, by the addition of clauses, which add little or nothing to the sense; and which have been compared to the false handles and keyholes with which furniture is decorated, that serve no other purpose than to *correspond to the real ones.* Much of Dr. Johnson's writing is chargeable with this fault.

Bacon, in his Rhetoric, furnishes, in his commonplaces, (i. e. heads of Arguments, *pro* and *contra*, on a variety of subjects,) some admirable specimens of compressed and striking Antitheses; many of which are worthy of being enrolled among the most approved proverbs: e. g. "He who dreads new remedies, must abide old evils." "Since things alter for the worse spontaneously, if they be not altered for the better designedly, what end will there be of the evil?" "The humblest of the virtues the vulgar praise, the middle ones they admire, of the highest, they have no perception." &c.

It will not unfrequently happen that an Antithesis may be even more happily expressed by the sacrifice of the Period, if the clauses are by this means made of a more convenient length, and a resting-place provided at the most suitable point: e. g. "The persecutions undergone by the Apostles, furnished both a trial to *their* faith, and a confirmation to *our's*:—a trial to them, because if human honours and rewards had attended them, they could not, even themselves, have been certain that these were not their object; and a confirmation to us, because they would not have encountered such sufferings in the cause of imposture." If this sentence were not broken as it is, but compacted into a Period, it would have more heaviness of effect, though it would be rather shorter: e. g. "The persecutions undergone by the Apostles, furnished both a trial of their faith, since if human honours, &c. &c. and also a confirmation of ours, because," &c. Universally, indeed, a complex sentence, whether Antithetical or not, will often have a degree of spirit and liveliness from the latter clause being made to *turn back*, as it were, upon the former, by containing, or referring to, some word that had there been mentioned: e. g. "The introducers of the now-established principles of political economy may fairly be considered to have made a great *discovery*; a *discovery* the more creditable, from the circumstance that the facts on which it was founded had long been well known to all." This kind of Style also may, as well as the Antithetical, prove offensive if carried to such an excess as to produce an appearance of affectation or mannerism.

Lastly, to the *Speaker* especially, the occasional employment of the *Interrogative* form, will often prove serviceable with a view to Energy. It calls the hearer's attention more forcibly to some important point, by a personal appeal to each, either to assent to what is urged, or to frame a reasonable objection; and it often carries with it an air of triumphant defiance of an opponent to refute the argument if he can. Either the Premiss or the Conclusion, or both, of any argument, may be stated in this form; but it is evident that if it be introduced too frequently, it will necessarily fail of the object of directing a *particular* attention to the most important points. To attempt to make every thing emphatic, is to make nothing emphatic. The utility, however, of this figure, to the Orator at least, is sufficiently established by the single

* Cowper.

consideration, that it abounds in the Speeches of Demosthenes.

§ 3. On the last quality of Style to be noticed, Elegance or Beauty, it is the less necessary to enlarge, both because the most appropriate and characteristic excellence of the class of compositions here treated of, is, that Energy of which we have been speaking, and also because many of the rules laid down under that head, are equally applicable with a view to Elegance; the same Choice, Number, and Arrangement of words, will, for the most part, conduce both to Energy and to Beauty. The two qualities however are by no means undistinguishable: a Metaphor, for instance, may be apt, and striking, and consequently conducive to Energy of expression, even though the new image, introduced by it, have no intrinsic beauty, or be even unpleasant; in which case it would be at variance with Elegance, or at least would not conduce to it. Elegance requires that all homely and coarse words and phrases should be avoided, even at the expense of circumlocution; though they may be the most apt and forcible that language can supply. And Elegance implies a smooth and easy flow of words in respect of the sound of the sentences; though a more harsh and abrupt mode of expression may often be, at least equally, energetic.

Accordingly, many are generally acknowledged to be forcible writers, to whom no one would give the credit of Elegance; and many others, who are allowed to be elegant, are yet by no means vigorous and energetic.

When the two excellencies of Style are at variance, the general rule to be observed by the Orator, is to prefer the energetic to the elegant. Sometimes, indeed, a plain, or even a somewhat homely expression, may have even a *more* energetic effect, from that very circumstance, than one of more studied refinement, since it may convey the idea of the Speaker's being thoroughly in earnest, and anxious to convey his sentiments, where he uses an expression that can have no other recommendation; whereas a strikingly elegant expression may sometimes convey a suspicion that it was introduced *for the sake* of its Elegance; which will greatly diminish the force of what is said. Universally, a writer or speaker should endeavour to maintain the appearance of expressing himself, not, as if he *wanted to say something*, but as if he *had something to say*: i. e. not as if he had a subject set him, and was anxious to compose the best essay or declamation on it that he could; but as if he had some ideas to which he was anxious to give utterance;—not as if he wanted to compose (for instance) a sermon, and was desirous of performing that task satisfactorily, but as if there was something in his mind which he was desirous of communicating to his hearers. This is probably what Dr. Butler means when he speaks of a man's writing "with simplicity and in earnest." *His* manner has this advantage, though it is not only inelegant, but often obscure: Dr. Paley's is equally earnest, and very perspicuous; and though often homely, is more impressive than that of many of our most polished writers. It is easy to discern the prevalence of these two different manners in different authors, respectively, and to perceive the very different effects produced by them; it is not so easy for one who is not really writing "with simplicity and in earnest," to assume the appearance of it. But certainly nothing is more adverse to this appearance than over-refinement. Any expression indeed that is vulgar, in bad taste, and unsuitable to the dignity of the subject, or of the occasion, is to be avoided; since, though it might have, with some hearers, an energetic effect, this would be more than counterbalanced by the disgust produced in others; and where a *small* accession of Energy is to be gained at the expense of a *great* sacrifice of Elegance, the latter will demand a preference. But still, the general rule is not to be lost sight of by him who is in earnest aiming at the true ultimate end of the Orator, to which all others are to be made subservient; viz. not the amusement of his hearers, nor their admiration of himself, but their Conviction or Persuasion. It is from this view of the subject that we have dwelt most on that quality of Style which seems most especially adapted to that object. Perspicuity is required in *all* compositions; and may even be considered as the ultimate end of a *Scientific* writer, considered as such; he may indeed practically increase his utility by writing so as to excite curiosity, and recommend his subject to general attention; but in doing so, he is, in some degree, superadding the office of the Orator to his own; as a Philosopher, he may *assume* the existence in his reader of a desire for knowledge, and has only to convey that knowledge in language that may be *clearly understood*. Of the Style of the Orator, (in the wide sense in which we have been using this appellation, as including all who are aiming at Conviction,) the appropriate object is to *impress* the meaning strongly upon men's minds. Of the Poet, as such, the ultimate end is to *give pleasure*; and accordingly Elegance or Beauty (in the most extensive sense of those terms,) will be the appropriate qualities of his language.

Some indeed have contended, that to give pleasure is not the ultimate end of Poetry;* not distinguishing between the object which the Poet may have in view, *as a man*, and that which is the object of Poetry, as Poetry. Many, no doubt, may have proposed to themselves the far more important object of producing moral improvement in their hearers through the medium of Poetry; and so have others, the inculcation of their own political or philosophical tenets, or, (as is supposed in the case of the *Georgics*,) the encouragement of Agriculture: but if the views of the individual are to be taken into account, it should be considered that the personal fame or emolument of the author is very frequently *his* ultimate object. The true test is easily applied: that which to competent judges affords the appropriate *pleasure* of Poetry, is *good* poetry, whether it answer any other purpose or not; that which does *not* afford this pleasure, however instructive it may be, is not good *Poetry*, though it may be a valuable *work*.

It may be doubted, however, how far these remarks apply to the question respecting Beauty of *Style*; since the chief gratification afforded by Poetry, arises, it may be said, from the Beauty of the *thoughts*; and undoubtedly if these be mean and common-place, the Poetry will be worth little; but still, it is not any quality of the thoughts that *constitutes* Poetry. Notwithstanding all that has been advanced by some

* Supported in some degree by the authority of Horace: *Aut prodesse volunt, aut delectare Poetæ.*

French critics,* to prove that a work, not in metre, may be a Poem, (which doctrine was partly derived from a misinterpretation of a passage in Aristotle's *Poetics*,†) universal opinion has always given a contrary decision. Any composition in *verse*, (and none that is not,) is always called, whether good or bad, a Poem, by all who have no favourite hypothesis to maintain. It is indeed a common figure of speech to say, in speaking of any work that is deficient in the qualities which Poetry *ought* to exhibit, that it is not a Poem; just as we say of one who wants the characteristic excellences of the species, or the sex, that he is *not a man*:‡ and thus some have been led to confound together the appropriate *excellence* of the thing in question, with its *essence*: but the use of such an expression as, an "*indifferent*," or "a *dull Poem*," shows plainly that the title of Poetry does not necessarily imply the requisite *Beauties* of Poetry.

Poetry is not distinguished from Prose by superior Beauty of thought or of expression, but is a distinct kind of composition;§ and they produce, when each is excellent in its kind, distinct kinds of pleasure. Try the experiment, of merely breaking up the metrical structure of a fine Poem, and you will find it *inflated and bombastic* Prose: remove this defect by altering the words and the Arrangement, and it will be *better* Prose than before; then arrange this again into metre, without any other change, and it will be *tame* and *dull* Poetry; but still it will *be* Poetry, as is indicated by the very censure it will incur; for if it were not, there would be no fault to be found with it; since, while it remained Prose, it was (as we have supposed,) unexceptionable. The circumstance that the same Style which was even required in one kind of composition, proved offensive in the other, shows that a different kind of language is suitable for a composition in metre.

Another indication of the essential difference between the two kinds of composition, and of the superior importance of the *expression* in Poetry, is, that a good *translation* of a Poem, (though, perhaps, strictly speaking, what is so called is rather an *imitation*,) is read with equal, or even superior pleasure by one well acquainted with the original; whereas the best translation of a Prose work, (at least of one not principally valued for beauty of Style,) will seldom be read by one familiar with the original. And for the same reason, a fine passage of Poetry will be reperused, with unabated pleasure, for the twentieth time, even by one who knows it by heart

According to the views here taken, good Poetry might be defined, "*Elegant and decorated language in metre, expressing* such and such thoughts;" and good Prose composition, "*such and such thoughts expressed in* good language;" that which is primary in each being subordinate in the other.

What has been said may be illustrated as fully, not as it might be, but as is suitable to the present occasion, by the following passages from Dr. A. Smith's admirable fragment of an *Essay on the Imitative Arts*: "Were I to attempt to discriminate between Dancing and any other kind of movement, I should observe, that though in performing any ordinary action,—in walking, for example, across the room, a person may manifest both grace and agility, yet if he betrays the least intention of showing either, he is sure of offending more or less, and we never fail to accuse him of some degree of vanity and affectation. In the performance of any such ordinary action, every one wishes to appear to be solely occupied about the proper purpose of the action; if he means to show either grace or agility, he is careful to conceal that meaning; and in proportion as he betrays it, which he almost always does, he offends. In Dancing, on the contrary, every one professes and avows, as it were, the intention of displaying some degree either of grace or of agility, or of both. The display of one or other, or both of these qualities, is, in reality, the proper purpose of the action; and there can never be any disagreeable vanity or affectation in following out the proper purpose of any action. When we say of any particular person, that he gives himself many affected airs and graces in Dancing, we mean either that he exhibits airs and graces unsuitable to the nature of the Dance, or that he exaggerates those which are suitable. Every Dance is, in reality, a succession of airs and graces of some kind or other, which, if I may say so, profess themselves to be such. The steps, gestures, and motions which, as it were, avow the intention of exhibiting a succession of such airs and graces, are the steps, gestures, and motions which are peculiar to Dancing. * * * * The distinction between the sounds or tones of Singing, and those of Speaking, seems to be of the same kind with that between the step, &c. of Dancing, and those of any other ordinary action. Though in Speaking a person may show a very agreeable tone of voice, yet if he seems to *intend* to show it,—if he appears to listen to the sound of his own voice, and as it were to tune it into a pleasing modulation, he never fails to offend, as guilty of a most disagreeable affectation. In Speaking, as in every other ordinary action, we expect and require that the speaker should attend only to the proper purpose of the action,—the clear and distinct expression of what he has to say. In Singing, on the contrary, every one professes the intention to please by the tone and cadence of his voice; and he not only appears to be guilty of no disagreeable affectation in doing so, but we expect and require that he should do so. To please by the Choice and Arrangement of agreeable sounds, is the proper purpose of all music, vocal, as well as instrumental; and we always expect that every one should attend to the proper purpose of whatever action he is performing. A person may appear to sing, as well as to dance, affectedly; he may endeavour to please by sounds and tones which are unsuitable to the nature of the song; or he may dwell too much on those which are suitable to it. The disagreeable affectation appears to consist always, not in attempting to please by a proper, but by some

* See Preface to *Telemaque*.

† Ψιλοὶ λόγοι has been erroneously interpreted language *without metre*, in a passage where it certainly means Metre *without music*: or, as he calls it in another passage of the same work, ψιλομετρία.

‡ "I dare do all that may become a man
Who dares do more is *none*."—*Macbeth*.

§ It is hardly necessary to remark, that we are not defending or seeking to introduce any *unusual* or *new* sense of the word Poetry; but, on the contrary, explaining and vindicating that which is the most customary among all men who have no particular theory to support. The mass of mankind often need, indeed, to have the meaning of a word (i. e. their *own* meaning,) *explained* and *developed*; but not, to have it determined *what* it shall mean, since *that* is determined by their use; the true sense of each word being, *that which is understood by it*.

Rhetoric. improper modulation of the voice." It is only necessary to add, (what seems evidently to have been in the author's mind, though the Dissertation is left unfinished,) that Poetry has the same relation to Prose, as Dancing to Walking, and Singing to Speaking; and that what has been said of *them*, will apply exactly, *mutatis mutandis*, to the other. It is needless to state this at length, as any one, by going over the passages just cited, merely substituting for "Singing," "Poetry," for "Speaking," "Prose," for "Voice," "Language," &c. will at once perceive the coincidence.

What has been said will not be thought an unnecessary digression, by any one who considers, (not to mention the direct application of Dr. Smith's remarks, to *Elocution*) the important principle thus established in respect of the decorations of Style: viz. that though it is possible for a poetical Style to be affectedly and offensively ornamented, yet the same degree and kind of decoration which is not only allowed, but required, in Verse, would in Prose be disgusting; and that the *appearance of attention* to the Beauty of the expression, and to the Arrangement of the words, which in Verse is essential, is to be carefully avoided in Prose. Chap. III.

Chap. IV. And since, as Dr. Smith observes, "such a design, when it exists, is almost always betrayed; the safest rule is, never, during the act of composition, to study Elegance, or think about it at all. Let an author study the best models—mark their beauties of Style, and dwell upon them, that he may insensibly catch the habit of expressing himself with Elegance; and when he has completed any composition, he may revise it, and cautiously alter any passage that is awkward and harsh, as well as those that are feeble and obscure: but let him never, *while writing*, think of any beauties of Style; but content himself with such as may occur spontaneously. He should carefully study *Perspicuity* as he goes along; he may also, though more cautiously, aim, in like manner, at Energy; but if he is endeavouring after Elegance, he will hardly fail to betray that endeavour; and in proportion as he does this, he will be so far from giving pleasure, to good judges, that he will offend more than by the rudest simplicity.

CHAPTER IV.

OF ELOCUTION.

On the importance of this branch, it is hardly necessary to offer any remark. Few need to be told that the effect of the most perfect composition may be entirely destroyed, even by a Delivery which does not render it unintelligible;—that one, which is inferior both in matter and style, may produce, if better spoken, a more powerful effect than another which surpasses it in both those points; and that even such an Elocution as does not spoil the effect of what is said, may yet fall far short of doing full justice to it. "What would you have said," observed Æschines, when his recital of his great rival's celebrated Speech on the Crown was received with a burst of admiration,—"what would you have said had you heard *him* speak it?"

The subject is far from having failed to engage attention: of the prevailing deficiency of this, more than of any other qualification of a perfect Orator, many have complained; and several have laboured to remove it: but it may safely be asserted, that their endeavours have been, at the very best, entirely unsuccessful. Probably not a single instance could be found of any one who has attained by the study of any system of instruction that has appeared, a really good Delivery; but there are many, probably nearly as many as have fully tried the experiment, who have by this means been totally spoiled;—who have fallen irrecoverably into an affected style of *spouting*, worse, in all respects, than their original mode of Delivery. Many accordingly have, not unreasonably, conceived a disgust for the subject altogether; considering it hopeless that Elocution should be *taught* by any rules; and acquiescing in the conclusion that it is to be regarded as entirely a gift of nature, or an accidental acquirement of practice. It is to counteract the prejudice which may result from these feelings, that we profess in the outset a dissent from the principles generally adopted, and lay claim to some degree of originality in our own. Novelty affords at least an opening for hope, and the only opening, when former attempts have met with total failure.

The requisites of Elocution correspond in great measure with those of Style: *Correct Enunciation*, in opposition both to *indistinct* utterance, and to *vulgar* and *dialectic* pronunciation, may be considered as answering to Purity, and Grammatical Propriety. These qualities of Style and of Elocution, being equally required in common conversation, do not properly fall within the province of Rhetoric. The three qualities, again, which have been treated of, under the head of Style, Perspicuity, Energy, and Elegance, may be regarded as equally requisites of Elocution; which, in order to be perfect, must convey the meaning *clearly*, *forcibly*, and *agreeably*. Requisites of Elocution.

Before however we enter upon any separate examination of these requisites, it will be necessary to premise a few remarks on the distinction between the two branches of Delivery, viz. *Reading* aloud, and *Speaking*. The object of *correct* Reading is, to convey to the hearers, through the medium of the ear, what is conveyed to the reader by the eye;—to put them in the same situation with him who has the book before him;—to exhibit to them, in short, by the voice, not only each word, but also all the stops, paragraphs, italic characters, notes of interrogation, &c.* Reading and Speaking.

* It may be said, indeed, that even tolerable Reading aloud supplies more than is exhibited by a book to the eye; since though Italics, e. g. indicate which word is to receive the emphasis, they do not point out the *tone* in which it is to be pronounced; which may be essential to the right understanding of the sentence; e. g. in such a sentence as in *Genesis*, ch. i. "God said, let there be light, and there *was* light:" here we can indicate indeed that the stress is to be upon "*was*;" but it may be pronounced in different tones; one of which would alter the sense, by implying that there *was* light *already*. This is true indeed; and it is also true, that the very words themselves are

which his sight presents to him. His voice seems to indicate to them, "thus and thus it is written in the book or manuscript before me." *Impressive* reading superadds to this, some degree of adaptation of the tones of voice to the character of the subject, and of the Style. What is usually termed *fine* Reading seems to convey, in addition to this, a kind of admonition to the hearers respecting the feelings which the composition ought to excite in them: it appears to say, "this deserves your admiration;—this is sublime;—this is pathetic, &c." But Speaking, i. e. *natural* speaking, when the Speaker is uttering his own sentiments, and is thinking exclusively of *them*, has something in it distinct from all this: it conveys, by the sounds which reach the ear, the idea, that what is said is the effusion of the Speaker's own mind, which he is desirous of imparting to others. A decisive proof of which is, that if any one overhears the voice of another, to whom he is an utter stranger—suppose in the next room—without being able to catch the sense of what is said, he will hardly ever be for a moment at a loss to decide whether he is *Reading* or *Speaking*; and this, though the hearer may not be one who has ever paid any critical attention to the various modulations of the human voice. So wide is the difference of the tones employed on these two occasions, be the subject what it may.*

The difference of effect produced is proportionably great: the personal *sympathy* felt towards one who appears to be delivering his own sentiments is such, that it usually rivets the attention, even involuntarily, though to a discourse which appears hardly worthy of it. It is not easy for an auditor to fall asleep while he is hearing even perhaps feeble reasoning clothed in indifferent language, delivered extemporaneously, and in an unaffected style; whereas it is common for men to find a difficulty in keeping themselves awake, while listening even to a good dissertation, of the same length, or even shorter, on a subject, not uninteresting to them, when *read*, though with propriety, and not in a languid manner. And the thoughts, even of those not disposed to be drowsy, are apt to wander, unless they use an effort from time to time to prevent it; while, on the other hand, it is notoriously difficult to withdraw our attention, even from a trifling talker,

not always presented to the eye with the same distinctions as are to be conveyed to the ear; as e. g. "abuse," "refuse," "project," and many others are pronounced differently, as nouns and as verbs. This ambiguity however in our written signs, as well as the other, relative to the emphatic words, are imperfections which will not mislead a moderately practised reader. Our meaning in saying that such Reading as we are speaking of, puts the hearers in the same situation as if the book were before them, is to be understood on the supposition of their being able not only to read, but to read so as to take in the full sense of what is written.

* "At every sentence let them ask themselves this question, How should I utter this, were I Speaking it as my own immediate sentiments?——I have often tried an experiment to show the great difference between these two modes of utterance, the natural and the artificial; which was, that when I found a person of vivacity delivering his sentiments with energy, and of course with all that variety of tones which nature furnishes, I have taken occasion to put something into his hand to read, as relative to the topic of conversation; and it was surprising to see what an immediate change there was in his Delivery, from the moment he began to read. A different pitch of voice took place of his natural one, and a tedious uniformity of cadence succeeded to a spirited variety; insomuch that a blind man could hardly conceive the person who Read to be the same who had just been Speaking." Sheridan, *Art of Reading*.

of whom we are weary, and to occupy the mind with reflections of its own.

Of the two branches of Elocution which have been just mentioned, it might at first sight appear as if one only, that of the Speaker, came under the province of Rhetoric. But it will be evident, on consideration, that both must be, to a certain extent, regarded as connected with our present subject; not merely because many of the same principles are applicable to both, but because any one who delivers (as is so commonly the case) a written composition of his own, may be reckoned as belonging to either class; as a Reader who is the author of what he reads, or as a Speaker who supplies the deficiency of his memory by writing. And again, in the (less common) case where a Speaker is delivering without book, and from memory alone, a written composition, either his own or another's, though this cannot in strictness be called Reading, yet the tone of it will be very likely to resemble that of Reading. In the other case, that where the author is actually reading his own composition, he will be still more likely, notwithstanding its being his own, to approach, in the Delivery of it, to the Elocution of a Reader; and, on the other hand, it is possible for him, even without actually deceiving the hearers into the belief that he is speaking extempore, to approach indefinitely near to that Style.

The difficulty however of doing this to one who has the writing actually before him, is considerable; and it is of course far greater when the composition is *not* his own. And as it is evident from what has been said, that this, as it may be called, Extemporaneous style of Elocution, is much the more impressive, it becomes an interesting inquiry, how the difficulty in question may best be surmounted.

Artificial style of Elocution.

Little, if any, attention has been bestowed on this point by the writers on Elocution; the distinction above pointed out between Reading and Speaking, having seldom or never been precisely stated and dwelt on. Several however have written elaborately on "good Reading," or on Elocution, *generally*; and it is not to be denied, that some ingenious and (in themselves) valuable remarks have been thrown out relative to such qualities in Elocution as might be classed under the three heads we have laid down, of Perspicuity, Energy, and Elegance: but there is one principle running through all their precepts, which being, according to our views, radically erroneous, must (if those views be correct) vitiate every system founded on it. The principle we mean is, that in order to acquire the best style of Delivery, it is requisite to study analytically the emphases, tones, pauses, degrees of loudness, &c. which give the proper effect to each passage that is well delivered—to frame *rules* founded on the observation of these—and then, in practice, deliberately and carefully to conform the utterance to these rules, so as to form a complete artificial system of Elocution.

That such a plan not only directs us into a circuitous and difficult path, towards an object which may be reached by a shorter and straighter, but also, in most instances, completely fails of that very object, and even produces, oftener than not, effects the very reverse of what is designed, is a doctrine for which it will be necessary to offer some reasons; especially as it is undeniable that the system here reprobated, as employed in the case of Elocution, is precisely that

recommended and taught in this very article, in respect of the conduct of *Arguments*. By analyzing the best compositions, and observing what kinds of arguments, and what modes of arranging them, in each case, prove most successful, general rules have been formed, which an author is recommended studiously to observe in Composition: and this is precisely the procedure which, in Elocution, we deprecate. The reason for making such a difference in these two cases is this: whoever (as Dr. A. Smith remarks in the passage lately cited*) appears to be attending to his own utterance, which will almost inevitably be the case with every one who is doing so, is sure to give offence, and to be censured for an affected delivery; because every one is expected to attend exclusively to the proper object of the action he is engaged in; which, in this case, is the expression of the thoughts—not the sound of the expressions. Whoever therefore learns and endeavours to apply in practice, any artificial rules of Elocution, so as deliberately to modulate his voice conformably to the principles he has adopted, (however sound they may be in themselves) will hardly ever fail to betray his intention; which always gives offence when perceived. Arguments, on the contrary, *must* be deliberately framed: whether any one's course of reasoning be sound and judicious, or not, it is necessary, and it is expected, that it should be the result of thought. No one, as Dr. Smith observes, is charged with affectation for giving his attention to the proper object of the action he is engaged in. As therefore the proper object of the Orator is to adduce convincing arguments, and topics of Persuasion, there is nothing offensive in his appearing deliberately to aim at this object. He may indeed weaken the force of what is urged, by *too great* an appearance of elaborate composition, or by exciting suspicion of Rhetorical *trick*; but he is so far from being expected to pay no attention to the sense of what he says, that the most powerful argument would lose much of its force, if it were supposed to have been thrown out casually, and at random. *Here* therefore the employment of a regular system (if founded on just principles) can produce no such ill effect as in the case of Elocution; since the habitual attention which that implies to the choice and arrangement of arguments, is such as *must* take place, at any rate; whether it be conducted on any settled principles or not. The only difference is, that he who proceeds on a correct system, will think and deliberate concerning the course of his Reasoning to *better purpose* than he who does not: be will do *well* and *easily*, what the other does ill, and with more labour. Both alike must bestow their attention on the *Matter* of what they say, if they would produce any effect; both are not only allowed, but expected to do so.

The two opposite modes of procedure therefore which are recommended in respect of these two points, (the Argument and the Delivery,) are, in fact, both the result of the same circumstance; viz. that the Speaker is expected to bestow his attention on the proper ultimate object of his Speech, which is, not the Elocution, but the Matter.†

Natural style of Elocution.

When however we protest against all artificial systems of Elocution, and all *direct* attention to Delivery, *at the time*, it must not be supposed that a *general* inattention to that point is recommended; or that the most perfect Elocution is to be attained by never thinking at all on the subject; though it may safely be affirmed that even this negative plan would succeed far better than a studied modulation. But it is evident that if any one wishes to *assume the Speaker* as far as possible; i. e. to deliver a written composition with some degree of the manner and effect of one that is extemporaneous, he will have a considerable difficulty to surmount: since though this may be called, in a certain sense, the NATURAL MANNER, it is far from being what he will naturally, i. e. *spontaneously*, fall into. It is by no means natural for any one to *read* as if he were *not* reading, but speaking. And again, even when any one is reading what he does not wish to deliver as his own composition, as, for instance, a portion of the Scriptures, or the Liturgy, it is evident that this may be done better or worse, in infinite degrees; and that though (according to the views here taken) a studied attention to the sounds uttered, at the time of uttering them, leads to an affected and offensive delivery, yet, on the other hand, an utterly careless reader cannot be a good one.

Reading.

With a view to Perspicuity then, the first requisite in all Delivery, viz. that quality which makes the meaning fully understood by the hearers, the great point is that the Reader (to confine our attention for the present to that branch) should appear to *understand* what he reads. If the composition be, in itself, intelligible to the persons addressed, he will make them fully understand it, by so delivering it. But to this end, it is not enough that he should himself *actually* understand it; it is possible, notwithstanding, to read it as if he did not. And in like manner with a view to the quality, which has been here called Energy, it is not sufficient that he should himself feel, and be impressed with the force of what he utters; he may, notwithstanding, deliver it as if he were unimpressed.

The remedy that has been commonly proposed for these defects, is to point out in such a work, for instance, as the Liturgy, which words ought to be marked as emphatic,—in what places the voice is to be suspended, raised, lowered, &c. One of the best writers on the subject, Sheridan, in his *Lectures on the Art of Reading*,* (whose remarks on many points coincide with the principles here laid down, though he differs from us on the main question—as to the System to be practically followed with a view to the proposed object,) adopts a peculiar set of marks for denoting the different pauses, emphases, &c. and applies these, with accompanying explanatory observations, to the greater part of the Liturgy, and to an Essay subjoined;†

* See ch. iii. sec. 3, p. 290.

† Style occupies in some respects an intermediate place between these two; in what degree each quality of it should or should not be made an object of attention *at the time of composing*, and how far the appearance of such attention is tolerated, has been already treated of in the preceding chapter.

* See note *, p. 292.

† "For the benefit of those who are desirous of getting over their bad habits, and discharging that important part of the Sacred office, the Reading the Liturgy with due decorum, I shall first enter into a minute examination of some parts of the Service, and afterwards deliver the rest, accompanied by such marks as will enable the Reader, in a short time, and with moderate pains, to make himself master of the whole.

"But first it will be necessary to explain the marks which you

Rhetoric.

recommending that the habit should be formed of regulating the voice by his marks; and that afterwards readers should "write out such parts as they want to deliver properly, without any of the usual stops; and, after having considered them well, mark the pauses and emphases by the new signs which have been annexed to them, according to the best of their judgment," &c.

To the adoption of any such artificial scheme there are three weighty objections; 1st, that the proposed system must necessarily be *imperfect*; 2dly, that if it were perfect, it would be a circuitous path to the object in view; and 3dly, that even if both those objections were removed, the object would not be effectually obtained.

Chap. IV.

Imperfection of the artificial system.

1st, Such a system must necessarily be imperfect, because though the *emphatic* word in each sentence may easily be pointed out in writing, no variety of marks that could be invented,—not even musical notation,—would suffice to indicate the different *tones** in which the different emphatic words should be pronounced; though on this depends frequently the whole force, and even sense of the expression. Take

will hereafter see throughout the rest of this course. They are of two kinds; one, to point out the emphatic words, for which I shall use the Grave accent of the Greek, (`).

"The other, to point out the different pauses or stops, for which I shall use the following marks:

"For the shortest pause, marking an incomplete line, thus ′.

"For the second, double the time of the former, two ″.

"And for the third or full stop, three ‴.

"When I would mark a pause longer than any belonging to the usual stops, it shall be by two horizontal lines, as thus =.

"When I would point out a Syllable that is to be dwelt on some time, I shall use this —, or a short horizontal over the Syllable.

"When a Syllable should be rapidly uttered, thus ˘, or a curve turned upwards; the usual marks of long and short in Prosody.

"The Exhortation I have often heard delivered in the following manner:

"'Dearly beloved brethren, the Scripture moveth us in sundry places to acknowledge and confess our manifold sins and wickedness. And that we should not dissemble nor cloke them before the face of Almighty God our Heavenly Father, but confess them with an humble lowly penitent and obedient heart, to the end that we may obtain, forgiveness of the same, by his infinite goodness and mercy. And although we ought at àll times humbly to acknòwledge our sins before God, yet ought we most chiefly sò to do, when we assemble and meet together. To render thànks for the great benefits we have received at his hands, to set forth his most worthy praise, to hear his most holy word, and to ask those things that are requisite and necessary, as well for the body as the soul. Wherefore I pray and beseech you, as many as are here present, to accompany me with a pure heart and humble voice to the throne of the heavenly grace, saying after me.'

"'In the latter part of the first period, 'but confess them with an humble lowly penitent and obedient heart, to the end that we may obtain, forgiveness of the same, by his infinite goodness and mercy,' there are several faults committed. In the first place, the four epithets preceding the word 'heart,' are huddled together, and pronounced in a monotone, disagreeable to the ear, and enervating to the sense; whereas each word rising in force above the other, ought to be marked by a proportional rising of the notes in the voice; and, in the last, there should be such a note used as would declare it at the same time to be the last—'with an humble′ lowly′ penitent and obèdient heart,' &c. At first view it may appear, that the words 'humble' and 'lowly,' are synonymous; but the word 'lowly,' certainly implies a greater degree of humiliation than the word 'humble.' The word 'penitent' that follows, is of stronger import than either; and the word 'obedient,' signifying a perfect resignation to the will of God, in consequence of our humiliation and repentance, furnishes the climax. But if the climax in the words be not accompanied by a suitable climax in the notes of the voice, it cannot be made manifest. In the following part of the sentence, 'to the ènd that we may obtain′ forgìveness of the sàme″' there are usually three emphases laid on the words, *end*, *obtain*, *same*, where there should not be any, and the only emphatic word, *forgiveness*, is slightly passed over; whereas it should be read—'to the end that we may obtain forgivèness of the same,' keeping the words, *obtain*, and *forgiveness*, closely together, and not disuniting them, both to the prejudice of the Sense and Cadence, &c. &c.

"I shall now read the whole, in the manner I have recommended; and if you will give attention to the marks, you will be reminded of the manner, when you come to practise in your private reading. 'Dearly belòved brethren!—The Scripture moveth us′ in sundry places′ to acknowledge and confess our manifold sins and wickedness″ and that we should not dissemble nor clòke them′ before the face of Almighty God′ our Heavenly Father″ but confèss them′ with an humble′ lowly′ penitent′ and obèdient heart′ to the end that we may obtain forgiveness of the same′ by his infinite goodness and mercy‴ And although we ought at àll times′ humbly to acknowledge our sins before Gòd″ yet ought we most chiefly so to dò′ when we assemble and meet together′ to render thanks′ for the great benefits we have received at his hands″ to set forth′ his most worthy pràise″ to hèar′ his most hòly wòrd and to àsk those things′ which are requisite and necessary′ as well for the bòdy′ as the sòul‴ Wherefore I pray and beseèch yòu′ as màny as are here present′ to accompany mè′ with a pùre heart′ and hùmble voice′ to the throne of the heavenly grace′ saying,' &c."

The generality of the remarks respecting the way in which each passage of the Liturgy should be read, are correct; though the mode recommended for attaining the proposed end, is totally different from what is suggested in the present chapter. In some points, however, the author is mistaken as to the emphatic words: e. g. in the Lord's Prayer, he directs the following passage to be read thus; "thy will′ bè done′ on earth′ as it ìs′ in Heaven," with the emphasis on the words "be" and "is;" these, however, are not the emphatic words, and do not even *exist, in the original Greek*, but are supplied by the translator; the latter of them might, indeed, be omitted altogether without any detriment to the sense; "thy will be done, as in Heaven, so also on earth," which is a more literal translation, is perfectly intelligible. A passage in the second Commandment again, he directs to be read, according indeed to the usual mode, both of reading and pointing it,—"visit the sins of the fathers′ upon the children′ unto the third and fourth generation of them that hàte me;" which mode of reading destroys the sense, by making a pause at "children," and none at "generation;" for this implies that the third and fourth generations, who suffer these judgments, are *themselves* such as hate the Lord, instead of being merely, as is meant to be expressed, the *children* of such; "*of* them that hate me," is a genitive not governed by "generation," but by "children:" it should be read (according to Sheridan's marks) "visit the sins of the fathers′ upon the children unto the third and fourth generation′ of them that hate me:" i. e. "visit the sins of the fathers who hate *me*, upon the third and fourth generations of their descendants." The same sanction is given to an equally common fault in reading the fifth Commandment; "that thy days′ may be long in the land′ which the Lord thy God giveth thee:" the pause should evidently be at "*long*" not at "*land*." No one would say in ordinary conversation, "I hope you will find enjoyment in the garden′ ——— which you have planted." He has also strangely omitted an emphasis on the word "covet," in the tenth Commandment. He has, however, in the negative or prohibitory commands avoided the common fault of accenting the word "*not*." And here it may be worth while to remark, that in some cases the Copula ought to be made the emphatic word; (i. e. the "*is*" if the proposition be affirmative, the "*not*," if negative) viz. where the proposition may be considered as in opposition to its *contradictory*. If, e. g. it had been a question, *whether* we ought to steal or not, the commandment, in answer to that, would have been rightly pronounced, "thou shalt *not* steal:" but the question being, *what* things we are forbidden to do, the answer is, that "to steal" is one of them, "thou shalt not *steal*." In such a case as this, the proposition is considered as opposed, not to its *contradictory*, but to one with a *different Predicate*. The question being not, *which Copula* (negative or affirmative) shall be employed, but *what* shall be affirmed or denied of the subject: e. g. "it is lawful *to beg*; but not *to steal*:" in such a case, the *Predicate* will usually be the emphatic word, not the *Copula*.

* See note *, p. 291.

as an instance the words of Macbeth in the witches' cave, when he is addressed by one of the Spirits which they raise, "Macbeth! Macbeth! Macbeth!" on which he exclaims, "Had I three ears I'd hear thee:" no one would dispute that the stress is to be laid on the word "three;" and thus much might be indicated to the reader's eye; but if he had nothing else to trust to, he might chance to deliver the passage in such a manner as to be utterly absurd; for it is possible to pronounce the emphatic word "three," in such a tone as to indicate that "since he has but *two* ears, he cannot hear." It would be nearly as hopeless a task to attempt adequately to convey, by any written marks, precise directions as to the *rate*,—the degree of rapidity or slowness,—with which each sentence and clause should be delivered. Longer and shorter pauses may indeed be easily denoted; and marks may be used, similar to those in music, to indicate, generally, quick, slow, or moderate time; but it is evident that the variations which actually take place are infinite;—far beyond what any marks could suggest; and that much of the force of what is said depends on the degree of rapidity with which it is uttered; chiefly on the *relative* rapidity of one part in comparison of another: for instance in such a sentence, as the following in one of the Psalms, which one may usually hear read at one uniform rate; "all men that see it shall say, this hath God done; for they shall perceive that it is his work;" the four words, "this hath God done," though monosyllables, ought to occupy very little less time in utterance than all the rest of the verse together.

Circuitousness of the artificial system.

2dly, But were it even possible to bring to the highest perfection the proposed system of marks, it would still be a circuitous road to the desired end. Suppose it could be completely indicated to the eye, in what tone each word and sentence should be pronounced according to the several occasions, the learner might ask, "but *why* should this tone suit the awful,—this, the pathetic, this, the narrative style? *why* is this mode of delivery adopted for a command,—this for an exhortation,—this, for a supplication?" &c. The only answer that could be given, is, that these tones, emphases, &c. are a part of the language;—that nature, or custom, which is a second nature, suggests, spontaneously, these different modes of giving expression to the different thoughts, feelings, and designs, which are present to the mind of any one who, without study, is speaking in earnest his own sentiments. Then, if this be the case, why not leave nature to do her own work! Impress but the mind fully with the sentiments, &c. to be uttered; withdraw the attention from the sound, and fix it on the sense; and nature, or habit, will spontaneously suggest the proper Delivery. That this will be the case, is not only true, but is the very supposition on which the artificial system proceeds; for it professes to teach the mode of delivery *naturally* adapted to each occasion. It is surely, therefore, a circuitous path that is proposed, when the learner is directed, first to consider how each passage ought to be read; i. e. what mode of delivering each part of it would *spontaneously* occur to him, if he were attending exclusively to the matter of it; then to observe all the modulations, &c. of voice, which take place in such a Delivery; then, to note these down by established marks, in writing; and, lastly, to pronounce according to these marks. This seems like recommending, for the purpose of raising the hand to the mouth, that he should first observe, when performing that action, without thought of any thing else, what muscles are contracted,—in what degrees,—and in what order; then, that he should note down these observations; and, lastly, that he should, in conformity with these notes, contract each muscle in due degree, and in proper order; to the end that he may be enabled, after all, to—lift his hand to his mouth; which, by supposition, he had already done. Such instruction is like that bestowed by Moliere's pedantic tutor upon his *Bourgeois Gentilhomme*, who was taught, to his infinite surprise and delight, what configurations of the mouth he employed in pronouncing the several letters of the alphabet, which he had been accustomed to utter all his life, without knowing how.*

Appearance of affectation resulting from the artificial system.

3dly, Lastly, waving both the above objections, if a person could learn thus to read and speak, as it were, *by note*, with the same fluency and accuracy as are attainable in the case of singing, still the desired object of a perfectly *natural* as well as correct Elocution, would never be in this way attained. The reader's attention being fixed on his own voice, the inevitable consequence would be that he would betray more or less, his studied and artificial Delivery; and would, in the same degree, manifest an offensive affectation.†

Natural manner.

The practical rule then to be adopted, in conformity with the principles here maintained, is, not only to pay no studied attention to the voice, but studiously to *withdraw* the thoughts from it, and to dwell as intently as possible on the Sense; trusting to nature to suggest spontaneously the proper emphases and tones.‡ He who not only understands fully what he is reading, but is earnestly occupying his mind with the matter of it, will be likely to read as if he understood it, and thus, to make others understand it;§ and in like manner, he who not only feels it, but is exclusively absorbed with that feeling, will be likely to read as if he felt it, and to communicate the im-

* "*Qu'est ce que vous faites quand vous prononcez O? Mais, je dis, O!*"

An answer which, if not savouring of Philosophical analysis, gave at least a good practical solution of the problem.

† It should be observed, however, that, in the reading of the Liturgy especially, so many gross faults are become quite familiar to many, from what they are accustomed to hear, if not from their own practice, as to render it peculiarly difficult to unlearn, or even detect them; and as an aid towards the exposure of such faults, there may be great advantage in studying Sheridan's observations and directions respecting the delivery of it; provided care be taken, *in practice*, to keep clear of his faulty principle, by withdrawing the attention from the sound of the voice, as carefully as he recommends it to be directed to that point.

‡ Many persons are so far impressed with the truth of the doctrine here inculcated, as to acknowledge that "it is a great fault for a reader to be *too much* occupied with thoughts respecting his own voice;" and thus they think to steer a middle course between opposite extremes: but it should be remembered that this middle course entirely nullifies the whole advantage proposed by the plan recommended. A reader is sure to pay *too much* attention to his voice, not only if he pays *any at all*, but if he does not strenuously *labour to withdraw* his attention from it altogether.

§ Who, for instance, that was really *thinking* of a resurrection from the dead, would ever tell any one that our Lord "rose *again* from the dead," (which is so common a mode of reading the Creed,) as if He had done so more than once?

pression to his hearers. But this cannot be the case if he is occupied with the thought of what their opinion will be of his reading, and, how his voice ought to be regulated;—if, in short, he is thinking of *himself*, and, of course, in the same degree, abstracting his attention from that which ought to occupy it exclusively.

It is not, indeed, desirable, that in reading the Bible, for example, or any thing which is not intended to appear as his own composition, he should deliver what are, avowedly, another's sentiments, in the same style, as if they were such as arose in his own mind; but it is desirable that he should deliver them as if he were *reporting* another's sentiments, which were both fully understood and felt in all their force by the reporter; and the only way to do this effectually,—with such modulations of voice, &c. as are suitable to each word and passage,—is to fix his mind earnestly on the meaning, and leave nature and habit to suggest the utterance.

Some may, perhaps, suppose that this amounts to the same thing as *taking no pains at all*; and if, with this impression, they attempt to try the experiment of a natural Delivery, their ill-success will probably lead them to censure the proposed method, for the failure resulting from their own mistake. In truth, it is by no means a very easy task, to fix the attention on the meaning, in the manner, and to the degree, now proposed. The thoughts of one who is reading any thing very familiar to him, are apt to wander to *other* subjects, though perhaps such as are connected with that which is before him; if, again, it be something new to him, he is apt (not indeed to wander to *another* subject,) but to get the start, as it were, of his readers, and to be thinking, while uttering each sentence, not of that, but of the sentence which comes next. And in both cases, if he is careful to avoid those faults, and is desirous of reading well, it is a matter of no small difficulty, and calls for a constant effort, to prevent the mind from wandering in another direction; viz. into thoughts respecting his own voice,—respecting the effect produced by each sound,—the approbation he hopes for from the hearers, &c. And this is the prevailing fault of those who are commonly said to take *great pains* in their reading; pains which will always be taken in vain, with a view to the true object to be aimed at, as long as the effort is thus applied in a wrong direction. With a view, indeed, to a very different object, the approbation bestowed on the reading, this artificial delivery will often be more successful than the natural. Pompous spouting, and many other descriptions of unnatural tone and measured cadence, are frequently admired by many as excellent reading; which admiration is itself a proof that it is not deserved; for when the Delivery *is really* good, the hearers (except any one who may deliberately set himself to observe and criticise,) never think about it, but are exclusively occupied with the sense it conveys, and the feelings it excites.

Advantages of imitation and of practice precluded by the adoption of the Natural manner.

Still more to increase the difficulty of the method here recommended, (for it is no less wise than honest to take a fair view of difficulties) this circumstance is to be noticed, that he who is endeavouring to bring it into practice, is in a great degree precluded from the advantages of *imitation*. A person who hears and approves a good *reader in the Natural manner*, may, indeed, so far imitate him with advantage, as to *adopt his plan*, of fixing his attention on the matter, and not thinking about his voice; but this very plan, evidently by its nature, precludes any further imitation; for if, while reading, he is thinking of copying the manner of his model, he will, for that very reason, be unlike that model; the main principle of the proposed method being, carefully to exclude every such thought. Whereas, any artificial system may as easily be learned by imitation as the notes of a song. Practice also, (i. e. private practice for the sake of learning,) is much more difficult in the proposed method; because the rule being to use such a Delivery as is suited, not only to the *matter* of what is said, but also, of course, to the *place*, and *occasion*, and this, not by any studied modulations, but according to the spontaneous suggestions of the matter, place, and occasion, to one whose mind is fully and exclusively occupied with these, it follows, that he who would practise this method in *private*, must, by a strong effort of a vivid imagination, figure to himself a place and an occasion which are not present; otherwise, he will either be *thinking of his Delivery*, (which is fatal to his proposed object,) or else will use a Delivery suited to the situation in which he actually *is*, and not, to that for which he would prepare himself. Any system, on the contrary, of studied emphasis and regulation of the voice, may be learned in private practice, as easily as singing.

Some additional objections to the method recommended, and some further remarks on the counterbalancing advantages of it, will be introduced presently, when we shall have first offered some observations on *Speaking*, and on that branch of Reading which the most nearly approaches to it.

When any one delivers a written composition, of which he is, or is supposed to profess himself the author, he has peculiar difficulties to encounter,* if his object be to approach as nearly as possible to the ex-

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* It must be admitted, however, that the difficulty of reading the Liturgy with spirit, and even with propriety, is something peculiar, on account of (what has been already remarked) the inveterate and long-established faults to which almost every one's ears are become familiar; so that such a delivery as would shock any one of even moderate taste, in any other composition, he will, in this, be likely to tolerate, and to practise. Some, e. g. in the Liturgy, read, "have mercy *upon* us, miserable sinners;" and others, "have mercy upon *us*, miserable sinners;" both, laying the stress on a wrong word, and making the pause in the wrong place, so as to disconnect "us" and "miserable sinners," which the context requires us to combine. Every one, in expressing his own natural sentiments, would say "have *mercy*, upon us-miserable-sinners."

Many are apt even to commit so gross an error, as to lay the chief stress on the words which denote the *most important things*; without any consideration of the emphatic *word* of each sentence: e. g. in the Absolution many read, "let us beseech Him to grant us *true repentance*;" because "forsooth true repentance" is an important thing; not considering that, as it has been just mentioned, it is not the *new idea*, to which the attention should be directed by the emphasis; the sense being, that since God pardoneth all that have true repentance, therefore, we should "beseech Him to *grant* it to *us*."

In addition to the other difficulties of reading the Liturgy well, it should be mentioned, that prayer, thanksgiving, and the like, even when avowedly not of our own composition, should be delivered as (what in truth they ought to *be*,) the genuine sentiments of our own minds at the moment of utterance; which is not the case with the Scriptures, or with any thing else that is read, not professing to be the speaker's own composition.

temporaneous style. It is indeed impossible to produce the *full* effect of that style, while the audience are aware that the words he utters are before him: but he may approach indefinitely near to such an effect; and in proportion as he succeeds in this object, the impression produced will be the greater. It has been already remarked, how easy it is for the hearers to keep up their attention,—indeed, how difficult for them to withdraw it,—when they are addressed by one who is *really speaking* to them in a natural and earnest manner; though perhaps the discourse may be incumbered with a good deal of the repetition, awkwardness of expression, and other faults incident to extemporaneous language; and though it be prolonged for an hour or two, and yet contain no more matter than a good *writer* could have clearly expressed in a discourse of half an hour; which last, if read to them, would not, without some effort on their part, have so fully detained their attention. The advantage in point of Style, Arrangement, &c. of written, over extemporaneous, discourses, (such at least as any but the most accomplished orators can produce,) is sufficiently evident:* and it is evident also that other advantages, such as have been just alluded to, belong to the latter. Which is to be preferred on each occasion, and by each orator, it does not belong to the present discussion to inquire: but it is evidently of the highest importance to *combine*, as far as possible, in each case, the advantages of both. A perfect familiarity with the rules laid down in the first Chapter of this Essay, would be likely, it is hoped, to give the extemporaneous orator that habit of *quickly* methodizing his thoughts on a given subject, which is essential (at least where no very long premeditation is allowed,) to give to a *speech* something of the weight of argument and clearness of arrangement which characterise good Writing.† In order to attain the corresponding advantage,—to impart to the delivery of a written discourse something of the vivacity and interesting effect of real, earnest, *speaking*, the plan to be pursued, conformably with the principles we have been maintaining, is, for the reader to draw off his mind as much as possible from the thought that he *is* reading, as well as from thought respecting his own utterance;—to fix his mind as earnestly as possible on the *matter*, and to strive to adopt as his *own*, and as his *own at the moment* of utterance, every sentiment he delivers;—and to *say* it to the audience, in the manner which the occasion and subject spontaneously suggest to him who has abstracted his mind both from all consideration of *himself*, and from the consideration that he is reading.

The advantage of this NATURAL MANNER, (i. e. the manner which one naturally falls into who is *really speaking*, in earnest, and with a mind *exclusively* intent on what he has to say,) may be estimated from this consideration; that there are few who do not *speak* so as to give effect to what they are saying: some, indeed, do this much better than others:—some have, in ordinary conversation, an indistinct or incorrect pronunciation,—an embarrassed and hesitating utterance, or a bad choice of words: but hardly any one fails to deliver,* (when speaking earnestly) what he does say, so as to convey the sense and the force of it, much more completely than even a good reader would, if those same words were written down and read. The latter might, indeed, be more *approved*; but that is not the present question; which is, concerning the *impression* made on the hearers' minds. It is not the polish of the blade, that is to be considered, nor the grace with which it is brandished, but the keenness of the edge, and the weight of the stroke.

On the contrary, it can hardly be denied that the elocution of most readers, when delivering their own compositions, is such as to convey the notion, at the very best, not that the preacher is expressing his own sentiments, but that he is making known to his audience what is written in the book before him: and, whether the composition is professedly the reader's own, or not, the usual mode of delivery, though grave and decent, is so remote from the energetic style of real Natural Speech, as to furnish, if one may so speak, a kind of running comment on all that is uttered, which says, "I do not mean, think, or feel, all this; I only mean to recite it with propriety and decorum:" and what is usually called *fine* Reading, only superadds to this, (as has been above remarked,) a kind of admonition to the hearers, that *they* ought to believe, to feel, and to admire, what is read.

It is easy to anticipate an objection which many will urge against, what they will call, a *colloquial* style of delivery; viz. that it is indecorous, and unsuitable to the solemnity of a serious, and especially, of a Religious discourse. The objection is founded on a mistake. Those who urge it, derive all their notions of a Natural Delivery from two, irrelevant, instances; that of ordinary conversation, the usual subjects of which, and consequently its usual tone,

* Practice in public speaking, generally,—practice in speaking on the particular subject in hand,—and (on each occasion) premeditation of the matter and arrangement, are, all, circumstances of great consequence to a speaker. Nothing but a *miraculous gift* can supersede these advantages. The Apostles accordingly were forbidden *to use any premeditation*, being assured that "it should be *given* them, in that same hour, what they should say:" and when they found, in effect, this promise fulfilled to them, they had experience, within themselves, of a sensible miracle. This circumstance may furnish a person of sincerity with a useful test for distinguishing (in his own case) the emotions of a fervid imagination, from actual inspiration. It is evident that an *inspired* preacher can have nothing to gain from practice, or study of any kind: he therefore who finds himself *improve* by practice, either in Argument, Style, or Delivery,—or who observes that he speaks more fluently and better on subjects on which he has been *accustomed* to speak, or better, with *premeditation*, than on a sudden, may indeed deceive his hearers by a pretence to inspiration, but can hardly deceive *himself*.

† Accordingly, it may be remarked, that, (contrary to what might at first sight be supposed,) though the preceding Chapters, as well as the present, are intended for general application, yet it is to the *extemporary speaker* that the rules laid down in the former part (supposing them correct,) will be the most peculiarly useful; while the suggestions offered in this last, respecting Elocution, are more especially designed for the use of the *reader*.

* There is, indeed, a wide difference between different men, in respect of the degrees of impressiveness with which, in earnest conversation, they deliver their sentiments; but it may safely be laid down, that he who delivers a written composition with the same degree of spirit and energy, with which he would naturally speak on the same subject, has attained, not indeed, necessarily, *absolute* perfection, but the utmost excellence attainable by *him*. Any attempt to out-do his own Natural manner, will inevitably lead to something worse than failure.

are comparatively light;—and, that of the coarse and extravagant rant of vulgar fanatical preachers. But to conclude that the objections against either of these styles, would apply to the Natural Delivery of a man of sense and taste, speaking earnestly, on a serious subject, and on a solemn occasion, or that he would naturally adopt, and is advised to adopt, such a style as those objected to, is no less absurd than if any one, being recommended to walk in a natural and unstudied manner, rather than in a dancing step, (to employ Dr. A. Smith's illustration,) or a formal march, should infer that the natural gait of a clown following the plough, or of a child in its gambols, were proposed as models to be imitated in walking across a room. It is evident, that what is *natural* in one case, or for one person, may be, in a different one, very unnatural. It would not be by any means natural, to an educated and sober-minded man, to speak like an illiterate enthusiast; nor to discourse on the most important matters in the tone of familiar conversation respecting the trifling occurrences of the day. Any one who does but notice the style in which a man of ability, and of good choice of words, and utterance, delivers his sentiments in *private*, when he is, for instance, earnestly and seriously admonishing a friend,—defending the doctrines of Religion,—or speaking on any other grave subject on which he is intent, may easily observe how different his tone is from that of *light* and familiar conversation,—how far from deficient in the decent seriousness which befits the case: even a stranger to the language might guess that he was not engaged in any frivolous topic: and when an opportunity occurs of observing how he delivers a written discourse, of his own composition, on perhaps the very same, or a similar subject, one may generally perceive how comparatively stiff, languid, and unimpressive is the effect. It may be said, indeed, that a sermon should not be preached before a congregation assembled in a place of worship, in the same style as one would employ in conversing across a table, with equal seriousness, on the same subject: this is undoubtedly true: and it is evident that it has been implied in what has been said; the Natural manner having been described as accommodated, not only to the *subject* but to the *place*, *occasion*, and all other circumstances: so that he who should *preach* exactly as if he were *speaking in private*, though with the utmost earnestness, on the same subject, would so far be *departing* from the genuine Natural manner; but it may be safely asserted, that even *this* would be by far the less fault of the two. He who appears unmindful, indeed, of the place and occasion, but deeply impressed with the *subject*, and utterly forgetful of himself, would produce a much stronger effect than one, who, going into the opposite extreme, is, indeed, mindful of the place and the occasion, but not fully occupied with the subject, (though he may strive to *appear* so;) being partly engaged in thoughts respecting his own voice. The latter would, indeed, be less likely to incur censure; but the other would produce the deeper impression. The object, however, to be aimed at, (and it is not unattainable,) is to avoid both faults;—to keep the mind impressed both with the matter spoken, and with all the circumstances also of each case, so that the voice may spontaneously accommodate itself to *all*; carefully avoiding all studied modulations, and, in short, all thoughts of *self*, which, in proportion as they intrude, will not fail to diminish the effect.

It must be admitted, indeed, that the different kinds of Natural Delivery of any one, on different subjects and occasions, various as they are, do yet bear a much greater resemblance to each other, than any of them does to the Artificial style usually employed in reading: a proof of which is, that a person familiarly acquainted with the Speaker, will seldom fail to *recognise his voice*, amidst all the variations of it, when he is *speaking* naturally and earnestly; though it will often happen that, if he have never before heard him *read*, he will be at a loss, when he happens accidentally to hear without seeing him, to know who it is that is reading; so widely does the artificial cadence and intonation differ in many instances from the natural. And a consequence of this is, that the Natural manner, however perfect,—however exactly accommodated to the subject, place, and occasion, will, even when these are the most solemn, in some degree *remind* the hearers of the tone of conversation: amidst all the differences that will exist, this one point of resemblance, that of the delivery being unforced and unstudied, will be likely, in some degree, to strike them. Those who are good judges will perceive at once, and the rest, after being a little accustomed to the Natural manner, that there is not necessarily any thing irreverent or indecorous in it; but that, on the contrary, it conveys the idea of the speaker's being deeply impressed with that which is his proper business. But, for a time, many will be disposed to find fault with such a kind of elocution. But even while this disadvantage continues, a preacher of this kind may be assured that the *doctrine* he delivers is much more forcibly impressed, even on those who censure his style of delivering it, than it could be in the other way. A discourse delivered in this style has been known to elicit the remark, from one of the lower orders, who had never been accustomed to any thing of the kind, that "it was an excellent sermon, and it was great pity it had not been *preached*:" a censure which ought to have been very satisfactory to the preacher: had he employed a pompous spout, or modulated whine, it is probable such an auditor would have admired his *preaching*, but would have known and thought little or nothing about the *matter* of what was taught. Which of the two objects ought to be preferred by a Christian minister, on Christian principles, is a question not hard to decide, but foreign to the present discussion: it is important, however, to remark, that an orator is bound, as such, not merely on moral, but, if such an expression may be used, on *rhetorical* principles, to be mainly, and indeed exclusively, intent on *carrying his point*; not, on gaining approbation, or even avoiding censure, except with a view to that point. He should, as it were, adopt as a motto, the reply of Themistocles to the Spartan commander, Eurybiades, who lifted his staff to chastise the earnestness with which his own opinion was controverted: "Strike, but hear me."

Besides the inconvenience just mentioned,—the censure to which the proposed style of elocution will be liable from perhaps the majority of hearers, till they shall have become somewhat accustomed to it,—this circumstance also ought to be mentioned, among what many, perhaps, would reckon, (or at least *feel*,)

as the disadvantages of it; that, after all, even when no disapprobation is incurred, no praise will be bestowed, (except by observant critics,) on a truly Natural Delivery: on the contrary, the more perfect it is, the more will it withdraw, from itself, to the arguments and sentiments delivered, the attention of all but those who are studiously directing their view to the mode of utterance, with a design to criticize or to learn. The credit, on the contrary, of having a very fine elocution, is to be obtained at the expense of a very moderate share of pains; though at the expense also, inevitably, of much of the force of what is said.

One inconvenience, which will at first be experienced by a person who, after having been long accustomed to the Artificial Delivery, begins to adopt the Natural, is, that he will be likely suddenly to feel an embarrassed, bashful, and, as it is frequently called, *nervous* sensation, to which he had before been comparatively a stranger. He will find himself in a new situation,—standing before his audience in a different character—stripped, as it were, of the sheltering veil of a conventional and Artificial Delivery;—in short, delivering to them his thoughts, as one man *speaking to* other men; not, as before, merely *reading in public*. And he will feel that he attracts a much greater share of their attention, not only by the novelty of a manner to which most congregations are little accustomed, but also, (even supposing them to have been accustomed to extemporary discourses,) from their perceiving themselves to be personally *addressed*, and feeling that he is not merely reciting something *before* them, but saying it *to* them. The speaker and the hearers will thus be brought into a new, and closer relation to each other: and the increased interest thus excited in the audience, will cause the Speaker to feel himself in a different situation,—in one which is a greater trial of his confidence, and which renders it more difficult than before to withdraw his attention from himself. It is hardly necessary to observe that this very change of feelings experienced by the speaker, ought to convince him the more, if the causes of it (to which we have just alluded,) be attentively considered, how much greater impression this manner is likely to produce. As he will be likely to feel much of the bashfulness which a really extemporary speaker has to struggle against, so, he may produce much of a similar effect.

After all, however, the effect will never be completely the same. A composition delivered from writing, and one actually extemporaneous, will always produce feelings, both in the hearer and the speaker, considerably different; even on the supposition of their being word for word the same, and delivered so exactly in the same tone, that by the ear alone no difference could be detected: still the audience will be differently affected, according to their knowledge that the words uttered, are, or are not, written down and before the speaker's eyes: and the consciousness of this, will produce a corresponding effect on the mind of the speaker. For were this not so, any one who, on any subject, can speak (as many can,) fluently and correctly in private conversation, would find no greater difficulty in saying the same things before a large congregation, than in reading to them a written discourse.

And here it may be worth while briefly to inquire into the causes of that remarkable phenomenon, as it may justly be accounted, that a person who is able with facility to express his sentiments in private to a friend, in such language, and in such a manner, as would be perfectly suitable to a certain audience, yet finds it extremely difficult to address to that audience the very same words, in the same manner; and is, in many instances, either completely struck dumb, or greatly embarrassed, when he attempts it.* It cannot be from any superior deference which he thinks it right to feel for their judgment; for it will often happen that the single friend, to whom he is able to speak fluently, shall be one whose good opinion he more values, and to whose wisdom he is more disposed to look up, than that of all the others together. The speaker may even feel that he himself has a decided and acknowledged superiority over every one of the audience; and that he should not be the least abashed in addressing any two or three of them, separately; yet still all of them, collectively, will often inspire him with a kind of dread.

Closely allied in its causes with the phenomenon we are considering, is, that other curious fact, that the very same sentiments expressed in the same manner, will often have a far more powerful effect on a large audience than they would have, on any one or two of these very persons, separately. That is in a great degree true of all men, which was said of the Athenians, that they were like sheep, of which a flock is more easily driven than a single one.

Another remarkable circumstance, connected with the foregoing, is the difference in respect of the style which is suitable, respectively, in addressing a multitude, and two or three even of the same persons. A much *bolder*, as well as less accurate, kind of language is both allowable and advisable, in speaking to a considerable number; as Aristotle has remarked,† in speaking of the *Graphic* and *Agonistic* styles,—the former suited to the closet, the latter to public speaking before a large assembly. And he ingeniously compares them to the different styles of painting; the greater the crowd, he says, the more distant is the view; so that in scene-painting, for instance, coarser and bolder touches are required, and the nice finish, which would delight a close spectator, would be lost. He does not, however, account for the phenomena in question.

The solution of them will be found by attention to a very curious and complex play of sympathies which takes place in a large assembly; and, (within certain limits,) the more, in proportion to its numbers. First, it is to be observed that we are disposed to sympathize with any emotion which we believe to exist in the mind of any one present; and hence, if we are at the same time otherwise disposed to feel that emotion, such disposition is in consequence heightened. In the next place, we not only ourselves feel this tendency, but we are sensible that others do the same; and thus, we sympathize not only with the other emotions of the rest, but also, with their sympathy towards us. Any emotion accordingly which we

* Most persons are so *familiar* with the fact, as hardly to have ever considered that it requires explanation: but attentive consideration shows it to be a very curious, as well as important one.

† *Rhetoric*, book iii.

feel, is still further heightened by the knowledge that there are others present who not only feel the same, but feel it the more strongly in consequence of their sympathy with ourselves. Lastly, we are sensible that those around us sympathize not only with ourselves, but with each other also; and as we enter into this heightened feeling of theirs likewise, the stimulus to our own minds is thereby still further increased.

The case of the *Ludicrous* affords the most obvious illustration of these principles, from the circumstance that the effects produced are so open and palpable. If any thing of this nature occurs, a man is disposed, by the character of the thing itself, to laugh: but much more, if any one else is known to be present whom he thinks likely to be diverted with it; even though that other should not know of the presence of the first; but much more still, if he does know it; because his companion is then aware that sympathy with his own emotion heightens that of the other: and most of all will the disposition to laugh be increased, if many are present, because each is then aware that they all sympathize with each other, as well as with himself. It is hardly necessary to mention the exact correspondence of the fact with the above explanation. So important, in this case, is the operation of the causes here noticed, that hardly any one ever laughs when he is quite alone: or if he does, he will find on consideration, that it is from a *conception* of the presence of some companion whom he thinks likely to have been amused, had he been present, and to whom he thinks of describing, or repeating, what had diverted himself. Indeed, in other cases, as well as the one just instanced, almost every one is aware of the *infectious* nature of any emotion excited in a large assembly. It may be compared to the increase of sound by a number of echoes, or of light, by a number of mirrors; or to the blaze of a heap of firebrands, each of which would have speedily gone out, if kindled separately, but which, when thrown together, help to kindle each other.

The application of what has been said to the case before us, is sufficiently obvious. The speaker who is addressing a large assembly, knows that each of them sympathizes both with his own anxiety to acquit himself well, and also with the same feeling in the minds of the rest. He knows also, that every slip he may be guilty of, that may tend to excite ridicule, pity, disgust, &c. makes the stronger impression on each of the hearers, from their mutual sympathy, and their consciousness of it. This augments his anxiety. Next, he knows that each hearer, putting himself, mentally, in the speaker's place,* sympathizes with this augmented anxiety; which is by this thought increased still further. And if he becomes at all embarrassed, the knowledge that there are so many to sympathize, not only with that embarrassment, but also with each other's feelings, on the perception of it, heightens the speaker's confusion to the utmost.

The same causes will account for a skilful orator's being able to rouse so much more easily, and more powerfully, the passions of a *multitude:* they inflame each other by mutual sympathy, and mutual consciousness of it. And hence it is that a bolder kind of language is suitable to such an audience: a passage which, in the closet, might just at the first glance tend to excite awe, compassion, indignation, or any other such emotion, but which would, on a moment's cool reflection, appear extravagant, may be very suitable for the *Agonistic* Style; because before that moment's reflection could take place in each hearer's mind, he would be aware that every one around him sympathized in that first emotion; which would thus become so much heightened as to preclude, in a great degree, the ingress of any counteracting sentiment.

If one could suppose such a case as that of a speaker, (himself aware of the circumstances,) addressing a multitude, each of whom believed himself to be the *sole* hearer, it is probable that little or no embarrassment would be felt, and a much more sober, calm, and finished style of language would be adopted.

The impossibility of bringing the delivery of a written composition *completely* to a level with real extemporary speaking, (though, as has been said, it may approach indefinitely near to such an effect,) is explained on the same principle. Besides that the audience are more sure that the thoughts they hear expressed, are the genuine emanation of the speaker's mind at the moment, their attention and interest are the more excited by their sympathy with one whom they perceive to be carried forward solely by his own unaided and unremitted efforts, without having any book to refer to: they view him as a swimmer supported by his own constant exertions; and in every such case, if the feat be well accomplished, the *surmounting of the difficulty* affords great gratification; especially to those who are conscious that they could not do the same. And one proof, that part of the pleasure conveyed does arise from this source, is, that as the spectators of an exhibition of supposed unusual skill in swimming, would instantly withdraw most of their interest and admiration, if they perceived that the performer was supported by corks, or the like; so would the feelings alter of the hearers of a supposed extemporaneous discourse, as soon as they should perceive, or even suspect, that the orator had it written down before him.

The way in which the respective inconveniences of both kinds of discourses may best be avoided, is evident from what has been already said. Let both the extemporary Speaker, and the Reader of his own composition, study to avoid, as far as possible, all thoughts of *self*, earnestly fixing the mind on the matter of what is delivered; and the one will feel the less of that embarrassment which arises from the thought of what opinion the hearers will form of him; while the other will appear to be speaking, because he actually *will be* speaking, the sentiments, not indeed which at that time first *arise* in his own mind, but, which are then really *present* to, and occupy his mind.

One of the consequences of the adoption of the mode of elocution here recommended, is that he who endeavours to employ it will find a growing reluctance to the delivery, as his own, of any but his own compositions. Doctrines, indeed, and arguments he will freely borrow; but he will be led to compose his own discourses, from finding that he cannot deliver those of another to his own satisfaction, without laboriously studying them, as an actor does his part, so as to *make* them, in some measure, his own. And with this view, he will generally find it advisable

* Hence it is that *shy* persons are, as is matter of common remark, the more distressed by this infirmity when in company with those who are subject to the same.

to introduce many alterations in the expression, not with any thought of improving the style, *absolutely*, but only with a view to his *own delivery*. And indeed, even his own *former* compositions, he will be led to alter, almost as much, in point of expression, in order to accommodate them to the Natural manner of delivery.* Much that would please in the closet,—much of the *Graphic* style described by Aristotle, will be laid aside for the *Agonistic*;—for a style somewhat more blunt and homely,—more simple and, apparently, unstudied in its structure, and, at the same time, more daringly energetic. And if again he is desirous of fitting his discourses for the press, he will find it expedient to reverse this process, and alter the style afresh. A mere *sermon-reader*, on the contrary, will avoid this inconvenience, and this labour; he will be able to preach another's discourses nearly as well as his own; and may send his own to the press, without the necessity of any great preparation: but to these advantages he will sacrifice more than half the force which might have been given to the sentiments uttered. And he will have no right to complain that his discourses, though replete perhaps with good sense, learning, and eloquence, are received with languid apathy, or that many are seduced from their attendance on his teaching, by the vapid rant of an illiterate fanatic. Much of these evils must, indeed, be expected, after all, to remain: but he does not give himself a fair chance for diminishing them, unless he does justice to his own arguments, instructions, and exhortations, by *speaking* them, in the only effectual way, to the hearts of his hearers, that is, as uttered naturally *from* his own.†

One important practical maxim resulting from the views here taken, is the decided condemnation of all *recitation of speeches* by school-boys; a practice so much approved and recommended by many, with a view to preparing youths for public Speaking in after-life. It is to be condemned, however, (supposing the foregoing principle correct,) not as useless merely, but absolutely pernicious, with a view to that object. The *justness*, indeed, of this opinion will, doubtless, be disputed; but its *consistency* with the plan we have been recommending, is almost too obvious to be insisted on. In any one who should think a Natural Delivery desirable, it would be an obvious absurdity to think of attaining it by practising that which is the most completely artificial. If there is, as is evident, much difficulty to be surmounted, even by one who is delivering, on a serious occasion, his own composition, before he can completely succeed in abstracting his mind from all thoughts of his own voice,—of the judgment of the audience on his performance, &c. and in fixing it on the Matter, Occasion, and Place,—on every circumstance which *ought* to give the character to his elocution,—how much must this difficulty be enhanced, when neither the sentiments he is to utter, nor the character he is to assume, are his own, or even supposed to be so, or in anywise connected with him:—when neither the place, the occasion, nor the audience, which are *actually present*, have any thing to do with the substance of what is said. It is therefore almost inevitable, that he will studiously form to himself an Artificial manner;* which, especially if he succeeds in it, will probably cling to him through life, even when he is delivering his own compositions on real occasions. The very best that can be expected, is, that he should become an accomplished *actor*,—possessing the plastic power of putting himself, in imagination, so completely into the situation of him whom he personates, and of adopting, for the moment, so perfectly, all the sentiments and views of that character, as to express himself exactly as such a person would have done, in the supposed situation. Few are likely to attain such perfection; but he who shall have succeeded in accomplishing this, will have taken a most circuitous rout to his proposed object, if that object be, not to qualify himself for the stage, but to deliver in public, on real and important occasions, his own sentiments. He will have been carefully learning to *assume*, what, when the real occasion occurs, need not be assumed, but only *expressed*. Nothing surely can be more preposterous than labouring to acquire the art of *pretending* to be, what he is *not*, and, to feel, what he does not feel, in order that he may be enabled, on a real emergency, to *pretend* to be and to feel just what the occasion itself requires and suggests.†

* In many instances accordingly, the perusal of a manuscript sermon, would afford, from the observation of its style, a tolerably good ground of conjecture as to the author's customary elocution.

† The principles here laid down may help to explain a remarkable fact, which is usually attributed to other than the true causes. The powerful effects often produced by some fanatical preachers, not superior in pious and sincere zeal, and inferior in learning, in good sense, and in taste, to men who are listened to with comparative apathy, are frequently considered as proofs of superior *eloquence*; though an eloquence tarnished by barbarism, and extravagant mannerism. But may not such effects result, not from any superior powers in the preacher, but merely from the intrinsic beauty and sublimity, and the measureless importance of the *subject*? Why then, it may be replied, does not the other preacher, whose subject is the same, produce the same effect? The answer is, because he is but *half-attended to*. The ordinary measured cadence of reading, is not only in itself dull, but is what men are *familiarly* accustomed to: Religion itself also, is a subject so *familiar*, in a certain sense, (familiar, that is, to the *ear*,) as to be *trite*, even to those who *know* and *think* little about it. Let but the attention be thoroughly roused, and intently fixed on such a stupendous subject, and that subject itself will produce the most overpowering emotion. And not only unaffected earnestness of manner, but, perhaps, even still more, any uncouth oddity, and even ridiculous extravagance, will, by the stimulus of *novelty*, have the effect of thus rousing the hearers from their ordinary lethargy. So that a preacher of little or no real eloquence, will sometimes, *on such a subject*, produce the effects of the greatest eloquence, by merely forcing the hearers (often, even by the excessively glaring *faults* of his style and delivery,) to *attend*, to a subject which no one can *really* attend to unmoved.

It will not of course be supposed that our intention is to *recommend* the adoption of extravagant rant. The good effects which it undoubtedly does sometimes produce, incidentally, in some, is more than counterbalanced by the mischievous consequences to others.

* Some have used the expression of "a *conscious* manner," to denote that which results, either in conversation,—in the ordinary actions of life,—or in public Speaking, from the anxious attention which some persons feel to the opinion which the company may form of them;—a *consciousness* of being watched and scrutinized in every word and gesture, together with an extreme anxiety for approbation, and dread of censure.

† The Barmecide, in the *Arabian Nights*, who amused himself by setting down his guest to an imaginary feast, and trying his skill in imitating, at an empty table, the actions of eating and drinking, did not propose this as an advisable mode of instructing him how to perform those actions in reality.

Let all studied recitation therefore,—every kind of speaking which, from its nature, must necessarily be artificial,—be carefully avoided, by one whose object is to attain the only truly impressive,—the Natural Delivery.*

The last circumstances to be noticed among the results of the mode of delivery recommended, is, that the speaker will find it much easier, in this Natural manner, to *make himself heard:* he will be heard, that is, much more distinctly,—at a greater distance,—and with far less exertion and fatigue to himself. This is the more necessary to be mentioned, because it is a common, if not a prevailing opinion, that the reverse of this is the fact. There are not a few who assign as a reason for their adoption of a certain unnatural tone and measured cadence, that it is necessary, in order to be heard by a large congregation. But though such an artificial voice and utterance will often appear to produce a *louder sound*, (which is the circumstance that probably deceives such persons,) yet a natural voice and delivery, provided it be clear, though it be less laboured, and may even seem low to those who are near at hand, will be distinctly heard at a much greater distance. The only decisive proof of this must be sought in experience; which will not fail to convince of the truth of the assertion, any one who will fairly make the trial.

The requisite degree of loudness will be best obtained, conformably with the principles here inculcated, not by thinking about the voice, but by *looking* at the most *distant* of the hearers, and addressing one's self especially to him. The voice rises *spontaneously*, when we are speaking to a person who is not very near

And that the organs of voice are much less strained and fatigued by the Natural action which takes place in real speaking, than by any other, (besides that it is, what might be expected, *a priori*,) is evident from daily experience. An extemporary Speaker will usually be much less exhausted in two hours, than an elaborate reciter, (though less distinctly heard,) will be, in one. Even the ordinary tone of *reading* aloud is so much more fatiguing than that of conversation, that feeble patients are frequently unable to continue it for a quarter of an hour without great exhaustion; even though they may feel no inconvenience from *talking*, with few or no pauses, and in no lower voice, for more than double that time.

He then who shall determine to aim at the Natural manner, though he will have to contend with considerable difficulties and discouragements, will not be without corresponding advantages in the course he is pursuing. He will be at first, indeed, repressed to a greater degree than another, by emotions of bashfulness; but it will be more speedily and more completely subdued: the very system pursued, since it forbids all thoughts of *self*, striking at the root of the evil. He will, indeed, on the outset, incur censure, not only critical but moral;—he will be blamed for using a *colloquial* delivery; and the censure will very likely be, as far as relates to his earliest efforts, not wholly undeserved; his manner will probably at first too much resemble that of conversation, though of serious and earnest conversation: but by perseverance he may be sure of avoiding deserved, and of mitigating, and ultimately overcoming, undeserved, censure. He will, indeed, never be praised for a very fine delivery; but his matter will not lose the approbation it may deserve; as he will be the more sure of being heard and attended to. He will not, indeed, meet with many who can be regarded as models of the Natural manner; and those he does meet with, he will be precluded, by the nature of the system, from minutely imitating; but he will have the advantage of carrying within him an INFALLIBLE GUIDE, as long as he is careful to follow the suggestions of nature, abstaining from all thoughts respecting his own utterance, and fixing his mind intensely on the business he is engaged in. And though he must not expect to attain perfection at once, he may be assured that, while he steadily adheres to this plan, he is in the right road to it; instead of becoming, as on the other plan, more and more artificial, the longer he studies: and every advance he makes will produce a proportional effect: it will give him more and more of that hold on the attention, the understanding, and the feelings, of the audience, which no studied modulation can ever attain. And though others may be more successful in escaping censure, and insuring admiration, he will far more surpass them, in respect of the proper object of the Orator, which is, *to carry his point*.

Much need not be said on the subject of *Action*, which is at present so little approved, or, designedly, employed, in this country, that it is hardly to be reckoned as any part of the Orator's art.

Action, however, seems to be natural to man, when speaking earnestly: but the state of the case at present seems to be, that the disgust excited, on the one hand, by awkward and ungraceful motions, and, on the other, by studied gesticulations, has led to the general disuse of Action altogether; and has induced men to form the habit (for it certainly is a *formed* habit,) of keeping themselves quite still, or nearly so, when speaking. This is supposed to be, and perhaps is, the more rational and dignified way of speaking: but so strong is the tendency to indicate strong internal emotion by some kind of outward gesture, that those who do not encourage or allow themselves in any, frequently fall unconsciously into some awkward trick of swinging the body,* folding a paper, twisting a string, or the like. But when any one is reading, or even speaking, in the Artificial manner, there is

* It should be observed, that the censure here pronounced on school-recitations, and all exercises of the like nature, relates, exclusively, to the effect produced on the style of *Elocution*. With any other objects that may be proposed, the present argument has, obviously, no concern. Nor can it be doubted that a familiarity with the purest forms of the Latin and Greek languages, may be greatly promoted by committing to memory, and studying, not only to understand, but to recite with propriety, the best orations and plays in those languages. But let no one seek to attain a natural, simple, and forcible *Elocution*, by a practice which, the more he applies to it, will carry him still the farther from the object he aims at.

* Of one of the ancient Roman Orators it was satirically remarked (on account of his having this habit,) that he must have learned to speak *in a boat*. Of some other Orators, whose favourite action is rising on tiptoe, it would perhaps have been said, that they had been accustomed to address their audience over a high wall.

 little or nothing of this tendency; precisely, because the mind is *not* occupied by that strong internal emotion which occasions it. And the prevalence of this manner may reasonably be conjectured to have led to the disuse of all gesticulation, even in extemporary speakers; because if any one, whose delivery is artificial, does use action, it will of course be, like his voice, studied and artificial; and savouring still more of disgusting affectation, from the circumstance that it evidently might be entirely omitted.* And hence, the practice came to be generally disapproved, and exploded.

It need only be observed, that in conformity with the principles maintained throughout this Chapter, no *care* should, in any case, be taken to use graceful or appropriate action; which, if not perfectly unstudied, will always be, (as has been just remarked,) intolerable. But if any one spontaneously falls into any gestures that are unbecoming, care should *then* be taken to break the habit; and that, not only in public speaking, but on all occasions. The case, indeed, is the same with utterance: if any one has, in common discourse, an indistinct, hesitating, dialectic, or otherwise faulty, delivery, *his* Natural manner certainly is not what he should adopt in public speaking; but he should endeavour, by care, to remedy the defect, not in public speaking only, but in ordinary conversation also. And so also, with respect to attitudes and gestures. It is in these points, principally, if not exclusively, that the remarks of an intelligent friend will be beneficial.

If, again, any one finds himself naturally and spontaneously led to use, in speaking, a moderate degree of action, which he finds from the observation of others, not to be ungraceful or inappropriate, there is no reason that he should study to repress this tendency.

It would be inconsistent with the principle just laid down, to deliver any *precepts* for gesture; because the *observance* of even the best conceivable precepts, would, by destroying the natural appearance, be fatal to their object: but there is a *remark*, which is worthy of attention, from the illustration it affords of the erroneousness, in detail, as well as in principle, of the ordinary systems of instruction in this point. Boys are generally taught to employ the prescribed action either *after*, or *during* the utterance of the words it is to enforce. The best and most appropriate action, must, from this circumstance alone, necessarily appear a feeble affectation. It suggests the idea of a person speaking to those who do not fully understand the language, and striving by signs to explain the meaning of what he has been saying. The very same gesture, had it come at the proper, that is, the *natural*, point of time, might perhaps have added greatly to the effect; viz. had it *preceded* somewhat the utterance of the words. *That* is always the natural order of action. An emotion,* struggling for utterance, produces a tendency to a bodily gesture, to express that emotion more *quickly* than *words* can be framed; the words follow, as soon as they *can* be spoken. And this being always the case with a real, earnest, unstudied speaker, this mode of placing the action foremost, gives (if it be otherwise appropriate,) the appearance of earnest emotion actually present in the mind. And the reverse of this natural order would alone be sufficient to convert the action of Demosthenes himself into unsuccessful and ridiculous mimicry.

* *——Gratas inter mensas symphonia discors,*
Et crassum unguentum, et Sardo cum melle papaver
Offendunt; poterat duci quia cœna sine istis.
Horace, *Ars Poet.*

* *Format enim Natura prius nos intus ad omnem*
Fortunarum habitum; juvat, aut impellit ad iram;
Aut ad humum mœrore gravi deducit, et angit;
Post *effert animi motus interprete linguâ.*
Horace, *Ars Poet.*

GEOMETRY.

History of the Science.

Probable origin of Geometry

The origin of Geometry, like that of the other ancient sciences, is involved in obscurity. Herodotus and Strabo inform us, that we owe the invention of it to the annual overflowings of the Nile; which, inundating the lands of Lower Egypt, and frequently carrying away the marks and boundaries by which every man's particular property was assigned, rendered it necessary to have some means of ascertaining the respective portions of land belonging to each individual, after the subsiding of the waters. In many cases also, the land was swallowed up in the Nile itself, which by increasing its boundaries in certain places, abstracted every year some portion of land from cultivation; and, according to the former historian, Sesostris, who had divided the country amongst his people, at a certain annual rent, in such cases sent proper persons to measure and value the property thus lost, that a corresponding reduction might be made in the yearly tribute. It has been however very properly observed, that supposing this to be the true state of the case, yet it by no means points out the origin of Geometry, it rather shows that this science had already attained to a certain state of maturity, and that it was merely employed then, as it would be now, in similar cases. At the same time it must be admitted, that the derivation of the word Geometry, which is from γῆ, *earth*, and μετρέω, *measure*, shows clearly that its principal application in the early ages of the world, was the measurement and the division of lands; and there is no doubt, whether Geometry had its origin in the circumstances alluded to or not, that they furnished a motive for its cultivation, and gave rise to various useful and important propositions. But with respect to its first origin, we can scarcely conceive a state of society, however rude, in which something like the first principles of Geometry did not exist. As soon as man began to relinquish his wandering and savage life, and taste the pleasures of social intercourse; as soon as laws were framed to secure to each individual the reward of his own industry and labour, the lands, which had before yielded spontaneously all that he required in his barbarous state, stood now in need of cultivation, in order to render their productions subservient to his more refined appetites, and to the necessity of his family, or the little society over which he presided; this refinement necessarily gave rise to the division of lands, and the partition of flocks and herds, and this again, to comparison of quantity and magnitude; which comparison on the one hand, laid the foundation of Arithmetic, and on the other, that of Geometry, and formed the first links in the chain of propositions which now constitute these two abstract sciences.

In the first instance, there can be little doubt that the attempts were rude and frequently inaccurate, but the science, even in this state, must be said to have commenced; the observations of the father were transmitted to the son; the son again with new acquisitions, passed them down to his children; each succeeding generation added improvements to the observations and experience of that which preceded it; till at length arose some superior genius, who collecting into one mass all the traditionary knowledge of his predecessors, formed them from the efforts of his own mind into a rude system; this was afterwards remodelled and improved by others; and thus by degrees, Geometry, which had, originally, nothing further in view than the mere division of property, became an independent and highly important science. And we think it very probable, that it had already assumed this first form of a system when it was employed by the Egyptians for the purposes that have been stated in the leading part of this article.

First traces of Geometry in Egypt.

At all events, it is in Egypt the first traces of the science are found; and whence it was transplanted into Greece by the celebrated philosopher, Thales. This distinguished sage was born about 640 years before the Christian era, and being unable to gratify his ardent desire for knowledge in his native country, he travelled into Egypt at an advanced period of life, where he conversed with the priests, who, in themselves, embodied all the learning of that country.

Thales, the first Grecian geometer. A. C. 640.

Diogenes Laertius relates, that Thales measured the height of the pyramids, or probably of the obelisks, by means of their shadows; and Plutarch says, that the king Amasis was astonished at this instance of sagacity in the Grecian philosopher. It would seem therefore, by this account, that if Thales actually went to Egypt as a student, he very soon surpassed his masters, whose knowledge of the science of geometry could be but little advanced, if this statement be correct. But whether this philosopher taught the Egyptians, or the latter taught him the method of measuring the heights of objects by their shadows, we see, at all events, that he returned to his own country, furnished at least with some elementary knowledge of geometry; and that it was he who laid the foundation of that science in Greece, and inspired his countrymen with a taste for its study. Various discoveries are attributed to Thales concerning the circle and the comparison of triangles, and in particular he is mentioned as the first who found that all angles in a semicircle are right angles: this discovery is said to have excited in his mind the most lively emotions, and foreseeing, probably, the many important consequences to which it might lead, he is said to have expressed his gratitude to the muses by a sacrifice. He is also stated to have first employed the circumference of the circle for the measure of angles; but this, from what we have stated relative to Archimedes in our History of Astronomy, seems to be incorrect.

Pythagoras. A. C. 550.

The next Grecian geometer of importance was Pythagoras, who flourished about 550 years before Christ, and who had been a pupil of Thales. Like his master he travelled into Egypt, and afterwards into India, and acquired from the priests of the former

country, and from the Brahmins in the latter, a great stock of learning, both in geometry and in astronomy; he did not however immediately transplant this acquisition of learned lore into his native country, but opened his first school in Italy, which was afterwards the most celebrated in antiquity. To this philosopher we are indebted for the discovery of that remarkable property in right angled triangles, which constitutes the forty-seventh proposition in the first book of Euclid's *Elements of Geometry*; namely, that the square described upon the hypothenuse is equal to the sum of the squares described upon the other two sides; a proposition equally curious from the peculiarity of the result, and important for the numerous applications it finds in every branch of mathematical science. This property of the sides of a right angled triangle gave rise to investigations relative to the incommensurability of certain lines, as for example the side of a square and its diagonal; and other properties, again laid the foundation of that part of solid geometry which relates to the five regular bodies. Pythagoras is also said to have first demonstrated that of all plane bodies, the circle is that which has the greatest area under a given circumference.

From this time, at least, therefore geometry had assumed the character of a regular science, and it was cultivated with more or less success, from this date to the destruction of the Alexandrian school, by all the most learned of the Grecian philosophers; we have indeed evident proof of the progress made in the science by the *Elements of Geometry* of Euclid; a work which has stood the test of so many ages without a rival, or at least without an equal for the closeness of its logical reasoning, and the accuracy of its demonstrations.

Hippocrates, Œnopides, &c.

Before this time, however, some geometers of note had cultivated the science in Greece, of whom Œnopides, of Chios, Zenodorus, and Hippocrates, are the most distinguished: to the two former we are said to be indebted for some practical geometrical problems, and to the latter, for the celebrated quadratore of the lunes which still bear his name. Having described on the three sides of an isoceles right angled triangle as diameters, three semicircles, placed all in the same direction, he observed, that the sum of the two equal lunes comprised between the two quadrants of the circumference on the hypothenuse, and the circumferences on the two equal sides, was equal in area to the triangle, and therefore each equal to half the triangle; and this was the first instance in which a curvilineal space had been shown to be equal to a rectilineal area. Hippocrates also attempted the quadrature of the circle, and seems to have deceived himself, with the belief that he had effected it; he was more successful, however, in some other points, and was the first to show that the duplication of the cube required the finding of two mean proportionals between two given lines. He wrote also *Elements of Geometry*, much esteemed at that time, but they are lost; and the only regret that can be entertained for the circumstance is, that they would enable us to understand what the state of that science then was. The date of Hippocrates is generally stated at about 450 years before Christ. Aristotle also mentions two other distinguished geometers of this period, viz. Brisoo and Antiphon, but we have no records of their particular discoveries.

Plato. A. C. 390

We come next to the school of Plato, founded about 390, A. C. This philosopher, as Thales and Pythagoras had done before, travelled into Egypt, and having acquired a great store of knowledge on various subjects, and particularly on geometry, he returned to Greece, and there established his school, over which was placed the celebrated inscription, "Let no one enter here who is ignorant of Geometry;" he, in fact, considered this as the first of all human sciences, and although we have no express work of his on the subject, there is every reason to believe that he was very profound in his geometrical knowledge. We have already mentioned the problem of the duplication of the cube, which about this time engaged so much attention, and which Hippocrates had, as we have seen, reduced to the finding of two geometrical means between the side of the given cube, and another line double of the same. Plato took up the problem at this point, and having in vain attempted to solve it geometrically, (viz. by the help of the ruler and compasses only,) he invented a method of solution by two rulers; but being a mechanical construction it could not be admitted as a geometrical solution, which indeed we now know to be impossible. The most important discovery, however, attributed to Plato was that of the geometrical analysis, to which we may also add, as very little inferior, the invention of what is now termed *geometrical loci*; but there is perhaps some doubt to what extent Plato himself advanced these doctrines, they, doubtless, both had their origin in his school, as had also the conic sections, but whether any of these were originally due to this philosopher is uncertain, although it is very usual to attribute the merit of the discoveries to him, particularly of the first.

Leon, Neoclis, and Eudoxus. A. C. 368.

Geometry had now made so great a progress that a new course of its elements became necessary, a task which was undertaken by Leon, a scholar of Neoclis or Neoclide, a philosopher, who had studied under Plato. To this author has been ascribed the invention of that part of the solution of a problem called its determination; that is to say, the part which points out the limits of possibility, or impossibility. Eudoxus, who was also one of the most celebrated friends of Plato, generalized many theorems, and thereby contributed greatly to the advancement of the science. To him has indeed been attributed the invention of the conic sections, which, at all events, he cultivated with great success; he has been also stated as the author of the doctrine of proportions, given in the fifth book of Euclid's *Elements*; and it seems unquestionable that he was the first who discovered that a cone or pyramid is equal to one-third of the prism of equal base and altitude. Some other important geometrical inventions and discoveries are attributed to Eudoxus, amongst which is that of the theory of curved lines generally. This distinguished geometer died in the year 368, A. C.

Division of the Platonic school.

The school of Plato was now divided into two, which upon some points maintained different opinions, but they both agreed in regarding the knowledge of mathematics, as absolutely necessary to every one who was desirous of studying philosophy. Thus the geometrical theories which had been so much cultivated during the life-time of the celebrated founder of this school, still continued to make great progress. Amongst those who most contributed to the advance-

Geometry. ment of the science at this period was Aristæus, who composed five books on the conic sections, and of which the ancients have spoken in the highest terms of approbation, but which are unfortunately lost. He composed likewise five books on solid loci, which shared the fate of his conic sections; this philosopher is said to have been the friend and preceptor of Euclid.

Euclid. A. C. 280. Euclid flourished under the first of the Ptolemies, about 280 years before Christ, and soon after the founding of the Alexandrian school. The place of his birth is not certainly known, but it appears that he had studied at Athens previously to his settling at Alexandria. Pappus, in the introduction to the seventh book of his *Collections*, gives him an excellent moral character, gentle and modest towards all, and particularly to those who cultivated the mathematical sciences. He composed treatises on various subjects, but he is best known by his *Elements*, a work on geometry and arithmetic, in thirteen books, which still exist; but of these, the first six, and the eleventh and twelfth, are those only which are now consulted, the other books on numbers being of no value in the present state of arithmetic; but of the other eight, it may be said, that notwithstanding the various attempts that have been made, either to improve or to surplant them, they have stood the test of more than 2000 years, and still maintain their preeminence in the schools and universities, not only in this country, but in every part of the world where the science of geometry is cultivated, which is such an instance of excellence and unvaried approbation as cannot be paralleled in any other scientific treatise whatever.

Commentaries on Euclid. The *Elements* of Euclid have had a great number of commentators, from the time of Theon, who was the first, to the present day; after Theon, who flourished about the middle of the fourth century, the *Elements* of Euclid, as well as most of the other scientific works of the Greeks, passed first under the persecution, and afterwards under the patronage of the Arabs, to whom we are mostly indebted for those that have been preserved. To an Arabic version of this work, we owe our first Latin editions by Athelard, in England, and by Campanus, in Italy, about the same time; that is, during the twelfth or thirteenth century. The former remains only in manuscript in some libraries, but the latter was made the foundation of some other Latin translations about the beginning of the sixteenth century, or rather at the latter end of the fifteenth. The Greek text appeared for the first time at Basle, in 1533, edited by Simon Grynæus; and this has been made the foundation of various other editions that have since appeared, particularly of the celebrated one of Commandine, in 1572, and again in 1619. It was this also that Gregory used in preparing the Oxford edition; and lastly, Simson's translation in 1756, is also drawn principally from the same source.

Archimedes. A. C. 250. We have now arrived at the period of our history which introduces us to the prince of Grecian mathematicians, Archimedes, who lived about 250 years before Christ. He was the first who discovered an approximate ratio between the diameter and the circumference of a circle, and which has been made the foundation of the numerous modern approximations which are not dependent on the doctrine of fluxions. It may therefore be interesting to many of our readers to be furnished with a brief sketch of this ingenious process. History. Having seen, that if he inscribed in and circumscribed about a circle two regular polygons of the same number of sides, the circumference of the circle, which will fall between their perimeters, will be greater than the one, and less than the other; and by continually augmenting the number of sides, the circle will at length differ less from the actual perimeter of either, by a quantity less than any that can be assigned; consequently, by computing the perimeter of the two polygons, whatever may be the number of their sides, we shall be certain that the circumference of the circle is comprised between these two limits. Archimedes first employed polygons of six sides; then by bisecting each, he obtained two others of twelve, then of twenty-four, forty-eight, and lastly of ninety-six, where he stopped; the exterior and interior polygons already approaching towards each, very nearly; and here, by taking the mean of the two, he found that the diameter was to the circumference as seven to some number between twenty-one and twenty-two, but much nearer to the latter; and in short, the approximation of seven to twenty-two, is near enough even in the present day, for most practical cases. The most interesting part of this process, however, was that by which he made every successive approximation a step towards the next, and which considering the very defective state of the Greek numeral notation at this time, displays an effort of genius which has certainly never been surpassed. The fluxional analysis has enabled us now to approach towards the actual ratio much more nearly, but the results are more curious than useful: such is the present approximation, that we might with the necessary data state correctly to the nearest unit, the number of grains of sand that would compose a sphere equal in diameter to the orbit of Saturn; a refinement which no practice can ever require.

This, however, is only one of the numerous discoveries with which Archimedes enriched the Grecian geometry; he wrote also treatises *On the Sphere and Cylinder*, that is to say, on the ratio between these two solids, when their diameters and altitudes were equal, and on the relation of their surfaces. He was the first to discover the elegant deduction, that the solidity of the sphere is to that of the cylinder as 2 to 3; and that their curvilinear surfaces are equal, or, which is the same thing, that the surface of the sphere is equal to four of its great circles.

His treatise *On Conoids and Spheroids* relates to the solids generated by the conic sections revolving about their axes; those produced by the rotation of the parabola and hyperbola, he called *conoids*; and such as are generated by the revolution of the ellipse about either axis, are his *spheroids*. Here he compares the area of an ellipse with that of a circle; he also proves that the sections of conoids and spheroids are conic sections, and he treats of their tangent planes. He proves, for the first time, that a parabolic conoid is equal to three times the half of a cone of the same base and altitude; and he also investigates the ratio of any segment of a hyperbolic conoid, or of a spheroid to a cone of the same base and altitude. His reasoning is a model of accuracy; and it exhibits the true spirit of the ancient synthetic method; it is, however, exceedingly prolix and difficult, so much so, indeed, that few will have patience to follow the steps

of the venerable mathematician, more especially as the same conclusion may be found with equal certainty by the modern analysis, at an infinitely less expense of thought and labour. His work *On Spirals* treats of a curve, which was the invention of his friend Conon, who, it seems, had found its properties, but he died before he had time to complete their demonstrations; these Archimedes has supplied; the whole subject is, however, so much his own, that what is properly the spiral of Conon, is usually called the spiral of Archimedes. He has also treated *Of the Equilibrium of Planes, or of their Centres of Gravity*, in two books; and next *Of the Quadrature of the Parabola*. This is the first complete quadrature of a curve that was ever found. He here shews that the area of any segment of a parabola cut off by a chord, is two-thirds of the circumscribing parallelogram; and this he proves by two different methods. His *Arenarius* was written to evince the possibility of expressing, by numbers, the grains of sand that might fill the whole space of the universe. Here he introduces a property of a geometrical progression, that has since been made the foundation of the theory of logarithms; but it would be going too far to suppose that Archimedes had made any approach to that noble invention. This tract is valuable, not on account of the subject on which he treats, but because of the information it contains respecting the ancient astronomy, and the application which it gives of the Greek arithmetic. In addition to the works we have enumerated, there is a treatise *On Bodies which are carried on a Fluid*, in two books, and a book of *Lemmas*, which is a collection of theorems and problems, curious in themselves, and useful in the geometrical analysis. These are all the writings of Archimedes now extant, but many have been lost.

The works of Archimedes are the most precious relict of ancient geometry; they shew to what an extent such a genius as his could carry its method of demonstration; but they likewise prove, that there were certain limits beyond which it became inapplicable, on account of the unwieldiness of the machinery. In general, the progress of discovery is slow; but Archimedes took up the subject where men of ordinary capacities were at a stand, and by the vigour of his mind, anticipated the labour of ages: he was, undoubtedly, the Newton of antiquity.

Apollonius, A. C. 240.

This was the most brilliant epoch in the history of Grecian science; such a philosopher as Archimedes would alone have given a character and eclat to the period when he flourished; but nearly at the same time we meet with Eratosthenes, Apollonius, Nicomedes, and some others, who are still admired for the elegance, depth, and ingenuity of their geometrical compositions; of these, however, Apollonius, undoubtedly stands next in fame to Archimedes.

This *Great Geometer*, as he was deservedly surnamed by his contemporaries, flourished about 240 years before the commencement of the Christian era. He composed a great number of works upon the higher branches of the science, most of which are unfortunately lost, or only small fragments of them remain; but we have, at least, nearly entire, his treatise *On the Conic Sections*, which is alone sufficient to justify the high reputation that he has acquired. This treatise was divided into eight books, of which the first four have reached us in their original language; but the three following have been only handed down to our time through the medium of an Arabic version, made about the year 1250, A. D. and which was rendered into Latin about the middle of the seventeenth century. The eighth book is entirely lost, but attempts have been made to supply it, by following out the plans of the author as far as they could be ascertained from the first seven. This task was first undertaken by the celebrated Dr. Halley, who also revised and corrected the translation that had been before made of the leading part; and in 1710 published the splendid Oxford edition of this noble monument of Grecian geometry. The first four books of Apollonius treat of the generation of the conic sections, and of their principal properties, with reference to their axes, foci, and diameters. The greater part of these properties were, indeed, known before the time of this author, and are merely given as preliminaries to his general and extended view of the subject. Before this time the right cone only had been considered; but Apollonius treats generally of every cone having a circular base, and presented many new theorems, or rendered those already known more general. The following books contain a great number of elegant and interesting propositions entirely new, but which it would be inconsistent with our plan to describe in detail. The most important of his other works were: 1. *On the Section of a Ratio*; 2. *On the Sections of a Space*; 3. *On Determinate Sections*; 4. *On Tangencies*; 5. *On Inclinations*; and, 6. *On Plane Loci*.

Eratosthenes and Nicomedes.

We must here pass over, with very brief notices, the names of several other distinguished geometers who lived about this time. We have already mentioned Eratosthenes and Nicomedes; the former was most distinguished as a geometer for his construction of the duplication of the cube, and for two books, entitled, *De Locis ad Medietates*, and the latter for the invention of the *conchoid*, a curve which still carries his name; and for the application that he made of it to the finding two mean proportionals between two given lines or numbers.

Conon, Trasideus, Nicoteles, and Dositheus, were also distinguished geometers about this period; but their labours have not been handed down to our time.

Decline of Grecian science.

We have now, unquestionably, passed the zenith of Grecian science, we find, indeed, many authors, but they added little, perhaps nothing, whatever to the discoveries of Archimedes and Apollonius. We must, however, except Theodosius, the author of an excellent treatise *On Spherics*, in three books, which have been preserved and justly admired; and Menelaus of Alexandria, who lived in the second century of the Christian æra; he was the author of a treatise *On Trigonometry*, in six books; and another *On Spherics*, in three books, which are still extant. He appears, also, to have treated of the geometry of curved lines. Ptolemy, also, the author of the *Almagest*, born in 70 A. D. must be considered, if not as an original genius, at least as a valuable promoter of geometrical science; his treatise *On Optics*, which is lost, is supposed to have contained some beautiful specimens and applications of geometry.

The next two or three centuries are entirely barren of any names, which in this brief sketch of the History of Geometry require to be particularized.

Geometry. Science in general was, indeed, now fast declining, and the only names of distinction between this time and the fall of Alexandria, which totally extinguished the faint light that still remained of Grecian learning, are very few. Pappus, Theon, and his accomplished daughter Hypatia, Diocles, and Proclus, are, perhaps, the only names to which it will, in our case, be requisite to call the attention of the reader.

Pappus. A. D. 380. Pappus flourished about the year 380, A. D., and was the author of a work which, although it does not possess so much originality as some we have referred to, is still extremely curious and interesting. We allude to his *Mathematical Collections*, in eight books, of which, however, the first and half of the second are lost. He seems to have intended to collect, into one body, several scattered discoveries, and to illustrate and complete, in many places, the writings of the most celebrated mathematicians, in particular, those of Apollonius, Archimedes, Euclid, and Theodosius; for this purpose he has given a multitude of lemmas, and curious theorems, which they had supposed known; and he has also described the different attempts which had been made to resolve the most difficult problems, as the duplication of the cube, and the trisection of an angle. The preface to his seventh book is highly valuable; having preserved from oblivion many analytical works on geometry, of which we should otherwise have been entirely ignorant. The abridgement which he has given of these is all that remains of the greater number; yet it has served to give a continuity to the History of Geometry, and to inspire modern mathematicians with a high opinion of the theories of the ancients. In fact, such of their geometrical writings as have descended to our times, are merely elementary; their more recondite works have either been entirely lost, or are only known by the account which Pappus has given of them. The books that remain of this author, have suffered much from the injuries of time; there are many inaccuracies, and some passages so mutilated as to be hardly intelligible. The original Greek, except some extracts, has never been published. The only translation that has been given, which is by Commandine, was published at Pesara in 1558, and again, with little variation, in 1660, at Bologna. Commandine appears to have had access to only one manuscript, which wanted the first two books, and which was, throughout, very faulty. There are, however, several manuscripts of Pappus in the libraries of some public institutions. The University of Oxford possesses two, one of which has half the second book: this part, which treats of arithmetic, was published by Dr. Wallis in 1688; it is, therefore, probable, that both these books treated on this subject. Amongst many other curious problems contained in this work, Pappus has some perfectly original, such as that of finding quadrable spaces on the surfaces of a sphere. He demonstrates, by means of the theorems of Archimedes, that if a moveable point, proceeding from the vertex of a hemisphere, passes over a quarter of the circumference, while this quadrant makes an entire revolution about the vertical axis of the hemisphere, the space included between the circumference of the base and the spiral of double curvature, described on the hemisphere by the moving point, is equal to the square of the diameter. Such a proposition as this, even with all the aid afforded by analysis, is far from elementary, and shews that the author, with the means of investigation which he possessed, must have been a very profound geometrician. This problem has since been generalised, it having been shown that, if instead of the quadrant making a complete revolution, it makes only n given part of a revolution, while the moveable point descends through it. The spherical space described between the quadrant, the corresponding arc of the base, and the spiral, is to the square of the radius, as the arc of the base to a quarter of the circumference. We shall have again to refer to this species of problems in speaking of the geometry of the moderns. History.

We shall only further add respecting this work of Pappus, that in the preface to the seventh book is given a sufficiently distinct idea of that beautiful theorem, commonly ascribed to the Pere Guldio, and which English mathematicians commonly call the *centrobaryc* problem; viz, the solidity of any solid, or the area of any surface described by the motion of an area or line, is equal to the product of the area, or length of the generatrix into the path of its centre of gravity.

Theon, Hypatia. Theon is principally distinguished for his *Commentaries, or Scholia on Euclid*, although, according to the statements and corrections of Dr. Simson in his translation, he rather darkened and bewildered the subject, than elucidated it. Theon was the father of the accomplished and unfortunate Hypatia, who had so much distinguished herself by her cultivation of the mathematical sciences generally, that she was deemed worthy to succeed her father in the Alexandrian school, where she shone o distinguished ornament to her sex and her country, till she fell a sacrifice to the blind fury of a bigoted and fanatical mob, about the beginning of the fifth century.

Proclus, Sporus, &c. After Theon and his daughter we meet with only two or three names of any note. Proclus, who was the chief of the Platonists at Athens, signalized himself by his *Commentaries on Euclid*; and Diocles has been principally remembered as the author of the *cissoid*, a curve still named after him. Eutocius also attributes to him the solution of a problem concerning the division of the sphere; Sporus and Philo also lived about this period; the former gave a solution to the problem of finding two mean proportionals, and the latter extended the approximation of the ratio between the diameter and the circumference of the circle to the ten thousandths part, or to four places of decimals, the diameter being unity. Some other names might also be mentioned, but they possess little interest, and we must now consider the light of Grecian science as about to be extinguished. What little remained up to this period, the commencement of the seventh century, had long taken refuge in the museum of Alexandria, where, destitute of support and encouragement, they could not fail to degenerate. Still, however, they preserved, at least by tradition or imitation, that strict and correct character bestowed upon them by the early Greeks; but before the date above mentioned, a tremendous political and religious storm arose which threatened their total destruction. Filled with all the enthusiasm a militant religion is calculated to inspire, the successors of Mohammed ravaged that vast extent of country which stretches from the east to the southern confines of Europe. All the cultivators of the arts and sciences, who Destruction of the Alexandrian library. A. D. 640.

from every part had taken refuge in Alexandria, were driven away with ignominy, or fell by the swords of their conquerors: the former fled into remote countries, to drag out the remainder of their lives in poverty and distress. The places, and the instruments which had been so useful in making observations on astronomy, which was then scarcely distinguishable from geometry, were involved with the records in one common ruin. The whole of the valuable library, which contained the works of so many eminent philosophers and geometers, and which was the common depository of every species of learning which does honour to the human mind, was devoted to the flames by the Arabs; the Caliph Omar observing, "that if they agreed with the Koran, they were useless, and if they did not they ought to be destroyed," a sentiment worthy of such a leader and of the cause in which he was engaged. This event happened in the year 640 of the Christian era.

The Arabs promote the science.

It has been said that a few were enabled to escape the blind fury of Omar and his followers by flight, and of course these carried with them some remnant of that general learning for which this school had been so celebrated; but still, destitute of books and instruments, and probably of the means of subsistence without manual labour, very little of that great mass of learning could have been preserved, and still less accumulated, had not the Arabians themselves, within less than two centuries of this fatal conflagration, become the admirers and supporters of those very sciences they had before, in their bigoted fury, so nearly annihilated. Fortunately for geometry and for the sciences in general, these men now studied the works of the Greeks with the greatest assiduity, and if they added little to the general stock of knowledge which they found contained in the few manuscripts which escaped from the general wreck, they became at least sufficiently masters of many of the subjects to comment upon them, and to set a due estimation upon these valuable relics of ancient science. It is by this means so many of them have been preserved, and that we are enabled to bestow our admiration on the transcendant talents and genius of Archimedes, Apollonius, and the other distinguished Greeks, whose names we have recorded. It is, however, principally for the preservation of the Greek authors that we are indebted to the Arabs, and not for any important improvements or discovery in geometry; for if we except the simplification they gave to trigonometry, we owe to them very little, and even this is by some supposed to have been derived by them from India with the numeral figures which we now employ in arithmetic; one perhaps of the most useful discoveries that was ever made, and that to which the mathematical sciences are more indebted than to any other whatever. It would be useless to quote here the names of the several Arabs who have translated, or ordered the translation, of the different Greek authors to whom we have referred, and still less so, those of the Persians and Turks; because in these two countries nothing appears to have been attended to but the most elementary parts; we shall therefore pass to a slight mention of the geometry of the Hindoos and Chinese, not that they, any more than the Persians and Turks, have pursued this science to any great length, but because it is a question whether they did not possess their knowledge on the subject at an earlier date than the Greeks, and whether the first knowledge which the latter nation obtained was not of Hindoo or Chinese origin. Opinions on this subject are much divided. The researches of the learned have brought to light tables in India which must have been constructed by geometry; but the period at which they were formed, although unquestional, a very early one, has not been completely ascertained.

Geometry of the Hindoos and Chinese.

The Hindoos have a treatise called the *Suryá Sidh'anta*, which they profess to be a revelation from heaven to Maya, a man of great sanctity, about four millions years ago; but notwithstanding the extravagance of this fable, there seems no question that it is of a very remote date; and although interwoven with many absurdities, it contains a rational system of trigonometry, which differs entirely from that first known in Greece and Arabia. It is, in fact, founded on theorems not known in Europe before the time of Vieta, not more than two centuries back; and it employs the sines of arcs, and not the chords of the double arcs, which was the practice of the Greeks. It is, therefore, questionable, whether the introduction of the sines into trigonometry, which is generally considered as an Arabic invention, may not have been, as well as their numerals, of Indian origin. The Chinese also, according to their romantic historians, were very early promoters of geometry and astronomy; but whatever may be the antiquity of these sciences amongst them, their extent has been very limited, and they have been long perfectly sterile in their hands.

Geometry of the Romans.

Before we enter upon the geometry of modern Europe, it may he proper to allude slightly to the state of geometry amongst the Romans. This warlike people were at no time distinguished by their knowledge in what have been termed the exact sciences; they studied astronomy, but not so much for the love of the science itself, as for its supposed relation with astrology, and their desire to pry into the secrets of futurity. With such ideas geometry was not likely to be much extended in their hands, and, in fact, the only authors of any note amongst them, were Boetius the senator and consul, and Vitruvius; which latter has displayed considerable knowledge of geometry, particularly in the ninth book of his architecture; and he seems to have had some general knowledge of most other mathematical subjects. A few other names might be mentioned, but they would answer no purpose but needlessly to lengthen this historical sketch.

State of geometry during the dark ages.

We are arrived now at what have been properly termed the dark ages; for from the fatal catastrophe which extinguished the last faint glimmerings of Grecian science in the middle of the seventh century, we pass over a space of nearly six hundred years without meeting with any discovery to arrest our attention for a moment, except those we have already spoken of as due to the Arabs; we might, indeed, mention the venerable Beda, 700 A. D. and Roger Bacon, 1240 A. D. as individuals who, during this long period, displayed some knowledge of the sciences; but we owe to them no discoveries. During the thirteenth century, indeed, we meet with several names of some note; in fact, the sun of science, which had been so long set, was now gradually advancing towards the horizon of Europe, and the twilight had already commenced of that brilliant day which now illuminates so great a

Geometry. Geometers of the thirteenth century.

portion of the globe. Amongst the mathematicians of this time, may be mentioned John de Sacro-Bosco, or John of Halifax, who wrote a treatise *On the Sphere*, and Campanus of Navarre, who translated Euclid, and composed a treatise *On the quadrature of the Circle*; Albertus Magnus wrote also on geometry during this century.

Of the fourteenth century.

The fourteenth century is still further distinguished by its geometers, and particularly in England; amongst whom we may mention Wallingfort and the poet Chaucer; but it is only in the fifteenth century that geometry shone forth with that splendour which was indicative of the sublime discoveries that were to follow.

Of the fifteenth century.

The principal promoters during this century were Purbach and Muller, or Regiomontanus, Lucus de Burgo; and the celebrated Copernicus, although he never wrote on this subject, was a learned geometrician. Purbach's first essay was to amend the Latin translation of Ptolemy's *Almagest*: he wrote a tract which he entitled, *An Introduction to Arithmetic*; a treatise *On Gnomonics and Dialing*; he corrected by the Greek text the ancient version of Archimedes made by Gerrard of Cremona; he translated the *Conics* of Apollonius; the *Cylinders* of Serenus; and gave a Latin version of the *Spherics* of Theodosius and Menelaus. He commented on certain books of Archimedes, which Eutocius had passed over; refuted a pretended quadrature of the circle by Cardinal Cusa; besides various important labours connected with astronomy, which was, indeed, his favourite science; one of the most useful of which was his rejection of the ancient sexagesimal division of the radius, instead of which he divided it, or supposed it divided, into 600,000 parts. Regiomontanus, who out-lived his friend and preceptor Purbach, made a still further improvement in this case, by carrying the division to 100,000, and calculating new tables for every degree and minute of the quadrant.

Lucus de Burgo revived Campanus's translation of Euclid, which, however, was only published in 1509. His work, *Summa de Arithmetica, Geometria*, &c. 1494, contains a treatise *On Geometry*. The progress which had now been made in the Greek tongue, and the invention of printing, contributed greatly to the dissemination of geometrical knowledge. The Greek mathematicians began to be known in Europe, and Euclid was printed for the first time at Venice in 1482, in a folio volume, by Erhard Ratdolt, one of the first printers of that age.

Of the sixteenth century.

About the beginning the sixteenth century several of the Greek authors were translated and published, as the *Spherics* of Theodosius, and such books of Apollonius as were then known; but the translators, although good Greek scholars, had but little knowledge of geometry, so that these translations were in many respects defective; at length Commandine, about the middle of the century, who possessed both the requisite qualifications, undertook a similar task. He translated into Latin, and published in 1558, a part of the works of Archimedes, with a commentary. He published, also, a translation of the first four books of Apollonius's *Conics*, with the *Commentary* of Eutocius, and the *Lemmas* of Pappus. His Latin translation of Euclid appeared in 1572. We owe to him also a treatise *On Geodisia*, or the division of figures, the work of an Arabian geometer. But his last and most important labour was his translation of the *Mathematical Collections* of Pappus, the only one that has yet appeared, and it is probable that but for the mathematical zeal of the author, this interesting work, so highly curious and valuable, might still have been nearly unknown to modern geometers.

History.

John Dee, a singular and eccentric English writer, wrote some mathematical works about this time, many of them connected with astrology and alchemy, and some on geometry. In 1570 he published a *Preface Mathematical to the English Euclid by Henry Billingsley*, "which," says Dr. Hutton, "is certainly a very curious and elaborate composition;" and the same year *Divers and many Annotations and Inventions dispersed and added after the tenth Book of the English Euclid*. During this century, Maurolycus published some works which were much esteemed at that time; and it was also in the same century that Tartaglia, who had translated Euclid into Italian, discovered the method of solving cubic equations, which were clandestinely published by Cardan, and still bear his name. He also translated a part of Archimedes, and demonstrated the rule for finding the area of a triangle when the three sides are given; but the rule itself was discovered by Hero the younger, some centuries before. We might, if our limits admitted of it, particularize the works of a number of other ingenious mathematicians of this period, but we can only name a few of the most distinguished; as Clavius, whose translation and commentary on Euclid is still esteemed; Metius, a mathematician of the Low Countries, the author of a very convenient approximation to the ratio between the diameter and circumference of a circle, viz. 113 to 355. This was soon after extended by Romanus to seventeen places of decimals. Nonius distinguished himself by the invention of a method of reading angles to a great degree of accuracy, something resembling what we still, sometimes, improperly attribute to him, but which is more properly called a *vernier*, or vernier scale. Wright, an English mathematician, was the author of the chart which we always improperly attribute to Mercator. But, perhaps, the man of most original genius, who wrote on mathematical subjects during this age, was Vieta, who flourished in France just before the commencement of the seventeenth century; his writings abound with marks of great originality and the finest genius; and his inventions and improvements in all parts of mathematics, were very considerable. He was, to a certain degree, the inventor and introducer of literal algebra; that is, in which letters are used instead of numbers, as well as of many beautiful theorems in that science. He made also very considerable improvements in geometry and trigonometry; his *Angular Sections* is a very ingenious and masterly performance; by these he was enabled to resolve the problem of Adrianus Romanus, proposed to all mathematicians, amounting to an equation of the 45th degree. His *Apollonius Gallus*, being a restoration of Apollonius's tract *On Tangencies*; and many other geometrical pieces to be found in his works, show the truest and finest taste for geometrical investigations. He gave some masterly tracts on trigonometry, both plane and spherical, which may be found in the collection of his works published at Leyden in 1646, by Schooten; besides another larger and separate volume in folio, published in the author's life-time at Paris in 1579; containing

Vieta, born 1540.

Geometry. extensive trigonometrical tables, with the construction and use of the same; these are particularly described in the introduction to Dr. Hutton's *Logarithms*. To this complete treatise, on trigonometry, plane and spherical, are subjoined several miscellaneous problems and observations; such as on the quadrature of the circle, the duplication of the cube, &c. Computations are here given of the ratio of the diameter of the circle to its circumference, and of the length of the sine of one minute, both to a great many places of figures.

Geometry of the seventeenth century. The seventeenth century gave birth to many illustrious geometers; but it was now found that analysis was a much more powerful and expeditious instrument, and many who commenced their mathematical career as geometers, were turned from their pursuit to follow the new analysis, which had its origin about this period; our business is, however, only with the geometrical writings of these authors. One of the earliest geometers of this century was Lucas Valerius, an Italian; he distinguished himself by his determination of the situation of the centre of gravity in conoids, spheroids, and their segments. Marinus Ghetaldus was well acquainted with the ancient geometry, and, guided by the indications of Pappus, attempted a restoration of the lost book of Apollonius *On Inclinations*; he also wrote a supplement to the *Apollonius Galus* of Vieta. Lodolph Van Ceulen distinguished himself by his laborious approximation to the circumference of a circle, when the diameter is unity, stating it to be 3·14159,2,6535,89793,23846,26433,83279,50238, or rather that this number is in defect; but that with the last number increased by unity, it is in excess, the true ratio lying between these two numbers.

Willebrod Snellius was another Dutch mathematician of this period; at an early age he undertook to restore the work of Apollonius on determinate sections, which was published under the title of *Apollonius Batavus*. He published also a work, *Cyclometria*, where he treated of the approximation between the diameter and circumference, and displayed in it some ingenuity and dexterity in his numerical operations.

Albert Girard, also a Fleming, possessed great originality and genius. He first gave a rule for finding the area of a spherical triangle, or of a polygon bounded by great circles on a sphere; he also offered some general theorems for measuring and comparing solid angles, and endeavoured to restore the porisms of Euclid.

Kepler, born 1571. Hitherto no new principle had been introduced into geometrical investigations; the models laid down by the Greek mathematicians were considered as standards of perfection, and no one had yet been bold enough to break the charm, till the celebrated Kepler, in his *Nova Stereometria*, ventured on this dangerous ground, and first introduced considerations of infinity into geometry: according to these new views a circle was conceived to be composed of an infinite number of indefinitely small triangles, having their vertex at the centre, and their bases at the circumference; cones, in like manner, were supposed to consist of an infinite number of small pyramids, &c. By this ingenious way of treating his subject, Kepler was enabled to go far beyond Archimedes with infinitely less labour. The latter conceived all the bodies that he had treated of, as formed by the rotation of different conic sections about an axis, and his investigations were limited to such bodies; but Kepler treated of solids generated by the rotation of these curves about any line whatever in their planes, and thus gave, as it were, to the problems of Archimedes, an almost indefinite extent; and what is of more importance, he thus laid the foundation of the modern doctrine of infinitesimals. History.

Cavallerius, born 1598. The next important innovation in the method of handling geometrical subjects, was made by Cavallerius in his work, *Geometria Indivisibilibus*, published in 1635. Here a line is conceived to be made up of an infinite number of points; a surface of an infinite number of lines; and a solid, as composed of an infinite number of surfaces, which elements of magnitude he called indivisibles. So bold an innovation was not likely to be received with universal approbation by men who had devoted themselves to the study of the ancients, and who knew no other standard for forming their taste and judgment; in fact, this work met with great opposition, and led to various controversies. In answer to some of the objections that had been urged, Cavalierius maintained that the hypothesis he had advanced, was by no means an essential part of his theory, which, in fact, was the same as the ancient method of exhaustions, but free from its tedious and indirect mode of reasoning. To effect their purposes, the ancients were under the necessity of inscribing and circumscribing polygons about circles, and polyhedra in the same way about spheres; and although with great ingenuity, it was also with great labour that they arrived at their conclusion. Cavallerius advanced more directly to his object. He considered, as we have stated, surfaces as composed of an infinite number of lines, and solids as made up of an infinite number of planes; and the principle he assumed was, that the ratio of these infinite sums of lines or planes, as compared with the unit of numeration, in each case, was the same as that of the surfaces or solids of which they were the measure. This work of Cavallerius is divided into seven books; in the first six the author applies his new theory to the quadrature of the conic sections, and the solidity of their solids of revolutions, and to other questions of a similar nature relative to spirals; the seventh is employed in demonstrating the same things by principles independent of indivisibles, and establishing by the agreement of the results, the exactitude of the new method.

Fermat, born 1665. The French geometers, during this time, were no less intent upon improving and extending geometry. The dates of the letters of Fermat, published in the *Commerce Epistolaire*, of this author, shew that his investigations preceded the year 1636, and therefore that his discoveries were independent of those of the Italian geometer. Archimedes had measured the area of the common parabola, and found the solidity of the conoid produced by the rotation of the plane about its axis. Fermat, by a new method, solved both these problems with great facility, and determined moreover the situation of the centre of gravity of the paraboloid as well as that of the solid generated by the parabola revolving about its base; and what was still more difficult, he found the quadrature of parabolas of all orders, and the value of their solids of revolution, made about either an absciss or an ordinate; he ascertained likewise the centres of gravity of these solids,

Geometry. and solved a number of other problems which marked him as a most profound geometer.

Roberval, born 1602. Roberval was also a geometer of high reputation, although inferior to Fermat; and he solved as soon as the problems were proposed to him by the latter, all the cases of the parabolas above mentioned. He employed considerations similar to those of Cavallerius, but under more guarded language; that is, he assumed surfaces to be made up with other surfaces of little breadth, and solids as composed of a number of indefinitely thin prisms, instead of calling them lines and sections, as Cavallerius had done. On these principles he solved a number of very difficult and curious problems in a work, entitled *Traité des Indivisibles*, which was not printed till after his death in 1693. Geometry is also indebted to Roberval for several curious investigations relative to the cycloid, and particularly for his method of tangents, which was an exceedingly near approach to the principles of fluxions, and will be more particularly noticed in our HISTORY OF ANALYSIS.

Descartes, born 1596. The celebrated Descartes was a contemporary with Fermat and Roberval, and much rivalry and, unfortunately, much of envy and petty jealousies subsisted between these great masters. They proposed to each other difficult problems, and both the question and answer were frequently couched in, or accompanied with language, which we are sorry to see employed between men whose talents it is impossible not to respect; this spirit however was very common at this period, and was cherished till the time of the Bernoullis, between whom even the fraternal relation, in which these great geometricians stood towards each other, was forgotten in their characters of scientific rivals. Descartes was unquestionably a man of distinguished talents; and as the author of a system of philosophy, which found able defenders for many years, he will always stand conspicuous in the annals of mathematical science; but in geometry, more is certainly attributed to him than is justly his due. He is, for example, always cited as the first who invented the application of algebra to geometry, which is not strictly the case. He certainly considerably extended the nature of this application, but the foundation had been already laid by Vieta, and practised to a certain extent by others. It was Descartes, however, who first solved, in general terms, the problem that had been proposed by the ancient geometers; namely, having any number of right lines given in position on a plane, to find a point, from which we may draw as many other right lines, one to each of the given lines, making with them given angles, and under the following conditions, viz. that the product of the two lines thus drawn shall have a given ratio, with the square of the third, if there be only three, or with the product of the two others, if there be four; or if there be five, that the product of the three shall have the given ratio with the product of the two lines remaining, and a third given line, &c. &c. Descartes was the author also of several other highly interesting geometrical problems which led the way to the establishment of the new analysis, and will therefore be more appropriately treated of in the history of that science. His work, containing the investigations alluded to above, was published in 1637.

Our limits will only admit of noticing in very concise terms the distinguished geometers who succeeded those last mentioned, in fact we are now nearly arrived at that period when the entire current of mathematical science took a new direction; every discovery in geometry is now leading us nearer to the invention of the new analysis, and they are so blended with it, that it is almost impossible to notice the one without referring also to the other; we shall therefore, in this place, confine our observations within a very limited space, referring the reader who is desirous of examining the progress of geometry at this time, to the History of Analysis to which we have already referred in the preceding page. The names which intervene between this time and the full developement of the new analysis, by Newton and Leibnitz, were Gregory St. Vincent, a Flemish mathematician, whose object was the quadrature of the circle, in which he thought he had succeeded; but, although mistaken in this, he arrived at such a multitude of curious and interesting properties and theorems, as fully to recompense him for his laborious research. History.

Another name which will ever be highly esteemed by every admirer of the exact sciences, occurs at this period. Huygens was one of the brightest ornaments of the seventeenth century; at a very early age he published his *Theoremata de Circuli et hyp. quad.*, and he afterwards found the surfaces of conoids and spheroids, a problem which had not been attempted before his time. He determined the measure of the cissoid, and showed how the problem of the rectification of curves might be reduced to that of their quadratures. It is also to him that we are indebted for the theory of evolutes and involutes. His treatise *De Horologio Oscillatorio* is a work of the highest merit, and contains some of the most beautiful applications of geometry to mechanics that had ever been made before his time. Huygens, born 1629.

Dr. Barrow, an English mathematician, and the tutor of the illustrious Newton, was highly distinguished at this period by his geometrical writings: his *Geometrical Lectures* are composed partly in the style of the ancient, and partly in that of the modern geometry. To him we are indebted for another step towards the new analysis. Dr. Barrow born 1630.

For the rest it will be sufficient to state the names of Tacquet, James Gregory, Borelli, Viviani, Simson, Stewart, and Horsley, each of whom has distinguished himself by his taste for geometrical pursuits, has added some perfections, and rendered some service to the science, but not such as to claim from us any particular notice in this brief sketch.

It only now remains for us to add a few remarks relative to a new species of geometry introduced into notice in France, by Monge, during the period of the revolution, under the designation of descriptive geometry. When any surface whatever penetrates another, there most frequently results from their intersections, curves of double curvature, the determination of which is necessary in many arts, as in groined vault work, cutting arch-stones, wood-cutting, for ornamental work, &c., the form of which is frequently very singular and complicated: it is in the solution of problems appertaining to these subjects that descriptive geometry is especially useful. Descriptive geometry.

Some architects, more versed in geometry than persons of that profession commonly are, have long ago thrown some light on the first principles of this kind of geometry. There is, for example, a work by a jesuit, named Courcier, who examined and showed

how to describe the curves resulting from the mutual penetration of cylindrical, spherical, and conical surfaces: this work was published at Paris in 1663. P. Deraud, Matheurin, Frezier, &c. had likewise contributed a little towards the promotion of this branch of geometry. But Monge has given it very great extension, not only by proposing and resolving various problems both curious and difficult, but by the invention of several new and interesting theorems. We can only mention in this place one or two of the problems and theorems. Among the problems are the following; first, Two right lines being given in space, and which are neither parallel nor in the same plane, to find in both of them the points of their least distance, and the position of the line joining these points; second, Three spheres being given in space, to determine the position of the plane which touches them. There are also some curious problems relative to lines of double curvature, and to surfaces, resulting from the application of a right line that leans continually upon two or three lines given in position in space. Among the theorems the following may be mentioned; if a plane surface given in space be projected upon three planes, the one horizontal, and the two others vertical, and perpendicular to each other, the square of that surface will be equal to the sum of the squares of the three surfaces of projection.

A few other works possessing some novelty in their manner of treating the subject, may be also here enumerated, as *Développement de Géométrie*, and *Applications de Géométrie*, by Baron Dupin, the celebrated author of *Travels in England*; the *Polygonometrie*, of L'Huillier; the *Géométrie du Compas*, by Mascheroni, in which no instrument but the compasses is employed; the *Géométrie de Position*, by Carnot; and Cresswell on *Geometrical Maximo et Minimo*.

As to the elementary works of the present day they are very numerous; those most approved of, however, are the translation of Euclid's *Elements*, by Simson; and the *Geometries* of Ingram, Playfair, Bonnycastle, and Leslie; and amongst the French writers we may mention the *Treatises of Geometry*, by La Croix, and Le Gendre; to which latter work we have been much indebted, in compiling the following treatise, although we have, in some instances, deviated widely from it.

BOOK I.

Properties of lines, angles, and triangles.

Definitions.

1. Geometry is that science which is applied to the measure of extension. Extension is comprised under three dimensions; namely, length, breadth, and depth or thickness.

2. A *line* is length, without breadth or thickness. The extremities of a line are called *points*. So that a point has no dimensions, but position only.

3. A *right* or *straight line* is the nearest distance between two points. In the following treatise when the word line is used, a right line is to be understood.

4. Every line which is not a right line, or composed of right lines, is a *curve*. Thus A B is a right line, A D a compound or crooked line, and A E a curve,
Fig. 1. or curved line, fig. 1.

5. A *surface* is that which has length and breadth, without thickness.

6. A *plane* is a surface in which any two points being taken within it, the right line which joins those points will be every where in the surface.

7. Every surface, which is neither a plane nor composed of planes, is a *curved* surface.

8. A *solid* is a body comprised under three dimensions; length, breadth, and thickness.

9. When two right lines, not in the same right line, meet each other, they form *an angle*, which is greater or less as the lines are more or less inclined or opened. The point of their meeting is called the *summit*, or angular point, and the two lines are its *sides*, fig. 2. Fig. 2.

An angle may be designated by the single letter at its *summit*, or by three letters; in which latter case that letter which is at the summit or angular point, is to be read in the middle. Thus the above angle may be called the angle A, or B A C, or C A B. Angles, like other quantities, may be added, subtracted, multiplied, and divided.

10. When one line, as C D, meets another, as A B, so that the angles on each side are equal to one another, each of them is called a *right angle*; and the line C D is said to be *perpendicular* to A B, fig. 3; and C D is said Fig. 3. to be the *perpendicular distance* of the point C from the line A B.

11. An *acute* angle is less than a right angle, as A B C; and an *obtuse* angle is greater than a right angle, as C B D, fig. 4. Fig. 4.

12. *Parallel lines* are those in which any point being taken in the one, and any point being taken in the other, the perpendicular distance of these points from the other line shall be equal to each other, fig. 5. Fig. 5.

The usual definition of parallel lines: that they are those, "which produced to any distance whatever will never meet," is not sufficiently specific. For in order to demonstrate the properties of those lines, as given in the 29th Proposition of the *Elements of Euclid*, or our 19th proposition, it is not sufficient to know that parallel lines will never meet, but also that they will never approach; and it cannot be demonstrated in this part of Geometry, that two right lines may not approach, although they never meet; a condition which Euclid takes for granted in his twelfth axiom. It is essential to the demonstration of the above proposition, that it be first shown that parallel lines do not approach towards each other, and it is therefore necessary to demonstrate the twelfth axiom by means of the previous proposition; or to give a definition of parallel lines which will comprehend their essential property of never approaching towards each other, or of being every where at the same perpendicular distance.

Simson, in his translation, has endeavoured, by means of two other definitions, five propositions, and corollaries, to demonstrate the twelfth axiom of Euclid; and after all he has failed, because he has not shown that two lines cannot approach without ultimately intersecting. He has shown that they cannot approach, and then recede again; but he has taken for granted, as Euclid himself has done, that if they do approach they will meet if produced, which is the very point in question.

We have, therefore, preferred the definition above given, and have made the property of parallel lines never meeting, a proposition instead of a definition.

13. A *plane figure* is a plane terminated on all sides

by lines. If the sides are right lines, it is called a *rectilineal figure*; it receives also particular denomination according to the number of its sides.

14. A rectilineal figure of three sides is a *triangle*, Fig. 6. fig. 6; of four sides a *quadrilateral*; of five sides a *pentagon*; but generally a figure of more than four sides is called a *polygon*.

15. A *triangle*, whose sides are all equal to each other, is called an *equilateral* triangle, fig. 7; when Fig. 7. only two of its sides are equal, it is an *isosceles* triangle, fig. 8; and when they are all unequal, it is Fig. 8. Fig. 9. called a *scalene triangle*, fig. 9.

Triangles also receive specific denominations from the nature of their angles.

16. When a triangle A B C, has one of its angles, as A, a right angle, it is called a *right angled triangle*, and the side B C opposite the right angle is called Fig. 10. the *hypothenuse*, fig. 10.

Fig. 11. When one of the angles, as B, fig. 11, is obtuse, it is an *obtuse angled* triangle; and when all the angles are acute, it is an *acute angled triangle*, Fig. 12. fig. 12.

Quadrilateral figures receive also particular denominations, as follow:

17. A *square* is a quadrilateral, having all its sides Fig. 13. equal, and all its angles right angles, fig. 13.

18. A *rectangle* has its opposite sides parallel, and its Fig. 14. angles right angles, fig. 14.

19. Every quadrilateral having its opposite sides Fig. 15. parallel, is a *parallelogram*, fig. 15.

20. A parallelogram which has all its sides equal, but its angles not right angles, is called a *rhombus*, Fig. 16. fig. 16. When only the opposite sides are equal, it Fig. 17. is a *rhomboid*, fig. 17.

21. A *trapezoid* is a quadrilateral, in which two Fig. 18. only of the opposite sides are parallel, fig. 18.

22. The *diagonal* of any rectilineal figure, is a right line joining any two of its angles which are not adjacent. In fig. 18, A C is the diagonal.

23. An *equilateral* polygon is one in which the sides are all equal; and an *equiangular* polygon is one which has all its angles equal.

24. Two polygons are said to be *equilateral to each other*, when the sides of the one are equal to those of the other, each to each, and are placed in the same order; that is, so that in following the perimeters in the same direction, the first side of the one is equal to the first side of the other, the second side of the one to the second side of the other, and so on; and in like manner polygons are said to be equiangular when their angles are equal, each to each, taken also in the same order.

In both the above cases the equal sides and angles which are alike situated, are called *homologous*.

Regular polygons, whose number of sides do not exceed twelve, receive specific denominations, as follow:

A polygon of three sides is called a triangle.
four sides..............a square.
five sides ....a pentagon.
six sides ....a hexagon.
seven sides ....a heptagon.
eight sides ....an octagon.
nine sides..............a nonagon.
ten sides ....a decagon.
twelve sides..........a dodecagon.

Definitions of other terms employed in Geometry.

An *axiom* is a self evident truth, and which therefore requires no demonstration.

A *proposition* is any thing proposed to be done or demonstrated.

A *theorem* is a proposition proposed to be demonstrated.

A *problem* is a proposition in which something is proposed to be done.

A *lemma* is a preliminary proposition intended to render what follows more obvious.

A *corollary* is a consequent truth drawn immediately from a preceding proposition.

A *scholium* is a remark applied to some preceding propositions, in order to point out their relative connection, or general utility and application.

An *hypothesis* is a supposition advanced either in the enunciation of a proposition, or in the course of the demonstration.

Illustration of the symbols to be employed.

In order to render the demonstration as concise as possible, mathematicians have agreed in the adoption of certain symbols, to signify particular terms of frequent recurrence, which, without in any degree weakening the force of the argument, bring the whole subject more immediately under the eye of the reader: thus,

The sign $+$ signifies addition, and is read *plus*, so that $A + B$ is read A *plus* B, and signifies that the quantity B is to be added to the quantity A.

The sign $-$ signifies subtraction, and is read *minus*: thus $A - B$ is read A *minus* B, and implies that the quantity B is to be taken from the quantity A.

The sign $\times$ signifies multiplication, and is read *multiplied by*: thus $A \times B$, is read A *multiplied by* B, and implies that the quantity A is to be multiplied by the quantity B.

The *parenthesis* or *vinculum*, is used to reduce a quantity compounded of several others into one only: thus $A + B - C$ is sometimes included in a parenthesis thus, $(A + B - C)$; and in this form it may be considered as a single quantity, and then $(A + B - C) \times D$, and $(A + B) \times (C + D)$, signify that the quantity expressed by $(A + B - C)$, is to be multiplied by D; and that the quantity $(A + B)$ is to be multiplied by $(C + D)$.

A number placed before any quantity as $3\,B$, or $5\,(A - B)$, signifies that the quantity is to be multiplied by that number, or that it is such a multiple of the quantity as is expressed by the number: thus, the above signify *three* times B, and *five* times $(A - B)$, although in this case the sign of multiplication does not appear. In the same way we express any part of a quantity by prefixing to the quantity the fraction expressing the part. As $\frac{1}{2}\,A$, $\frac{1}{3}\,(A + B)$, &c. which signify *half* A, one-*third* of $(A + B)$, &c.

The *square* of any line AB, is denoted by $\overline{AB}^2$; the *cube* of a line by $\overline{AB}^3$, and so on.

The sign $\surd$ signifies the square *root* of a quantity: thus $\surd 2$, $\surd \overline{A \times B}$, &c. denote the square root of the number 2, or of the product $A \times B$, or which is the same, the *mean proportional* between A and B.

The sign $=$ placed between any two quantities, denotes that these quantities are equal to each other:

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thus $A = B$ and $(A - B) = B \times C$, signify that A is equal to B, and that the difference $A - B$, is equal to the product $B \times C$.

The sign $\angle$ placed between two quantities, denotes that the first of those quantities is less than the second: thus $A \angle B$, is read A is *less than* B; but when the sign is inverted, as A ⦣ B, it signifies and is to be read A is *greater than* B.

The above are all the conventional signs employed in the following book; what further symbols of this kind may be required as we proceed, will be explained in their proper places.

Axioms.

1. Things which are equal to the same thing, are equal to one another.
2. If equals be added to equals, the wholes are equal.
3. The whole is greater than its part.
4. The whole is equal to the sum of all its parts.
5. A right line may be drawn from any point to another point, and there can be but one such right line joining those two points.
6. Magnitudes, whether lines, surfaces, or solids, which coincide or fill the same space, are equal.
7. All right angles are equal to each other.

PROPOSITION I.—*Theorem.*

If one right line meet another right line, it makes the two adjacent angles taken together equal to two right angles, fig. 19.

Fig. 19 Let the line A B meet C D, the two angles A B D, A B C together, are equal to two right angles.

Let E B be perpendicular to C D, then the two angles E B C and E B D are both right angles, (def. 10:) and if A B coincide with B E, the two angles A B C, A B D, will also be both right angles; but if not, and A B falls otherwise, as in the figure, then, since A B D is equal to the sum of E B A and E B D, the two angles E B C and E B D, are equal to the three A B C, E B A and E B D; but C B E is equal to the two A B C and E B A, therefore the two angles C B A and A B D are equal to the two E B C and E B D; but these are both right angles, therefore C B A and A B D are, together, equal to two right angles.

Otherwise, by employing the conventional symbols.

Let E B be perpendicular to C D, then E B C and E B D are each right angles; consequently E B C + E B D = two right angles; and if A B coincide with E B, then A B C + A B D = two right angles.

But if not, because

$$EBC = EBA + ABC$$

we shall have

$$EBD + EBA + ABC = EBC + EBD,$$

but $EBD + EBA = ABD$;

therefore $ABC + ABD = EBC + EBD$,

but $EBC + EBD =$ two right angles;

therefore $ABC + ABD =$ two right angles.

Corollary. Hence, also, the sum of all the angles made by any number of lines meeting C D in B on the same side, is equal to two right angles.

PROPOSITION II.—*Theorem.*

If two right lines meet the extremity of another right line, so as to make the adjacent angles equal to two right angles, these two lines are in one and the same right line, fig. 20.

Fig. 20. Let the lines C B, B D meet the line A B at the point B, so as to make A B C + A B D equal to two right angles, then will C B, B D be in one and the same right line.

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For if B D be not in the same right line with C B, let B E* be in a right line with it; then by prop. 1, the two angles A B C + A B E = two right angles; but by hypothesis A B C + A B D = two right angles; therefore A B C + A B E = A B C + A B D; taking away the common angle A B C we shall have A B E = A B D; a part equal to the whole, which is impossible; therefore B E is not in the same right line with C B; and the same may be demonstrated of every line but B D. Therefore B D is in the same right line with C B.

PROPOSITION III.—*Theorem.*

If two right lines cut each other, the vertical or opposite angles are equal, fig. 21.

Let A B and C D cut each other in E, then will A E C = B E D and C E B = A E D. Because the right line C E meets the right line A B, the two angles, Fig. 21.

$AEC + CEB =$ two right angles, (prop. 1,)

so also $CEB + BED =$ two right angles;

therefore $AEC + CEB = CEB + BED$;

taking away the common angle C E B, there remains the angle A E C equal to the angle B E D; and in the same way it may be shown that C E B is equal to A E D.

Cor. 1. The sum of the four angles formed about the point E is equal to four right angles; for C E A + A E D = two right angles, and C E B + D E B = two right angles; therefore the four angles C E A + A E D + C E B + D E B = four right angles.

Cor. 2. Hence, also, the sum of all the angles that can be made about any given point, is equal to four right angles.

Cor. 3. When one of the four angles formed by the intersection of two right lines is a right angle, the other three angles are also right angles.

PROPOSITION IV.—*Theorem.*

If two triangles have two sides of the one equal to two sides of the other, each to each, and have the angles included by these sides also equal; the triangles will be equal, and have all the corresponding sides and angles equal, fig. 22.

Let the two triangles A B C, D E F have the side A C = D F, C B = E F, and the angle C = the angle F; then will the side A B = D E, &c. as stated in the proposition. Fig. 22.

For the triangle A B C may be conceived to be applied to D E F, so that the point C falls upon F, and the side A C upon F D to which it is equal; consequently the point A will coincide with the point D. And, because the angle C = F, the side C B will fall upon F E, and being equal to it, the point B will coincide with E, and therefore the side A B with D E (ax. 5;) thus the two triangles coinciding, will be equal to each other (ax. 6,) and have A B = D E, A = D and B = E.

PROPOSITION V.—*Theorem.*

If two triangles have two angles of the one equal to two angles of the other, each to each, and the side adjacent

* The line B E is omitted in the plate by the engraver.

to these angles also equal, the triangles will be equal, and have the other corresponding sides and angles equal, fig. 22.

Let the angle A = D, B = E, and the side A B = D E, then will the triangle A B C = D E F. For the side A B may be applied to the side D E, so that A falls on D, and B on E; and since the angle A = D, the side A C will fall upon D F and B C on E F, and consequently the point C will fall upon F, and the two triangles will coincide or fill the same space, and will therefore be equal to each other, (ax. 6;) that is, the side A C = D F, B C = E F, and the angle C = F.

Proposition VI.—*Theorem.*

If two of the sides of a triangle are equal to each other, the angles opposite those sides will also be equal to each other, fig. 23.

Fig. 23. That is, if A C = C B, then will A = B. Conceive the angle C to be bisected by the line C D; then in the two triangles A C D and C B D, there are two sides A C, C D, equal to the two C B, C D, each to each, and the included angles equal; consequently the two triangles A C D and B C D are also equal, and the angle A = the angle B, (prop. 4.)

Cor. 1. The triangles A C D, D C B are equal, and the side A D = D B, and the angle C D A = C D B, (prop. 4;) hence the line which bisects the vertical angle of an isosceles triangle also bisects the base, and is perpendicular to it, (def. 10.)

Cor. 2. If the three sides of a triangle are equal to each other, the three angles will also be equal to each other.

Proposition VII.—*Theorem.*

If a triangle have two of its angles equal, the sides opposite to those angles will also be equal, fig. 45.

Fig. 45. That is, if the angle A = B, then will A C = B C. First let it be granted that a point may be found in B C, or B C produced, fig. 45, such that a line drawn from it to A shall be equal to its distance from B, and if C be not that point, let it be some other point as D; join D A, then because A D = D B; the angle D A B = D B A; but D B A or C B A = C A B; therefore D A B = C A B, a part to the whole, which is impossible; and the same may be shown of every point in B C except C: therefore C A = B C.

Cor. Hence if the three angles of a triangle be equal to each other, the three sides will also be equal.

Proposition VIII.—*Theorem.*

Any two sides of a triangle are greater than the third side, fig. 24.

Fig. 24. Let A B C be a triangle, any two of its sides (A C + C B) $>$ A B. For A B being a right line, it is the shortest distance between the two points A and B, (def. 3;) therefore (A C + C B) $>$ A B; and the same may be demonstrated of any other two sides.

Proposition IX.—*Theorem.*

If from a point within a triangle, there be drawn two right lines to the extremities of one of its sides, these two lines taken together will be less than the sum of the other two sides of the triangle, fig. 25.

Fig. 25. Let A B C be a triangle, and O a point taken within it; join A O, O B, then will (A O + O B) $<$ (A C + C B).

Produce A O to E; then by the above proposition (O E + E B) $>$ O B; add to each A O, then (A E + E B) $>$ (A O + O B), but (A C + C E) $>$ A E. Much more therefore is (A C + C E + E B) or (A C + C B) $>$ (A O + O B).

Proposition X.—*Theorem.*

If there be two triangles which have two sides of the one equal to two sides of the other, each to each, but the angle contained by the two sides of the one, greater than the angle contained by the two sides of the other, that which has the greater angle will have the greater base, fig. 26, 27, 28.

Let A B = D E, and A C = D F, but B A C $>$ E D F; then will B C $>$ E F. Make the angle C A G = F D E and A G = D E; then will G C = E F, (prop. 4.) Now the point G will either fall without the triangle A B C, or in the side B C, or within the triangle A B C. Fig. 26, 27, 28.

First let it fall without the triangle A B C, as in fig. 26, then (A I + B I) $>$ A B
and (C I + I G) $>$ G C } prop. 8;
therefore (A I + B I + C I + I G), or (A G + B C) $>$ (A B + G C), but A G = A B; therefore B C $>$ G C, or B C $>$ E F.

If the point G fall in B C, as in fig. 27, it is obvious that B C $>$ G C, or greater than its equal E F.

Lastly, if G fall within the triangle A B C, as in fig. 28, (B A + B C) $>$ (A G + G C) (prop. 9;) but B A = A G, therefore B C $>$ G C, or greater than its equal E F.

Proposition XI.—*Theorem.*

The greater side of every triangle is opposite the greater angle; and the greater angle is opposite the greater side, fig. 29.

Let the angle C B A of the triangle A B C be $>$ A, then will A C $>$ B C. Make the angle A B D = B A D, then will A D = B D, (prop. 7.) Now (B D + D C) $>$ B C, (prop. 8,) but (B D + D C) = (A D + D C) = A C; therefore A C $>$ B C. Fig. 29.

Next, let C A be greater than B C, then A B C $>$ B A C. For if it be not greater, it must be either equal to it or less; but it is not less, because then B C $>$ A C, (by the above,) which it is not, neither can it be equal; because then A C = B C, (prop. 7,) which it is not; being therefore neither equal nor less it must be greater.

Proposition XII.—*Theorem.*

If the three sides of one triangle are equal to the three sides of another triangle, each to each, the triangles will be equal, fig. 30.

Let A B = D E, A C = D F, and B C = E F; then will A = D; for if A $>$ D, then B C $>$ E F, (prop. 10,) but it is not; and if A $<$ D, then B C $<$ F E, but it is not; therefore A being neither greater nor less than D, it must be equal to it; and since A B, A C are equal to D E, D F, each to each, and the included angles being also equal, the triangles are equal, and have all their corresponding angles also equal, (prop. 4.) Fig. 30.

Proposition XIII.—*Theorem.*

If one side of a triangle be produced, the exterior angle

will be greater than either of the interior and opposite angles, fig. 31.

Fig. 31. Let the side A B be produced to D,* the angle C B D is greater than either of the angles A C B or C A B. Conceive C B to be bisected in E; join A E, and produce it to F, making E F = E A, and join F B; then the two sides C E, E A are equal to the two B E, E F, each to each, and the angle A E C = F E B, (prop. 3;) therefore the angle E B F = E C A, (prop. 4;) but C B D 7 E B F, therefore it is also greater than E C A or B C A; and in the same way if C B be produced, and A B bisected, it may be shown that A B G, or its equal C B D, is greater than C A B.

Proposition XIV.—*Theorem.*

Any two angles of a triangle are together less than two right angles, fig. 32.

Fig. 32. That is, A + C, or A + B, or B + C, are together, less than two right angles. Let A C be produced to D, then by the last proposition, the angle C B A $\angle$ B C D; to each add B C A, then (C B A + B C A) $\angle$ (B C D + B C A); but B C D + B C A = two right angles, (prop. 1;) therefore C B A + B C A $\angle$ two right angles; and the same may be shown of any other two angles of the triangle A B C.

Proposition XV.—*Theorem.*

Of all lines that can be drawn from a point to a line, the perpendicular is the shortest, and of the others, that which is nearer to the perpendicular is less than the one more remote, and from the same point to the same line there can be drawn but two lines equal to each other, one on each side of the perpendicular, fig. 33.

Fig. 33. Let I B be any right line, and C a point beyond it; and let C D be perpendicular to I B; let also C E, C G be any other lines, then will C D be the shortest, and E C less than C G. Produce C D to H, making D H = C D, and join E H, G H. Because C D E, is a right angle, E D H is a right angle, (prop. 3, cor. 3;) and in the triangles C E D, H E D, the two sides E D, C D are equal to the two E D, D H, each to each, and the angle C D E = H D E; therefore E H = E C, (prop. 4,) and in the same manner it may be shown that G H = G C. Now (E C + E H) 7 C H, (prop. 6;) and (C G + G H) 7 (E C + E H) (prop. 9;) or since C D = D H, E C = E H, and C G = G H; 2 E C 7 2 C D, and 2 C G 7 2 E C; consequently E C 7 C D, and C G 7 E C; but E C is any line except the perpendicular, therefore the perpendicular is shorter than any other line drawn from C to the line I B; and of the rest, E C is less than C G: C G than C I, and so on. Take F D = D E, and join C F, then C F = C E, (prop. 4,) and it is the only line that can be drawn from C to I B, that is equal to C E. For any line falling between D and F, will be less than C F or C E, (by the foregoing,) and any line falling beyond F, will be greater than C F or C E; therefore C F is the only line that can be drawn from C to the line I B, that is equal to C E; that is, there can be but two equal lines, one on each side of the perpendicular.

* The line should have been produced on the side towards B, instead of A as in the figure.

Proposition XVI.—*Theorem.*

If two right angled triangles have their hypothenuses, and one of their other sides equal, each to each, the triangles will be equal, or have their other sides and angles equal, fig. 34.

Fig. 34. Let the triangles A B C, D E F be right angled at B and E, and have A C = D F, and C B = E F, then will the triangles be equal. For apply A B C to D E F, so that A B may fall on D E, and the point B upon E; because the angle B = E, the line B C will fall upon E F, and because C B = E F, the point C will fall on F, and the line C A upon F D. For if A C do not fall on D F, let it fall in some other direction, and meet the base D E, then will this line be equal to D F; therefore we shall have two lines drawn from a point above a line, to that line, on the same side of the perpendicular, equal to each other, which is impossible by the last proposition; A C therefore cannot but coincide with F D, and consequently the two triangles are equal to each other, or they have all their corresponding sides and angles equal.

Proposition XVII.—*Theorem.*

Parallel lines will not meet when produced to any distance whatever, fig. 35.

Fig. 35. Let A B and C D be two parallel lines; they will not meet when produced. In one of them as A B, take any two points E, F, and let fall the perpendiculars E G, F H: then E G will be equal to F H, (def. 12;) in the same way it may be shown, that any point whatever being taken in A B, its perpendicular distance from C D will be equal to F H; consequently no point in A B can fall in C D; that is these lines can never meet, however far they may be produced.

Proposition XVIII.—*Theorem.*

A line which is perpendicular to one of two parallel lines is also perpendicular to the other, fig. 36.

Fig. 36. Let E F be perpendicular to A B, one of two parallel lines A B and C D; it will also be perpendicular to the other: for if E F be not perpendicular to C D, let F G* be perpendicular to it; then E F and F G are equal to each other, (def. 12.) Hence a line drawn from a point to a line, and perpendicular to it as E F, is equal to F G more remote, which is impossible, (prop. 16.) F G therefore is not perpendicular to C D, and the same may be shown of every line except F E; therefore F E, which is perpendicular to A B, is also perpendicular to C D.

Proposition XIX.—*Theorem.*

If two parallel lines be cut by a third line, the two alternate angles will be equal to each other, and the outward angle will be equal to the inward angle on the same side, and the two interior angles on the same side will be together equal to two right angles, fig. 37.

Fig. 37. Let the parallel lines A B, C D be cut by the line G H, then will E F C = B E F, and G E B = E F D, also B E F + E F D = two right angles. First if G H be perpendicular to A B, then the truth of the proposition is manifest from the last; and if it be not, draw E I perpendicular to C D, and F K perpendicular to

* The letter G is omitted by the engraver.

A B: then will E I F and E K F be two right angled triangles, in which the hypothenuse E F is common, and the side E I of the one equal to K F of the other, (def. 12:) therefore these triangles are equal, and the angle I F E = K E F, (prop. 16;) but these are alternate angles: also I E F = E F K, to each of these add I E A and K F D, which are equal, being both right angles, and we shall have A E I + I E F = E F K + K F D, or A E F = E F D, which are the two other alternate angles.

Again A E F = G E B, (prop. 3;) therefore G E B = E F D, that is, the outward angle is equal to the inward angle on the same side: to each of these add B E F, then will the two G E B + B E F = B E F + E F D; but G E B + B E F = two right angles, (prop. 1;) therefore B E F + E F D = two right angles; that is, the interior angles on the same side are together equal to two right angles.

PROPOSITION XX.—*Theorem.*

If a line falling upon two other lines make the alternate angles equal to each other, those lines are parallel, fig. 38.

Fig. 38. Let H I fall on the two lines A B, C D, and make the angle B E F = E F C, then will A B and C D be parallel; for if A B be not parallel to C D, let some other line E G be parallel to C D; then, because E G is parallel to C D, the angle G E F = E F C, (prop. 19;) but B E F = E F C; therefore G E F = B E F, a part equal to a whole, which is impossible; therefore E G is not parallel to C D, and the same may be shown of every line passing through E, except A B; consequently A B is parallel to C D.

PROPOSITION XXI.—*Theorem.*

If a line falling upon two other lines make the outward angle equal to the interior angle on the same side, these two lines are parallel, fig. 38.

Let H I fall upon A B, C D, and make the angle H E B = E F D, then A B is parallel to C D. For if not, let some line, as E G be parallel to C D, then H E G = E F D, (prop. 19;) but H E B = E F D; therefore H E B = H E G, a part to the whole, which is impossible; consequently E G is not parallel to C D, and the same may be shown of every other line passing through E, except A B; therefore A B is parallel to C D.

PROPOSITION XXII.—*Theorem.*

If one line falling upon two others make the sum of the two interior angles upon the same side equal to two right angles, these lines are parallel, fig. 38.

Since, by hypothesis B E F + E F D = two right angles, and since B E F + A E F = two right angles, (prop. 1;) it follows that E F D = A E F, which are alternate angles; therefore A B is parallel to C D, (prop. 20.)

PROPOSITION XXIII.—*Theorem.*

Lines which are parallel to the same line are parallel to each other, fig. 39.

Fig. 39. Let A B and C D be both parallel to I H; they will be parallel to each other: draw any line cutting each of the three lines as E F K; because A B is parallel to I H the angle B E F = E F I, (prop. 19;) and, because I H and C D are parallel, the angle H F K = F K C; but H F K = E F I, (prop. 3;) therefore F K C and B E F are both equal to E F I: they are therefore equal to each other, and they are alternate angles, therefore A B and C D are parallel, (prop. 20.)

PROPOSITION XXIV.—*Theorem.*

The three angles of every triangle taken together are equal to two right angles, fig. 40.

Let A B C be a triangle, the three angles A + B + C = two right angles. Produce A C to D, and draw C E parallel to A B: then since A B, C E are parallel and B C meets them, the alternate angles A B C and B C E are equal, (prop. 19;) and because these parallels are also cut by A C, the angle B A C = E C D, (prop. 19;) consequently A B C + B A C = B C E + E C D = B C D; to each of these equals, add B C A; then A B C + B A C + B C A = B C D + B C A; but B C D + B C A = two right angles; therefore A B C + B A C + B C A = two right angles. Fig. 40.

Cor. It follows from this, that if two lines A B, A C are cut by a third line B C, so as to make the two interior angles on the same side as A B C + B C A less than two right angles, these lines produced will meet and form a triangle, of which the third angle A, shall be equal to the difference between two right angles, and the sum of the two interior angles B and C.

PROPOSITION XXV.—*Theorem.*

In every polygon the sum of all the interior angles is equal to twice as many right angles as the figure has sides, wanting four right angles, fig. 41.

Let A B C D E be any polygon; from a point O within it, draw the lines O A, O B, O C, &c., to every angle of the figure which will divide the polygon into as many triangles as the figure has sides; now the sum of the three angles of every triangle being equal to two right angles, (prop. 24;) the sum of all the angles of all the triangles is equal to twice as many right angles as the figure has sides; but of these angles those about the point O, which are equal to four right angles, (prop. 3, cor. 1,) are angles of the triangles, but are not angles of the polygon; therefore the angles of the polygon alone are equal to twice as many right angles as the figure has sides, wanting four right angles. Fig. 41.

PROPOSITION XXVI.—*Theorem.*

If each of the sides of any polygon be produced, the sum of all the outward angles is equal to four right angles, fig. 42.

Let the sides A B, B C, C D, &c. of the polygon A B C D, &c. be produced, then the sum of each inward and outward angle is equal to two right angles, (prop. 1;) therefore the sum of all the outward and inward angles, is equal to twice as many right angles as the figure has sides. But the sum of all the inward angles and four right angles, is equal to twice as many right angles as the figure has sides, (by the last proposition;) therefore the sum of all the inward and outward angles is equal to all the inward angles, and four right angles; taking away all the inward angles from each sum, there remains the sum of all the outward angles equal to four right angles. Fig. 42.

PROPOSITION XXVII.—*Theorem.*

The opposite sides and angles of any parallelogram are respectively equal to each other: that is, the angles to the angles, and the sides to the sides; and the diagonal divides the parallelogram into two equal triangles, fig. 43.

Fig. 43. Let A B C D be a parallelogram, then will A D = B C, A B = D C, the angle A = C, the angle A B C = A D C; and the triangle B A D = B C D. Draw the diagonal D B: because A D is parallel to B C, the angle A D B = C B D: and for the same reason the angle C D B = A B D, (prop. 19;) therefore A D B + B D C = A B D + D B C, or the angle A B C = A D C which are two of the opposite angles. Again, because the two triangles A B D, and C D B, have two angles equal, each to each, and the side adjacent to them common; they will be equal in all respects, (prop. 5,) and will have the angle A = C: which are the other two opposite angles, also the side A D = B C, A B = D C; and the triangle A D B = the triangle D C B.

Cor. Hence two parallels comprised between two other parallels are equal to each other.

PROPOSITION XXVIII.—*Theorem.*

Lines which join the extremities of two equal and parallel lines towards the same parts, are themselves equal and parallel, fig. 43.

Let A B, D C join the extremities of the equal and parallel lines A D, B C, then will D C be equal and parallel to A B. Draw the diagonal D B, then the angle A D B = D B C, (prop. 19,) and A D = B C, and B D is common; therefore A B = D C, (prop. 4,) and the angle A B D = C B D, (prop. 4.;) but these last are alternate angles, therefore A B is parallel to D C, (prop. 20.)

PROPOSITION XXIX.—*Theorem.*

A quadrilateral whose opposite sides are equal, is a parallelogram, that is, if A D = B C *and* A B = D C; *the figure* A B C D *is a parallelogram,* fig. 43.

Draw the diagonal B D: then in the two triangles A B D and D B C, the three sides of the one are equal to the three sides of the other, each to each; therefore the corresponding angles are equal, (prop. 12;) that is, A D B = D B C, and C D B = A B D; therefore A D is parallel to B C, and A B to D C, (prop. 20.)

PROPOSITION XXX.—*Theorem.*

The two diagonals of any parallelogram bisect each other, fig. 44.

Fig. 44. For the diagonals being drawn, the angle D A O = B C O, and A D O = C B O, (prop. 19;) also A D = B C: therefore A O = O C, and D O = O B, (prop. 5,) the diagonals are therefore bisected in O.*

BOOK II.

On Ratios and Proportions.

DEFINITIONS.

1. RATIO is the relation of two magnitudes of the same kind to each other, with respect to quantity.

* The letter O is omitted in the figure.

The relations of magnitudes, with respect to quantity, may be expressed by numbers, either exactly or approximatively; and in the latter case, the approximation may be brought within less than any assignable difference.

Thus, of two magnitudes, one of them may be conceived to be divided into some number of equal parts, each of the same kind as the whole; and one of those parts being considered as an unit, of measure, the magnitude may be expressed by the number of units it contains. If then the other magnitude contain a certain number of those units, this also may be expressed by the number of its units, and the two quantities are said to be *commensurable*. But if, whatever unit be assumed for the measure of the first magnitude, the second magnitude do not contain an exact number of such units, then the two magnitudes are said to be *incommensurable*, and their relation, with respect to quantity, cannot be correctly expressed in numbers; but the relation between the first magnitude and a third, may be expressed in numbers, and the third magnitude be such as to differ from the second, by a quantity less than any that can be assigned; for it is obvious, that a third magnitude may be found commensurable with the first, which shall differ from the second, by less than the measuring unit; and as the measuring unit may be less than any assignable quantity, the difference between the second magnitude, (which is incommensurable with the first,) and the third, (which is commensurable with it,) may be so taken as to differ from each other, by less than any assignable quantity.*

Hence, it appears, that when magnitudes are commensurable, we may always express their relation, or ratio, numerically; and that when they are incommensurable, we may still approximate so nearly to their correct ratio, by means of numbers, that the ratio assumed shall differ from the actual ratio of the incommensurable magnitudes, by a quantity less than any that can be assigned. Therefore, of two magnitudes, A and B, we shall conceive A to be divided into some number, M units, each equal to A', or $A = M A'$; and B as equal to N such units, or $B = N A'$, M and N being integral numbers; and, consequently, the ratio of A to B will be expressed by the ratio of $M A'$ to $N A'$. In the same manner the ratio of any other two magnitudes C and D, may be expressed by $P C'$ to $Q C'$, P and Q being also integral numbers;

* In order to connect the doctrine of commensurable quantities with incommensurables, or magnitudes generally with numbers, it must be assumed that whatever relations subsists between A, B, C, D (in which A and B, C and D are commensurables;) subsists also between A, M, C, N, (in which A and M, and C and N are incommensurables) provided B and D be such as to differ from M and N respectively, by quantities which are less than any quantity that can be assigned. Authors have invented a variety of ingenious devices to hide this transition; but, however the defect may be concealed on a superficial view of the subject, it will always be found, upon a closer investigation, to be admitted or taken for granted, and we have preferred stating the full amount of the defect to hiding it under a specious disguise. Euclid's doctrine of ratios and propositions is perhaps unobjectionable, and applies equally to commensurables and incommensurables; but as soon as we have occasion to apply it to numbers, the difficulty again appears. It cannot, for example, be shown that the proper numerical measure of a rectangle is the product of its two sides, without admitting the principle advanced above, or one tantamount to it, and equally objectionable.

 and since A′ and C′ are each units of their respective kinds, these ratios are simply those of M to N, and of P to Q.

2. Ratios are said to be equal to each other, when the number expressing the second term divided by the first, is equal to the number expressing the fourth term divided by the third; thus, if $\frac{N}{M} = \frac{Q}{P}$; then the ratio of M to N is said to be equal to the ratio of P to Q; and these four quantities are then said to be *proportional*.

3. When magnitudes or quantities are in proportion, they are expressed thus, $M : N :: P : Q$, and they are read, "M is to N as P is to Q."

4. Of four proportional quantities, the first and third are called *antecedents*, and the second and fourth *consequents*.

5. Three magnitudes are proportionals, when the first has the same ratio to the second, that the second has to the third, and then the middle term is said to be a *mean proportional* between the other two.

6. Of four proportional quantities, the last is said to be a *fourth proportional* to the other three taken in their order.

7. Magnitudes are said to be in proportion, by *inversion* or *inversely*, when the consequents are taken as antecedents, and the antecedents as consequents.

8. Magnitudes are in proportion, by *alternation* or *alternately*, when the antecedent is compared with the antecedent, and the consequent with the consequent.

Proposition I.—*Theorem.*

When four quantities are in proportion, the product of the two extremes is equal to the product of the two means.

Let A, B, C, D be four quantities in proportion, and $M : N :: P : Q$ be their numerical representatives; then will $M \times Q = N \times P$; for since they are in proportion $\frac{Q}{P} = \frac{N}{M}$, therefore $Q = \frac{N \times P}{M}$, and $M \times Q = N \times P$.

Cor. Hence if there be three proportional quantities, the product of the extremes is equal to the square of the mean, (def. 5.)

Proposition II.—*Theorem.*

If the product of two quantities be equal to the product of two other quantities, two of them will be the extremes, and the other two the means of a proportion.

Let $M \times Q = N \times P$; then will $M : N :: P . Q$. For if P have not to Q the ratio which M has to N, let P have to Q′, (a number less than Q,) the same ratio that M has to N; that is, let $M : N :: P : Q'$; then $M \times Q' = N \times P$, or $Q' = \frac{N \times P}{M}$; but $Q = \frac{N \times P}{M}$, therefore Q′ is not less than Q; and in the same way it may be shown, that it is not greater; consequently $Q' = Q$, and the four quantities are proportional; that is, $M : N :: P : Q$.

Proposition III.—*Theorem.*

If four quantities be in proportion, they will be in proportion when taken alternately.

Let M, N, P, Q be the numerical representatives of the four quantities in proportion; so that

$$M : N :: P : Q, \text{ then will also}$$
$$M : P :: N : Q.$$

Because $M : N :: P : Q$, $M \times Q = N \times P$, or $M \times Q = P \times N$; but $M \times Q$, and $P \times N$, are the products of the extremes and means of the terms M, P, N, Q; and they are equal to each other; therefore

$$M : P :: N : Q.$$

Proposition IV.—*Theorem.*

If four quantities be in proportion, they will be in proportion when taken inversely.

Let $M : N :: P : Q$, then will also

$$N : M :: Q : P;$$

for the first four terms being in proportion,

$$M \times Q = N \times P, \text{ or } N \times P = M \times Q.$$

But $N \times P$, and $M \times Q$, are the products of the extremes and means of the four quantities N, M, Q, P; and these products being equal,

$$N : M :: Q : P.$$

Proposition V.—*Theorem.*

If four quantities be in proportion, they will be in proportion by composition or division.

Let, as before, M, N, P, Q be the numerical representatives of the four quantities, so that

$$M : N :: P : Q, \text{ then will}$$
$$\overline{M \pm N} : M :: P \pm Q : P;$$

for by the first $M \times Q = N \times P$, or $N \times P = M \times Q$, to each add $M \times P$; then

$$\overline{M \times P} \pm \overline{N \times P} = \overline{M \times P} \pm \overline{M \times Q},$$

or $$\overline{M \pm N} \times P = \overline{P \pm Q} \times M.$$

But $\overline{M \pm N}$ and P, are the extremes, and $\overline{P \pm Q}$ and M, the means, of the four quantities in the second line, and the product of these being equal, the quantities are in proportion; that is,

$$\overline{M \pm N} : M :: \overline{P \pm Q} : P.$$

Proposition VI.—*Theorem.*

Equimultiples of any two quantities, have the same ratio as the quantities.

Let M and N be any two quantities, and m any integral number; then will

$$m M : m N :: M : N;$$

for $$m M \times N = m N \times M = m M N.$$

Proposition VII.—*Theorem.*

Of four proportional quantities, if there be taken any equimultiples of the two antecedents, and any equimultiples of the two consequents, the four resulting quantities will be proportionals.

Let M, N, P, Q be the numerical representatives of four quantities in proportion; and let m and n be any numbers whatever, then will

$$m M : n N :: m P : n Q.$$

Because $M : N :: P : Q$, $M \times Q = N \times P$

therefore $m M \times n Q = n N \times m P$;

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and these being the product of the extremes and means, of mM, nN, mP, nQ, they are proportionals, or $mM : nN :: mP : nQ$.

PROPOSITION VIII.—*Theorem.*

Of four proportional quantities, if the two consequents be either augmented or diminished by quantities that have the same ratio as the antecedents, the resulting quantities and the antecedents will be proportionals.

Let $M : N :: P : Q$ be the four quantities; and let $M : P :: m : n$, then will

$$M : N \pm m :: P : Q \pm n.$$

Because $M : N :: P : Q$, $M \times Q = N \times P$;
and because $M : P :: m : n$, $M \times n = m \times P$;
therefore $M \times Q \pm M \times n = N \times P \pm m \times P$;
or $M \times \overline{Q \pm n} = P \times \overline{N \pm m}$;
hence $M : N \pm m :: P : \overline{Q \pm n}$.

PROPOSITION IX.—*Theorem.*

If any number of quantities be proportionals, any one antecedent will be to its consequent as the sum of all the antecedents is to all the consequents.

Let $M : N :: P : Q :: R : S$, &c. be quantities in proportion, then will

$$M : N :: \overline{M + P + R} : \overline{N + Q + S}.$$

Because $M : N :: P : Q$, $M \times Q = N \times P$;
and because $M : N :: R : S$, $M \times S = N \times R$;
therefore $M \times Q + M \times S = N \times P + N \times R$,
to each add $M \times N$, or $N \times M$,
then $M \times N + M \times Q + M \times S = N \times M + N \times P + N \times R$,
or $M \times \overline{N + Q + S} = N \times \overline{M + P + R}$;
therefore $M : N :: \overline{M + P + R} : \overline{N + Q + S}$.

PROPOSITION X.—*Theorem.*

If two magnitudes be each increased or diminished by like parts of each, the resulting quantities will have the same ratios as the first two.

Let M and N be any magnitudes, and $\frac{M}{m}$ and $\frac{N}{m}$ be like parts of each, then will

$$M \pm \frac{M}{m} : N \pm \frac{N}{m} :: M : N.$$

For it is obvious that

$$\left(M \pm \frac{M}{m}\right) \times N = \left(N \pm \frac{N}{m}\right) \times M,$$

each being equal to $M \times N \pm \frac{M \times N}{m}$. Consequently the four quantities are proportional.

PROPOSITION XI.—*Theorem.*

If four quantities be proportionals, their squares or cubes will also be proportionals.

Let $M : N :: P : Q$,
then will $M^2 : N^2 :: P^2 : Q^2$,
and $M^3 : N^3 :: P^3 : Q^3$.
For since $M \times Q = N \times P$
$M^2 \times Q^2 = N^2 \times P^2$
$M^3 \times Q^3 = N^3 \times P^3$, &c.

and, therefore,

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$$M^2 : N^2 :: P^2 : Q^2$$
$$M^3 : N^3 :: P^3 : Q^3, \text{ \&c.}$$

Book III.

Cor. In the same way it may be shown, that any power or roots of proportional quantities are proportionals.

PROPOSITION XII.—*Theorem.*

If there be four proportional quantities, and four other proportional quantities, the product of the corresponding terms will be proportionals.

Let $M : N :: P : Q$
and $R : S :: T : V$,
then will $\overline{M \times R} : \overline{N \times S} :: \overline{P \times T} : \overline{Q \times V}$;
for since $M \times Q = N \times P$, and $R \times V = S \times T$
$M \times Q \times R \times V = N \times P \times S \times T$,
or $\overline{M \times R} \times \overline{Q \times V} = \overline{N \times S} \times \overline{P \times T}$;
therefore $\overline{M \times R} : \overline{N \times S} :: \overline{P \times T} : \overline{Q \times V}$.

BOOK III.

Of the circle, and the measure of angles.

DEFINITIONS.

1. THE *circumference* of a circle is a curved line A B D, every where equally distant from a point within C, called the centre, fig. 46. Fig. 46.

2. The *circle* is the superficial space, included within the circumference. These terms are frequently confounded; the circumference being sometimes called the circle. Thus, we say, describe a circle from a given point, &c., and not describe the circumference of a circle; but the distinction is easily made, the one being a line, and the other the space included within it.

3. The *radius* of a circle is any right line drawn from the centre and terminated in the circumference, as C A, C B, C D; consequently all the radii of the same circle are equal to each other: and

The *diameter* of a circle is any right line passing through the centre and terminating at each extremity in the circumference, as A D. Hence, a diameter is equal to double the radius; and hence the radius is sometimes called the *semi-diameter*.

4. An *arc* of a circle is any portion of the circumference, as A B or B D.

5. The *chord* or *subtence* of an arc is any right line, as A B, joining the extremities of the arc; and the space included within the chord and the arc is called a *segment*. The same chord is common to two arcs and two segments; but unless the contrary be stated, it is always to be understood that the less arc, or less segment, is spoken of in these cases.

6. A *sector* of a circle is the space included between any two radii and the arc comprised between them, as A C B or B C D.

7. A *line* is said to be *inscribed in a circle* when its two extremities are in the circumference, as A B.

8. An *angle* is *inscribed in a circle*, or *contained* in it, when it is comprised between two chords meeting at a point in the circumference, as B A D.

9. A *triangle*, or any right lined figure, is said to be *inscribed in a circle*, when all the angular points of the

Geometry. former are in the circumference of the latter, as ABC, ABCD, fig. 48.

Fig. 48. 10. A *secant* is any line which cuts the circumference of the circle in two points, as AB, fig. 49.

Fig. 49. 11. A *tangent* is any right line which touches the circumference in one point only, as CD, fig. 49; and the touching point M is called the *point of contact*.

12. A *rectilineal figure* is said to be *circumscribed* about a circle, or the *circle inscribed* in it, when all the sides of the former are tangents to the circle,

Fig. 50. fig. 50.

PROPOSITION I.—*Theorem.*

A diameter divides the circle and its circumference into two equal parts, fig. 51.

Fig. 51. Let AB be a diameter, it divides the circle into two equal parts. For conceive the semicircle ABD to be applied upon ABC, so that the diameter AB may be common; then will the circumferences also coincide. For if they do not, from the centre O draw the line OEF; then OF $>$ OE; but OF and OE, being both radii, are equal, (def. 3, book iii.:) they are therefore both equal and unequal at the same time, which is impossible: that is, OE is not unequal to OF; and the same may be shown of any other points; the circumferences therefore coincide, and are equal to each other, as are also the two segments; and each of them is called a semicircle.

PROPOSITION II.—*Theorem.*

Any chord in a circle which does not pass through the centre is less than the diameter, fig. 52.

Fig. 52 Let AB be a diameter, and DE a chord not passing through the centre, DE $<$ AB. Let C be the centre of the circle; and join CD, CE: then DE $<$ (DC + CE) (prop. 8, book i.) but DE + CE = AB; therefore DE $<$ AB.

PROPOSITION III.—*Theorem.*

A right line cannot cut a circle in more than two points.

If it were possible for a right line to cut a circle in more than two points, lines drawn from the centre to each of these points would be equal to each other, (def. 3, book iii.) which is impossible; because from a point there cannot be drawn to a line more than two lines which are equal to one another, (prop. 15, book i.) Therefore a right line cannot cut a circle in more than two points.

PROPOSITION IV.—*Theorem.*

In the same, or in equal circles, equal arcs are subtended by equal chords, and equal chords by equal arcs, fig. 53.

Fig. 53. Let AMB, DNF, be equal circles, and AB, DF equal chords; then will the arc AMB = DNF. Let C and E be the centres of the two circles, and join AC, CB, DE and EF; then in the two triangles ACB, DEF, the three sides of the one are equal to the three sides of the other, each to each, and consequently the triangles also are equal, (prop. 12, book i.;) and if the circle AMB be applied to the circle DNF, so that the point or centre C falls upon E, and the line or radius AC upon ED, the radius CB will fall upon EF, (because the angle C = E,) and the points A and B will coincide with E and F; the line AB with DF, and are AMB with DNF. Book III. For if these latter do not coincide let them be situated in some other way, as in the figure, and join EG, cutting the arc DNF in H. Then DE = EG, being radii of the same circle; and for the same reason DE = EH; therefore EG = EH a part to the whole, which is absurd: and the same may be shown of any point that is not in the arc DNF: that is, no point in the arc AMB, falls out of the arc DNF; consequently these arcs coincide and are equal to each other. Next let the arc AMB = DNF; then will the chord AB = DF.

For if AB be not equal to DF, let AI be equal to DF; then, because DF and AI are equal chords in equal circles, the arcs subtended by these chords are equal, that is the arc AMI = DNF, but AMB = DNF; therefore AMI = AMB, the less to the greater, which is absurd. Therefore the arc AMB is not unequal to DNF; that is, it is equal to it.

PROPOSITION V.—*Theorem.*

In equal circles equal angles at the centre are subtended by equal arcs, and equal arcs subtend equal angles; and when the arcs are unequal the angles will have the same ratio to each other which the arcs have, fig. 54.

Let AMB and DNF be equal arcs of equal circles, Fig. 54. and let C and E be the centres; then if the angle C = E, the arc AMB = DNF. Because the circles are equal, AC = DE, and CB = EF, and the angle ACB is equal to DEF, therefore the base or chord AB = DF, (prop. 4, book i.;) and therefore also the arc AB = DF, (prop. 4, book iii.;) that is equal angles at the centre, in equal circles, are subtended by equal arcs.

Again, if the arc AB = DF, then will the angle C = E.

Because the arc AB = DF, the chord AB = DF, (prop. 4, book iii.;) and the three sides of the triangle ACB are equal to three sides of the triangle DEF, each to each, and therefore the angle C = E; that is, in equal circles equal arcs subtend equal angles. Next, let the arcs MN and PQ of the equal circles MON, PQR be unequal, then will the arc MN be to PQ, as the angle MON to PRQ. Conceive the arc MN to be divided into any number of equal parts Ma, ab, bc, cN, making Mo the measuring unit of the arc MN, and join Oa, Ob, Oc. Then because the arcs Ma, ab, &c. are equal, the angles Moo, aob, &c. are all equal to each other, and any one of them may be taken as the measuring unit of the angle MON. From P towards Q, on the arc PQ, apply the measuring unit $Pa = Ma$ till it at length either coincide with Q, or fall beyond it as at f, making Qf less than Pa or Mo; and join Ra, Rb, Rc, &c., dividing PRf into the equal angles PRa, oRb, &c. each equal to the angle MOa.

Thus the angles MON will be the same multiple of MOa, as the arc MN is of Ma; and in the same manner the angle PRf is the same multiple of MOa as PF is of Ma; these quantities will therefore be to each other as the number of units in each; that is,

$$MN : Pf :: MON : PRf.$$

But the arc Pf may be made to approach nearer to PQ, and the angle PRf nearer to PRQ than any assignable difference, by reducing the magnitude of the measuring unit; and hence it follows, that whatever ratio subsists between MN and Pf, and MON

Geometry. and PR*f*, subsists also between M N and P Q, and M O N and P R Q;*

that is M N : P Q :: M O N : P R Q.

Scholium. Since the arcs have always to each other the ratio which the angles at the centres have, it follows that the arcs may be assumed as the measure of the angles at the centre; and as all the angles that can be formed about the centre of a circle, or any other point, are together equal to four right angles, (prop. 3, cor. 1, book i.) the whole circumference will be the measure of four right angles; the semicircle the measure of two right angles, and a quadrant or quarter of the circumference the measure of one right angle.

Proposition VI.—*Theorem*.

If a right line drawn through or from the centre of a circle bisect a chord, it will be perpendicular to it, or if it be perpendicular to the chord it will bisect it, fig. 55.

Fig. 55. Let A B be any chord in a circle, and C D a line drawn from the centre C, bisecting A B in D, then will C D be perpendicular to A B.

Draw the two radii A C, C B: in the two triangles A C D, B C D, the two sides A C, A D, are equal to the two, B C, B D, and C D is common; hence the triangles are equal, and have their corresponding angles equal, (prop. 12, book i.;) therefore each at D is a right angle, and C D is perpendicular to A B, (def. 10, book i.)

Again, let C D be perpendicular to A B, then will A B be bisected in D. For in the two right angled triangles A C D, B C D, the hypothenuses are equal, and the side C D is common; therefore the third sides A D, D B are also equal, (prop. 16, book i.:) that is the chord A B is bisected in D.

Cor. 1. Hence a line bisecting any chord in a circle at right angles passes through the centre.

Cor. 2. It follows also from the above, that the line which bisects and is perpendicular to a chord, bisects also the arc of that chord; for the angles at C being equal, the arcs which subtend them, A E, E B, are also equal, (prop. 4, book iii.) or the arc A B is bisected in E.

Proposition VII.—*Theorem*.

If more than two equal lines can be drawn from any point within a circle to the circumference, that point will be the centre, fig. 56.

Fig. 56. Let A B C be a circle, and D a point within it; then if any three lines D A, D B, D C, drawn from the point D to the circumference, be equal to each other, that point will be the centre. Join A B, B C, bisect A B in E, and B C in F, and join E D, D F. In the triangles A E D, B E D, the two sides A D, A E, are equal to the two D B, B E, each to each, and E D is common; therefore these two triangles are equal, and the angles at D are equal, (prop. 12, book i.;) consequently each of them is a right angle, (def. 10, book i.;) E D therefore bisects the chord E D at right angles, and therefore passes through the centre, (prop. 6, cor. 1, book iii.) In the same way D F passes through the centre, consequently the point D is the centre.

Proposition VIII.—*Theorem*.

If two circles touch each other internally, the centres of the circles and the point of contact are in the same right line, fig. 57.

Let the two circles A C B, E A D, touch each other internally in the point A; then will the point A and the centres of the circles be in the same right line. Fig. 57 Let F be the centre of the circle A B C, and draw the diameter A F C; the centre of the circle A D E will be also in this line. For if not, let it be in some other point, as G; join F G, and produce it to meet A B C in B, and join also A G. Then G being the centre of the circle A E D, A G = G D; but A G + F G 7 A F, (prop. 8, book i.;) therefore G D + F G, or F D 7 A F; but A F = F B; hence also F D 7 F B, a part greater than the whole, which is absurd; therefore G is not the centre of the circle A E D, and the same may be shown of every point that is not in A C. The centre of the circle A E D is therefore in A C; that is, the centres of the circles and the point of contact are in the same right line.

Proposition IX.—*Theorem*.

If two circles touch each other externally, the centres of the circles and the points of contact are in the same right line, fig. 58.

Let A E D and A C B touch each other externally in A; then will the centres of the circles, and the point A be in the same right line. Fig. 58.

Let F be the centre of A B C, join A F and produce it to E; the centre of the circle A E D is in this line. For if it be not, let it be in some other point as G, and join A G, F G: then A F + A G 7 G F, (prop. 8, book i.;) but A G = G D, and A F = F B; therefore G D + F B 7 G E; a part greater than the whole, which is impossible; and the same may be shown of any point not in F E: therefore the centre of the circle E A D is not out of the line F E; that is, it is in it.

Proposition X.—*Theorem*.

Chords in a circle which are equally distant from the centre are equal to each other; and if they are equal to each other they are equally distant from the centre, fig. 59.

Let the chord A B = C D; they are equally distant from the centre. Fig. 59. Let G be the centre of the circle, and G F, G E two perpendiculars from the centre upon the chords A B, C D; then E G = G F: join A G, C G. Now E G, being perpendicular to A B, it bisects it in E, (prop. 6, book iii.;) and for the same reason G F bisects C D in F: therefore A E = C F; also A G = C G: hence the two right angled triangles A E G, G F C are equal to each other, (prop. 16, book i.;) and consequently E G = F G: that is the equal chords A B, C D are equally distant from the centre.

* It is here taken for granted, that if four quantities, A B C D, be proportionals, and that N and M be two other quantities incommensurable with B and D, but which latter are still such that they may be made to approach nearer to N and M than any assignable quantities, that then also A : N :: B : M. It must be acknowledged, that this conclusion is not so strictly geometrical as could be wished, but it is a defect which necessarily attends the transition from magnitude to number; and which, however it may be disguised, is still to be found upon a minute and strict inquiry. In the first six books of Euclid, magnitudes only are considered, and the difficulty does not appear; but it presents itself the moment we attempt to apply his propositions to the purposes of mensuration. See note to Definitions, Book II.

Geometry. Next let them be equally distant from the centre; that is, let $EG = FG$; then will also $AB = CD$: for drawing the lines as above; in the two right angled triangles AEG, CFG: the hypothenuses are equal, and the side $EG = FG$: therefore also $EA = CF$, (prop. 16, book i.;) but AB is double of AE, and CD is double of CF; consequently $AB = CD$.

Proposition XI.—*Theorem.*

A right line perpendicular to the extremity of a radius is a tangent to the circle, fig. 60.

Fig. 60. Let the line AB be perpendicular to the extremity of the radius CD; then will AB be a tangent to the circle, or touch it in the point D only.

For take any other point E in AB, and join CE, C being the centre: then will CE (prop. 15, book i.) $>$ DC, or than CF; therefore the point E is beyond the circumference, and the same may be shown of every point in the line AB, except the point D; consequently AB touches the circle in no one point except at D; and is therefore a tangent (def. 11, book iii.) to it at that point.

Proposition XII.—*Theorem.*

If a right line be a tangent to a circle, a radius drawn to the point of contact will be perpendicular to the tangent, fig. 61.

Fig. 61. Take any point E, as before: then it is obvious, since the line is wholly without the circle, that $CE >$ CF, or than CD; consequently CD is the shortest line from the centre C to AB; therefore CD is perpendicular to AB, (prop. 15, book i.)

Proposition XIII.—*Theorem.*

The angle formed by a tangent and chord is measured by half the arc of that chord, fig. 62.

Fig. 62 Let AB be a tangent to a circle, and CD a chord drawn from the point of contact C; then is the angle BCD measured by half the arc CFD, and the angle ACD by half the arc AGD.

For draw the radius EC to the point of contact, and the radius EF perpendicular to the chord at H. Then the radius EF, being perpendicular to the chord CD, bisects the arc CFD, (prop. 6, cor. book iii.;) therefore CF is half the arc CFD.

In the triangle CEH, the angle H being a right angle, the sum of the two remaining angles E and ECH is equal to a right angle, (prop. 24, book l.) which is equal to the angle BCE, because the radius CE is perpendicular to the tangent, (prop. 12, book iii.) From each of the equals take away the common part or angle ECH, and there remains the angle CEF equal to the angle BCD. But the angle E is measured by the arc CF, (prop. 5, book iii.) which is half CFD; therefore the equal angle BCD must also have the same measure, half the arc CFD of the chord CD.

Again the line GEF, being perpendicular to the chord CD, bisects the arc CGD. Therefore CG is half the arc CGD. Now since the line CE meeting FG makes the sum of the two angles at E equal to two right angles, and the line CD makes with AB the sum of the two angles at C equal to two right angles; if from these two equal sums there be taken away the parts or angles ECH and BCH, which have been proved equal, there remains the angle CEG equal to the angle ACH. But the former of these, CEG, being an angle at the centre, is measured by the arc CG, (see prop. 5, book iii.;) consequently the equal angle ACD must also have the same measure CG, which is half the arc CGD. Book III

Proposition XIV.—*Theorem.*

An angle at the circumference of a circle is measured by half the arc that subtends it, fig. 63.

Let BAC be an angle, at the circumference it has for its measure half the arc BC which subtends it. Fig. 63.

For let the tangent DE pass through the point of contact A; then the angle DAC, being measured by half the arc ABC, and the angle DAB by half the arc AB, (prop. 13, book iii.;) it follows by equal subtraction, that the difference or angle BAC must be measured by half the arc BC which it stands upon.

Proposition XV.—*Theorem.*

All angles in the same segment of a circle, or standing upon the same arc, are equal to each other, fig. 64.

Let ACB, ADB be two angles in the same segment AC, DB, or which is the same, standing upon the same arc AEB; then will the angle ACB be equal to the angle ADB. Fig. 64.

For each of these angles is measured by half the arc AEB, (prop. 14, book iii.:) and thus having equal measures, they are equal to each other.

Proposition XVI.—*Theorem.*

An angle at the centre of a circle is double the angle at the circumference, when both of them stand upon the same arc, fig. 65.

Let ACB be an angle at the centre C, and ADB an angle at the circumference, both standing upon the same arc or same chord AB, then will the angle C be double of the angle D, or the angle D equal to half the angle C. Fig. 65.

For the angle at the centre C is measured by the whole arc AEB, (prop. 5, book iii.;) and the angle at the circumference D is measured by half the same arc AEB, (prop. 14;) the angle D is only half the angle C, or the angle C double the angle D.

Proposition XVII.—*Theorem.*

An angle in a semicircle is a right angle, fig. 66.

Let ABC or ADC be a semicircle, then any angle ABC in that segment is a right angle. Fig. 66.

For the angle B at the circumference is measured by half the arc ADC, (prop. 14, book iii.;) that is by a quadrant of the circumference. But a quadrant is the measure of a right angle; therefore the angle B is a right angle.

Cor. It follows from this, that an angle in an arc that is greater than a semicircle, is less than a right angle; and an angle in an arc less than a semicircle is greater than a right angle.

Proposition XVIII.—*Theorem.*

The angle formed by a tangent to a circle and a chord drawn from the point of contact, is equal to the angle in the alternate segment, fig. 67.

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Fig. 67. If A B be a tangent, A C a chord, and D any angle in the alternate segment A D C; then will the angle D be equal to the angle B A C made by the tangent and the chord of the arc A E C.

For the angle D at the circumference is measured by half the arc A E C, (prop. 13 and 14, book iii.;) and the angle B A C, made by the tangent and chord, is also measured by the same half arc A E C: therefore these two angles are equal.

Proposition XIX.—*Theorem.*

The sum of any two opposite angles of a quadrangle inscribed in a circle is equal to two right angles, fig. 68.

Fig. 68. Let A B C D be a quadrangle inscribed in a circle; then shall the sum of the two opposite angles, A and C, or B and D, be equal to two right angles.

For the angle A is measured by half the arc D C B, which it stands upon, and the angle C by half the arc D A B, (prop. 14, book iii.;) therefore the sum of the two angles, A and C, is measured by half the sum of these two arcs, that is by half the circumference. But half the circumference is the measure of the two right angles, (prop. 5, schol. book iii.;) therefore the sum of the two opposite angles, A and C, is equal to two right angles. And in like manner it is shown the sum of the other two opposite angles, B and D, is equal to two right angles.

Proposition XX.—*Theorem.*

If any side of a quadrangle inscribed in a circle be produced out, the outward angle will be equal to the inward opposite angle, fig. 69.

Fig. 69. If the side A B of the quadrangle A B C D, inscribed in a circle, be produced to E, the outward angle D A E will be equal to the inward opposite angle C.

For the sum of the two adjacent angles D A E, D A B is equal to two right angles, (prop. 1, book i.;) and the sum of the two opposite angles, C and D A B, is equal to two right angles, (prop. 19, book iii.;) therefore the sum of the two right angles, D A E and D A B, is equal to the sum of the two, C and D A B; from each of these equals, taking away the common angle D A B, there remains the angle D A E equal the angle C.

Proposition XXI.—*Theorem.*

Two parallel chords intercept equal arcs, fig. 70.

Fig. 70. Let the chords A B, C D be parallel, then will the arcs A B, C D be equal, or A B = C D.

For draw the line B C; then because the lines A B C D are parallel, the alternate angles B and C are equal, (prop. 20, book i.) But the angle at the circumference B is measured by half the arc A C, (prop. 14, book iii.;) and the other angle at the circumference C is measured by the arc B D; hence the halves of the arcs A C, B D, and consequently the arcs themselves are equal.

Proposition XXII.—*Theorem.*

If a tangent and chord be parallel to each other, they intercept equal arcs, fig. 71.

Fig. 71. Let the tangent A B C be parallel to the chord D E; then are the arcs B D, B E equal; that is, B D = B E.

For draw the chord B D; then because the lines A B, D E are parallel, the alternate angles D and B are equal; but the angle B, formed by a tangent and a chord, is measured by half the arc B D, (prop. 18, book iii.;) and the angle at the circumference D, is measured by half the arc B E; the arcs B E, B D are therefore equal. Book III.

Proposition XXIII.—*Theorem.*

The angle formed within a circle by the intersection of two chords, is measured by half the sum of the two arcs intercepted by those chords, fig. 72.

Let the two chords A B, C D intersect at the point E; the angle A E C, or D E B, is measured by half the sum of the two arcs A C, D B. Fig. 72.

For draw the chord A F parallel to C D; then because the lines A F, C D are parallel, and A B cuts them, the angles on the same side, A and D E B, are equal; but the angle at the circumference A is measured by half the arc B F, (prop. 14, book iii.) or of the sum of F D and D B; therefore the angle F is also measured by half the sum of F D and D B. Again, because the chords A F, C D are parallel, the arcs A C, F D are equal, (prop. 21;) therefore the sum of the two arcs A C, D B is equal to the sum of the two F D, D B; and consequently the angle E, which is measured by half the latter sum, is also measured by half the former.

Proposition XXIV.—*Theorem.*

The angle formed without a circle by two secants, is measured by half the difference of the intercepted arcs, fig. 73.

Let the angle E be formed by two secants, A B and C D. This angle is measured by half the difference of the two arcs, A C, D B, intercepted by the two secants. Fig. 73.

Draw the chord A F parallel to C D; then because the lines A F, C D are parallel, and A B cuts them, the angles on the same side, A and D E B, are equal, (prop. 21, book i.) But the angle A, at the circumference, is measured by half the arc B F, or of the difference of D F and D B; therefore the equal angle E is also measured by half the difference of D F, D B.

Again because the chords A F, C D are parallel, the arcs A C, F D are equal, (prop. 21, book iii.;) therefore the difference of the two arcs, A C, D B, is equal to the difference of the two D F, D B; consequently the angle E, which is measured by half the latter difference, is also measured by half the former.

Proposition XXV.—*Theorem.*

The angle formed by two tangents, is measured by half the difference of the two intercepted arcs, fig. 74.

Let E B, E D be two tangents to a circle at the points A, C: then the angle E is measured by half the difference of the two arcs C F A, C G A. Fig. 74.

For draw the chord A F parallel to E D; then because the lines A F, E D are parallel, and E B meets them, the angles on the same side, A and E, are equal, (prop. 21, book iii.;) but the angle A, formed by the chord A F and tangent A B, is measured by half the arc A F: therefore the equal angle E is also mea-

sured by half the same arc A F, or half the difference of the arcs C F A and C F.

Again, because the tangent E D and chord A F are parallel, the intercepted arcs (prop. 21, book iii.) C G E, C F are equal; the arc A F therefore is equal to the difference of C F A and C G A; consequently the angle E, which is measured by half the former, is also measured by half the latter.

Cor. In like manner it is proved that the angle E (fig. 74) formed by a tangent E C D, and a secant E A B, is measured by half the difference of the two intercepted arcs, C A and C F B.

Problems relative to Books II. and III.

Problem I.

To divide a given right line A B *into two equal parts*, fig. 75.

Fig. 75. From the two extremities, A and B, and with any equal radii greater than half A B, describe arcs of circles intersecting each other in C and D, and draw the line C D, which will bisect the given line A B in the point E.

Join A C, C B, A D, D B, which are all equal to each other; consequently the triangles D A C, D B C, which have the three sides of the one equal to the three sides of the other, each to each, will have their corresponding angles also equal; therefore the angle A C E $=$ B C E. And because A C $=$ C B, the angle C A E $=$ C B E; hence the angles A C E, C A E, being equal to the two E C B, C B E, each to each, and the side A C $=$ B C, the two triangles A C E, B C E, are equal; and will have the base A E $=$ E B, that is the right line A B has been bisected in E, as was required to be done.

Problem II.

To bisect a given angle, B A C, *fig.* 76.

Fig. 76. From the summit A, with any radius, describe an arc cutting off the equal parts A D, A E; and from D and E, with any radius greater than half D E, describe the two arcs intersecting in F; and join A F, which will bisect the angle A, as required. Join D F, E F, then the two triangles A D F, A E F, will have the sides A D, D F equal to A E, E F, each to each, and the base A F common; therefore the triangles will be equal, and the angle D A F $=$ E A F; that is, the angle A has been bisected by the line A F.

Problem III.

At a given point C *in a line* A B *to raise a perpendicular*, fig. 77.

Fig. 77. From the given point C, set off the equal distances C D, C E, on the line A B, and from D and E as centres, with any radius greater than D C or E C, describe arcs intersecting each other in F; join C F, which will be the perpendicular required.

Join D F, F E, then in the two triangles D F C, E F C, the sides D F, D C are equal to E F, E C, each to each, and the base F C is common; therefore the triangles are equal, and the angle D C F $=$ E C F: they are therefore right angles, and F C is perpendicular to A B.

Scholium. As it is assumed that a given line may be produced, if the point C were at the extremity of the line A B, the line might be produced and the construction remain as above; but it is sometimes a convenience in practice to erect a perpendicular without producing the line beyond the point at which it is to be erected. In such cases we may proceed as follows:

Fig. 78. Take any point D (fig. 78) out of the line A B, and from D as a centre, and with the radius D C, describe a circle, E C F, cutting A B in E; join E D, and produce it, to cut the circumference in F, draw F C; it will be the perpendicular required. For E C F being a semicircle, the angle C in it is a right angle, and consequently C F is perpendicular to A B.

Problem IV.

From a given point A, *to let fall a perpendicular upon a given line* B C, *fig.* 79.

Fig. 79. From the point A, with any radius greater than the perpendicular distance, describe an arc cutting B C in two points, D and E; from D and E as centres, with any radius, describe arcs intersecting in F; join A F, cutting B C in G, then will C G be the perpendicular sought.

For join D A, D F, A E, E F: the triangles A D F, A G F, having the three sides equal, each to each; the angle D A F $=$ E A F: and the triangle D A E, being isosceles, the angle A D E $=$ A E D: hence in two triangles D A G, E A G, the two angles A D G, D A G are equal to the two A E G, E A G, each to each, and the side A D $=$ A E; therefore the triangles are equal, and the angles at G are equal; they are therefore right angles, and A G is perpendicular to A B.

Fig. 80. *Scholium.* As in the last problem this construction supposes the line A B (fig. 80) of unlimited length. If the point be nearly opposite the end of the line the following construction may be employed: From any point D in A B, and with the radius D C, describe an arc C A F; and from A, with the radius A C, describe an arc cutting the former in C and F; join C F, and it will be the perpendicular sought.

Join A C, A F, which being equal chords, the arcs A C, A F will be also equal: hence D A bisects the arc A F, and consequently also the chord of the arc: but the line drawn from the centre to bisect a chord is perpendicular to it. Hence C G is perpendicular to A G, and consequently A G is perpendicular to D G or to A B.

Problem V.

At a point A, *in a given line* A B, *to make an angle equal to a given rectilineal angle* C, fig. 81.

Fig. 81. From the centres A and C, with any radius, describe the arcs D E and F G; join E D, and from F, with the distance D E, describe an arc cutting F G in G; draw A G, so will the angle A $=$ C.

For the chords D E, F G, being equal, the arcs D E and F G are also equal; and consequently the angles C and A.

Problem VI.

Through a given point A, *to draw a line parallel to a given line*, B C, fig. 82.

Fig. 82. From the given point A, draw any line A D to the line A B; and at the point A make the angle D A F $=$ A D C, produce A F, and it will be parallel to B C.

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For the alternate angles A D C, and F A D being equal, the lines E F and B C are parallel.

PROBLEM VII.

To describe a triangle when there are given the two sides and the included angle, fig. 83.

Fig. 83. Draw the indefinite line A D, and at the point A make the angle B A C equal to the given angle; take also A B, and A C equal to the given sides, and join C B, and A B C will be the triangle required, as is obvious.

PROBLEM VIII.

Given two angles, and any side of a triangle to construct the triangle, fig. 83.

There are two cases to this problem, accordingly as the given side is adjacent to one only, or to both, the given angles.

1. When the given side is adjacent to both the given angles.

Let A B be the given side, and A and B the given angles. At A and B, make angles equal to the given angles, and produce the lines till they intersect in C, A B C will be the triangle required.

2. Let A B be the given side, and A and C the given angles. Produce A B to D, and at B make the angle C B D equal to the sum of the two angles A and C; and at A make the angle A equal to one of the given angles, meeting B C in C, then will A B C be the triangle required.

The first case requires no demonstration; and in the second, since C B D is equal to the sum of the given angles, and since the three angles are equal to two right angles, A B C must be equal to the third angle; which reduces the problem to the former case.

PROBLEM IX.

Given two sides of a triangle, and an angle opposite to one of them to construct the triangle, fig. 85.

Fig. 85. Let A B be one of the given sides, and C A the other, and B the given angle. At the point B make the angle A B C equal to the given angle; and from A, with A C as a radius, describe an arc cutting B D in C and C′; join A C, A C′, and A B C or A B C′ will be the triangle required, as is obvious.

Scholium. It appears from the above, that when A C is greater than the perpendicular A E, let fall from A to B D, there are two triangles answering the required conditions. If A C be equal to that perpendicular distance, there is but one, and in that case the triangle will be right angled; and if A C is less than the perpendicular distance A E, the construction is impossible.

PROBLEM X.

To describe a triangle that shall have its three sides equal to three given lines, A, B, C, fig. 86.

Fig. 86. Draw D E equal to C, and from D and E as centres, and with radii equal to A and C, describe arcs intersecting in F; join D F, E F, and D E F will have its three sides equal to the three given lines A, B, and C, as is obvious.

It is necessary in this case that any two of the sides be greater than the third.

PROBLEM XI.

Problems relative to Books II. and III.

Given the two adjacent sides, A and B, of a parallelogram, and the angle they include, to describe the parallelogram, fig. 87.

Draw D E equal to B, one of the given sides, and at D make the angle F D E equal to the given angle; take $DF = A$, and through F draw F G parallel to D F, and through E, E G parallel to D F; so shall E G F D be the parallelogram required. For $DE = B$, and $DF = A$; by the construction and the sides being parallel the opposite sides are equal, and the figure is a parallelogram. Fig. 87.

Cor. This construction comprehends the construction of the square and rectangle. It is only necessary in these cases, that the angle D be made equal to a rectangle.

PROBLEM XII.

To make a square equal to the sum of two given squares, fig. 88.

Let A B, C B be the sides of the given squares: on A B, at the point B, erect the perpendicular B C, equal to the other given line, and join A C, so will A C be the side of the square required. For Fig. 88.

$$AC^2 = AB^2 + BC^2.$$

Cor. Hence also we may make a square equal to three or more squares: for produce B A and B C towards D and E, (fig. 89,) and let G H be the side of a third square; take $BE = GH$, and $BD = AC$, and join D E; so shall $DE^2 = AB^2 + BC^2 + GH^2$; for $DE^2 = DB^2 + BE^2$; and $DB^2 = AC^2 = AB^2 + BC^2$ and $BE^2 = GH^2$: therefore $DE^2 = AB^2 + BC^2 + GH^2$; and we may proceed in like manner with any number of squares.

PROBLEM XIII.

To make a square equal to the difference of two given squares, fig. 90.

Let A B, B C be the sides of the given squares: on A B, the greater, describe the semicircle A B C; and from B, with the radius C B, describe the arc $m\,n$, cutting the semicircle in C; join C B, C A; and C A will be the side of the square required. For by the construction C B is equal to the lesser given side B C, and A B to A B; and the angle C, being in a semicircle, is a right angle: therefore Fig. 90.

$$AC^2 = AB^2 - BC^2.$$

PROBLEM XIV.

To describe a circle through any three given points, A, B, C, *not in a right line*, fig. 91.

From the middle point B draw the lines B A, B C to the other two given points; and bisect these by the perpendiculars D O, E O, which will intersect in some point O; then from the centre O, and with the distance O B, describe a circle which will pass through the other two points A and C. For the two right angled triangles O A D, O B D, having the side A D, D B equal, and O D common; also the angles at D right angles, will have their third sides likewise equal, that is $OA = OB$; and in the same way it may be shown, that $OC = OB$; hence the three lines O A, O B, O C, being all equal, are radii of the same circle. Fig. 91.

Problem XV.

To find the centre of any given circle, or of any arc of a given circle, fig. 92.

Fig. 92. Take any three points in the given arc or circle, and find the centre of the circle passing through them by the last problem, and it will be the centre sought, as is obvious.

Problem XVI.

To draw a tangent to a given circle, through a given point A, *either in or beyond the circumference*, fig. 93.

Fig. 93. Find the centre of the circle, and then first, if the given point is in the circumference, join A and the centre O, and at A draw B C perpendicular to A O, and it will be the tangent required.

But if A be beyond the circumference, then also join A and the centre O, and upon A O describe the semicircle A D O; then from A, through D, draw the line B C, and it will be the tangent sought. For A D O, being an angle in a semicircle, is a right angle; consequently B C is perpendicular to D O, and is therefore a tangent to the circle.

Problem XVII.

Upon a given line A B, *to describe a segment that may contain a given angle* C, fig. 94.

Fig. 94. At the ends of the given line make the angles D A B, D B A, each equal to the given angle C; and draw A E, B E, perpendicular to A D, B D, and with the centre E and radius E A, or B E, describe a circle, so shall A F B be the segment required; that is, any angle F in it will be equal to the given angle C.

For the two lines A D, B D, being perpendicular to the radii E A, E B, are tangents to the circle; and the angle A or B, which is made equal to the given angle C, is equal to the angle in the alternate segment A F B.

Problem XVIII.

To cut off a segment from a given circle that shall contain an angle equal to a given angle C, fig. 95.

Fig. 95 Draw any tangent A B, to the given circle; and a chord A D, making the angle $DAB = C$; so shall D E A be the segment required.

For the angle A, made by the tangent and chord, being equal to the angle C; the angle E in the alternate segment is also equal to the angle C.

Problem XIX.

To inscribe a circle in a given triangle A B C, fig. 96.

Fig. 96 Bisect the angles A and B with the two lines A D, B D; from the intersection D, draw the perpendiculars D E, D F, D G, and they will be radii of the circle required. For in the two triangles A D G, A E D, the angle $DAG = EAG$, and the angle $DGA = DEA$; therefore also $GDA = ADE$, because the sum of the three angles of every triangle is equal to two right angles. Hence the side A D, being common, and the angles adjacent to it equal, the triangles are equal, and the side $DG = DE$; in the same manner it may be shown, that $DF = DE$; consequently a circle described from D, with the radius D E, will pass through G and F; and the sides AB, BC, CA, being perpendiculars to these radii, will be tangents to the circle; which is therefore inscribed in the triangle.

Problem XX.

To circumscribe a circle about a given triangle A B C, fig. 97.

Bisect any two sides with two perpendiculars, as D F, D E, and D will be the centre: from D, with the radius D A, describe a circle, which will pass through A B C. The demonstration is the same as in the last problem. Fig. 97.

BOOK IV.

Of the proportions of figures, and the measure of areas.

1. Similar figures, are those which have the angles of the one equal to the angles of the other, each to each, and the sides about the equal angles in each proportional.

2. *Homologous* sides and angles, are those sides and angles which have the same situation in any two similar figures.

3. In different circles *similar* arcs, *similar* segments, and *similar* sectors, are those which correspond to equal angles at the centre.

4. The *base* of any rectilineal figure is any side on which the figure is supposed to stand.

5. The *altitude* of a parallelogram, or trapezoid, is the perpendicular distance between the side taken for a base and the side opposite.

6. The *altitude* of a triangle is the perpendicular distance of its vertex from the base.

7. The *area*, or *surface* of a figure, is its superficial content: and it is estimated numerically by the number of times it contains some other area which is assumed for its measuring unit.

8. Figures having equal areas, that is figures which contain the same measuring unit the same number of times, are said to be *equal*.

Hence figures may be equal to each other, although they are not similar.

Some authors distinguish between figures which are both equal and similar, and those which are only equal according to the above definition. In this case the former are called *identical*, and the latter *equal*; or the former *equal*, and the latter *equivalent*.

Proposition I.—*Theorem.*

The complements about the diagonal of any parallelogram are equal to each other, fig. 98.

Let A C be a parallelogram and B D its diagonal: and let E F be parallel to D C, and G H to A D, both passing through any common point I in the diagonal; then the figures A I, I C are called the complements of the parallelograms E G, H F, and it is to be demonstrated that they are equal to each other. Because the diagonals of parallelograms bisect them, (prop. 27, book i.;) the triangles D G I, and D E I are equal; for the same reason I H B and I F B are equal; as are likewise D A B and D C B: if therefore from these last equal triangles there be taken on one side the two triangles D G I and I F B, and on the other the two triangles D E I and I H B, there will remain the complement I C equal to the complement A I. Fig. 98.

PROPOSITION II.—*Theorem.*

Parallelograms on the same base, and between the same parallels are equal to each other, fig. 99.

Fig. 99. Let A B C D, A B E F be two parallelograms on the common base A B, and between the same parallels A B, D E, then will the parallelogram A B C D = A B E F; because A B = D C, and A B = F E, (prop. 27, book i.) to each add C F, then will D F = C E; also D A = C B, and A F = B E, (prop. 27, book i.;) therefore, in the two triangles D A F and C B E, the three sides D F, D A, and A F are equal to the three C E, C B, and B E, each to each; therefore the triangles themselves are also equal, (prop. 12, book i.;) and if each of them be taken from the whole figure A B D E, there will remain the parallelogram A B C D in the one case, equal to the parallelogram A B E F in the other.

Cor. 1. Parallelograms on equal bases, and between the same parallels are equal; because the bases being equal, the one figure may be applied to the other, so that their bases shall coincide; and they may then be considered as standing on the same base; which thus reduces itself to the case above demonstrated.

Cor. 2. Because parallel lines have every where the same perpendicular distance, which in this case is the altitude of the parallelograms; it follows then that parallelograms of equal bases and altitudes are equal to each other.

Cor. 3. Every parallelogram is equal to a rectangle of the same base and altitude.

PROPOSITION III.—*Theorem.*

Triangles on the same base and between the same parallels are equal to each other, fig. 99.

Let A B C, A B F, be triangles upon the same base and between the same parallels; the triangle A B C = A B F; produce C F, and draw A D parallel to B C and B E to A F; then will A B C D and A B E F, be parallelograms upon the same base and between the same parallels; therefore, by the last prop. A B C D = A B E F; but the triangle A B C is half the parallelogram A B C D, and the triangle A B F is half the parallelogram A B E F, (prop. 27, book i.;) therefore the triangle A B C = A B F.

Cor. 1. Hence also triangles on equal bases and between the same parallels are equal, for the equal bases may be made to coincide, and the case thus reduced to the above.

Cor. 2. Because the perpendicular distance from C and F to the base A B, or A B produced, are equal, (def. 12, book i.) which are the altitudes of the triangles: it follows that triangles of equal bases and altitudes are equal to each other.

Cor. 3. Since the triangle A B C is half the parallelogram A B C D; or A B F half the parallelogram A B E F; and that these are parallelograms of equal bases and altitudes with the triangle; it follows that every triangle is equal to half a parallelogram of the same base and altitude.

Cor. 4. Hence a triangle is equal to half the rectangle of equal base and altitude.

PROPOSITION IV.—*Theorem.*

A trapezoid is equal to half a parallelogram, whose base is equal to the sum of the two parallel sides, and its altitude the perpendicular distance between them, fig. 100.

Let A B C D be a trapezoid whose two parallel sides are A B, D C; produce A B to E, till B E = D C, and D C to F, till C F = A B, and join F E, so shall A B C D be equal to half the parallelogram A E F D, which has for its base the sum of the two parallel sides, and for its altitude the perpendicular distance between them. Draw C G, B H parallel to A D or F E; then because B E = D C, or A G, the two parallelograms A C, B F, are upon equal bases, and between the same parallels, therefore they are equal, (by cor. 1, last prop.) and because C B is the diagonal of the parallelogram G H; the triangle C G B = C H B, (prop. 27, book i.) consequently A C + C G B = B F + C H B, or A B C D = B E F C, or A B C D = half the parallelogram A E F D. Fig. 100.

PROPOSITION V.—*Theorem.*

Triangles having the same altitude, are to each other in the same ratio as their bases, fig. 101.

Let the two triangles A D C, D E F have the same altitude, they will have to each other the ratio of their bases; that is, A D C : D E F :: A D : D E. Fig. 101.

Conceive the base A D of the triangle A D C divided into any number of equal parts, or units of measure, as A B, B D; and let the same unit be repeated on the base D E, till it either coincides with E, or fall beyond it by a quantity less than the measuring unit, as at M, and join C B, F G, F H, F M; thus dividing the triangles A C D and D F M into a number of triangles, A C B, B C D, D F G, G F H, &c. which are equal to each other, having equal bases and altitudes, (prop. 3, cor. 2, book iv.;) therefore the same number of units which there are in the base A D, the same number of equal triangles, or units, are there in the triangle A C D; and the same number of units there are in the base D M, equal to those in A D, the same number of triangles are there in D F M, each also equal to those in A D C, therefore as

A D : D M :: A C D : D F M.

But D M may be made to differ from D E, by a quantity less than the measuring unit; and the unit itself may be taken less than any assignable quantity; therefore D M may be made to differ from D E, by a quantity less than any that can be assigned, and at the same time the triangle D F M will differ from D F E by less than any quantity that can be assigned; consequently, (see note to def. 1, book ii.)

A D : D E :: A D C : D F E

or A D C : D F E :: A D : D E.

PROPOSITION VI.—*Theorem.*

Parallelograms of equal altitude, are to each other as their bases, fig. 102.

Let A D K I, D E F K be parallelograms of equal altitude, they are to each other as their bases: for join A K, D F; then by the last proposition, Fig. 102.

A K D : D E F :: A D : D E;

but the parallelogram A K is double of the triangle A K D, and the parallelogram D F is double of the triangle D E F, (prop. 27, book i.;) and equimultiples of quantities have the same ratio as the quantities; therefore as

A D K I : D E F K :: A D : D E.

Proposition VII.—*Theorem.*

Triangles and parallelograms having equal bases, are to each other as their altitudes, fig. 103.

Fig. 103. Let A B C, B E F be two triangles, having the bases A B, B E equal, and whose altitudes are the perpendiculars C G, F H; then will the triangle A B C : the triangle B E F :: C G : F H.

For, let B K be perpendicular to A B, and equal to C G, in which let there be taken B L = F H; and draw A K and A L.

Then triangles of equal bases and altitudes being equal, the triangle A B K = A B C, and A B L = B E F. But considering now A B K, A B L as two triangles on the bases B K, B L, and having the same altitude A B, these will be as their bases, namely, the triangle A B K : A B L :: B K : B L. But A B K = A B C, and A B L = B E F, also B K = C G, and B L = F H. Therefore A B C : B E F :: C G : F H.

And since parallelograms are the doubles of triangles, having the same bases and altitudes, these when their bases are equal, will likewise have to each other the same ratios as their altitudes.

Proposition VIII.—*Theorem.*

Triangles and parallelograms are to each other in the ratio of the products of their bases and altitudes, fig. 104.

Fig. 104. Let A B C, E F G be any two triangles whose altitudes are C D, G H, and bases A B, E F, then will

trian. ABC : trian. EFG :: AB × CD : EF × GH.

Let K L M be another triangle whose base K L = A B, and altitude M N = G H.

Because A B C and K L M have equal bases, they are to each other as their altitudes; and because E F G and K L M have equal altitudes, they are to each other as their bases: that is, in the former, A B C : K L M :: D C : M N (prop. 6, book iv.;) in the latter, K L M : E F G :: K L : E F (prop. 7, book iv.) Hence by (prop. 12, book ii.)

ABC × KLM : EFG × KLM :: DC × KL : MN × EF.

Or since quantities have the same ratio, their equimultiples have

A B C : E F G :: D C × K L : M N × E F.

But K L = A B, and M N = G H; therefore

A B C : E F G :: A B × D C : E F × G H.

And since every parallelogram is double of a triangle of equal base and altitude, and that equimultiples of quantities have the same ratio as the quantities, (prop. 6, book ii.) it follows that parallelograms are also to each other as the product of their bases and altitudes.

Scholium. Since the area of parallelograms, and consequently of rectangles, are to each other as the product of their bases and altitudes, this product may be assumed as the proper measure of such areas; by which is to be understood, that as many units as there are in the product of the base and altitude of any rectangle, the same number of units are there in the area of the rectangle; the latter unit being the square described upon the linear unit, by which the sides of the figure are measured.

In the same way the area of a triangle is measured by half the product of its base and altitude; and the area of a trapezoid by the product of its altitude, by half the sum of its two parallel sides.

Proposition IX.—*Theorem.*

The sum of all the rectangles contained under one whole line, and the several parts of another line is equal to the rectangle contained under the two whole lines, fig. 105.

Let A D be one line, and A B another, divided into the parts A E, E F, F B; the rectangles contained under D A and A E, D A and E F, D A and F B are together equal to the rectangle D A, A B. Fig. 105.

Let D A be perpendicular to A B, and A C, the rectangle contained under D A, A B; conceive also E G, F H, to be perpendicular to A B; then because D C is parallel to A B, (def. 18, book i.) A D, E G, F H and C B are all equal to each other, and the whole figure or the rectangle of A B, and A D is divided into the three rectangles A G, E H, F C; of which A G is equal to the rectangle of A D and A E; E H = the rectangle of E F and E G, or E F and A D, because E G = A D; and F C = the rectangle of F B and F H, or F B and A D, because H F = A D; therefore the rectangle $AB \times AD = \overline{AE \times AD} + \overline{EF \times AD} + \overline{FB \times AD}$. (schol. to last prop.)

Proposition X.—*Theorem.*

The square of the sum of two lines is greater than the sum of their squares, by twice the rectangle of those lines, fig. 106.

Let A B be the sum of any two lines A C and B C, or A B = A C + B C; then will $AB^2 = AC^2 + BC^2 + 2\,AC \times BC$. Let A B D E be the square on the line A B, and A C F G the square on the line A C. Produce C F and G F to the other sides at H and I. From C H and G I which are equal, being each equal to the side of the square A B, or B D, (prop. 27, cor. 1, book i.) take the parts C F, G F which are also equal, being sides of the square on A C, and there remains F H = F I, which are equal to D I, H D, being opposite sides of a parallelogram, (prop. 27, book i.:) the figure F I D H has therefore all its sides equal, and its angles are right angles; it is therefore a square on the line F I, or on its equal C B, (def. 17.) Again I C is a rectangle contained by A C and C B, for C F = A C; and G H is a rectangle contained by A C and B C; for G F = A C, and F H = F I = B C; therefore the whole square A B D E, which is made up of the four figures, that is of the two squares A F, F D, and the two rectangles F B, and G H, is equal to the squares on A C and B C and twice the rectangle A C × B C. Fig. 106

Cor. Hence if a line be divided into two equal parts, the square of the whole line is equal to four times the square of half the line.

Proposition XI.—*Theorem.*

The square of the difference of two lines is less than the sum of their squares, by twice the rectangle of the said lines, fig. 107.

Let A C, B C be any two lines, and A B their difference, then will $AB^2 = AC^2 + BC^2 - 2AC \times CB$. Fig. 107.

For let A B D E be the square on the difference A B, and A C F G the square on the line A C. Produce E D to H, also produce D B and H C, and draw K I, making B I the square of the other line B C

Geometry. Now the square A D is less than the two squares A F, B I, by the two rectangles E F, D I; but G F = A C, and G E or F H = B C; consequently the rectangle E F contained under E G and G F is equal to the rectangle of BC and A C. Again, F H being equal to C I or B C or D H, by adding the common part H C, the whole H I will be equal to the whole F C, or equal to A C, and consequently the figure D I is equal to the rectangle contained by A C and B C.

Fig. 107.

Hence the two figures E F, D I are two rectangles on the lines A C, B C, and consequently the square of A B is less than the square of A C, B C by twice the rectangle $AC \times BC$.

PROPOSITION XII.—*Theorem.*

The difference of the squares of any two unequal lines is equal to the rectangle under the sum and difference of the same lines, fig. 108.

Fig. 108. Let A B, A C be any two unequal lines, then will $AC^2 = \overline{AB + BC} \times \overline{AB - AC}$.

For let A B D E be the square of A B, and A C F G the square of A C, produce D B till B H is equal to A C, and let H I be parallel to A B or E D, and produce F C both ways to I and K.

Then the difference of the two squares A D, A F, is evidently the two rectangles E F, K B; but the rectangles E F, B I are equal, being contained under equal lines; for E K and B H are each equal to A C, and G E is equal to C B, being each equal to the difference between A B and A C, or their equals A E and A G; therefore the two E F, K B are equal to the two K B and B I, or to the whole K H; and consequently K H is equal to the difference of the squares A D, A F; but K H is a rectangle contained under D H (or the sum of A B and A C,) and K D (or the difference of A B and A C.) Therefore the difference of the squares A B and A C, is equal to the rectangle contained under the sum and difference of those lines.

PROPOSITION XIII.—*Theorem.*

In every right angled triangle, the square of the hypothenuse is equal to the sum of the square of the other two sides, fig. 109.

Fig. 109. Let A B C be a right angled triangle, having the right angle C; then will the square on the hypothenuse A B, be equal to the two squares on A C and B C, or $AB^2 = AC^2 + BC^2$.

Let A E be the square on A B, A G the square on A C, and C I the square on C B, and let C K be parallel to A D or B E; join A I, B F, C D, and C E. Then because the lines C G, C B meet the line A C, so as to make two right angles, these two C G, C B are in the same right line (prop. 2, book i.) and for the same reason A C, C H are in the same right line. Because the angle $FAC = DAB$, being each a right angle; to each add the angle B A C, then will the angle F A B be equal to the angle C A D; and the two triangles F A B, C A D will have the sides F A, A B, equal to the two C A, A D, each to each, and the angles included by these sides equal, therefore the triangles are equal (prop. 4, book i.;) that is, the triangle $FAB = CAD$; but $FAB =$ half the square A G, being on the same base and between the same parallels (prop. 3, cor. 3, book iv.;) and, for the same reason $= CAD$ half the rectangle A K, and as the doubles of equal things are equal, the square A G is equal to the rectangle A K; and in like manner it may be shown, that the square C I is equal to the rectangle B K; consequently the two squares A G, C I, are together equal to the whole square on A B; that is, $AB^2 = AC^2 + BC^2$.

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Cor. 1. Hence the square on either side of a right angled triangle, is equal to the difference of the squares on the hypothenuse and other side; that is, $AC^2 = AB^2 - BC^2$, or $BC^2 = AB^2 - AC^2$.

Cor. 2. Because the rectangle under the sum and difference of any two unequal lines, is equal to the difference of their squares; therefore the square on either side of a right angled triangle is equal to the rectangle under the sum and difference of the hypothenuse and the other side.

PROPOSITION XIV.—*Theorem.*

In any triangle the difference of the squares of the two sides is equal to the difference of the square of the two lines or distances, included between the extremes of the base and perpendicular, fig. 110.

Let A B C be any triangle, having C D perpendicular to A B, then will the difference of AC^2, BC^2, be equal to the difference of AD^2, BD^2; that is, $AC^2 - BC^2 = AD^2 - BD^2$. Fig. 110

$$\left.\begin{array}{ll}\text{For since} & AC^2 = AD^2 + CD^2 \\ \text{and} & BC^2 = BD^2 + CD^2\end{array}\right\} \text{(prop. 13, book iv.)}$$

the difference between $AD^2 + CD^2$ and $BD^2 + CD^2$ is equal to the difference between AD^2 and BD^2 by taking away the common square CD^2. That is, $AC^2 - BC^2 = AD^2 - BD^2$.

Cor. Since $AC^2 - BC^2 = \overline{AC + BC} \times \overline{AC - BC}$ (prop. 12, book iv.)

and $AD^2 - BD^2 = \overline{AD + BD} \times \overline{AD - BD}$

it follows that the rectangle under the sum and difference of the sides of a triangle, is equal to the rectangle under the sum and difference of the segments of the base, or the whole base and the sum and difference of the segments, according as the perpendicular fall without or within the base.

PROPOSITION XV.—*Theorem.*

In an obtuse angled triangle, the square of the side subtending the obtuse angle, is greater than the sum of the square of the other two sides, by twice the rectangle of the base and distance of the perpendicular from the obtuse angle, fig. 111.

Let A B C be a triangle obtuse angled at B, and C D perpendicular to A B; then will $AC^2 = AB^2 + BC^2 + 2AB \times BD$. Fig 111.

For $AD^2 = AB^2 + 2AB \times BD$ (prop. 10, book. iv.) and if we add CD^2 to each, their results $AD^2 + CD^2 = AB^2 + BD^2 + CD^2 + 2AB \times BD$. But $AD^2 + CD^2 = AC^2$ and $BD^2 + CD^2 = BC^2$; therefore $AC^2 = AB^2 + BC^2 + 2AB \times BD$.

PROPOSITION XVI.—*Theorem.*

In any triangle, the square of the side subtending an acute angle is less than the squares of the other two sides by twice the rectangle of the base, and the distance of the perpendicular from the acute angle, fig. 112.

Geometry. Let ABC be a triangle, having the angle at A acute, and CD perpendicular to AB, then will
Fig. 112. $BC^2 = AB^2 + AC^2 - 2AB \times AD$.

For in fig. 1, $AC^2 = BC^2 + AB^2 + 2AB \times BD$ by the last proposition, to each of these equals add the square of AB,

then $AB^2 + AC^2 = BC^2 + 2AB^2 + 2AB \times BD$

or $= BC^2 + 2AB \times \overline{AB + BD} = BC^2 + 2AB \times AD$

that is $BC^2 = AB^2 + AC^2 - 2AB \times AD$.

Again, in fig. 2, $AC^2 = AD^2 + DC^2$ (prop. 13, book iv.) and $AB^2 = AD^2 + DB^2 + 2AD \times BD$ (prop. 10, book iv.) therefore $AB^2 + AC^2 = BD^2 + DC^2 + 2AD^2 + 2AD \times DB$;

but $BD^2 + DC^2 = BC^2$;

therefore $AB^2 + AC^2 = BC^2 + 2AD^2 + 2AD \times DB$,

or $= BC^2 + 2AB \times AD$;

that is, $BC^2 = AB^2 + AC^2 - 2AB \times AD$.

Proposition XVII.—*Theorem.*

In any triangle, the double of the square of a line drawn from the vertex to the middle of the base, together with double the square of the half base, is equal to the sum of the squares of the other two sides, fig. 113.

Fig. 113. Let ABC be a triangle, and CD the line drawn from the vetex to the middle of the base, dividing it ioto two equal parts AD, DB, then will $AC^2 + CB^2 = 2CD^2 + 2DB^2$.

For $AC^2 = DC^2 + AD^2 + 2AD \times DE$, (prop. 15, book iv.) $= DC^2 + BD^2 + 2BD \times DE$

and $BC^2 = DC^2 + BD^2 - 2DB \times DE$, (prop. 16, book iv.) therefore, by equal additions,

$$AC^2 + BC^2 = 2DC^2 + 2DB^2.$$

Proposition XVIII.—*Theorem.*

In an isosceles triangle, the square of the line drawn from the vertex to any point in the base, together with the rectangle of the segments of the base, is equal to the square of one of the equal sides of the triangle, fig. 114.

Fig. 114. Let ABC be an isosceles triangle, and CD a line drawn from the vertex to any point in the base; then will the square on AC be equal to the square on CD, together with the rectangle of AD and DB; that is, $AC^2 = CD^2 + AD \times DB$.

Let CE bisect the vertical angle, and it also bisect the base perpeodicularly (prop. 6, cor. 1, book i.) making $AE = BE$.

Now in the triangle ACD obtuse angled, as D, we have $AC^2 = CD^2 + AD^2 + 2AD \times DE$ (prop 15)

$$= CD^2 + AD \times \overline{AD + 2DE}$$
$$= CD^2 + AD \times \overline{AE + DE}$$
$$= CD^2 + AD \times \overline{BE + DE}$$
$$= CD^2 + AD \times DB.$$

Proposition XIX.—*Theorem.*

In any parallelogram the sum of the squares of the two diagonals is equal to the sum of the squares of the four sides, fig. 115.

Fig. 115. Let ABCD be a parallelogram, and AC, DB its diagonals, then will

$AC^2 + DB^2 = AD^2 + DC^2 + CB^2 + AB^2$.

For since the diagonals of parallelograms bisect each other (prop. 30, book i.) $DE = EB$, and $AE = EC$;

therefore $2EC^2 + 2EB^2 = DC^2 + CB^2$ Book IV.

and $2AE^2 + 2EB^2 = DA^2 + AB^2$.

But $AC^2 = EC^2$, and $2EC^2 = 2AE^2$, therefore $4AE^2 + 4BE^2 = AD^2 + DC^2 + BC^2 + AB^2$.

But $4AE^2 = AC^2$ and $4BE^2 = DB^2$; therefore

$$AC^2 + DB^2 = AD^2 + DC^2 + BC^2 + AB^2.$$

Proposition XX.—*Theorem.*

A line drawn parallel to the base of a triangle, divides the other two sides proportionally, fig. 116.

Let DE be drawn parallel to the side BC of the Fig. 116. triangle ABC, then

$$AD : DB :: AE : EC.$$

Join BE and DC. The two triangles BDE, DEC having the same base DE, and the same altitude, sioce both their vertices lie in a plain parallel to the base, are equal, (prop. 3, book iv.)

The triangles ADE, BDE, whose common vertex is E, have the same altitnde, and are to each other as their bases, (prop. 5, book iv.;) hence we have

$$ADE : BDE :: AD : DB.$$

The triangles ADE, DEC, whose common vertex is D, have also the same altitude, and are to each other as their bases; hence

$$ADE : DEC :: AE : EC.$$

But the triangles BDE, DEC are equal; and therefore, since those proportions have a common ratio, we obtaio

$$AD : DB :: AE : EC.$$

Cor. 1. Hence, (prop. 5, book ii.) we have $AD + DB : AD :: AE + EC : AE$, or $AB : AD :: AC : AE$; and also $AB : BD :: AC : CE$.

Cor. 2. If between two straight lines AB, CD, (fig. 117,) any oumber of parallels AC, EF, GH, Fig. 117 BD, &c. be drawn, those straight lines will be cut proportionally, and we shall have $AE : CF :: EG : FH :: GB : HD$.

For, let O be the point where AB and CD meet. In the triangle OEF, the line AC being drawn parallel to the base EF, we shall have $OE : AE :: OF : CF$, or $OE : OF :: AE : CF$. In the triangle OGH, we shall likewise have $OE : EG :: OF : FH$, or $OE : OF :: EG : FH$. And by reason of the common ratio $OE : OF$, those two proportions give $AE : CF :: EG : FH$. It may be proved in the same manner, that $EG : FH :: GB : HD$, and so on; hence the lines AB, CD are cut proportionally by the parallels AC, EF, GH, &c.

Proposition XXI.—*Theorem.*

If the sides of a triangle are cut proportionally by any line DE, so that we have $AD : DB :: AE : EC$, the line DE will be parallel to the base BC, fig. 118.

For if DE is not parallel to BC, suppose that DO Fig. 118. is parallel to it. Then, by the preceding theorem, we shall have $AD : BD :: AO : OC$. But, by hypothesis, we have $AD : DB :: AE : EC$; hence we must have $AO : OC :: AE : EC$, or $AO : AE :: OC : EC$; an impossible result, since AO, the one antecedent, is less than its consequent AE, and OC, the other antecedent, is greater than its consequent EC. Hence the parallel to BC, drawn from the point D, cannot differ from DE; hence DE is that parallel.

Scholium. The same conclusion would be true, if

the proportion : AB AD :: AC : AE were the proposed one. For this proportion would give us AB − AD : AD :: AC − AE : AE, or BD : AD :: CE : AE, (prop. 5, book ii.)

Proposition XXII.—*Theorem.*

The line which bisects any angle of a triangle, divides the base into two segments, which are proportional to the adjacent sides, fig. 119.

Fig. 119. Let AD bisect the angle BAC of the triangle ABC, then AD divide CB in the proportion of CA to BA, or

CA : BA :: CD : DB.

Through the point C, draw CE parallel to AD till it meet BA produced.

In the triangle BCE, the line AD is parallel to the base CE; hence (prop. 20, book iv.) we have the proportion BD : DC :: AB : AE.

But the triangle ACE is isosceles: for since AD, CE are parallel, we have the angle ACE = DAC, and the angle AEC = BAD, (prop. 19, book i.;) and, by hypothesis, DAC = BAD; hence the angle ACE = AEC, and consequently AE = AC. In place of AE in the above proportion, substitute AC, and we shall have BD : DC :: AB : AC.

Proposition XXIII.—*Theorem.*

Two equiangular triangles have their homologous sides proportional, and are similar, fig. 120.

Fig. 120. Let ABC, CDE be two triangles which have their angles equal, each to each, namely, BAC = CDE, ABC = DCE, and ACB = DEC; then the homologous sides, or the sides adjacent to the equal angles, will be proportional, so that we shall have BC : CE :: AB : CD :: AC : DE.

Place the homologous sides BC, CE in the same straight line; and produce the sides BA, ED till they meet in F.

Since BCE is a straight line, and the angle BCA is equal to CED, it follows (prop. 19, book i.) that AC is parallel to DE. In like manner, since the angle ABC is equal to DCE, the line AB is parallel to DC. Hence the figure ACDF is a parallelogram.

In the triangle BFE, the line AC is parallel to the base FE; hence (prop. 20, book iv.) we have BC : CE :: BA : AF; or, putting CD in the place of its equal AF,

BC : CE :: BA : CD.

In the same triangle BEF, if BF be considered as the base, CD is parallel to it; and we have the proportion BC : CE :: FD : DE; or putting AC in the place of its equal FD,

BC : CE :: AC : DE.

And finally, since both those proportions contain the same ratio BC : CE, we have

AC : DE :: BA : CD.

Thus the equiangular triangles BAC, CDE have their homologous sides proportional. But two figures are similar when they have their angles respectively equal, and their homologous sides proportional; consequently the equiangular triangles BAC, CDE, are two similar figures.

Cor. For the similarity of two triangles it is enough that they have two angles equal, each to each; since the third will also be equal in both, and the two triangles will be equiangular

Proposition XXIV.—*Theorem.*

Two triangles which have their homologous sides proportional, are equiangular and similar, fig. 121.

Let BC : EF :: AB : DE :: AC : DF; then will the triangles ABC, DEF have their angles equal, namely, A = D, B = E, C = F. Fig. 121.

At the point E, make the angle FEG = B, and at F, the angle EFG = C; the third G will be equal to the third A, and the two triangles ABC, EFG will be equiangular. Therefore, by the last theorem, we shall have BC : EF :: AB : EG; but, by hypothesis, BC : EF :: AB : DE; hence EG = DE. By the same theorem, we shall also have BC : EF :: AC : FG; and, by hypothesis, BC : EF :: AC : DF; hence FG = DF. Hence (prop. 12, book i.) the triangles EGF, DEF, having their three sides respectively equal, are themselves equal. But, by construction, the triangles EGF and ABC are equiangular; hence DEF and ABC are also equiangular and similar.

Scholium. By the last two propositions, it appears that in triangles, equality among the angles is a consequence of proportionality among the sides, and conversely; so that one of those conditions sufficiently determines the similarity of two triangles The case is different with regard to figures of more than three sides . even in quadrilaterals, the proportion between the sides may be altered without altering the angles, or the angles be altered without altering the proportion between the sides; and thus proportionality among the sides cannot be a consequence of equality among the angles of two quadrilaterals, or *vice versa*. It is evident, for example, that by drawing EF (fig. 122) parallel to BC, the angles of the quadrilateral AEFD, are made equal to those of ABCD, though the proportion between the sides is different; and, in like manner, without changing the four sides AB, BC, CD, AD, we can make the point B approach D or recede from it, which will change the angles. Fig. 122.

Proposition XXV.—*Theorem.*

Two triangles which have an equal angle included between proportional sides, are similar, fig. 123.

Let the angles A and D be equal; if AB : DE :: AC : DF, the triangle ABC is similar to DEF. Fig. 123.

Take AG = DE, and draw GH parallel to BC. The angle AGH (prop. 19, book i.) will be equal to the angle ABC; and the triangles AGH, ABC will be equiangular: hence AB : AG :: AC : AH. But, by hypothesis, AB : DE :: AC : DF; and, by construction, AG = DE: hence AH = DF. The two triangles AGH, DEF have an equal angle included between equal sides; therefore they are equal; but the triangle AGH is similar to ABC; therefore DEF is also similar to ABC.

Proposition XXVI.—*Theorem.*

Two triangles which have their homologous sides parallel, or perpendicular to each other, are similar, fig. 124 and 125.

First. If the side AB is parallel to DE, and BC to EF, the angle ABC will be equal to DEF; for ABC = AHC = DEC, (prop. 19, book i.;) and if AC is Fig. 124 and 125.

parallel to DF, the angle ACB will be equal to DFE, and also BAC to EDF; hence the triangles ABC, DEF are equiangular; hence they are similar.

Secondly. If the side DE is perpendicular to AB, and the side DF to AC, the two angles I and II of the quadrilateral AIDH will be right; and since all the four angles are together equal to four right angles, the remaining two IAH, IDH will be together equal to two right angles. But the two angles EDF, IDH are also equal to two right angles: hence the angle EDF is equal to IAH or BAC. In like manner, if the third side EF is perpendicular to the third BC, it may be shown that the angle DFE is equal to C, and DEF to B; hence the triangles ABC, DEF, which have the sides of the one perpendicular to the corresponding sides of the other, are equiangular and similar.

Scholium. In the case of the sides being parallel, the homologous sides are the parallel ones: in the case of their being perpendicular, the homologous sides are the perpendicular ones. Thus in the latter case DE is homologous with AB, DF with AC, and EF with BC.

The case of the perpendicular sides might present a relative position of the two triangles different from that exhibited in the diagram; but the equality of the respective angles might still be demonstrated, either by means of quadrilaterals like AIDH having two right angles, or by the comparison of two triangles having two of their angles vertical, and a right angle in each. Besides, we may always conceive a triangle DEF to be constructed within the triangle ABC, and such that its sides shall be parallel to those of the triangle compared with ABC; and then the demonstration given in the text will apply.

Proposition XXVII.—*Theorem.*

Any lines drawn through the vertex of a triangle, will divide the base, and a line parallel to the base, in the same proportion, fig. 126.

Fig. 126. Let AF, AG, AH be drawn from the vertex A to the base BC of the triangle ABC, and let DE be parallel to BC; then will DI : DF :: IK : FG :: KL : GH, &c.

For since DI is parallel to BF, the triangles ADI and ABF are equiangular; and DI : BF :: AI : AF; also, since IK is parallel to FG, we have in like manner AI : AF :: IK : FG; hence, the ratio AI: AF being common, DI : BF :: IK : FG. In the same manner IK : FG :: KL : GH; and so with the other segments: hence the line DE is divided at the points I, K, L, as the base BC at the points F, G, H.

Cor. Therefore if BC were divided into equal parts at the points F, G, H, the parallel DE would also be divided into equal parts at the points I, K, L.

Proposition XXVIII.—*Theorem.*

If from the right angle of a right angled triangle, a perpendicular be let fall on the hypothenuse; the two triangles thereby made, will be similar to the whole triangle, and to one another. Each side of the triangle will be a mean proportional between the whole base and the adjacent segment, and the perpendiculars will be a mean proportional between the two segments, fig. 127.

The triangles BAD and BAC have the common angle B, the right angle BDA = BAC, and therefore the third angle BAD of the one equal to the third C of the other; hence those two triangles are equiangular and similar. In the same manner it may be shown, that the triangles DAC and BAC are similar; hence all the three triangles are similar and equiangular. Fig. 127.

Again, the triangles BAD, BAC being similar, their homologous sides are proportional. But BD in the triangle ABD, and BA in the triangle ABC are homologous, because they lie opposite the equal angles BAD, BCA; the hypothenuse BA of the former is homologous with the hypothenuse BC of the latter: hence the proportion BD : BA :: BA : BC. By the same reasoning, we should find DC : AC :: AC : BC; hence each of the sides AB, AC is a mean proportional between the hypothenuse and the segment adjacent to that side.

Further, since the triangles ABD, ADC are similar, by comparing their homologous sides, we have BD : AD :: AD : DC; hence, the perpendicular AD is a mean proportional between the segments DB, DC of the hypothenuse.

Scholium. Since BD : AB :: AB : BC, the product of the extremes will be equal to that of the means, or $AB^2 = BD.BC$. For the same reason we have $AC^2 = DC.BC$; therefore $AB^2 + AC^2 = BD.BC + DC.BC = (BD+DC.)BC = BC.BC = BC^2$; or the square described on the hypothenuse BC is equal to the squares described on the two sides AB, AC. Thus we again arrive at the property of the square of the hypothenuse, by a path very different from that which formerly conducted us to it; and thus it appears, that the property of the square of the hypothenuse is a consequence of the more general property, that the sides of equiangular triangles are proportional. Thus the fundamental propositions of geometry are reduced, as it were, to this single one, that equiangular triangles have their homologous sides proportional.

It happens frequently, as in this instance, that by deducing consequences from one or more propositions, we are led back to some proposition already proved. In fact, the chief characteristic of geometrical theorems, and one indubitable proof of their certainty is, that, however we combine them together, provided only our reasoning be correct, the results we obtain are always perfectly accurate. The case would be different, if any proposition were false or only approximately true; it would frequently happen that on combining the propositions together, the error would increase and become perceptible. Examples of this are to be seen in all the demonstrations, in which the *reductio ad absurdum* method is employed. In such demonstrations, where the object is to show that two quantities are equal, we proceed by showing that if there existed the smallest inequality between the quantities, a train of accurate reasoning would lead us to a manifest and palpable absurdity; from which we are forced to conclude that the two quantities are equal.

Cor. If from a point A, (fig. 128,) in the circumference of a circle, two chords AB, AC be drawn to the extremities of a diameter BC, the triangle BAC (prop. 17, book iii.) will be right angled at A; hence, first, *the perpendicular* AD *is a mean proportional be-* Fig. 128.

Geometry. *tween the two segments* B D, D C, *of the diameter*, or what amounts to the same, $AD^2 = BD.DC$.

Hence also, in the second place, *the chord* A B *is a mean proportional between the diameter* B C *and the adjacent segment* B D, or what amounts to the same, $AB^2 = BD.BC$. In like manner, we have $AC^2 = CD.BC$; hence $AB^2 : AC^2 :: BD : DC$; and comparing AB^2 and AC^2 to BC^2, we have $AB^2 : BC^2 :: BD : BC$, and $AC^2 : BC^2 :: DC : BC$.

Proposition XXIX.—*Theorem.*

Two triangles having an equal angle, are to each other as the rectangles of the sides which contain that angle, fig. 129.

Fig. 129. That is, the triangle A B C is to the triangle A D E, as the rectangle A B . A C is to the rectangle A D . A E.

Draw B E. The triangles A B E, A D E, having the common vertex E, have the same altitude, and consequently (prop. 5, book iv.) are to each other as their bases: that is,

$$ABE : ADE :: AB : AD.$$

In like manner,

$$ABC : ABE :: AC : AE.$$

Multiply together the corresponding terms of those proportions, omitting the common term A B E; we have $ABC : ADE :: AB.AC : AD.AE$.

Proposition XXX.—*Theorem.*

Two similar triangles are to each other as the squares of their homologous sides, fig. 130.

Fig. 130. Let the angle A be equal to D, and the angle B = E. Then, first, by reason of the equal angles A and D, according to the last proposition, we shall have

$$ABC : DEF :: AB.AC : DE.DF.$$

Also, because the triangles are similar,

$$AB : DE :: AC : DF.$$

And multiplying the terms of this proportion by the corresponding terms of the identical proportion,

$$AC : DF :: AC : DF,$$

there will result

$$AB.AC : DE.DF :: AC^2 : DF^2.$$

Consequently,

$$ABC : DEF :: AC^2 : DF^2.$$

Therefore two similar triangles A B C, D E F are to each other as the squares of the homologous sides A C, D F, or as the squares of any other two homologous sides.

Proposition XXXI.—*Theorem.*

Two similar polygons are composed of the same number of triangles similar, each to each, and similarly situated, fig. 131.

Fig. 131. From any angle A, in the polygon A B C D E, draw diagonals A C, A D to the other angles. From the corresponding angle F, in the other polygon F G H I K, draw diagonals F H, F I to the other angles.

These polygons being similar, the angles A B C, F G H, which are homologous, will be equal, (def. 1 and 2,) and the sides A B, B C will also be proportional to F G, G H; that is, $AB : FG :: BC : GH$. Wherefore the triangles A B C, F G H have each an equal angle, contained between proportional sides; they are therefore similar; hence the angle B C A is equal to G H F. Take away these equal angles from the equal angles B C D, G H I; there remains A C D = F H I. But since the triangles A B C, F G H are similar, $AC : FH :: BC : GH$; and (def. 1) since the polygons are similar, $BC : GH :: CD : HI$; hence $AC : FH :: CD : HI$. But the angle A C D is equal to F H I; hence the triangles A C D, F H I have an equal angle in each, included between proportional sides, and are consequently similar. In the same manner all the remaining triangles may be shown to be similar; therefore two similar polygons are composed of the same number of triangles similar and similarly situated.

Book IV.

Scholium. The converse of the proposition is equally true: *If two polygons are composed of the same number of triangles similar and similarly situated, those two polygons will be similar.*

For the similarity of the respective triangles will give the angles A B C = F G H, B C A = G H F, A C D = F H I; hence B C D = G H I, likewise C D E = H I K, &c. Moreover we shall have $AB : FG :: BC : GH :: AC : FH :: CD : HI$, &c. hence the two polygons have their angles equal and their sides proportional; hence they are similar.

Proposition XXXII.—*Theorem.*

The contours or perimeters of similar polygons are to each other as the homologous sides; and the surfaces are to each other as the squares of those sides, fig. 131.

By the nature of similar figures, we have $AB : FG :: BC : GH :: CD : HI$, &c.; therefore (prop. 9, book ii.) the sum of the antecedents A B + B C + C D, &c. (the perimeter of the first polygon) is to the sum of the consequents F G + G H + H I, &c. (the perimeter of the second polygon) as any one antecedent is to its consequent, that is as the side A B is to its corresponding side F G. Fig. 131.

Again, since the triangles A B C, F G H are similar, we have (prop. 30, book iv.) the triangle $ABC : FGH :: AC^2 : FH^2$; and in like manner, from the similar triangles A C D, F H I, we shall have $ACD : FHI :: AC^2 : FH^2$; therefore, by reason of the common ratio, $AC^2 : FH^2$, it follows that

$$ABC : FGH :: ACD : FHI.$$

By the same mode of reasoning,

$$ACD : FHI :: ADE : FIK;$$

and so on, if there were more triangles. Consequently (prop. 9, book ii.) the sum of the antecedents A B C + A C D + A D E, or the polygon A B C D E, is to the sum of the consequents F G H + F H I + F I K, or to the polygon F G H I K, as one antecedent A B C is to its consequent F G H, or as AB^2 is to FG^2; hence the surfaces of similar polygons are to each other as the squares of the homologous sides.

Cor. If three similar figures were constructed, on the three sides of a right angled triangle, the figure on the hypothenuse would be equal to the sum of the other two: for the three figures are proportional to the squares of their homologous sides; but the square of the hypothenuse is equal to the sum of the squares of the two other sides; hence, &c.

Proposition XXXIII.—*Theorem.*

The segments of two chords which intersect each other in a circle, are reciprocally proportional, fig. 132.

That is, $AO : DO :: CO : OB$.

Fig. 132. Join AC and BD. In the triangles ACO, BOD the angles at O are equal, being vertical; the angle A is equal to the angle D, because both are inscribed in the same segment, (prop. 15, book iii.;) for the same reason the angle $C=B$; the triangles are therefore similar, and the homologous sides give the proportion, $AO : DO :: CO : OB$.

Cor. Therefore $AO \cdot OB = DO \cdot CO$; hence the rectangle under the two segments of the one chord is equal to the rectangle under the two segments of the other.

Proposition XXXIV.—*Theorem.*

If from the same point without a circle secants be drawn terminating in the concave arc, the whole secants will be reciprocally proportional to their external segments, fig. 133.

Fig. 133. That is, $OB : OC :: OD : OA$.

For, join AC, BD, then the triangles OAC, OBD have the angle O common; likewise the angle $B = C$ (prop. 15, book iii.;) these triangles are therefore similar; and their homologous sides give the proportion, $OB : OC :: OD : OA$.

Cor. The rectangle $OA \cdot OB$ is hence equal to the rectangle $OC \cdot OD$.

Scholium. This proposition bears a great analogy to the preceding, and differs from it only as the two chords AB, CD, instead of intersecting each other within the circle, cut each other externally. The following proposition may also be regarded as a particular case of the proposition just demonstrated.

Proposition XXXV.—*Theorem.*

If from a point without a circle, a tangent and a secant be drawn, the tangent will be a mean proportional between the secant and its external segment, fig. 134.

Fig. 134. That is, $OC : OA :: OA : OD$,

or $OA^2 = OC \cdot OD$.

For, joining AD and AC, the triangles OAD, OAC have the angle O common; also the angle OAD, formed by a tangent and a chord, has for its measure (prop. 18, book iii.) half of the arc AD; and the angle C has the same measure: hence the angle $OAD = C$; and the two triangles are similar, and we have the proportion,

$$OC : OA :: OA : OD,$$

which gives $OA^2 = OC \cdot OD$.

Proposition XXXVI.—*Theorem.*

If any angle of a triangle be bisected by a line which cuts the base; the rectangle of the segments of the base, together with the square of the bisecting line, is equal to the rectangle of the sides, including the bisected angle, fig. 135.

Fig. 135. Let AD bisect the angle BAC of the triangle ABC; then $BA \cdot AC = BD \cdot DC + DA^2$.

Describe a circle through the three points A, B, C; produce AD till it meets the circumference, and join CE.

The triangle BAD is similar to the triangle EAC; for, by hypothesis, the angle $BAD = EAC$; also the angle $B = E$, since they both have for measure half of the arc AC; hence these triangles are similar, and the homologous sides give the proportion, $BA : AE :: AD : AC$; hence $BA \cdot AC = DE \cdot AD$; but $AE = AD + DE$, and multiplying each of these equals by AD, we have $AE \cdot AD = AD^2 + AD \cdot DE$; now $AD \cdot DE = BD \cdot DC$, (prop. 33, book iv.;) hence finally, $BA \cdot AC = AD^2 + BD \cdot DC$.

Proposition XXXVII.—*Theorem.*

In any triangle, the rectangle of any two of its sides is equal to the rectangle of the perpendicular let fall on its third side, and the diameter of its circumscribing circle, fig. 136.

Let AD be the perpendicular upon BC, and EC the diameter of the circumscribing circle; then Fig. 136.

$$BA \cdot AC = AD \cdot EC.$$

For, joining AE, the triangles ABD, AEC are right angled, the one at D, the other at A; also the angle $B=E$; these triangles are therefore similar, and they give the proportion, $AB \cdot CE :: AD : AC$; and hence $AB \cdot AC = CE \cdot AD$.

Cor. If these equal quantities be multiplied by the same quantity BC there will result $AB \cdot AC \cdot BC = CE \cdot AD \cdot BC$; now $AD \cdot BC$ is double of the surface of the triangle, (prop. 8, book iv.;) therefore *the product of the three sides of a triangle is equal to its surface multiplied by twice the diameter of the circumscribed circle.*

Scholium. It may also be demonstrated, that the surface of a triangle is equal to its perimeter multiplied by half the radius of the inscribed circle.

For the triangles AOB, BOC, AOC, (fig. 137) Fig. 137. which have a common vertex at O, have for their common altitude the radius of the inscribed circle; hence the sum of these triangles will be equal to the sum of the bases AB, BC, AC, multiplied by half the radius OD; hence the surface of the triangle ABC is equal to the perimeter multiplied by half the radius of the inscribed circle.

Proposition XXXVIII.—*Theorem.*

In every quadrilateral inscribed in a circle, the rectangle of the two diagonals is equal to the sum of the rectangles of the opposite sides, fig. 138.

That is, $AC \cdot BD = AB \cdot CD + AD \cdot BC$. Fig. 138.

Take the arc $CO = AD$, and draw BO meeting the diagonal AC in I.

The angle $ABD = CBI$, since the one has for its measure half of the arc AD, and the other half of CO equal to AD; the angle $ADB = BCI$, because they are both inscribed in the same segment AOB; hence the triangle ABD is similar to the triangle IBC, and we have the proportion $AD : CI :: BD : BC$; hence $AD \cdot BC = CI \cdot BD$. Again, the triangle ABI is similar to the triangle BDC; for the arc AD being equal to CO, if OD be added to each of them, we shall have the arc $AO = DC$; hence the angle ABI is equal to DBC; also the angle BAI to BDC, because they are inscribed in the same segment; hence the triangles ABI, DBC are similar, and the homologous sides give the proportion, $AB : BD :: AI : CD$; hence $AB \cdot CD = AI \cdot BD$.

Adding the two results obtained, and observing that $AI \cdot BD + CI \cdot BD = (AI + CI.) BD = AC \cdot BD$, we shall have $AD \cdot BC + AB \cdot CD = AC \cdot BD$.

Scholium. Another theorem concerning the in-

Geometry scribed quadrilateral may be demonstrated in the same manner:

The similarity of the triangles A B D and B I C gives the proportion $BD : BC :: AB : BI$; hence $BI \cdot BD = BC \cdot AB$. If C O be joined, the triangle I C O, similar to A B I, will be similar to B D C, and will give the proportion $BD : CO :: DC : OI$; hence $OI \cdot BD = CO \cdot DC$, or, because $CO = AD$; $OI \cdot BD = AD \cdot DC$. Adding the two results, and observing that $BI \cdot BD + OI \cdot BD$ is the same as $BO \cdot BD$, we shall have $BO \cdot BD = AB \cdot BC + AD \cdot DC$.

If B P had been taken equal to A D, and C K P been drawn, a similar train of reasoning would have given us

$$CP \cdot CA = AB \cdot AD + BC \cdot CD.$$

But the arc B P being equal to C O, if B C be added to each of them, it will follow that $CBP = BCO$; the chord C P is therefore equal to the chord B O, and consequently $BO \cdot BD$ and $CP \cdot CA$ are to each other as B D is to C A; hence,

$$BD : CA :: AB \cdot BC + AD \cdot DC : AD \cdot AB + BC \cdot CD.$$

Therefore *the two diagonals of an inscribed quadrilateral are to each other, as the sums of the rectangles under the sides which meet at their extremities.*

These two theorems may serve to find the diagonals when the sides are given.

Proposition XXXIX.—*Theorem.*

Let P be a given point within a circle upon the radius A C, and let a point Q be taken externally upon the same radius produced, so that $CP : CA :: CA : CQ$; if from any point M of the circumference straight lines M P, M Q be drawn to the two points P and Q, these straight lines will every where have the same ratio, or $MP : MQ :: AP : AQ$, fig. 139.

Fig. 139. For by hypothesis, $CP : CA :: CA : CQ$; or substituting C M for C A, $CP : CM :: CM : CQ$; hence the triangles C P M, C Q M, have each an equal angle O contained by proportional sides; hence they are similar; and hence the third side M P is to the third side M Q, as C P is to C M or C A. But (prop. 5, book ii.) the proportion $CP : CA :: CA : CQ$ gives $CP : CA :: CA - CP : CQ - CA$, or $CP : CA :: AP : AQ$; therefore $MP : MQ :: AP : AQ$.

Problems relating to book IV.

Problem I.

To divide a given straight line into any number of equal parts, or into parts proportional to given lines, fig. 140.

Fig. 140. Let it, for example, be proposed to divide the line A B into five equal parts. Through the extremity A, draw the indefinite straight line A G; and taking A C of any magnitude, apply it five times upon A G; join the last point of division G, and the extremity B, by the straight line G B; then draw C I parallel to G B: A I will be the fifth part of the line A B; and thus, by applying A I five times upon A B, the line A B will be divided into five equal parts.

For, since C I is parallel to G B, the sides A G, A B (prop. 20, book iv.) are cut proportionally in C and I. But A C is the fifth part of A G, hence A I is the fifth part of A B.

Problems relative to Book IV. Again, let it be proposed to divide the line A B (fig. 141) into parts proportional to the given lines P, Q, R. Through A, draw the indefinite line A G; make $AC = P$, $CD = Q$, $DE = R$; join the extremities E and B; and through the points C, D draw C I, D F parallel to E B; the line A B will be divided into parts A I, I F, F B proportional to the given lines P, Q, R. Fig. 141.

For, by reason of the parallels C I, D F, E B, the parts A I, I F, F B are proportional to the parts A C, C D, D E; and, by construction, these are equal to the given lines P, Q, R.

Problem II.

To find a fourth proportional to three given lines A, B, G, fig. 142.

Draw the two indefinite lines D E, D F, forming any angle with each other. Upon D E take $DA = A$, and $DB = B$; upon D F take $DC = G$; join A C; and through the point B, draw B X parallel to A C; D X will be the fourth proportional required: for, since B X is parallel to A C, we have the proportion $DA : DB :: DC : DX$; now the three first terms of that proportion are equal to the three given lines; consequently D X is the fourth proportional required. Fig. 142.

Cor. A third proportional to two given lines A, B, may be found in the same manner, for it will be the same as a fourth proportional to the three lines, A, B, B.

Problem III.

To find a mean proportional between two given lines A and B, fig. 143.

Upon the indefinite line D F, take $DE = A$, and $EF = B$; upon the whole line D F, as a diameter, describe the semicircle D G F; at the point E, erect upon the diameter the perpendicular E G meeting the circumference in G; E G will be the mean proportional required. Fig. 143.

For the perpendicular E G, let fall from a point in the circumference upon the diameter, is a mean proportional between D E, D F, the two segments of the diameter, (prop. 28, book iv.;) and these segments are equal to the given lines A and B.

Problem IV.

To divide a line in extreme and mean ratio, that is into two parts, such that the greater part shall be a mean proportional between the whole line and the other part, fig. 144.

At the extremity B of the line A B, erect the perpendicular B C equal to the half of A B; from the point C as a centre, with the radius C B describe a semicircle; draw A C cutting the circumference in D; and take $AF = AD$: the line A B will be divided at the point F in the manner required; that is, we shall have $AB : AF :: AF : FB$. Fig. 144.

For A B, being perpendicular to the radius at its extremity, is a tangent; and if A C be produced till it again meets the circumference in E, we shall have (prop. 35, book iv.) $AE : AB :: AB : AD$; hence, (prop. 5, book ii.) $AE - AB : AB :: AB - AD : AD$. But since the radius is the half of A B, the diameter D E is equal to A B, and consequently A E

Geometry. — $AB = AD = AF$; also, because $AF = AD$, we have $AB - AD = FB$; hence $AF \cdot AB :: FB : AD$ or AF; whence (prop. 4, book ii.) $AB : AF :: AF : FB$.

Problem V.

Through a given point A, in the given angle B C D, to draw the line B D, so that the segments A B, A D, comprehended between the point A and the two sides of the angle, shall be equal, fig. 145.

Fig. 145. Through the point A draw A E parallel to C D, make $BE = CE$, and through the points B and A draw B A D; this will be the line required.

For, A E being parallel to C D, we have $BE : EC :: BA : AD$; but $BE = EC$; therefore $BA = AD$.

Problem VI.

To describe a square that shall be equal to a given parallelogram, or to a given triangle, fig. 146 and 147.

Fig. 146. Let A B C D be the given parallelogram, A B its base, D E its altitude; between A B and D E find a mean proportional X Y; then will the square constructed upon X Y be equal to the parallelogram A B C D.

For, by construction, $AB : XY :: XY : DE$; therefore $XY^2 = AB \cdot DE$; but $AB \cdot DE$ is the measure of the parallelogram, and XY^2 that of the square; consequently they are equal.

Fig. 147. Again, let A B C (fig. 147) be the given triangle, B C its base, A D its altitude: find a mean proportional between B C and the half of A D, and let X Y be that mean; the square constructed upon X Y will be equal to the triangle A B C.

For, since $BC : XY :: XY : \frac{1}{2} AD$, it follows that $XY^2 = BC \cdot \frac{1}{2} AD$; hence the square constructed upon X Y is equal to the triangle A B C.

Problem VII.

Upon a given line to describe a rectangle that shall be equal to a given rectangle, fig. 148.

Fig. 148. Let A D be the given line, and A B F C the given rectangle.

Find a fourth proportional to the three lines A D, A B, A C, and let A X be that fourth proportional; a rectangle constructed with the lines A D and A X will be equal to the rectangle A B F C.

For, since $AD : AB :: AC : AX$, it follows that $AD \cdot AX = AB \cdot AC$; hence the rectangle A D E X is equal to the rectangle A B F C.

Problem VIII.

To find two lines which shall have the same ratio to each other, as the rectangle of the two given lines A and B has to the rectangle of the two given lines C and D, fig. 149.

Fig. 149. Let X be a fourth proportional to the three lines B, C, D; then will the two lines A and X have the same ratio to each other as the rectangles A B and C D.

For, since $B : C :: D : X$, it follows that $C \cdot D = B \cdot X$; hence $A \cdot B : C \cdot D :: A \cdot B : B \cdot X :: A : X$.

Cor. Hence to obtain the ratio of the squares constructed upon the given lines A and C, find a third proportional X to the lines A and C so that $A : C :: C : X$; then $A^2 : C^2 :: A : X$.

Problems relative to Book IV.

Problem IX.

To find two lines that shall have the same ratio to each other as the product of the three given lines A, B, C, has to the product of the three given lines, P, Q, R, fig. 150.

Fig. 150. Find a fourth proportional X to the three given lines A, B, C: find also a fourth proportional Y to the three given lines, P, Q, R. The two lines X, Y will be to each other as the products $A \cdot B \cdot C$, $P \cdot Q \cdot R$.

For, since $P : A :: B : X$, it follows that $A \cdot B = P \cdot X$; and multiplying each of these equals by C, we have $A \cdot B \cdot C = C \cdot P \cdot X$. In like manner, since $C : Q :: R : Y$, it follows that $Q \cdot R = C \cdot Y$; and multiplying each of these equals by P, we have $P \cdot Q \cdot R = P \cdot C \cdot Y$: hence the product $A \cdot B \cdot C$ is to the product $P \cdot Q \cdot R$ as $C \cdot P \cdot X$ is to $P \cdot C \cdot Y$, or as X is to Y.

Problem X.

To find a triangle that shall be equal to a given polygon, fig. 151.

Fig. 151. Let A B C D E be the given polygon. Draw first the diagonal C E cutting off the triangle C D E; through the point D, draw D F parallel to C E, and meeting A E produced; join C F: the polygon A B C D E will be equal to the polygon A B C F, which has one side less than the original polygon.

For the triangles C D E, C F E have the base C E common; they have also the same altitude, since their vertices D, F, are situated in a line D F parallel to the base; these triangles are therefore equal. Add to each of them the figure A B C E, and there will result the polygon A B C D E equal to the polygon A B C F.

The angle B may in like manner be cut off, by substituting for the triangle A B C the equal triangle A G C, and thus the pentagon A B D E will be changed into an equal triangle G C F.

The same process may be applied to every other figure; for, by successively diminishing the number of its sides, one being retrenched at each step of the process, the equal triangle will at last be found.

Scholium. We have already seen that every triangle may be changed into an equal square; and thus a square may always be found equal to a given rectilineal figure, which operation is called *squaring* the rectilineal figure, or finding the *quadrature* of it.

The problem of *the quadrature of the circle* consists in finding a square equal to a circle whose diameter is given.

Problem XI.

To find the side of a square which shall be equal to the sum or the difference of two given squares, fig. 152.

Fig. 152 Let A and B be the sides of the given squares.

First. If it is required to find a square equal to the sum of these squares, draw the two indefinite lines E D, E F at right angles to each other; take $ED = A$, and $EC = B$; join D G: this will be the side of the square required.

For the triangle DEG being right angled, the square constructed upon DG is equal to the sum of the squares upon ED and EG.

Secondly. If it is required to find a square equal to the difference of the given squares, form in the same manner the right angle FEH; take GE equal to the shorter of the sides A and B; from the point G as a centre, with a radius GH, equal to the other side, describe an arc cutting EH in H: the square described upon EH will be equal to the difference of the squares described upon the lines A and B.

For the triangle GEH is right angled, the hypothenuse $GH = A$, and the side $GE = B$; hence the square constructed upon EH, &c.

Scholium. A square may thus be found equal to the sum of any number of squares; for the construction which reduces two of them to one, will reduce three of them to two, and these two to one, and so of others. It would be the same, if any of the squares were to be subtracted from the sum of the others.

Problem XII.

To construct a square which shall be to a given square ABCD *as the line* M *is to the line* N, fig. 153.

Fig. 153. Upon the indefinite line EG, take $EF = M$, and $FG = N$; upon EG as a diameter describe a semicircle, and at the point F erect the perpendicular FH. From the point H, draw the chords HG, HE, which produce indefinitely: upon the first take HK equal to the side AB of the given square, and through the point K draw KI parallel to EG; HI will be the side of the square required.

For, by reason of the parallels KI, GE, we have $HI : HK :: HE : HG$; hence $HI^2 : HK^2 :: HE^2 : HG^2$; but in the right angled triangle EHG (prop. 28, book iv.) the square of HE is to the square of HG as the segment EF is to the segment FG, or as M is to N; hence $HI^2 : HK^2 :: M : N$. But $HK = AB$; therefore the square described upon HI is to the square described upon AB as M is to N.

Problem XIII.

Upon the side FG, *homologous to* AB, *to describe a polygon similar to the given polygon* ABCDE, fig. 154.

Fig. 154. In the given polygon, draw the diagonals AC, AD; at the point F make the angle $GFH = BAC$, and at the point G the angle $FGH = ABC$; the lines FG, GH will cut each other in H, and FGH will be a triangle similar to ABC. In the same manner upon FH, homologous to AC, construct the triangle FIH similar to ADC; and upon FI, homologous to AD, construct the triangle FIK similar to ADE. The polygon FGHIK will be similar to ABCDE, as required.

For, these two polygons are composed of the same number of triangles, which are similar and similarly situated, (prop. 3, book iv.)

Problem XIV.

Two similar figures being given, to construct a figure which shall be similar to one of them, and equal to their sum or their difference.

Let A and B be homologous sides of the two given figures. Find a square equal to the sum or to the difference of the squares described upon A and B; let X be the side of that square; then will X be the side of the figure required, be the side which is homologous to the sides A and B in the given figures. The figure itself may then be constructed on X, by the last problem.

For, the similar figures are as the squares of their homologous sides; now the square of the side X is equal to the sum, or to the difference, of the squares described upon the homologous sides A and B; therefore the figure described upon the side X is equal to the sum, or to the difference, of the similar figures described upon the sides A and B.

Problem XV.

To construct a figure similar to a given one, and bearing to it any given ratio of M *to* N.

Let A be a side of the given figure, X the homologous side of the figure required. The square of X must be to the square of A as M is to N; hence X will be found by problem 12; and knowing X, the rest will be accomplished by problem 13.

Problem XVI.

To construct a figure similar to one given figure, and equal to another, fig. 156.

Find M the side of a square equal to the figure P, and N the side of a square equal to the figure Q. Let X be a fourth proportional to the three given lines M, N, AB; upon the side X, homologous to AB, describe a figure similar to the figure P; it will also be equal to the figure Q. Fig. 156.

For, calling Y the figure described upon the side X, we have $P : Y :: AB^2 : X^2$; but, by construction, $AB : X :: M : N$, or $AB^2 : X^2 :: M^2 : N^2$; hence $P : Y :: M^2 : N^2$. But by construction also, $M^2 = P$ and $N^2 = Q$; therefore $P : Y :: P : Q$; consequently $Y = Q$; hence the figure Y is similar to the figure P, and equal to the figure Q.

Problem XVII.

To construct a rectangle equal to a given square C, *and having its adjacent sides together equal to a given line* AB, fig. 157.

Upon AB as a diameter, describe a semicircle; draw the line DE parallel to the diameter, at a distance AD equal to the side of the given square C; from the point E, where the parallel cuts the circumference, draw EF perpendicular to the diameter; AF and FB will be the sides of the rectangle required. Fig. 157.

For their sum is equal to AB, and their rectangle $AF . FB$ is equal to the square of EF, or to the square of AD; hence that rectangle is equal to the given square C.

Scholium. To render the problem possible, the distance AD must not exceed the radius; that is, the side of the square C must not exceed the half of the line AB.

Problem XVIII.

To construct a rectangle that shall be equal to a given

 square C, and the difference of whose adjacent sides shall be equal to a given line A B, fig. 158.

Fig. 158. Upon the given line A B as a diameter, describe a semicircle; at the extremity of the diameter draw the tangent A D, equal to the side of the square C; through the point D and the centre O draw the secant D F: then will D E and D F be the adjacent sides of the rectangle required.

For, first, the difference of their sides is equal to the diameter E F or A B; secondly, the rectangle D E, D F is equal to AD^2, (prop. 35, book iv.;) hence that rectangle is equal to the given square C.

BOOK V.

Of regular polygons, and the measure of the circle.

Definition.

A regular polygon is one having all its angles and sides equal.

Proposition I.—*Theorem.*

All regular polygons of the same number of sides are similar, fig. 159.

Fig. 159. Let ABCDEF, *abcdef*, be two regular polygons, (in this case hexagons.) The sum of all the angles is the same in both figures, being each equal to eight right angles, (prop. 25, book i.) and the number of angles in each are also equal, and equal to each other; that is, each is equal to one-sixth of eight right angles. Again, since the polygons are regular, by hypothesis, the sides A B, B C, C D, &c. are all equal, as are also *a b*, *b c*, *cd*, &c. Whence A B : *a b* : : B C : *b c* : : C D : *c d*, &c. That is, the two figures have their angles equal, and the sides about those angles proportional; they are therefore similar.

Proposition II.—*Theorem.*

To inscribe a square in a given circle, fig. 160.

Fig. 160. Draw two diameters A C, B D, cutting each other at right angles; join their extremities, A, B, C, D; the figure A B C D will be the square required. For the angle A O B, B O C, &c. being equal, the chords A B, B C, &c. are also equal; and the angles A B C, B C D, &c. being in semicircles, are right angles. The figure is therefore equilateral, and its angles right angles; it is therefore a square.

Scholium. Since the triangle is right angles, $BD^2 = BC^2 + DC^2$ or $2DC^2 = BD^2$ or $DC\sqrt{2} = BD$ or $DC : BD :: 1 : \sqrt{2}$; and in the same way, since $BC^2 = 2BO^2$; $BC : BO :: \sqrt{2} : 1$; that is, the side of the inscribed square is to radius; as the diameter is to the side of the inscribed square, the ratio in both cases being as the square root of 2 to unity.

Proposition III.—*Theorem.*

To inscribe an equilateral triangle and a regular hexagon in a given circle, fig. 161.

Fig. 161. *First.* To inscribe the regular hexagon in a circle. From any point A in the circle apply the line A B equal to the radius, and join BO, O being the centre; then because A B O is an equilateral triangle, each of its angles is one-third of two right angles, (prop. 24, book i.) or one-sixth of four right angles; consequently the arc A B is one-sixth of the whole circumference, because it is the measure of the angle A O B, (prop. 14, book iii.) Therefore the line A B, applied six times in the circumference from A to B, from B to C, from C to D, will be the regular hexagon required.

Join now A C, C E, E A, and A E C will be the equilateral triangle, as is obvious.

Scholium. The figure A B C O is a parallelogram, and a rhombus, since $AB = BC = CO = AO$, (prop. 19, book iv.;) the sum of the squares of the diagonals $AC^2 + BO^2$ is equal to the sum of the squares of the sides; that is, to $4AB^2$, or $4BO^2$; and taking away BO from both, there will remain $AC^2 = 3BO^2$; hence $AC^2 : BO^2 :: 3 : 1$, or $AC : BO :: \sqrt{3} : 1$; hence *the side of the inscribed equilateral triangle is to the radius, as the square root of three is to unity.*

Proposition IV.—*Problem.*

In a given circle, to inscribe a regular decagon; then a pentagon, and a pentedecagon, fig. 162.

Divide the radius A O in extreme and mean ratio (prop. 4, book iv.) at the point M; take the chord A B equal to O M the greater segment; A B will be the side of the regular decagon, and will require to be applied ten times to the circumference. Fig. 162.

For, joining M B we have, by construction, $AO : OM :: OM : AM$; or, since $AB = OM$, $AO : AB :: AB : AM$; hence the triangles A B O, A M B have a common angle A, included between proportional sides; hence (prop. 25, book iv.) they are similar. Now the triangle O A B being isosceles, A M B must be isosceles also, and $AB = BM$; besides $AB = OM$; hence also $MB = OM$; hence the triangle B M O is isosceles.

Again, the angle A M B being exterior to the isosceles triangle B M O, is double of the interior angle O, (prop. 24, book i.;) but the angle $AMB = MAB$; hence the triangle O A B is such, that each of the angles at its base, O A B or O B A, is double of O the angle at its vertex; hence the three angles of the triangles are together equal to five times the angle O, which consequently is the fifth part of the two right angles, or the tenth part of four; hence the arc A B is the tenth part of the circumference, and the chord A B is the side of the regular decagon.

Cor. 1. By joining the alternate angles of the regular decagon, the regular pentagon A C E G I will also be formed.

Cor. 2. A B being still the side of the decagon, let A L be the side of the hexagon; the arc B L will then, with reference to the whole circumference, be $\frac{1}{6} - \frac{1}{10}$, or $\frac{1}{15}$; hence the chord B L will be the side of the pentedecagon or regular polygon of fifteen sides. It is evident, also, that the arc C L is the third of C B.

Scholium. Any regular polygon being inscribed, if the arcs subtended by its sides be severally bisected, the chords of those semi-arcs will form a new regular polygon of double the number of sides: thus, it is plain, the square may enable us successively to inscribe regular polygons of 8, 16, 32, &c. sides. And in like manner, by means of the hexagon, regular polygons of 12, 24, 48, &c. sides may be inscribed; by means of the decagon, polygons of 20, 40, 80, &c. sides; by

 means of the pentedecagon, polygons of 30, 60, 120, &c. sides.*

Proposition V.—*Problem.*

A regular inscribed polygon A B C D, &c. *being given, to circumscribe a similar polygon about the same circle,* fig. 163.

Fig. 163. At T, the middle point of the arc A B, apply the tangent G H, which (prop. 22, book iii.) will be parallel to A B; do the same at the middle point of each of the arcs B C, C D, &c.; those tangents, by their intersections, will form the regular circumscribed polygon G H I K, &c., similar to the inscribed one.

It is evident, in the first place, that the three points O, B, H, lie in the same straight line; for the right angled triangles O T H, O H N, having the common hypothenuse O H, and the side O T = O N, must be equal; and consequently the angle T O H = H O N, wherefore the line O H passes through the middle point B of the arc T N. For a like reason, the point I is in the prolongation of O C; and so with the rest. But since B H is parallel to A B, and H I to B C, the angle G H I = A B C; in like manner, H I K = B C D; and so with all the rest: hence the angles of the circumscribed polygon are equal to those of the inscribed one. And farther, by reason of the same parallels, we have G H : A B : : O H : O B, and H I : B C : : O H : O B; therefore G H : A B : : H I : B C. But A B = B C, therefore G H = H I. For the same reason, H I = I K, &c.; hence the sides of the circumscribed polygon are all equal; and this polygon is regular, and similar to the inscribed one.

Cor. 1. Reciprocally, if the circumscribed polygon G H I K, &c. were given, and the inscribed one A B C, &c. were required to be deduced from it, it would only be necessary to draw from the angles G, H, I, &c. of the given polygon, straight lines O G, O H, &c. meeting the circumference in the points A, B, C, &c.; then to join those points by the chords A B, B C, &c.; which would form the inscribed polygon. An easier solution of this problem would be simply to join the points of contact T, N, P, &c. by the chords T N, N P, &c. which likewise would form an inscribed polygon similar to the circumscribed one.

Cor. 2. Hence we may circumscribe about a circle any regular polygon, which can be inscribed within it; and conversely.

Proposition VI.—*Theorem.*

The area of a regular polygon is equal to its perimeter multiplied by half the radius of the inscribed circle, fig. 163.

Fig. 163. Let the regular polygon be G H I K, &c. the triangle G O H will be measured by G H × ½ O T; the triangle O H I by H I × ½ O N: but O N = O T; hence the two triangles taken together will be measured by (G H + H I) × ½ O T. And, by continuing the same operation for the other triangles, it will appear that the sum of them all, or the whole polygon, is measured by the sum of the bases G H, H I, I K, &c. or the perimeter of the polygon, multiplied into ½ O T, or half the radius of the inscribed circle.

Scholium. The radius O T of the inscribed circle is obviously the perpendicular let fall from the centre to one of the sides; and is sometimes named the *apothem* of the polygon.

Proposition VII.—*Theorem.*

The perimeters of two regular polygons, having the same number of sides, are to each other as the radii of the circumscribed circles, and also as the radii of the inscribed circles; and their areas are to each other as the squares of those radii, fig. 163.

Let A B be a side of the one polygon, O the centre, and consequently O A the radius of the circumscribed circle, and O D, perpendicular to A B, the radius of the inscribed circle; let *a b*, in like manner, be a side of the other polygon, *o* its centre, *o a* and *o d* the radii of the circumscribed and the inscribed circle. The perimeters of the two polygons are to each other as the sides A B and *a b*; but the angles A and *a* are equal, being each half of the angle of the polygon; so also are the angles B and *b*; hence the triangles A B O, *a b o* are similar, as are likewise the right angled triangles A D O, *a d o*; also A B : *a b* : : A O : *a o* : : D O : *d o*; therefore the perimeters of the polygons are to each other as the radii A O, *a o* of the circumscribed circles, and also as the radii D O, *d o* of the inscribed circles. Fig. 163.

Again the areas of those polygons are to each other as the squares of the homologous sides A B, *a b*; they are therefore likewise to each other as the squares of A O, *a o* the radii of the circumscribed circles, or as the squares of O D, *o d* the radii of the inscribed circles.

Proposition VIII.—*Lemma.*

Any curve, or any polygonal line, which envelopes the convex line A M B *from one end to the other, is longer than* A M B *the enveloped line,* fig. 164.

By the term convex line is to be understood a line, polygonal or curve, or partly curve and partly polygonal, such that a straight line cannot cut it in more than two points. If in the line A M B there were any sinuosities or re-entring portions, it would cease to be convex, because a straight line might evidently cut it in more than two points. The arcs of a circle are essentially convex; but the present proposition extends to any line which fulfils the required condition. Fig. 164.

This being premised, if the line A M B be not shorter than any of those which envelope it, there will be found among the latter a line shorter than all the rest, which is shorter than A M B, or, at most, equal to it. Let A C D E B be this enveloping line; any where between those two lines draw the straight line P Q, not meeting, or at least only touching, the line A M B. The straight line P Q is shorter than P C D E Q; hence if, instead of the part P C D E Q, we substitute the straight line P Q, the enveloping line A P Q B will be shorter than A P D Q B. But, by hypothesis, this latter was shorter than any other; hence that hypothesis was false; consequently all of the enveloping lines are longer than A M B.

* It was long supposed, that, besides the polygons here mentioned, no other could be inscribed by the operations of elementary geometry, or what amounts to the same, by the resolution of equations of the first and second degree. But M. Gauss, of Goettingen, at length proved, in a work entitled *Disquisitiones Arithmeticæ*, Lipsiæ, 1801, that by the method in question, a regular polygon of 17 sides might be inscribed, and generally a regular polygon of $2^n + 1$ sides, provided $2^n + 1$ be a prime number. See also Barlow's *Essay on the Theory of Numbers.*

PROPOSITION IX.—*Lemma.*

Two concentric circles being given, a regular polygon may always be inscribed within the greater, the sides of which shall not meet the circumference of the less; and likewise, a regular polygon may always be described about the less, the sides of which shall not meet the circumference of the greater, fig. 165.

Fig. 165. Let C A, C B be radii of the given circles. At the point A, apply the tangent D E, terminating in the greater circumference at D and E; inscribe within this greater circumference any regular polygons, by the methods already explained; next bisect the arcs subtended by its sides, and draw the chords of those half arcs; a polygon will thus be found, having twice as many sides. Continue the bisection, till an arc is obtained less than D B E. Let M B N be that arc, the middle point of it being supposed to lie at B: it is plain that the chord M N will be farther from the centre than D E; and that consequently the regular polygon, of which M N is a side, cannot meet the circumference, of which C A is the radius.

Now, the same construction remaining, join C M and C N, meeting the tangent D E in P and Q; P Q will be the side of a polygon described about the less circumference, similar to that polygon inscribed within the greater, of which the side is M N. And it is evident, that this circumscribed polygon having P Q for its side, can never meet the greater circumference, C P being less than C M.

Hence, by the same operation, a regular polygon may be inscribed within the greater circumference, and a similar one described about the less, both of which shall have their sides included between the two circumferences.

Scholium. If two concentric sectors F C G, I C H be given, a *portion of a regular polygon* may, in like manner, be inscribed in the greater, or circumscribed about the less, so that the perimeters of the two polygons shall be included between the two circumferences. For this purpose, it will be sufficient to divide the arc F B G successively into 2, 4, 8, 16, &c. equal parts, till a part smaller than D B E is obtained.

By the expression, *portion of a regular polygon*, is here meant the figure terminated by a series of equal chords inscribed in the arc F G, from one of its extremities to the other. This portion has all the principal properties of regular polygons; it has its angles equal, and its sides equal, it can be inscribed in a circle, or circumscribed about one: yet, properly speaking, it forms part of a regular polygon only in those cases where the arc subtended by one of its sides is an aliquot part of the circumference.

PROPOSITION X.—*Theorem.*

The circumferences of circles are to each other as their radii, and the surfaces as the squares of those radii, fig. 166.

Fig. 166. For the sake of brevity, let us designate the circumference whose radius is C A by *circ.* C A; we are to show that *circ.* C A : *circ.* O B : : C A : O B.

If this proposition is not true, C A must be to O B as *circ.* C A is to a fourth term less or greater than *circ* O B: suppose it less; and that, if possible, C A : O B : : *circ.* C A : *circ.* O D.

In the circle of which O B is the radius inscribe a regular polygon E F G K L E, such that the sides of it shall not meet the circumference of which O D is the radius by the last proposition; inscribe a similar polygon, M N P F M, in the circle of which A C is the radius.

Then, since those polygons are similar, their perimeters M N P S M, E F G K E will be to each other (prop. 7, book v.) as C A, O B, the radii of the circumscribed circles, that is M N P S M : E F G K E : : C A : O B. But, by hypothesis, C A : O B : : *circ.* C A : *circ.* O D; therefore M N P S M : E F G K E : : *circ.* C A : *circ.* O D; which proportion is false, because (prop. 8, book v.) the perimeter M N S P M is less than *circ.* C A, while on the contrary E F G K E is greater than *circ.* O D; therefore it is impossible that C A can be to O B as *circ.* C A is to a circumference less than *circ.* O B: or, in more general terms, it is impossible that one radius can be to another, as the circumference described with the former radius is to a circumference less than the one described with the latter radius.

Hence, too, we conclude it to be equally impossible that C A can be to O B as *circ.* C A is to a circumference greater than *circ.* O B; for if this were the case, by reversing the ratios, we should have O B to C A as a circumference greater than *circ.* O B is to *circ.* C A; or, what amounts to the same thing, as *circ.* O B is to a circumference less than *circ.* C A; and therefore one radius would be to another as the circumference described with the former radius is to a circumference less than the one described with the latter radius; a conclusion shown above to be erroneous.

And since the fourth term of this proportion C A : O B : : *circ.* C A : x can neither be greater nor less than *circ.* O B, it must be equal to *circ.* O B: consequently the circumference of circles are to each other as their radii.

By the same construction, a similar train of reasoning would show, that the surfaces of circles are to each other as the squares of their radii. We need not enter upon any farther details respecting this proposition, particularly as it forms a corollary of the following theorem:

Cor. The similar arcs A B, D E (fig. 167) are to each other as their radii A C, D O; and the similar sectors A C B, D O E are to each other as the squares of those radii. Fig. 167.

For, since the arcs are similar, the angle C (def. 1, book iv.) is equal to the angle O; but C is to four right angles (prop. 5, book. iii.) as the arc A B is to the whole circumference described with the radius A C; and O is to four right angles, as the arc D E is to the circumference described with the radius O D; hence the arcs A B, D E are to each other as the circumferences of which they form part; but these circumferences are to each other as their radii AC, DO; therefore *arc* A B : *arc* D E : : A C : D O.

For a like reason, the sectors A C B, D O E are to each other as the whole circles; which again are as the squares of their radii; therefore *sect.* A C B : *sect.* D O E : : $AC^2 : DO^2$

PROPOSITION XI.—*Theorem.*

The area of a circle is equal to the product of its circumference by half the radius, fig. 168.

Geometry. Fig. 168.

Let us designate the surface of the circle whose radius is C A by *surf.* C A; we shall have *surf.* C A $= \frac{1}{2}$ C A $\times$ *circ.* C A.

For if $\frac{1}{2}$ C A $\times$ *circ.* C A be not the area of the circle whose radius is C A, it must be the area of a circle either greater or less. Let us first suppose it to be the area of a greater circle; and, if possible, that $\frac{1}{2}$ C A $\times$ *circ.* C A $=$ *surf.* C B.

About the circle whose radius is C A describe a regular polygon D E F G, &c. such (prop. 9, book v.) that its sides shall not meet the circumference whose radius is C B. The surface of this polygon will be equal (prop. 6, book v.) to its perimeter D E + E F + F G + &c. multiplied by $\frac{1}{2}$ A C; but the perimeter of the polygon is greater than the inscribed circumference enveloped by it on all sides; hence the surface of the polygon D E F G, &c. is greater than $\frac{1}{2}$ A C $\times$ *circ.* A C, which by the supposition is the measure of the circle whose radius is C B; thus the polygon must be greater than that circle. But in reality it is less, being contained wholly within the circumference; hence it is impossible that $\frac{1}{2}$ C A $\times$ *circ.* A C can be greater than *surf.* C A; in other words, it is impossible that the circumference of a circle multiplied by half its radius can be the measure of a greater circle.

In the second place, we assert it to be equally impossible that this product can be the measure of a smaller circle. To avoid the trouble of changing our figure, let us suppose that the circle in question is the one whose radius is C B; we are to show that $\frac{1}{2}$ C B $\times$ *circ.* C B cannot be the measure of a smaller circle, of the circle, for instance, whose radius is C A. Grant it to be so; and that, if possible, $\frac{1}{2}$ C B $\times$ *circ.* C B $=$ *surf.* C A.

Having made the same construction as before, the surface of the polygon D E F G, &c. will be measured by (D E + E F + F G + &c.) $\times$ $\frac{1}{2}$ C A; but the perimeter D E + E F + F G + &c. is less than *circ.* C B, being enveloped by it on all sides; hence the area of the polygon is less than $\frac{1}{2}$ C A $\times$ *circ.* C B, and still more than $\frac{1}{2}$ C B $\times$ *circ.* C B. Now, by the supposition, this last quantity is the measure of the circle whose radius is C A; hence the polygon must be less than the inscribed circle, which is absurd; it is therefore impossible that the circumference of a circle multiplied by half its radius, can be the measure of a smaller circle.

Hence, finally, the circumference of a circle multiplied by half its radius is the measure of that circle itself.

Cor. 1. The surface of a sector is equal to the arc of that sector multiplied by half its radius.

Fig. 169. For (fig. 169) the sector A C B is to the whole circle as the arc A M B is to the whole circumference A B D, or as A M B $\times$ $\frac{1}{2}$ A C is to A B D $\times$ $\frac{1}{2}$ A C. But the whole circle is equal to A B D $\times$ $\frac{1}{2}$ A C; hence the sector A C B is measured by A M B $\times$ $\frac{1}{2}$ A C.

Cor. 2. Let the circumference of the circle whose diameter is unity be denoted by π; then, because circumferences are to each other as their radii or diameters, we shall have the diameter 1 to its circumference π as the diameter 2 C A is to the circumference whose radius is C A, that is, $1 : \pi :: 2$ C A : *circ.* C A, therefore *circ.* C A $= 2\pi \times$ C A. Multiply both terms by $\frac{1}{2}$ C A; we have $\frac{1}{2}$ C A $\times$ *circ.* C A $= \pi \times$ C A^2, or *surf.* C A $= \pi \times$ C A^2; hence the surface of a circle is equal to the product of the square of its radius by the constant number π, which represents the circumference whose diameter is 1, or the ratio of the circumference to the diameter.

Book V.

In like manner, the surface of the circle, whose radius is O B, will be equal to $\pi \times$ O B^2; but $\pi \times$ C A^2 : $\pi \times$ O B^2 :: C A^2 : O B^2; hence *the surfaces of circles are to each other as the squares of their radii*, which agrees with the preceding theorem.

Scholium. It is of course understood, that the problem of the quadrature of the circle consists in finding a square equal in surface to a circle, the radius of which is known. Now it has just been proved, that a circle is equal to the rectangle contained by its circumference and half its radius; and this rectangle may be changed into a square, by finding (prop. 3, book v.) a mean proportional between its length and its breadth. To square the circle, therefore, is to find the circumference when the radius is given; and for effecting this, it is enough to know the ratio of the circumference to its radius or its diameter.

Hitherto the ratio in question has never been determined except approximately; but the approximation has been carried so far, that a knowledge of the exact ratio would afford no real advantage whatever beyond that of the approximate ratio. Accordingly, this problem, which engaged geometers so deeply, when their methods of approximation were less perfect, is now sunk to the rank of those useless questions, with which no one possessing the slightest tincture of geometrical science will occupy any portion of his time.

Archimedes showed that the ratio of the circumference to the diameter is included between $3\frac{10}{70}$ and $3\frac{10}{71}$; hence $3\frac{1}{7}$ or $\frac{22}{7}$ affords at once a pretty accurate approximation to the number above designated by π; and the simplicity of this first approximation has brought it into very general use. *Metius*, for the same number, found the much more accurate value $\frac{355}{113}$. At last the value of π, developed to a certain order of decimals, was found by other calculators to be 3.1415926535897932, &c.; and some have had patience enough to continue these decimals to the hundred and twenty-seventh, or even to the hundred and fortieth place. Such an approximation is evidently equivalent to perfect correctness: the root of an imperfect power is in no case more accurately known.

The following problems will exhibit two of the simplest elementary methods of obtaining those approximations.

PROPOSITION XII.—*Problem.*

The surface of a regular inscribed polygon, and that of a similar polygon circumscribed, being given; to find the surfaces of the regular inscribed and circumscribed polygons having double the number of sides, fig. 170.

Let A B be a side of the given inscribed polygon; E F, parallel to A B, a side of the circumscribed polygon; C the centre of the circle. If the chord A M and the tangents A P, B Q be drawn, A M will be a side of the inscribed polygon, having twice the number of sides; and (prop. 5, book v.) P Q, double of P M, will be a side of the similar circumscribed polygon. Now, as the same construction will take place at each of the angles equal to A C M, it will be sufficient to consider A C M by itself, the triangles Fig. 170.

connected with it being evidently to each other as the whole polygons of which they form part. Let A, then, be the surface of the inscribed polygon whose side is A B, B that of the similar circumscribed polygon; A′ the surface of the polygon whose side is A M, B′ that of the similar circumscribed polygon; A and B are given; we have to find A′ and B′.

First. The triangles A C D, A C M, having the common vertex A, are to each other as their bases C D, C M; they are likewise to each other as the polygons A and A′, of which they form part; hence A : A′ : : C D : C M. Again, the triangles C A M, C M E, having the common vertex M, are to each other as their bases C A, C E; they are likewise to each other as the polygons A′ and B of which they form part; hence A′ : B : : C A : C E. But since A D and M E are parallel, we have C D : C M : : C A : C E; hence A : A′ : : A′ : B; hence the polygon A′, one of those required, is a mean proportional between the two given polygons A and B, and consequently $A' = \sqrt{A \times B}$.

Secondly. The altitude C M being common, the triangle C P M is to the triangle C P E as P M is to P E; but (prop. 21, book iv.) since C P bisects the angle M C E, we have P M : P E : : C M : C E : : C D : C A : : A : A′; hence C P M : C P E : : A : A′; and consequently C P M : C P M + C P E or C M E : : A : A + A′. But C M P A or 2 C M P and C M E are to each other as the polygons B′ and B, of which they form part; hence B′ : B : : 2 A : A + A′. Now A′ has already been determined; this new proportion will serve for determining B′, and give us $B' = \dfrac{2\,A\,.\,B}{A + A'}$; and thus by means of the polygons A and B, it is easy to find the polygons A′ and B′, which have double the number of sides.

PROPOSITION XIII.—*Problem.*

To find the approximate ratio of the circumference to the diameter.

Let the radius of the circle be 1; the side of the inscribed square will be $\sqrt{2}$, (prop. 2, book v.) that of the circumscribed square will be equal to the diameter 2; hence the surface of the inscribed square is 2, and that of the circumscribed square is 4. Let us therefore put A = 2, and B = 4; by the last proposition, we shall find the inscribed octagon $A' = \sqrt{8} = 2.8284271$, and the circumscribed octagon $B' = \dfrac{16}{2 + \sqrt{8}} = 3.3137085$. The inscribed and the circumscribed octagon being thus determined, we shall easily, by means of them, determine the polygons having twice the number of sides. We have only in this case to put A = 2.8284271, B = 3.3137085; we shall find $A' = \sqrt{A\,.\,B} = 3.0614674$, and $B = \dfrac{2\,A\,.\,B}{A + A'} = 3.1825979$. These polygons of 16 sides will in their turn enable us to find the polygons of 32; and the process may be continued, till there remains no longer any difference between the inscribed and the circumscribed polygon, at least so far as that place of decimals where the computation stops,—so far as the seventh place, in this example. Being arrived at this point, we shall infer that the last result expresses the surface of the circle, which, since it must always lie between the inscribed and the circumscribed polygon, and since those polygons agree as far as a certain place of decimals, must also agree with both as far as the same place.

We have subjoined the computation of those polygons, carried on till they agree as far as the seventh place of decimals.

Number of sides.	Inscribed polygon.	Circumscribed polygon.
4	2.0000000	4.0000000
8	2.8284271	3.3137085
16	3.0614674	3.1825979
32	3.1214451	3.1517249
64	3.1365485	3.1441184
128	3.1403311	3.1422236
256	3.1412772	3.1417504
512	3.1415138	3.1416321
1024	3.1415729	3.1416025
2048	3.1415877	3.1415951
4096	3.1415914	3.1415933
8192	3.1415923	3.1415928
16384	3.1415925	3.1415927
32768	3.1415926	3.1415925

The area of the circle, therefore, is equal to 3.1415926. Some doubt may exist perhaps about the last decimal figure, owing to errors proceeding from the parts omitted; but the calculation was carried on with no additional figure, that the final result here given might be absolutely correct even to the last decimal place.

Since the surface of the circle is equal to half the circumference multiplied by the radius, the half circumference must be 3.1415926, when the radius is 1; or the whole circumference must be 3.1415926, when the diameter is 1; hence the ratio of the circumference to the diameter, formerly expressed by π, is equal to 3.1415926.

PROPOSITION XIV.—*Lemma.*

The triangle C A B *is equal to the isosceles triangle* D C E, *which has the same angle* C, *and one of its equal sides* C E *or* C D *a mean proportional between* C A *and* C B. *And if the angle* C A B *is right, the perpendicular* C F, *drawn to the base of the isosceles triangle will be a mean proportional between the side* C A *and half the sum of the sides* C A, C B, fig. 171.

First. Because of the common angle C, the triangle A B C is to the isosceles triangle D C E as A C × C B is to D C × C E or D C² (prop. 29, book iv.;) hence those triangles will be equal, if D C² = A C × C B, or if D C is a mean proportional between A C and C B. Fig. 171.

Secondly. Because the perpendicular C G F bisects the angle A C B, we shall have A G : G B : : A C : C B (prop. 21, book iv.;) and therefore (prop. 5, book ii.) A G : A G + G B or A B : : A C : A C + C B; but A G is to A B as the triangle A C G is to the triangle A C B, or 2 C D F; besides if the angle A is right, the right angled triangles A C G, C D F must be similar, and give A C G : C D F : : A C² : C F²; hence,

A C² : 2 C F² : : A C : A C + C B.

Multiply the second pair by A C; the antecedents will be equal, and consequently we shall have 2 C F² = A C . (A C + C B) or $CF^2 = AC\,.\left(\dfrac{AC \times CB}{2}\right)$;

hence if the angle A is right, the perpendicular C F will be a mean proportional between the side A C and the half sum of the sides A C, C B.

PROPOSITION XV.—*Problem.*

To find a circle differing as little as we please from a given regular polygon, fig. 172.

Fig 172. Let the square B M N P be the proposed polygon. From the centre C, draw C A perpendicular to M B, and join C B.

The circle described with the radius C A is inscribed in the square, and the circle described with the radius C B circumscribes this same square; the first will in consequence be less than it, the second greater: it is now required to compress those limits.

Take C A and C E, each equal to the mean proportional between C A and C B, and join E D; the isosceles triangle C D E will, by the last proposition, be equal to the triangle C A B. Perform the same operation on each of the eight triangles which compose the square; you will thus form a regular octagon equal to the square B M N P. The circle described with the radius C F, a mean proportional between C A and $\frac{CA + CB}{2}$, will be inscribed in this octagon, and the circle whose radius is C D will circumscribe it. The first of them will therefore be less than the given square, the second greater.

If the right angled triangle C D F be, in like manner, changed into an equal isosceles triangle, we shall by this means form a regular polygon of sixteen sides, equal to the proposed square. The circle inscribed in this polygon will be less than the square; the circumscribed circle will be greater.

The same process may be continued, till the ratio between the radius of the inscribed and that of the circumscribed circle, approach as near to equality as we please. In that case, both circles may be regarded as equal to the square.

Scholium. The investigation of the successive radii is reduced to this. Let a be the radius of the circle inscribed in one of the polygons, b the radius of the circle circumscribing the same polygon; let a' and b' be the corresponding radii for the next polygon, which is to have twice the number of sides. From what has been demonstrated, b' is a mean proportional between a and b, and a' is a mean proportional between a and $\frac{a+b}{2}$; so that $b' = \sqrt{a.b}$, and $a' = \sqrt{a.\frac{a+b}{2}}$: hence a and b the radii of one polygon being known, we may easily discover the radii a' and b' of the next polygon; and the process may be continued till the difference between the two radii become insensible; then either of those radii will be the radius of the circle equal to the proposed square or polygon.

This method is easily practised with regard to lines; for it implies nothing but the finding of successive mean proportionals between lines which are given: it is still more easily practised with regard to numbers, and forms one of the most commodious plans which elementary geometry can furnish, for discovering speedily the approximate ratio of the circumference to the diameter. Let the side of the square be 2; the first inscribed radius C A will be one, and the first circumscribed radius C B will be $\sqrt{2}$ or 1.4142136.

Hence, putting $a = 1$, $b = 1.4142136$, we shall find $b' = 1.1892071$, and $a' = 1.0986841$. These numbers will serve for computing the rest, the law of their combination being known.

Radii of the circumscribed circles.	*Radii of the inscribed circles.*
1.4142136	1 0000000
1.1892071	1.0986841
1.1430500	1.1210863
1.1320149	1.1265639
1.1292862	1.1279257
1.1286063	1.1282657

Since the first half of these ciphers is now become the same on both sides, it will occasion little error to assume the arithmetical means instead of the mean proportionals or geometrical means, which differ from the former only in their last figures. By this method, the operation is greatly abridged, the results are:

1.1284360	1.1283508
1.1283934	1.1283721
1.1283827	1.1283774
1.1283801	1.1283787
1.1283794	1.1283791
1.1283792	1.1283792

Thus 1.1283792 is very nearly the radius of a circle equal in surface to the square whose side is 2. From this, it is easy to find the ratio of the circumference to the diameter: for it has already been shown that the surface of the circle is equal to the square of its radius multiplied by the number π; hence if the surface 4 be divided by the square of 1.1283792 the radius, we shall get the value of π, which by this computation is found to be 3.1415926, &c. as was formerly determined by another method.

BOOK VI.

Of planes and solid angles.

DEFINITIONS.

1. THE common *section* of two planes is the line in which they meet to cut each other.

2. A line is *perpendicular to a plane*, when it is perpendicular to any two lines in that plane which meet it.

3. One *plane is perpendicular* to another, when every line in the one which is perpendicular to their common section is perpendicular to the other plane.

4. The *inclination of two planes to each other*, or *the angle they form between them*, is the angle contained by two lines drawn from any point in the common section, and at right angles to the same, one of these lines in each plane.

5. A line is *parallel to a plane*, when, if both are produced to any distance, they do not meet; and conversely, the plane is then also parallel to the line.

6. Two *planes are parallel* to each other, when both being produced to any distance they do not meet.

7. A *solid angle* is the angular space included between three or more planes which meet at the same point.

Proposition I.—*Theorem.*

A straight line cannot be partly in a plane and partly out of it, fig. 173.

Fig. 173. For the part of the line which is in the plane may be produced in the plane, as for example to D; and if a part of the line were also out of the plane, then two straight lines might have a common segment A B, which is impossible.

Proposition II.—*Theorem.*

Two straight lines which intersect each other lie in the same plane, and determine its position, fig. 174.

Fig. 174. Let A B, A C be two straight lines which intersect each other in A; and conceive some plane passing through one of the lines as A B, and if also A C be in this plane, then it is clear that the two lines, according to the terms of the proposition, are in the same plane; but if not, let the plane passing through A B be supposed to be turned round A B till it passes through the point C, then the line A C, which has two of its points A and C in this plane, lies wholly in it; and hence the position of the plane is determined by the single condition of containing the two straight lines A B . A C.

Cor. 1. A triangle A B C, or any three points not in a straight line, determines the position of a plane.

Cor. 2. Hence, also, two parallels A B, C D (fig. 2) determines the position of a plane. For drawing the secant E F, the plane of the two straight lines A E . E F is that of the parallels A B . C D.

Proposition III.—*Theorem.*

The common section of two planes is a right line, fig. 175.

Fig. 175. Let A C B D A, and A E B F A be two planes cutting each other, and A, B two points in which the planes meet. Draw the line A B, this line is the common intersection of the two planes.

For, because the right line touches the two planes in the points A and B, it lies wholly in both these planes, or is common to both of them. That is, the common intersection of the two planes is in a right line.

Proposition IV.—*Theorem.*

If a straight line A P *be perpendicular to two other straight lines* P B, PC, *which cross each other at its foot in the plain* M N, *it will be perpendicular to any straight line* P Q *drawn through its foot in the same plane, and thus it will be perpendicular to the plane* M N, *fig.* 176.

Fig. 176. Through any point Q in PQ, draw (prop. 5, book iv.) the straight line B C in the angle B P C, so that $BQ = QC$; join A B, AQ, A C.

The base B C being divided into two equal parts at the point L, the triangle B P C (prop. 17, book iv.) will give

$$PC^2 + PB^2 = 2PQ^2 + 2QC^2.$$

The triangle B A C will, in like manner, give

$$AC^2 + AB^2 = 2AQ^2 + 2QC^2.$$

Taking the first equation from the second, and observing that the triangles A P C, A P B, which are both right angled at P, give

$AC^2 - PC^2 = AP^2$, and $AB^2 - PB^2 = AP^2$;

we shall have

$$AP^2 + AP^2 = 2AQ^2 - 2PQ^2.$$

Therefore, by taking the halves of both, we have $AP^2 = AQ^2 - PQ^2$, or $AQ^2 = AP^2 + PQ^2$; hence the triangle A P Q is right angled at P; and therefore A P is perpendicular to P Q.

Scholium. Thus it is evident, not only that a straight line may be perpendicular to all the straight lines which pass through its foot in a plane, but that it always must be so, whenever it is perpendicular to two straight lines drawn in the plane.

Cor. 1. The perpendicular A P is shorter than any oblique line A Q; therefore it measures the true distance from the point A to the plane P Q.

Cor. 2. At a given point P on a plane, it is impossible to erect more than one perpendicular to that plane; for if there could be two perpendiculars at the same point P, draw along these two perpendiculars a plane, whose intersection with the plane M N is P Q; then those two perpendiculars would be perpendicular to the line P Q, at the same point, and in the same plane, which is impossible.

It is also impossible to let fall from a given point out of a plane two perpendiculars to that plane; for let A P, A Q be these two perpendiculars; then the triangle A P Q would have two right angles A P Q, A Q P, which is impossible.

Proposition V.—*Theorem.*

Oblique lines equally distant from the perpendicular to a plane are equal; and, of two oblique lines unequally distant from the perpendicular, that which is nearer is less than that more remote, fig. 177.

For the angles A P B, A P C, A P D being right, if we suppose the distances P B, P C, P D to be equal to each other, the triangles A P B, A P C, A P D will have each an equal angle contained by equal sides; therefore they will be equal; therefore the hypothenuses, or the oblique lines A B, A C, A D will be equal to each other. In like manner, if the distance F E be greater than P D or its equal PB, the oblique line A E will evidently be greater than A B, or its equal A D; that is A B will be less than A E. Fig. 177

Cor. All the equal oblique lines AB, A C, A D, &c. terminate in the circumference of a circle B C D, described from P the foot of the perpendicular as a centre; therefore a point A being given out of a plane, the point P at which the perpendicular let fall from A would meet that plane, may be found by marking upon that plane three points B, C, D, equally distant from the point A, and then finding the centre of the circle which passes through these points; this centre will be P, the point sought.

Scholium. The angle A B P is called the *inclination of the oblique line* A B *to the plane* M N; which inclination is evidently equal with respect to all such lines A B, A C, A D, as are equally distant from the perpendicular; for all the triangles A B P, A C P, A D P, &c. are equal to each other.

Proposition VI.—*Theorem.*

Let A P *be a perpendicular to the plane* M N, *and* B C *a line situated in that plane; if from* P, *the foot of the perpendicular,* P D *be drawn at right angles to* B C, *and* A D *joined,* A D *will be perpendicular to* B C, *fig.* 178.

Geometry. Fig. 178. Take $DB = DC$, and join PB, PC, AB, AC; since $DB = DC$, the oblique line $PB = PC$; and with regard to the perpendicular AP, since $PB = PC$, the oblique line $AB = AC$ by the last proposition; therefore the line AD has two of its points A and D equally distant from the extremities B and C; therefore AD is a perpendicular at the middle of BC.

Cor. It is evident likewise, that BC is perpendicular to the plane APD, since BC is at once perpendicular to the two straight lines AD, PD.

Scholium. The two straight lines AE, BC afford an instance of two lines which do not meet, because they are not situated in the same plane. The shortest distance between these lines is the straight line PD, which is perpendicular both to the line AP and to the line BC. The distance PD is the shortest between these two lines; for if we join any other two points, such as A and B, we shall have $AB > AD$, $AD > PD$; therefore $AB > PD$.

The two lines AE, CB, though not situated in the same plane, are conceived as forming a right angle with each other, because AD and the line drawn through one of its points parallel to BC would make with each other a right angle. In the same manner, the line AB and the line PD, which represent any two straight lines not situated in the same plane, are supposed to form with each other the same angle, which would be formed by AB and a straight line parallel to PD drawn through one of the points of AB.

Proposition VII.—*Theorem.*

If the line AP *be perpendicular to the plane* MN, *any line* DE *parallel to* AP *will be perpendicular to the same plane*, fig. 179.

Fig. 179. Along the parallels AP, DE, extend a plane; its intersection with the plane MN will be PD; in the plane MN draw BC perpendicular to PD, and join AD.

By the corollary of the preceding theorem, BC is perpendicular to the plane $APDE$; therefore the angle BDE is right; but the angle EDP is right also, since AP is perpendicular to PD, and DE parallel to AP; therefore the line DE is perpendicular to the two straight lines DP, DB; therefore it is perpendicular to their plane MN.

Cor. 1. Conversely, if the straight lines AP, DE are perpendicular to the same plane MN, they will be parallel; for if they be not so, draw through the point D a line parallel to AP, this parallel will be perpendicular to the plane MN; therefore through the same point D more than one perpendicular might be erected in the same plane, which (prop. 4, book vi.) is impossible.

Cor. 2. Two lines A and B, parallel to a third C, are parallel to each other; for, conceive a plane perpendicular to the line C, the lines A and B, being parallel to C, will be perpendicular to the same plane; therefore, by the preceding corollary, they will be parallel to each other.

When the three lines are in the same plane the case falls under prop. 23, book i.

Proposition VIII.—*Theorem.*

If the line AB *be parallel to a straight line* CD *drawn in the plane* MN, *it will be parallel to that plane*, fig. 180.

Book VI. Fig. 180. For if the line AB, which lies in the plane $ABCD$, could meet the plane MN, this could only be in some point of the line CD, the common intersection of the two planes; but AB cannot meet CD, since they are parallel; hence it will not meet the plane MN; hence (def. 5) it is parallel to that plane.

Proposition IX.—*Theorem.*

Two planes MN, PQ *perpendicular to the same straight line* AB, *are parallel to each other*, fig. 181.

Fig. 181. For, if they can meet anywhere, let O be one of their common points, and join OA, OB; the line AB, which is perpendicular to the plane MN, is perpendicular to the straight line OA drawn through its foot in that plane; for the same reason AB is perpendicular to BO; therefore OA and OB are two perpendiculars let fall, from the same point O, upon the same straight line; which is impossible: therefore the planes MN, PQ, cannot meet each other; therefore they are parallel.

Proposition X.—*Theorem.*

The intersections EF, GH *of two parallel planes* MN, PQ, *with a third plane* FG, *are parallel*, fig. 182.

Fig. 182. For, if the lines EF, GH, lying in the same plane, were not parallel, they would meet each other when produced; therefore the planes MN, PQ, in which those lines lie, would also meet; therefore the planes would not be parallel.

Proposition XI.—*Theorem.*

The line AB, *which is perpendicular to the plane* MN, *is also perpendicular to the plane* PQ, *parallel to* MN, fig. 181.

Having drawn any line BC in the plane PQ, by the last proposition, along the lines AB and BC, extend a plane ABC, intersecting the plane MN in AD; the intersection AD will be parallel to BC; but the line AB, being perpendicular to the plane MN, is perpendicular to the straight line AD; therefore also to its parallel BC: hence the line AB being perpendicular to any line BC drawn through its foot in the plane PQ, is consequently perpendicular to that plane.

Proposition XII.—*Theorem.*

The parallels EG, FH, *comprehended between two parallel planes* MN, PQ, *are equal*, fig. 182.

Through the parallels EG, FH, draw the plane $EGHF$ to meet the parallel planes in EF and GH. The intersections EF, GH (prop. 10, book vi.) are parallel to each other; so likewise are EG, FH; therefore the figure $EGHF$ is a parallelogram; and $EG = FH$.

Cor. Hence it follows that *two parallel planes are every where equidistant*; for if EG and FH are perpendicular to the two planes MN, PQ, they will be parallel to each other, (prop. 7, cor. 1, book vi.;) and therefore equal.

PROPOSITION XIII.—*Theorem.*

If two angles C A E, D B F, *not situated in the same plane, have their sides parallel and lying in the same direction, those angles will be equal, and their planes will be parallel*, fig. 183.

Fig. 183. Make $AC = BD$, $AE = BF$; and join C E, D F, A B, C D, E F. Since A C is equal and parallel to B D, the figure A B D C is a parallelogram, (prop. 28, book i.;) therefore C D is equal and parallel to A B. For a similar reason, E F is equal and parallel to A B; hence also C D is equal and parallel to E F; the figure C E F D is therefore a parallelogram, and the side C E is equal and parallel to D F; therefore the triangles C A E, D B F have their corresponding sides equal; consequently the angle $CAE = DBF$.

Again, the plane A C E is parallel to the plane B D F. For suppose the plane parallel to B D F, drawn through the point A, were to meet the lines C D, E F, in points different from C and E, for instance in G and H; then, (prop. 12, book vi.) the three lines A B, G D, F H would be equal: but the lines A B, C D, E F are already known to be equal; hence $CD = GD$, and $FH = EF$, which is absurd; hence the plane A C E is parallel to B D F.

Cor. If two parallel planes M N, P Q are met by two other planes C A D B, E A B F, the angles C A E, D B F, formed by the intersections of the parallel planes will be equal; for (prop. 10, book vi.) the intersection A C is parallel to B D, and A E to B F, therefore the angle $CAE = DBF$.

PROPOSITION XIV.—*Theorem.*

If three straight lines A B, C D, E F, *not situated in the same plane, are equal and parallel, the triangles* A C E, B D F *formed by joining the extremities of these straight lines will be equal, and their planes will be parallel*, fig. 183.

For since A B is equal and parallel to C D, the figure A B C D is a parallelogram; hence the side A C is equal and parallel to B D. For a like reason the sides A E, B F are equal and parallel, as also C E, D F; therefore the two triangles A C E, B D F, are equal; and, consequently, as in the last proposition, their planes are parallel.

PROPOSITION XV.—*Theorem.*

Two straight lines, included between three parallel planes, are cut proportionally, fig. 184.

Fig. 184. Suppose the line A B to meet the parallel planes M N, P Q, R S, at the points A, E, B; and the line C D to meet the same planes at the points C, F, D; then $AE : EB :: CF : FD$.

Draw A D meeting the plane P Q in G, and join A C, E G, G F, B D; the intersections E G, B D, of the parallel planes P Q, R S, in the plane A B D, are parallel, (prop. 10, book vi.;) therefore $AE : EB :: AG : GD$; in like manner, the intersections A C, G F being parallel, $AG : GD :: CF : FD$; the ratio $AG : GD$ is the same in both; hence

$$AE : EB :: CF : FD.$$

PROPOSITION XVI.—*Theorem.*

If the line A P *be perpendicular to the plane* M N, *any plane* A P B *drawn along* A P *will be perpendicular to the plane* M N, fig. 185.

Let the two planes A B, M N intersect each other in the line B C. In the plane M N draw D E perpendicular to B P; then the line A P, being perpendicular to the plane M N, will be perpendicular to each of the two straight lines B C, D E; but the angle A P D, formed by the two perpendiculars P A, P D at their common intersection B P, is the measure of the angle of the two planes, (def. 4;) and since in the present case the angle is a right angle, the two planes are perpendicular to each other. Fig. 185

Scholium. When the three lines such as A P, B P, D P are perpendicular to each other, each of these lines is perpendicular to the plane of the other two; and the planes themselves are perpendicular to each other.

PROPOSITION XVII.—*Theorem.*

If the plane A B *be perpendicular to the plane* M N, *and if in the plane* A B *the line* P A *be perpendicular to the common intersection* B P, *then will* A P *be perpendicular to the plane* M N, fig. 185.

For in the plane M N draw P D perpendicular to P B; then because the planes are perpendicular, the angle A P D is a right angle; therefore the line A P is perpendicular to the two straight lines P B, P D; and is therefore perpendicular to their plane M N.

Cor. If the plane A B be perpendicular to the plane M N, and if at a point P of the common intersection a perpendicular be erected to the plane M N, that perpendicular will be in the plane A B; for if not, then in the plane A B we might draw A P perpendicular to P B, their common intersection, and this A P at the same time would be perpendicular to the plane M N; therefore at the same point P there would be two perpendiculars to the plane M N, which is impossible.

PROPOSITION XVIII.—*Theorem.*

If two planes be perpendicular to a third plane, their common intersection will be also perpendicular to the third plane, fig. 185.

Let A B, A D be perpendicular to M N, then will their common intersection A P be perpendicular to the same plane M N.

For at the point P erect the perpendicular to the plane M N; then that perpendicular must be in the plane A D, and also in A B, (by the last prop.;) therefore it is their common intersection A P.

PROPOSITION XIX.—*Theorem.*

If a solid angle is formed by three plane angles, the sum of any two of these angles will be greater than the third, fig. 186.

The proposition requires demonstration only when the plane angle, which is compared to the sum of the other two, is greater than either of them. Therefore suppose the solid angle S to be formed by three plane angles A S B, A S C, B S C, whereof the angle A S B is the greatest; we are to show that $ASB \angle ASC + BSC$. Fig. 186

In the plane A S B make the angle $BSD = BSC$, draw the straight line A D B at pleasure; and having taken $SC = SD$, join A C, B C.

The two sides B S, S D are equal to the two B S, S C; the angle $BSD = BSC$; therefore the triangles B S D, B S C are equal; therefore $BD = BC$. But $AB \angle AC + BC$; taking B D from the one side, and from the other its equal BC, there remains $AD \angle AC$. The two sides A S, S D are equal to the two A S, S C; the third side A D is less than the third side A C; therefore (prop. 8, book i.) the angle $ASD \angle ASC$. Adding $BSD = BSC$, we shall have $ASD + BSD$ or $ASB \angle ASC + BSC$.

Proposition XX.—*Theorem.*

The sum of the plane angles which form a solid angle, is always less than four right angles, fig. 187.

Fig. 187. Conceive the solid angle S to be cut by any plane A B C D E; from O, a point in that plane, draw to the several angles straight lines A O, O B, O C, O D, O E.

The sum of the angles of the triangles A S B, B S C, &c. formed about the vertex S, is equivalent to the sum of the angles of an equal number of triangles A O B, B O C, &c. formed about the point O. But at the point B the angles A B O, O B C, taken together make the angle A B C (prop. 19, book vi.) less than the sum of the angles A B S, S B C; in the same manner, at the point C we have $BCO + OCD \angle BCS + SCD$; and so with all the angles of the polygon A B C D E: whence it follows, that the sum of all the angles at the bases of the triangles whose vertex is in O, is less than the sum of the angles at the bases of the triangles whose vertex is in S; hence to make up the deficiency, the sum of the angles formed about the point O, is greater than the sum of the angles about the point S. But the sum of the angles about the point O is equal to four right angles, (prop. 3, book i.;) therefore the sum of the plane angles, which form the solid angle S, is less than four right angles.

Scholium 1. This demonstration is founded on the supposition that the solid angle is convex, or that the plane of no one surface produced can ever meet the solid angle; if it were otherwise, the sum of the plane angles would no longer be limited, and might be of any magnitude.

Proposition XXI.—*Theorem.*

If two solid angles are composed of three plane angles respectively equal to each other, the planes which contain the equal angles will be equally inclined to each other, fig. 188.

Fig. 188. Let the angle $ASC = DTF$, the angle $ASB = DTE$, and the angle $BSC = ETF$; then will the inclination of the planes A S C, A S B, be equal to that of the planes D T F, D T E.

Having taken S B at pleasure, draw B O perpendicular to the plane ASC; from the point O, at which that perpendicular meets the plane, draw O A, O C perpendicular to S A, S C; join A B, B C; next take $TE = SB$; draw E P perpendicular to the plane D T F; from the point P draw P D, P F, perpendicular to T D, T F; lastly, join D E, E F.

The triangle S A B is right angled at A, and the triangle T D E at D; and since the angle $ASB = DTE$, we have $SBA = TED$. Likewise $SB = TE$; therefore the triangle S A B is equal to the triangle T D E; therefore $SA = TD$, and $AB = DE$. In like manner it may be shown, that $SC = TF$, and $BC = EF$. That granted, the quadrilateral S A O C is equal to the quadrilateral T D P F; for, place the angle A S C upon its equal D T F; because $SA = TD$, and $SC = TF$, the point A will fall on D, and the point C on F; and at the same time, A O, which is perpendicular to S A, will fall on P D which is perpendicular to TD, and in like manner O C on P F; wherefore the point O will fall on the point P, and A O will be equal to D P. But the triangles A O B D P E, are right angled at O and P; the hypothenuse $AB = DE$, and the side $AO = DP$; hence those triangles are equal; therefore the angle $OAB = PDE$. The angle O A B is the inclination of the two planes A S B, A S C; the angle P D E is that of the two planes D T E, D T F; hence those two inclinations are equal to each other.

It must, however, be observed, that the angle A of the right angled triangle O A B is properly the inclination of the two planes A S B, A S C, only when the perpendicular B O falls on the same side of S A as S C falls; for if it fell on the other side, the angle of the two planes would be obtuse, and joined to the angle A of the triangle O A B it would make two right angles. But in the same case, the angle of the two planes T D E, T D F would also be obtuse, and joined to the angle D of the triangle D P E, it would make two right angles; and the angle A being thus always equal to the angle at D, it would follow in the same manner that the inclination of the two planes A S B, A S C, must be equal to that of the two planes T D E, T D F.

Scholium 2, relative to the measure of solid angles.

A more general definition of solid angles than that given at the commencement of this book is, that a solid angle is the angular space included between several plane surfaces, or one or more curved surface meeting in the point which forms the summit of the angle.

According to this definition, solid angles bear just the same relation to the surfaces which comprise them, as plane angles do to the lines by which they are included; so that, as in the latter, it is not the magnitude of the lines, but their mutual inclination which determines the angles; so, in the former, it is not the magnitude of the planes, but their mutual inclination which determine the solid angles. According to this view of the subject, the spherical surface described about the summit of any solid angle as a centre, will become a measure of that angle; as the circular arc is employed to measure and to compare rectilinear angles. Let us imagine, in the first place, such a sphere to be described about any given solid angle comprised under three plane angles, and that those planes are produced till they cut the surface of the sphere; then will the surface of the spherical triangle included between those planes be the measure, or may be assumed as the measure, of the solid angle, made by the planes at the common point of meeting; for no change can be conceived in the relative position of the bounding planes, that is, in the magnitude of the solid angle, without a corresponding and proportional mutation in the surface of the spherical

Geometry. triangle; and if, in like manner, the three or more plane surfaces comprising another solid angle be produced till they cut the surface of the same, or if an equal sphere, whose centre coincides with the summit of the angle, the surface of the spherical triangle or polygon included between the planes which determine the angle, will in like manner be a correct measure of that angle; and the ratio which subsists between the areas of these triangles and polygons, or other surfaces thus formed, will be accurately the ratio which subsists between the solid angles, constituted by the meeting of the several planes or surfaces at the centre of the sphere.

It will, of course, be understood, that this measurement has only a relation to the magnitude of the angles. It has no reference to their geometrical properties, which may be very different, although their magnitudes, as above estimated, may be the same.

BOOK VII.

Of solids bounded by planes.

Definitions.

1. A solid is that which has length, breadth, and thickness.

2. A *prism* is a solid contained by plane figures, of which two that are opposite are equal, similar, and parallel to one another; and the others are parallelograms. To construct this solid, let A B C D E, (fig. 189,) be any rectilineal figure. In a plane parallel to ABC draw the lines FG, GH, HI, &c. parallel to the sides AB, BC, CD, &c.; thus there will be formed a figure FGHIK, similar to ABCDE. Now let the vertices of the corresponding angles be joined by the lines AF, BG, CH, &c. the faces ABGF, BCHG, &c. will evidently be parallelograms, and the solid thus formed will be a prism. Fig. 189.

3. The equal and parallel plane figures A B C D E, FGHIK are called the *bases* of the prism. The other planes or parallelograms taken together constitute the *lateral* or *convex* surface of the prism.

4. The *altitude* of a prism is the perpendicular distance between its *bases*; and its *length* is a line equal to any one of its lateral edges, as AF, or BG, &c.

5. A *right prism* is one in which the lateral edges AF, BG, &c. are perpendicular to the planes of its bases; then each of them is equal to the altitude of the prism; in every other case the prism is oblique.

6. A prism is *triangular*, *quadrangular*, *pentagonal*, &c. according as the base is a triangle, a quadrilateral, a pentagon, &c.

7. A prism which has a parallelogram for its base has all its faces parallelograms, and is called a *parallelopiped*, or *parallelopipedon*, (fig. 190.) A parallelopiped is *rectangular*, when all its faces are rectangles. Fig. 190.

8. When the faces of a rectangular parallelopiped are squares, it is called a *cube*.

9. A *pyramid* is a solid formed by several triangular planes which meet in a point, as V, (fig. 191,) and terminate in the same plane rectilineal figures ABCDE. Fig. 191.

The plane figure ABCDE is called the *base* of the pyramid; the point V is its *vertex*; and the triangles AVB, BVC, &c. taken together, form the *convex* or *lateral* surface of the pyramid. Book VII.

10. The *altitude* of a pyramid is the perpendicular drawn from the vertex to the plane of its base, produced if necessary.

11. A pyramid is *triangular*, *quadrangular*, &c. according as its base is a *triangle*, a *quadrangle*, &c.

12. A pyramid is *regular*, when its base is a regular figure, and the perpendicular from its vertex passes through the centre of its base; that is, through the centre of a circle which may be conceived to circumscribe its base.

13. Two solids are *similar*, when they are contained by the same number of similar planes, similarly situated, and having like inclinations to one another.

Proposition I.—*Theorem.*

Two prisms are equal when a solid angle in each is contained by three planes, which are equal in both and similarly situated, fig. 192.

Let the base ABCDE be equal to the base *a b c d e*; the parallelogram ABGF equal to the parallelogram *a b g f*; and the parallelogram BCHG equal to the parallelogram *b c h g*; then will the prism ABCI be equal to the prism *a b c i*. Fig. 192.

For apply the base ABCDE upon its equal *abcde*, so that the bases (being equal) may coincide. But the three plane angles which form the solid angle B, are respectively equal to the three plane angles which form the solid angle *b*, that is, ABC = *a b c*, ABG = *abg*, and GBC = *gbc*, and they are also similarly situated; therefore the solid angles B and C are equal, and therefore BG will fall on its equal *b g*; and it is likewise evident, because the parallelograms ABGF and *a b g f* are equal, that the side GF will fall on its equal *g f*, and in the same manner GH on *g h*; therefore the upper base FGHIK will coincide with its equal *f g h i k*, and the two solids will be identical, since their vertices are the same.

Cor. Two right prisms which have equal bases and equal altitudes, are equal. For, since the side AB is equal to *a b*, and the altitude BG to *b g*, the rectangle ABGF will be equal to *a b g f*; and in the same way the rectangle BGHC will be equal to *b g h c*; and thus the three planes, which form the solid angle B, will be equal to the three which form the solid angle *b*. Hence the two prisms are equal.

Proposition II.—*Theorem.*

In every parallelopipedon the opposite planes are equal and parallel, fig. 193.

By the definition of this solid, the bases ABCD, EFGH are equal parallelograms, and their sides are parallel: it remains only to show, that the same is true of any two opposite lateral faces, such as AEHD, BFGC. Now AD is equal and parallel to BC, because the figure ABCD is a parallelogram; for a like reason, AE is parallel to BF; hence the angle DAE is equal to the angle CBF, and the planes DAE, CBF are parallel; hence also the parallelogram DAEH is equal to the parallelogram CBFG. In the same way it might be shown that the opposite parallelograms ABFE, DCGH are equal and parallel. Fig. 193.

Cor. Since the parallelopipedon is a solid bounded by six planes, whereof those lying opposite to each other

 are equal and parallel, it follows that any face and the one opposite to it may be assumed as the bases of the parallelopipedon.

Scholium. If three straight lines A B, A E, A D, passing through the same point A, and making given angles with each other, are known, a parallelopipedon may be formed on those lines. For this purpose, a plane must be extended through the extremity of each line, and parallel to the plane of the other two; that is, through the point B a plane parallel to D A E, through D a plane parallel to B A E, and through E a plane parallel to B A D. The mutual intersections of those planes will form the parallelopipedon required.

Proposition III.—*Lemma.*

In every prism A B C I, *the sections* N O P Q R, S T V X Y, *formed by parallel planes, are equal polygons,* fig. 194.

Fig. 194. For the sides S T, N O are parallel, being the intersections of two parallel planes with a third plane A B G F; moreover the sides S T, N O, are included between the parallels N S, O T, which are sides of the prism; hence N O is equal to S T. For like reasons, the sides O P, P Q, Q R, &c. of the section N O P Q R, are respectively equal to the sides T V, V X, X Y, &c. of the section S T V X Y. And since the equal sides are at the same time parallel, it follows that the angles N O P, O P Q, &c. of the first section are respectively equal to the angles S T V, T V X of the second. Hence the two sections N O P Q R, S T V X Y are equal polygons.

Cor. Every section in a prism, if drawn parallel to the base, is also equal to that base.

Proposition IV.—*Theorem.*

The two symmetrical triangular prisms A B D H E F, B C D F G H, *into which the parallelopipedon* A G *may be decomposed, are equal to each other,* fig. 195.

Fig. 195. Through the vertices B and F, draw the planes B *a d e*, F *e h g* at right angles to the side B F, and meeting A E, D H, C G, the three other sides of the parallelopipedon, in the points *a*, *d*, *e* towards one direction, and in *e*, *h*, *g* towards the other; then the sections B *a d e*, F *e h g* will be equal parallelograms; being equal because they are formed by planes perpendicular to the same straight line, and consequently parallel; and being parallelograms, because *a* B, *d e*, two opposite sides of the same section, are formed by the meeting of one plane with two parallel planes A B F E, D C G H.

For a like reason, the figure B *a e* F is a parallelogram; so also are B F *g e*, *c d h g*, and *a d h e*, the other lateral faces of the solid B *a d c* F *e h g*; hence that solid is a prism, (def. 5;) and that prism is a right one, because the side B F is perpendicular to its base.

This being proved, if the right prism B*h* be divided by the plane B F H D into two right triangular prisms *a* B *d e* F *h*, B *d c* F *h g*; it will remain to be shown that the oblique triangular prism A B D E F H will be equal to the right triangular prism *a* B *d e* F *h*. And since those two prisms have a part A B D *h e* F in common, it will only be requisite to prove that the remaining parts, namely, the solids B *a* A D *d*, F *e* E H *h* are equal.

Now, by reason of the parallelograms A B F E, *a* B F *e*, the sides A E, *a e*, being equal to their parallel B F, are equal to each other; and taking away the common part A *e*, there remains A *a* = E *e*. In the same manner we could prove D *d* = H *h*.

Let us now place the base F *e h* on its equal B *a d*; the point *e* falling on *a*, and the point *h* on *d*, the sides *e* E, *h* H will fall on their equals *a* A, *d* D, because they are perpendicular to the same plane B *a d*. Hence the two solids in question will coincide exactly with each other, and the oblique prism B A D F E H is therefore equal to the right one B *a d* F *e h*.

In the same manner might the oblique prism B D C F H G be proved equal to the right prism B *d c* F *h g*. But (prop. 1, book vii.) the two right prisms B *a d* F *e h*, B *d c* F *h g* are equal, since they have the same altitude B F, and since their bases B *a d*, B *d e* are halves of the same parallelogram. Hence the two triangular prisms B A D F E H, B D C F H G, being equal to the equal oblique prisms, are equal to each other.

Cor. Every triangular prism A B D H E F is half of the parallelopipedon A G described on the same solid angle A, with the same edges A B, A D, A E.

Proposition V.—*Theorem.*

If two parallelopipedons A G, A L *have a common base* A B C D, *and if their upper bases* E F G H, I K L M *lie in the same plane and between the same parallels* E K, H L, *those two parallelopipedons will be equal to each other,* fig. 196.

There may be three cases to this proposition, according as E I is greater, less than, or equal to E F; but the demonstration is the same for all. In the first place, then, we shall show that the triangular prism A E I D H M is equal to the triangular prism B F K C G L. Fig. 196.

Since A E is parallel to B F, and H E to G F, the angle A E I = B F K, H E I = G F K, and H E A = G F B. Of these six angles the first three form the solid angle E, the last three the solid angle F; therefore, the plane angles being respectively equal, and similarly arranged, the solid angles F and E must be equal. Now, if the prism A E M be laid on the prism B F L, the base A E I being placed on the base B F K will coincide with it because they are equal; and since the solid angle E is equal to the solid angle F, the side E H will fall on its equal F G; and nothing more is required to prove the coincidence of the two prisms throughout their whole extent, for (prop. 1, book vii.) the base A E I and the edge E H determine the prism A E M, as the base B F K and the edge F G determine the prism B F L; hence these prisms are equal.

But if the prism A E M is taken away from the solid A L, there will remain the parallelopipedon A I L; and if the prism B F L is taken away from the same solid, there will remain the parallelopipedon A E G; hence those two parallelopipedons A I L, A E G are equal.

Proposition VI.—*Theorem.*

Two parallelopipedons having the same base and the same altitude are equal to each other, fig. 197.

Let A B C D be the common base of the two parallelopipedons A G, A L; since they have the same altitude, their upper bases E F G H, I K L M will be in the same plane. Also the sides E F and A B will be Fig. 197.

equal and parallel, as well as I K and A B; hence E F is equal and parallel to I K; for a like reason G F is equal and parallel to L K. Let the sides E F, H G be produced, and likewise L K, I M, till by their intersections they form the parallelogram N O P Q; this parallelogram will evidently be equal to either of the bases E F G H, I K L M. Now if a third parallelopipedon be conceived, having A B C D for its lower base, and N O P Q for its upper, this third parallelopipedon will (prop. 5, book vii.) be equal to the parallelopipedon A G, since with the same lower base, their upper bases lie in the same plane and between the same parallels G Q, F N. For the same reason this third parallelopipedon will also be equal to the parallelopipedon A L; hence the two parallelopipedons A G, A L, which have the same base and the same altitude, are equal.

Proposition VII.—*Theorem.*

Any parallelopipedon may be changed into an equal rectangular parallelopipedon having the same altitude and an equal base, fig. 197 and 198.

Fig. 197. Let A G be the parallelopipedon proposed. From the points A, B, C, D, draw A I, B K, C L, D M, perpendicular to the plane of the base; and we shall thus form the parallelopipedon A L equal to A G, and having its lateral faces A K, B L, &c. rectangular. Hence if the base A B C D be a rectangle, A L will be the rectangular parallelopipedon equal to A G, the

Fig. 198. parallelopipedon proposed. But if A B C D (fig. 198) is not a rectangle, draw A O and B N perpendicular to C D, and O Q and N P perpendicular to the base; then the solid A B N O I P Q will be a rectangular parallelopipedon: for, by construction, the base A B N O and its opposite I K P Q are rectangles; so also are the lateral faces, the edges A I, O Q, &c. being perpendicular to the plane of the base; hence the solid A P is a rectangular parallelopipedon. But the two parallelopipedons A P, A L may be conceived as having the same base A B K I and the same altitude A O; hence the parallelopipedon A G, which was at first changed into an equal parallelopipedon A L, is again changed into an equal rectangular parallelopipedon A P, having the same altitude A I, and a base A B N O equal to the base A B C D.

Proposition VIII.—*Theorem.*

Two rectangular parallelopipedons A G, A L, *which have the same base* A B C D, *are to each other as their altitudes* A E, A I, *fig.* 199.

Fig. 199. First, suppose the altitudes A E, A I, to be to each other as two whole numbers, for example as 15 is to 8. Divide A E into 15 equal parts; whereof A I will contain 8; and through x, y, z, &c. the points of division, draw planes parallel to the base. These planes will cut the solid A G into 15 partial parallelopipedons, all equal to each other, having equal bases and equal altitudes,—equal bases, because every section M I K L, made parallel to the base A B C D of a prism, is equal to that base,—equal altitudes because these altitudes are the same divisions A x, xy, yz, &c. But of those 15 equal parallelopipedons, 8 are contained in A L; hence the solid A G is to the solid A L as 15 is to 8, or generally, as the altitude A E is to the altitude A I.

But if the two altitudes are incommensurable with each other, divide one of them into any number of equal parts or units, and the other into parts equal to the former; then, as is shown in our second book, the remainder (if the second altitude be not exactly commensurable with the first) will be less than the measuring unit; and this unit may be taken less than any assignable quantity. Whatever ratio therefore obtains between the commensurable parts, differing by less than any assignable quantity from the incommensurable, obtains also between the incommensurable; but when the altitudes are commensurable, the prisms are as the altitudes; they are therefore so also when the altitudes are incommensurable.

Proposition IX.—*Theorem.*

Two rectangular parallelopipedons A G, A K, *having the same altitude* A E, *are to each other as their bases* A B C D, A M N O, fig. 200.

Fig. 200. Having placed the two solids by the side of each other, as the figure represents, produce the plane O N K L till it meets the plane D C G H in P Q; we shall thus have a third parallelopipedon A Q, which may be compared with each of the parallelopipedons A G, A K. The two solids A G, A Q, having the same base A E H D, are to each other as their altitudes A B, A O; in like manner the two solids A Q, A K, having the same base A O L E, are to each other as their altitudes A D, A M. Hence we have the two proportions,

$$sol.\ AG : sol.\ AQ :: AB : AO,$$
$$sol.\ AQ : sol.\ AK :: AD : AM.$$

Multiply together the corresponding terms of those proportions, omitting in the result the common multiplier *sol.* A Q; we shall have

$$sol.\ AG : sol.\ AK :: AB \times AD : AO \times AM.$$

But $AB \times AD$ represents the base A B C D; and $AO \times AM$ represents the base A M N O; hence two rectangular parallelopipedons of the same altitude are to each other as their bases.

Proposition X.—*Theorem.*

Any two rectangular parallelopipedons are to each other as the products of their bases by their altitudes, that is to say, as the products of their three dimensions, fig. 201.

Fig. 201. For, having placed the two solids A G, A Z, so that their surfaces have the common angle B A E, produce the interior planes necessary for completing the third parallelopipedon A K, having the same altitude with the parallelopipedon A G. By the last proposition, we shall have

$$sol.\ AG : sol.\ AK :: ABCD : AMNO.$$

But the two parallelopipedons A K, A Z having the same base A M N O, are to each other as their altitudes A E, A X; hence we have

$$sol.\ AK : sol.\ AZ :: AE : AX.$$

Multiply together the corresponding terms of those proportions, omitting in the result the common multiplier *sol.* A K; we shall have

$$sol.\ AG : sol.\ AZ :: ABCD \times AE : AMNO \times AX.$$

Instead of the bases A B C D and A M N O, put $AB \times AD$ and $AO \times AM$; it will give

$$sol.\ AG : sol.\ AZ :: AB \times AD \times AE : AO \times AM \times AX$$

Hence any two rectangular parallelopipedons are to each other, &c.

Scholium. We are consequently authorized to assume, as the measure of a rectangular parallelopipedon, the product of its base by its altitude, in other words, the product of its three dimensions.

In order to comprehend the nature of this measurement, it is necessary to reflect, that by the product of two or more lines is always meant the product of the numbers which represent them, those numbers themselves being determined by their linear unit, which may be assumed at pleasure. Upon this principle, the product of the three dimensions of a parallelopipedon is a number, which signifies nothing of itself, and would be different if a different linear unit had been assumed. But if the three dimensions of another parallelopipedon are valued according to the same linear unit, and multiplied together in the same manner, the two products will be to each other as the solids, and will serve to express their relative magnitude.

The magnitude of a solid, its volume or extent, form what is called its *solidity*; and this word is exclusively employed to designate the measure of a solid: thus we say the solidity of a rectangular parallelopipedon is equal to the product of its base by its altitude, or to the product of its three dimensions.

As the cube has all its three dimensions equal, if the side is 1, the solidity will be $1 \times 1 \times 1 = 1$; if the side is 2, the solidity will be $2 \times 2 \times 2 = 8$; if the side is 3, the solidity will be $3 \times 3 \times 3 = 27$; and so on: hence, if the sides of a series of cubes are to each other as the numbers 1, 2, 3, &c. the cubes themselves or their solidities will be as the numbers 1, 8, 27, &c. Hence it is, that in arithmetic, the *cube* of a number is the name given to the product which results from three factors each equal to this number.

If it were proposed to find a cube double of a given cube, the side of the required cube would have to be to that of the given one, as the cube root of 2 is to unity. Now, by a geometrical construction, it is easy to find the square root of 2; but the cube root of it cannot be so found, at least not by the simple operations of elementary geometry, which consist in employing nothing but straight lines, two points of which are known, and circles whose centres and radii are determined.

Owing to this difficulty the problem of the *duplication of the cube* became celebrated among the ancient geometers, as well as that of the *trisection of an angle*, which is nearly of the same species. The solutions of which such problems are susceptible have, however, long since been discovered; and though less simple than the constructions of elementary geometry, they are not, on that account, less rigorous or less satisfactory.

Proposition XI.—*Theorem.*

The solidity of a parallelopipedon, and generally of any prism, is equal to the product of its base by its altitude.

For, in the first place, any parallelopipedon (prop. 7, book vii.) is equal to a rectangular parallelopipedon, having the same altitude and an equal base. Now the solidity of the latter is equal to its base multiplied by its height; hence the solidity of the former is, in like manner, equal to the product of its base by its altitude.

In the second place, and for a like reason, any rectangular prism is half of the parallelopipedon so constructed as to have the same altitude and a double base. But the solidity of the latter is equal to its base multiplied by its altitude; hence that of a triangular prism is also equal to the product of its base (half that of the parallelopipedon) multiplied into its altitude.

In the third place, any prism may be divided into as many triangular prisms of the same altitude, as there are triangles capable of being formed in the polygon which constitutes its base. But the solidity of each triangular prism is equal to its base multiplied by its altitude; and since the altitude is the same for all, it follows that the sum of all the partial prisms must be equal to the sum of all the partial triangles, which constitute their bases, multiplied by the common altitude.

Hence the solidity of any polygonal prism is equal to the product of its base by its altitude.

Cor. Comparing two prisms, which have the same altitude, the products of their bases by their altitudes will be as the bases simply; hence *two prisms of the same altitude are to each other as their bases.* For a like reason, *two prisms of the same base are to each other as their altitudes.*

Proposition XII.—*Theorem.*

Similar prisms are to one another as the cube of their homologous sides, fig. 203.

Let P and p be two prisms of which BC, bc are homologous sides; the prism P is to the prism p as the cube of BC to the cube of bc. From A and a, homologous angles of the two prisms, draw AH, ah perpendicular to the bases BCD, bcd. Join BH, take $Ba = ba$, and in the plane BHA draw ah perpendicular to BH; then ah shall be perpendicular to the plane CBD, (prop. 16, book vi.) and equal to ah, the altitude of the other prism; for if the solid angles B and b were applied the one to the other, the planes which contain them, and consequently the perpendiculars ah, ah would coincide. Fig. 203.

Now because of the similar triangles ABH, abh, and the similar figures AC, ac we have

$$AH : ah :: AB : ab :: BC : bc;$$

and because of these similar bases, the

$\textit{base}\ BCD : \textit{base}\ bcd :: BC^2 : bc^2$, (prop. 32, book iv.)

From these two proportions, by considering all the quantities as represented by numbers, we get (prop. 12, book ii.)

$AH \times \textit{base}\ BCD : ah \times \textit{base}\ BCD :: BC^3 : bc \times BC^2$;

$ah \times \textit{base}\ BCD : ah \times \textit{base}\ bcd :: bc \times BC^2 : bc^3$;

therefore (prop. 12, book ii.) and cancelling the like terms.

$$AH \times \textit{base}\ BCD : ah \times \textit{base}\ bcd :: BC^3 : bc^3.$$

But $AH \times \textit{base}$ BCD expresses the solidity of the prism P; and $ah \times \textit{base}$ BCD expresses the solidity of the other prism p; therefore

$$\textit{prism}\ P : \textit{prism}\ p :: BC^3 : bc^3.$$

Cor. Similar prisms are to one another in the triplicate ratio of their homologous sides. For let Y and Z be two lines, such that $BC : bc :: bc : Y$, and $bc : Z :: Y : Z$; then the ratio of BC to Z is triplicate of the ratio of BC to bc.

But since $BC : bc :: be : Y$, therefore $BC^2 : bc^2 :: bc^2 : Y^2$, (prop. 11, book ii.;) and, multiplying the antecedents by BC, and the consequents by bc, $BC^3 : bc^3 :: BC \times bc^2 : bc \times Y^2 :: BC \times bc : Y^2$; but $Y^2 = bc \times Z$; therefore $BC^3 : bc^3 :: BC \times bc : bc \times Z :: BC : Z$. But $BC^3 : bc^3 ::$ *prism* P : *prism* p; therefore the prisms have to each other the ratio of BC to Z, that is the triplicate ratio of BC to bc.

Proposition XIII.—*Theorem.*

If a triangular pyramid A—BCD *be cut by a plane parallel to its base, the section* FGH *is similar to the base*, fig. 204.

Fig. 204. For because the parallel planes BCD, FGH are cut by a third plane ABC, the sections FG, BC are parallel, (prop. 10, book vi.) In like manner it appears, that FH is parallel to BD; therefore the angle HFG is equal to the angle DBC, (prop. 13, book. vi.;) and because the triangle ABC is similar to the triangle AFG, and the triangle ABD is similar to the triangle AFH, we have

$$BC : BA :: FG : FA,$$
and
$$BA : BD :: FA : FH.$$

Therefore $BC : BD :: FG : FH$; now the angle DBC has been shown to be equal to the angle HFG; therefore the triangle DBC, HFG are equiangular, (prop. 25, book iv.)

Proposition XIV.—*Theorem.*

If two triangular pyramids A—BCD, a—bcd, *which have equal bases, and equal altitudes, be cut by planes that are parallel to the bases, and at equal distances from them; the sections* FGH, fgh *will be equal*, fig. 205.

Fig. 205. Draw AKE, ake perpendicular to the bases BCD, bcd, meeting the cutting planes in K and k; then, because of the parallel planes, we have (prop. 15, book vi.)

$$AE : AK :: AB : AF, \text{ and } ae : ak :: ab : af,$$

but, by hypothesis, $AE = ae$, and $AK = ak$; therefore $AB : AF :: ab : af$; again, because of similar triangles, $AB : AF :: BC : FG$, and $ab : af :: bc : fg$; therefore, $BC : FG :: bc : fg$; and hence, $BC^2 :: FG^2 :: bc^2 : fg^2$, (prop. 11, book ii.;) but because of the similar triangles BDC, FGH, $BC^2 : FG^2 ::$ *trian.* BDC : *trian.* FHG, and in like manner $bc^2 : fg^2 ::$ *trian.* bcd : *trian.* fgh, therefore *trian.* BCD : *trian.* FGH :: *trian.* bcd : *trian.* fgh. Now *trian.* BCD = *trian.* bcd, (by hypothesis,) therefore the triangle FHG is equal to the triangle fhg.

Scholium. It is easy to see, that what is proved in this and the preceding proposition, is also true of polygonal pyramids.

Proposition XV.—*Theorem.*

A series of prisms of the same altitude may be inscribed in a pyramid, and another series may be circumscribed about it, which shall exceed the other by less than any given solid, fig. 206.

Fig. 206. Let ABCD be a pyramid, and let AC, one of its lateral edges, be divided into some number of equal parts, at the points F, G, H; through these let planes pass parallel to the base BCD, making with the sides of the pyramids the sections QPF, SRG, UTH; which will be similar to one another and to the base, (prop. 13, book vii.) From B in the plane of the triangle ABC, draw BK parallel to CF, meeting FP produced in K; in like manner from D draw DL parallel to CF meeting FQ produced in L; join KL, and the solid CBD—FKL will evidently be a prism. By the same construction let the prisms PM, RO, TV be described: also let the straight line IP, which is in the plane of the triangle ABC, be produced till it meet BC in h, and let MQ be produced till it meet DC in g; join hg, then Chg, FPQ will be a prism, and be equal to the prism PM. In the same manner is described the prism mS equal to the prism RO, and the prism qU equal to the prism TV. Therefore the sum of all the inscribed prisms hQ, mS, and qU is equal to the sum of the prisms PM, RO, and TV; that is, to the sum of all the circumscribed prisms, except the prism BL; wherefore BL is the excess of the prisms circumscribed about the pyramid above the prisms inscribed within it.

Let us now suppose that Z denotes some given solid equal to a prism, which has the same base CBD as the pyramid, and its altitude equal to a perpendicular from E (a point in AC) upon the base. Then, however near E may be to C, it will evidently be possible to divide AC into such a number of equal parts, that one of them, CF, shall be less than CE; and this being the case, the prism BL will evidently be less than the prism whose base is the triangle CBD and altitude, a perpendicular from E on the base BCD; that is less than the given solid Z; therefore the excess of the circumscribed above the inscribed prisms may be less than the solid Z.

Cor. Since the difference between the circumscribed and inscribed prisms may be less than any given magnitude, and the pyramid is greater than the latter, and less than the former, it follows that a series of prisms may be circumscribed about the pyramid, and also a series of prisms may be inscribed in it, which shall differ from the pyramid itself by less than any given solid.

Proposition XVI.—*Theorem.*

Pyramids that have equal bases and altitudes are equal to one another, fig. 207.

Let A—BCD, a—bcd be two pyramids that have equal bases BCD, bcd, and equal altitudes; viz. the perpendiculars drawn from the vertices A and a upon the planes BCD, bcd, the pyramid ABCD is equal to the pyramid $abcd$. Fig. 207.

For, if they are not equal, let Z represent the solid which is equal to the excess of one of them, a—bcd above the other A—BCD; and let a series of prisms CE, FG, HK, LM, of the same altitude, be circumscribed about the pyramid ABCD, so as to exceed it by a solid less than Z, which is always possible; (prop. 15, book vii.) also let a series of prisms $cefgh$ klm, equal in number to the other and of the same altitude, be circumscribed about the pyramid a—bcd. And because the pyramids have equal altitudes, and the number of prisms described about each is the same, the altitudes of the prisms will be all equal, and the bases of the corresponding prisms in the two pyramids, in EF, ef, will be sections of the pyramids at equal distances from their bases; therefore they are equal (prop. 14, book vii.) and the prisms themselves are equal, (prop. 1, book vii.) and the sum of all the prisms described about the one pyramid is equal

Geometry. to the sum of all the prisms described about the other pyramid. For the sake of abridging, let P and p denote the pyramids A B C D, and $a\,b\,c\,d$, respectively, and Q and q express the sums of the prism described about them. Then, because by the hypothesis $Z = p - P$, and by construction $Z \gtrdot Q - P$, therefore $(p-P) \gtrdot (Q-P)$; hence p must be greater than Q; but Q is equal to q; therefore p must be greater than q; that is the pyramid p is greater than q, the sum of the prisms described about it, which is impossible; therefore the pyramids P, p are not unequal, that is they are equal to each other.

Proposition XVII.—*Theorem.*

Every triangular pyramid is the third of the triangular prism having the same base and altitude, fig. 208.

Fig. 208. Let F A B C be a triangular pyramid, A B C D E F a triangular prism of the same base and altitude: the pyramid will be equal to one-third of the prism. Conceive the pyramid F A B C to be cut off from the prism by a section made along the plane F A C, and there will remain the solid F A C D E, which may be considered as a quadrangular pyramid whose vertex is F, and whose base is the parallelogram A C D E. Draw the diagonal C E, and extend the plane F C E, which will cut the quadrangular pyramid into two triangular ones F A C E, F C D E. These two triangular pyramids have for their common altitude the perpendicular let fall from F on the plane A C D E; they have equal bases, the triangles A C E, C D E being halves of the same parallelogram; hence the two pyramids F A C E, F C D E are equal. But the pyramid F C D E and the pyramid F A B C, have equal bases A B C, D E F; they have also the same altitude, namely, the distance of the parallel planes A B C, D E F; hence the two pyramids are equal. Now the pyramid FCDE has already been proved equal to F A C E; hence the three pyramids F A BC, F C D E, F A C E, which compose the prism A B D are all equal. Hence the pyramid F A B C is the third part of the prism A B D, which has the same base and the same altitude.

Cor. The solidity of a triangular pyramid is equal to a third part of the product of its base by its altitude.

Proposition XVIII.—*Theorem.*

Any pyramid S A B C D E *is measured by the third part of the product of its base by its altitude*, fig. 209.

Fig. 209. For, extending the planes S E B, S E C through the diagonals E B, E C, the polygonal pyramid S A B C D E will be divided into several triangular pyramids all having the same altitude S O. But (prop. 17, book vii.) each of these pyramids is measured by multiplying its base A B E, B C E, or C D E by the third part of its altitude S O; hence the sum of these triangular pyramids, or the polygonal pyramid S A B C D E will be measured by the sum of the triangles A B E, B C E, C D E, or the polygon A B C D E, multiplied by S O; hence every pyramid is measured by a third part of the product of its base by its altitude.

Cor. 1. Every pyramid is the third part of the prism which has the same base and the same altitude.

Cor. 2. Two pyramids having the same altitude are to each other as their bases.

Scholium. The solidity of any polyedral body may be computed, by dividing the body into pyramids; and this division may be accomplished in various ways. One of the simplest is to make all the planes of division pass through the vertex of one solid angle; in that case, there will be formed as many partial pyramids as the polyedron has faces, *minus* those faces which form the solid angle whence the planes of division proceed. Book VII. Book VIII.

Proposition XIX.—*Theorem.*

Two similar pyramids are to each other as the cubes of their homologous sides, fig. 210.

For two pyramids being similar, the smaller may be placed within the greater, so that the solid angle S shall be common to both. In that position the bases A B C D E, $a\,b\,c\,d\,e$ will be parallel; because, since the homologous faces are similar, the angle S $a\,b$ is equal to S A B, and S $b\,c$ to S B C; hence the plane A B C is parallel to the plane $a\,b\,c$. This granted, let S O be the perpendicular drawn from the vertex S to the plane A B C, and o the point where this perpendicular meets the plane $a\,b\,c$; from what has already been shown we shall have S O : S o : : S A : S a : : A B : $a\,b$; and consequently, Fig. 210.

$\tfrac{1}{3}$ S O : $\tfrac{1}{3}$ S a : : A B : $a\,b$.

Let H represent the altitude of the *frustum of a pyramid*, having parallel bases A and B; $\surd$ A B will be the mean proportion. But the bases A B C D E, $a\,b\,c\,d\,e$ being similar figures, we have

A B C D E : $a\,b\,c\,d\,e$: : AB^2 : ab^2.

Multiply the corresponding terms of these two proportions; there results the proportion,

A B C D E $\times$ $\tfrac{1}{3}$ S O : : $a\,b\,c\,d\,e$ $\times$ $\tfrac{1}{3}$ S o : : AB^3 : ab^3.

Now A B C D E $\times$ $\tfrac{1}{3}$ S O is the solidity of the pyramid S A B C D E, and $a\,b\,c\,d\,e$ $\times$ $\tfrac{1}{3}$ S o is that of the pyramid S$a\,b\,c\,d\,e$, (prop. 17 and 18, book vii.;) hence two similar pyramids are to each other as the cubes of their homologous sides.

BOOK VIII.

The three round bodies.

Definitions.

1. A cylinder is a solid produced by the revolution of a rectangle A B C D, conceived to turn about the immovable side A B, fig. 211. Fig. 211.

In this rotation, the sides A D, B C, continuing always perpendicular to A B, describe equal circular planes D H P, C G Q, which are called the *bases of the cylinder*, the side C D at the same time describing the *convex surface*.

The immovable line A B is called the *axis of the cylinder*.

Every section K L M, made in the cylinder, at right angles to the axis, is a circle equal to either of the bases; for, whilst the rectangle A B C D revolves about A B, the line K I, perpendicular to A B, describes a circular plane, equal to the base, which is a section made perpendicular to the axis at the point I.

Every section P Q G H, passing through the axis,

 is a rectangle, and is double of the generating rectangle A B C D.

2. A *cone* is a solid produced by the revolution of a right angled triangle S A B, conceived to turn about
Fig. 212. the immovable side S A, fig. 212.

In this rotation, the side A B describes a circular plane B D C E, named the *base of the cone*; and the hypothenuse S B its *convex surface*.

The point S is named the *vertex of the cone*, S A its *axis* or *altitude*.

Every section H K F I, formed at right angles to the axis, is a circle; every section S D E passing through the axis is an isosceles triangle double of the generating triangle S A B.

3. If from the cone S C D B, the cone S F K H be cut off by a section parallel to the base, the remaining solid C B H F is called a *truncated cone*, or the *frustum of a cone*.

We may conceive it to be described by the revolution of a trapezium A B H G, whose angles A and C are right, about the side A G. The immovable line A G is called the *axis* or *altitude of the frustum*, the circles B D C, H F K are its *bases*, and B H is its *side*.

4. Two cylinders, or two cones, are *similar*, when their axes are to each other as the diameters of their bases.

Fig. 213. 5. If in the circle A C D, (fig. 213,) which forms the base of a cylinder, a polygon A B C D E is inscribed, a right prism, constructed on this base A B C D E, and equal in altitude to the cylinder, is said to be *inscribed in the cylinder*, or the cylinder to be *circumscribed about the prism*.

The edges A F, B G, C H, &c. of the prism, being perpendicular to the plane of the base, are evidently included in the convex surface of the cylinder; hence the prism and the cylinder touch one another along these edges.

Fig. 214. 6. In like manner, if A B C D (fig. 214) is a polygon, circumscribed about the base of a cylinder, a right prism, constructed on this base A B C D, and equal in altitude to the cylinder, is said to be *circumscribed about the cylinder*, or the cylinder to be *inscribed in the prism*.

Let M, N, &c. be the points of contact in the sides A B, B C, &c.; and through the points M, N, &c. let M X, N Y, &c. be drawn perpendicular to the plane of the base: those perpendiculars will evidently lie both in the surface of the cylinder, and in that of the circumscribed prism; hence they will be their lines of contact.

Note. The cylinder, the cone, and the sphere, are the *three round bodies* treated of in the elements of geometry.

Proposition I.—*Theorem.*

The solidity of a cylinder is equal to the product of its base by its altitude, fig. 215.

Fig. 215. Let C A be a radius of the given cylinder's base; H the altitude; let *surf.* C A, represent the area of the circle whose radius is C A; we are to show that the solidity of the cylinder is *surf.* C A × H. For, if *surf.* C A × H is not the measure of the given cylinder, it must be the measure of a greater cylinder, or of a smaller one. Suppose it first to be the measure of a smaller one; of a cylinder, for example, which has C D for the radius of its base, H being the altitude.

About the circle whose radius is C D, circumscribe a regular polygon G H I P, (prop. 9, book v.) the sides of which shall not meet the circumference whose radius is C A. Imagine a right prism, having the regular polygon G H I P for its base, and H for its altitude; this prism will be circumscribed about the cylinder, whose base has C D for its radius. Now, (prop. 11, book vii.) the solidity of the prism is equal to its base G H I P, multiplied by the altitude H; the base G H I P is less than the circle, whose radius is C A; hence the solidity of the prism is less than *surf.* C A × H. But, by hypothesis, *surf.* C A × H is the solidity of the cylinder inscribed in the prism; hence the prism must be less than the cylinder; whereas in reality it is greater, because it contains the cylinder; hence it is impossible that *surf.* C A × H can be the measure of the cylinder whose base has C D for its radius, H being the altitude; or, in more general terms, *the product of the base, by the altitude of a cylinder, cannot measure a less cylinder.*

We must now prove that the same product cannot measure a greater cylinder. To avoid the necessity of changing our figure, let C D be a radius of the given cylinder's base; and, if possible, let *surf.* C D × H, be the measure of a greater cylinder, for example, of the cylinder whose base has C A for its radius, H being the altitude.

The same construction being performed as in the first case, the prism, circumscribed about the given cylinder, will have G H I P × H for its measure; the area G H I P is greater than *surf.* C D; hence the solidity of this prism is greater than *surf.* C D × H; hence the prism must be greater than the cylinder, having the same altitude, and *surf.* C A for its base. But on the contrary the prism is less than the cylinder, being contained in it; therefore *the base of a cylinder, multiplied by its altitude, cannot be the measure of a greater cylinder.*

Hence, finally, the solidity of a cylinder is equal to the product of its base by its altitude.

Cor. 1. Cylinders of the same altitude are to each other as their bases; and cylinders of the same base are to each other as their altitudes.

Cor. 2. Similar cylinders are to each other as the cubes of their altitudes, or as the cubes of the diameters of their bases. For the bases are as the squares of their diameters; and the cylinders being similar, the diameters of their bases (def. 4) are to each other as the altitudes: hence the bases are as the squares of the altitudes; hence the bases, multiplied by the altitudes, or the cylinders themselves, are as the cubes of the altitudes.

Scholium. Let R be the radius of a cylinder's base; H the altitude: the surface of the base (prop. 11, book v.) will be πR^2; and the solidity of the cylinder will be $\pi R^2 \times H$, or $\pi R^2 H$.

Proposition II.—*Lemma.*

The convex surface of a right prism is equal to the perimeter of its base multiplied by its altitude, fig. 213.

For this surface is equal to the sum of the rectangles A F G B, B G H C, C H I D, &c. (fig. 213) which compose it. Now the altitudes A F, B G, C H, &c. of those rectangles, are equal to the altitude of the prism; their bases A B, B C, C D, &c. taken together, make up the perimeter of the prism's

Geometry. base. Hence the sum of these rectangles, or the convex surface of the prism, is equal to the perimeter of its base, multiplied by its altitude.

Cor. If two right prisms have the same altitude, their convex surfaces will be to each other as the perimeters of their bases.

Proposition III.—*Lemma.*

The convex surface of a cylinder is greater than the convex surface of any inscribed prism, and less than the convex surface of any circumscribed prism, fig. 213.

For (fig. 213) the convex surface of the cylinder and that of the prism may be considered as having the same length, since every section made in either parallel to AF is equal to AF; and if these surfaces be cut, in order to obtain the breadths of them, by planes parallel to the base, or perpendicular to the edge A F, the one section will be equal to the circumference of the base, the other to the contour of the polygon A B C D E, which is less than that circumference: hence, with an equal length, the cylindrical surface is broader than the prismatic surface; hence the former is greater than the latter.

By a similar demonstration, the convex surface of the cylinder might be shown to be less than that of any circumscribed prism B C D K L K H, fig. 214. Fig. 214.

Proposition IV.—*Theorem.*

The convex surface of a cylinder is equal to the circumference of its base multiplied by its altitude, fig. 216.

Fig. 216. Let C A be the radius of the given cylinder's base, H its altitude; the circumference whose radius is CA, being represented by *circ.* C A, we are to show that *circ.* $CA \times H$ will be the convex surface of the cylinder. For, if this proposition be not true, then *circ.* $CA \times H$ must be the surface of a greater cylinder, or of a less one. Suppose it first to be the surface of a less cylinder; of the cylinder, for example, the radius of whose base is C D, and whose altitude is H.

About the circle whose radius is C D, circumscribe a regular polygon G H I P, the sides of which shall not meet the circle whose radius is C A; conceive a right prism having H for its altitude, and the polygon G H I P for its base. The convex surface of this prism will be equal (prop. 2, book viii.) to the contour of the polygon G H I P multiplied by the altitude H: this contour is less than the circumference whose radius is C A; hence the convex surface of the prism is less than *circ.* $CA \times H$. But, by hypothesis, *circ.* $CA \times H$ is the convex surface of the cylinder whose base has C D for its radius; which cylinder is inscribed in the prism: hence the convex surface of the prism must be less than that of the inscribed cylinder; but, by hypothesis (prop. 3, book viii.) it is greater: hence, in the first place, *the circumference of a cylinder's base multiplied by its altitude cannot be the measure of a smaller cylinder.*

Neither can this product be the measure of a greater cylinder. For, retaining the present figure, let C D be the radius of the given cylinder's base; and, if possible, let *circ.* $CD \times H$ be the convex surface of a cylinder, which with the same altitude has for its base a greater circle, the circle, for instance, whose radius is C A. The same construction being performed as above, the convex surface of the prism will again be equal to the contour of the polygon G H I P multiplied by the altitude H. But this contour is greater than *circ.* C D; therefore the surface of the prism must be greater than *circ.* $CD \times H$, which, by hypothesis, is the surface of the cylinder having the same altitude, and C A for the radius of its base. Hence the surface of the prism must be greater than that of the prism. Now if this prism were inscribed in the cylinder, its surface (prop. 3, book viii.) would be less than the cylinder's; much more then is it less when the prism does not reach so far as to touch the cylinder. Consequently also, in the second place, *the circumference of a cylinder's base multiplied by the altitude cannot measure the surface of a greater cylinder.* Book VIII.

The product in question being, therefore, neither the measure of the convex surface of a less nor greater cylinder, must be the measure of the cylinder itself.

Proposition V.—*Theorem.*

The solidity of a cone is equal to the product of its base by the third of its altitude, fig. 217.

Let S O be the altitude of the given cone, A O the radius of its base; the surface of the base being designated by *surf.* A O, it is to be demonstrated that *surf.* $AO \times \frac{1}{3} SO$ is equal to the solidity of the cone. Fig. 217.

Suppose, first, that *surf.* $AO \times \frac{1}{3} SO$, is the solidity of a greater cone; for example, of the cone whose altitude is also S O, but whose base has O B, greater than A O, for its radius.

About the circle whose radius is A O, circumscribe a regular polygon M N P T (prop. 9, book v.) so as not to meet the circumference whose radius is O B; imagine a pyramid having this polygon for its base, and the point S for its vertex. The solidity of this pyramid (prop. 18, book vii.) is equal to the area of the polygon M N P T multiplied by a third of the altitude S O. But the polygon is greater than the inscribed circle represented by *surf.* A O; hence the pyramid is greater than *surf.* $AO \times \frac{1}{3} SO$, which, by hypothesis, measures the cone having S for its vertex and O B for the radius of its base: whereas, in reality, the pyramid is less than the cone, being contained in it; hence, first, the base of a cone multiplied by a third of its altitude cannot be the measure of a greater cone.

Neither can this same product be the measure of a smaller cone. For now let O B be the radius of the given cone's base; and, if possible, let *surf.* $OB \times \frac{1}{3} SO$ be the solidity of the cone having S O for its altitude, and for its base the circle whose radius is A O. The same construction being made, the pyramid S M N P T will have for its measure the area M N P T multiplied by $\frac{1}{3}$ S O. But the area M N P T is less than *surf.* O B; hence the measure of the pyramid must be less than *surf.* $OB \times \frac{1}{3} SO$, and consequently it must be less than the cone whose altitude is S O and whose base has A O for its radius. But, on the contrary, the pyramid is greater than the cone, because the cone is contained in it; hence, in the second place, the base of a cone multiplied by a third of its altitude cannot be the measure of a smaller one.

Consequently the solidity of a cone is equal to the product of its base by the third of its altitude.

Cor. A cone is the third of a cylinder having the same base and the same altitude; whence it follows,

1. That cones of equal altitudes are to each other as their bases;

2. That cones of equal bases are to each other as their altitudes;

3. That similar cones are as the cubes of the diameters of their bases, or as the cubes of their altitudes.

Scholium. Let R be the radius of a cone's base, H its altitude; the solidity of the cone will be $\pi R^2 \times \frac{1}{3} H$, or $\frac{1}{3} \pi R^2 H$.

PROPOSITION VI.—*Theorem.*

The convex surface of a cone is equal to the circumference of its base multiplied by half its side, fig. 218.

Fig. 218. Let A O be a radius of the base of the given cone, S its vertex, and S A its side; the surface will be *circ.* A O × ½ S A. For, if possible, let *circ.* A O × S O be the surface of a cone having S for its vertex, and for its base a circle whose radius O B is greater than A O.

About the smaller circle describe a regular polygon M N P T, the sides of which shall not meet the circle whose radius is O B; and let S M N P T be the regular pyramid, having this polygon for its base, and the point S for its vertex. The triangle S M N, one of those which compose the convex surface of the pyramid, has for measure its base M N multiplied by half its altitude S A, or half the side of the given cone; and since this altitude is the same in all the other triangles S N P, S P Q, &c. the convex surface of the pyramid must be equal to the perimeter M N P T M multiplied by ½ S A. But the contour M N P T M is greater than *circ.* A O; hence the convex surface of the pyramid is greater than *circ.* A O × ½ S A, and consequently greater than the convex surface of the cone having the same vertex S, and the circle whose radius is O B for its base. But the surface of this cone is greater than that of the pyramid; because, if two such pyramids are adjusted to each other base to base, and two such cones base to base, the surface of the double cone will envelope on all sides that of the double pyramid, and therefore be greater than it, as is evident; hence the surface of the cone is greater than that of the pyramid, whereas by the hypothesis it is less: hence, in the first place, the circumference of the cone's base multiplied by half the side cannot measure the surface of a greater cone.

Neither can it measure the surface of a smaller cone; for let B O be the radius of the base of the given cone; and, if possible, let *circ.* B O × ½ S B be the surface of a cone having S for its vertex, and A O less than O B for the radius of its base.

The same construction being made as above, the surface of the pyramid S M N P T will still be equal to the perimeter M N P T × ½ S A. Now this perimeter M N P T is less than *circ.* O B; likewise S A is less than S B; consequently, for a double reason, the convex surface of the pyramid is less than *circ.* O B × ½ S B, which, by hypothesis, is the surface of the cone having S A for the radius of its base; hence the surface of the pyramid must be less than that of the inscribed cone. But it is obviously greater; for, adjusting two such pyramids to each other, base to base, and two such cones, base to base, the surface of the double pyramid will envelope that of the double cone, and will be greater than it. Hence, in the second place, the circumference of the base of the given cone multiplied by half the side cannot be the measure of the surface of a smaller cone.

Therefore, finally, the convex surface of a cone is equal to the circumference of its base multiplied by half its side.

Scholium. Let L be the side of a cone, R the radius of its base; the circumference of this base will be $2 \pi R$, and the surface of the cone will be $2 \pi R \times \frac{1}{2} L$, or $\pi R L$.

PROPOSITION VII.—*Theorem.*

The convex surface of a truncated cone A D E B *is equal to its side* A D *multiplied by half the sum of* A B, D E, *the circumferences of its two bases*, fig. 219

In the plane S A B which passes through the axis S O, draw the line A F perpendicular to S A, and equal to the circumference having A O for its radius; join S F; and draw D H parallel to A F. Fig. 219.

From the similar triangles S A O, S D C we have A O : D C : : S A : S D; and by the similar triangles S A F, S D H, A F : D H : : S A : S D; hence A F : D H : : A O : D C, or (prop. 10, book v.) as *circ.* A O is to *circ.* D C. But, by construction, A F = *circ.* A O; hence D H = *circ.* D C. Hence the triangle S A F, measured by A F × ½ S A, is equal to the surface of the cone S A B which is measured by *circ.* A O × ½ S A. For a like reason, the triangle S D H is equal to the surface of the cone S D E. Therefore the surface of the truncated cone A D E B is equal to that of the trapezium A D H F. But the latter (prop. 4, book iv.) is measured by $AD \times \left(\frac{AF + DH}{2}\right)$; hence the surface of the truncated cone A D E B, is equal to its side A D multiplied by half the sum of the circumferences of its two bases.

Cor. Through I, the middle point of A D, draw I K L parallel to A B, and I M parallel to A F; it may be shown as above that I M = *circ.* I K. But the trapezium A D H F = A D × I M = A D × *circ.* I K. Hence it may also be asserted, that *the surface of a truncated cone is equal to its side multiplied by the circumference of a section at equal distances from the two bases.*

Scholium. If a line A D, lying wholly on one side of the line O C, and in the same plane, make a revolution around O C, the surface described by A D will have for its measure $AD \times \left(\frac{circ.\, AO + circ.\, DC}{2}\right)$, or A D × *circ.* I K; the lines A O, D C, I K being perpendiculars, let fall from the extremities and from the middle of the axis O C.

For, if A D and O C are produced till they meet in S, the surface described by A D is evidently that of a truncated cone having A O and D C for the radii of its bases, the vertex of the whole cone being S. Hence this surface will be measured as we have said.

This measure will always hold good, even when the point D falls on S, and thus forms a whole cone; and also when the line A D is parallel to the axis, and thus forms a cylinder. In the first case D C would be nothing; in the second, D C would be equal to A O and to I K.

Geometry.

Proposition VIII.—*Lemma.*

Let A B, B C, C D *be several successive sides of a regular polygon,* O *its centre, and* O I *the radius of the inscribed circle; if that portion of the polygon* A B C D, *which lies wholly on one side of the diameter* F G, *be supposed to make a revolution about this diameter, the surface described by* A B C D *will have for its measure* M Q × circ. O I, M Q *being the altitude of that surface, or the axis included between* A M *and* D Q *the extreme perpendiculars,* fig. 220.

Fig. 220 The point I being the middle of A B, and I K a perpendicular let fall from the point I upon the axis, the surface described by A B by the last proposition will have for its measure A B × *circ.* I K. Draw A X parallel to the axis; the triangles A B X, O I K will have their sides perpendicular, each to each, namely, O I to A B, I K to A X, and O K to B X; hence these triangles are similar, and give the proportion A B : A X, or M N : : O I : I K, or as circ. O I to circ. I K; hence A B × *circ.* I K = M N × *circ.* O I. Whence it is plain that the surface described by the partial polygon A B C D is measured by (M N + N P + P Q) × *circ.* O I, or by M Q × *circ.* O I; hence it is equal to the altitude multiplied by the circumference of the inscribed circle.

Cor. If the whole polygon has an even number of sides, and if the axis F G passes through two opposite vertices F and G, the whole surface described by the revolution of the half polygon F A C G will be equal to its axis F G multiplied by the circumference of the inscribed circle. This axis F G will at the same time be the diameter of the circumscribed circle.

Proposition IX.—*Theorem.*

The surface of a sphere is equal to its diameter multiplied by the circumference of a great circle, fig. 221.

Fig. 221. It is first to be shown, that the diameter of a sphere multiplied by the circumference of its great circle cannot measure the surface of a larger sphere. If possible, let A B × *circ.* A C be the surface of the sphere whose radius is C D.

About the circle whose radius is C A, circumscribe a regular polygon having an even number of sides, so as not to meet the circumference whose radius is C D; let M and S be the two opposite vertices of this polygon; and about the diameter M S let the half polygon M P S be made to revolve. The surface described by this polygon will be measured (prop. 7, book viii.) by M S × *circ.* A C; but M S is greater than A B; hence the surface described by this polygon is greater than A B × *circ.* A C, and consequently greater than the surface of the sphere whose radius is C D; but the surface of the sphere is greater than the surface described by the polygon, since the former envelopes the latter on all sides. Hence, in the first place, the diameter of a sphere multiplied by the circumference of its great circle cannot measure the surface of a larger sphere.

Neither can this same product measure the surface of a smaller sphere. For, if possible, let D E × *circ.* C D be the surface of that sphere whose radius is C A. The same construction being made as in the former case, the surface of the solid generated by the revolution of the half polygon will still be equal to M S × *circ.* A C. But M S is less than D E, and *circ.* A C is less than *circ.* C D; hence, for these two reasons, the surface of the solid described by the polygon must be less than D E × *circ.* C D, and therefore less than the surface of the sphere whose radius is A C. But the surface described by the polygon is greater than the surface of the sphere whose radius is A C, because the former envelopes the latter; hence, in the second place, the diameter of a sphere multiplied by the circumference of its great circle, cannot measure the surface of a smaller sphere. Book VIII.

Therefore the surface of a sphere is equal to its diameter multiplied by the circumference of its great circle.

Cor. The surface of the great circle is measured by multiplying its circumference by half the radius, or by a fourth of the diameter; hence *the surface of a sphere is four times that of its great circle.*

Proposition X.—*Theorem.*

The surface of any spherical zone is equal to its altitude multiplied by the circumference of a great circle, fig. 222 and 223.

Let E F be any arc less or greater than a quadrant; and let F G be drawn perpendicular to the radius E C; the zone with one base, described by the revolution of the arc E F about E C, will be measured by E G × *circ.* E C. Fig. 222.

For, suppose, first, that this zone is measured by something less; if possible, by E G × *circ.* C A. In the arc E F, inscribe a portion of a regular polygon E M N O P F, whose sides shall not reach the circumference described with the radius C A; and draw C I perpendicular to E M. By proposition 8, book viii. the surface described by the polygon E M F turning about E C will be measured by E G × *circ.* C I. This quantity is greater than E G × *circ.* A C, which by hypothesis is the measure of the zone described by the arc E F. Hence the surface described by the polygon E M N O P F must be greater than the surface described by E F the circumscribed arc; whereas this latter surface is greater than the former, which it envelopes on all sides; hence, in the first place, the measure of any spherical zone with one base cannot be less than the altitude multiplied by the circumference of a great circle.

Secondly, the measure of this zone cannot be greater than its altitude multiplied by the circumference of a great circle. For suppose the zone described by the revolution of the arc A B about A C to be the proposed one; and, if possible, let *zone* A B $>$ A D × *circ.* A C. The whole surface of the sphere composed of the two zones A B, B H, is measured by A H × *circ.* A C, (prop. 9, book viii.) or by A D × *circ.* A C + D H × *circ.* A C; hence, if we have *zone* A B $>$ D H × *circ.* A C, we must also have *zone* B H $>$ D H × *circ.* A C; which cannot be the case, as is shown above. Therefore, in the second place, the measure of a spherical zone with one base, cannot be greater than the altitude of this zone multiplied by the circumference of a great circle.

Hence, finally, every spherical zone with one base is measured by its altitude multiplied by the circumference of a great circle.

Let us now examine any zone with two bases, described by the revolution of the arc F H (fig. 223) about the diameter D E. Draw F O, H Q perpendi- Fig. 223.

Geometry. cular to this diameter. The zone described by the arc FH is the difference of the two zones described by the arcs DH and DF; the latter are respectively measured by $DQ \times circ. CD$ and $DO \times circ. CD$; hence the zone described by FH has for its measure $(DQ - DO) \times circ. CD$, or $OQ \times circ. CD$.

That is, any spherical zone, with one or two bases, is measured by its altitude multiplied by the circumference of a great circle.

Cor. Two zones, taken in the same sphere or in equal spheres, are to each other as their altitude; and any zone is to the surface of the sphere as the altitude of that zone is to the diameter.

PROPOSITION XI.—*Theorem.*

If the triangle BAC *and the rectangle* BCEF, *having the same base and the same altitude, turn simultaneously about the common base* BC, *the solid described by the revolution of the triangle will be a third of the cylinder described by the revolution of the rectangle*, fig. 224 and 225.

Fig. 224. On the axis, let fall the perpendicular AD; the cone described by the triangle ABD is the third part of the cylinder described by the rectangle AFBD (prop. 5, book viii.;) also the cone described by the triangle ADC is the third part of the cylinder described by the rectangle ADCE; hence the sum of the two cones, or the solid described by ABC, is the third part of the two cylinders taken together, or of the cylinder described by the rectangle BCEF.

Fig. 225 If the perpendicular AD (fig. 225) falls without the triangle; the solid described by ABC will, in that case, be the difference of the two cones described by ABD and ACB; but, at the same time, the cylinder described by BCEF will be the difference of the two cylinders described by AFBD and AECD. Hence the solid, described by the revolution of the triangle, will still be a third part of the cylinder described by the revolution of the rectangle having the same base and the same altitude.

Scholium. The circle of which AD is radius has for its measure $\pi \times AD^2$; hence $\pi \times AD^2 \times BC$ measures the cylinder described by BCEF, and $\frac{1}{3}\pi \times AD^2 \times BC$ measures the solid described by the triangle ABC.

PROPOSITION XII.—*Problem.*

The triangle CAB *being supposed to perform a revolution about the line* CD, *drawn at will without the triangle through its vertex* C, *to find the measure of the solid so produced*, fig. 226.

Fig. 226. Produce the side AB till it meets the axis CD in D; from the points A and B, draw AM, BN perpendicular to the axis.

The solid described by the triangle CAD is measured (prop. 11, book viii.) by $\frac{1}{3}\pi \times AM^2 \times CD$; the solid described by the triangle CBD is measured by $\frac{1}{3}\pi \times BN^2 \times CD$; hence the difference of those solids, or the solid described by ABC, will have for its measure $\frac{1}{3}\pi(AM^2 - BN^2) \times CD$.

To this expression another form may be given. From I the middle point of AB, draw IK perpendicular to CD; and through B, draw BO parallel to CD; we shall have $AM + BN = 2IK$, (prop. 4, book iv.) and $AM - BN = AO$; hence $(AM + BN) \times (AM - NB)$, or $AM^2 - BN^2 = 2IK \times AO$, (prop. 12, book iv.) Hence the measure of the solid in question is expressed by $\frac{2}{3}\pi \times IK \times AO \times CD$. But if CP is drawn perpendicular to AB, the triangles ABO, DCP will be similar, and give the proportion $AO : CP :: AB : CD$; hence $AO \times CD = CP \times AB$; which $CP \times AB$ is double the area of the triangle ABC; hence we have $AO \times CD = 2ABC$; hence the solid described by the triangle ABC is also measured by $\frac{4}{3}\pi \times ABC \times IK$, or which is the same thing, by $ABC \times \frac{2}{3} circ.\ IK$, $circ.\ IK$ being equal to $2\pi \times IK$. Hence *the solid described by the revolution of the triangle* ABC, *has for its measure the area of this triangle multiplied by two-thirds of the circumference traced by* I, *the middle point of the base.* Book VIII.

Cor. If the side $AC = CB$, (fig. 227,) the line CI will be perpendicular to AB, the area ABC will be equal to $AB \times \frac{1}{2}CI$, and the solidity $\frac{4}{3}\pi \times ABC \times IK$ will become $\frac{2}{3}\pi \times AB \times IK \times CI$. But the triangles ABO, CIK are similar, and give the proportion $AB : BO$ or $MN :: CI : IK$; hence $AB \times IK = MN \times CI$; hence the solid described by the isosceles triangle ABC will have for its measure $\frac{2}{3}\pi \times MN \times CI^2$. Fig. 227.

Scholium. The general solution appears to include the supposition that AB produced will meet the axis; but the results would be equally true, though AB were parallel to the axis.

Thus, the cylinder described by AMNB (fig. 228) is equal to $\pi . AM^2 . MN$; the cone described by ACM is equal to $\frac{1}{3}\pi . AM^2 . CM$, and the cone described by BCN to $\frac{1}{3}\pi . AM^2 . CN$. Add the first two solids and take away the third; we shall have the solid described by ABC equal to $\pi . AM^2 . (MN + \frac{1}{3}CM - \frac{1}{3}CN)$: and since $CN - CM = MN$, this expression is reducible to $\pi . AM^2 . \frac{2}{3}MN$, or $\frac{2}{3}CP^2 . MN$; which agrees with the conclusion drawn above. Fig. 228.

PROPOSITION XIII.—*Theorem.*

Let AB, BC, CD *be several successive sides of a regular polygon*, O *its centre, and* OI *the radius of the inscribed circle; if the polygonal sector* AOD, *lying all on one side of the diameter* FG *be supposed to perform a revolution about this diameter, the solid so described will have for its measure* $\frac{2}{3}\pi . OI^2 . MQ$, MQ *being that portion of the axis which is included by the extreme perpendiculars* AM, DQ, fig. 229.

For, since the polygon is regular, all the triangles AOB, BOC, &c. are equal and isosceles. Now, by the last corollary, the solid produced by the isosceles triangle AOB has for its measure $\frac{2}{3}\pi . OI^2 . MN$; the solid described by the triangle BOC has for its measure $\frac{2}{3}\pi . OI^2 . NP$; and the solid described by the triangle COD has for its measure $\frac{2}{3}\pi . OI^2 . PQ$; hence the sum of those solids, or the whole solid described by the polygonal sector AOD, will have for its measure $\frac{2}{3}\pi . OI^2 . (MN + NP + PQ)$ or $\frac{2}{3}\pi OI^2 . MQ$. Fig. 229.

PROPOSITION XIV.—*Theorem.*

Every spherical sector is measured by the zone which forms its base, multiplied by a third of the radius; and the whole sphere has for its measure a third of the radius, multiplied by its surface fig. 230.

Let ABC be the circular sector, which, by its re- Fig. 230

volution about AC, describes the spherical sector; the zone described by AB being $AD \times$ circ. AC, or $2\pi . AC . AD$, and it is to be shown that this zone multiplied by $\frac{1}{3}$ of AC, or that $\frac{2}{3}\pi . AC^2 . AD$, will measure the sector.

First, suppose, if possible, that $\frac{2}{3}\pi . AC^2 . AD$ is the measure of a greater spherical sector, say of the spherical sector described by the circular sector ECF similar to ACB.

In the arc EF, inscribe ECF, a portion of a regular polygon, such that its sides shall not meet the arc AB; then imagine the polygonal sector $ENFC$ to turn about EC, at the same time with the circular sector ECF. Let CI be a radius of the circle inscribed in the polygon; and let FC be drawn perpendicular to EC. The solid described by the polygonal sector will, by the last proposition, have for its measure $\frac{2}{3} CI^2 . EG$; but CI is greater than AC by construction; and EG is greater than AD; for joining AD, EF, the similar triangles EFG, ABD give the proportion $EG : AD :: FG : BD :: CF : CB$; hence $EG > AD$.

For this double reason, $\frac{2}{3}\pi CI^2 . EG$ is greater than $\frac{2}{3}\pi . CA^2 . AD$. The first is the measure of the solid described by the polygonal sector; the second, by hypothesis, is that of the spherical sector described by the circular sector ECF; hence the solid described by the polygonal sector must be greater than the spherical sector; whereas, in reality, it is less, being contained in the latter: hence our hypothesis was false; therefore, in the first place, the zone or base of a spherical sector multiplied by a third of the radius, cannot measure a greater spherical sector.

Secondly, it is to be shown, that it cannot measure a less spherical sector. Let CEF be the circular sector, which, by its revolution, generates the given spherical sector; and suppose, if possible, that $\frac{2}{3}\pi . CE^2 . EG$ is the measure of some smaller spherical sector, say of that produced by the circular sector ACB.

The construction remaining as above, the solid described by the polygonal sector will still have for its measure $\frac{2}{3}\pi . CI^2 . EG$. But CI is less than CE; hence the solid is less than $\frac{2}{3}\pi . CE^2 . EG$, which, according to the supposition, is the measure of the spherical sector described by the circular sector ACB. Hence the solid described by the polygonal sector must be less than the spherical sector described by ACB; whereas, in reality, it is greater, the latter being contained in the former; therefore, in the second place, it is impossible that the zone of a spherical sector, multiplied by a third of the radius, can be the measure of a smaller spherical sector.

Hence every spherical sector is measured by the zone which forms its base, multiplied by a third of the radius.

A circular sector ACB may increase till it becomes equal to a semicircle: in which case, the spherical sector described by its revolution is the whole sphere. Hence *the solidity of a sphere is equal to its surface multiplied by a third of the radius*

Cor. The surfaces of spheres being as the squares of their radii, these surfaces multiplied by the squares of the radii must be as the cubes of the latter. Hence *the solidity of two spheres are as the cubes of their radii, or as the cubes of their diameters.*

Scholium. Let R be the radius of a sphere, its surface will be $4\pi R^2$; its solidity $4\pi R^2 \times \frac{1}{3}R$, or $\frac{4}{3}\pi . R^3$. If the diameter is named D, we shall have $R = \frac{1}{2}D$, and $R^3 = \frac{1}{8}D^3$; hence the solidity may likewise be expressed by $\frac{4}{3}\pi \times \frac{1}{8}D^3$, or $\frac{1}{6}\pi D^3$.

Proposition XV.—*Theorem.*

The surface of a sphere is to the whole surface of the circumscribed cylinder (including its bases) as 2 is to 3; and the solidities of these two bodies are to each other in the same ratio, fig. 231.

Fig. 231. Let $MNPQ$ be a great circle of the sphere; $ABCD$ the circumscribed square; if the semicircle PMQ and the half square $PADQ$ are at the same time made to revolve about the diameter PQ, the semicircle will generate the sphere, while the half-square will generate the cylinder circumscribed about that sphere.

The altitude AD of that cylinder is equal to the diameter PQ; the base of the cylinder is equal to the great circle, its diameter AB being equal to MN; hence (prop. 4, book viii.) the convex surface of the cylinder is equal to the circumference of the great circle multiplied by its diameter. This measure (prop. 9, book viii.) is the same as that of the surface of the sphere; hence *the surface of the sphere is equal to the convex surface of the circumscribed cylinder.*

But the surface of the sphere is equal to four great circles; hence the convex surface of the cylinder is also equal to four great circles; and adding the two bases, each equal to a great circle, the total surface of the circumscribed cylinder will be equal to six great circles; hence the surface of the sphere is to the total surface of the circumscribed cylinder as 4 is to 6, or as 2 is to 3; which is the first branch of the proposition.

In the next place, since the base of the circumscribed cylinder is equal to a great circle, and its altitude to the diameter, the solidity of the cylinder (prop. 1, book viii.) will be equal to a great circle multiplied by its diameter. But (prop. 14, book viii.) the solidity of the sphere is equal to four great circles multiplied by a third of the radius; in other terms, to one great circle multiplied by $\frac{4}{3}$ of the radius, or by $\frac{2}{3}$ of the diameter; hence the sphere is to the circumscribed cylinder as 2 to 3, and consequently the solidities of these two bodies are as their surfaces.

BOOK IX.

Of the sphere, and spherical triangles.

Definitions.

1. The *sphere* is a solid terminated by a curve surface, all the points of which are equally distant from a point within, called the *centre*.

The sphere may be conceived to be generated by the revolution of a semicircle DAE (fig. 223) about its diameter DE; for the surface described in this movement, by the curve DAE, will have all its points equally distant from the centre C. Fig. 223.

2. The *radius of a sphere* is a straight line drawn from the centre to any point in the surface; the *diameter* or *axis* is a line passing through this centre, and terminated on both sides by the surface.

Geometry. All the *radii* of a sphere are equal; all the *diameters* are equal, and double of the radius.

3. A *great circle of the sphere* is a section which passes through the centre; a *small circle*, one which does not pass through it.

4. A *plane* is a *tangent* to a sphere, when their surfaces have but one point in common.

5. The *pole of a circle* of a sphere is a point in the surface equally distant from all the points in the circumference of this circle.

6. A *spherical triangle* is a portion of the surface of a sphere, bounded by three arcs of great circles.

Those arcs, named the *sides* of the triangle, are always supposed to be each less than a semicircumference. The angles, which their planes form with each other, are the *angles of the triangle*.

7. A spherical triangle takes the name of *right-angled, isosceles, equilateral*, in the same cases as a rectilineal triangle.

8. A *spherical polygon* is a portion of the surface of a sphere terminated by several arcs of great circles.

9. A *lune* is that portion of the surface of a sphere, which is included between two great semicircles meeting in a common diameter.

10. A *spherical wedge* or *ungula* is that portion of the solid sphere, which is included between the same great semicircles, and has the lune for its base.

11. A *spherical pyramid* is a portion of the solid sphere, included between the planes of a solid angle whose vertex is the centre. The *base* of the pyramid is the spherical polygon intercepted by the same planes.

12. A *zone* is the portion of the surface of the sphere, included between two parallel planes, which form its *bases*. One of those planes may be a tangent to the sphere; in which case, the zone has only a single base.

13. A *spherical segment* is the portion of the solid sphere, included between two parallel planes which form its bases.

One of those planes may be a tangent to the sphere; in which case, the segment has only a single base.

14. The *altitude of a zone* or *of a segment* is the distance of the two parallel planes, which form the bases of the zone or segment.

15. Whilst the semicircle D A E (see def. 1) revolving round its diameter D E, describes the sphere; any circular sector, as D C F or F C H, describes a solid, which is named a *spherical sector*.

PROPOSITION I.—*Theorem.*

Every section of a sphere made by a plane is a circle, fig. 234.

Fig. 234. Let A M B be the section, made by a plane in the sphere whose centre is C. From the point C, draw C O perpendicularly to the plane A M B; and draw lines C M, C M to different points of the curve A M B, which terminates the section.

The oblique lines C M, C M, C B being equal, being radii of the sphere, they are equally distant from the perpendicular C O, (prop. 5, book vi.;) hence all the lines O M, M O, O B are equal; hence the section A M B is a circle, whose centre is O.

Cor. 1. If the section passes through the centre of the sphere, its radius will be the radius of the sphere; hence all great circles are equal.

Cor. 2. Two great circles always bisect each other; for their common intersection, passing through the centre, is a diameter. Book IX.

Cor. 3. Every great circle divides the sphere and its surface into two equal parts; for, if the two hemispheres were separated, and afterwards placed on the common base, with their convexities turned the same way, the two surfaces would exactly coincide, no point of the one being nearer the centre than any point of the other.

Cor. 4. The centre of a small circle, and that of the sphere, are in the same straight line perpendicular to the plane of the little circle.

Cor. 5. Small circles are the less the farther they lie from the centre of the sphere; for the greater C O is, the less is the chord A B, the diameter of the small circle A M B.

Cor. 6. An arc of a great circle may always be made to pass through any two given points in the surface of the sphere; for the two given points and the centre of the sphere make three points, which determine the position of a plane. But if the two given points were at the extremities of a diameter, these two points and the centre would then lie in one straight line, and an infinite number of great circles might be made to pass through the two given points.

PROPOSITION II.—*Theorem.*

In every spherical triangle A B C, *any side is less than the sum of the other two*, fig. 235.

Let O be the centre of the sphere; and draw the radii O A, O B, O C. Imagine the planes A O B, A O C, C O B; those planes will form a solid angle at the point O; and the angles A O B, A O C, C O B will be measured by A B, A C, B C, the sides of the spherical triangle. But (prop. 19, book vi.) each of the three plane angles composing a solid angle is less than the sum of the other two; hence any side of the triangle A B C is less than the sum of the other two. Fig. 235.

PROPOSITION III.—*Theorem.*

The shortest distance between one point to another, on the surface of a sphere, is the arc of the great circle which joins the two given points, fig. 236.

Let A N B be the arc of the great circle which joins the points A and B; and without this line, if possible, let M be a point in the line of the shortest distance between A and B. Through the point M, draw M A, M B, arcs of great circles; and take B N = M B. Fig. 236.

By the last theorem, the arc A N B is shorter than $AM + MB$; take $BN = BM$ respectively from both; there will remain $AN < AM$. Now, the distance of B from M, whether it be the same with the arc B M or with any other line, is equal to the distance of B from N; for by making the plane of the great circle B M to revolve about the diameter which passes through B, the point M may be brought into the position of the point N; and the shortest line between M and B, whatever it may be, will then be identical with that between N and B; hence the two lines from A to B, one passing through M, the other through N, have an equal part in each, the part from M to B equal to the part from N to B. The first line is the shorter, by hypothesis; hence the distance from A to M must

be shorter than the distance from A to N; which is absurd, the arc A M being proved greater than A N: hence no point of the shortest line from A to B can lie out of the arc A N B; consequently this arc is itself the shortest distance between its two extremities.

Proposition IV.—*Theorem.*

The sum of all the three sides of a spherical triangle is less than the circumference of a great circle, fig. 237.

Fig. 237. Let A B C be any spherical triangle; produce the sides A B, A C till they meet again in D. The arcs A B D, A C D will be semicircumferences, since (prop. 1, book ix.) two great circles always bisect each other. But in the triangle B C D, we have (prop. 2, book ix.) the side $BC \angle BD + CD$; add $AB + AC$ to both; we shall have $AB + AC + BC \angle ABD + ACD$, that is to say, less than a circumference.

Proposition V.—*Theorem.*

The sum of all the sides of any spherical polygon is less than the circumference of a great circle, fig. 238.

Fig. 238. Let us take, for example, the pentagon A B C D E. Produce the sides A B, D C, till they meet in F; then since B C is less than $BF + CF$, the perimeter of the pentagon A B C D E will be less than that of the quadrilateral A E D F. Again, produce the sides A E, F D, till they meet in G; we shall have $ED \angle EG + DG$; hence the perimeter of the quadrilateral A E D F is less than that of the triangle A F G; which last is itself less than the circumference of a great circle; hence *a fortiori* the perimeter of the polygon A B C D E is less than this same circumference.

Proposition VI.—*Theorem.*

The diameter D E *being drawn perpendicular to the plane of the great circle* A M B, *the extremities* D *and* E *of this diameter will be the poles of the circle* A M B, *and of all the little circles, as* F N G, *which are parallel to it*, fig. 223.

Fig. 223. For, D C being perpendicular to the plane A M B, is perpendicular to all the straight lines C A, C M, C B, &c. drawn through its foot in this plane; hence all the arcs D A, D M, D B, &c. are quarters of the circumference. So likewise are all the arcs E A, E M, E B, &c.; hence the points D and E are each equally distant from all the points of the circumference A M B; therefore (def. 5) they are the poles of that circumference.

Again, the radius D C, perpendicular to the plane A M B, is perpendicular to its parallel F N G; hence (prop. 1, book ix.) it passes through O the centre of the circle F N G; therefore, if the oblique lines D F, D N, D G be drawn, these oblique lines will diverge equally from the perpendicular D O, and will themselves be equal. But, the chords being equal, the arcs are equal; hence the point D is the pole of the small circle F N G; and for like reasons the point E is the other pole.

Cor. 1. Every arc D M, drawn from a point in the arc of a great circle A M B to its pole, is a quarter of the circumference, which, for the sake of brevity, is usually named a *quadrant*; and this quadrant at the same time makes a right angle with the arc A M. For (prop. 16, book vi.) the line D C being perpendicular to the plane A M C, every plane D M C passing through the line D C is perpendicular to the plane A M C; hence the angle of these planes, or the angle A M D, is a right angle.

Cor. 2. To find the pole of a given arc A M, draw the indefinite arc M D perpendicular to A M; take M D equal to a quadrant; the point D will be one of the poles of the arc A M D; or thus, at the two points A and M, draw the arcs A D and M D perpendicular to A M; their point of intersection D will be the pole required.

Cor. 3. Conversely, if the distance of the point D from each of the points A and M be equal to a quadrant, the point D will be the pole of the arc A M, and also the angles D A M, A M D will be right angles.

For, let C be the centre of the sphere; and draw the radii C A, C D, C M. Since the angles A C D, M C D are right, the line C D is perpendicular to the two straight lines C A, C M; it is therefore perpendicular to their plane; hence the point D is the pole of the arc A M; and consequently the angles D A M, A M D are right.

Scholium. The properties of these poles enable us to describe arcs of a circle on the surface of a sphere, with the same facility as on a plane surface. It is evident, for instance, that by turning the arc D F, or any other line extending to the same distance, round the point D, the extremity F will describe the small circle F N G; and by turning the quadrant D F A round the point D, its extremity A will describe the arc of the great circle A M.

If the arc A M were required to be produced, and nothing were given but the points A and M through which it was to pass, we should first have to determine the pole D, by the intersection of two arcs described from the points A and M as centres, with a distance equal to a quadrant; the pole D being found, we might describe the arc A M and its prolongation, from D as a centre, and with the same distance as before.

Lastly, if it be required from a given point P to let fall a perpendicular on the given arc A M; produce this arc to S, till the distance P S be equal to a quadrant; then from the pole S, and with the same distance, describe the arc P M, which will be the perpendicular required.

Proposition VII.—*Theorem.*

Every plane perpendicular to a radius at its extremity is a tangent to the sphere, fig. 240.

Let F A G be a plane perpendicular to the radius O A. Any point M in this plane being assumed, and O M, A M being joined, the angle O A M will be right, and hence the distance O M will be greater than O A. Hence the point M lies without the sphere; and as the case is similar with every other point in the plane F A G, this plane can have no point but A common to it with the surface of the sphere; it is therefore a tangent, def. 4. Fig. 240

Scholium. In the same way it may be shown, that two spheres have but one point in common, and therefore touch each other, when the distance between their centres is equal to the sum or the difference of

Geometry. their radii; in which case, the centres and the point of contact lie in the same straight line.

PROPOSITION VIII.—*Theorem.*

The angle B A C, *formed by* A B, A C *two arcs of great circles, is equal to the angle* F A G *formed by the tangents of these arcs at the point* A; *and is therefore measured by the arc* D E *described from the point* A *as a pole between the sides* A B, A C, *produced if necessary*, fig. 240 and 241.

For the tangent A F, drawn in the plane of the arc A B, is perpendicular to the radius A O; and the tangent A G, drawn in the plane of the arc A C, is perpendicular to the same radius A O. Hence (book vi. def. 4) the angle F A G is equal to the angle contained by the planes O A B, O A C; which is that of the arcs A B, A C, and is named B A C.

In like manner, if the arcs A D and A E are both quadrants, the lines O D, O E will be perpendicular to A O, and the angle D O E will still be equal to the angle of the planes A O D, A O E; hence the arc D E is the measure of the angle contained by these planes, or of the angle C A B.

Cor. The angles of spherical triangles may be compared together, by means of the arcs of great circles described from their vertices as poles and included between their sides; hence it is easy to make an angle of this kind equal to a given angle.

Scholium. Vertical angles, such as A C O and B C N
Fig. 241. (fig. 241) are equal; for either of them is still the angle formed by the two planes A C B, O C N.

It is farther evident, that, in the intersection of two arcs A C B, O C N, the two adjacent angles A C O, O C B taken together are equal to two right angles.

PROPOSITION IX.—*Theorem.*

The triangle A B C *being given, if from the points* A, B, C *as poles, the arcs* E F, F D, D E *be described to form the triangle* D E F; *then, conversely, the three points* D, E, F *will be the poles of the sides* B C, A C, A B, fig. 242.

Fig. 242 For, the point A being the pole of the arc E F, the distance A E is a quadrant; the point C being the pole of the arc D E, the distance C E is likewise a quadrant: hence the point E is removed the length of a quadrant from each of the points A and C; hence (prop. 6, cor. 3, book ix.) it is the pole of the arc A C. It might be shown, by the same method, that D is the pole of the arc B C, and F that of the arc A B.

Cor. Hence the triangle A B C may be described by means of D E F, as D E F may by means of A B C.

PROPOSITION X.—*Theorem.*

The same supposition being made as in the last theorem, each angle in the one of the triangles, A B C, D E F *will be measured by the semicircumference* minus *the side lying opposite to it in the other triangle,* fig. 242 and 243.

Produce the sides A B, A C, if necessary, till they meet E F in G and H. The point A being the pole of the arc G H, the angle A will be measured by that arc. But the arc E H is a quadrant, and likewise G F, E being the pole of A H, and F of A G; hence E H + G F is equal to a semicircumference. Now, Book IX. E H + G F is the same as E F + G H; hence the arc G H, which measures the angle A, is equal to a semicircumference *minus* the side E F. In like manner, the angle B will be measured by $\frac{1}{2}$ *circ.* — D F; the angle C by $\frac{1}{2}$ *circ.* — D E.

And this property must be reciprocal to the two triangles, since each of them is described in a similar manner by means of the other. Thus we shall find the angles D, E, F of the triangle D E F to be measured respectively by $\frac{1}{2}$ *circ.*—B C, $\frac{1}{2}$ *circ.* — A C, $\frac{1}{2}$ *circ.* — A B. Accordingly the angle D, for example, is measured by the arc M I; but M I + B C = M C + B I = $\frac{1}{2}$ *circ.*; hence the arc M I, the measure of D, is equal to $\frac{1}{2}$ *circ.*—B C; and so of all the rest.

Scholium. It must farther be observed, that besides the triangle D E F (fig. 243) three others might be Fig. 243. formed by the intersection of the three arcs D E, E F, D F. But the proposition immediately before us is applicable only to the central triangle, which is distinguished from the other three by the circumstance (see fig. 242) that the two angles A and D lie on the same side of B C, the two B and E on the same side of A C, and the two C and F on the same side of A B.

Various names have been given to the triangles A B C, D C F; but they are now more generally denominated *polar triangles.*

PROPOSITION XI.—*Lemma.*

The triangle A B C *being given, if from the pole* A, *with a distance* A C, *the arc* D E C *of a small circle be described; if from the pole* B, *with a distance* B C, *the arc* D F C *be described in like manner; and if from the point* D, *where the arcs* D E C, D F C *intersect each other,* A D, D B *two arcs of great circles be drawn; then will* A D B, *the triangle thus formed, have all its parts equal to those of the triangle* A C B, fig. 244.

For, by construction, the side A D = A C, D B = Fig. 244. B C, and A B is common; hence those two triangles have their sides equal, each to each, and it is to be shown that the angles opposite these equal sides are also equal.

If the centre of the sphere is supposed to be at O, a solid angle may be conceived as formed at O by the three plane angles A O B, A O C, B O C; likewise another solid angle may be conceived as formed by the three plane angles A O B, A O D, B O D. And because the sides of the triangle A B C are equal to those of the triangle A D B, the plane angles forming the one of these solid angles must be equal to the plane angles forming the other, each to each. But in this case the planes, in which the equal angles lie, are equally inclined to each other; hence all the angles of the spherical triangle D A B are respectively equal to those of the triangle C A B, namely, D A B = B A C, D B A = A B C, and A D B = A C B; therefore the sides and the angles of the triangle A D B are equal to the sides and the angles of the triangle A C B.

PROPOSITION XII.—*Theorem.*

Two triangles on the same sphere, or on equal spheres, are equal in all their parts, when they have each an equal angle included between equal sides, fig. 245.

Suppose the side A B = E F, the side A C = E G, Fig. 245. and the angle B A C = F E G; the triangle E F G may

 be placed on the triangle A B C, or on A B D symmetrical with ABC, just as two rectilineal triangles are placed upon each other, when they have an equal angle included between equal sides. Hence all the parts of the triangle E F G will be equal to all the parts of the triangle A B C; that is, besides the three parts equal by hypothesis, we shall have the side B C = FG, the angle A B C = EF G, and the angle A C B = E F G.

Proposition XIII.—*Theorem.*

Two triangles on the same sphere, or on equal spheres, are equal in all their parts, when two angles and the included side of the one are equal to two angles and the included side of the other.

For one of those triangles, or the triangle symmetrical with it, may be placed on the other, and be made to coincide with it, as is obvious.

Proposition XIV.—*Theorem.*

If two triangles on the same sphere, or on equal spheres, have all their sides respectively equal, their angles will likewise be all respectively equal, the equal angles lying opposite the equal sides, fig. 246.

Fig. 246. The truth is evident by prop. 11, book ix., where it was shown that, with three given sides A B, A C, B C, there can only be two triangles A C B, A B D, different as to the position of their parts, and equal as to the magnitude of those parts. Hence those two triangles, having all their sides respectively equal in both, must either be absolutely equal, or at least *symmetrically* so; in both of which cases, their corresponding angles must be equal, and lie opposite to equal sides.

Proposition XV.—*Theorem.*

In every isosceles spherical triangle, the angles opposite the equal sides are equal; and conversely, if two angles of a spherical triangle are equal, the triangle will be isosceles, fig. 247.

Fig. 247. *First.* Suppose the side A B = A C; we shall have the angle C = B. For, if the arc A D be drawn from the vertex A to the middle point D of the base, the two triangles A B D, A C D will have all the sides of the one respectively equal to the corresponding sides of the other, namely, A D common, B D = D C, and A B = A C; hence, by the lost proposition, their angles will be equal; therefore B = C.

Secondly. Suppose the angle B = C; we shall have the side A C = A B. For, if not, let A B be the greater of the two; take B O = A C, and join O C. The two sides B O, B C are equal to the two A C, B C; the angle O B C, contained by the first two, is equal to A B C contained by the second two. Hence (prop. 12, book ix.) the two triangles B O C, A C B have all their other parts equal; hence the angle O C B = A B C; but, by hypothesis, the angle A B C = A C B; hence we have O C B = A C B, which is absurd; therefore A B is not different from A C; that is, the sides A B, A C, opposite to the equal angles B and C, are equal.

Scholium. The same demonstration proves that the angle B A D = D A C, and the angle B D A = A D C. Hence the two last are right angles; consequently *the arc drawn from the vertex of an isosceles spherical triangle to the middle of the base, is at right angles to that base, and bisects the opposite angle.*

Proposition XVI.—*Theorem.*

In a spherical triangle ABC, *if the angle* A *is greater than the angle* B, *the side* BC *opposite to* A *will be greater than the side* A C *opposite to* B; *and conversely, if the side* B C *is greater than* A C, *the angle* A *will be greater than the angle* B, fig. 248.

First. Suppose the angle A $>$ B; make the angle B A D = B; then (prop. 15, book ix.) we shall have A D = D B; but A D + D C is greater than A C; hence, putting D B in place of A D, we shall have D B + D C or B C $>$ A C. Fig. 248.

Secondly. If we suppose B C $>$ A C, the angle B A C will be greater than A B C. For, if B A C were equal to A B C, we should have B C = A C; if B A C were less than A B C, we should then, as has just been shown, find B C $<$ A C. Both these conclusions are false; hence the angle B A C is greater than A B C.

Proposition XVII.—*Theorem.*

If the two sides A B, A C *of the spherical triangle* A B C, *are equal to the two sides* D E, D F *of the triangle* D E F, *drawn upon an equal sphere; and if at the same time the angle* A *is greater than the angle* D, *then will the third side* B C *of the first triangle be greater than the third side* E F *of the second,* fig. 249.

The demonstration is every way similar to that of prop. 10, book i. Fig. 249.

Proposition XVIII.—*Theorem.*

If two triangles on the same sphere, or on equal spheres, are mutually equiangular, they will also be mutually equilateral, fig. 250.

Let A and B be the two given triangles; P and Q their polar triangles. Since the angles are equal in the triangles A and B, the sides will be equal in the polar triangles P and Q, (prop. 10, book ix.;) but since the triangles P and Q are mutually equilateral, they must also (prop. 14, book ix.) be mutually equiangular; and, lastly, the angles being equal in the triangles P and Q, it follows (prop. 10, book ix.) that the sides are equal in their polar triangles A and B. Hence the mutually equiangular triangles A and B are at the same time mutually equilateral. Fig. 250.

This proposition may also be demonstrated, without the aid of polar triangles, as follows:

Let A B C, D E F be two triangles mutually equiangular, having A = D, B = E, C = F; we are to show that A B = D E, A C = D F, A C = D F, B C = E F.

On the prolongations of the sides A B, A C, take AG = D E, and A H = D F; join G H; and produce the arcs B C, G H, till they meet in I and K.

The two sides A G, A H are equal, by construction, to the two D F, D E; the included angle G A H = B A C = E D F; hence (prop. 12, book ix.) the triangles A G H, D E F, are equal in all their parts; hence the angle A G H = D E F = A B C, and the angle A G H = D F E = A C B.

In the triangles I B G, K B G, the side B G is common; the angle I B G = G B K; and, since I G B

Geometry. + B G K is equal to two right angles, and likewise G B K + I B G, it follows that B G K = I B G. Hence (prop. 13, book ix.) the triangles I B G, G B K are equal; hence I G = B K, and I B = G K.

In like manner, the angle A H G being equal to A C B, we can show that the triangles I C H, H C K have two angles and the interjacent side in each equal; they are therefore themselves equal; and I H = C K, and H K = I C.

Now if the equals C K, I H be taken away from the equals B K, I G, the remainders B C, G H will be equal. Besides, the angle B C A = A H G, and the angle A B C = A G H. Hence the triangles A B C, A H G have two angles and the interjacent side in each equal; and are therefore themselves equal. But the triangle D E F is equal in all its parts to A H G; hence it is also equal to the triangle A B C, and we have A B = D E, A C = D F, B C = E F; therefore if two spherical triangles are mutually equiangular, the sides opposite their equal angles will also be equal.

Scholium. This proposition is not applicable to rectilineal triangles; in which, equality among the angles indicates only proportionality among the sides. Nor is it difficult to account for the difference observable, in this respect, between spherical and rectilineal triangles. In the proposition now before us, as well as in the four last, which treat of the comparison of triangles, it is expressly required that the arcs be traced on the same sphere, or on equal spheres. Now similar arcs are to each other as their radii; hence, on equal spheres, two triangles cannot be similar without being equal. Therefore it is not strange that equality among the angles should produce equality among the sides.

The case would be different, if the triangles were drawn upon unequal spheres; there, the angles being equal, the triangles would be similar, and the homologous sides would be to each other as the radii of their spheres.

Proposition XIX.—*Theorem.*

The sum of all the angles in any spherical triangle is less than six right angles, and greater than two, fig. 251.

Fig. 251. For, in the first place, every angle of a spherical triangle is less than two right angles, (see the following scholium;) hence the sum of all the three is less than six right angles.

Secondly, the measure of each angle in a spherical triangle (prop. 10, book ix.) is equal to the semicircumference *minus* the corresponding side of the polar triangle; hence the sum of all the three is measured by three semicircumferences *minus* the sum of all the sides of the polar triangle. Now (prop. 4, book ix.) this latter sum is less than a circumference; therefore, taking it away from three semicircumferences, the remainder will be greater than one semicircumference, which is the measure of two right angles; hence, in the second place, the sum of all the angles in a spherical triangle is greater than two right angles.

Cor. 1. The sum of all the angles in a spherical triangle is not constant, like that of all the angles in a rectilineal triangle; it varies between two right angles and six, without ever arriving at either of these limits. Two given angles therefore do not serve to determine the third.

Cor. 2. A spherical triangle may have two or even three angles right, two or three obtuse. Book IX.

If the triangle A B C have two right angles B and C, the vertex A will (prop. 6, book ix.) be the pole of the base B C; and the sides A B, A C will be quadrants.

If the angle A is also right, the triangle A B C will have all its angles right, and its sides quadrants. The tri-rectangular triangle is contained eight times in the surface of the sphere; as is evident by fig. 252, supposing the arc M N to be a quadrant.

Scholium. In all the preceding observations, we have supposed, in conformity with definition 6, that our spherical triangles have always each of their sides less than a semicircumference; from which it follows that any one of their angles is always less than two right angles. For (see fig. 237) if the side A B is less than a semicircumference, and A C is so likewise, both those arcs will require to be produced before they can meet in D. Now the two angles A B C, C B D, taken together, are equal to two right angles; hence the angle A B C itself is less than two right angles.

We may observe, however, that some spherical triangles do exist, in which certain of the sides are greater than a semicircumference, and certain of the angles greater than two right angles. Thus, if the side A C is produced, so as to form a whole circumference A C E, the part which remains, after subtracting the triangle A B C from the hemisphere, is a new triangle also designated by A B C, and having A B, B C, A E D C for its sides. Here, it is plain, the side A E D C is greater than the semicircumference A E D; and, at the same time, the angle B opposite to it exceeds two right angles, by the quantity C B D.

The triangles whose sides and angles are so large have been excluded from our definition; but the only reason was, that the solution of them, or the determination of their parts, is always reducible to the solution of such triangles as are comprehended by the definition. Indeed, it is evident enough, that if the sides and angles of the triangle A B C are known, it will be easy to discover the angles and sides of the triangle which bears the same name, and is the difference between a hemisphere and the former triangle.

Proposition XX.—*Theorem.*

The lune A M B N A is to the surface of the sphere, as M A N, the angle of this lune, is to four right angles, or as the arc M N, which measures that angle, is to the circumference, fig. 252.

Suppose, in the first place, the arc M N to be to the circumference M N P Q as some one rational number is to another, as 5 to 48, for example. The circumference M N P Q being divided into 48 equal parts, M N will contain 5 of them; and if the pole A were joined with the several points of division, by as many quadrants, we should in the hemisphere A M N P Q have 48 triangles, all equal, because having all their parts equal. Hence the whole sphere must contain 96 of those partial triangles, the lune A M B N A will contain 10 of them; hence the lune is to the sphere as 10 is to 96, or as 5 to 48, in other words, as the arc M N is to the circumference. Fig. 252

If the arc M N is not commensurable with the circumference, we may still show, by the mode of

reasoning exemplified in book ii., that in this case also, the lune is to the sphere as M N is to the circumference.

Cor. 1. Two lunes are to each other as their respective angles.

Cor. 2. It was shown (prop. 19, book ix.) that the whole surface of the sphere is equal to eight tri-rectangular triangles; hence, if the area of one such triangle is taken for unity, the surface of the sphere will be represented by 8. This granted, the surface of the lune, whose angle is A, will be expressed by 2 A (the angle A being always estimated from the right angle assumed as unity) since 2 A : 8 : : A : 4. Thus we have here two different unities; one for angles, being the right angle, the other for surfaces, being the tri-rectangular spherical triangle, or the triangle whose angles are all right, and whose sides are quadrants.

Scholium. The spherical ungula, bounded by the planes A M B, A N B, is to the whole solid sphere as the angle A is to four right angles. For, the lunes being equal, the spherical ungulas will also be equal; hence two spherical ungulas are to each other, as the angles formed by the planes which bound them.

PROPOSITION XXI.—*Theorem.*

Two symmetrical spherical triangles are equal in surface, fig. 253.

Fig. 253. Let A B C, D E F be two symmetrical triangles, that is to say, two triangles having their sides A B = D E, A C = D F, C B = E F, and yet incapable of coinciding with each other; we are to show that the surface A B C is equal to the surface D E F.

Let P be the pole of the little circle passing through the three points A, B, C;* from this point draw (prop. 6, book ix.) the equal arcs P A, P B, P C; at the point F, make the angle D F Q = A C P, the arc F Q = C P; and join D Q, E Q.

The sides D F, F Q are equal to the sides A C, C P; the angle D F Q = A C P; hence (prop. 12, book ix.) the two triangles D F Q, A C P are equal in all their parts; hence the side D Q = A P, and the angle D Q F = A P C.

In the proposed triangles D F E, A B C, the angles D F E, A C B, opposite to the equal sides D E, A B, being equal, (prop. 11, book ix.) if the angles D F Q, A C P, which are equal by construction, be taken away from them, there will remain the angle Q F E, equal to P C B. Also the sides Q F, F E are equal to the sides P C, C B; hence the two triangles F Q E, C P B are equal in all their parts; hence the side Q E = P B, and the angle F Q E = C P B.

Now, observing that the triangles D F Q, A C P, which have their sides respectively equal, are at the same time isosceles, we shall see them to be capable of mutual adaptation, when applied to each other; for, having placed P A on its equal Q F, the side P C will fall on its equal Q D, and thus the two triangles will exactly coincide: hence they are equal, and the surface D Q F = A P C. For a like reason, the surface F Q E = C P B, and the surface D Q E = A P B; hence we have D Q F + F Q E − D Q E = A P C + C P B − A P B, or D F E = A B C; therefore the two symmetrical triangles A B C, D E F are equal in surface.

Scholium. The poles P and Q might lie within the triangles A L C, D E F; in which case it would be requisite to add the three triangles D Q F, F Q E, D Q E together, in order to make up the triangle D E F; and in like manner, to add the three triangles A P C, C P B, A P B together, in order to make up the triangle A B C; in all other respects, the demonstration and the result would still be the same.

PROPOSITION XXII.—*Theorem.*

If two great circles A O B, C O D *intersect each other anyhow in the hemisphere* A O C B D, *the sum of the opposite triangles* A O C, B O D *will be equal to the lune whose angle is* B O D, fig. 241.

For, producing the arcs O B, O D in the other hemisphere, till they meet in N, the arc O B N will be a semicircumference, and A O B one also; and taking O B from both, we shall have B N = A O. For a like reason, we have D N = C O, and B D = A C. Hence the two triangles A O C, B D N have their three sides respectively equal; besides, they are so placed as to be symmetrical; hence (prop. 21, book ix.) they are equal in surface, and the sum of the triangles A O C, B O D is equal to the lune O B N D O whose angle is B O D. Fig. 241.

Scholium. It is likewise evident that the two spherical pyramids, which have the triangles A O C, B O D for bases, are together equal to the spherical ungula whose angle is B O D.

PROPOSITION XXIII.—*Theorem.*

The surface of any spherical triangle is measured by the excess of the sum of its three angles above two right angles, fig. 254.

Let A B C be the proposed triangle: produce its sides till they meet the great circle D E F G drawn anywhere without the triangle. By the last theorem, the two triangles A D E, A G H are together equal to the lune whose angle is A, and which is measured (prop. 20, book ix.) by 2 A. Hence we have A D E + A G H = 2 A; and for a like reason, B G F + B I D = 2 B, and C I H + C F E = 2 C. But the sum of those six triangles exceeds the hemisphere by twice the triangle A B C, and the hemisphere is represented by 4; therefore twice the triangle A B C is equal to 2 A + 2 B + 2 C − 4; and consequently once A B C = A + B + C − 2; hence every spherical triangle is measured by the sum of all its angles *minus* two right angles. Fig. 254.

Cor. 1. However many right angles there be contained in this measure, just so many tri-rectangular triangles, or eighths of the sphere, which (prop. 20, book ix.) are the unit of surface, will the proposed triangle contain. If the angles, for example, are each equal to $\frac{4}{3}$ of a right angle, the three angles will amount to 4 right angles, and the proposed triangle will be represented by 4 − 2 or 2; therefore it will be equal to two tri-rectangular triangles, or to the fourth part of the whole surface of the sphere.

* The circle which passes through the three points A, B, C, or which circumscribes the triangle A B C, can only be a little circle of the sphere; for if it were a great circle, the three sides A B, B C, A C would lie in one plane, and the triangle A B C would be reduced to one of its sides.

Cor. 2. The spherical triangles A B C is equal to the lune whose angle is $\frac{A+B+C}{2}-1$; likewise the spherical pyramid, which has A B C for its base, is equal to the spherical ungula whose angle is $\frac{A+B+C}{2}-1$.

Scholium. While the spherical triangle A B C is compared with the tri-rectangular triangle, the spherical pyramid, which has A B C for its base, is compared with the tri-rectangular pyramid, and a similar proportion is found to subsist between them. The solid angle at the vertex of the pyramid is, in like manner, compared with the solid angle at the vertex of the tri-rectangular pyramid. These comparisons are founded on the coincidence of the corresponding parts. If the bases of the pyramids coincide, the pyramids themselves will evidently coincide, and likewise the solid angles at their vertices. From this, the following consequences are deduced.

First. Two triangular spherical pyramids are to each other as their bases; and since a polygonal pyramid may always be divided into a certain number of triangular ones, it follows that any two spherical pyramids are to each other, as the polygons which form their bases.

Second. The solid angles at the vertices of those pyramids are also as their bases; hence, for comparing any two solid angles, we have merely to place their vertices at the centres of two equal spheres, and the solid angles will be to each other as the spherical polygons intercepted between their planes or faces; see scholium 2, prop. 21, book vi.

The vertical angle of the tri-rectangular pyramid is formed by three planes at right angles to each other; this angle, which may be called a *right solid angle*, will serve as a very natural unit of measure for all other solid angles. And if so, the same number that exhibits the area of a spherical polygon, will exhibit the measure of the corresponding solid angle. If the area of the polygon is $\frac{3}{4}$, for example, in other words, if the polygon is $\frac{3}{4}$ of the tri-rectangular polygon, then the corresponding solid angle will also be $\frac{3}{4}$ of the right solid angle.

PROPOSITION XXIV.—*Theorem.*

The surface of a spherical polygon is measured by the sum of all its angles, minus the product of two right angles by the number of sides in the polygon minus two, fig. 255.

From one of the vertices A, let diagonals A C, A D be drawn to all the other vertices; the polygon A B C D E will be divided into as many triangles *minus* two as it has sides. But the surface of each triangle is measured by the sum of all its angles *minus* two right angles; and the sum of the angles in all the triangles is evidently the same as that of all the angles in the polygon; hence the surface of the polygon is equal to the sum of all its angles diminished by twice as many right angles as it has sides *minus* two. Fig. 255.

Scholium. Let s be the sum of all the angles in a spherical polygon, n the number of its sides; the right angle being taken for unity, the surface of the polygon will be measured by $s-2(n-2,)$ or $s-2n+4$.

ARITHMETIC.

History of the Science.

Definition.

(1.) Arithmetic may be defined to be the science of numbers and their notation, and of the different operations to which they are subject.

With the exception of the theory of arithmetical notation, we shall not include under the head of Arithmetic any portion of what is commonly called the Theory of Numbers, the complete discussion of which would require a very extensive knowledge of Algebra, and which will be afterwards considered in a separate treatise. We shall confine ourselves in the following treatise to the consideration of the common operations of Arithmetic, and to those common rules for the solution of numerical questions, which are of such frequent occurrence in the ordinary business of life, and whose principles may be established and understood without the aid of algebraical investigations.

Idea of number, how acquired?

(2.) The idea of number is one of those which are first presented to the mind, and which indeed may be considered as nearly coexistent with the exercise of our natural faculties; and the mode in which it is acquired, considered as a metaphysical question, forms a natural introduction to an historical notice of the different methods of numeration, which have been adopted by different nations at different periods of the world.

If objects of various kinds be placed before a child, he will be struck with the more marked peculiarities by which they are severally distinguished; but the idea of their multitude will probably escape his observation, or, if in any way excited, will leave no distinct impression on the mind: the case would be somewhat different, if different objects of the same kind were placed before him, as under such circumstances the very first idea which would succeed to his perception of their resemblance, would be that of their multitude. But the passage from the vague idea of multitude to the more definite one of number, is one of great difficulty in the infant state of the reflex operations of the mind. It requires an analysis of the individual units of which a number is composed, which can only be effected by the comparison of different numbers with each other; and the process of the mind by which such comparisons are made is slow and difficult, unless the numbers are small, and our attention powerfully directed to them by the excitement of our appetites, or other circumstances: thus place before a child different sets of toys, or fruits, or other objects naturally desirable, in the selection of which a choice is left to him, and he will rapidly acquire the habit of comparing them with each other; and, as the result of such a comparison, and of the examination of the individuals of which each set is composed, he will gradually acquire the idea of number.

Abstraction is the creature of language, and without the aid of language he will never separate the idea of any number from the qualities of the objects with which it is associated. He will have a distinct idea of four cows, as distinguished from five cows; but it by no means follows, that the idea of the number four, as connected with four cows, will be perfectly identical in his mind with the idea of the number four as connected with four horses; as they would in both cases be blended with his ideas of the individual qualities of the objects themselves: but if his idea of the number four be *registered* in the memory by a specific *word*, independent of the qualities of the objects with which it was in the first instance associated, he will become accustomed, after a more enlarged experience, to pronounce the word without reference to such associations, though they must necessarily spring up in one form or other in the mind, upon a farther analysis of the idea, of which the word is the general symbol.*

* We are thus lead to the distinction of numbers into abstract and concrete, though the abstraction exist merely in the word by which any number is designated, or in the equivalent symbol by which it is represented in different arithmetical systems. In Arithmetic we consider both kinds of numbers, though the operations are in all cases the same as if the numbers were perfectly abstract; the association of qualities being merely of use in directing us to the particular operations or reductions to be performed, and in assisting us in the proper interpretation of the result: thus in the statement of the Rule of Three question, " If 1lb. 6oz. of tea cost 8s. 4d. what is the price of 3lb. 8oz." We say,

	lb. oz.		lb. oz.		s. d.		
	1 6	:	3 8	::	8 4	:	x.

And after reducing the two first terms to units of one denomination, and the third term to units of another denomination, we have

oz.		oz.		d.		
20	:	50	::	100	:	x.

The last term is 250, and the same number results, whether we suppose the terms of the proportion to be abstract or concrete; but there is an obvious advantage in considering them as concrete, as we are thus guided not merely to the previous reduction of the terms of the ratios, but likewise to the interpretation and reduction of the result.

Most writers on Arithmetic would state this question in the following manner:

lb. oz.		s. d		lb. oz.		
1 6	:	8 4	::	3 8	:	x.

In this statement, however, there is a manifest violation of propriety, as the terms of the ratios are not homogeneous; and the practice is not justified by any corresponding advantage. It is obvious, however, from the preceding observations, as well as from other considerations, that the result will be the same as is obtained from the correct proportion.

Some authors have defined *abstract* or *discrete* numbers to be those which have no denomination annexed to them; considering all others as *contract* or *concrete*. Upon this definition, a difficulty arose about the class to which fractional numbers were to be referred; the units of the numerator being limited in value by the denominator, and consequently being in this respect different from abstract whole numbers. The solution of the difficulty is to be found in the meaning of the word *denomination*, which in our definition would be confined to designate a quality of the subject which the unit is supposed to denote. This question is very well discussed in the *Whetstone of Witte*, the first work on Algebra published in England, by Robert Recorde, in 1557.

This latter distinction of *abstract* and *concrete* would nearly answer to their meaning in ordinary language, when applied to any general term and its corresponding adjective: thus, in the well known epigram,

Mentitur qui te vitiosum, Zoile, dicit,
Non vitiosus homo es, Zoile, sed vitium.

Arithmetic.

Names for numbers not perfectly arbitrary.

(3.) We might suppose this process for the formation of abstract numbers, to be completely effected by attaching names to the series of natural numbers, beginning from unity; but if such names were perfectly arbitrary and independent of each other, our progress in numeration would be extremely limited, as the memory would be overwhelmed with a multitude of disconnected words; and the performance of the most simple operations of addition, subtraction, multiplication, or division, would require an insight into the constitution of numbers, to which the mind, particularly in the infancy of society, would be altogether unequal: under such circumstances, we might readily credit the narrations of travellers who have limited the powers of numeration of some savage tribes to five, or to ten; but it will be found, upon an examination of the numerical words of different languages, that they have been formed upon regular principles, subordinate to those methods of numeration which have been suggested by nature herself, and which we may suppose to have been more or less practised amongst all primitive people; for in what other manner can we account for the very general adoption of the decimal system of notation, and what other origin can we assign to it than the very natural practice of numbering by the fingers on the two hands.*

Theoretical origin of the decimal scale of notation.

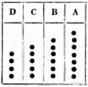
(4.) Assuming such an hypothesis as true, it would not be difficult to give a probable theory of the formation of the decimal scale of numeration, and of the adaptation of language to it; for suppose a number of counters, or pebbles, or objects of any other kind were placed before a person accustomed to count upon his fingers; in making his tale, he would first place his fingers in succession upon ten counters; and let us suppose him to reject nine of them, and to put the tenth apart as a *register* of the completion of one operation. Again, let him repeat the same operation, rejecting nine counters each time, and preserving the tenth, until the number of counters remaining is less than ten: let them be preserved by themselves in a place, which, for greater distinctness, we will call A, whilst the place for the counters which were separated from the original heap, to mark the completion of each operation, is called B. We may now suppose the same process to be repeated upon the counters in the place B, rejecting nine and preserving the tenth in a place C, until the number of counters remaining in B is less than ten: if the number of counters in C exceed ten, the same process may be repeated upon them, and every tenth counter may be placed in D; and so on, until the number of counters remaining in the last place is less than ten. We shall thus get a series of sets of counters A, B, C, D, &c. where each counter in B corresponds to ten counters in A; every counter in C to ten in B, and so on; every counter in a superior place corresponding to ten in a place next inferior: in this arrangement the counters acquire a representative value dependent upon their position, and the number itself may be considered as expressed by a comparatively small number of counters, particularly when the number is large.

History.

By a little variation of the process we should be enabled, by means of nine counters only corresponding to each place, to effect a similar resolution of any number whatever of objects, and consequently to express it: it would be merely necessary, whenever ten counters were required for any one place, to remove the nine which were previously there, and to place one counter in the next superior place; we shall thus possess a natural *abacus*, representing very distinctly the principles and formation of the decimal scale of numeration.

Nomenclature of numbers in the decimal scale.

(5.) The discovery of this mode of breaking up numbers into classes, the units in each class increasing in a decuple proportion, would lead, very naturally, to the invention of a nomenclature for numbers thus resolved, which is more simple and equally comprehensive. By giving names to the first nine natural numbers, or *digits*,* and also to the units of each class in the ascending series by ten, we shall be enabled, by combining the names of the digits with those of the units possessing local or representative value, to express in words any number whatsoever: thus the number resolved by means of counters in the following manner,



would be expressed, (supposing seven, six, five, and four, denote the numbers of the counters, in A, B, C, D, and ten, hundred, and thousand, the value of each unit in B, C, and D,) by seven, six tens, five hundreds, four thousands; or, inverting the order, and making the slight changes required by the existing form of the language, by four thousand, five hundred, and sixty-seven.

It is quite unnecessary for us to exhibit this transition from the expression of a number by artificial methods, to its expression in words either for other numbers or for other languages than our own; the one just given being abundantly sufficient for the illustration of our hypothesis.

The advantages of this resolution of numbers are not confined to the expression of large numbers by

Vitium is a general abstract term for every species of vice; but *vitiosus*, though equally general in its application, is concrete, as designating a quality of a subject.

* There is a curious passage in Ovid on the origin of the decimal scale of numeration. Speaking of the ancient Roman year, he says,

Annus erat decimum cum Luna repleverat orbem,
Hic numerus magno tunc in honore fuit;
Seu quia tot digiti, per quos numerare solemus,
Seu quia bis quino fœmina mense parit;
Seu quod ad usque decem numero crescente venitur
Principium spatiis sumitur inde novis.

Fasti, lib. iii. 124.

* The earlier writers on Arithmetic distinguished numbers into digital, articulate, and compound; the first denoting the first nine natural numbers, which were counted upon the *digiti*, or fingers; the second multiples of ten, of a hundred, &c. which might be counted upon the *articuli*, or joints of the fingers; the third, all numbers which arise from adding digital and articulate numbers together. The Arabs denoted the second class of numbers by a word which means *knots*.

Arithmetic. few words which are easily remembered; for we thus become familiar with the superior units, such as ten, a hundred, a thousand, as well from frequent repetition as from our knowledge of their relation to each other and to unity; and are thus enabled to form clear and distinct conceptions of large numbers, whose composition we discover, in the words by which they are expressed, or in the symbols by which they are represented.

Other natural scales of numeration.

(7.) But the decimal scale of numeration is not the only one which may be properly characterised as a natural scale. In numbering with the fingers we might very naturally pause at the completion of the fingers on one hand; and registering this result by a counter, or by any other means, we might proceed over the fingers of the same hand again, or with the fingers of the second hand, and register the result by another counter, or replace the former by a new counter, which should become the representative of ten. If the first process were adopted, we should be led to the formation of a scale of numeration which is strictly quinary: by pursuing the second process, we should end in the formation of the denary scale, with the quinary scale subordinate to it; and in adopting language to such a practical mode of numeration, we should give independent names to the first five digits, and subsequently express the digits between five and ten by combining the name of five, considered as a superior unit with the names of the first four digits: in the first system the name for ten would be expressed by a word equivalent to twice five; in the second, it would be expressed by a simple and independent word.

Again, the scale of numeration by twenties has its foundation in nature, equally with the quinary and denary scales. In a rude state of society, before the discovery of other methods of numeration, men might avail themselves for this purpose, not merely of the fingers on the hands but likewise of the toes of the naked feet; such a practice would naturally lead to the formation of a *vicenary* scale of numeration, to which the denary, or the denary with the quinary, or the quinary alone, might be subordinate: in the first case we must have single and independent names for the first nine digits, for ten, and for twenty; in the second for the first four digits, for five, for ten, and for twenty; in the last, for the first four digits, for five, and for twenty. Such are the principles of a philosophical nomenclature adapted to suit these different scales of numeration, subject of course to such variations as may be required by the genius of the language to which they are applied.

Of other systems of numeration the binary might be considered as natural, from the use of the two hands in separating objects into pairs, and from the prevalence of binary combinations in the members of the human body;* but the scale of its superior units increases too slowly, to embrace within moderate limits the numbers which are required for the ordinary wants of life, even in the infancy of society. It requires twelve orders of superior units in the binary scale for numbers, which are expressible in the quinary scale by five orders, in the denary by three, and in the vicenary by two; the adoption of this scale therefore would require a more complete knowledge of the classification of numbers, upon which numerical systems depend, than we could expect to find at the period of society when such systems are formed. History.

There are no members of the human body, and no use of those members, which could naturally lead to the adoption of any other scale of numeration than those above mentioned; the senary scale possesses some advantages over the quinary, and the duodenary over the denary, but the perception of those advantages belongs to an advanced state of arithmetical knowledge, and they form, therefore, no argument for the adoption of such scales at the period of society to which our argument refers.

Methods of numeration have preceded the formation of numerical language.

(8.) As the necessity of numeration is one of the earliest and most urgent of those wants, which are not essential to the support and protection of life, we might naturally expect that the discovery of expedients for that purpose should precede the epoch of civilisation, and the full developement and fixing of language. That such has been the case, we shall find very fully and clearly established, by an examination of the numerical words of different languages; for without any exception, which can be well authenticated, they have been formed upon regular principles, having reference to some one of those three systems of numeration, which we have characterised as natural; the quinary scale, whenever any traces of it appear, being generally subordinate to the denary, and in some cases both the quinary and denary scales being subordinate to the vicenary. In some cases also we shall find from an examination of primitive numerical words, conveying traces of obsolete methods of numeration, that the quinary, and even the vicenary scales have been superseded altogether by the denary, either from a sense of its superior advantages in the progress of society and civilisation, or introduced from other nations through commercial intercourse, colonization, or conquest.

Besides the general proposition contained in the preceding statement, *that the natural scales of numeration alone have ever met with general adoption*; there is another proposition which is in some degree a consequence of the former, but which an examination of the structure of numerical language will in many cases more completely establish; which is, *that amongst all nations practical methods of numeration have preceded the formation of numerical language.*

Numerical words are not always primitive.

(9.) It is in the language of people far removed from civilized life, that the connection existing between practical methods of numeration and numerical words will generally be most clearly exhibited; for in such languages words are more immediately the transcripts of things, and are less diverted from their primitive meaning and application, than in those which have been expanded by the culture necessary to fit them for the multiplied wants of civilized life, and to enable them to express the infinite variety of ideas introduced by an enlarged exercise of the reflex operations of the mind; it might be contended, indeed, that numerical words, being of early use, and therefore primitive, are likely to remain unaltered amidst the fluctuations to which all languages are subject, before they become fixed and permanent by literature; but the changes of language depend less upon internal than upon external causes, being less affected by the progress of the arts of life, than by the introduction of new words, or even of portions of new languages, through intercourse with other nations,

* Monboddo, *on the Origin and Progress of Language*, p. 551.

Arithmetic. colonization, or conquest: different languages become in this manner incorporated with each other, and primitive languages either altogether disappear, or lose much of their original character. In this union of the languages of different people with each other, possessing different numerical systems, as well as different numerical words, it is natural to suppose that the most perfect system of numeration, or the best constructed numerical language, should be adopted in whole or in part; and in those cases where a change of grammatical structure is a consequence of this union, the numerals, particularly such as are compound, may be different from those of either of the component languages, and may become more or less expressive of practical methods of numeration, whether primitive or not; it is the combination of all these circumstances, that renders it extremely difficult in such languages to trace the existence of primitive methods of numeration in numerical words, and to show the connection which subsists between them.

Use of numerals for tracing the affinities of languages.

(10.) Extensive collections have been made of the numerals of different people, for the purpose of ascertaining the affinities of languages, and perhaps few classes of words could be selected which are better calculated to answer this object; but the preceding, as well as other considerations, show that their authority is not in all cases to be depended upon. The more philosophical of modern Philologists, indeed, have ceased to regard affinity of roots as a decisive proof of the affinity of languages; it may arise from the mere mixture of languages, or from the intercourse of the people by whom they are spoken, but it by no means demonstrates them to be of common origin, unless accompanied also by a corresponding affinity of grammatical structure. Thus the numerals of nearly all the languages of Europe, and of many of those of Asia, are nearly the same, or very slightly different from each other; and some authors have attempted from this circumstance, supported by the analogy of other roots, to refer all those languages to a common origin;* the essential diversity, however, of their grammatical structure, would show such a classification to be much too comprehensive; and even after referring them to three great classes, the Indo Pelasgic,† including the Sanscrit, Greek, and Latin, the Persian and German, with their immediate derivatives; the Slavonic, including the Armenian, Russian, Polish, and Bohemian; and the Celtic, including the Welsh, the Erse, the Gaelic, the Armorican, and the Basque of Biscay; we shall still find some reasons for thinking that we have associated together, and particularly in the last of these classes, some languages which are essentially distinguished from each other.

It has long been a favourite theory of Philologists to trace up all existing languages to a small number of others, which they consider as primitive; but the reasonings by which such theories have commonly been supported, are founded upon an assumption of an order in the occurrence of facts which is directly contrary to experience; it being the constant tendency of civilisation, and the certain influence of extensive empires to diminish, and not to increase the number of languages; the numerous languages of Greece and Italy, of the former existence of many of which we have evidence, have been reduced to mere dialects of two; the only trace of any of the languages which we know from the authority of Strabo existed formerly in Spain, is to be found in the mountains of Biscay: it is only at the base of the Pyrenees, and in the remote parts of Brittany, where the influence and authority of the Romans were little felt or known, that we can discover any remains of the languages of the numerous tribes of ancient Gaul: the mountains of Wales and of Scotland have alone prevented the exclusive use of a common language in Great Britain: the Arabic and its derivatives have nearly superseded, or greatly affected all other languages, where the authority of the Koran has been long acknowledged: the commercial activity and enterprising character of the Malays, have propagated their language, in whole or in part, throughout the islands of the Indian Archipelago and the South Sea: in North America the numerous tribes who were driven from their settlements by European colonists, have disappeared with their languages; and the same effect, in perhaps a still greater degree, has attended the progress of the Spanish dominion in the South.

History.

The immense number of languages, radically different from each other, which are spoken by the tribes of barbarous countries which have never been subject to a common empire, establishes the same proposition in a still more striking manner. The Jesuit Missionary Dobrizhoffer* says, that there are upwards of thirty known languages spoken in Paraguay alone. Father Lasuen† observed no fewer than seventeen languages in an extent of only 500 miles on the coast of California. More than 150 other languages have been observed in other parts of that vast continent, and farther researches would probably greatly increase that number. Mr. Bowdich‡ has given the numerals of thirty-one languages, most of which are spoken within a district of small extent upon the western coast of Africa; and Mr. Salt§ has given those of fifteen others on the eastern coast, between Mozambique and Abyssinia. That continent, indeed, may be almost said to swarm with languages, so numerous do they appear in almost every part of the small portion of it, which has hitherto been subject to examination.

In judging of the proper uses of numerals for ascertaining the affinity of languages, it is particularly necessary to consider whether they exist under their original and unaltered form, or have been mixed up with others without a more intimate union, or have become mere dialects of a predominant language. In the languages of barbarous and primitive people, possessing a general affinity of grammatical structure, as in those of the tribes of South America,‖ they will generally form a just measure of the affinities of the languages themselves; in the absence of such a common structure, and in cases where languages from different causes are greatly altered from their primitive form, the affinity of numerals may serve as a monument of the communication of the people by whom they are used, and even of the present intermixture of their

* Parsons, *Remains of Japhet*; Vallancey, *Collectanea de Rebus Hibernicis*, vol. iii. No. 11.

† Frederick Schlegel, *Ueber die Sprache und Weisheit der Indier*; Vater, *Mithridates*.

* *History of the Abipones*.

† Humboldt, *Essai politique sur le royaume de Nouvelle Espagne*.

‡ *Mission to Ashantee*, Appendix.

§ *Travels in Abyssinia*, Appendix.

‖ Humboldt, *Personal Narrative*, vol. iii. p. 244. English Trans.

Arithmetic. languages, but furnishes no proof of their primitive affinity with each other.

There are some circumstances, particularly in the numerals of African languages, which are extremely difficult to explain. In the languages of Bornon and Cashna,* two neighbouring African tribes, *okoo* is the name for *five* in the first, and for *three* in the second, all the other numerals being different from each other; and Mr. Bowdich† has remarked other instances of a similar interchange of the names for *four* and *five* in the numerals of tribes, geographically remote from each other, in which all the rest are different; again, the name for *four* in the Inta language is the same as that for the same number in the language of Empoonga, at the distance of 1000 miles; and the name for five in the first of these languages, is the same as that for five in the language of Kamsallahoo. Barton‡ has given from the records of the first settlers in North America, the numerals of the Nanticocks, an extinct tribe, who inhabited the south bank of the Chesapeak, which are nearly identical with those of the Mandingoes of Africa, as will be immediately seen upon examination of them.

Nanticocks.	Mandingoes.§
1. Killi.	1. Killim.
2. Filli.	2. Foola.
3. Sabo.	3. Sabba.
4. Nano.	4. Nani.
5. Turo.	5. Loolo.
6. Woro.	6. Woro.
7. Wollango.	7. Oronglo.
8. Secki.	8. Sec.
9. Collango.	9. Conanto.
10. Ta.	10. Tang.

The resemblance of these numerals is apparently too remarkable to be accidental, yet the people by whom they were used, belonged to races essentially different, and between whom it is difficult to imagine that any intercourse could have taken place; the examination of their languages, if it were now possible, might perhaps throw some light upon this very curious and very embarrassing fact.

There are perhaps some cases, where an affinity exists between languages, which is in no respect borne out by the affinity of their numerals. Voyagers and others have remarked the resemblance between the languages of Nootka Sound and its neighbourhood on the north-west coast of America and the Aztec of Ancient Mexico; and Humboldt,‖ though he supposes that the resemblance is more apparent than real, arising in a great measure from the frequent use of the same very peculiar combination of consonants, yet admits the existence of some affinity between them; it will be found, however, that they have not one numeral in common, or between which the most distant resemblance can be traced. History.

In the classification of the languages of Europe, the Lapponian, Finnish, Esthonian, and Hungarian, have been usually associated together, as belonging to the same family.* The following are the numerals in the first and last of these languages:

Lapponian.†	Hungarian.‡
1. Anft.	1. Egi.
2. Gouft.	2. Ketto.
3. Golm.	3. Harum.
4. Nielja.	4. Negy.
5. Vit.	5. Et.
6. Gut.	6. Hat.
7. Zhieczhia.	7. Hêt.
8. Kantze.	8. Nyoltz.
9. Aotze.	9. Kilentz.
10. Lange.	10. Tiz.

If the affinity of these languages, which so many authors have attempted to prove, really exist, it is quite clear that little or no trace of it is discoverable in a comparison of their numerals.

The extraordinary coincidences as well as diversities of numerals, which are given above, show how dangerous it is to form any general conclusions respecting the relations of languages from the comparison of a small number of their roots, however apparently well chosen for the purpose.§

Scales and methods of numeration. (11.) We shall now quit the philological discussion of numerals, and proceed to the consideration of them as records of systems of numeration; in this inquiry we shall not pretend to embrace those systems in all known languages, which would lead into very extensive details, but shall confine ourselves to such as may be requisite to establish our two general propositions, (Art. 8;) noticing occasionally remarkable examples of the adaptation of language to systems of numeration, and other facts which may illustrate the process followed by the human mind in the formation of such systems; imperfect as this notice must necessarily be, it will enable us to give some degree of arrangement to a great multitude of very interesting facts, and will show in a very remarkable manner, how near an approach is made, in a great many instances, to the simplicity of the most philosophical language.

Numerals of Thibet. Of all the systems of numerical words with which we are acquainted, that of Thibet possesses the most simple structure, and makes the nearest approach to arithmetical notation by local value; the first twenty-nine numerals are as follows:

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* Horneman, *Proceedings of the African Society*, p. 148—156.

† *Mission to Ashantee.*

‡ *On the Origin of the American Tribes and Indians.*

§ Park's *First Travels in Africa*, p. 61.

‖ There is considerable difficulty in collecting materials for an inquiry of this kind, as travellers have usually contented themselves with giving the simple numerals as far as ten, without noticing the formation of the expressions for higher numbers; or where such are given, they have seldom added an explanation either of the meaning or grammatical connection of the terms in compound expressions, which is highly necessary, in order to deduce from them an idea of their arithmetical systems. Amongst other exceptions to this remark, (which is only generally true,) we ought in justice to mention Mr. Crawford, who has given an excellent account of the numeral systems of the Islanders in the Indian Archipelago, in drawing up which he professes to have been guided by the excellent observations of Professor Leslie in his *Philosophy of Arithmetic*.

There is a work of the Abbé Hervas, expressly on this subject, frequently referred to by Humboldt and Vater, entitled *Idea del' Arithmetica di tutte le Nationi conosciute*, a copy of which we have not been able to procure. The materials of this work must be of great interest and value, as the author was in possession of a large collection of American vocabularies, which only exist in manuscript.

* Schubert's *Reise durch Schweden, Norwegen, Lappland, Finnland, &c.* vol. iii. p. 453.

† Kand Lecms, *De Lapponibus Finmarchiæ.*

‡ Kalmar, *Prodromus Idiomatis Scythico-Magorico, Chazo-Avarici, sive apparatus Criticus in Linguam Hungaricam*, p. 79.

§ Klaproth, *Asia Polyglotta*, p. 40.

Arithmetic.

1.	Cheie.	16.	Chutru.
2.	Gnea.	17.	Chutoon.
3.	Soom.	18.	Chughe.
4.	Zea.	19.	Chugoo.
5.	Gna.	20.	Gnea chutumbha.
6.	Tru.	21.	Gnea cheic.
7.	Toon.	22.	Gnea gnea.
8.	Ghe.	23.	Gnea soom.
9.	Goo.	24.	Gnea zea.
10.	Chutumbha.	25.	Gnea gna.
11.	Chucheic.	26.	Gnea tru.
12.	Chugnea.	27.	Gnea toon.
13.	Chusum.	28.	Gnea ghe.
14.	Chuzea.	29.	Gnea goo.
15.	Chugna.		

In this system, the numerals from ten to nineteen, are formed by the combination of the first syllable of the word for ten, with the names of the first nine numbers, in the same manner as in most of the languages adapted to the decimal scale; but from twenty-one to twenty-nine, the name for two acquires a value from position, in a manner which bears the closest analogy to our ordinary arithmetical notation. Turner,* who has only given these numerals incidentally in his observations on the Thibetan month and calendar, has added no explanation whatever of their mode of expressing higher numbers.†

Our arithmetical notation probably due to Thibet.

If the same simplicity of structure prevails throughout the numerical language of Thibet, (and it is difficult to imagine that this happy idea when once started should not have been pursued to a much greater extent,) it would give great weight to the opinion that we are indebted to this country for our system of arithmetical notation; as of the two great difficulties attending its invention, namely, local value and the zero, one at least was overcome, at the period when their numerical language was fixed.‡

Considered by the Hindoos of Divine origin.

(12.) The Hindoos consider this method of numeration as of Divine origin, "*the invention of nine figures with device of place being ascribed to the beneficent Creator of the universe.*"§ Of its great antiquity amongst them there can be no doubt, having been used at a period certainly anterior to all existing records. Most other memorable inventions they have attributed to human authors; but this, in common with the invention of letters, they have ascribed to the Divinity, agreeably to the practice of the Egyptians, Greeks, and most other nations, with respect to the more important inventions in the arts of life, whose origin is lost in the remoteness of antiquity.

The intimate analogy in the grammatical structure, and in many of the roots of the classical languages of Europe with the Sanskrit, combined with the evidence furnished by historical and other monuments, point out the East as the origin of those tribes, whose progress to the west was attended by civilisation and empire, and amongst whom the powers of the human

* *Embassy to Thibet*, p. 321.

† See also Klaproth, *Asia Polyglotta*, p. 353, where the numerals are given under a somewhat different form; and Remusat, *Récherches sur les Langues Tartares*, p. 364.

‡ The numerals in a kindred language, and where the compound expressions for numbers are nearly similar to those of Thibet, may be seen in Kirkpatrick's *Vocabulary of the Newar Dialect of Nepaul*, p. 243.

§ Bhascara; Vasara, and Crishna's *Commentary on the Vija Ganita*, quoted by Mr. Colebrooke in his *Hindoo Algebra*, p. 4.

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mind have received their highest degree of developement; and it may, perhaps, be not altogether unfair to form some inference respecting the extent of the arithmetical system of those tribes at the period of their separation, by the numeral words which those languages possess in common. The Sanskrit names of the ten numerals, which are

Sanskrit numerals.

1.	Eca,	6.	Shata,
2.	Dwau,	7.	Sapta,
3.	Traya,	8.	Ashta,
4.	Chatur,	9.	Nava,
5.	Panca,	10.	Dasa,

have been adopted with slight variations, as we have before remarked, not merely in all languages of the same class and origin, but likewise in many others which are radically different from them. If we proceed to the expressions for higher numbers, we find the same general law of their formation, by the combination of the names of the articulate numbers with those of the nine digits. In the Sanskrit also, as well as in its immediate descendant the Hindostanee, it is more elegant to make use of a word which is equivalent to *less twenty*, rather than of the one which would naturally express *nineteen*, and similarly for other numbers in the next series below the articulate numbers:* precisely as in the Latin, we say *unus de viginti* for *novemdecim*, *unus de triginta* for *viginti novem*; and the same form of expression is observable in the Greek:† these are points of resemblance in the construction of their numerical terms which deserve to be remarked, though not without example in other languages. If we pursue our comparison of the other and higher numerical terms of those languages, we shall find few other points of resemblance; the names for twenty, a hundred, a thousand, are completely different: making it probable at least, that at the epoch of which we are speaking, their Arithmetic was confined within very narrow limits.

Great extent of Sanskrit numeral language.

The Sanskrit numeral language assigns names to seventeen orders of superior units in the decimal scale, as will be immediately seen from the following list:

1.	Eca.	10^9.	Abja or padma.
10.	Dasá.	10^{10}.	C'harva.
10^2.	Sáta.	10^{11}.	Nic'harva.
10^3.	Sahasra.	10^{12}.	Mahadpadma.
10^4.	Ayuta.	10^{13}.	Sáncu.
10^5.	Lacsha.	10^{14}.	Jaludhi or samudra.
10^6.	Prayuta.	10^{15}.	Antya.
10^7.	Cóti.	10^{16}.	Madhya.
10^8.	Arbuda.	10^{17}.	Parard'ha.‡

This luxury of names for numbers, much greater than what are required for the ordinary uses of life, or even for the most extended astronomical calculations, is entirely without example in any other language, whether ancient or modern; and implies a familiarity with the classification of numbers according to the decimal scale and the power of indefinite extension which it possesses, which could only arise from some very perfect system of numeration, such as that "with device of place." Indeed there is no circumstance which so strongly characterises Hindoo science, as this very extraordinary facility of dealing with high numbers: witness their enormous astrono-

* Halhed's *Grammar of the Bengal Language*, p. 160.

† Matthiæ, *Greek Grammar*, vol. i. p. 174.

‡ Colebrooke, *Hindoo Algebra*, p. 4.

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mical periods, and the extravagant dates of their chronology; and this at a period when the most scientific people of the western world were incapable by any refinement of arithmetical notation, of expressing numbers beyond one hundred millions.[*]

There is an epoch in the languages of all civilized people, at which they acquire a fixed and permanent character, and after which the admission of new terms, not arising from those natural combinations which the genius of the language sanctions, is effected with great difficulty: this takes place whenever a national literature, whether oral or written, is so generally diffused, as to form a standard of reference or a test of purity, which, whilst it enforces a legitimate character upon all existing terms, watches over the introduction of all others with extreme jealousy; from this consideration alone, independently of other evidence, we should be inclined to assign to the Sanskrit terms for high numbers, and consequently to the system of numeration, upon which they are founded, an antiquity at least as great as their most ancient literary monuments; as the arbitrary imposition of so many new names, for the most part independent of each other, and in number also so much greater than could possibly be required for any ordinary application of them, would be a circumstance entirely without example in any language which had already acquired a settled and generally recognised character.

Chinese numerals.

(13.) There is another eastern people, remarkable at once for the great antiquity and unchangeable character of their existing institutions, who possess a numeral language of great extent, connected with a very perfect system of numeration. The following is the list of Chinese numerals:[*]

1. Yih.	10. Shih.
2. Irr.	100. Puh.
3. San.	1000. Ts'hyen.
4. Sè.	10000. Wan.
5. Ngoo.	10^5. Eè.
6. Lyeu.	10^6. Chaò.
7. Ts'hih.	10^7. King.
8. Pah.	10^8. Kyai.
9. Kyeú.	

The very peculiar character of the Chinese language, a language, in short, of symbols and their combinations, which is addressed to the eye and not to the ear, connects these numeral terms inseparably with the seventeen *figures*, or characters, which are made use of in Chinese Arithmetic. In alphabetical languages, there is no connection between numerical words and numerical symbols, the latter being, in almost all cases where they exist, of subsequent invention to the former; but the Chinese numeral symbols, being either simple elements, or *keys*, or composed of them like other characters, are transferred to the oral language upon those arbitrary yet regular principles by which *monosyllabic* sounds are attached to all their characters, however complicated they may be.

Three kinds of Chinese figures.

In Plate I., we have given three series of Chinese numeral characters, the first being those which commonly occur in historical and scientific works; the second are the characters made use of in bonds and formal instruments, in order to avoid frauds, to which the first series of numerals are very liable, from their simplicity of form: they are likewise characters to which other meanings are attached, and which are only *conventionally* used for the purpose of numerals. Thus the character used in such documents for one, means *perfection*; that for two, is a verb meaning *to assist*, *to separate*; for three, *an accusation*; for four, *to expose publicly*; for five, *to associate*; for six, *a mound of earth*; for seven, *a certain tree*; for eight, *to divide*; for nine, *a peculiar stone*; for ten, *to collect*; and similarly for the characters of the other superior units. The use of such characters for numbers corresponds to our use of numeral words at full length instead of figures, for such purposes; but the analogy exists in the application only, the Chinese expressions for numbers being in all cases symbolical. The third set of figures are used for mercantile purposes, and are said to have been introduced by the Catholic Missionaries; they have been adopted in consequence of their greater simplicity of form, and from their admitting of being rapidly and easily written.

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In the same Plate, (I.) the reader will find examples of the actual mode of expressing particular numbers; such as 22, 100, 1100, 1010, 1001, and 1923000, according to these three methods of notation:[*] in the two first, the numbers are written in vertical columns, the value decreasing downwards, the *digital* symbol being placed immediately over the symbol of the superior unit: thus, to express the number seventy, the symbol for seven is placed over that for ten, and similarly in other cases. There is also another character denominated *ling*, which means *residue* or *remainder*, which in some respects may be considered as filling the place of a *zero* in notation by local value: thus, in expressing 1001, the symbols denominated *yih*, *ts'hyen*, *ling*, *ling*, *yih*, are written successively underneath each other. It is clear that if the symbol called *ts'hyen* for 1000 were omitted, this notation would strictly coincide with our ordinary arithmetical notation; the use of this character, however, is certainly superfluous, though it affords a very remarkable approximation to a more perfect system of numeration. In the use of the symbols of the third series, obviously founded upon the principle of approximating the system of Chinese Arithmetic to that of the Hindoos, the symbols are written from right to left, and the character *ling* is replaced by the European zero.

Their great antiquity.

The essential distinction of Chinese arithmetical notation and our own, clearly consists in the use of symbols for the superior units in one case, which are expressed by *position* alone in the other. In the last of their three methods of notation, those systems would become identical by the entire omission of the second of the two lines of symbols.

The Chinese consider the symbols of the first class for numbers which are below 10,000, as coeval with the invention of their other characters, and consequently as possessing an antiquity of at least 3000 years. The symbols for higher numbers are of later date, having been introduced at different times to meet the increasing wants of their Arithmetic: it would follow, therefore, if full credit can be attached to their annals, that the claims of the Chinese to the first invention of arithmetical figures, are equal, if not superior, to those of any other people. Independently,

* Marshman, *Clavis Sinica*, p. 209.

* Morrison's *Chinese Grammar*, p. 84.

indeed, of direct historical evidence, we might venture to infer, from the universal prevalence of the decimal scale throughout the empire, not merely in the classification of numbers, but also in the divisions of their coins, their weights, and their measures; from the great number of superior units, expressed by their symbols; and from the great perfection of their practical Arithmetic, for which they have long been celebrated throughout the neighbouring countries, and the Indian Archipelago, that they have been in possession of a very perfect system of numeration during many ages: an opinion which derives additional support from observing, that amongst them literature, science and the arts of life have long reached a stationary point; and that, from the very nature of their government and institutions, a limit is put to the progress of improvement, and, apparently, even to the powers and speculations of the human mind.

Not the inventors of notation by local value.

As the Chinese are not in possession of the method of arithmetical notation by nine figures and zero, they clearly can have no proper claim to its invention, however nearly in some respects they may have approximated to it; for it is next to impossible that a system of numeration, so much more perfect and commodious than their own, if once generally known or practised, could ever have been lost or abandoned.

In considering the claims of other nations to this great invention, which is, unquestionably, of eastern origin, if our decision is to be determined by the *known* antiquity of possession, we must certainly refer it to Hindostan; though some circumstances in the construction of the numerical language of Thibet have induced us to express a suspicion, that it may have originated in that country; an opinion which derives some support from the frequent and intimate communication between these countries from very early periods: and whilst from Hindostan they derived the doctrines of Bouddha, the Sanskrit alphabet, under the form in which it is seen in the most ancient inscriptions, and the polysyllabic portion of their language, which is otherwise intimately allied with the monosyllabic colloquial medium of China, it is not improbable that they may have communicated in return the elements of the system of arithmetical notation by local value. +

Economy of numerical words.

(14.) The economy of numerical words, which is observable in most languages, affords a very strong confirmation of the truth of our proposition, that they have been in all cases adapted to systems of numeration previously in use; thus, it is a very rare case in any language to find two different words to express the same number; and when such do occur, they are usually the vestiges of primitive methods of numeration which have been superseded by others adapted to the denary scale, where the new terms which have accompanied its introduction are either of foreign origin, or formed from the natural combinations of the language in such a manner as to be more expressive of the process of numeration itself.

Malay and Javanese numerals.

We shall find many examples of this circumstance in the languages of the islanders of the Indian Archipelago; their primitive systems of numeration, which were in ancient times for the most part quinary, subordinate to the vicenary, have been superseded by the more perfect arithmetical system of Hindostan, transmitted to them, either immediately or indirectly, through the Malays of Malacca and Sumatra. The

following list of Malay numerals, with those corresponding to them of the ordinary language of Java, will assist us in generalizing some remarks, not merely on this subject,* but likewise others which arise immediately from an examination of them:

Malay numerals.	Javanese numerals.
1. Sa, satu, sūatu.	1. Sa, siji.
2. Dūa.	2. Loro.
3. Tīga.	3. Tālu.
4. Ampat.	4. Papat.
5. Lima.	5. Limo.
6. Anăm.	6. Nānăm.
7. Tūjah.	7. Pitu.
8. Dūlāpan, salāpan.	8. Wolu.
9. Sambīlan.	9. Songo.
10. Pūluh, sa-pūluh.	10. Pūluh.
11. Sa hlas.	11. Sawālas.
12. Dūa hlas.	12. Rolas.
13. Tīga hlas.	13. Tālulas.
20. Dūa pūluh.	20. Rong pūluh, or līkur.
21. Dūa pūluh satu.	21. Rong pūluh siji, or sa-likur.
25. Dūa pūluh lima, or taugah tīga pūluh.	25. Rong pūluh limo, or limo likur, or lawe.
30. Tīga pūluh.	30. Tălung pūluh.
35. Tīga pūluh lima, or taugah ampat pūluh.	35. Tălung puluh limo.
50. Lima pūluh.	50. Limo pūluh, sekăt.
60. Anăm pūluh.	60. Nānăm pūluh, swidak
65. Anăm pūluh lima, or taugah anăm pūluh.	65. Pitusasor.
100. Rātus, sa-rātus.	100. Hatus.
200. Dūa rātus.	200. Rongātus.
400. Ampat rātus.	400. Papat-ratus, samas.
800. Dūlāpan rātus.	800. Wulung-atus, domas.
1000. Rihu.	1000. Hewu.
10^4. Laksa.	10^4. Lăkso.
10^5. Sa-pūluh laksa.	10^5. Kăti.
10^6. Sa-yūta.	10^6. Yūto.
10^7. ———	10^7. Wăndro.
10^8. ———	10^8. Boro.
10^9. ———	10^9. Părti.
10^{10}. ———	10^{10}. Partomo.
10^{11}. ———	10^{11}. Gulmo.
10^{12}. ———	10^{12}. Kerno.
10^{13}. ———	10^{13}. Wurdo.

In most of the islands of the Indian Archipelago, there is a ceremonial dialect, as well as the one in ordinary use, and as might be expected, the numerals are not always the same in both: thus in the ceremonial dialect of Java, the term for one is *satungil*, compounded of *sa, one*, and *tungil, alone by itself*; for two, the word *kaleh* is used, which is the preposition *with*; for three, *tigo*; for four, *kawan, a flock or herd of animals*; for five, *gangsal*, a term of unknown derivation; and for ten, the Sanskrit term *doso*; the other terms, excepting where those above mentioned are used in expressions for compound and articulate numbers, are the same as in the ordinary dialects.

The influence of the Malays in the Indian Archipelago is, comparatively, of modern date; and we consequently find, every where, remains of ancient dialects, very different from those at present in use. That of Java is an immediate derivative of the Sanskrit, pos-

* Crawford's *Indian Archipelago*, vol. i. p. 264; Marsden's *Malay Grammar and Dictionary*, p. 37.

Arithmetic. sessing likewise the Sanskrit numerals, with slight variations, and those chiefly in the names of the superior units, which have been transmitted unchanged from the ancient to the modern dialects. The great number of the names of those units, unexampled in the languages of any other of those islands, which possess no native term for a number beyond *onethousand*, and no borrowed term for a number beyond *one million*, is a circumstance strongly confirmatory of our argument respecting the great antiquity of those names and of the arithmetical system connected with them, amongst the people from whence they were derived.

Confusion in borrowed numerical terms. (15.) Throughout the islands of the Indian Archipelago, with the exception of the Lampungs, an inland people of Sumatra, the Sanskrit term *laksha* for 100,000 has been borrowed to express, not the same number, but 10,000; a circumstance which frequently causes mistakes in their commercial transactions with the people of Hindostan. In a similar manner the Javanese use the term *kăti* for 10^5, which is the same as the Sanskrit term *koti* for 10^7, and the term *yŭto* for 10^6 the same as the Sanskrit *ayuta* for 10^4: this confusion of the terms for high numbers, which are evidently borrowed from each other, is a very remarkable circumstance, and can only be accounted for by supposing that amongst a rude people, little accustomed to the use and contemplation of such numbers, the terms by which they were expressed would convey no distinct impression to the mind, and consequently in making use of them more reliance would be placed upon the uncertain testimony of the memory, than the surer guidance of the understanding.

Other examples may easily be produced of a similar change in the value of borrowed numerical terms. In the Newar dialect of Nepaul, we find *lak-sehee* borrowed to express *a million*;* in the language of the Mantischeoo Tatars, immediately bordering on the north of China, in which the numerals are taken generally from the Chinese, though they have lost their monosyllabic form, we find the term *iwuan* for 1000, obviously derived from the Chinese term *wan*, which expresses 10,000.† Again, *alp*, the term for 1000 in most of the languages which modern philologists have agreed to call Semitic,‡ and which prevails in those of Upper Egypt, Abyssinia, and Darfur, signifies 10,000 in the Amharic, a language intimately allied with them; the term *she* for 1000 having been interpolated between it and the term *meto* for 100, derived immediately from the common Semitic term for that number.§

Small number of native terms for high numbers. (16.) The poverty of languages in native terms for high numbers, arises either from the limited extent of their Arithmetic, or from the difficulty in all established languages of inventing new words, even when the want of them is felt: it is from this latter reason chiefly, that the extent of numerical language is no just measure of advancement in the arts of life, or even in the art of numeration itself; and we shall find many examples of barbarous people who possess terms for higher numbers than the Greeks or Romans, or of other nations incomparably more civilized than themselves. The same remark may be extended to languages generally, which neither in the perfection of their grammar, nor even in their copiousness, appear to bear any certain relation to the state of civilisation of the people by whom they are spoken. History.

Some authors have asserted, that many nations possess a numerical language more extensive than their powers of numeration, and have referred, in proof of their assertion, to the numeral words of many South American tribes, which are sufficiently comprehensive, though the people by whom they are used cannot without great difficulty count beyond twenty. When people are descended from a people more civilized than themselves, from whose monuments of whatever nature such terms are collected, such an opinion may be entitled to credit; but in all other cases it seems to involve its own refutation, as the very existence and interpretation of the word implies that its meaning is understood by some one at least, if not generally.

The statements of travellers respecting the languages and customs of people with which they have not become familiar from long intercourse, must always be received with extreme caution; and there are few subjects upon which greater mistakes have been made, than on those which respect the extent and methods of numeration of barbarous nations. In most instances such errors have consisted in greatly understating the extent to which such people are able to count; but in other cases, they have been of a completely opposite character, as the following example will show: we had long been embarrassed with the account given by Labillardière,* of the enormous extent of the numeral language of the natives of Tongataboo, one of the Friendly Islands, proceeding as far as 10^{14}, a fact in apparent contradiction to our theory, and not to be explained by their intercourse with the Malays, from whom much of their numeral language is derived, but who possessed no terms for numbers equally great; and it was only by referring to the account given of these islands by Mariner,† that we found that their highest numerical term was *mano* for 100,000, and that the other terms which he has put down for higher numbers, have significations of a very different nature, imposed upon the poor Naturalist from a species of revenge, more remarkable for its humour than decency, for the persevering and annoying efforts which he made to extract from them the names of numbers of which they had no knowledge.

Names for superior units. If we examine the limits of the numeral terms of different languages, we shall find few which possess terms for numbers beyond a thousand; and the cases are extremely rare in which they reach a million. The instances are still rarer where such terms are native, having been introduced, as in some cases we have seen already, by intercourse with other nations; and we frequently find the same terms for such numbers where the lower numerals, as well as the languages to which they belong, are essentially different from each other: such examples are not without a considerable historical interest, as monuments of the communications of nations with each other, and as indicating the channels through which improvements not merely in arithmetic, but likewise in the other arts of life, have been conveyed. We have already given examples of facts

* Kirkpatrick's *Nepaul*, p. 243.
† Klaproth, *Asia Polyglotta*, p. 360. ‡ Ibid. p. 107.
§ Salt's *Travels in Abyssinia*, App.

* *Voyage in Search of La Perouse*, vol. ii. p. 408. English edition.
† Mariner's *Account of the Tonga Islands*.

of this kind among Eastern languages, and it would be very easy to multiply their number: a few more instances will establish the truth of our assertions, respecting the invention and transmission of numeral terms in a still more striking manner.

Greek. The Greeks possessed a term, μυρία, for 10,000; and, notwithstanding the increasing wants of their Arithmetic, they never attempted to proceed beyond it: it appears originally to have signified an indefinite number, and in this sense it is always used in Homer; but in later times they gave it a new and restricted meaning without abandoning the old, and distinguished between its definite and indefinite signification by a difference of accent or tone. This term in its later sense, at least, was unknown to the Æolic tribes at the time of the colonization of Latium, as no traces of it appear in the Latin language, though the terms for 100 and 1000 were transmitted through them with very slight alterations. **Latin.** The characteristic contempt of the Romans for whatever was connected with science or the arts, may sufficiently account for their not attempting to extend their numeral language as far as the Greek, by borrowing or inventing an additional word; and the improvement which was not effected during the zenith of their empire, could not be looked for during its decline, and that long period of darkness and barbarism, which ended in the extinction of the Latin as a living language. At the beginning of the fourteenth century, when the modern **Italian.** Italian, its legitimate successor, was beginning from the revival of learning and the writings of native authors, to assume a settled character, and when the introduction of the Hindoo arithmetical notation, through the Arabians, was bringing into familiar use numbers much greater than were expressible by the Roman numerical symbols, we find a great addition to their former numerical language, by the use of the word *millione*, which properly signifies *great thousand*, to denote the square of one thousand, and which was followed by the words *billione*, *trillione*, deduced immediately from the former by pursuing the natural analogies of the language: a series of numeral terms were thus formed, proceeding not by tens, but by millions, like the *monads* of Archimedes, which proceed by myriads of myriads. In a numeral language thus constituted there is clearly no limit to the expression of numbers, the composition of the names for the monads or superior units being once understood.

German. These terms were at different periods adopted in almost every language of Europe: the Germans, who adhere as much as possible to the combinations of their own language in the formation of new words, resisted the introduction of the term million, forming no natural succession to their native words *hundert* and *tausend*, until the commencement of the sixteenth century. The Poles,* the most cultivated of Sclavonic nations, admitted it at a still later period; and it was introduced into Russia, along with the Hindoo notation, by Peter the Great, at the commencement of the last century.

Spanish. The Spanish term for a million is *cuento*, which in ordinary language means *a tale or fable for children*; it most probably originated from *cubo ciento*, the cube

* The Polish word for 100 is *sto*, and for 1000 *tisiacz*; the Russian word for 100 is also *sto*, but there is no native word for 1000, which is expressed by *desat sto*, or ten hundred.

of a hundred. Though without any certain means of judging of its antiquity, we have probable reasons for thinking it nearly, if not quite, as old as the corresponding Italian word; for the Arabian notation was known at least as early in Spain as in Italy; and as the consequent necessity for an extended numeral language, must have been equally felt in both countries, it is not likely that one word could have long existed in one language without being communicated to the other, particularly when the intimate alliance of the languages with each other, and the frequent intercourse at that period of the people by whom they are spoken, is considered.

Welsh. There are two different series of names for superior units in the Welsh language; one ancient, and the other used in its more modern and latinized form:* in the last of these, we have *cant*, 100; *mil*, 1000; *myrz*, 10,000; *can mil*, 100,000, *myrzcan*, or *milmil*, 1,000,000; *milcan mil*, 10,000,000; and similarly for higher numbers. The selection of the word *myrz* for 10,000, which is clearly the Greek μυριὰς, and the deriving of the rest from the Latin, would appear to show that they had been introduced at a late date by some monk or other person who was familiar with the classical languages. The ancient and more native superior numerals are chiefly remarkable for their redundancy, and an extent greater than amongst any other European people: thus we have three names, *muni*, *catyrva*, *rhiolla*, for 100,000; and other three, *myata*, *bewa*, *catyrva vawr*, for 1,000,000. The appearance of the Latin word *catyrva* as a numeral is a very extraordinary circumstance, and we are not aware of any hypothesis by which it may be explained.

Erse. Of other Celtic languages the Erse,† and its descendant the Gaelic, have no native term beyond *ciad*, or 100; the expression for 1000 being *deichciad*, ten hundred, or more commonly the Latin term *mile*. We believe the same remark applies to the Armoric language, and the Basque of Biscay.

Hebrew. In Rabbinical Hebrew only, we find the term *ribbo* for 10,000, which is never found in any of the kindred dialects. The term *aleph*, or *alph*, for 1000, as we have before remarked, prevails very extensively; being found, with slight variations, in Arabic, Persian, Abyssinian, the ancient Punico-Maltese, and in many of the languages in the north of Africa; and we very frequently find it, as well as the term *meah*, for 100, terminating the numeral systems of languages which have no other terms in common.

Amharic. In the Amharic, and some neighbouring dialects, where the term *alph* has been misapplied to denote 10,000, we find likewise the term *ilef* denoting 1,000,000, a solitary example amongst Semitic languages of a term for so great a number.

Arabian and Persian. Neither the Arabians nor Persians, though the notation by numeral figures possessing local value was known amongst the former at least as early as the ninth or tenth centuries, have attempted to make any addition to their numeral language; a circumstance which may be accounted for, partly by the advanced state of their literature at the period when it was first known, and partly by the genius of those languages not admitting the formation of terms like the *millione* and *cuento* of the Italian and the Spanish: thus to express a

* Owen's *Welsh Dictionary*.
† Vallancey's *Irish Grammar*.

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million, they are obliged to repeat the term for a *thousand* twice; a *thousand millions*, to repeat it three times, and similarly for other numbers in the same series.* We recollect in an old German author on Arithmetic to have seen a similar expedient adopted to express the number 10^{63}, made use of by Archimedes in his *Arenarius*, which is given as follows:

Ein tausend.

tau tau tao tau tau tan tao tan tau tao tau tau tau tau tau tau tau tau tausend mahl tausend.†

Gothic.

There are many other examples of the formation of expressions for superior units by the repetition of the names for its factors, as often as they are contained in it. In the *Codex Argenteus*, preserved at Upsal, and which is a translation of the four Gospels made by Bishop Ulphilas in the fourth century, into Mœso-Gothic, we find *taihun taihund*,‡ or *ten ten* for 100. In the language of the Knistenenux, one of the principal hunting tribes of North America,§ who inhabit the northern shores of Lake Superior, we find 100 expressed by *mitana mitenah*, or *ten ten*; and 1000 by *mitana mitena mitanah*, or *ten ten ten*.‖ The Sapibocones, a South American tribe, express 10, 100, 1000, by *tunca, tunca tunca, tunca tunca tunca*, respectively:¶ such a mode of expression, indeed, is one of the most simple and obvious expedients for denoting numbers, which are not immediately within the compass of any numerical language.

North American numeral systems.

(17.) We shall find in general, that the numeral languages of the tribes in the central parts of North America are more complete, both in structure and extent, than could be expected from their low state of civilisation: they are almost universally adapted to the decimal scale, and in most instances extend as far as 1000. The Algonquins a kindred tribe of the Knisteneaux, speaking a dialect of the same language, and possessing many numerals in common, have simple terms *ningontwack* and *kitchiwack*, both for 100 and 1000. The Hurons, once a numerous and powerful tribe, living in Upper Canada, around the lake of that name, who speak a language** singularly rude and inartificial, without adjectives, abstract nouns, or verbs of action, and incapable of expressing a negation, without an absolute change of the word, possess a numeral language sufficiently regular; the name for 10 being *assou*, for 100 *egyo-tiwoissan*, and for 1000 *assou attevoignavoy*. We shall find numeral systems equally complete among the Iroquois, and the rest of the tribes of Upper Canada; amongst the Indians on the Delaware, and those who formerly occupied the neighbourhood of New York; amongst the ancient inhabitants of Virginia;* and most of the tribes of Central North America of whose languages we possess any records.

History.

The decimal scale not very common in South America.

The decimal scale is much less generally prevalent among the numerous tribes of South America than among those of North, and their numeral systems much less perfect, rarely proceeding beyond a hundred, and frequently limited to much smaller numbers: there are not wanting, however, numeral systems adapted to the decimal scale, which are sufficiently complete and comprehensive; but in most cases the names for numbers, particularly for those which are compound, are of such extraordinary length and complexity, as to appear to exceed the powers of human utterance. In one language, however, namely the Qquichua, or ancient Peruvian, we find a numeral system equally simple and more extensive than that of the Greeks or Romans, as the following names or expressions for the series of superior units will show:

10. Chunca.
100. Pachac.
1,000. Huaranca.
10,000. Chunca huaranca.
100,000. Pachac huaranca.
1,000,000. Hunu.†

The New World is not without its examples also of names for superior units borrowed by one people from another more civilized than themselves: thus the Molluches, a tribe who inhabit a district to the South of Chili, have adopted the Peruvian term *pataca* for 100, and *huaranca* for 1000; though the languages of these people, as well as their other numerals, have nothing further in common.‡

It is quite unnecessary to pursue this inquiry into the extent and developement of numeral systems farther, as the examples which we have adduced will sufficiently demonstrate the truth of the assertions which we made at its commencement. We shall now proceed to the consideration of some peculiarities in the expression of numbers, which illustrate in a very striking manner the very regular and artificial manner in which numeral language has in most cases been constructed.

Numbers sometimes expressed with reference to the next superior articulate numbers.

(18.) We have before noticed the method of expressing some numbers, such as nineteen, twenty-nine, &c. by their defect from the next superior articulate numbers, which is usual in the Sanskrit, Greek, and Latin; and we shall find the same peculiarity in the Malay and other languages. Thus, instead of saying *sambilan pūlah sambilan*, or *ninety-nine*,§ they more frequently use the expression *kōrang asa sa-rātus*, or *wanting one of a hundred*. The word *sambīlan*,‖ or *nine* itself, means *one taken*, that is, taken from the heap or

* Chardin, *Voyages en Perse, par Langlès*, tom. iv. p 293.

† *Rechenbuch auf den Linien und mit zifern durch* Simon Jacob von Coburgk, Rechenmeister zu Frankfurt am. Mayn, 1559.

‡ Hickes, in his *Thesaurus Linguarum Veterum Septentrionalium*, considers our term *hundred* to have originated in the custom of writing the last syllable *hund* of this expression only for greater brevity, particularly when combined with other numbers. From the same principle of abbreviation, we have got the term *thousand*, contracted from *taihun hund*, or *tigus hund*, *ten hundred*. The reader may see other etymologies of these words, many of them extremely absurd, in the *Etymologicum Anglicanum* of Junius.

§ Dr. Richardson, in Franklin's *Journey*.

‖ Mackenzie's *Journey to the North Sea*, Introduction.

¶ Humboldt, *Vues des Cordillères et des Monumens de l'Amerique*, p. 251.

** Monboddo, *Origin and Progress of Language*, p. 543. The numerals are given in a very curious and rare work by a Franciscan monk, G. Sagard, published in 1632, entitled *Le Grand Voyage des Hurons, situé en Amerique vers la mer douce es dernières confins de la nouvelle France*; with a dedication, "*Au roy des roys et tout puissant monarque du ciel et de la terre* Jesus Christ, *Sauveur du monde*," written in a very quaint style, but describing with considerable force and eloquence the efforts of the missionaries to bring these rude people under the dominion of Christ.

* *Account of Virginia*, by Captain Smith, 1624. Their names for 100 and 1000 are of very formidable length; for the first being *necuttaughysinough*, and for the second, *necuttweunguavagh*.

† Humboldt, *Vues des Cordillères*, &c. p. 252.

‡ There is a grammar of the language of this tribe published by Robert Falkner, an English Jesuit, who resided as a missionary in Patagonia for upwards of forty years.

§ Marsden's *Malay Grammar*, p. 39.

‖ Crawfurd's *Indian Archipelago*, vol. i. p. 256.

whole; and *sakŏrang*, which in Malay means *one wanting*, is the term for nine in the Achinese dialects.

Numerals of the Oedh-Ostiaks.

The numeral language of the Oedh-Ostiaks, or Sable Fur Ostiaks, a Siberian tribe living on the banks of the Jenesei, exhibits this peculiarity of construction in a very remarkable manner, and we shall therefore give it, with more than ordinary detail, as follows:*

1. Chusem.
2. Ynem.
3. Dôgom.
4. Syjem.
5. Chôjem.
6. Ahjem, or Chôjem-ehosem, 5 and 1.
7. Ohnem, or Chôjem-ynem, 5 and 2.
8. Chôjem-dôgom, 5 and 3; or ynem hotsche chojum, 2 from 10.
9. Chôjem-syjem, 5 and 4; or Chusem botsche chojum, 1 from 10.
10. Chojum.
11. Chusem chujum.
18. Ynem botsche agem, 2 from 20.
20. Agem.
50. Cholepky-scha.
70. Ohna-chojum.
80. Ynem botsche chojum chojum, 2 from 10 times 10.
90. Chusem hotsche chojum chojum, 1 from 10 times 10.
100. Kyschash, or ky.
1000. Chujum-kyschash, 10 times 100.

Such is the numeral language which we might expect to be formed by a people labouring under extreme poverty of numeral words, and who endeavoured to adapt them to a system of numeration previously known.

The other tribes who inhabit the banks of the Jenesei and its tributary streams, whose languages constitute a distinct class, being intimately allied with each other, but different from those of other Siberian people, whether of the Samoeid, Tatar, or Mongol race, possess numeral systems which are generally formed in the same manner.†

Kamtschatka and Kurile Islands.

The same construction is observable in the languages of the Kamtschatkans, and the inhabitants of the Kurile Islands, which are opposite to the mouth of the Amur, as will be readily seen from an examination of their expressions for one, two, eight, nine, and ten.‡

Kamtschatka.	Kurile Islands.
1. Syhnâp.	1. Sinezb.
2. Dûpk.	2. Zuzb.
8. Dûhpyhs, 2 from 10.	8. Zujemambe, 2 from 10.
9. Syhnâppyhs, 1 from 10.	9. Sinesambe, 1 from 10.
10. Upyhs.	10. Fambe.

Other peculiarities.

(19.) There is another peculiarity in the construction of numeral language, of very general prevalence both in Asiatic and European languages, which we shall now proceed to notice. Every student in Greek literature is acquainted with the phrase, apparently so remarkable, of ἕβδομον ἡμιτάλαντον, which, literally translated, means the *seventh half talent*, but which in all cases denotes *six talents and a half*.* Vestiges of the same construction are observable in the Latin word *sestertius*, which is the contracted form of *semis tertius*, and signifies *two whole asses and a half*, a meaning distinctly expressed in its original symbol L.L.S. which in later times became H.S. Of a similar description is the Anglo-Saxon phrase, *threo healf*, or *thridde healfe*, *two and a half*;† and the German, *anderthalb*, for *one and a half*; *viertehalb*, for *three and a half*; *elftehalb*, for *ten and a half*; and similarly in other cases.‡

In Danish numerals.

So prevalent was this mode of expressing numbers amongst the ancient Cimbri and their Danish descendants, that we find it combined with the vicenary scale, for the expression of the alternate articulate numbers between forty and a hundred, as will be immediately seen from what follows:

10. Tie.
20. Tyve.
30. Tredeve, 3 times 10.
40. Fyrteve, 4 times 10.
50. Halv tredie sinds tyve, half the third time 20.
60. Tre sinds tyve, 3 times 20.
70. Halv fierte sinds tyve, half the fourth time 20.
80. Fire sinds tyve, 4 times 20.
90. Halv femte sinds tyve, half the fifth time 20.
100. Hundrede.§

Icelandic.

We find examples of expressions precisely similar to those for 50, 70, and 90, in the Icelandic language. Thus *halft fiorda hundrada* means *three hundred and fifty*; and in expressing the age of a person, half way between two articulate numbers, instead of saying thirty-five, fifty-five, &c. they use the phrase *halft fertugr*, which means *half the fourth ten*; *halft sextogr*, or *half the sixth ten*; and similarly in other cases.‖

In Javanese and Malay.

Exact parallels to such expressions are to be found in the Malay, Javanese, and other Eastern languages. Thus in the first of these languages, instead of *dūa pūluh lima*, or *twenty-five*, it is more usual to say *tangah tiga pūluh*, or, literally, *half of thirty*; and similarly for thirty-five, forty-five, fifty-five, and so on. Again, for one hundred and fifty, they use the expression *tangah dūa rātus*, which is *half of two hundred*; that is, *of the second hundred*.¶ In the same manner in Javanese, *ewūlak sasor*, or *half sixty*, means *fifty five*; *pitusasor*, or *half seventy*, means *sixty-five*; and similarly in other cases.**

It is needless to add instances from other languages of a mode of expression which is so common that it hardly can be considered as peculiar, but which exhibits evidence, in the latter cases at least, of that constant reference to the articulate numbers, which is so generally characteristic of numeral language.

Expressions for compound numbers.

(20.) The mode of expressing numbers intermediate to articulate numbers, in the language of Lapland, is very peculiar and very significant; the first ten numerals

* Klaproth, *Asia Polyglotta*, p. 171.
† Ibid.
‡ Ibid. p. 315. * La Perouse's *Voyage*, vol. ii. p. 85. English edition.

* Matthiæ's *Greek Grammar*, p. 176.
† Hickesii *Thesaurus Linguarum Septentrionalium*, *Grammatica Mœso-Gothica*, p. 33.
‡ Noehden's *German Grammar*, p. 198.
§ Parson's *Remains of Japhet*, c. x. p. 317.
‖ Hickesii *Thesaurus: Grammatica Islandica*, p. 42. Hickes says that the Scotch, when asked the hour of the day, instead of saying *half past nine*, *half past eleven*, prefer answering *it is half ten*, *it is half twelve*; and he considers this mode of expression as a vestige of the Danish dominion in that country.
¶ Marsden's *Malay Grammar*, p. 40.
** Crawfurd's *Indian Archipelago*, vol. i. p. 266.

In the language of Lapland.

are given above, (Art. 10:) to express 11, they say *oust nubbe lokkai*, which is *one to the second ten*; for 12, *goust nubbe lokkai*, *two to the second ten*; for 23, *golm goaalmad lokkai*, *three to the third ten*. They proceed in this manner, combining the cardinal with the ordinal numbers, as far as *zhioette*, or 100, which is the limit of their numeral system.*

Of the Knisteneaux.

In the numeral language of the Knisteneaux, the numbers from 10 to 20 are expressed by the first nine numerals with the preposition *osap*, or *with*, the term for ten being omitted.†

1. Peyac.	11. Peyac osap.
2. Nishew.	12. Nishew osap.
3. Nishtou.	13. Nishton osap.
4. Neway.	14. Neway osap.
5. Ni-annan.	15. Niannan osap.
6. Negoutawoisic.	16. Negoutawoisic osap.
7. Nishwoisic.	17. Nishwoisic osap.
8. Jannawew.	18. Jannawewosap.
9. Shack.	19. Shack osap.
10. Mitatat.	20. Nishew mitenah.

The expression for 21 is *nishew mitenah peyac osap*, the omission of the preceding articulate number being no longer allowable, on account of the ambiguity which it would occasion.

In Malay and Javanese.

In the Malay and Javanese languages the expressions for numbers between 10 and 20, as may be seen from our list of their numerals, are formed by adding to the digit the particle *blas* in one case, and *wălas* in the other; probably identical with the Javanese term *tălas*, which means *done* or *finished*, that is with reference to the end of the scale. For numbers beyond 20, the expressions are formed in the regular way, excepting those cases which are included in some of the peculiarities above mentioned, or in which *absolute* terms, the remains of former methods of numeration, are used: thus *lawe* is used to denote 25, and denotes also a *thread* or *string*; and *sekăt* or *ekăt*, which means a *skein of thread*, also denotes 50; *swidak*, a term of unknown derivation, is used for 60; and *samas* and *domas*, denoting respectively "*one bit of gold* and *two bits of gold*," are used, the first for 400 and the last for 800.

In English.

Our term *eleven*, and the Anglo-Saxon *endlufon* means *leave one*, that is *above ten*, the point from which the numeration commences again as it were anew; and in the same manner *twelve* means *leave two*, with reference to the same number; beyond this number‡ the terms are formed in the way which is usual in most languages, by the combination of the nine digits with the preceding articulate number; and this departure from a very general rule in the expression of these two numbers, which is observable in all languages of Gothic origin, is, as far as we know, peculiar to them. It must be considered, however, as a variation and not as a violation of a general principle; the point of departure from which the numeration recommences being equally kept in view in both cases.

It might be imagined that this distinction in the formation of the expression for eleven and twelve, had its origin in the frequent use of the latter number amongst Scandinavian nations.§ Thus amongst the inhabitants of Iceland and Norway, the addition of the word *tolfræd* (that is *duodena ratio*) to the symbols or expressions for *ten* and a *hundred*, made the one signify *twelve* units, and the other *twelve decads*, and similarly for higher numbers: thus, CC *vætra tolfræd*, or *ducenti anni tolfræd*, means 240 years; CCC *dago tolfræd ae fim dagar*, or *three hundred days tolfræd and five days*, means 365 days, and similarly in other cases.

Preference of the number twelve amongst Scandinavian nations.

Traces of this preference of the number 12 amongst ourselves, as well as amongst other Gothic nations, are to be found not merely in the very frequent use of the term *dozen* in the classification and parcelling out of many objects of barter and trade, but likewise in our primary divisions of money, weights, and measures. In some cases, even the technical meaning attached by merchants to the word *hundred*, associated with certain objects, is *six score*; a usage which is commemorated, though perhaps in too sweeping and general a form, in the popular distich,

> Five score of men, money, and pins,
> Six score of all other things.

Though the influence of this division by twelve upon the customs and languages of northern nations is very remarkable, yet it hardly can be considered as indicating the existence of a duodenary scale of notation, properly so called; for, in the first place, the name for twelve is dependent upon the radix of the decimal scale; and in the second place, though there is a simple name *gross* for 12^2 or 144, yet in no case is that number, or even the former considered as an articulate number, or as a point of departure for a new numeration. The partition indeed of numbers and concrete units by 12, probably suggested in the first instance by the natural divisions of the year, is of very general use; but has no natural connection, in its origin at least, with the methods of classifying whole numbers: it being a refinement long posterior to the formation of numeral systems, to consider an *abstract* unit as capable of division at all, and still less that the results of such successive divisions should constitute a series of inferior units, admitting of classification in the same manner as abstract whole numbers themselves.

There are few other circumstances in the formation of expressions for compound numbers as distinguished from those which are articulate, which deserve to be remarked. In no one respect is the general economy of numeral language more strikingly exemplified, than in the terms for such numbers; for we not only hardly ever find two names for 11, 12, 13, and so on, but in no one instance do we find them expressed by an *arbitrary* and *independent* name, that is by a name which has no reference to the radix of the scale of numeration; a proof amounting nearly to demonstration, that words have been expressly adapted to such scales and are consequently subsequent to them.

Formation of expressions for articulate numbers.

(21.) The names of the articulate numbers are usually formed by the *incorporation* of the term, for the radix of the scale, with the names of the nine digits; and in almost all cases the etymology of such names is sufficiently obvious. We frequently, however, find two names for 20, one of them arbitrary and independent, and the other adapted in the usual manner to the decimal scale; the former are very generally vestiges of the vicenary scale, which has been superseded by the denary, and will be noticed hereafter; the latter commonly admit of a resolution into their

* Knud (Canutus) Leems, *De Lapponibus Finmarchiæ*.
† Mackenzie's *Travels*, Introduction.
‡ Junii *Etymologicum Anglicanum*, on word eleven.
§ Hickesii *Thesaurus: Grammatica Islandica*, p. 43.

component parts, as readily as the terms for the other articulate numbers. Thus, in our own language, *score* is a term of the former kind, reminding us of an ancient and extinct method of numeration; whilst *twenty* is immediately derived from the Gothic *twentig*, compounded of *twa* and *tig*, the latter signifying *ten* equally with *taihun*,* and generally used in preference to the latter in all compound words.

In Greek. The Greek word *εἴκοσι*† seems to defy all probable etymology, and may therefore most properly be referred to the class of arbitrary terms; whilst the terms *τριάκοντα*, *τεσσαράκοντα*, &c. are regular and simple in their formation, though the origin of the term *κοντα* for ten, corresponding to the Latin *ginta*, in *triginta*,

In Latin. *quadraginta*, &c. is extremely difficult to explain. The Latin word *viginti* is equivalent to *biginti*, or *twice ten*, and is not derived, as some authors have imagined, from the Celtic term *ugent*, or *ugain*, for the same number.‡

An accurate etymological examination of the expressions for articulate numbers in different languages, would frequently lead to results of great interest, not merely as exhibiting traces of ancient methods of numeration, but likewise as showing the limits to which they have proceeded, or the changes which they have undergone. In many cases, however, the etymologies of such words are extremely difficult, exhibiting very obscure traces of the digital numbers merely, with no discoverable reference to the radix of the scale; and in others, they may be considered as arbitrary and independent terms, which it is impossible in any way to connect with any system of numeration.

Turkish and Mandschen numerals. In the Oigour, of the elevated plain of Turfan, the most pure of the numerous class of Turkish dialects, and in the Mandscheu, one of the principal of the languages denominated Tungusic, we shall find examples which illustrate these observations, as will be seen from the following list of their numerals:

Oigour, or Eastern Turkish.§	Mandscheu.‖
1. Bir.	1. Emu.
2. Iki.	2. Dschus.
3. Utsch.	3. Ilan.
4. Töst.	4. Duin.
5. Bisch.	5. Sundscha.
6. Alty.	6. Ningan.
7. Yidi.	7. Nadan.
8. Sekis.	8. Dschakûn.
9. Tochus.	9. Ujun.
10. On.	10. Dschuan.
20. Igirmi.	20. Orin.
30. Otus.	30. Gutschin.
40. Chirch.	40. Dechi.
50. Ellik.	50. Sousai.
60. Altmisch.	60. Nindscheu.
70. Yitmisch.	70. Nadandscheu.
80. Sekis on.	80. Dschakûndscheu.
90. Tochus on.	90. Ujundscheu.
100. Yus.	100. Tanu.
1000. Ming.	1000. Mingan.
10^4. Tümen.	
10^5. Kuldy.	
10^6. Niut.	

In the first of these systems, the names for 20, 30, 40, and 50, have no common principle of formation, and with the exception of the first may be considered as perfectly arbitrary: those for 60 and 70, involve the names of the digital numbers 6 and 7, without any apparent reference to the radix of the scale; whilst those for 80 and 90 are formed in the ordinary manner. The name for 100 prevails not merely amongst all Turkish tribes, but has likewise been borrowed by some Siberian people,* who speak languages belonging to an actually different class; whilst the term *ming* for 1000 has been communicated not merely to the Mandscheu, but to the Mongol and all Tungusic languages, from one extremity of the continent of Asia to the other, though their numeral systems have nothing more in common. The other terms, as far as a *million*, are apparently arbitrary, and certainly native; there being no terms for such high numbers amongst any neighbouring or kindred nations.

In the second system of numerals which we have given above, the names for 20, 30, 40, and 50 are arbitrary, whilst those for the subsequent articulate numbers are formed in the ordinary manner. In the other Tungusic dialects, we generally find the greatest regularity in the formation of their numeral systems; the names for the articulate numbers being formed by the combination of the name for ten with that of the digital number, excepting in the one which follows from the dialect of Nertschinsk, where we find the Mandscheu names for 20 and 30, and all the others, expressed by a modified form of the names of the corresponding digits.†

Nertschinsk numerals.

1. Omin.	10. Dschón.
2. Dschur.	20. Orin.
3. Ilán.	30. Gotin.
4. Dygin.	40. Dyginni.
5. Tonna.	50. Tonnanni.
6. Njemun.	60. Njamnnnni.
7. Nôddan.	70. Nodanni.
8. Dschópkun.	80. Dschópkunni.
9. Jagyn.	90. Jaginni.

The same principle of formation of the expressions for articulate numbers is observable in the Semitic and many Asiatic languages. Where examples are so numerous, we shall content ourselves with the following list of Mongol numerals, which are found with slight variations amongst all Tatar nations, from the Wolga to the Wall of China.‡

Mongol numerals.

1. Nige.	7. Dolohn.
2. Gojer.	8. Naiman.
3. Churban.	9. Jisun.
4. Dürbän.	10. Arban.
5. Tahan.	20. Chorin.
6. Djirohn.	30. Chutschin.

* Our word *ten* is derived from the word *taihun* or *tehan*, or perhaps from the old German word (Franco-Theotiscan,) *zehen*, *to draw*, i. e. one from the heap or number: and the participle *tig* or *tigas* in one case, and *zogh* or *zug* in the other, are used in compound terms, as in those for 20, 30, &c. which signify *drawn twice*, *drawn thrice*, and so on: thus *sibuntig* and *sibunzug* are used indifferently in ancient German for 70, in which language we also find *zehenzogh*, or *zehenzug* for 100, equivalent to *taihun taihund*, noticed above.

† Jamieson's *Hermes Scythicus*, p. 199.

‡ Parson's *Remains of Japhet*.

§ Klaproth, *Sprache und Schrift der Uiguren*, Paris, 1820; *Asia Polyglotta*, p. 214. Remusat, *Récherches sur les Langues Tatares*, p. 267.

‖ Klaproth, *Sprachatlas*; *Asia Polyglotta*.

* Klaproth, *Asia Polyglotta*, p. 159.

† Ibid. *Sprachatlas*.

‡ Ibid. *Asia Polyglotta*, p. 284.

Arithmetic.

40. Dütschia	80. Najan.
50. Tabin.	90. Jaran.
60. Djiran.	100. Djan.
70. Dalan.	

Systems adapted to the quinary and vicenary scales.

(22) We could very easily extend to a much greater length our observations upon numeral systems adapted to the decimal scale, as there are few cases in which they may not be made the foundation of some remarks of interest and importance, illustrative of general principles concerned in their formation; but the limits to which we are confined by the very nature of this work, compel us to bring them to a conclusion. We shall now proceed, therefore, to the consideration of the other natural scales of notation, the quinary and the vicenary, which, though incomparably less generally prevalent than the denary, yet are very frequently met with amongst savage and rude people, and sometimes also amongst people considerably advanced in the arts of life; and even amongst civilized people we find traces of their former existence, though subsequently they have been partly or wholly superseded by systems adapted to the decimal scale.

"Aristotle," says Sir Thomas Herbert, "not without good reason admired, that both Greeks and barbarians used a like numeration unto ten; which, seeing it was so universal, could not rationally be concluded accidental, but rather a number that had its foundation in nature."* The passage of the Greek Philosopher, to which this admirable old traveller refers, is found in his *Problems*; and is in every respect so curious, and contains so correct a description of what constitutes a scale of numeration, that we shall give it entire:

Reasons according to Aristotle of the universality of the decimal scale.

Διατί πάντες ἄνθρωποι καὶ βάρβαροι καὶ ἕλληνες, εἰς τὰ δέκα καταριθμοῦσι, καὶ οὐκ εἰς ἄλλον ἀριθμόν, οἷον β, γ, δ, ε· εἶτα πάλιν ἐπαναδιπλοῦσιν, ἓν πέντε, δύο πέντε, ὥσπερ ἕνδεκα, δώδεκα· οὐδ᾽ ἂν ἐξωτέρω παυσάμενοι τῶν δέκα, εἶτα ἐκεῖθεν ἐπαναδιπλοῦσιν· ἔστι γὰρ ἕκαστος τῶν ἀριθμῶν ὁ ἔμπροσθεν, καὶ ἓν ἢ δύο, καὶ εἶτ᾽ ἄλλος τίς· ἀριθμοῦσι δ᾽ ὅμως ὁρίσαντες ἄχρι τῶν δέκα· οὐ γὰρ δὴ ἀπὸ τύχης γε αὐτὸ ποιοῦντες φαίνονται, καὶ ἀεί· τὸ δὲ ἀεὶ καὶ ἐπὶ πάντων, οὐκ ἀπὸ τύχης, ἀλλὰ φυσικόν. Πότερον ὅτι τὰ δέκα τέλειος ἀριθμός, ἔχων γὰρ πάντα τὰ τοῦ ἀριθμοῦ εἴδη, ἄρτιον, περιττὸν, τετράγωνον, κύβον, μῆκος, ἐπίπεδον, πρῶτον σύνθετον; ἢ ὅτι ἀρχὴ ἡ δεκάς; ἓν γὰρ καὶ δύο καὶ τρία καὶ τέτταρα, γίνεται δεκάς· ἢ ὅτι τὰ φερόμενα σώματα ἐννέα; ἢ ὅτι ἐν δέκα ἀναλογίαις τέτταρες κύβικοι ἀριθμοὶ ἀποτελοῦνται· ἐξ ὧν φάσι ἀριθμῶν οἱ Πυθαγόρειοι τὸ πᾶν συνεστάναι; ἢ ὅτι πάντες ὑπῆρξαν ἄνθρωποι ἔχοντες δέκα δακτύλους; οἷον οὖν ψήφους ἔχοντες τοῦ οἰκείου ἀριθμοῦ, τούτῳ τῷ πλήθει καὶ τἆλλα ἀριθμοῦσι.†

The universality of the decimal scale proves, according to Aristotle, that its adoption was not accidental, but had its foundation in some general law of nature: *τὸ δὲ ἀεὶ καὶ ἐπὶ πάντων, οὐκ ἀπὸ τύχης, ἀλλὰ φυσικόν*. This is a most philosophical principle of reasoning, which leads in the present instance to the correct conclusion, notwithstanding the Pythagorean and Platonic dreams about the perfection and properties of the number *ten*, which are thrown out as conjectures to account otherwise for its general adoption. But were there any traces in the time of Aristotle, in the Greek language itself, (we speak not of others,) of the quinary scale, a case to which he alludes? We shall state some reasons for answering this question in the affirmative.

History.

Traces of the quinary scale amongst the Greeks.

In the *Odyssey* of Homer we find the word *πεμπάζεσθαι*, to count by *fives* (*quasi per quinos digitos*,) used as equivalent to *ἀριθμεῖν*. Calypso, speaking of Proteus, making the *tale* of his phocæ, says,

Φώκας μέν τοι πρῶτον ἀριθμήσει καὶ ἔπεισιν
Αὐτὰρ ἐπὴν πάσας πεμπάσσεται, ἠδὲ ἴδηται
Λέξεται ἐν μέσσῃσι, νομεὺς ὣς πώεσι μήλων.

Ὀδυσσ. δ'. 411.

The familiar use of this word, whose derivation is so very obvious, would seem to indicate that the method of counting by fives was common, at least in the time of Homer; and the introduction of the same word by Apollonius in his *Argonautics*,* would prove that the use of it had continued in the poetical, if not in the ordinary, language of Greece to a much later period.

But we have other evidence besides the existence of a word, to show their tendency at least to follow this quinary classification of numbers. In ancient Greek inscriptions, (and some authors assign an antiquity to this practice as remote as the laws of Solon,)† we have 5 and 10 expressed by Π and Δ, the initials of the words Πεντε and Δεκα; 50 was denoted by inscribing the Δ within the Π, and 500, by inscribing within it Η, the initial of Ηεκατον. In other respects this symbolical notation corresponded entirely with the Latin, and in common with it constituted a system for the representation of numbers, which might be considered as quinary subordinate to the denary. With the Greeks this rude method of notation was superseded, except for inscriptions, at a very early period by the more perfect system derived from the Hebrews; but with the latter it remained unchanged to the end of their empire.

Romans.

Phœnician and Palmyrene numeral symbols

We can discover no other trace of the existence of the vicenary scale amongst the Greeks and Romans, either in their numeral language or symbols. If, however, we refer to the East, from whence their alphabets originated, we shall find amongst the Phœnicians a system of numerals, first ascertained by Dr. Swinton‡ from coins found at Sidon, which possess simple symbols for *ten* and *twenty*; by the *latter* of which they proceed as for 100. An examination of Palmyrene inscriptions§ furnishes likewise a system of numerals of great extent, with simple symbols for *five*, *ten*, and *twenty*; but in other respects intimately allied with the former, and proceeding like it according to the *vicenary* scale, within the same limits. The reader will find both these systems in Plate I. Nos. 2 and 3, which are of great interest, not merely from their analogy to the Roman numeral symbols, but likewise as furnishing the key to the numeral systems of the Celtic nations.

The intercourse of the Phœnicians with Spain, Cornwall, and Wales, and more particularly with

* *Some Years Travels into Africa and Asia the Great, &c.* 1677.

† Ἀριστοτέλους Προβλήματων τμημα τοι: ιε.

* Speaking of the streams which flow from the Thermodon, he says,

Τετράκις εἰς ἑκατὸν δεύοντα κεν, εἴ τις ἕκαστα
Πεμπάζοι. *Αργοναυτικων*, β. 976.

† Gatterer, *Artis Diplomaticæ Elementa*, p. 64. Beveridge, *Arithmetices Chronologicæ*, lib. i. 1705; Rose, *Inscriptiones Græcæ vetustissimæ*.

‡ *Philosophical Transactions*, 1758, p. 791.

§ Ibid. 1754, p. 690.

Celtic numeral systems vicenary.

Ireland, is an historical fact attested by innumerable monuments; and the general affinity of structure between the Celtic and Semitic languages, however altered by subsequent intercourse with other people, is of all monuments of their ancient communication with each other, the most permanent and unquestionable. Amongst all the nations of the Celtic race, the numeral language is constructed in conformity with the Phœnician numerals, proceeding by *twenties* as far as 100, and no farther. The following is a list of Welsh, Erse, and Gaelic numerals:

Welsh.	Erse.	Gaelic.
1. Un.	1. Aon.	1. Aon.
2. Dau.	2. Do.	2. Da.
3. Tri.	3. Tri.	3. Tri.
4. Pedwar.	4. Ceatair, or ceitre.	4. Ceithar.
5. Pump.	5. Cùig.	5. Coig.
6. Cwec.	6. Sè.	6. Sia.
7. Saith.	7. Seact.	7. Seachd.
8. Wyth.	8. Oct.	8. Ochd.
9. Nau.	9. Naoi.	9. Nai.
10. Deg.	10. Deic.	10. Deich.
11. Unarzeg.	11. Aon deag.	11. Aon deng.
15. Pymtheg.	15. Cùig deag.	15. Coig deag.
16. Unarpymtheg.	16. Seact deag.	16. Sin deag.
20. Ugain or ugaint.	20. Fitce.	20. Fichid.
21. Un ar ugain.	21. Aon is fitce.	21. Aon thar fichid.
30. Deg ar ugain.	30. Deic ar ficead.	30. Deich thar fichid.
36. Unarpymtheg ar ugain.	36. Seact deag is fitce.	36. Sin deng thar fichid.
40. Deugain.	40. Da fitcead.	40. Da fichid.
50. Deg ar deugain.	50. Deic is da fitcead.	50. Deich thar da fichid.
60. Trigain.	60. Tri fitcead.	60. Tri fichid.
70. Deg ar trigain.	70. Deic is tri fitcead.	70. Deich thar tri fichid.
80. Pedwar ugain.	80. Ceitre fitcead.	80. Ceithar fichid.
90. Deg ar pedwar ugain.	90. Deic is ceitre fitcead.	90. Deich thar ceithar fichid.
100. Cant.	100. Cead.	100. Coig fichid, or ciad.
1000. Mil.*	1000. Mile.†	1000. Deichciad, or mile.‡

All these systems possess much of a common character, and the two last are nearly identical; a circumstance which might be expected, as the Gaelic is a mere dialect of the Erse and an immediate descendant of it. Amongst the Welsh numerals we find a peculiarity, without any corresponding example in any other Celtic dialect; which consists in making *pymtheg* (15) an articulate number, and a point of departure for a new numeration: thus 16 is *un ar pymtheg, one over fifteen*; 17 is *dou ar pymtheg, two over fifteen*; 38 is *tri ar pymtheg ar ugain, three over fifteen over twenty*; 59 is *pedwar ar pymtheg ar deugain, four over fifteen over twice twenty*; and similarly in other cases. The origin of this solitary vestige of the quinary scale in this class of languages is extremely difficult to explain, unless we suppose that their primitive methods of numeration were quinary, subordinate to the vicenary, and that this was a monument of the resistance made by popular habits or prejudices to the partial introduction of the denary scale, from a people more civilized than themselves.

The numeral systems in the Armorican and Basque languages possess a general conformity with those above given, as a small number of their numerals will readily show:

Armorican.	Basque.
1. Unan.	1. Bat.
2. Daou.	2. Bi.
3. Tri.	3. Iru.
20. Hugent.	20. Oguei.
40. Daou hugent.	40. Berroguei.
60. Tri hugent.	60. Iruroguei.

The first of these systems resembles the Welsh, a language with which the Armorican is closely allied: the second, though differing considerably from the former, yet possesses a greater analogy to it than could be expected from the peculiar and insulated nature of this language, so difficult to associate even with the Celtic languages, and still less with those of any other class.

Amongst Scandinavian nations.

The vicenary scale appears to have prevailed very extensively amongst Scandinavian nations, if we may judge from the numerous vestiges of it, not merely amongst them, but likewise amongst those people whose languages are partly derived from them. We have before noticed the curious construction of the Danish numerals between 40 and 100, adapted to this system; and also the preference given to the numbers *twelve* and *twenty* by the inhabitants of Iceland. In our own language also, the word *score*, which originally meant a *notch* or *incision*, has become equivalent to *twenty*, a long mark being made on a tally to signify the successive completion of such a number; a plain indication that such a mode of *scoring** or *counting*, was of all others the most familiar to the habits of our ancestors. In expressing numbers beyond 40, though we do not copy the Danish form of expression for 50, 70, 90, yet in popular language we more readily say *three score*, than *sixty*, *three score and ten* than *seventy*, *four score* than *eighty*, and so on, particularly when such numbers are associated in such a manner, as to be frequently and familiarly used by the humbler and less latinized classes of society. The French have given a still more striking proof of the influence of national habits of thinking and acting upon language; they have made *soixante* a point of departure for a new system of numeration by twenties, expressing 70 by *soixante dix*, 80 by *quatre vingt*, and 90 by *quatre vingt dix*, instead of *septante*, *octante*, *nonante*, the terms which sometimes have been, and which in conformity with the general analogies of the language should be used to express those numbers.

* Owen's *Welsh Grammar and Dictionary*.
† Vallancey's *Irish Grammar*. Neilson's *Irish Grammar*.
‡ Shaw's *Analysis of the Gaelic Language*.

* Amongst other reproaches to Lord Say, which Shakspeare has put into the mouth of Jack Cade, it is said, "and whereas, before, our forefathers had no other books but the *score* and the *tally*, thou hast caused printing to be used: and contrary to the king, his crown and dignity, thou hast built a paper mill." *Henry VI. Second Part.*

Arithmetic.

Other instances.

The examples which we have given, are not the only ones in which the decimal scale has not entirely succeeded in obliterating all traces of the primitive existence of quinary and vicenary systems of numeration, which are so extensively used amongst people in a rude state of civilisation. The Persian term *pendje* signifies *five*, and *péntcha*, the *expanded hand*; and the corresponding terms in the Sanskrit are said to have a similar meaning. The term *lima*, which with very slight modifications is used for *five* throughout the Indian Archipelago and the Islands of the South Sea, means *hand* in the language of the Celebes, Formosa, Otaheite, and many other Islands. Among the ancient Javanese numerals, we find very distinct traces of both these scales; for besides the Sanskrit term *poncho* for five, we find also a simple term *lawe* for twenty-five, the only instance with which we are acquainted of a *secondary* articulate number in the quinary scale, it being usually superseded before it reaches that point by one or other of the other natural scales, again, in the same ancient dialect, we find *likor*, an arbitrary term for twenty, which is frequently used in expressions for *compound* numbers; and also terms for two *secondary* articulate numbers in the vicenary scale; namely, *sa-mas*, *one four hundred*, *do-mas*, *two four hundred*, a circumstance of rather unusual occurrence: the only instance of a *ternary* articulate number in this scale, is to be found in the Azteck, or ancient language of Mexico.

Ende numerals.

The following numerals in the Enda language, a dialect of the Flores in the same group of Islands, shows the operation of the same principle in their formation, though partly derived from the ordinary Polynesian numerals.*

1. Sa	7. Limazua.
2. Zua.	8. Ruabútn.
3. Têln.	9. Trása.
4. Wátn.	10. Sahúln.
5. Lima.	20. Buluzna.
6. Limasa.	100. Sang'asn.

The terms for *six* and *seven*, are equivalent to *five one*, *five two*, in strict conformity with the quinary scale; the term for *eight* is *two four*, a remarkable circumstance, which ought rather to be attributed to the poverty of the language of a rude people, who felt great difficulties in the numeration and expression of very small numbers, than to any natural tendency to proceed by the quaternary scale.†

Numerals of the Islands of the South Seas.

(23.) In examining the numerals of the islanders of the South Seas, we shall find that they very generally exhibit traces of a Malay origin, and that in some cases the denary scale has completely prevailed, and superseded the other natural scales; of this kind are the numerals of the Friendly or Tonga Islands, which are otherwise remarkable for their great extent; in general, however, we shall find that their systems of numeration are denary, subordinate to the vicenary, as may be seen from the following numerals of Otaheite: ‡

In Otaheite.

1. Tahai.	3. Toroo.	5. Rima.	30. Tahai-taon-mara-hourou.
2. Rua.	4. Ita.	6. Wheneu.	32. Tahai-taou-ma-rua.
		7. Hetu.	40. Rua-taou.
		8. Warou.	50. Rua-taou-mara-hourou.
		9. Iva.	60. Torou-taou.
		10. Hourou.	80. Ita-taou.
		11. Ma-tahai.	100. Rima-taon.
		12. Ma-rua.	200. Aou-manna.
		20. Tahai-taou.	2,000. Manna-tine.
		21. Tahai-taou-mara-tahai.	20,000. Torou-tine.

History.

The expression for eleven means *one more*, for twelve *two more*, and so on as far as twenty, which is the true basis of their numeral system. The names for 200, 2000, 20.000, were given by Sir Joseph Banks to Lord Monboddo, and would indicate the resumption of the denary scale beyond 200. But Forster,* the most judicious and philosophical of the observers of the South Sea Islanders, declares that the *teachers* alone can count as far as 200, and that few others can proceed beyond 10; we shall hereafter notice many examples of powers of numeration which are equally confined.

The inhabitants of Otaheite and the Society Islands, the Sandwich Islands, the Friendly Islands, the Marquesas, the Easter Islands and New Zealand, New Guinea, and other Islands in the neighbourhood, belong to the same race, and possess nearly the same numerals, at least for low numbers, differing chiefly in the extent to which the decimal scale has superseded the vicenary. If we turn our attention to the different and less favoured race who inhabit New Caledonia, Tannà, Mallicollo, and the other Islands of the New Hebrides,† we find a difference in their languages and numerical systems, which are chiefly quinary, as will be seen from the following examples:

In New Caledonia, &c.

New Caledonia.	Tannà.	Mallicollo.
1. Pārai.	1. Rettee.	1. Thkai.
2. Pā-ròo.	2. Carroo.	2. Ery.
3. Par-ghen.	3. Kāhār.	3. Erey.
4. Par-bai.	4. Kafā.	4. Ebàts.
5. Pā-nim.	5. Karirrom.	5. Erihm.
6. Pānim-gha.	6. Ma-riddee.	6. Tsukài.
7. Pānim-roo.	7. Ma-carroo.	7. Goory.
8. Pānim-ghen.	8. Ma-kāhār.	8. Goorey.
9. Pānim-bai.	9. Ma-kafā.	9. Goodbàts.
10. Pārooneek.	10. Karirrom-karirrom.	10. Seneàm.

In the first of these systems, six, seven, eight, and nine, are expressed by *five one*, *five two*, *five three*, and *five four*; in the second, by *more one*, *more two*, *more three*, *more four*; in the third, by the combination of the word *goo*, of which we do not know the meaning, with *one*, *two*, *three*, *four*; in the second, ten is expressed by the repetition of the term for five, an example of which we recollect to have seen somewhere in the numerals of a tribe in Africa. In every respect, indeed, the formation of these quinary systems, as far as they proceed, is as regular and systematic, as any of the denary systems which we have examined; and are equally, if not more completely derived from practical methods of numeration.

Labillardière † has given the numerals of New

* Raffles, *History of Java*, vol. ii. App. F.

† Crawfurd's *Indian Archipelago*, vol. i. p. 256.

‡ Monboddo, *Origin and Progress of Language*, vol. i. p. 544; Cook's *Voyages*.

* *Observations made during a Voyage Round the World*, by John Reinhold Forster, p. 528.

† Ibid. p. 284.

‡ *Voyages d'Entrecasteaux*, vol. ii. App.

Arithmetic.

Caledonia as far as forty, though it is evident from an examination of them, that they are little more than a repetition of the first ten numerals; the form also under which they appear in his work, is so very different from that given above, that it is extremely difficult to recognise in them a common character, farther than that of being adapted to the same scale; an instance, amongst a thousand others which might be produced, of the impossibility of forming correct vocabularies of languages, by persons who have not been habituated, from long intercourse, with the native sounds.

Numerals in North-eastern Asia.

(24.) We shall find many examples of numerals adapted to this scale amongst the miserable tribes who inhabit the north-eastern parts of Asia. Of the following examples, the first are the numerals of the continental Koriaks to the north of Kamtschatka; the second, of the Koriaks of the Island of Karaga; the third, of the Tschutki, on the Anadyr, who inhabit the western part of the north-eastern angle of the continent of Asia.*

1. Onnen.	1 Ingsing.	1. Innen.
2. Hyttaka.	2. Gnitag.	2. Nirach.
3. Ngroka.	3. Gnasog.	3. N'roch.
4. Ngraka.	4. Gnasag.	4. N'rach.
5. Myllanga.	5. Monlon.	5. Myllygen.
6. Onnan-myllanga.	6. Ingsinagasit.	6. Innan-myllygen.
7. N'jettan-myllanga.	7. Gnitagasit.	7. Nirach-myllygen.
8. Ngrok-myllanga.	8. Gnasogasit.	8. Anwrotkin.
9. Ngrak-myllanga.	9. Gnasagasit.	9. Chonatschinki.
10. Myngytkan.	10. Damalagnos.	10. Myngyten.

Of these numeral systems, which possess much of a common character, the first is formed in the most regular manner; in the second the name *monlon* for *five* is replaced by *gasit* in the compound words; in the last, the expressions for numbers according to the quinary scale, is interrupted after 7, and 8 and 9 are expressed by words which have no connection with those which precede them; in all these cases the name for *ten* is an independent word; in these instances, as well as in many which will follow, we are deprived of much interesting information respecting the methods of numeration of these primitive people, by our entire ignorance of the etymology and grammatical construction of their languages.

In Kamtschatka.

The following numerals of the inhabitants of the north and south of the peninsula of Kamtschatka are remarkable, as the names for 8 and 9 alone are adapted to the quinary scale, whilst those for other numbers, with the exception perhaps of that for 7, in the first decad, are apparently independent.

1. Konni.	1. Dischak.
2. Kuscha.	2. Kascha.
3. Tschok.	3. Tschook.
4. Tschak.	4. Tschaaka.
5. Koshleh.	5. Kumnaka.
6. Kylkoch.	6. Ky'lkoka.
7. Ngtonok.	7. Itátyk.
8. Tschook-tonok.	8. Tschookotuk.
9. Tschaak-tonok.	9. Tschaktuk.
10. Tuta.	10. Komechtuk.*

* Klaproth, *Sprachatlas*, 56.

History.

If the following account of the method of counting of these people be correct, it would appear that they adopt the method which would naturally lead to the vicenary scale, and which in every instance may be considered as its foundation. "It is very amusing to see them attempt to reckon above ten: for having reckoned the fingers of both hands, they clasp them together, which signifies ten: they then begin at their toes and count to twenty; after which they are quite confounded, and cry *matcha*, that is, where shall I take more."†

Numerals of the Polar American tribes.

(25.) The Greenlanders, the Esquimaux, the inhabitants of Norton Sound, of the Aleutian Islands, of Kadjak and the other Fux Islands, and of the sea coast of the north-east angle of Asia, bordering on the Tschutki of the Anadyr, constitute a distinct and common race, who may be properly termed Polar Americans, equally remarkable for their very limited powers of numeration, and for the extreme poverty of their numeral language.

In Greenland.

The Greenlanders, according to the relation of the Moravian Missionary Crantz,‡ who resided for many years amongst them, in counting commence with the fingers on the left hand, and thence proceed to those of the right, naming the first ten numerals as follows:

1. Attausek.	6. Arbennek.
2. Arlaek.	7. Arlæk.
3. Pingajuah.	8. Pingajuah.
4. Sissamat.	9. Sissamat.
5. Tellimat.	10. Tellimat, or Kollit.

They afterwards proceed to the toes of the feet, and the second series as far as 19 are expressed as follows:

11. Arkanget.	16. Arbasanget.
12. Arlæk.	17. Arlæk.
13. Pingajuah.	18. Pingajuah.
14. Sissamat.	19. Sissamat.
15. Tellimat.	

These names are mere repetitions of the names of the first five digits, with a slight variation in those of six, eleven, sixteen, to distinguish the series of which they form successively the commencement: the term for 20, the completion of those members of the human body which are employed in this natural process of numeration, is *innuk* or *man*; for 40, they use the expression *innuk arlæk, two men*; for 100, *innuk tellimat, five men*; but beyond 20 they proceed with great difficulty and reluctance, and generally apply to such numbers a term which signifies *innumerable*.

There are other examples of the identity of the terms for *man* and for *twenty* amongst the tribes of South America, originating in the same method of numeration. Thus, in the numerals of the Jaruroes *canipume, man*, is the term for 20, and *noenipume* (*noeni* 2) *two men* is the term for 40.§

The Esquimaux, according to the relation of Captain Parry,‖ are still more limited in their power of nume-

* Klaproth, *Sprachatlas*, p. 16.

† *Account of Russian Discoveries in Annual Register for* 1764, App. 4.

‡ *Account of Greenland*, vol. i. p. 208.

§ Humboldt, *Vues des Cordillères*, p. 253.

‖ *Second Voyage*, p. 556.

Arithmetic.

Amongst the Esquimaux.

ration than the inhabitants nf Greenland; the first five mmnerals are,

1. Attôwsak.
2. Mädleroke, nr Ardlek.
3. Pingabake.
4. Sittamat.
5. Tèd-leē-mà.

They nsually express the remaining nnmerals of the deead by the repetition nf the first five; in some cases they use the term Argwĕnrŭk for 6, and Argwĕnrŭk tōwa for 7; and when reference is made to the fingers nn the right hand, they express 8, 9, and 10, by

Kittŭklee-moot,
Mikkeelnkka-moot,
Eĕrkit-koke,

which are derived from the names for 2d, 3d, and 4th fingers, which are,

2d, Keituk-lie-rak.
3d, Mikkēe-lie-rak.
4th, Irkit-kōb.

In counting as far as three, they make use of their fingers, and generally make some mistake before they reach 7; beyond 9, they hnld np both hands; and if 15 or 20 are required, they make another person do the same, but never resort to the toes nf the feet; they feel greatly distressed to go beyond 10, and generally ery out *oonōoktoot*, which may mean any oumber between 10 and 10,000.

The numerals of the Eastern Tschutki, of the inhabitants of Kadjak, the principal of the Fox Islands, and of Nurton Sonnd, sufficiently resemble the preceding to prove them to be the same people.

Eastern Tschutki.*	Kadjak.†	Norton Sound.
1. Atashek.	1. Ataudsen.	1. Adowjnk.
2. Malgok.	2. As'loka.	2. Arba.
3. Pigajut.	3. Pingaswak.	3. Piogashook.
4. Ishtamat.	4. Itamik.	4. Sissamat.
5. Tatlimat.	5. Tallmik.	5. Dallamik.
6. Sewinlak.	6. Aghoiljujnn.	
7. Malgok.	7. Mall'ehonghin.	
8. Pigajuk.	8. Pengtjujun.	
9. Aglinlikn.	9. Kuln'ghaen.	
10. Kulla.	10. Kulen.	

Numerals of the central tribes of North America.

(26.) If we advance southwards from the Pale, from the fishing to the hunting tribes of North America, wa shall find, as we have before remarked, the decimal scale generally prevalent, and in most eases their numeral systems perfectly regular, and eumprehending large oumbers; in some lnstances, however, we may discover traces of the qninary scale in the formation of the numerals between five and ten; thus, amongst the following numerals of the Delaware Indians, those for 6, 7, 8, are modified forms nf those for 1, 2, 3.

1. Cintta.	6. Cinttas.
2. Nissa.	7. Nissas.
3. Naha.	8. Naas.
4. Nuec-oo.	9. Paes-chnn.
5. Pa-reen-ach.	10. Thae-raen.

Numerals of African Tribes.

(27.) Amongst the innumerable languages of Africa, we find many examples of quinary numeral systems terminnting, as they always do, in the denary or vicenary scales; of the first kind are the nnmerals of the Jaloffs, one nf the nations visited by Park in his first jnurney.

History.

Jaloffs.

1. Ben, or Benna.	15. Fook agh juorom.
2. Niar.	16. Fook agh juorom ben.
3. Nyet.	20. Nitt, or Niar fook.
4. Nianet.	30. Fanever, or Nyet fook.
5. Juorom.	40. Nianet fook.
6. Juorom ben.	50. Jnorom fook.
7. Juorom niar.	100. Temier.
8. Juorom nyet.	200. Niar temier.
9. Juorom nianet.	1000. Djoone.
10. Fook.	1100. Djoone agh temier.*
11. Fook agh ben.	

The wnrd for 5, *juorom*, likewise signifies hand, and the system is in every respect a perfect example of the union of the qulnary and denary scales, the first being subordinate to the other.

Foulahs.

The numerals nf the Fonlahs, a neighbouring tribe, though essentially different from the preeeding, are of the same eharacter.

1. Go.	6. Jego.
2. Deeddee.	7. Jedeeddee.
3. Tettee.	8. Je tettee.
4. Nee.	9. Je nee.
5. Jonee.	10. Sappo.

In ordinary cases, says Winterbnttom,† they reckon by the fingers of the hands, first on the right hand, and secondly nn the left; hnt in trading and in other occasions, where accorate nnmeration is important, they nse small pebbles, gun flints, nr the kernels of the palm nut, which they dispose in heaps of 5 and 10; thus showing that their practical methods of eounting accurately coineide with their numeral language.

Of the same kind are the numerals of the Jallonkas and Feilnps, two tribes visited by Park, and of the inhabitants of the coast of Lagoa Bay.

Jallonka.	Feilups.	Lagoa Bay.
1. Kidding.	1. Enory.	1. Chingea.
2. Fidding.	2. Cookaba.	2. Seberey.
3. Sarra.	3. Sisajee.	3. Triarou.
4. Nani.	4. Sibakeer.	4. Moonau.
5. Sooin.	5. Footuck.	5. Thanou.
6. Seni.	6. Footuck enory.	6. Thanou-na-chingea.
7. Soolo ma fidding.	7. Footuck cookaba.	7. Thanou-na-seberey.
8. Soolo ma sarra.	8. Footuck sisajee.	8. Thanou-na-triarou.
9. Sooln ma nani.	9. Footuck sibakeer.	9. Thanou-na-moonau.
10. Foo.	10. Sibankonyen.	10. Koomoo.

It is very seldnm that their numerals are given to a sufficient extent to enable us to judge whether they proceed by the denary or vicenary scale. We know but of one case nf the latter kind, in the numerals of the Mandingoes, the first ten nf which we have given before. (Art. 10.)

* Klaproth, *Sprachatlas*, p. 56.
† Ibid. *Asia Polyglotta*, p. 325.

* *Classical Journal*, vol. v.
† *Account of Sierra Leone*, vol. i. p. 174.

Arithmetic. Mandingoes.

11. Tang killin.
20. Mulu.
30. Mulu nintang.
40. Mulu foola.
50. Mulo foola nintang.
60. Mulu sabba.
70. Mulu sabba nintang.
80. Mulu nani.
90. Mulu nani nintang.
100. Kemi.
1000. Ali.*

Azteck numerals.

(28.) Of all numeral systems adapted to the vicenary scale, the most perfectly developed is the Azteck, or ancient Mexican, proceeding as far as an articulate number of the third order; the numerals are as follow:

1. Ce.	15. Matlactli oz chicuace.
2. Ome.	16. Matlactli oz chicome.
3. Jei.	20. Pohualli, or cem-pohualli.
4. Nahui.	30. Cem-pohualli oz matlactli.
5. Macuilli.	40. Om-pohualli.
6. Chicuace.	50. Om-pohualli oz matlactli.
7. Chicome.	60. Jei-pohualli.
8. Chicuei.	80. Nahui-pohualli.
9. Chicuhnahui.	100. Macuilli-pohualli.
10. Matiactli.	400.
11. Matlactli oz ce.	800. Xiquipilli.†
12. Matlactli omome.	
13. Matlactli oz jei.	
14. Matlactli oz nahui.	

We are obliged to omit the name for *four hundred*, as it is not mentioned by Humboldt, from whose splendid works these numerals are taken; and we have in vain searched for a Mexican grammar, or vocabulary, in many of the principal libraries of this country. In the same author we find an account of the symbols employed for numbers in their hieroglyphical writing, which exactly corresponded with their numeral language. A small standard, or flag, denoted 20; if divided by two cross lines, and half coloured, it represented half twenty, or 10; and if three quarters coloured, it denoted 15. The square of twenty, or 400, was denoted by a *feather*, because grains of gold enclosed in a quill, were used in some places as money, or a sign for the purposes of exchange. The figure of a sack indicated the cube of twenty, or 8000, and bore the name of *Xiquipilli*, given also to a kind of purse that contained 8000 grains of cacao. These symbols were repeated twice, thrice, four times, &c. to denote multiples of them by 2, 3, 4, &c.; and grouped together, like the common symbols, to denote any compound number.

Their numerical hieroglyphics.

Muysca numerals.

(29.) The Chibcha or Moysca language, of the Indians of Bogota, in New Grenada, exhibits a numeral system adapted to the same scale, to which the denary alone is subordinate, and which merits consideration on more accounts than one. The following are the numerals:

1. Ata	4. Muyhica.	7. Cahupqua.	21. Guetas asaqui ata.
2. Bosa.	5. Hisca.	8. Suhuzu.	22. Guetas asaqui bosa.
3. Mica.	6. Ta.	9. Aca.	30. Guetas asaqui ubchica.
		10. Ubchica.	40. Goe-bosa.
		11. Quicha ata.	60. Gue-mica.
		12. Quicha bosa.	80. Gue-muyhica.
		13. Quicha mica.	100. Gue-hisca.
		15. Quicha hisca.	
		20. Quicha ubchica, or gueta.	

History.

The term *ubchica*, after the first decad of numerals, is replaced by *quicha* in the second decad, which means *foot*; thus the expressions for 11, 12, &c. mean *foot one, foot two*, &c. being accurately significant of their primitive methods of numeration. Twenty is expressed either by *quicha ubchica, foot ten*, or by *gueta*, which signifies *house*; forty, by *two houses*; sixty, by *three houses*; and similarly for higher articulate numbers in the same series.

Their meaning.

Humboldt has given from the researches of Duquesne, a Canon of the Metropolitan Church of Santa Fé de Bogota, the etymological significations of most of these numerals. Thus *ata* signifies *water*; *bosa*, *an enclosure*; *mica*, *changeable*; *muyhica*, *a cloud threatening a tempest*; *hisca*, *repose*; *ta*, *harvest*; *cahupqua*, *deaf*; *suhuzu*, *a tail*; and *ubchica*, *resplendent moon*. No meaning has been discovered of *aca*, the numeral for 9. It is impossible amidst meanings so various, to recognise any principle which may seem to have pointed out the use of these terms as numerals; and it is making little advance towards an explanation of the difficulty to say, with Duquesne, that the words relate either to the phases of the moon in its increase or wane, or to objects of agriculture or worship; as far as their signification as numerals are concerned, they may be considered as perfectly arbitrary; and it is in vain to attempt any probable theory for the explanation of a fact, where there is no analogy to guide us, except perhaps the very imperfect one which is furnished by the ordinary meanings of the second series of Chinese numeral symbols.

Muysca numerical hieroglyphics.

The same people possessed hieroglyphical symbols for the first ten numbers, and for twenty, which are given in Plate I. fig. 5. In the Mexican numeral symbols, there is an intelligible connection between the sign and the thing signified; but if the following explanations given to Duquesne, by some Indians who were instructed in the calendar of their ancestors, be correct, it is impossible to conceive any association which is more perfectly arbitrary. Thus the hieroglyphic for *one*, is *a frog*; for *two*, *a nose with extended nostrils*, part of the lunar disk, figured as a face; for *three*, *two eyes open*, another part of the lunar disk; for *four*, *two eyes closed*; for *five*, *two figures united*, the nuptials of the sun and moon, conjunction; for *six*, *a stake with a cord*, alluding to the sacrifice of Guesa tied to a pillar; for *seven*, *two ears*; for *eight*, no meaning assigned; for *nine*, *two frogs coupled*; for *ten*, *an ear*; for *twenty*, *a frog extended*. It would be difficult, for a common observer, to discover in these symbols the objects mentioned in the preceding explanations of them; but, in answer, it may be said, that their forms have degenerated from long use, and consequently furnish no decisive argument against the correctness of their traditional interpretation; and that the same objections would apply

* Jackson's *Account of Marocco*, p. 226.
† Humboldt, *Vues des Cordillères*, p. 141 and 251.

Arithmetic. to the present explanation of the *keys* of the Chinese symbols, however certainly derived, in many instances at least, from rude imitations of natural objects.

Chandra Sangkala of Java.

It might be imagined that there existed some analogy between this use of words as numerals, which have other significations, and the custom which has prevailed among the Javanese from very remote antiquity, denominated *chándra sangkála*, "*reflections of royal times*," or *the light of royal dates*.* It consists in attaching the names of various objects, or things, or their representations, to the nine digits and zero, twenty or more being assigned to each of them; and in expressing a date, to select such of them as may form a sentence, significant of the event which it commemorates. Thus the date (1400) of one of the most calamitous events of their history, is expressed thus:

Sirna ilang Kertaning Búmi.
Lost *and* gone *is* the pride *of* the land.
0 0 4 1

Thus *Búmi* is one of the words significant of unity; *kertaning*, of four; *ilang* and *sirna*, of zero. Again, the date (1313) on the tomb of the Princess Chermai is thus stated:

Káya wulan putri iku.
Like *unto* the moon was *that* princess.
3 1 3 1

Where *iku* and *wulan* are significant of unity, and *putri* and *kaya* of three. This practice constitutes a technical memory of a very elegant and amusing nature, and reminds us rather of the literary luxury of a refined people, than of the efforts of a primitive nation, to pass from practical methods of numeration to numerical language.

The Mexicans, Muyscas, and Peruvians, constituted the only three nations of ancient America, who possessed governments regularly organized, and who had made considerable progress in many of the arts of civilized life, in architecture, sculpture, and painting. They were the only people, in short, in that vast continent, who could be considered as possessing literary or historical monuments. On this account alone their numeral systems would merit very particular attention; but still more so from their perfect developement. The first presents the most complete example that we possess of the vicenary scale, with the quinary and denary subordinate to it. The second, of the same scale, with the denary alone subordinate to it; whilst the third, or Peruvian, is strictly denary, and is equally remarkable for its great extent and regularity of construction.

Of other South American tribes.

(30.) It is the latter scale which is of rare occurrence amongst American tribes, the vicenary being much more generally prevalent in their numeral systems; so much so indeed as to be almost characteristic of them. In proceeding to a farther consideration of them, we must again lament our inability to procure access to vocabularies, or grammars, of these languages, in consequence of which we are compelled to pass over a subject of very great interest in a very cursory and imperfect manner, having been only able to collect a very small number of disconnected facts which have reference to it.

Dobrizhoffer† has given an account of the numeral systems of the Abipones and Guaranies, amongst whom he resided for many years, and with whose habits and language he was intimately acquainted: the first are an equestrian people of Paraguay, whose predatory habits long made them formidable to the Spaniards and neighbouring tribes. The first five numerals are expressed by

History.

Abipones.

1. Iñitara.
2. Iñoaka.
3. Iñoaka yekaim.
4. Geyenk ñate.
5. Neenhalek.

The names for 1, 2, 3, have no reference to natural objects; the expression for 4, means the fingers of the *emu*, a bird extremely common in Paraguay, possessing *four* claws on each foot, *three* before and *one* turned back; whilst that for *five* is the name of a beautiful skin with *five* different colours. The same number, however, is more commonly expressed by *hanam begem*, *the fingers of one hand*; to express numbers between *five* and *ten*, they combine the name for five with the inferior units; *ten* is expressed by *lanam rihegem*, *the fingers of both hands*; and for *twenty*, they say *hanam rihegem cat grachaha'ka anamicheri hegem*, *the fingers of both hands and feet*.

Guaranies.

The Guaranies are another tribe of Paraguay, who speak a language which is the mother of many other dialects, yet they possess only four independent numerals.

1. Petey.
2. Mokoy.
3. Irbohossi.
4. Irundy.

Tupi.

If we pass further north to the Tupi, a very numerous tribe in Brazil, speaking a kindred language to the former, we find only five independent numerals.*

1. Auge-pe
2. Mocouein.
3. Mossapnt.
4. Oioicoudie.
5. Ecoinbo.

Maco Macoes.

Humboldt interrogated a native of the Maco Macoes, a tribe on the Orinoco, who knew no names for numbers beyond four.

1. Niante.
2. Tojus.
3. Percotahuja.
4. Inantegroa.†

Caribbees and Gali

The Caribbees who constituted the native population of Barbadoes, St. Christopher's, Antigua, and the other Islands of the Caribbean Sea, and who, under the name of Galibi, are dispersed extensively over the adjoining continent, and form one of the finest of the American tribes, are equally limited in their names for numbers.‡

1. Aban.
2. Bean.
3. Eleoan.
4. Beambourl.

In all these cases, the numeration beyond five is carried on by means of the fingers and toes, and their numeral language becomes generally, as in the case of

* Raffles, *Java*, vol. i. p. 372. and vol. ii. App. G.
† *History of the Abipones.*

* Southey's *History of Brazil*, vol. i. p. 226.
† Humboldt's *Personal Narrative*, vol. v. p. 125. English edition.
‡ Raymond, *Histoire des Caraibes*, 1665.

Arithmetic.

the Abipones, descriptive of their practical methods of counting; thus amongst the last mentioned people, to express five, they show the fingers of one hand, and for ten, the fingers of both hands; "for twenty, their expression is pleasant," says Davies,* "being obliged to show all the fingers of their hands and the toes of their feet."

In the languages of these rude tribes, abstract terms are almost entirely unknown, and their expressions from mere poverty, in many cases assume a highly figurative form, being obliged to refer to natural objects and the most common relations of life, to express ideas which do not otherwise come within the compass of their languages; thus in the Caribbean language, the fingers are termed the children of the hand, and the toes the children of the feet; and the phrase for ten, *chon oucabo raim, all the children of the hands.*

Achaguas.

There is no difficulty in producing other examples of numeral language constructed in this manner, and equally descriptive of practical methods of numeration. The Achaguas, a tribe on the Orinoco, express *five* by *abacaje*, or *the fingers of one hand*; *ten*, by *tucha macaje*, *all the fingers*; *twenty*, by *abacaytacay*, or *all the fingers and toes*; *forty*, by *incha matacacay*, or *the fingers and toes of two men*; and so on for very large numbers;† among the Zamucoes, as well as the Muyscas, *five*, is *the hand finished*; *six*, *one of the other hand*; *ten*, *the two hands finished*; *eleven*, *foot one*; *twelve*, *foot two*; *twenty*, *the feet finished*.‡ It is evident that this absence of abstract and independent terms for numbers, and the tedious circumlocutions which it occasions, must form an insuperable obstacle to the expression of large numbers in such languages.

Zamucoes.

Brazilians of Pernambuco.

In the collection of Theodore de Bry, there is an account of the inhabitants in the neighbourhood of Pernambuco in Brazil, by a German Jesuit of the name of Stadius, containing the following statement of their methods of numeration, which is applicable to many other American tribes: *numeros non ultra quinarium notant: si res numerandæ quinarium excedant, indicant eos digitis pedum et manuum pro numeris demonstratis: quod si numeros et horum multitudinem excedat, conjungunt aliquot personas et pro multitudine digitorum in illis res notant et numerant.*§

Practical methods of counting among the Guaranies.

(31.) The practical methods of counting of American tribes, however, are not in all cases restricted to the fingers and toes, and their numeration is not necessarily confined to twenty, the radix of their scale, when destitute of the aid of names, whether arbitrary or not, for higher numbers, or when they cannot call in the assistance of other persons. The Guaranies make heaps of maize, each consisting of twenty grains, two, three, four, &c. of which are used to denote 40, 60, 80, &c. the excess above any one of this series of articulate numbers being reckoned in the ordinary way: the same custom prevails in other parts of that continent, and we are reminded of it in the Mexican hieroglyphical symbols.

The ancient Peruvians possessed practical methods of numeration equally perfect with those of the Greeks and Romans, and incomparably superior to those of any other American nation: the *Quipus* were *knots*, nine in number, movable upon a string like the beads of a rosary, which was attached by one end to a rod; of these strings there was one for units, and one for each of the successive orders of superior units as far as one hundred millions. The use of the *quipus* was nearly the same as that of the Roman abacus; and it not only enabled them to express any number, but likewise to perform the ordinary arithmetical operations of addition, subtraction, multiplication, and division. *Knots* of peculiar and different colours appear to have been used in the numeration of different objects, whether of gold, silver, &c. and to have been appropriated to them.*

History. Peruvian Quipus

The whole business of calculation appears to have been confided to the *Quinpucamoya*, or *guardians of the quipus*; and the reports of the early historians of this empire bear testimony to the rapidity and accuracy of their operations. We are not aware of the existence of any similar practice among other American nations. Marsden, in his account of Sumatra, has noticed a practice which bears some analogy to it, where it is usual to denote the completion of a *tale* of one hundred, by making a knot in a string, which is repeated as often as necessary; such *knots*, or *quipus*, are made use of not merely as an assistance to the memory in the process of numeration, but likewise as *records* or *accounts* of numbers.†

(32.) It was an opinion maintained by that singularly paradoxical writer De Pauw, that no indigenous nation of America could reckon in their own idiom beyond *three*;‡ the facts, however, given above, are more than sufficient to refute such an assertion; though it must be allowed, that the numeral systems of the South American tribes are remarkably limited in absolute extent, and still more so in arbitrary and independent words: it is to the latter chiefly that De Pauw refers, and there are some examples which might appear to bear out his assertion: of this kind are the numerals of the Abipones mentioned above, and the celebrated example of the Yancos on the Amazon, whose name for three is

The arithmetic of South American tribes extremely limited.

Poettarrarorincoaroac,

of a length sufficiently formidable to justify the remark of La Coudamine: *Heureusement pour ceux qui ont à faire avec eux, leur Arithmetique ne va pas plus loin.*§

All travellers have borne testimony to the extreme difficulty which these South American tribes usually experience in attempting to count even small numbers; they are indolent from constitution and habit, and are reluctant to enter upon any exercise of the mind which requires the least effort of abstraction. Dobrizhoffer relates of the Abipones, that they could rarely count as far as ten. When attempting, upon their return from their expeditions, to give an idea of the number of their enemies, or of the horses they had captured, they would mark out a space, and say that they were as many as could stand within it.

* *History of Barbadoes, St. Christopher's, Antego, Martinico, Montserrat, and the rest of the Caribby Islands*: Englished by John Davies, of Kedwilly, 1666.
† Southey's *History of Brazil*, note, p. 638.
‡ Humboldt, *Vues des Cordillères*, &c. p. 253.
§ *Americæ Descriptio*, vol. i. part iii. p. 128.

* *Histoire des Yncays Roys de Peru*, p. 680. 1633.
† Marsden's *Sumatra*, p. 192. In counting money, each tenth and sometimes also each hundredth piece is put aside.
‡ *Récherches Philosophiques sur les Americains*, vol. ii. p. 162.
§ La Condamine, *Voyage de la Riviere des Amazons*, p. 64.

Arithmetic.

On one occasion, when he accompanied a party of ten upon a defensive expedition, he mentions the following dialogue as having taken place between them: "*Are we many?*" "*Yes, you are many.*" "*Are we innumerable?*" "*Yes, you are innumerable.*" So sensible, indeed, were the Missionaries throughout Paraguay and Brazil, of this deficiency of the natives, that it is a general practice in the churches of the several *Reductions*, to teach, or attempt to teach them to count as far as two hundred in the Spanish or Portuguese language.

In the account of the Caribbees which we have referred to above, it is said, that in counting numbers beyond ten, they generally get confused, and exclaim, "in their gibberish," as Davies expresses it, *tamigati cati nitibouri bali, they are as many as the hairs of my head, or the sand on the sea shore.*

The general testimony of Humboldt is decisive of the same fact; he declares that he never met with a native Indian who, if asked his age, would not answer indifferently 16 or 60:* he at the same time observes, that this is the case even amongst tribes who possess a numeral language which embraces very high numbers; may we not, however, reasonably suspect, that the existence of such terms rests in general upon very insufficient authority? or that the individuals whom he interrogated were less skilled than others of their countrymen in the practice and language of numeration? For it is absurd to suppose, that terms exist among such rude people to which they can attach no meaning.

We have given examples of people whose powers of numeration are equally confined with those who are the subject of our present discussion, particularly amongst the Polar Americans; and it would not be difficult to produce other instances which are equally remarkable. The natives of New South Wales possess no numerals beyond those which follow:

Numerals of natives of New Holland.

1. Wagul.
2. Boola.
3. Brewy.†

When a number exceeds *three*, they use the phrase *murray-loolo*, which signifies an indefinite number. We know, however, from the authority of a gentleman who has long filled an official situation in that colony, that they count to higher numbers by means of the fingers. For five, they hold up the expanded hand; for ten, both the hands; for greater numbers, they avail themselves of the hands of another person, in the same manner as the Esquimaux, and in this manner they are enabled to proceed as far as twenty or thirty.

Konssa Caffres & Hottentots.

The Konssa Caffres, as well as the Hottentots, according to the authority of Lichtenstein,‡ have no numeral beyond ten, though some authors have extended it to 100; whenever they express a number, they raise up the like number of fingers; so indistinct and imperfect is the impression conveyed to the minds of these rude people by an abstract term, unaided by an appeal to the senses.

It is mentioned by Suidas,§ that the ancient comic poets, amongst other marks of stupidity which they attributed to one Melitides, asserted that it was only after long and diligent teaching that he counted as far as *five*; and Aristotle, at the conclusion of the passage which we have quoted above, on the universality of the decimal scale, says that a certain tribe of Thrace formed the only exception, whose numeration was limited to four: *Μόνοι δὲ ἀριθμοῦσι τῶν Θρακῶν γένος τι εἰς τέτταρα, διὰ τὸ, ὥσπερ τὰ παιδία, μὴ δύνασθαι μνημονεύειν ἐπιπολὺ, μηδὲ χρῆσιν μηδενὸς εἶναι πολλοῦ αὐτοῖς.*

History.

A Thracian tribe mentioned by Aristotle.

This passage is curious, as showing that even amongst the Greeks some attention was paid to the methods of numeration of barbarous nations; and though we might admit the fact, however contrary to modern observation, yet we certainly must dispute the correctness of the conclusion, that their powers of numeration were limited to four, because they never felt either the want or the use of higher numbers.

The natural scales alone are national.

(33.) The mention of this passage of Aristotle naturally leads us to the consideration of the question, whether in any modern instance, any other than the natural scales of notation have ever prevailed in any nation whatsoever? whether, in short, there is any limitation to the first of the general propositions which are stated in Art. 8? The examples which we have hitherto produced, are strongly confirmatory of its being universally true; and show, that though in some cases numerical language may fail in reaching even the radix of the lowest of these scales, yet that there is no exception to the existence of practical methods by which the numeration is extended, at least as far as ten, if not much farther; and that these methods are essentially adapted to the natural scales, and furnish indeed the foundation of them.

Alleged instances of other scales.

In parcelling out certain objects, it very commonly happens, that a particular number of them are united or associated together, and the lot designated by a peculiar name: thus, *pair, couple, brace* are synonymous terms: but the associations which our habits have long connected with them, would not allow of their being interchanged with propriety in the expressions, *a pair of horses, a couple of dogs*, and *a brace of partridges*. The term *leash* is of still more restricted application; whilst *warf*, (from the German *wurfen*, to cast,) or *cast*, is appropriated to the *four herrings* which the fisherman throws at a time, two in each hand, in making his tale.* Terms of this kind, which are not perfectly abstract, afford no proper evidence of the existence of the binary, ternary, or quaternary scales of notation, as the process of classification is generally terminated at the very first step, and does not proceed to articulate numbers of the second or higher orders. We may sometimes hear such an expression as *pair of pair, couple of couple*, but never *brace of brace, leash of leash*, a *warf of warf*; as the last set of expressions would indicate a degree of abstraction in the terms which they never possess. If men were all sportsmen or fishermen, and the only objects which required numeration were birds or fish, one might possibly conceive that the accidental circumstances which lead to this primary classification of such objects, might have been followed to a sufficient extent to form a ternary or quaternary scale; but in no other manner could we conceive such scales to be generally adopted, which have no foundation in those practical methods of numeration which are pointed out by nature herself.

* *Personal Narrative*, vol. v. p. 125. English edition.
† Collins's *New South Wales*, App.
‡ *Travels in Southern Africa*, vol. i. App.
§ In voce γέλοιος.

* Leslie's *Philosophy of Arithmetic*, p. 3.

Arithmetic.

It is mentioned by Crawfurd,* that the woolly haired races who inhabit the mountains of the peninsula of Malacca, have no native terms for numbers beyond two; that for *one*, being *nai*, and for *two*, *bu*, which likewise signifies *second born*: for higher numbers they use the common Polynesian numerals; such an example furnishes no proof of the existence of the binary scale amongst these people; and even granting that native terms for higher numbers never existed, and were not superseded by those of a predominant language, the case is merely analogous to many others which we have mentioned, where numeral language had not kept pace with practical methods of numeration.

Binary Arithmetic of Leibnitz.

(34.) Though it is in vain to look for the binary Arithmetic amongst the primitive institutions of nations, yet its adoption has been recommended in later times by the celebrated Leibnitz, as presenting many advantages, from its enabling us to perform all the operations in symbolical Arithmetic, by mere addition and subtraction: it requires the use but of two symbols for zero and unity, which are adequate to the expression of all numbers.

As unity was considered the symbol of the Deity, this formation of all numbers from zero and unity was considered in that age of metaphysical dreaming, as an apt image of the creation of the world by God from chaos. It was with reference to this view of the binary Arithmetic, that a medal was struck bearing on its obverse, as an inscription, the Pythagorean distich,

Numero Deus (1) *impari gaudet;*

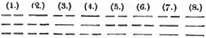
and on its reverse, the appropriate verse descriptive of the system which it celebrated,

Omnibus ex nihilo ducendis sufficit Unum.†

This invention was studiously circulated by its author by means of the scientific journals, and his extensive correspondence;‡ it was communicated by him to Bouvet, a Jesuit Missionary at Pekin, at that time engaged in the study of Chinese antiquities, and who imagined that he had discovered in it a key to the explanation of the Cova, or *lineations* of Fohi, the founder of the Empire. They consist of eight sets of three lines, either entire or broken, arranged in the following manner, or in a circle.

The cova or suspended symbols of Fohi.

(1.) (2.) (3.) (4.) (5.) (6.) (7.) (8.)



If we suppose the broken lines to represent *zero*, and the entire line *unity*, and that it possesses value from its position, increasing as it descends, these *lineations*, would severally become in the binary arithmetical notation, 0, 1, 10, 11, 100, 101, 110, 111, or 0, 1, 2, 3, 4, 5, 6, 7, respectively. The explanation of this system is certainly thus far consistent; and if the assertion made by Leibnitz be true, that it applies likewise to the great Cova of Fohi, consisting of 64 characters, and 384 lines, embracing six places of figures in this system, and representing therefore all the natural numbers in order between 0 and 63, it would afford a strong presumption that this theory was correct, and would thus furnish an example of a species of Arithmetic with *device of place*, possessing an antiquity of more than three thousand years.

History.

These figures of *eight cova* are held in great veneration, being suspended in all their temples, and though not understood, are supposed to conceal great mysteries, and the true principles of all philosophy both human and divine. The good Jesuit who seems to have caught the very spirit of Chinese belief, is triumphant at his discovery, and seems to consider these symbols of the binary Arithmetic of Fohi, as a most mysterious testimony to the unity of the Deity, and as containing within it the germ of all the sciences. *Cette figure*, says he, *est une des figures de Fohi, qui par l'art admirable d'une science consommée, avoit sçu renfermer, comme sous deux symboles, généraux et magiques, les principes de toutes les sciences de la vraie sagesse; et ce grand Philosophe, dont la physiognomie n'a rien de Chinois, quoique cette nation le réconnoisse pour l'auteur des sciences et pour le fondateur de la monarchie, avoit bâti ce systéme de sa figure circulaire, ce semble, pour calculer et reconnoitre exactement toutes les periodes et les mouvemens des corps célestes et donner les connoissances claires de tous les changemens, qui par leur moyen arrivent continuellement et successivement dans la nature.**

We have been induced to make this digression on the subject of the binary Arithmetic, chiefly for the purpose of noticing this very curious and very ancient monument of its existence; if, however, we make every concession in favour of the explanation above given, and many serious doubts might easily be started, we can at most consider it but as a solitary instance of its adoption not by a nation, but by an individual who surpassed his contemporaries in knowledge, and who left this, amongst other memorable inventions, to his successors, who begun by venerating it as a relic of the founder of their science and their monarchy, and concluded by regarding it as a mystical symbol, which contained the hidden principles of the most sublime and important truths.

Duodenary scale.

(35.) Of scales, different from those which are properly called natural, the existence of the binary and duodenary alone have been supported by probable arguments; the first, under any circumstances, could claim a philosophical existence only, and could hardly therefore be considered as militating against the universality of our proposition; the second we have noticed before, and have stated our reasons for thinking that the preference shown amongst Scandinavian nations for the number *twelve*, and its very general use in the division of concrete numbers, furnish no sufficient ground for considering it as having been used as the radix of a scale of notation, however nearly in some respects it may have approximated to it.

Conclusion.

(36.) We shall now conclude this examination of numeral systems, which has perhaps proceeded to a greater length than is consistent with the design of a work of this nature. We think we have fully established the propositions which we proposed as the objects of our investigation; and have shown that the principles which are concerned both in the origin and formation of numeral systems and numeral languages, are not only remarkably consistent with the most philosophical theory, but possess an universality of application, which is seldom to be met with, except in the physical

* *Indian Archipelago*, vol. i. p. 255.
† Leibnitzii *Opera*, tom. iii. p. 346.
‡ Ibid. tom. ii. p. 349, 391. tom. iv. p. 152, 207.

* Leibnitzii *Opera*, tom. iv. p. 153.

 sciences. We shall add one more instance of this extraordinary accordance between theory and observatioo.

Io Art. 4, we have given what we considered a probahle theory of the origin of the classification of numbers by successive decimation, and we have since discovered the following passage in a history of the Island of Modagascar, hy which it is illustrated in a very remarkable manner. After noticing their numeral language, which coincides with that of the Indian Archipeiago, and refuting the assertions of some authors who have limited their powers of numeratioo to ten, he adds the following account of their mode of counting. "*Lorsqu'ils veulent compter les hommes d'une armée, ils obligent les hommes de passer un à un par un passage etroit en presence des principaux chefs et de poser une pierre chacun en une place; et quand ils ont tous passés, ils comptent toutes les pierres de dix en dix, qu'ils adjoutent ensemble: puis les dizaines de dix en dix et les centaines jusques à ce qu'ils soient à la fin de leur nombre.*"*

Methods of indigitation.

(37.) Before we proceed to give an account of symbolical Arithmetic, as it exists, or has existed amongst different nations, we shall notice a species of digital Arithmetic very generally practised amongst the ancients, and to which frequent allusion is made io classical authors. It consisted io denoting the nine digits and the articulate nnmbers as far as 100, hy inflections of the fingers of the *left* hand, whilst the hundreds were marked on the *right* hand, hy the same inflections which were used to denote the articulate numbers oo the *left*, and the thousands were a repetition on the *right* hand of the ioflections used for the digits; they were thus enabled to denote all nombers which were less than ten thousand. This is the extent to which this system of digital Arithmetic appears to have been carried in ancient times, at least if we may judge from the work of Nicholas, a Monk of Smyrna,† the earliest of all those with which we are acquainted, in which it is distinctly described. But the venerable Bede, in a short Tract, *de Computo vel de Loquela per Gestum Digitorum*, has exteoded this method of numeratioo as far as a million, hy placing the left hand for lower numbers and the right hand for higher, either expanded or closed, with the fingers upwards or downwards, upon the hreast, thighs, and other parts of the body; ten variations only heing required to answer this purpose. The same illustrious author has proposed another application of this system, for the purpose of holding conversations hy means of the fingers of one hand, and which may be done hy making the natural numbers in their order the representatives of the successive letters of the alphabet, when the indication of the nnmber would likewise be made the indication of the letter; thus, to convey the cautioo "*caute age*" to a friend amongst thieves or sharpers, it would be merely requisite to make tha signs of the oumbers 3, 1, 20, 19, 5, 1, 7, 5.

It is quite necessary to refer to this method of nomeratiou, in order to explain many passages in classical authors. Juvenal states it as a peculiar felicity of Nestor, that he counted the years of his age oo the right hand:

Felix nimirum, qui tot per secula mortem
Distulit, atque suos jam dextra computat annos.

Sat. x. 248.

The image of Janus was represented, according to Pliny, with his fingers so placed as to represeot 365, the number uf days in the year:

Janus geminus a Numâ rege dicatus, qui pacis bellique argumento colitur, digitis ita figuratis, at trecentorum sexaginta quinque dierum nota per significationem anni temporis et ævi se Deum indicaret. *Hist. Nat.* lib. xxxiv. 7.*

Tha same custom must he kept io view io order to comprehend the sarcastic exaggeration in the Greek epigram of Nicarchus, *in vetulam annosam*:

Η φάιον ἀθρήσασα ἐλάφου πλέον, ἡ χερὶ λαιῇ
Γῆρας αριθμεισθαι δεύτερον αρξαμένη.†

The following passages are a few out of a great number whieh contain similar aliusions:

Alii igitur digitis complicatis numerum, alii constrictis significabantur. Quinctilian, lib. ii. ch. iii.

Componit vultum, intendit oculos, movet labra, agitat digitos, computat nihil. Caii Plioii *Epist.* 20. lib. ii.

Numerum docet me arithmetica, avaritiæ accomodare digitos. Seneca, *Epist.* 88. lib. i.

Ecce autem avortit nixus læva, in femore habet, manum
Dextera digitis rationem computat, feriens femur.

Plauti *Miles Gloriosus*, act ii. sc. 2.‡

From the first and last of these passages, we should be incloed to suspect, that, however general this practice may have been among the ancients, it varied both at differeot times and with different persons, in the particular mode in which the numbers were denoted.

Henischius§ and other authors have discovered some reference to this practiee, in the description of Wisdom in the *Proverbs* of Solomon:

Length of days is in her right hand, and in her left hand riches and honour. *Prov.* iii. 16.

However fanciful such an explanation may appear to he, it is both simple and natural, compared with that which has been given of the follawing verse in the Parable of the Seed, and which Bede has quoted with approbation:

But others fell into good ground and brought forth fruit, some an hundred fold, some sixty fold, some thirty fold.

Matt. xiii. 8.

"*Centesimus,*" says St. Jerome, "*et sexagesimus et trigesimus fructus, quanqam de una terra et de una semente noscatur, tamen multum differt in numero. Triginta referuntur ad nuptias, nam et ipsa digitorum conjunctio, quasi molli se complexans osculo et fæderans, maritum pingit et conjugem. Sexaginta vero ad viduas, eo quod in angustia et tribulotione sunt positæ, unde et superiori digito deprimuntur: quantoque major est difficultas expertæ quondam voluptatis illecebris abstinere, tanto majus est*

* *Histoire de la grande Isle de Madagascar, par de* Flacourt, ch. xxviii. 1661.

† Νικολάου Σμυρναίου περὶ δακτυλίκου μέτρου. It is published in the *Spicilegium Evangelicum* of Posinus; an Appendix to, or rather a Commentary on, the *Catena Græcorum Patrum*, Rome, 1683, where representations are given of hands with the fingers in the several positions which are required: the same may be seen also in Henischius, *de Numeratione Multiplici*, and with the additional positions of Bede, in the *Theatrum Arithmeticum* of Leopold, 1727.

* Henischius, *de Numeratione Multiplici*, 1605; Leslie's *Philosophy of Arithmetic*, p 223.

† Host, *de Numeratione emendatâ veteribus Latinis et Græcis usitatâ*, Antwerp, 1582

‡ Vallancey, *Collectanea de rebus Hibernicis*, vol. iii. p. 567.

§ *De Numeratione Multiplici.*

Arithmetic. *præmium. Porrò centesimus numerus (diligenter quæso, Lector, attende) de sinistra transfertur ad dexteram; et iisdem quidem digitis, quibus in lævâ nuptæ significantur et viduæ, circulum faciens exprimet Virginitatis coronam.*" It is necessary to refer to the configurations of the fingers themselves, in order to understand the allusions to the numbers in this very singular commentary, which, at all events, shows how very familiar and common this practice must have been at the time it was written.

The Chinese have a system of indigitation, by which they can express on one hand all numbers less than a a hundred thousand; the thumb nail of the right hand touches each joint of the little finger, passing first up the external side, then down the middle, and afterwards up the other side of it, in order to express the nine digits; the tens are denoted in the same way, on the second finger; the hundreds on the third; the thousands on the fourth; and the ten thousands on the thumb. It would be merely necessary to proceed to the right hand, in order to be able to extend this system of numeration much farther than could be required for any ordinary purposes.

The common phrases *ad digitos redire, in digitos mittere*, have the same meaning as *computare*, and distinctly refer to digital numeration; there is also another phrase, *micare digitis*, of frequent occurrence, which alludes to a game extremely popular among the Romans, and which was most probably the same as the *morra* of modern Italy. This noisy game is played by two persons, who stretch out a number of their fingers at the same moment, and instantly call out a number, and he is the winner who names a number expressing the sum of the number of fingers thrown out.* The same game is found amongst the Sicilians, Spaniards, Moors, and Persians; and, under the name *tsoimoi*,† is practised also in China.

There exists a species of digital Arithmetic amongst nearly all eastern nations. The Bengalese‡ count as far as fifteen by touching in succession the joints of the fingers; and merchants, in concluding bargains, the particulars of which they wish to conceal from the bystanders, put their hands beneath a cloth, and signify the prices they offer or take by the contact of the fingers. The same custom is prevalent also in Barbary,§ and Arabia;‖ when they conceal their hands beneath the folds of their cloaks, and possess methods which are probably peculiar and national, of conveying the expression of numbers to each other.

Greek arithmetical notation. (38.) In considering different systems of symbolical Arithmetic, we shall commence with that of the Greeks; a preference which it merits, as well from the superior developement which it received from the hands of the people of antiquity, who cultivated the sciences with the greatest success, as also from its being absolutely essential to the understanding of the ancient astronomical and other writings, in which numbers and calculations are involved.

* Cadell's *Travels in Istria and Carniola*, vol. ii. p. 118; Blunt's *Vestiges of Ancient Manners and Customs in Italy and Sicily*, p. 230. When played in the night it required the utmost confidence in the honour of the parties; and it is an expression of Cicero to designate a perfectly honest man, that he is *dignus, quocum in tenebris mices. Off.* lib. iii.

† Barrow's *Travels in China.*

‡ Halhed's *Bengalee Grammar.*

§ Shaw, *Travels in Barbary.*

‖ Niebuhr's *Travels in Arabia.*

History. The Greeks expressed the natural numbers below 10,000, or a myriad, by means of the twenty-four letters of the alphabet, together with three interpolated symbols, ϛ, ϟ, ϡ, which denoted 6, 90, 900, respectively. The following table exhibits the four classes of digits and articulate numbers of the 1st, 2d, and 3d order, into which the numerical symbols were distributed:

(1.)	α	β	γ	δ	ε	ϛ	ζ	η	θ
	1	2	3	4	5	6	7	8	9
(2.)	ι	κ	λ	μ	ν	ξ	ο	π	ϟ
	10	20	30	40	50	60	70	80	90
(3.)	ρ	σ	τ	υ	φ	χ	ψ	ω	ϡ
	100	200	300	400	500	600	700	800	900
(4.)	$\alpha_{,}$	$\beta_{,}$	$\gamma_{,}$	$\delta_{,}$	$\epsilon_{,}$	ϛ,	$\zeta_{,}$	$\eta_{,}$	$\theta_{,}$
	1000	2000	3000	4000	5000	6000	7000	8000	9000.

The fourth class is a repetition of the first, each letter having a subscribed ι, or dot, by which its value was augmented one thousand fold.

First limit. The limit of Greek Arithmetical notation, as far as it was dependent upon the symbols in the preceding table, was 9999, which was expressed in symbols by $\theta_{,}$ϡϟθ, and in words by *εννεα χιλιαδες ἐννεακοσια ἐννενηκοντα ἐννεα.*

Their language, however, contained a term *μυριας* for the next superior unit, and consequently their numeration by words proceeded farther than their numeration by symbols; by making use, however, of the letter M or Mυ subscribed or postscribed to the symbols for any number within the limits of the preceding table, its value was augmented ten thousand fold, in the same manner as the values of the digital symbols were augmented one thousand fold by the subscribed ι: thus

$$\underset{M}{\alpha} \text{ or } \alpha\,.\,M\upsilon = 10000.$$

$$\underset{M}{\lambda\xi} \text{ or } \lambda\xi\,.\,M\upsilon = 370000.$$

$$\underset{M}{\eta_{,}\phi\mu\gamma} \text{ or } \eta_{,}\phi\mu\gamma\,.\,M\upsilon = 85430000.^*$$

Second limit. By this means the Arithmetical notation of the Greeks was made coextensive with the powers of expression of their numeral language, embracing eight places of figures, its limit being 99999999, which was expressed by $\theta_{,}$ϡϟθ, $\theta_{,}$ϡϟθ (the first with M beneath), or by $\theta_{,}$ϡϟθ . Mυ, $\theta_{,}$ϡϟθ; such is the notation which is found, with many variations, which we shall afterwards notice, in the commentaries of Eutocius, and in the works of Diophantus and Pappus. Without considering further at present the period when this notation was introduced, or the person by whom it was suggested, we shall assume it as the second limit of Greek symbolical Arithmetic.

The extent to which the Greeks were thus enabled to proceed, was sufficient for all the ordinary purposes of life; at all events, the inconveniences which might sometimes arise from its being confined within such narrow limits, were greatly lessened by the very considerable value of their primary units of length, weight, and capacity, and particularly of money.

The speculations, however, of philosophers, which were called forth by the progress of science in the

* Delambre, *Arithmétique des Grecs; Histoire de l'Astronomie Ancienne*, vol. ii. p. 1.

Improvements of Archimedes in the *Arenarius*.

decline of their literary and military glory, led to the consideration of greater numbers than were comprehended in this limited symbolical Arithmetic, more particularly when the increasing accuracy of astronomical observations was giving enlarged views of the extent of the universe; and it was a difficulty of this nature, which suggested to the greatest of the geometers of antiquity the necessity of inventing some expedient by which any numbers, however considerable, might be brought within the compass of their language. The work of Archimedes, in which this method is explained, is entitled Ψαμμιτης,* or *Arenarius*, from the nature of the question which is primarily proposed to be considered: it is addressed to Gelo, King of Syracuse, and commences by noticing the opinion entertained by some people, that the number of the sand is infinite;† "not of that merely which is about Syracuse and Sicily, but which is contained in the whole earth; whilst others deny that this number is infinite, but greater than what can be expressed by any method of numeration." In order to prove more completely the negative of both these propositions, he enlarges the hypothesis, and proposes to express a number which shall exceed the number of the sand, even in the universe (*κοσμον*)‡ of Aristarchus.

His Octades.

In order to effect this object, he forms a scale of numeration whose radix is a myriad of myriads, or the limit of the ordinary Arithmetical language. All numbers comprehended in this *radix* are called *primary* numbers, and the radix itself becomes a unit, or monad, of *secondary numbers*; he thus proceeds to *ternary*, *quaternary*, and other numbers of higher orders, forming successive classes; and the classes themselves are called *octades*, or *periods of eight*, from their requiring eight symbols, or in modern Arithmetic eight places of figures, to express the numbers which are included in each of them;§ and he then shows, without actually finding or assigning the number itself, that a number requiring for its expression not more than eight of these octades, or, in modern notation, not exceeding sixty-three places of figures, will exceed the number of the sand in the sphere of Aristarchus.

The method which he has made use of, for determining the number of places in this number, is very remarkable: he assumes the series

1, 10, 100, 1000, 10,000, &c.

commencing by unity and proceeding by powers of ten, of which the first eight terms are primary numbers, the next eight secondary numbers, and so on; and the question proposed is, to determine the term in this series which is equal to the product of any two assigned terms, such as the $(m+1)^{\text{th}}$ and the $(n+1)^{\text{th}}$, or the m^{th} and n^{th} terms, omitting the first: this is obviously the $(m+n)^{\text{th}}$, or the term whose place in the series, omitting the first, is the sum of the numbers which determine the corresponding places of the two factors. If this number be 8, the product is a monad of secondary numbers; if 16, it is a monad of ternary numbers; if 20, it is a myriad of monads of ternary numbers; and similarly in all other cases. Thus the product of 10 monads of secondary numbers into 100 myriads, which are the 9th and 6th terms in the decuple series respectively, after the first, is a number corresponding to the 15th term of the series under the same circumstances, and is consequently a thousand myriads of monads of secondary numbers; and in this manner he proceeded to assign the successive products of terms in this series, to the extent required by the conditions of his problem.

Supposed anticipation of logarithms.

Some authors have discovered in this process of Archimedes an anticipation of the principle and use of logarithms; and in one sense we may allow that such an opinion is not without foundation; the index of the power of ten, in any assigned term, is identical with the number of the terms omitting the first; and the number which determines the position of the term which is the product of any two, is the sum of their indices; in other words, this sum is the logarithm of their product; and so far the method of Archimedes, and the principles of logarithms, are identical with each other. But there was nothing in the state of knowledge at that time, nor in the nature of the question which be considered, that could lead to the invention of logarithms, properly so called, or to the interpolation of a series of fractional or decimal numbers between the integral indices 1, 2, 3, &c. which should correspond respectively to the series of natural numbers, and which should possess the fundamental property, that the sum of any two of these *logarithms*, or *interpolated* numbers, should be equal to the *logarithm* corresponding to the product of the numbers to which the others corresponded. In this ascription, therefore, of any portion of the credit of this great invention to Archimedes, we only observe another example of a practice which is much too common in the history of the sciences, where the accidental and unconscious possession of some fragment of a great and general truth, or important invention, is made a ground for detracting from the honour which is due to their proper authors.

It is hardly possible to speak with certainty of the actual state of Greek Arithmetic in the time of Archimedes; in his Κυκλου Μετρησις, or *Treatise on the Measure of the Circle*, we find examples of the symbolical representation of *primary* numbers exceeding a myriad, as well as of methods of denoting fractions whose numerators are unity: thus 14688 is denoted by $\overset{\alpha}{\mathrm{M}}\,\delta_{,}\chi\pi\eta'$: 349450 by $\overset{\lambda\delta}{\mathrm{M}}\,\theta_{,}\upsilon\nu$; the fraction $\frac{1}{3}$ is denoted

* There was another work of Archimedes on the subject of this extended Arithmetic, entitled Αρχαι, or *principles*, addressed to Zeuxippus, which is frequently referred to in the Ψαμμιτης.

† Τοῦ ψάμμου τὸν ἀριθμὸν ἄπειρον εἶναι τῷ πλάθει.

‡ The *κοσμος* of the ancient Greek astronomers was the sphere whose centre was that of the earth, and whose radius was the distance of the sun, and therefore also, in their opinion, of the sphere of the fixed stars. Aristarchus, however, by a memorable anticipation of the knowledge of the true system of the universe, placed the sun and not the earth in the centre of his mundane system, the dimensions of which were determined by the following arbitrary proportion: "The sphere of the earth was to the *κοσμος*, or universe of other astronomers, as that universe to the sphere of the fixed stars, or *κοσμος* of Aristarchus. Archimedes, after noticing a geometrical inaccuracy in the mode of stating this proportion, considers the system itself as destitute of foundation; and assumes it an hypothesis merely which is convenient for the more striking developement of his method of expressing very large numbers.

§ Ἔστω οὖν ἁμῖν οἱ μὲν νῦν εἰρημένοι ἀριθμοὶ ἐς τὰς μυρίας μυριάδας, πρῶτοι καλούμενοι. Τῶν δὲ πρώτων ἀριθμῶν αἱ μύριαι μυριάδες, μονὰς καλείσθω Δευτέρων ἀριθμῶν καὶ ἀριθμείσθωσαν δευτέρων ἀριθμῶν μονάδες καὶ ἀπὸ τᾶν μονάδων δεκάδες καὶ ἑκατοντάδες καὶ χιλιάδες καὶ μυριάδες ἐς τὰς μυρίας μυριάδας. Πάλιν δὲ καὶ αἱ μύριαι μυριάδες τῶν δευτέρων ἀριθμῶν, μονὰς καλείσθω τρίτων ἀριθμῶν καὶ ἀριθμείσθωσαν τρίτων ἀριθμῶν μονάδες, καὶ ἀπὸ τᾶν μονάδων δεκάδες καὶ ἑκατοντάδες καὶ χιλιάδες καὶ μυριάδες, ἐς τὰς μυρίας μυριάδας. κ. τ. λ.

Arithmetic.

by a peculiar symbol resembling *K*, the form of which, however, varies in different manuscripts; other fractions, whose numerators are unity, are denoted by simply writing the denominator, after the monads, or whole numbers: thus $591\frac{1}{6}$ is denoted by $\phi\varsigma\alpha'\varsigma'$; $1009\frac{1}{4}$ by $\alpha_{,}\theta'\delta'$; $4673\frac{1}{7}$ by $\delta_{,}\chi o\gamma K$; $3013\frac{1}{2}+\frac{1}{4}$ by $\gamma_{,}\iota\gamma K'\delta'$; and the fraction $\frac{10}{71}$ is denoted by δέκα οα', the numerator being expressed in words. We should not, however, be justified in asserting, that such was the notation employed by Archimedes himself. Successive copyists of manuscripts appear to have altered the notation of numbers to suit the practice which was common in their age; and the notation of which we have just given examples, is precisely the same as that which is found in the *Commentaries* of Eutocius of Ascalon upon this Treatise, which were written six hundred years after the death of Archimedes; and it is most probable that the notation in the text was supplied by the commentator.

Notation of Archimedes or Eutocius.

Whatever, however, was the state in which Greek symbolical Arithmetic was left by Archimedes, it is quite clear that the speculations contained in the *Arenarius* excited the attention of succeeding geometers, and particularly of the celebrated Apollonius of Perga in Pamphylia, who flourished towards the conclusion of the second century before the birth of Christ. Though the work of Apollonius has perished, and we have no record even of its name, except in an obscure allusion to it by Eutocius,* yet the substance of it formed the second Book of the *Mathematical Collections* of Pappus; a great part of this also has shared the fate of the original, the unique manuscript of it, which was left by Sir Henry Saville to the University of Oxford, wanting the first fourteen out of the twenty-seven propositions of which it originally consisted.†

Improvements of Apollonius.

The improvements introduced by Apollonius were of various kinds; and are, many of them, of great importance. In the first place, he appears to have adopted the plan proposed by Archimedes, of classifying numbers, only reducing his *octades* to *tetrads*, or reducing the radix of the geometric series, by which the units of these classes increased in value, from a myriad of myriads to a simple myriad; the units in each class, after the first, being severally denominated μυριαι απλη, διπλη, τριπλη, τετραπλη, and so on; and were denoted by M. α, M. β. M. γ, M. δ, &c. the digital number which designated the order of the myriad, being written after the initial letter M; in making this change, Apollonius was probably as much influenced by the increased convenience of the numeral language, which was formed by means of it, as well as from its giving greater facilities to the symbolical notation of large numbers: it will be soon seen, from an example, to what extent he succeeded.

The chief object, however, of the work of Apollonius appears to have been, the simplification of the process of the multiplication of articulate numbers; as the articulate numbers in Greek Arithmetic were represented by distinct symbols and in practice, a multiplication table was required of the different combinations not of the nine digits, but of the thirty-six symbols of which their notation was composed. In our notation we are directed to the product of such numbers as 50 and 70, from the very nature of the notation itself, by our knowledge of the product of 5 and 7; but with them, the symbols ν and ο for 50 and 70, though connected, have nothing immediately in common with ε and ζ for 5 and 7, and the product of the first $\gamma_{,}\phi$ nothing in common with λε the product of the two last. The researches of Apollonius appear to have been directed to the removal of this great defect, and to make the multiplication of all numbers dependent upon the combination of the nine digits merely, with the aid of a few supplemental propositions.

History.

Theory of bases and analogous numbers.

The nine digits, α, β, γ, &c. were called by him πυθμενες, or *bases*; and the numbers which are found in the geometrical series, whose radix is 10, and of which any one of these *bases* is the first term, are called *analogous* to them: thus ι, ρ, α, or 10, 100, 1000, are *analogous* numbers to the *base* α, or 1; ξ, χ, $\varsigma_{,}$ or 60, 600, 6000, are *analogous* to the *base* ς, or 6; and similarly in other cases. In performing multiplications, he replaces articulate or *analogous* numbers by their *bases*, finds their product, and then, by means of other propositions, which are in some measure equivalent to the addition of the requisite number of zeros, he passes to the proper result. A few examples will furnish the best explanation of this process:

Example 1. To multiply together ν, ν, ν, μ, μ, λ, or 50, 50, 50, 40, 40, 60

ν ν ν μ μ λ	ξ Mυ . Mυ	60,0000,0000.
ι ι ι ι ι ι	ρ Mυ	100,0000.
ε ε ε δ δ γ	$\varsigma_{,}$ Mο	6000.*

Example 2. To multiply together σ, τ, υ, φ, or 200, 300, 400, 500.

σ τ υ φ	ρ κ Mυ . Mο	120,0000,0000.
ρ ρ ρ ρ	α Mυ . Mυ	1,0000,0000.
β γ δ ε	ρ κ	120.†

Example 3. To multiply together ι, κ, λ, κ, κ, σ, τ, υ, φ, or 10, 20, 30, 20, 20, 200, 300, 400, 500.

ικλκκστυφ	$\kappa_{,}\eta_{,}\omega$ Mυ . Mυ . Mο	28800,0000,0000,0000.
ιιιιιρρρρ	ι Mυ . Mυ . Mυ	10,0000,0000,0000.
αβγββ βγδε	$\beta_{,}\omega\pi$	2880.‡

Example 4. Multiply together σ, τ, κ, λ, μ, ι, β, γ, δ, or 200, 300, 20, 30, 40, 10, 2, 3, 4.

σ τ κ λ μ ι β γ δ	$\gamma_{,}\upsilon\nu\varsigma$ Mυ . Mυ	3456,0000,0000.
ρ ρ ι ι ι ι	α Mυ . Mυ	1,0000,0000.
β γ β γ δ α β γ δ	$\gamma_{,}\upsilon\nu\varsigma$	3456.§

The process appears to have been as follows; first, write down the numbers to be multiplied together; secondly, the 100s or 1000s by which the *bases* are multiplied to produce those numbers; and, lastly, the *bases* themselves. Form the product of the bases; and afterwards of the 10s, and 100s, which would be done by allowing 1 for every 10, and 2 for every 100: for

* At the end of his *Commentary on the Measure of the Circle*.

† This fragment of Pappus was discovered by Wallis, and published in 1688, and afterwards in the third volume of his works. There is no doubt of its containing the substance of the work of Apollonius, as he frequently refers to him by name, and quotes the specific examples which Apollonius had given.

* Pappi *Collectanea Mathematica*, lib. ii. prop. 15.

† Ibid. prop. 16.

‡ Ibid. prop. 18.

§ Ibid. prop. 26.

every four contained in the sum of them, there will be a corresponding *myriad* as a factor in the prodnct. In the first example this sum is 6, and the result ρ . Μυ, or 100 myriads; in the second it is 8, and the result therefore α . Μυ . Μυ, ur a myriad of myriads;* in the third it is 13, and the result is ι . Μυ . Μυ . Μυ, or ten myriads of myriads of myriads. In order to form, therefore, the first product, it remained only to multiply the product of the hases with the number α, ι, ρ, or $\alpha_{,}$, which preceded the Μυ in the second product; the rules by which this was effected were contained in those propositions which are lost; but as there were only four cases, we may readily conceive what they were; thus if $\gamma_{,}\upsilon\nu\varsigma_{,}$ as in the first example, was the product of the bases, we should find the prodnct of

α and $\gamma_{,}\upsilon\nu\varsigma = \gamma_{,}\upsilon\nu\varsigma$.

ι and $\gamma_{,}\upsilon\nu\varsigma = \gamma_{,}\delta_{,}\phi\xi$ or γ Μυ και Μο . $\delta_{,}\phi\xi$.

ρ and $\gamma_{,}\upsilon\nu\varsigma = \tau_{,}\mu_{,}\epsilon_{,}\chi$ or λδ Μυ και Μο $\epsilon_{,}\chi$.

$\alpha_{,}$ and $\gamma_{,}\upsilon\nu\varsigma = \tau\mu\epsilon$ Μυ και Μο . $\varsigma_{,}$.

Artifices of notation. There are many of the artifices of notation employed in this work, which if pursued and properly generalized, would have given inereased symmetry as well as extent to their symbolical Arithmetie; amongst these we ought particularly to notice the accentuation by the subscribed ι, of the symbols of articulate numbers of the second and third order, increasing their value, as in the case of the nine digits, one thousand fold.† The only reason which can easily be assigned why this extension of their notation had not been generally adopted for all the symbols, when once applied to those of the nine digits, appears to have been, that as they merely proposed by it, in the first instance, to make their notation coextensive with the terms of their numeral language, they paused when that object was effected; and, however simple its extension to all the other symbols may have been, it was not likely to be adopted when the utility of it was not felt; the advantages indeed of a simple and expressive notation addressed to the eye, as distinct from language, were in no respect understood by the ancient geometers; and it is only in modern times that the powers of symbolical language have been completely appreciated.

The use of the initial letters Μυ in such expressinns as $\iota\beta_{,}$ Μυ . Μυ for 12000,0000,0000, might at first sight appear to resemble the modern zero, in a scale of notation proceeding hy myriads; but it can only be considered in this case as the ahhreviated expression of Μυριαι Μυριαδων, and is never employed to give value from position, without reference to its value as a factor: thus 347900006008 is expressed hy $\gamma_{,}\upsilon o\theta$. Μυ . Μυ και Μυ . $\varsigma_{,}\eta$, and not by $\gamma_{,}\upsilon o\theta$. Μυ . $\varsigma_{,}\eta$. There is nothing, in short, in Greek Arithmetical notation, whieh, in the slightest degree, resembles our own; and nothing in the ohject proposed in the researches of Archimedes and Apollonius, which could naturally lead to its invention, with the exception of the discovery of the very important fact, that the multiplications of the articulate numbers depended upon that of their bases.

Pappus, at the conclnsion of this fragment, has given from the work of Apollonius two examples, to prove the facility of multiplying any numbers, however large, hy means of the process which he had explained in the preceding propositions. In one case it is proposed to find the continued prodnct of the numbers expressed hy the several letters in the verse:

Αρτέμιδος κλεῖτε κράτος ἔξοχον ἐννέα κοῦραι,

and in the other, in the verse,

Μῆνιν ἄειδε θεὰ Δημήτερος ἀγλαόκαρπον.

In the first example, and the only one which we think it necessary to notice, he multiplies the bases successively, and the resulting product

19,6036,8480,0000,0000 is expressed hy

Μδ . $\overline{\iota\theta}$ και Μγ . $\overline{\varsigma_{,}\lambda\varsigma}$ και Μβ . $\overline{\eta_{,}\upsilon\pi}$, where Μδ, Μγ, Μβ denote myriads of the fourth, third, and second orders respectively. If this number be multiplied into the contiuued product of the decads and centuries (ἑκατόντάδες) which is 10 myriads of the ninth order, the final product, or 196,0368,4800,0000,0000,0000, 0000,0000,0000,0000,0000,0000,0000,0000, is expressed Μιγ . $\overline{\rho\mathrm{ϟ}\varsigma}$ και Μιβ . $\overline{\tau\xi\eta}$ και Μια . $\delta_{,}\omega$.

Notation of Diophantus Delamhre has noticed forms of Greek notation, which appear to favour the notion that the principle of value from position was in later times in some measure understood; thus in Diophantus, we find the fraction $\frac{3069000}{331776}$ * expressed hy $\tau\varsigma . \theta_{,}^{\lambda\gamma\,.\,\alpha_{,}\psi o\varsigma}$, where the numerator is $\tau\varsigma . \theta_{,}$ and the denominator $\lambda\gamma . \alpha_{,}\psi o\varsigma$. Great and important as this simplification of the ordinary notation certainly is, it is seldom nsed by Diophantus, except in his fourth Book, and very rarely, if ever, hy later authors; in other parts of his works, the abbreviation Μυ is either prefixed or postfixed to the symbols which denote myriads, and the abbreviation Μο is sometimes prefixed to monads and sometimes omitted; thus, in one place we find 12768 denoted hy μυ . α . $\beta_{,}\psi\xi\eta$, and in the next line the same number is denoted hy μυ . α . μο . $\beta_{,}\psi\xi\eta$;† in another place 17136600 is denoted hy μυ . $\alpha_{,}\psi\iota\gamma$. μοναδες $\varsigma_{,}\chi$; and again 163021824, the only example in his works where a myriad of the second order is involved, is expressed by μμ . α . μ . $\varsigma_{,}\tau\beta$. μο . $\alpha_{,}\omega\kappa\delta$.‡ Amidst such a total want of uniformity of notation, we may fairly infer, that Diophantus was insensible to the value of his own discovery. Theon, who lived at a later period than Diophantus, and who was well acquainted with his writings, expresses the oumher of cubical stadia in the earth, or 38406364469497, hy μυριαντάδικων τριπλων ϟη, μυριαντάδικων διπλων $\delta_{,}\xi\gamma$, μυριαδες απλαι $\varsigma_{,}\upsilon\mu\epsilon$, και $\theta_{,}\upsilon\mathrm{ϟ}\zeta$, after the manner of

* Μυριαι Μυριαδων.

† Some Lexicographers and writers on Greek Arithmetic have mentioned another extension of this notation, and have quoted Herodian the Grammarian for their authority, though it is not noticed by him; it consists in increasing the value of the first twenty-seven symbols 1,000,000 times, by adding two accents to them, 1000,000,000 times by adding three accents, and so on, thus α̤ is 1,000,000, α̤̈ 1000,000,000, and so on.

* Diophanti *Arith.* lib. iv. prop. 46.

† Ibid. lib. iii. prop. 22.

‡ Ibid. prop. 86.

 Apollonius, and in no case does he adopt the notation in question; no notice of it is discoverable in the *Commentaries* of Eutocius, who lived at a still later period; under such circumstances, we should feel strongly inclined to ascribe this form of notation to the omissions of the successive transcribers of the manuscripts.

Numeral powers of words. It appears to have been a favourite practice with the Greeks of the later ages to form words in which the sum of the numbers expressed by their component letters should be equal to some remarkable number; of this kind were the words *αβρασαξ* and *αβρασαξα*; the letters in which express numbers, which, added together, are equal to 365 and 366, the number of days in the common and bissextile years respectively; and it was also remarked that the word *νειλος* possessed the same property with the first of these words.* Observations like these, however trifling, are not without their portion of curiosity; but the same indulgence cannot be shown to the absurdities of those Pythagorean philosophers who, amongst other extraordinary powers which they attributed to numbers, maintained that of two combatants, he would conquer, the sum of the numbers expressed by the characters of whose name exceeded the sum of those expressed by the other. It was upon this principle that they explained the relative prowess and fate of the Heroes in Homer, Πατροκλος, Ἑκτωρ, and Αχιλλευς, the sum of the numbers in whose names are 861, 1225, and 1276 respectively.*

Arithmetical operations. It is not very easy to give a complete account of Greek Arithmetical operations; there is no work of antiquity extant in which they are specifically detailed, and it is only in the *Commentaries* of Eutocius on the measure of the circle of Archimedes, that we can find any considerable number of examples of multiplications exhibited at full length; and even in this case the variations which are found in different manuscripts, in the order and form in which the different steps and symbols in the processes are written, prevents our speaking in a positive manner at least with respect to them.

The following examples are taken from the *Commentaries* of Eutocius, on the third and last proposition of Archimedes on the measure of the circle, in which it is chiefly required to find the squares of two numbers, and to assign the square root of their sum:

Example 1.

To find the square of $\rho\nu\gamma$, or 153.

$\rho\nu\gamma$	153
$\rho\nu\gamma$	153
$^{\alpha}\mathrm{M}\,{}_{,}\epsilon\,\tau$	10000 or $^{\alpha}\mathrm{M}$
${}_{,}\epsilon\,{}_{,}\beta\,\phi\,\rho\,\nu$	5000 or ${}_{,}\epsilon$
$\tau\,\rho\,\nu\,\theta$	300 or τ
$\beta_{\mathrm{M}}\,{}_{,}\gamma\,\nu\,\theta$	5000 or ${}_{,}\epsilon$
	2500 or ${}_{,}\beta\,\phi$
	150 or $\rho\nu$
	300 or τ
	150 or $\rho\nu$
	9 or θ
	23409

Process of multiplication of integers. In performing the operation they proceeded from the right to the left, and the successive products are written down separately, without any incorporation with those which precede or follow them; they do not appear to have adhered with much strictness to any order of magnitude in writing down the successive results, or to have been very solicitous about writing them underneath each other, as they are sometimes in the same line. They then performed the additions much in the same way

* Words in which the sums of the numbers expressed by the letters were equal, were called *ὀνόματα ἰσόψηφα*: and we have an example in the Greek Anthology, where the Poet, wishing to express his dislike of a pestilent fellow of the name of Δαμαγορας, says, that having heard that his name was equivalent in numeral power to λοιμὸς, proceeded to weigh them in a balance, when the latter was found to be the lighter of the two.

> *Δαμαγόραν καὶ λοιμὸν ἰσόψηφον τις ἀκούσας,*
> *Ἔστησ' ἀμφοτέρων τὸν τρόπον ἐκ κανόνος·*
> *Εἰς τὸ μέρος δὲ καθείλκετ' ἀνελκυσθὲν τὸ τάλαντον*
> *Δαμαγόρου, λοιμὸν δ' εὗρεν ἐλαφρότερον.*
>
> *Histoire de l'Académie des Inscriptions*, vol. v. p. 209.

* This very singular superstition continued in force as late as the sixteenth century, and was transferred from the Greek to the Roman numeral letters, I, U or V, X, L, C, D and M, which correspond to the numbers 1, 5, 10, 50, 100, 500, and 1000: thus the numeral power of the name of Maurice (Mauritius) of Saxony was considered as an index of his success against Charles V. It was the fashion also to select or form memorial sentences or verses to commemorate remarkable dates. Thus the year of the Reformation of religion in Germany (1517) was found to be expressed by the numeral letters of the verse of the *Te Deum*; *Tibi Cherubin et Seraphin incessabili voce proclamant*, in which there is one M, four Cs, two Ls, two Us or Vs, and seven Is. In a similar manner, the defeat of Francis at Pavia (1525) is commemorated in both the following verses:

> *Regia succumbunt pugnacis lilia Galli;*
>
> and *Captus erat Gallus, coeunt cum rure cohortes.*

See Henischius, *de Numeratione Multiplici*; and Hostus, *de Numeratione emendatâ, veteribus Latinis et Græcis usitatâ*, 1632.

Arithmetic. as in the addition of concrete numbers of different denominations in common Arithmetic, beginning with the digits and advancing in succession through the different orders of articulate numbers. The scheme which is given of this multiplication in common figures will render this process perfectly clear: the product of ρ and ρ is α_{M}, or 10,000; of ρ and ν is $\epsilon_{,}$ or 5000; of ρ and γ is τ, or 300; of ν and ρ is $\epsilon_{,}$, or 5000; of ν and ν is $\beta_{,}\phi$, or 2500; of ν and γ is $\rho\nu$, or 150; of γ and ρ is τ, or 300; of γ and ν is $\rho\nu$, or 150; and of γ and γ is θ, or 9. In Greek notation it is clearly a matter of indifference in what order these successive products are written; whilst in our notation the value of the digits depends on the number of places which follow them. History.

EXAMPLE 2.

To find the square of $\alpha_{,}\rho\xi\beta'\eta'$ or 1162 ⅛.

$\alpha_{,}\rho\xi\beta'\eta'$	1162 ⅛
$\alpha_{,}\rho\xi\beta'\eta'$	1162 ⅛
$\rho_{M}\ \iota_{M}\ \varsigma_{M}\ \beta_{,}\rho\kappa\epsilon$	1000000 or ρ_{M}
$\iota_{M}\ \alpha_{M}\ \varsigma_{,}\sigma\iota\beta K$	100000 or ι_{M}
$\varsigma_{M}\ \varsigma_{,}\gamma_{,}\chi\rho\kappa\zeta K$	60000 or ς_{M}
$\beta_{,}\alpha\rho\kappa\delta'\delta'$	2000 or $\beta_{,}$
$\rho\mu\epsilon\delta'\xi\delta'$	125 or $\rho\kappa\epsilon$
$\underset{M}{\rho}\lambda\epsilon\phi\lambda\delta K\xi\delta'$	100000 or ι_{M}
	10000 or α_{M}
	6000 or $\varsigma_{,}$
	200 or σ
	12½ or $\iota\beta K$
	60000 or ς_{M}
	6000 or $\varsigma_{,}$
	3600 or $\gamma_{,}\chi$
	120 or $\rho\kappa$
	7½ or ζK
	2000 or $\beta_{,}$
	200 or σ
	120 or $\rho\kappa$
	4 or δ
	¼ or δ
	145 or $\rho\mu\epsilon$
	$\frac{1}{64}$ or $\xi\delta'$
	1350534½ $\frac{1}{64}$

Notation of fractions. This example involves fractions, and the process will be sufficiently explained by the scheme with which it is accompanied. The fraction ½ is denoted by the peculiar symbol K; and the other fractions, whose numerators are unity, by writing the numbers in the denominator immediately after the integers, the distinction between them being marked by an accent; thus, in Ptolemy, we find 34 ½ ⅓ $\frac{1}{12}$ denoted by $\lambda\bar{\delta}\ \beta'\ \gamma''\ \iota\beta''$. In the following example the fraction $\frac{9}{11}$ has its numerator and denominator written immediately after the integers, thus $\theta'\ \iota\alpha'$; but when mixed up with integers in a manner which might lead to some confusion, the denominator is placed above the numerator to the right hand, in the manner of an index in Algebra.

EXAMPLE 3.

To find the square of $\alpha_{\prime}\,\omega\,\lambda\,\eta'\,\theta'\,\iota\,\alpha'$ or 1838 $\frac{9}{11}$.

$\alpha_{\prime}\,\omega\,\lambda\,\eta\,\theta'\iota\,\alpha'$	1838 $\frac{9}{11}$
$\alpha_{\prime}\,\omega\,\lambda\,\eta\,\theta'\iota\,\alpha'$	1838 $\frac{9}{11}$
$\underset{M}{\rho}\,\underset{M}{\pi}\,\underset{M}{\gamma}\,\eta_{\prime}\,\omega\,\iota\,\eta'\,\beta'^{\iota\alpha}$	1000000 or ρ_M
$\underset{M}{\pi}\,\underset{M}{\xi}\,\underset{M}{\delta}\,\beta\,\delta_{\prime}\,\varsigma_{\prime}\,\upsilon\,\chi\,\sigma\,\delta'\,\varsigma'^{\iota\alpha}$	800000 or π_M
$\underset{M}{\gamma}\,\underset{M}{\beta}\,\delta_{\prime}\,ϡ\,\sigma\,\mu\,\kappa\,\delta'\,\varsigma'^{\iota\alpha}$	30000 or γ_M
$\eta_{\prime}\,\varsigma_{\prime}\,\upsilon\,\sigma\,\mu\,\xi\,\delta\,\varsigma'\,\varsigma'^{\iota\alpha}$	8000 or $\eta_{\prime}$
$\omega\,\iota\,\eta'\,\beta'^{\iota\alpha}\,\chi\,\nu\,\delta'\,\varsigma'^{\iota\alpha}$	818 $\frac{2}{11}$ or $\omega\,\iota\,\eta'\,\beta'^{\iota\alpha}$
$\kappa\,\delta'\,\varsigma'^{\iota\alpha}\,\varsigma'\,\varsigma'^{\iota\alpha}\,\pi\,\alpha'^{\rho\kappa\alpha}$	800000 or π_M
	640000 or $\underset{M}{\xi\,\delta}$
$\underset{M}{\tau\,\lambda}\,\eta\,\alpha_{\prime}\,\sigma\,\nu\,\alpha'\,\zeta'^{\iota\alpha}\,\pi\,\alpha'^{\rho\kappa\alpha}$	24000 or $\beta_M\,\delta_{\prime}$
or $\underset{M}{\tau\,\lambda}\,\eta\,\alpha_{\prime}\,\sigma\,\nu\,\beta'\,\lambda\,\xi'^{\rho\kappa\alpha}$	6400 or $\varsigma_{\prime}\,\upsilon$
	654 $\frac{6}{11}$ or $\chi\,\nu\,\delta'\,\varsigma'^{\iota\alpha}$
	30000 or γ_M
	24000 or $\beta_M\,\delta_{\prime}$
	900 or ϡ
	240 or $\sigma\,\mu$
	24 $\frac{6}{11}$ or $\kappa\,\delta\,\varsigma^{\iota\alpha}$
	8000 or $\eta_{\prime}$
	5400 or $\varsigma_{\prime}\,\upsilon$
	240 or $\sigma\,\mu$
	64 or $\xi\,\delta$
	6 $\frac{6}{11}$ or $\varsigma'\,\varsigma'^{\iota\alpha}$
	818 $\frac{2}{11}$ or $\omega\,\iota\,\eta'\,\beta'^{\iota\alpha}$
	654 $\frac{6}{11}$ or $\chi\,\nu\,\delta'\,\varsigma'^{\iota\alpha}$
	24 $\frac{6}{11}$ or $\kappa\,\delta'\,\varsigma'^{\iota\alpha}$
	6 $\frac{6}{11}$ $\frac{9}{121}$ or $\varsigma\,\varsigma'^{\iota\alpha}\,\pi\,\alpha'^{\rho\kappa\alpha}$
	3381251 $\frac{7}{11}$ $\frac{81}{121}$
	or 3381252 $\frac{4}{121}$

Difficulty of multiplying numbers beyond the first limit of Greek Arithmetic

Eutocius, in the conclusion of his *Commentary*, states that Philo, of Gadara, had brought the approximation to the length of the circle to greater accuracy than Archimedes, in consequence of extending his multiplications and divisions to numbers involving myriads, which, he says, are difficult to follow, unless by a person well versed in the Logistics of Magnus. The term Λογιστικη is applied to the whole science of arithmetical calculation; and we may suppose that the work to which Eutocius refers, expressly treated on these subjects to an extent which they rarely attained in other books. The examples which we have given, show how very difficult and embarrassing these operations must have been, particularly when fractions were involved; and it is this reason which is expressly assigned by Ptolemy for his preference of sexagesimals.*

Eutocius has given no example of division; and in the repeated instances in which the square root of a number is required, he assumes the root, and then shows that its square coincides with the proposed number or nearly so; thus, in extracting the square root of $\underset{M}{\phi\,\mu\,\zeta}\,\beta_{\prime}\,\rho\,\lambda\,\beta'\,\epsilon\,\varsigma$, or 5472131 $\frac{1}{16}$, he assumes it to be $\beta_{\prime}\,\tau\,\lambda\,\theta'\,\delta'$, or 2339 $\frac{1}{4}$, finds its square or $\underset{M}{\phi\,\mu\,\zeta}\,\beta_{\prime}$ ϟ $K\,\epsilon\,\varsigma'$, or 5472090 $\frac{1}{2}$ $\frac{1}{16}$, which differs from the number whose root is required by Mo. $\mu\,\alpha$ K, or 41 $\frac{1}{2}$.

* Μεγαλη Συνταξις, lib. i. ch. ix.

In the *Commentaries* of Theon on the *Almagest* (Μεγαλη Συνταξις) of Ptolemy, we find a statement of the rule for extracting the square root, which corresponds in essential points with the one in common use, but it is not accompanied by any example exhibiting a type of the operation. In the same author we find also many examples of division, performed upon sexagesimals. Before we proceed, however, to notice them further, it may be as well to premise a few observations on the origin and design of this species of Arithmetic.

Sexagesimal Arithmetic.

(39.) The division of the circle into 360° seems to have been pointed out to the earlier astronomers, by its being an articulate number nearly equal to the days in the year; and, consequently, one of these degrees was nearly equal to the portion of the ecliptic described by the sun in one day. Whatever, however, were the grounds upon which this division was adopted in the first instance, it was adhered to afterwards in the most improved periods of ancient and modern astronomy, from a sense of the convenience presented by the number 360 in the great number of its divisors. The angle subtended by the side of a hexagon inscribed in a circle was therefore 60°, and the corresponding portions of the circumference were called μοιραι, parts or degrees; each μοιρα was divided into 60 λεπτα, or minutes, or primes, or sexagesimals of the first order; each minute into 60 seconds, or sexagesimals of the second order; and so on proceeding to trines, quaternes, &c. in a descending series. But this sexagesimal division was not confined to degrees of the circle: the side of the inscribed hexagon itself, which is equal to the radius, was likewise divided into 60 μοιραι, and the same series of sexagesimal subdivisions were applied to these rectilineal degrees (μοιραι ευθειων) as to those of the circumference (μοιραι περιφερειων;) and as the whole business of calculation in ancient astronomy was reduced to the arcs and chords of circles, this sexagesimal Arithmetic superseded every other in works on that subject.

By whom invented.

The invention of this species of Arithmetic is attributed to Ptolemy by his commentator Theon, and later authors; though, if we might judge from the language of Ptolemy himself,* when explaining the principles upon which his table of chords was constructed, we might be inclined to think, that the sexagesimal division of the degrees of the circle was known before his time, and that he only applied it to the division of the radius. Whoever, however, was its author, it must be considered as the greatest improvement in the science of calculation which preceded the introduction of the Hindoo notation; it enabled astronomers at once to get rid of fractions, the treatment of which, in their ordinary Arithmetic, was so extremely embarrassing; and enabled them to extend their approximations, particularly in the construction of tables, to any required degree of accuracy.

Notation.

The notation of sexagesimals, as it appears in Ptolemy and his Commentator, is nearly the same as that which is made use of in modern astronomical writings; the degrees, or μοιραι, were considered as units, and written in the ordinary manner, a stroke being placed over the last symbol, as in $\mu\bar{\delta}$, or 44°. The successive orders of sexagesimals, primes, seconds, trines, &c. were denoted by one, two, three, &c. accents, as in modern astronomical notation; thus

$$\iota\bar{\delta}\ \eta'\ \nu\zeta''\ \mu\beta'''\ \kappa\varsigma''''\ \lambda\theta'''''$$ is equivalent to

$$14^\circ\ 8'\ 57''\ 42'''\ 26^{iv}\ 39^{v}.$$

Symbol for zero.

It is quite clear, that in this notation all symbols beyond ξ, or 60, were superfluous; and, as in many cases a zero was necessary to signify the absence of any one term in the series of sexagesimals, the symbol ο next in order to it was taken for this purpose, as it could not be confounded with any of those which this notation made significant; thus $\bar{o}\ \kappa\delta'\ \iota\varsigma''$ denotes 0° 24′ 16″; $\iota\bar{\varsigma}\ o'\ \mu''$ denotes 16° 0′ 40″. It is a curious circumstance, that this symbol for zero, transmitted from the Greeks to the Arabians, and from thence to Europe, was adopted as the zero in the Hindoo notation, having superseded the simple dot which was generally used for that purpose amongst the people from whom it was derived. - - -

We shall now give a few examples of Arithmetical operations on sexagesimal quantities, in doing which we shall take for our guide the Commentary of Theon on the 9th Chapter of the *Almagest* of Ptolemy, of which the chief object is the exposition of the principles and practice of this Arithmetic.

Example of multiplication.

EXAMPLE 1.

To find the square of $\lambda\overset{\circ}{\zeta}\ \delta'\ \nu\epsilon''$ or 37° 4′ 55″.

$\lambda\overset{\circ}{\zeta}\ \delta'\ \nu\epsilon''$	37° 4′ 55″
$\lambda\overset{\circ}{\zeta}\ \delta'\ \nu\epsilon''$	37° 4′ 55″
$\alpha_{,}\tau\xi\overset{\circ}{\theta}\ \rho\mu\eta'\ \beta_{,}\lambda\epsilon''$	1369° 148′ 2035″
$\rho\mu\eta'\ \ \iota\varsigma''\ \sigma\kappa'''$	148′ 16″ 220‴
$\beta_{,}\lambda\epsilon''\ \sigma\kappa'''\ \gamma_{,}\kappa\epsilon''''$	2035″ 220‴ 3025⁗
$\alpha_{,}\tau\xi\overset{\circ}{\theta}\ \sigma\varsigma\varsigma'\ \delta_{,}\pi\varsigma''\ \upsilon\mu'''\ \gamma_{,}\kappa\epsilon''''$	1369° 296′ 4086″ 440‴ 3025⁗.

The multiplications are performed in the same manner as in duodecimals in our common books of Arithmetic, only proceeding from the right to the left; and it is probable that the multiplications were rendered more easy by means of a sexagesimal table, containing the products of all numbers with each

* Μεγαλη Συνταξις, *bibl.* A. κεφ. θ.

Arithmetic.

other as far as 60;* another question also which Theon has considered, was to determine the order of the product of sexagesimals of the same or different orders; thus, the product of two primes are seconds, of seconds and primes are trines, of seconds and trines are quinquines, and generally the order of the product of any sexagesimals will be the sum of the orders of the component factors; a fact, which will be very evident to any one who understands the theorem $\frac{p}{60^{m}} \times \frac{q}{60^{n}} = \frac{pq}{60^{m+n}}$

History.

It now remains to divide the successive sums of these sexagesimals by 60, so as to reduce them within the proper limits of the sexagesimal notation.

$\gamma_{,}\kappa\epsilon''''$ =	$\dot{o}$ o' o'' ν''' $\kappa\epsilon''''$	
$\upsilon\mu'''$ =	ζ'' κ'''	
$\delta_{,}\pi\varsigma''$ =	$\dot{\alpha}$ η' ς''	
$\sigma\varphi\varsigma'$ =	$\dot{\delta}$ $\nu\varsigma'$	
$\alpha_{,}\tau\xi\dot{\theta}$ =	$\alpha_{,}\tau\xi\dot{\theta}$	
	$\alpha_{,}\tau o\dot{\epsilon}$ δ' $\iota\delta''$ ι''' $\kappa\epsilon''''$	

3025'''' =	0°	0'	0''	50'''	25''''	
440''' =	0°	0'	7''	20'''		
4086'' =	1°	8'	6''			
296' =	4°	56'				
1369° =	1369°					
	1375°	4'	14''	10'''	25''''	

Additions, as well as subtractions and other arithmetical operations, appear to have been performed from right to left; a method which was subject to considerable inconvenience, particularly in the two first cases, from their requiring a constant reference to the numbers in the subsequent columns. Theon has proposed the following example of division, and detailed the process. He gives no scheme of the operation, which may, however, be easily supplied.

EXAMPLE 2.

Of division.

To divide $\alpha_{,}\phi\iota\dot{\epsilon}$ κ' $\iota\epsilon''$ by $\kappa\dot{\epsilon}$ $\iota\beta'$ ι'' or 1515° 20' 15'' by 25° 12' 10''.

$\alpha_{,}\phi\iota\dot{\epsilon}$ κ' $\iota\epsilon''$	$\kappa\dot{\epsilon}$ $\iota\beta'$ ι''
$\alpha_{,}\dot{\phi}$	ξ° ζ' $\lambda\gamma''$
$ϡ\kappa'$	
$\psi\kappa'$	
σ'	
ι'	
$\rho ϟ'$	
$\rho o\epsilon'$	
$ϡ\iota\epsilon''$	
$\pi\delta''$	
$\omega\lambda\alpha''$	
α'' ι'''	
$\omega\kappa\theta''$ ν'''	
$\omega\kappa\epsilon''$	
$\sigma ϟ'''$	
$\tau ϟ\varsigma'''$	

* Such tables were in genera. use when operations in this Arithmetic were required amongst astronomers before the decimal division of the radius, and may be found in many works, both astronomical and arithmetical; and, amongst others, in Wallis's *Algebra*.

(Example 2, *continued.*)

	1515° 20′ 15″	25° 12′ 10″	
25° × 60	1500°	60°	1st quotient.
	15° = 900′		
	920′		
12′ × 60	720′		
	200′		
10″ × 60	10′		
	190′	25° 12′ 10″	
25° × 7′	175′	7′	2nd quotient.
	15′ = 900″		
	915″		
12′ × 7′	84″		
	831″		
10″ × 7′	1′ 10‴		
	829 50‴	25° 12′ 10″	
25° × 33″	825	33″	3d quotient.
	4″ 50‴ = 290‴		
12′ × 33″	396‴		
	− 106		

The quotient is nearly, therefore, 60° 7′ 33″.*

The operation requires no farther illustration than what is afforded by the preceding schemes, and accurately resembles our processes for compound division. This example forms a natural introduction to one for extracting the square root, which Theon afterwards subjoins, referring for the proof of the operation to the figure and result of the fourth Proposition of Euclid's *Elements*. Of extracting the square root.

Example 3.

To extract the square root of $\delta_{,}\dot{\phi}$, or 4500.

$\delta_{,}\dot{\phi}$	$\xi\,\dot{\zeta}\;\vartheta'\;\nu\,\epsilon''$
$\delta_{,}\nu\,\pi\,\theta$	$\rho\,\lambda\,\delta$
$\chi\,\xi'$	
$\phi\,\nu\,s'\;\iota\,s''$	
$\zeta_{,}\nu\,\kappa\,\vartheta''$	$\rho\,\lambda\,\dot{\delta}\;\eta'$
$\zeta_{,}\,\tau\,\sigma''$	
$\zeta''\;\kappa'$	
$\nu'''\;\kappa\,\epsilon''''$	
$\mu\,\epsilon''\;\mu\,\theta'''\;\lambda\,\epsilon''''$	

* Delambre, *Histoire de l'Astronomie Ancienne*, tom. ii. p. 25.

(Example 3, *continued.*)

$$
\begin{array}{llll}
4500^\circ & & \multicolumn{2}{l}{|\ 67^\circ\ 4'\ 55''} \\
\underline{4489} & & \multicolumn{2}{l}{|\ 134^\circ = \ 2 \times 67^\circ} \\
\quad 11^\circ = 660' & & & \\
\qquad \underline{536'\ 16''} & & & \\
\qquad 123'\ 44'' = 7424' & & & |\ 134^\circ\ 8' \\
134^\circ \times 55 & 7370'' & & \\
\ \ 8' \times 55' & \quad 7''\ 20''' & & \\
55'' \times 55'' & \qquad \underline{50'''\ 25^{\text{iv}}} & & \\
 & 45''\ 49'''\ 35^{\text{iv}} & &
\end{array}
$$

Process for extracting the square root.

The process is as follows: the greatest number whose square is less than 4500 is 67; subtract the square of 67 from 4500, and the remainder is 11°, or 660′; double 67, which makes 134; the next term in the root is 4′, which, multiplied into 134° 4′, produces 536′ 16″; subtract this, and the remainder is 123′ 44″, or 7424″; the double of the root already obtained is 134° 8′, and the next term in the root obtained by trial is 55″, which, multiplied into 134° 8′ 55″, and the result subtracted, leaves a remainder 45″ 49‴ 35$^{\text{iv}}$. It is clear that the same process may be continued to any required degree of accuracy. The scheme of the operation, which we have copied from Delambre, agrees substantially with the process given by Theon; at its conclusion he has stated the rule with perfect distinctness in the following manner: "Find the root of the nearest square to the whole number; subtract this square, convert the remainder into primes, and divide it by the double of the first root, and thus determine the next term in the root; square the sum of the terms found, subtract this square, convert the remainder into seconds, and divide it by the double of the root already found, and you will have the square root very nearly."

Reasons for continuing the sexagesimal division of the circle.

(40.) The sexagesimal division of the circle has continued to our times, and is likely to continue, notwithstanding the attempt made in France, at the same period that they altered their measures of length, weight, and capacity, to replace it by the decimal division, or rather centesimal, and which has been sanctioned by the authority of Laplace. If the alteration had commenced with the centesimal division of the degree which should itself have remained unchanged, it would probably have met with general adoption, as it would have produced a considerable simplification of logarithmic tables, and would have assimilated trigonometrical with all other processes of calculation; but, by attempting to change the primary divisions of the circle, they not only abandoned the advantages presented by the number of divisors of 60, 90, 360, of which artists employed in the division of circles are very sensible, but likewise proposed to render useless the whole mass of existing tables, unless they had been calculated anew.

We have likewise retained the sexagesimal division of time, and have not merely retained the accentual notation, but likewise the names, such as minutes and seconds, which are connected with this division.* The sexagesimal division of the radius continued until the year 1464, when Regiomontanus, in his *Opus Palatinum de Triangulis*, divided the radius in ten millions; he at first proposed to divide it into sixty millions of parts, but abandoned his intention upon farther consideration, as we learn from the relation of Valentine Otho, in his Preface to that work.

Recapitulation.

(41.) In reviewing the history of Greek Arithmetic, we find it indebted for its greatest improvements to the same persons who contributed most to their geometrical and astronomical science; to Archimedes, for his indefinite extension of their numeral language; to Apollonius, for his distinction of bases and analogous numbers, and the practical methods of multiplication which were founded upon it; and, most of all, to Ptolemy, for his refined invention of sexagesimals, by which fractions and integers were brought within the compass of a common and uniform notation, and subjected to the same arithmetical operations. To this list we might, upon the authority of the learned and accurate Delambre, add the name of Diophantus, for the artifice of denoting myriads from position merely, by interposing a dot between symbols for myriads and monads, omitting the initial Mυ, or M, which are usually attached to the former; but we have given some reasons above for inducing us to believe, that if this artifice was really made use of by him, he was insensible of its advantages, as this important principle was nearly barren in his own hands, and is never noticed by subsequent writers.

Delambre considers it a fact humiliating to the pride of human genius, that the discovery of the notation by nine digits and zero, should have escaped the sagacity of these illustrious men, especially when engaged in researches connected with the improvement of arithmetical language and notation. To us, with whom this notation has been familiar from our boyhood, the invention of it may appear simple and easy; but with them it ran counter to all their associations. They had been accustomed to the use of twenty-seven independent symbols, which all appeared equally necessary for arithmetical notation; and it was not a very simple investigation which showed that nine of them only were necessary in arithmetical operations. In order to pass from this conclusion to their use in the expression of all numbers, there was required the invention of the zero and the device of place, both of them refinements of a nature not easily discovered. The Greeks also were altogether ignorant of the advantages of notation as distinct from language; and were unacquainted both with the powers of algebraical symbols, in exhibiting

* The primes were called λεπτα, that is *minuta*, or small portions of the μοιρα, or integral part.

at once to the eye and to the mind the most complicated relations of quantity, and such as language is incapable of expressing without extreme difficulty; they, in consequence, always appear to have considered numerical notation as of secondary importance to numerical language, and never attempted to make them independent of each other.

If Ptolemy had found the degree of the circle divided into 10 minutes instead of 60, and similarly in all further subdivisions, he would have been led to the invention of the decimal instead of the sexagesimal Arithmetic, with the zero, and much at least that is most essential in the device of place; for the accentual marks which distinguish the several orders of sexagesimals, though they made the zero unnecessary, did not supersede it; and the order in which these quantities were written gave their relative value with respect to each other, and their absolute value with respect to the primary unit. It might be objected, indeed, that the sexagesimal division was applied in a descending and not in an ascending scale; the units themselves being written in the ordinary notation, and not classified according to ascending powers of 60:* but we must keep in mind that this notation was introduced for avoiding the inconveniences of the common notation in the treatment of fractions, and not for the purpose of superseding it; and that its inventor and his successors naturally terminated their innovations, when they had fully answered the purpose for which they were introduced.

Ancient Greek arithmetical notation.

(42.) It is impossible, from any existing records or monuments, to fix the date of the origin of Greek arithmetical notation. We may assume it to have been introduced subsequently to their alphabet, and that it was unknown also at the period of the colonization of Latium, as no traces of it are discoverable among the Romans;† and we have before mentioned (Art. 22,);‡ a notation mentioned by Herodian the Grammarian, as made use of by Solon, in writing his laws, and which is frequently observed in ancient coins, and in monumental and other inscriptions. Thus in the Arundelian marbles we have the inscription, (in modern characters,) Ἀφ᾽ οὗ Κέκροψ Ἀθηνῶν ἐβασίλευσε, καὶ ἡ χώρα Κεκροπία ἐκλήθη τὸ πρότερον καλουμένη Ακτικὴ ἀπὸ Ακταίου Αυτοχθόνος ἔτη ΧΗΗΗΔΓΙΙΙ, (1318;) and again, Ἀφ᾽ οὗ Ὁμήρος ὁ ποιήτης ἐφανήθη, ΠΗΔΔΔΔΙΙΙ, (643.)* But it by no means follows from the use of these numerals on such occasions, that the other were unknown; it is sufficient that the one were more ancient than the other, to induce engravers and others to make use of them, whether from respect to, or affectation of, antiquity. Such at least may be easily imagined to have been a prevalent feeling, if we may judge from the practice of modern times.

Arithmetical notation of Semitic nations.

(43.) The Greeks derived their alphabet from the Phoenicians, and from a similar source they derived also the use of their ordinary numerical notation; for we find the same system in use amongst the Hebrews, Syrians, and in short amongst all Semitic nations. An enumeration of some of those systems, combined with some observations on the names and positions of the three interpolated symbols, will render their origin perfectly clear.

Of the Hebrews.

The following is the system of Hebrew numerals:

1. א Aleph.	60. ס Samech.
2. ב Beth.	70. ע Ain.
3. ג Gimel.	80. פ Pe.
4. ד Daleth.	90. צ Tsadi.
5. ה He.	100. ק Koph.
6. ו Vau.	200. ר Resch.
7. ז Zain.	300. ש Schin.
8. ח Chet.	400. ת Thau.
9. ט Teth.	500. ך Caph final.
10. י Jod.	600. ם Mem final.
20. כ Caph.	700. ן Nun final.
30. ל Lamed.	800. ף Pe final.
40. מ Mem.	900. ץ Tsadi final.
50. נ Nun.	

The ancient Hebrew and Samaritan alphabets consisted of only twenty-two letters, and the simple numeral symbols proceeded no farther than 400; to denote 500, they combined the symbols for 400 and 100, thus, תק; 600, תר; 700, תש; 800, תת; 900, תתק.† The same is the case also in the Syriac characters;‡ and, according to the statement of De Sacy,§ with the alphabet of the ancient Arabs. It was only in later times that they appear to have added the five final letters, to bring their numeral notation up to the limits of their numeral language.

Greek alphabet and numeral symbols of Semitic origin.

(44.) The comparison of the Hebrew numeral characters with those of the Greeks, will show at once their common origin, particularly when combined with the names which were given by the Greeks to their interpolated symbols; thus Alpha, Beta, Gamma, Delta, Epsilon, correspond with Aleph, Beth, Gimel,

* We find, however, in the *Commentaries* of Theon on the fourth Book of the *Almagest*, examples of superior sexagesimals, though the highest order of the sexagesimals are considered, as far as the notation is concerned, as the primary units: thus, in reducing ζυιβ ι′ μδ″ να‴ μ⁗, or 7412° 10′ 44″ 51‴ 40ⁱᵛ to the sexagesimal notation, he divides 7412 twice by 60, and writes down the result after the first division under the form

ρκγ λβ′ ι″ μδ‴ να⁗ μ′′′′′

and after the second as follows,

β̄ γ′ λβ″ ι‴ μδ⁗ να′′′′′ μ′′′′′′

which is equivalent to

$2^{\text{sec. sex.}}$ $3^{\text{sex.}}$ 32^{d} 10′ 44″ 51‴ 40ⁱᵛ.

It is clear from these examples, that the accents had reference to relative value from position only; and the quantities to which they were attached varied with the variation of the value of the primary units.

† In the later ages of the Roman Empire, the Greek numeral notation was sometimes made use of; the digits were denoted by *a, b, c, d, e, f, g, h, i;*

the articulate numbers of the first order by

k, l, m, n, o, p, q, r, s:

and those of the second order by

t, u, x, y, z, I, V, hi, hu, &c.

Henischius, *de Numeratione Multiplici.*

‡ It is found in a short Tract περὶ τῶν ἀριθμῶν amongst the *Grammatici Veteres.*

* See also Rose's *Inscriptiones Græcæ Vetustissimæ*, p. 41 and 137—140.

† Professor Leslie, in his *Philosophy of Arithmetic*, has characterised the numerical system as well as language of the ancient Hebrews, as equally remarkable for their poverty and rudeness. It is difficult, however, to discover upon what grounds this reproach is founded, in one respect at least, when we find that system adopted, with very few changes, by the most improved nation of antiquity; and that even under this form it was superior to that which continued to be employed by the Romans throughout their empire.

‡ Beveridge, *Arithmetices Chronologicæ*, lib. i.

§ *Grammaire Arabe*, vol. i. p. 74.

Arithmetic. Daleth, He, which denote 1, 2, 3, 4, 5; and also Zeta, Eta, Theta, with Zain, Chet, Tet, for 7, 8, 9; but in the Greek there is no letter corresponding to the Hebrew Vau, which denotes 6; and they consequently interpolated the symbol ϛ for this number, bearing as much resemblance in form to the corresponding Hebrew letter as is found amongst other letters of the alphabet, and expressly denominated by them *ἐπίσημον βαῦ*, that is, *indicating Vau*, to show its place in the system from which it was taken. The other two symbols were ϟ and ϡ, denominated *ἐπίσημον κοππὰ* and *ἐπίσημον σανπί*, that is, *indicating Koph* and *Tsadi*.* It is observable also that the symbol *Koph* has receded one place in its transmission to the Greek system, whilst the other symbol, *Tsadi*, or *σανπί*, may, or may not, be in its proper place, according as it is used for the final or initial letter of that name. Under any circumstances, the names as well as the positions in the system of these interpolated symbols, are more than sufficient to ascertain their origin, particularly when the discordance in the second half of the second, and the whole of the third *ennead* of symbols is considered, which arises from the diversity of the alphabets; from the vowels in one, and the compound letters in both.

Hebrew Arithmetic. (45.) In returning again to the Hebrew Arithmetic, we find little which distinguishes it from the Greek. Compound numbers were denoted by the combination of the symbols of the component numbers: thus כא is 21; ברץ is 932; the number 15 they denoted by טו, or 9 and 6, and not by יה, *Jah*, one of the names of the Deity, which could not be pronounced without profanation. In some cases they denoted thousands, by denoting the number of thousands by its proper symbol, and the other numbers after it thus, אתל, 1430, where א denotes 1000; ה רמב, 5242, where ה denotes 5000; but this is seldom done, unless succeeded by an articulate number of the second order; thus 1030 is hardly ever denoted by אל. It is not our object, however, to describe all the artifices of notation to which they resorted; it is sufficient for us to exemplify a system of Arithmetic made use of in the most ancient of languages, and which has been from thence transmitted, either directly or indirectly, to so many nations.†

Arabic. (46.) The ancient, or Cufic Arabic characters, were derived from the Syriac, and were only twenty-two in number. The modern characters were introduced about the year 800 after Christ, and are twenty-eight in number, though six of these are only different forms of the same letters when they appear in the middle and end of a word, like the Hebrew finals.‡ The Arabians were thus possessed, not only of the three *enneads* of symbols, which were used by the Hebrews, but likewise of a simple symbol for 1000. The same system was found also amongst the Persians, the Copts, and every other people whose language was in any considerable degree of a Semitic character.

Russian. (47.) The Russians derived their alphabet from the Greeks, amplified, however, so as to embrace the greater variety of sounds which their language required; the thirty-six letters* of their alphabet gave them numeral characters for all numbers below 10,000, and the system of accentuation extended their notation as far as any number less than 100,000,000. This notation continued as late as the time of Peter the Great, who introduced the Hindoo numerals; and in public and formal documents is sometimes made use of even at this time. We find it used also among Gothic and Scandinavian, as well as Sclavonic nations; and it was only abandoned when the influence of the Latin language, in the first place, made way for the Roman numeral characters; and, lastly, by that notation which has superseded every other. History.

Roman arithmetical notation. (48.) It would be a vain and idle task to attempt to enumerate all the conjectures which have been made to account for the origin of the Latin numeral symbols; it is sufficient for us to say, that they are obviously connected with the same numeral systems which gave rise to the more ancient Greek numeral characters, (Art. 22.) In fig. 6, we have given from Gruter and Beveridge a table containing the principal forms which their numeral characters are found to assume in ancient inscriptions; the first five of them are subject to very few variations. The character for 500 is IↃ, or under an abbreviated form D; its value is doubled, or becomes 1000, by prefixing a C to it, as in CIↃ; 5000 is denoted by IↃↃ, and 100,000 by CCIↃↃ; and the value becomes increased in a decuple proportion, by the successive addition of pairs of C, on each side of the line I; thus 100,000 is denoted by CCCIↃↃↃ; 10,000,000 by CCCCIↃↃↃↃ, and so on.

Though 6 is usually denoted by VI. yet in some inscriptions we find it expressed by six lines; thus we find IIIIIIVIR for *sevir*, or *sextumvir*; 20 is mostly denoted by XX, but sometimes by ⋇, and 30 by ⋇⋇; but V and L are never repeated, and X and C never more than four times. By placing a line over these numeral characters, their values were increased one thousand fold; thus $\overline{\text{I}}$ is 1000, $\overline{\text{V}}$ is 5000, $\overline{\text{X}}$ is 10,000, $\overline{\text{L}}$, 50,000, $\overline{\text{C}}$, 100,000; 2000 was usually denoted by CIↃCIↃ, but sometimes also by IICIↃ, or IIM; and in the same manner 4000 was represented by IVCIↃ, 7000 by VIICIↃ, and similarly in other cases.

Examples. Examples without number of these notations are every where to be found in classical authors and in inscriptions; we shall merely give the following,

Nam ferme ante annos DCCCCL (950) floruit
Homerus, intra ∞ (1000) *natus est.*

Velleius Paterculus.

Homo qui primus factus est ante annos (ut tradunt) IIIMDC (3600.) Plin. *Hist. Nat.* lib. xxxvi. c. 13.

Proh deûm atque hominum fidem! qui H-S CCCIↃↃↃ CCCIↃↃↃ CCCIↃↃↃ (300,000 *sestertia*) *quæstus facere noluit*, (*nam certè H-S CCCIↃↃↃ CCCIↃↃↃ CCCIↃↃↃ merere potuit et debuit, si potest Dionysio H-S CCCIↃↃↃ CCCIↃↃↃ* (200,000 *sestertia*) *merere*) *is per summam fraudem et malitiam et perfidiam H-S IↃↃↃ* (50,000 *sestertia*) *appetiit!*

Cicero, *pro Roscio Comœdo.*

Fig. 7. In fig. 7 is given an inscription, which will illustrate some other forms of numeral characters, which are of frequent occurrence.

These examples will sufficiently exhibit the cum-

* Seyffarth, *de Sonis Literarum Græcarum*, p. 592.
† Beveridge, *Arithmeticæ Chronologicæ*, lib. i.
‡ De Sacy, *Grammaire Arabe*, p. 74.

* Vater, *Grammatik der Russischen Sprache.*

brous structure of the Roman arithmetical notation, and will also account for the total absence of all arithmetical operations amongst them, which were not performed by means of the Abacus; and it is one of the many proofs of their extreme indifference to all scientific improvements, that a system so incommodious was not abandoned and replaced by the more perfect and comprehensive notation of the Greeks.

Its inconveniences.

In this instance the simplicity of arithmetical notation suffered from its being perfectly symbolical, and altogether independent of language, as all numbers were expressed by the mere apposition of the symbols for numbers in a series commencing from unity, and formed by successive and alternate multiplications by five and two; a mode of forming compound numbers much less simple than what is followed in nearly all languages, of expressing them as the sum of the digits, and the several articulate numbers which they contain.

The numeral notation of the Romans was adopted in almost every part of their extensive empire, and continued to be employed wherever the Latin language was used, long after the introduction of the Arabic numerals, from a feeling of respect to antiquity, and a desire of conforming in every particular to the practice of classical authors. The sexagesimal Arithmetic indeed prevailed amongst astronomers, and was used in astronomical tables and calendars; but it was clothed in Roman numerals, notwithstanding the inconvenience of such a practice for the purposes of calculation, and the knowledge of a better and more commodious notation.

Palmyrene and Phœnician numeral symbols

(49.) We have before mentioned the extraordinary analogy which exists between the Roman numerals and those which are found upon Phœnician and Palmyrene coins and inscriptions, (fig. 2 and 3.) In the last of these systems we find 2, 3, 4, denoted by the repetitions of the symbol for 1; 5 by a symbol very nearly resembling the Roman V in an inclined position; 6, 7, 8, 9, in the same manner as in Roman numerals, the symbols being written in an inverted order, conformably with the Eastern practice of writing; between 20 and 100, the numeration proceeds by the vicenary scale; the symbol for 100 is the same as that for 10, with the symbol of 1 preceding it; that for 200, is the symbol for 10, preceded by two units; the symbol for 300 is preceded by three units; for 500, by the symbol for 5, and so on to 1000, which is formed by repeating the symbol for 10 twice, and placing a unit before it. The Phœnician numerals generally agree with the Palmyrene, except that they possess no symbol for 5; the nine digits being formed by the repetition of the symbol for unity as often as it may be required.*

Egyptian hieroglyphical symbols.

(50.) In the hieroglyphical symbols for numbers made use of by the ancient Egyptians, as ascertained by the researches of Dr. Young,† we find the digits denoted by the repetition of the symbol for unity, with simple symbols for 10, 100, 1000, all the intermediate articulate numbers being denoted by the repetitions of those symbols, (fig. 8.) This system is decimal throughout, without any intermixture of any other scale, whether quinary or vicenary.

Fig. 8.

Palpable Arithmetic.

(51.) The existence of systems of symbolical Arithmetic implies some considerable progress in the arts of life; and we, consequently, cannot expect that such systems should be numerous, particularly when we consider how few are the nations with whom civilization has been of native growth. Amongst ancient people, we may refer all those systems to Egypt and Syria for their origin, however much modified in later times by the habits and languages of the people to whom they were transmitted. In passing from ancient to modern nations we shall find, that with the exception of China, possessing both a literature and institutions so different from all other nations, the Hindoo Arithmetic has superseded every other species of numeral symbols, both in Asia and Europe. Before we proceed, however, to the notice of the gradual advance of this Arithmetic from the East to the West, or the circumstances which accompanied its introduction, we shall premise a few remarks on the practice of Arithmetic by means of the Abacus, which was so much used by the ancients, and which was in general use amongst the nations of Europe until the end of the XVth century.

Roman abacus.

Fig. 9.

In the *Theatrum Arithmeticum* of Leopold, we have a representation of a Roman Abacus, which was preserved in the library of St. Geneviève, at Paris, and which is copied in fig. 9; in this, the numbers are denoted by small round counters moving in parallel grooves. There are seven divisions for whole numbers, representing units, tens, hundreds, thousands, ten thousands, hundred thousands, millions; the value of each superior unit being denoted by the numerical symbols which are placed between the long and the short grooves respectively. The counters in the longer grooves represent units, and in the shorter five; thus to denote 6, we put one counter in the longer and one in the shorter groove, between which I is placed; to denote 70, we put two counters in the longer and one in the shorter groove, between which X is placed; and similarly in other cases, the principle of denoting numbers by means of this instrument being too simple to require further explanation.

Below the place of units, there is a pair of grooves appropriated to the division of the *as*;* the counters in the long groove denote *unciæ*, or the twelfth part of the pound, and those in the short groove one half of it; thus five counters in the long, and one in the short groove, would denote 11 ounces. In order to denote the divisions of an *uncia*, there are three short grooves added; to the first is attached the symbol ·S·, or ʒ, which denotes *semuncia*, or half an ounce, which is the value of the counter placed in it; to the second is attached the symbol Ɔ, which denotes *sicilicum*, or *sicilicus*, the fourth part of an ounce; and to the last, to which two counters are appropriated, belongs the symbol 2, designating, according to Velser, a *duella*, or third part of an ounce, but, more probably, a *duodecima*, or twelfth part of an ounce, a supposition which would enable them to denote all the duodecimal parts of an ounce, by means of the four counters in the three grooves.†

In some cases the grooves were replaced by wires

* A minute examination of the forms of the symbols for 5, 10, 20, 100, might, very probably, show that they were modified forms of the letters of the alphabet which represented the same numbers in the different and strictly alphabetical Arithmetic which the Greeks derived from them.

† *Discoveries in Hieroglyphical Literature.*

* Leslie's *Philosophy of Arithmetic.*

† Weidler and Ward, *Philosophical Transactions for* 1744.

Arithmetic.

upon which were strung perforated beads, four on the longer and one on the shorter of each of them; in order to represent numbers, the requisite number of beads were moved on to the end of the wires, leaving the remainder in each case, if any, on the other extremity.*

Chinese Swan Pan. Fig. 10

(52.) Under this form, the Roman Abacus resembled the Swan Pan of the Chinese, which travellers have so frequently described, and a representation of which is given in fig. 10; it consists of ten parallel wires, unequally divided, with four beads in the longer and two on the shorter portions, and embraces numbers as far as *ten billions*. In representing numbers upon it, the wires are placed horizontally, the Abacus itself being vertical, and the values of the beads increase in descending, the greater numbers being placed underneath the smaller, in the same manner as in expressing numbers by their symbols, (Art. 13.) As the decimal division applies to their coins and to all their measures of weight, length, and capacity, this instrument is adapted to arithmetical operations of every kind; and so great is their dexterity in the use of it, that they have become celebrated throughout the Indian Archipelago and the neighbouring countries for their skill in practical Arithmetic.†

Classical allusions to the Tabula Logistica.

(53.) The Abacus, or Tabula Logistica, which was generally used, was merely a rectangular tablet, strewed with sand, in which grooves were made by the hand; the counters, (*calculi*, or *lapilli*,) were contained in little boxes, (*loculi*,) and Horace alludes to the custom in his time, of boys marching to school with the Abacus and its furniture suspended on their left arm:

> *Quo pueri magnis ex centurionibus orti,*
> *Lævo suspensi loculos tabulamque lacerto.*
> *Sat.* i. vi. 75.

Persius alludes to the custom of strewing the tablet with sand, in the following passage:

> *Nec qui abaco numeros et secto in pulvere metas*
> *Scit risisse vafer.* *Sat.* i. 131.

This sand, according to Martianus Capella, was of a sea-green colour:

> *Sic abacum perstare jubet, sic tegmine glauco*
> *Pandere pulvereum formosum ductibus æquor.*‡

Cicero makes use of the phrase *eruditum attigisse pulverem*,§ in a metaphorical sense to denote a person who is skilled in the science of numbers and calculation; and Tertullian applies the terms *primi numerorum arenarii*,‖ to the teachers of the first rudiments of arithmetical knowledge.

The counters which were made use of were of various kinds; and in the progress of Roman luxury were formed of the most precious materials. Thus Juvenal alludes to the employment of counters of ivory in the following lines,

> *Adeo nulla uncia nobis*
> *Est eboris, nec Tessellæ, non calculus ex hâc*
> *Materiâ.* *Sat.* ii. 131.

* This is the form of the Abacus, a drawing of which is given by Velser, and which is copied by Gruter, vol. i. p. 224.

† *Philosophical Transactions for* 1686, No. 180; Smethurst, in the same *Transactions for* 1749; *Leslie's Philosophy of Arithmetic*, p. 221; and Crawford's *Indian Archipelago*, vol. ii.

‡ *De nuptiis Philologiæ et Mercurii et de septem artibus liberalibus*, lib. vii. *de Arithmetica*; Leslie's *Philosophy of Arithmetic*, p. 221.

§ *De Natura Deorum*, lib. ii. 18.

‖ Mahudel, *Académie des Inscriptions*, vol. v. p. 261.

History.

And from a passage in Petronius Arbiter we may suppose that in later times they were sometimes made of silver, and even of gold: *Notavi rem omnium delicatissimam, pro calculis albis aut nigris, aureos argenteosque habebat denarios.**

The familiar use of these counters gave rise to numerous metaphorical phrases amongst classical authors, which have reference to arithmetical operations on the Abacus; thus *calculos ponere*, or *movere*, to state an argument; *hic calculus accedat*, to signify the addition of a proof to others which have preceded; *calculum detrahere*, or *subducere*, to suppress a proof, or step in an argument; *calculum reducere*, to change a line of conduct or reasoning, with which you are dissatisfied; and many other phrases, the proper force of which can only be understood by a reference to the use of this instrument.

Greek Abacus.

(54.) The same instrument was likewise made use of by the Greeks, and most other ancient nations; their counters were called ψῆφοι, and the process of calculation by their means ψηφοφορία. Amongst other distinctions which Herodotus has mentioned between the customs of the Greeks and Egyptians, it is said, "that in writing and in calculating with counters, the Greeks move the hand from the left to the right, but the Egyptians from the right to the left."† Some authors have attempted to trace the derivation of the use of this instrument from Abraham to the Egyptians, Phœnicians, and from thence to the Greeks; without, however, venturing upon so minute an examination of its history, we may certainly infer that its use was very general amongst the nations of antiquity; and that in almost every instance it preceded the use of symbolical Arithmetic.

Its use to a late period in Europe.

(55.) The use of counters was general throughout Europe as late as the end of the XVth century; about that period they had ceased to be used in Italy and Spain, where the early introduction of the Arabian figures, and the number of treatises of practical Arithmetic by means of them, had rendered them unnecessary. They were used to a still later period in France, and had not disappeared in England and Germany before the middle of the XVIIth century. Shakspeare, who may be considered as correctly representing the customs and opinions of his times, exhibits the clown in the play as embarrassed with an arithmetical question, and declaring that he could not do it without counters:‡ and Iago, to express his contempt of Michael Cassio, *forsooth, a great arithmetician*, terms him a *counter caster*.§ So general, indeed, appears to have been the practice of this species of Arithmetic, that its rules and principles formed an essential part of the arithmetical treatises of that day: thus Robert Record, in his *Arithmetick*, or *the Ground of Arts*,‖ prefaces his second dialogue, entitled *The Accounting by Counters*, by observing, "Now that you have learned Arithmetick with the pen, you shall see the same art in counters; which feat doth not onely serve for them

* Mahudel, *Académie des Inscriptions*, vol. v. p. 261.

† Γράμματα γράφουσι καὶ λογίζονται ψήφοισιν, Ἕλληνες μὲν ἀπὸ τῶν ἀριστερῶν ἐπὶ τὰ δέξια φέροντες τὴν χεῖρα. Αἰγύπτιοι δὲ ἀπὸ τῶν δεξιῶν ἐπὶ τὰ ἀριστερὰ, lib. ii.

‡ *The Winter's Tale*, act iv. sc. 3. "CLOWN. Let me see, every eleven weather tods, every tod yields—pounds and odd shillings, fifteen hundred shorn, what comes the wool to? I can not do it, without compters."

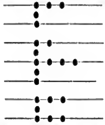
§ *Othello*, act i. sc. 1.

‖ First printed in 1540.

Arithmetic. that cannot write and read, but also for them that can do both; but have not at the same time their pen or tables with them;" and in a *Treatise on Arithmetic*, published in Germany as late as 1662, we find a chapter devoted to *Arithmetica Calcularis*, which is said to be of such common and general use amongst merchants, that it might more properly be termed *Arithmetica Mercatoria*.*

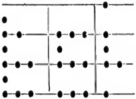
Calcular Arithmetic. (56.) We shall now proceed to give some account of the method of performing operations by this *calcular Arithmetic*. They commenced by drawing seven lines with a piece of chalk, or other substance, on a table, board, or slate, or by a pen on paper; the counters† on the lowest line represented units, on the next tens, and so on as far as a million on the last and uppermost line; a counter placed between two lines was equivalent to five counters on the lower line of the two;

Notation.



thus the disposition of counters in the annexed example represents the number 3629638; and it is clearly very easy to increase the number of lines so as to comprehend any number that might be required to be expressed.

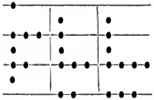
Addition. Suppose it was required to add together 788 and 383; express the numbers to be added in the two



first columns. The sum of the counters on the lowest line is 6; write, therefore, one on that line in the third column; carry one to the first space, which, added to the one already there, is equal to one on the second line; place a counter there, and add all the counters on that line together, the sum is 7; leave, therefore, two counters on that line, and pass one to the next space; add the counters on that space together, which are 3; leave one there, and place one also on the next line; add all the counters in that line together; the sum is 6. Leave one counter, and pass another to the next space; add all the counters in that space together, which are 2; leave no counter in the space, but pass one to the next, or fourth line; we thus represent the sum, which is 1171.

The principle of this operation is extremely simple, and the process itself, after a little practice, would clearly admit of being performed with great rapidity. History. In giving a scheme of this operation, we have made use of three columns; but in practice no more would be required than are sufficient to represent the sums to be added, the counters on each line being removed as the addition proceeds, and being replaced by the counters which are requisite to denote the sum.

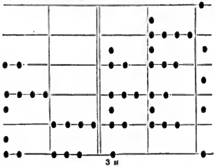
Subtraction. We shall now proceed to a second example: namely to subtract 682 from 1375.



Write the numbers in the first and second column. The two counters on the last line have none corresponding to them in the minuend; bring down the counter in the first space, and suppose it replaced by 5 counters; take away 2, and 3 remain on the lowest line of the remainder. Again, the three counters on the second line must be subtracted from 7, (bringing down 5,) and therefore leaving 4 on the second line of the remainder. The counter in the second space has now no counter corresponding to it in the minuend; remove one counter from the next line, and replace it by two counters in the next inferior space; there will remain, therefore, one counter for that space in the remainder. There is now one counter on the third line to subtract from two in the minuend, and there remains one for the remainder. The counter in the next space has nothing corresponding to it, and we must therefore bring down the counter on the highest line and replace it by two counters in the space below it; if one counter be subtracted from them, there will remain one, and the whole remainder will be 693.

Recorde writes the smaller number in the first column, and commences the subtraction with the highest counters; a very little consideration will show in what manner the operation must be performed, with such a change in the process.

Multiplication. We shall now give an example of multiplication, and let it be proposed to multiply 2457 by 43:

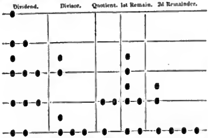


* Gasparis Schotti, *Arithmetica Practica*, Herbop. 1662.

† These counters were usually of brass.

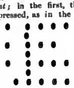
 Write the multiplicand in the first column, and the multiplier in the second; multiply first by 3, and write the product in the third column, and then by 4 in a superior place, and write the result in the fourth column; add the numbers in these two columns together, and the sum is the product required.

Division. We shall conclude with an example of division, and let it be required to divide 12,832 by 608:



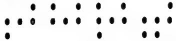
Write the dividend in the first column, and the divisor in the second, reserving the third for the quotient; then since 6 is contained twice in 12, in the line above that in which 6 is written, we may put down 2, in the last line but one in the column for the quotient; multiply 6 by 2, and subtract; there is no remainder; multiply 8 by 2, and subtract 16 from the number expressed by the counters remaining in the dividend in the line above the last; first take one counter from the three in the third line, and two remain; next take 6, which is done by taking 1 from the second line from the bottom, and bringing 1 from the third line, replacing it by 2 in the space below, and then subtracting one of them, thus leaving 67 as the remainder to be denoted in the second and third lines, and the spaces above them; the remaining two counters in the dividend are transferred to the corresponding line in the column for the first remainder; the operation is now repeated, the next figure in the quotient, or 1, being written on the lowest line; it is now merely necessary to subtract the divisor from the first remainder, and we get 64 for the second and last remainder. It is evident that the same process may be repeated to any extent that may be required; and that the complication of the process, as exhibited in a *scheme*, is much greater than in practice, where the dividend is replaced by the first remainder, and so on successively, until the remainder is zero, or less than the divisor.

Manner of using counters in merchant's accounts. (57.) Recorde* has mentioned two different ways of representing sums of money by means of counters, one of which he calls the *Merchant's*, and the other the *Auditor's Account*; in the first, the sum of £198. 19*s.* 11*d.* is expressed, as in the annexed scheme:



the lowest is the line of pence, the second of shillings, the third of pounds, and the fourth of scores of pounds; the single counters in the spaces denote half of the units in the next superior line: sixpence on the first space; ten shillings on the second; ten pounds on the third; and the detached counters to the left are equivalent to five counters to the right; the lowest of them, therefore, representing five shillings, the second five pounds, and the highest one hundred pounds; and the whole sum is expressed by being resolved into the following parts: £100. + £80. + £10. + £5. + £3. + 10*s.* + 5*s.* + 4*s.* + 6*d.* + 5*d.* The principle of this notation being once understood, it is unnecessary to give examples of addition, subtraction, multiplication, or division, which present no difficulty after the examples which we have given for abstract whole numbers. **History.**

Auditor's accounts. The mode of denoting the same sum *for the accompt of auditors*, is as follows:



The counters on the two lowest lines denote units of their respective classes; on the upper line, when placed to the left, they denote one quarter, and on the right one-half of the next superior unit.

In both these cases, we may consider the resolution of the number of pounds, into *twenties*, as a vestige of the preference for the vicenary scale, which was so general with our ancestors.

Bank. (58.) In ancient times, it was the custom for merchants, bankers, or money changers, auditors of accounts, and judges in affairs of revenue, to appear on a *bank*, or bench, and before them on a board, or table, were arranged the counters which were necessary in making their calculations; and the name of the Court of Exchequer was derived from *scaccarium*, a quadrangular table, about ten feet long and five broad, with an elevated ledge, around which the judges, tellers, and other officers were seated; it was covered with black cloth, divided by white lines at right angles to each other; they used small coins for counters, those on the lowest bar denoting pence, the second shillings, the third pounds, and the upper bars tens, twenties, hundreds, thousands, and ten thousands of pounds. The teller sat about the middle of the table; on his right hand, *eleven* pennies were heaped on the first bar, and a pile of *nineteen* shillings on the second; while a quantity of pounds was collected opposite to him on the third bar; for the sake of expedition, he sometimes employed a silver penny to represent ten shillings, and a gold penny for ten pounds.* **Exchequer.**

Algorithm, its meaning. (59.) The term *algorithmus*, or *algorismus*, was employed universally in the XIVth and XVth centuries, to denote the science of calculation by nine figures and zero; and its composition clearly shows the source from which it was derived in our own language. Algorism was corrupted into *Augrym*, or *Awgrym*, and the counters which were used in calculation were called *augrym* stones. Thus in Chaucer's description of the chamber of Clerk Nicholas,

* *Arithmetic*, p. 213.

* Leslie's *Philosophy of Arithmetic*, p. 97.

His climageste, and bokes grete and smale,
His astrelabre, longing for his art,
His *augrim* stones, layen faire apart,
On shelves couched at his beddes head.*
Millers Tale, v. 22—25.

Modern Palpable Arithmetic.

(60.) There are not wanting in our own times examples of persons who have attempted to revive the practice of Arithmetic by counters. Professor Leslie, in his *Philosophy of Arithmetic*, considers this method as better calculated than any other to give a student a philosophical knowledge of the classification of numbers, and the theory of their notation; and with this view he has given, in great detail, examples of the representation of numbers in different scales of notation by counters, as well as of arithmetical operations by means of them. With every feeling of respect for the opinions of this very distinguished author, we shall venture in this instance, on more grounds than one, to dissent from them. In the first place, in this mode of denoting numbers, the values of the counters depend upon their position, as well as in the notation by nine figures and zero, and as several counters correspond to one digit only, the first method is, on this account, much less simple than the other, when viewed as a representation addressed to the eye as well as to the mind; and, in the second place, arithmetical operations by counters are not so easily reducible, as in the case of figures, to rules which are few in number, simple in principle, and rapid in practice.

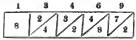
Saunderson's calculating board.

(61.) There are other species of Palpable Arithmetic, some of which have been adopted especially for the use of blind people; the celebrated Saunderson invented an instrument for this purpose, with which he is said to have worked arithmetical questions with extraordinary rapidity. His abacus, or calculating board, was about a foot square, divided into small squares, one hundred in each square inch, by sets of parallel lines at right angles to each other. At every point of intersection the board was perforated by small holes, capable of receiving a pin, of which he used two sorts, one with a large head, which denoted zero, and the other with small heads; and to every figure was appropriated a square, consisting of four smaller and contiguous squares. Zero was denoted by a large pin in the centre of the square; to denote unity this was removed and replaced by a small one; for other digits, the large pin was placed in the centre, and a small one either in the angle or middle of one of the sides of the square; and the position used to denote the several digits are given in fig. 11. The scheme in that figure represents a portion of the board, upon which are denoted the several sums which are appended. It is quite evident that with such an instrument any arithmetical operations might be performed, the sums being placed as in common figurate Arithmetic, and the successive steps of the process being recorded in the same manner.†

Arithmetical instruments of the kind which we have just described, possess considerable interest and importance from their use in lessening the privations consequent upon one of the greatest human calamities.

(62.) Many other arithmetical instruments or machines have been invented for the purpose of either shortening arithmetical operations, or otherwise for relieving the operator from any troublesome or difficult exertion of the memory. Of this description are the *virgulæ*, or *rods* of Nepier, which were formerly much celebrated and very generally used. The work in which they were first described was published in 1617, under the title of *Rabdologia*.* In the dedication to Chancellor Seton, he says, that the great object of his life had been to shorten and simplify the business of calculation; and the invention of logarithms, which he had just promulgated, was a noble proof that he had not laboured in vain. These *virgulæ*, *rods*, or *bones*, as they were often called, were thin pieces of brass, ivory, bone, or any other substance, about two inches in length and a quarter of an inch in breadth, distributed into ten sets, generally of five each; at the head of each of these, in succession, was inscribed the nine digits and zero, and underneath them in each piece the products of the digit at the top with each of the nine digits in succession, in a series of eight squares divided by diagonals, in the upper part of which were put the digits in the place of tens, and in the lower the digits in the place of units. In order to multiply any two numbers together, such as 3469 into 574, those rods are to be placed in contact which are headed by the digits 1, 3, 4, 6, 9; and the successive products of the figures of the multiplier into the multiplicand are found by adding successively together the digit on the upper half of the square to the right, so that in the lower half of the square to the left, in the line of squares which are opposite to the figure of the multiplier which is employed in the operation: thus to multiply 3469 by 8, we take the line of squares opposite to 8, which is represented by

Napier's rods



and the product is 27752, being found by writing first 2, the sum of 8 and 7, 2 and 4, 4 and 3, and 2, carrying tens when necessary, as in ordinary Arithmetic. In the case of division, those rods are arranged in contact which are headed by the figures of the divisor; and from thence we are enabled to form the products which the quotient forms with the successive figures of the divisor.

In the case which contains these rods, which Napier calls *multiplicationis promptuarium*, there are usually found also two pieces with broader faces, one consisting of three longitudinal divisions, and the other of four; one of which is adapted to the extraction of the square, and the other of the cube root; in the first, one column contains the nine digits, the second their doubles, and the third their squares; in the second, the first column contains the digits, the second their squares, and the third and

* Leslie's *Philosophy of Arithmetic*, p. 221.
† Saunderson's *Algebra*, vol. i. p. xxi.

* *Rabdologiæ seu Numerationis per virgulas libri duo, authore et inventore* Joanne Nepero Baroci Merchistonii et Scoto. The subject appears to have attracted immediate attention, and the invention was circulated throughout Europe with extraordinary rapidity, forming the subject of many separate publications, and a part of almost every book on Arithmetic which was published between 1625 and 1660.

fourth their cubes, two columns being necessary for this purpose, when the cube consists of three places: thus the last division but one in the first is represented by

6/4	16	8

and in the second by

5/1	2	64	8

In our times, when students in Arithmetic are more perfectly acquainted with their multiplication table than our ancestors appear to have been, we may feel some degree of surprise at the eagerness with which this invention was welcomed at its first publication, when its only object was to relieve the memory from so slight and trivial a burden; we shall afterwards, however, have occasion to notice examples in the books of Arithmetic of that and the preceding age, of the extreme anxiety of their authors to devise expedients to simplify the processes of multiplication and division; and we shall also find, that the arrangement of the half squares in Napier's rods agrees exactly with the method of multiplying numbers, which was adopted in Hindostan, Persia, and Arabia. At the conclusion of this work of Napier is added a short Tract, entitled *Arithmetica Localis*, which is merely entitled to notice from its being the production of so great a man. It is very ill adapted, however, to any practical use, and altogether unworthy of the genius of its author.

Napier's *Arithmetica Localis.*

Other arithmetical machines.

(63.) Leibnitz invented an arithmetical machine by which any numbers might be multiplied together;* and Leopold, in his *Theatrum Arithmeticum*, has recorded many others, including two of his own. The limits of this work will not allow us to enter into any description of these inventions, which would necessarily lead to great details. With respect to all of them, however, it may be remarked, that as they merely propose to multiply numbers together, the importance of the object to be attained bears no proper or reasonable proportion to the difficulty and refinement of the means which are required to attain it. In our own times, however, a gentleman of profound knowledge of practical mechanics and general science, and distinguished for the uncommon inventive powers of his mind, is engaged in the construction of an arithmetical machine of a very extraordinary character. It is adapted to the performance of all arithmetical calculations which depend upon differences; and, consequently, to the construction of logarithmic and many classes of astronomical tables; and is not limited to the mere work of calculation, but distributes the types which are required to record and register the result of its operations without the possibility of error.

Origin, antiquity, and period of the introduction of Arabic numerals.

(64.) There are few subjects which have given rise to more frequent controversies, than the invention of the notation by nine figures and zero; whether we regard the country which gave it birth, the channels through which it was communicated to Europe, and the period at which it was first known and generally adopted. The total revolution which this invention

* Leibnitzii *Opera*, vol. iii. p. 413.

effected in the practice of arithmetical calculation, whether for scientific or ordinary purposes, gives it an uncommon degree of importance in the history of the progress of human knowledge; and we shall therefore make no apology for discussing its origin and progress at considerable length.

Antiquity of this notation among the Hindoos.

(65.) We have before mentioned our reasons for thinking that the Hindoos had possessed a very perfect system of Arithmetic from great antiquity, from the internal evidence of their numeral language, independent of any external testimony. The assertion, however, of Anquetil du Perron,* that the ancient Sanscrit alphabet was distributed like the Greek into three classes of numeral letters, would greatly invalidate such an opinion, as such a notation must have preceded that with nine figures and zero, it being extremely improbable that a system of notation so inconvenient as the first, could have been adopted, when the other was already known and practised; but the opinions of this very fanciful and learned author have not been corroborated by the late researches of oriental scholars incomparably better acquainted with the antiquities of the Sanscrit language than himself; and we may, therefore, venture to consider it in the light of one of his numerous other dreams which have been found to have no foundation in fact. It remains to consider to what extent the antiquity of this invention may be ascertained from the testimony of Sanscrit authors.

Age of Bhascara.

(66.) We have two translations of the *Lilávati* and *Vija-ganita* of Bhascara, works on Arithmetic, Mensuration, and Algebra, which enjoy the highest reputation in Hindostan; of the first by Dr. Taylor, of the second by Mr. Strachey, and of both by Mr. Colebroke,† an author equally remarkable for his profound knowledge of oriental literature, and for his great scientific acquirements; to the last is prefixed a dissertation on the state of algebraic knowledge among the Hindoos, Arabs, and the Greeks, in which the respective claims of these people to originality in the possession and invention of the rules of this science are discussed with uncommon learning. He has established beyond controversy that Bhascara, the author of the *Siddhanta siromani*, of which these works are a part, lived about the middle of the XIIth century of the Christian era. He has also shown that Brahmegupta, an author frequently quoted by Bhascara, and portions of whose works, containing treatises on Arithmetic and Mensuration are extant, lived in the early part of the VIIth century; and again, that Arya-bhatta, who is referred to by Brahmegupta, and considered the oldest of their uninspired and merely human writers, and the subject of part of whose works was Algebra and Arithmetic, flourished at least as early as the Vth century, and probably at a much earlier period.

Brahmegupta.

Arya bhatta.

Hindoo Arithmetic and Algebra at least as old as Diophantus.

From these facts, which appear to be established upon very satisfactory evidence, it appears that Hindoo Algebra and Arithmetic are at least as ancient as Diophantus, and preceded, by four centuries, the introduction of those sciences among the Arabs; and in no case is the original invention of the notation by nine digits and zero referred to by any of these authors, but is always stated to be one of the benefactions of the Deity, which is the best proof of its possessing an

* *Zendavesta*, vol. i. p. 172. It is also asserted that this system exists among some of the alphabets on the coast of Malabar.

† *Algebra, with Arithmetic and Mensuration from the Sanscrit.*

Grant of land dated twenty-three years before Christ.

antiquity antecedent to all existing records. If the royal grant of land engraved on a copper plate, found in the ruins of Mongueer, and translated by Dr. Wilkins,* be not a forgery, it would furnish evidence of the existence of this notation at a much earlier period than any which we have mentioned; as it is dated in these figures in the thirty-third of Sambat,† corresponding to the twenty-third year before the birth of Christ; under any circumstances, however, whatever importance we may attach to this document, there can be no doubt of the Hindoos possessing this notation long before the Persians, Arabs, or any western people.

Its antiquity amongst the Arabs.

(67.) The testimony to the same fact derived from the Arabs, is completely decisive of the source from which they derived it. The first Arabian who wrote upon Algebra and the Indian mode of computation, is stated, with the common consent of Arabic authors, to have been Mohammed ben Musa, the Khowarezmite, who flourished about the end of the IXth century;‡ an author who is celebrated as having made known to his countrymen other parts of Hindoo science, to which he is said to have been very partial. Before the end of the Xth century, these figures, which are called *Hindasi* from their origin, were in general use throughout Arabia; amongst others is mentioned the celebrated Alkindi, who was nearly contemporary with Ben Musa, and who, amongst his numerous other works, wrote one on the Indian mode of computation, (*Hisabu l'Hindi*.) The same testimony is repeated in almost every subsequent author on Arithmetic or Algebra, and is completely confirmed by their writing those figures from left to right, after the manner of the Hindoos, but which is directly contrary to the order of their own writing.§

Its use general in the Xth century.

The use of this notation became general amongst Arabic writers, not merely on Arithmetic and Algebra, but likewise on Astronomy, about the middle of the Xth century. We find it in the works of the astronomer Ebo Younis, who died in the year 1008,‖ and it is found likewise in all subsequent astronomical tables. It was, of course, communicated to all those countries where their language and science were known. In the XIth century, the Moors were not merely in possession of the southern provinces of Spain, but had established a flourishing kingdom, where the favourite sciences of their eastern ancestors were cultivated with

* *Asiatic Researches*, vol. i. p. 127.

† The present year (1826) is the 1882d year of the Hindoo period Sambat.

‡ Colebrooke, *Dissertation*, p. 69. He has also mentioned an Arabic author of the latter part of the XIIth century, who is quoted by Casiri, in his account of the Arabic manuscripts in the Escurial, as mentioning among other works derived from the Hindoos, "A book on numerical computation which Abu Jâfr Mohammed ben Musa Al Khuwárezmi amplified, and which is a most expeditious and concise method, and testifies the acuteness and ingenuity of the Hindoos." Another testimony of a similar kind, which has been frequently quoted, is from the Commentaries of Alsephadi on a poem of Tograi, who remarks that the Hindoos have three inventions of which they boast, *the game of chess, numerical figures*, and *the book called Golaila Vadamna*, or the Fables of Bidpai

§ Silvester de Sacy, *Gram. Arab.* vol. i. p. 76.

‖ Delambre, *Histoire de l'Astronomie du moyen age*, p. 140. It is stated by Dr. Bevis, in a letter to Mr. Ames, that in the manuscript of this author in the Bodleian library, the Hindoo numerals are used throughout; and that when any number is given, it is afterwards expressed in words at full length. *Selections from Gentleman's Magazine*, vol. ii. p. 162.

uncommon activity and success, and from that quarter and from the Moors in Africa they chiefly appear to have been communicated to the Spaniards and other Europeans.

This notation used in Spain in 1136.

(68.) The learned Abbé Andrea* considers that the earliest example of the use of these figures which is to be found in Spain or in Europe, is a translation of Ptolemy in the year 1136; fac similes of the forms of these figures are said to be given in the XIIth plate of the *Paleografia Spagnuola* of Terreros, who found them in all the mathematical manuscripts subsequent to that period, but in no other books or documents, nor even in accounts, which were kept in the Castilian, which differed little from the Roman numerals; the calendars which were chiefly constructed in Spain, both in that age and until the end of the XIVth century, and were sent from thence to other parts of Europe, continued to be written in the old notation.

Hypothesis of Kircher.

Alphonsine Tables.

(69.) Kircher, in his *Arithmologia*, has advanced an hypothesis which is not destitute of probability, that the knowledge of these numerals was communicated to Christian Europe by means of the celebrated astronomical tables which were formed under the direction of Alphonso, King of Castile, and published at Toledo about the year 1252. These tables were chiefly computed by Arabian astronomers, and we should naturally expect that they would adhere to the notation which had so long been in general use in the writings of their countrymen; this question, however, cannot be decided, unless by an examination of the earlier manuscripts of these tables.†

Known in Italy at the beginning of the XIIIth century.

Leonardo Pisano.

(70.) But we have positive evidence of the existence of a work, written expressly for the purpose of communicating to Europe a knowledge of Arabic numerals and Algebra, at an earlier period than the publication of the Alphonsine Tables. About the middle of the last century, Targioni Tozzetti‡ found in the Magliabecchian Library at Florence a manuscript, entitled *Liber Abbaci compositus a Leonardo filio Bonacci Pisano in anno 1202*: and another work, by the same person, on square numbers, inserted in an anonymous Tract on computation, (an *Trattato d'Abbaco*,) in the library of the Royal Hospital in the same place. A transcript of another Treatise of his was also found in the Magliabecchian library, entitled *Leonardi Pisani de filiis Bonacci Practica Geometriæ composita anno 1220*. The subject of this work is the mensuration of land, and it is mentioned by the author, in the preface to a revised copy of the *Liber Abbaci* in 1228; Tozzetti met with a second, though somewhat mutilated copy of the *Liber Abbaci*, in the same library; a third has since been discovered in the Riccardi collection at Florence; and a fourth, but imperfect one, was communicated by Nelli to Cossali.

His life.

It appears from a short account of himself and his travels, which Leonardo has introduced into his Preface to the *Liber Abbaci*, that he travelled into Egypt, Barbary, Syria, Greece, and Sicily; that being in his youth at Bugia in Barbary, where his father, by appointment of the merchants of Pisa who resorted there, was scribe to the Custom-house, he learned the method

* *Dell' origine, dei progressi et dello stato attuale d'ogni litteratura*, tom. iv. p. 57.

† These tables were first printed in 1483.

‡ *Viaggi nella Toscana*, vol. i. 11; Cossali, *Origine e primi progressi dell'Algebra*, c. i. sec. 5. ch. ii. sec. 1.

of accounting by nine figures and zero; that finding it much more commodious and far preferable to that which was used in the other countries which he had visited, he pursued the study,* and with some additions of his own, and some propositions from Euclid, he composed the treatise in question, that "the Latin race might no longer be found deficient in the complete knowledge of that method of computation."† In the epistle, also, which is prefixed to the revised copy of his work, he professes to have taught‡ the complete doctrine of number according to the Indian method.

Date of his work.

The preceding facts would refer the studies and travels of Leonardo to the close of the XIIth century, and the date of his first work, and consequently of the introduction of the Arabic numerals through his means, to the second year of the following century, and fifty years before the publication of the Alphonsine Tables. That this work was the first on this subject which appeared in Italy, we know from other authority than that of Leonardo himself, as nearly all subsequent Italian writers on Arithmetic and Algebra ascribe the honour of priority to him, and particularly Lucas Paccioli, or Lucas de Burgo Sancti Sepulchri, whose work, entitled *Summa de Arithmetica*, &c. was published in 1484, being the first work which was printed on this subject; and a succession of writers on Algebra, and therefore on Arithmetic, are mentioned by Cossali, from the beginning of the XIVth century.

Leonardo considered by some authors as having flourished at a later period.

Blancanus, in his *Chronologia Mathematica*, referred Leonardo to the beginning of the XVth century, and this date was adopted by Vossius, by Wallis, and by Montucla in the first edition of his work. Professor Leslie appears also to favour the same opinion, and founds much of his argument upon the probability that the readers of the manuscripts of Leonardo mistook the 4 for a 2, making the date 1220 instead of 1420, those figures being easily confounded in the older forms; but we see no reason whatever for doubting the judgment or authority of the numerous persons by whom these manuscripts were examined, and the frequent occurrence of those figures in a work whose subject is Arithmetic and Algebra, would appear to prevent the possibility of a mistake of this nature; but independently of internal evidence, there are other reasons which render it altogether improbable that Leonardo could have published his work at the latter period, at least if we may place any reliance upon the testimony of Paccioli and all the writers on these subjects of the preceding and following age, that he was the first person who introduced the knowledge of Algorithm and Algebra to his countrymen; for, in the first place, Paccioli appears to have taught those sciences at Venice about the year 1460; and he speaks of three persons who successively filled the professorship expressly dedicated to their exposition, who had been his predecessors in it; namely, Paolo della Pergola, Demetrio Bragadini, and Antonio Cornaro, the latter of whom had been his fellow disciple; and in the century preceding the invention of printing, innumerable Treatises *de Algorithmo*

had been written, and manuscripts of them, of that age, are now found in great numbers, not merely in the manuscript collections of Italy, but likewise in those of every part of Europe. Again, Paolo de Dagomari died about the year 1350, and obtained the surname of *Dell' Abbaco* for his skill in the science of numbers,* and Villani, the earliest Florentine historian and his contemporary, speaks of him as a *great geometer, and most skilful Arithmetician, and who surpassed both ancients and moderns in the knowledge of equations.* Raffaello Caracci, a Florentine Arithmetician of the XIVth century, also wrote a Treatise, entitled *Ragionamento di Algebra*, in which he speaks of Guglielmo di Lunis, who before his time had translated a treatise on Algebra from the Arabic into Italian;† and even Professor Leslie himself refers to a date (1355,) written in these characters in the hand-writing of Petrarch, upon a manuscript of St. Augustin on the *Psalms*, which was given him by Boccacio.‡ The inference to be drawn from these facts is, that *algebra* and *algorithm*, terms of contemporaneous introduction into Europe, and the latter of which was always applied to treatises of Arithmetic with Arabic numerals, were perfectly well known in Italy throughout the whole of the XIVth century, and consequently could not have been introduced by Leonardo, if he flourished at the beginning of the XVth; instead, therefore, of clearing away many difficulties by the adoption of the latter date, we introduce others which it is impossible to explain upon any hypothesis which is consistent with facts, and the authority of the authors of that age.

Again, the work of Leonardo was written in Latin, and he speaks of the Italians as the Latin race, a circumstance which makes it probable, that in his time the Italian had not assumed the dignity of a written language; now we know on the authority of Muratori, one of the most profound and accurate of literary antiquaries, that there is no authentic example of Italian prose before the year 1264; but that after the year 1300 it came into general use, and nearly superseded the use of the Latin in writings on ordinary subjects; we may consider this circumstance, therefore, as furnishing a strong presumption at least, that Leonardo wrote before the middle of the XIIIth century; and it would likewise prove, that the translation into *Italian* of the work of Mohammed ben Musa by Guglielmo di Lunis, which some authors have considered as furnishing the first source of their knowledge of Algorithm, was made at a later period. The Tuscans generally, and the Florentines in particular, whose city was the cradle of the literature and arts of the XIIIth and XIVth centuries, were celebrated for their knowledge of Arithmetic: the method of book-keeping, which is called especially Italian, was invented by them; and the operations of Arithmetic, which were so necessary

Early proficiency of the Tuscans in Arithmetic.

* *Quære complectens strictius ipsum modum Yndorum, et actentius studens in eo, ex proprio sensu quædam addens et quædam ex subtilitatibus Euclidis geometriæ artis apponens.*

† *Ut gens Latina de cætero absque illa minime inveniatur.*

‡ *Plenam numerorum doctrinam edidi Yndorum, quem modum in ipsa scientia præstantiorem elegi.*

* Cossali, vol. i. p. 9.

† This was most probably the Treatise of Mohammed ben Musa, a translation of which was well known in Italy, as we know from the testimony of Bombelli, who refers to it as if it were perfectly familiar to his readers.

‡ Mabillon, in his noble work *De re Diplomatica*, has given a *fac simile* of this record of Petrarch, which is as follows: *Hoc immensum opus donavit mihi vir egregius Johannes Boccacius de Certaldo, poeta nostri temporis, quod de Florentia Mediolanum ad me pervenit* 1355, *Aprilis* 10. The figure of 3 is nearly the same as in modern times, but that of the 5 is the same as is generally found in manuscripts of the XIVth and XVth centuries.

Arithmetic. to the proper conduct of their extensive commerce, appear to have been cultivated and improved by them with particular care; to them we are indebted for our present processes for the multiplication and division of whole numbers, and also for the formal introduction into books of Arithmetic, under distinct heads, of questions in the single and double rule of three, loss and gain, fellowship, exchange, simple interest, discount, compound interest, and so on: in short, we find in those books, every evidence of the early maturity of this science, and of its diligent cultivation; and all these considerations combine to show that the Italians were in familiar possession of *Algorithm* long before the other nations of Europe.

If, therefore, we should found our decision upon the evidence already adduced, of the question, What nations in Europe were in the first possession of the notation by nine figures and zero? we must certainly answer, Spain in the first instance, and Italy in the second: in one case, it was introduced in the translations of astronomical works from Arabic into Latin, and appears to have been long confined to mathematical works alone; in the other, the algorithm itself is made the subject of a distinct treatise, written for the purpose of making it generally known. In the first case, it appears to have been chiefly confined to the Moors, by whom it was introduced, and its general propagation checked by the contests which distracted that country, until their final expulsion; in the other, it passed rapidly from the writings of arithmeticians into general use; and in less than a century and a half, it assumed a form much more adapted to practice, than that which it possessed amongst the people with whom it originated.

Claims of Pope Sylvester the Second. (71.) A much earlier date, however, has been assigned by some authors to the introduction of these numerals into Europe, than those which we have mentioned. In the latter part of the Xth century flourished Gerbert, a monk of Aurillac, in Auvergne, who was afterwards Archbishop of Rheims and of Ravenna, and finally Pope, under the name of Sylvester II.* In early life he travelled into Spain, and is represented as having made himself master of all the learning of his time, and as one consequence of these numerous acquirements, was accused of dealing with the powers of magic: he wrote largely on Arithmetic and Geometry, and in the opinion of Wallis,† Leibnitz,‡ and many subsequent writers, was the first European who acquired a knowledge of the Arabic numerals from the Saracens in Spain. This opinion is chiefly founded upon a passage in our English historian, William of Malmsbury;§ when speaking of Gerbert, he says, *Abacum certè primus a Saracenis rapiens, regulas dedit quæ a sudantibus abacistis vix intelliguntur.* This sentence, however, contains no certain intimation of the knowledge of the notation by nine figures and zero, as the rules which would be thence derived, would tend rather to relieve than increase the labours of the sweating calculators.‖ Other passages have been quoted by Wallis, from his letters to his fellow disciple Constantine, and others, which are supposed by him and Kæstner,¶ to give indications of his knowledge of that system: in a letter to the Emperor Otho, he History. styles himself *extremus numerorum abaci*; and in another to his friend, he says, *Nam quomodo rationes abaci explicare contenderemus, nisi te adhortante O mi dulce solamen laborum Constantine?* In another epistle to the son of the Bishop of Geneva, he says, *De multiplicatione et divisione numerorum Joseph sapiens sententias quasdam edidit. Eas pater Adelbero Remorum archiepiscopus habere cupit.* This was a work, celebrated in that age, by Joseph of Spain, which is again referred to in the following passage, in a letter to the Abbot of Orleans: *De multiplicatione et divisione numerorum libellum a Joseph Hispano editum Abbas Garnerius penes vos reliquit; at exemplar in commune sit rogamus, sc. ego et Adelbero.* If this book contained an exposition of the Hindoo notation, it is impossible that the knowledge of it could have been lost, when communicated to so many persons; and in supposing that the *abacus* referred to in preceding extracts meant the *mensa Pythagorica*, or common multiplication table, which may or may not have been the case, there is no reason why it should not have been expressed in Roman numerals, as the same is found in the works of Boethius.* Again, when in another passage he speaks of digital, compound, and articulate numbers: *Quid cum idem numerus modo simplex, modo compositus; nunc digitus, nunc constituatur ut articulus*, it must be kept in mind that these distinctions originated with the Arithmeticians of the Pythagorean school, and that there is no reason for us to interpret this sentence, as was done by Wallis, as if it was meant to assert that the same *figure* was sometimes employed to denote a digit, and sometimes an articulate number, according to its position. The observation which immediately follows is remarkable: *Habes ergo (talium diligens investigator) viam rationis (sc. abaci;) brevem quidem verbis sed prolixam sententiis: et ad collectionem intervallorum et distributionem, in actualibus Geometrici Radii secundum inclinationem et erectionem, in speculationibus et actualibus simul dimensiones Cœli et Terræ plenâ fide comparatam.* It is difficult, however, to conceive in what manner this character, *brevem quidem verbis sed prolixam sententiis*, could apply to the system in question; and the remainder of the sentence is so obscure, that no inference respecting the method to which it referred can properly be deduced from it.

In the third Volume of the *Thesaurus Anecdotorum novissimus* of Bernard Pez, which was printed at Augsburg in 1721, there is a notice of a manuscript of the Geometry of Gerbert, which was found in the monas- of St. Peter, at Salzburg, in which the Arabic figures are found: the author considered the manuscript to have been written in the year 1100, and he supposes that the transcriber would not have inserted these figures in his copy, if he had not found them in the original; it appears, however, that no practice has been more common than alterations of this nature, and that those figures have sometimes been inserted

* He died in 1003 † *Algebra*, ch. iv.

‡ *Opera Mathematica*, p. 254.

§ He flourished about the year 1150.

‖ North, *On the introduction of Arabic numerals*, *Archæologia*, vol. x. p. 366.

¶ *Geschichte der Mathematik*, vol. ii p. 366.

* Weidler, the historian of Astronomy, discovered a manuscript of the Geometry of Boethius in the Public Library at Altdorf, in which the *Mensa Pythagorica* is given in Arabic numerals; and in a Dissertation, *de characteribus numerorum vulgaribus et eorum ætatibus*, he attempted to show that they necessarily formed a part of the original work: it is a sufficient answer to show that they do not appear in the most ancient manuscripts of Boethius, where all the numbers are expressed by the Roman characters, and that consequently, in the later manuscript which Weidler saw, they had been inserted by the transcriber.

Arithmetic. at a later period than the date of the manuscript itself. That such has been the case with the manuscript in question, we may infer from the existence of other manuscripts of Gerbert, of nearly his own age, in which those figures are not found, even on occasions where they might most naturally have been looked for. Thus William of Malmsbury mentions an Epistle *Quam Adelbold fecit ad Gerbertum de questione diametri super Macrobium*, which Mr. North* found in Parker's Library, in Corpus Christi College, at the end of *Macrobii Opera*, together with Gerbert's answer: in this manuscript, which is of greater antiquity than the one of Salzburg, the Roman numerals are constantly used by both; a circumstance which affords a strong presumption, when the nature of the subject discussed is considered, that those figures were at that time unknown.

In the account given of Gerbert by Trithemius,† is the following passage:

Gerbertus docuit Fulbertum, hic etiam Fulbertus Berengarium, qui iterum Brunonem Remensem et alios multos hæredes Philosophiæ reliquit.

It would be a very extraordinary circumstance if Gerbert had known and taught this notation, that it should have been lost notwithstanding this regular chain and succession of his disciples; and it is no sufficient answer to the presumption, that he did not teach this system because he did not know it, to contend with Kæstner, that there is no necessity for a tutor to communicate the whole of his knowledge to his pupils.

We have been more particular in our examination of the claims of Gerbert to the knowledge of this system, because the arguments of Wallis on the subject appears to have convinced Mr. Colebrooke,‡ whose opinion and judgment are entitled to so much respect; and though it must certainly be allowed that it was possible for him to have acquired the knowledge of it from the Saracens in Spain, it is more probable, however, that they, who were recent conquerors of that country, amongst whom the arts of peace had hardly begun to take root, were at that time ignorant of most of the scientific improvements which had taken place in the preceding century amongst some of their countrymen in the east: at all events, it would certainly follow from the facts which we have mentioned, that if the system was known to Gerbert, it was barren in his hands, as no certain traces of it are discernible, either in his own writings or in those of his contemporaries or successors. We may, therefore, fairly deny him the merit of having introduced the knowledge of it into Europe.

Of John of Halifax. (72.) Wallis, who seemed to consider the figures in every manuscript which he saw, to be of the age of the author, and that they had never been introduced by subsequent copyists, has given a long list of authors, to whom this notation was known in the XIIth and XIIIth centuries. Amongst others, in particular, is mentioned Johannes de Sacro Bosco, the Latinized name of John of Halifax or Holywood, who died in 1256, and who wrote a Tract *de Sphæra*, which for a long time was a work of standard authority; another *De Computo Ecclesiastico*, in the later manuscripts only of which the Arabic figures appear; and another which is attributed to him, though apparently on very insufficient authority, *De Algorithmo*. Wallis speaks of two Tracts of his, one in prose and the other in verse, which are found together in a manuscript at Oxford; the second of these commences with the verses:* History.

Hæc Algorismus ars præsens dicitur: in quâ
Talibus Indorum fruimur bis quinque figuris;

which are there given, and are the same in form as those which are used in the XVth century, the age to which we should be inclined to refer this manuscript; in short, there is no sufficient reason to attribute either of these works to Sacro Bosco, as no notice of these figures appear in the older manuscripts of the two former works, where we should naturally have expected to meet with them, particularly in the latter.

Of Bishop Grosshead. (73.) The second person whom we shall mention is Robert Grossetête, or Grosshead, Bishop of Lincoln, and the contemporary of John of Halifax. He also wrote *De Computo Ecclesiastico;* and his calendar constructed by means of it, under the name of *Kalendarium Lincolniense*, appears to have long continued in repute, and numerous copies of it of the XVth century are found in manuscript libraries. It is impossible, however, to judge readily of the age of the earlier manuscripts of this work, and it is only in the later copies that the Arabic numerals are found.

Roger Bacon's calendar. (74.) There is a copy of the Calendar of the celebrated Roger Bacon in the British Museum, which Mr. Ames considers, upon the authority of Mr. Casley, to be of the date 1292; upon an examination of it, however, we found it headed by the following notice: *Kalendarium sequens extractum est tabulis Tholetanis Anno dm. 1292, factis ad meridianum civitatis Tholeti;* or in other words, that the calendar had been formed from the Toledo tables published in 1292, and calculated for the meridian of that city. In this case, as in many others, the name of Roger Bacon had been attached to the calendar by the monks who composed it; either for the purpose of recommending it by the authority of a name so distinguished for abstruse, and what was in that age deemed magical learning; or perhaps in the arrangement and composition of it, they had availed themselves of some of the rules of the Treatise *de Computo Ecclesiastico*, of which he was the author.

(75.) We find it unnecessary to give other examples of Astronomical tables, or of Treatises *de Algorithmo*, which are assigned by Wallis to the XIIIth century, either from the character of the manuscripts, or from the names of their authors, as the instances which we have given would show that he was too much devoted to his theory, to be inclined to subject his documents to a very accurate or critical examination. There is one hypothesis which he has made respecting the period at which the Arabic figures were introduced into England, which deserves a more particular examination, from the numerous discussions to which it has given rise. He supposes that they were brought from Spain into England about the year 1130; and to account for this very early introduction he has referred to several Englishmen who travelled in that country about that period: amongst others, he mentions Adelard, the Monk of Bath, who translated Euclid from Arabic into Latin; Robert of Reading, who translated the *Alcoran* into Latin in 1143; Daniel Morley, who studied Mathematics and the Arabic lan- English travellers in Spain in the XIIth century.

* *Archæologia*, vol. x. p. 368. † Ibid.
‡ *Hindoo Algebra*, Introduction, p. 54.

* There is a copy of this manuscript also in the Public Library at Cambridge.

Arithmetic.

guage at Toledo about 1180. By such persons it was natural to expect that the knowledge of these numerals should not only be acquired, but likewise communicated upon their return;* as a proof that such was the case, he refers to the date on the Mantel-piece in a room in the parsonage house at Helmdon in Northamptonshire, a description of which he has given in the *Philosophical Transactions*,† and afterwards in his *Algebra*.‡ The date which is there given, is supposed to be expressed partly in Roman and partly in Arabic figures, and is equivalent to A° DO^{i} M° 133.

Helmdon inscription.

Dr. Ward, in the same Transactions for 1735, resumed the subject; and as he had satisfied himself, though upon reasons which might have equally answered for a much later date, that the knowledge of these numerals had been introduced by the Crusaders, he will allow no date to be genuine which is before the year 1200; he therefore boldly corrects the date to 1233. Later observers, including the celebrated antiquary Gough, found great difficulty in making out the A° DO^{i}, which appeared so plain to Wallis, and still greater in identifying the other figures; and, lastly, Mr. Denne§ has shown that the fleur de lys and dragon volant with which this rude piece of rustic sculpture is adorned, are more appropriate to the reign of the last than the first or third of our Henries; and that the date, if the inscription be really meant for one, is designed for 1533 rather than 1133 or 1233.

Different opinions respecting it.

(75.) The Helmdon inscription, and the conclusions founded upon it, produced in a short time a number of others of equal or greater antiquity, which have all however yielded to a more sober and critical examination; of this kind was the inscription at Colchester,‖ which was said to be 1090, but which further investigation extended to 1490; and one at Widgel Hall, near Buntingford in Hertfordshire, which appeared to be M 16, but which was found to be M. I. G ¶ the initial letters of a proper name having been mistaken for numerals: on a barn belonging to Preston Hall, in Kent,** where the date 1102 is put between two armorial shields with the cyphers T. C. attached to each of them, but which were shown to be the initials of Thomas Colepepper, the owner of the estate, who lived about 1587, and who, most probably, commemorated in this manner the time when his family first got possession of the property: on a beam in a very ancient gateway near the great bridge in Cambridge, where the date which Dr. Warren represented to Dr. Ward as 1332,†† in reality should be read 1552:‡‡ on a stone found in digging up the foundation of the Black Swan Inn, in Holborn, with the date 1144,§§ though it is difficult to make out that such rude marks represented any numerals whatsoever, as they have no resemblance to such as were used before the end of the XVth century; but it is needless to extend this list, as in all the cases which have hitherto been produced, their pretensions to uncommon antiquity have been refuted by further investigation.‖‖

Other inscriptions with dates which have been mistaken.

History.

(76.) The real fact appears to be, that the Roman numerals were used throughout Europe long after the Arabic figures were in common and general use. The earliest example of a monumental date in these figures in England is in the church of Ware, on a brass plate commemorating the death of Ellen Wood in 1454; and the second is in the same church, and is dated 1484.*

Monumental dates in the church at Ware.

Kæstner considers that the earliest monumental date in Germany is A. D. 1497, on the wall of the church of Grossalmerode, in Hesse.† Gatterer says that these figures rarely ever appear in public documents during the XVth century, and only became common for such purposes at the close of the XVIth; and that the earliest date which he has observed amongst more than 1000 documents in his possession was 1527.‡ Calmet, in the *Mémoires de Trevoux*, has noticed a series of inscriptions, chiefly monumental, from 1445 to 1519.§ He also says, that at Turkheim there is an inscription on a stone of the date 1332, which we might venture, however, to convert into 1552, upon the same principle that was applied to a similar inscription noticed above. Mabillon, in the whole course of his loquiries, and after the examination of more than six thousand documents, found no authentic date earlier than 1355, in the hand-writing of Petrarch;‖ and the learned Benedictines, the authors of the *Nouveau Traité de Diplomatique*, declare their conviction that the appearance of a date in Arabic numerals before the XIVth century, would be sufficient to vitiate its authority.¶ The conclusion to be drawn from these facts unquestionably is, that it is not in dates that we must look for the first appearance of these numerals; but in astronomical works and tables, in calendars, and in Treatises of Arithmetic and Algebra.

In Germany.

Diplomatic, legal, and other records.

(77.) The earliest example of the use of these numerals which Montfaucon discovered, was in a chronicle in the Strozzi library at Rome. It is a manuscript written by different hands. The first and most ancient concludes in 1250, and is written in the Roman numerals; the second, and more modern, in which Arabic numerals are used, commences in 1268, and finishes in 1317, and is therefore posterior to that date. In another manuscript, there is found written the date 1245, in which the forms of the figures 2 and 5 are different from those on the chronicle, and possibly more ancient; but it is impossible to judge of the real antiquity of a date which is written in such a manner, and not embodied in the work itself.**

The Arabic numerals observed by Montfaucon in a very ancient manuscript in the Strozzi library.

(78.) The author of the *Göttwich Chronicle* is anxious to secure for his countrymen the honour of an earlier acquaintance with these numerals.†† He speaks of a manuscript at Fulda, in which they appear, which is said to be 1300 years old; of the same age, therefore, with Weidler's manuscript of Boethius; and of a

Manuscripts at Fulda and at Warsaw.

* *Algebra*, ch. xi.

† No. 178, for December, 1683. ‡ Ch. ix.

§ *Archæologia*, vol. xiii. p. 142.

‖ *Philosophical Transactions for* 1735, No. 439, p. 131.

¶ Ibid. p. 136.

** Hasted's *History of Kent*, vol. ii. p. 175.

†† *Philosophical Transactions for* 1744, No. 474.

‡‡ *Archæologia*, vol. x. p. 372.

§§ Ibid. vol. i. p. 149.

‖‖ Of this kind is the date 1102, said to be found on a brick building at Shalford in Bucks, though it may be satisfactorily proved, that there was no brick building in this country before the end of the XIVth century.

* *Archæologia*, vol. xiii. p. 148.

† *Geschichte der Mathematik*, vol. i. p. 36.

‡ *Elementa rei Diplomaticæ*, vol. i. p. 64.

§ *Mémoires pour l'Histoire des Sciences et beaux Arts à Trevoux, pour l'an* 1707, p. 1624.

‖ *De re Diplomaticâ*, p. 213, and tab. xiii. p. 373.

¶ Tom. iii. p. 537.

** Calmet, *Mémoires de Trevoux*, for 1753, p. 1692.

†† *Chronicon Gottwicense, seu Annales Liberi et exempti Monasterii Gottwicensis ordinis S. Benedicti*, p. 114.

Arithmetic. calendar in the library at Warsaw for the year 1268, which is expressed in these figures. In order, however, to found his argument upon facts of less disputable character, he asks how it is possible that the use of these figures should be unknown in Germany four hundred years after the Saracens had established their dominion in Spain; more particularly after the Alkoran and many works of Arabian physicians had been translated in Germany in the time of Conrad the Third, and Frederick Barbarossa in the XIIth century. It is a sufficient answer, however, to observations like these, to state, that the knowledge of these figures among the Arabs in the Xth and XIth centuries was not, properly speaking, popular, but confined to the few who had leisure for the study of science and philosophy; and that it was only in the XIIth century that those arts were much cultivated in Spain; and that in all discussions on this subject too much stress is laid upon the facility with which the knowledge and practice of this notation is acquired, and transmitted amongst a people who have been accustomed to the use of a system essentially different in its nature. The following fact will show that this system has been in some cases introduced without being perfectly understood.

Mistakes in the use of Arabic numerals.

In a manuscript of the XIVth century, an extract from which Mabillon has given a fac simile,[a] we find the Roman and the Arabic numerals mixed up together in a very curious manner; thus 10, 11, 12, 13, 14, &c. are denoted by X, X 1, X 2, X 3, X 4, &c.; 20 by XX; 31 by XXXI, or by 301; 34 by 304; 40 by XXXX; 41 by 401; 42 by 402. It is clear from hence, that the writer did not understand the proper force of the zero, and had but very imperfectly comprehended the principle of value from position.

Their early use in calendars.

(79.) A critical examination of the calendars which exist in different libraries in Europe, would lead to the most certain determination of the periods at which these numerals were generally introduced, as they contain within themselves the data from which the year in which they were composed may be very nearly ascertained; and there are few inquiries which would lead to the knowledge of more curious facts respecting the history of the human mind, as they generally contain all those topics of medical, astrological, and astronomical science, which were most popular in their time, and which were best adapted to the superstitions and prejudices of the people for whose use they were formed. The following are the contents of a calendar in the British Museum, consisting of eight vellum leaves folded up in a portable form, and which may serve as a specimen of others of the same date, about 1403; the leaves are marked from A to K.

Contents of a calendar in the British Museum.

A, contains a canon for the calculation of the movable feasts, subjoined to which is an astrological scheme.

B, C, D, E, the calendar, properly so called, three months in each.

F. *Tabula lunæ cum canone et imagine signorum*; the figure of a man with the signs of the zodiac on different parts of his body.

G. Eclipses of the sun, with their phases, from 1403 to 1462.

H. Eclipses of the moon, with their phases, from 1398 to 1448.

I. Eclipses of the moon from 1448 to 1462; to these is subjoined a Tract, entitled *Opera Apuleii, de indiciis urinarum*, with the figures of six urinals differently coloured, with the affections of the body which they severally indicate.

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K. *Tabula ad calculandum pro futuris.*

This calendar exhibits the Arabic numerals, with the figures which they usually present before the end of the XVth century; and the abstract of its contents which we have given, may be taken as a sample of the most popular scientific knowledge of those times.

Mr. Denne* has given an account of another calendar containing a table of eclipses from 1406 to 1462, which is nearly similar to the last; where, instead of enlarging on the indications of urinals, those days are particularly marked, upon which it is expedient to abstain from flebotomy; they amount to 133 in the course of the year, and the enumeration of them at the end of the year is terminated by the following formidable warning, *Isti sunt dies mali observandi ab incisione in anno, et qui homines vel pecora inciderint inde morientur.* In the last page of this calendar, there is the following short and very clear account of the Arabic numerals, which appears to have formed a common appendix to them in that and the preceding age, when their use was becoming general: *Nota quod quælibet figura algorismi in primo loco signat se ipsam, et in secundo decies se. Tertio loco centies se ipsam. Quarto loco millesies se. Quinto loco decies millesies se. Sexto loco centies millesies se. Septimo loco mille millesies se. Et semp. incipiendum est computare a parte sinistra more Judaico.* The following page contains the Roman and Arabic numerals from 1 to 100 placed opposite each other; and in the last page many numbers are given from 20 to 1,000,000, being severally specified in words, Roman numerals, and Arabic figures; thus, Viginti, XX, 20; mille milia, M[c] M[a] 1,000,000.

Of the calendars which contain a notice of Algorithm.

In the manuscript library of Corpus Christi College, Cambridge, there is a table of eclipses from 1330 to 1348, to which is also subjoined a table of the Arabic numerals, which is extremely interesting from its great antiquity.† In fig. 12 we have given a copy of this table arranged in three columns; the first for digits, the second for articulate, and the third for compound numbers; each of these columns is separated into three divisions, in the first of which we find the Roman numerals, in the second the Arabic, and in the third a peculiar notation nearly identical with the Roman in principle, though different in form. After the table is subjoined the following explanation: *Omnis numerus vel omnis figura in algorismo primo loco se ipsum significat; secundo loco, decies se ipsum significat; tertio loco, centies se; quarto loco, milesies se; quinto loco, decies milesies se; sexto loco, centies milesies se; septimo loco, mille milesies se; octavo loco, decies mille milesies se; nono loco, centies mille milesies se; decimo loco, mille milesies milesies se. Et sic multiplicando per decem centum et mille usque in infinitum computando versus sinistram.*

Manuscript in the library of Corpus Christi College.

There is no longer any difficulty in discovering in what manner the knowledge of Arabic notation was propagated throughout Europe, when we find such simple and popular expositions of its principles in those productions which were expressly formed for the most general circulation; and from which the

Means of propagating the knowledge of Arabic numerals.

[a] *De re Diplomatica*, p. 373, Plate 15.

* *Archæologia*, vol. xiii. p. 153.

† North, *Archæologia*, vol. x. p. 373.

majority even of the better informed of our ancestors derived so considerable a portion of their knowledge; so common indeed does the use of them appear to have been, and so frequent were the occasions of reference to them, that they were in some cases triply folded up in such a manner that they might be suspended by a knot at the girdle, so that persons might peruse them without removal. Several examples of such calendars may be seen in the British Museum, and, amongst others, one in ten leaves, for the year 1431, and which merits particular attention, from its being the earliest which we have discovered in which the English language is used. Its contents are of the usual kind, containing the calendar in Arabic figures, with rules for the calculation of eclipses, leap-years, indictions, &c. and the usual portion of astrological information. The author of it says that it was formed for the use of his sovereign mistress, though it is not easy to find a personage to whom this title would apply during the minority of Henry VI.

Calendar in the English language in 1431.

We may safely infer, that when calendars containing these figures began to be circulated, that the use and advantages of this notation must have been in some degree understood by the persons for whose use they were composed; or that, at all events, under such circumstances, the knowledge of it must have been rapidly propagated. The addition of the *Rules de Algorismo*, which was made to so many of them between the early part of the XIVth and the beginning of the XVth centuries, would seem to show that their composers were employing a notation which was not universally known or used, and that there were some of their readers for whom some explanation was necessary. After the commencement of the XVth century, however, we find few examples of such additions; and we may consider, that from such a period this notation was known and admitted in every part of Europe, where monasteries and other establishments of the Church of Rome were to be found.

The *figures newe* of Chaucer.

(80.) In a passage which has been quoted from the *Dreme* of Chaucer,* these numerals are called the *figures newe*; and, therefore, it is presumed, that they were not known long before his time.

THE WEDDE.

Shortly it was so full of bestes
That though Argos the noble counter
Yaste to reckin in his contour
For by the *figures newe* all hen.

If they be crafty, reckin and nombre
And tell of every thing the nombre,
Yet shullde fail to reckin even
The wonders we met in my sweven.

If we assign the date 1375 for the writing of this poem, the epithet *new* might yet be appropriate to these figures, as distinguished from the Roman numerals, even though they had been known from the beginning of the century. A term of this kind, indeed, is so indefinite in its application, that it is impossible to found any argument upon it; the passage, however, is in other respects interesting, as expressing the opinion of the author upon the power and extent of this new Arithmetic, in every way so superior to the old. Chaucer had been in Italy, and might there have enjoyed opportunities of witnessing this amongst all other arts and sciences in a more perfect and matured state than in other parts of Europe.

Appearance of these numerals in the dates of printed books.

(81.) It is not a matter of much difficulty or importance, to trace the progress of these numerals after the commencement of the XVth century: the manuscripts on astronomical and arithmetical subjects which were written in that century, and which are found in such numbers in all manuscript libraries, show how general the use of these numerals had become, where custom and respect for antiquity had not opposed their introduction: it was for these reasons that they were excluded until a late period for dates, from all public and formal deeds and documents, from registers, inscriptions, and so on. They appear neither in the dates, nor pages of works printed by Caxton, but in the *Myrrour*, or *Ymage of the World*, printed in 1480, when treating of Arsmetrike, or Algorithm, amongst other sciences, he has given a wood-cut of an Arithmetician sitting before a table, on which are slates or tablets with these figures upon them.

Early printed works on Arithmetic.

Amongst the earliest books printed at St. Albans, is one *Rhetorica nova Gulielmi de Soona*, in which there is the date 1478;* and the *Myrrour of the World* was reprinted in 1506, under an enlarged and altered form by Laurence Andrews, in which the common operations of Arsmetrike, or Algorism, are treated of with great clearness, and in which the figures appear under their present form; the treatise of Cuthbert Tonstall, Bishop of Durham, *de Arte Supputandi*, one of the most beautiful productions of Pynson's press, was published in 1522, and showed its author to be perfectly well acquainted with the most improved state of the science at that period.†

In the year 1537, there was printed at St. Alban's "*An Introduction for to lerne to reckon with the Pen and with the Counters after the true cast of Arsmetyke, or Awgrym, in hole numbers, and also in broken;*" and in 1542, was printed by Robert Recorde, Doctor in Physic, the first edition of "*The Grounde of Artes, teaching the Woorke and Practice of Arithmetyke, booth in whole numbers and fractions, after a more easy and exacte sorte, than any hitherto hath been set forth.*" This is a work which we have had frequent occasion to quote, and we do not find it necessary to notice at present any other contemporary or subsequent publications on this subject.

Their use in merchants' accounts, &c.

(82.) The accounts of merchants were kept in Roman numerals until the middle of the XVIth century,‡ and they continued the use of the Abacus in their calculations to a still later period; nay, even so late as the year 1595 the use of this instrument was a subject

* Deane, *Archæologia*, vol. xiii. p. 123.

* These numerals appear in the *Fasciculus Temporum Antiquorum*, printed at Louvain in 1476; and in 1484 was published the great work of Lucas de Burgo, entitled *Summa de Arithmetica, &c.* in which the numerals appear under very nearly their present form, and which was adopted in all books printed in the following century.

† In his dedication to Sir Thomas More, he says, that the reason which induced him to study Arithmetic was to protect himself from the frauds of money-changers and stewards, who availed themselves of the ignorance of their employers; he professes to have read, for the purpose of his work, all the books which had ever been written, whether *eruditos, ineptos, Latinos, barbaros, quorum callerem linguas*; and he adds, that there was hardly any nation which did not possess books on this subject in their own language; and by selecting what was excellent in each, and arranging his materials, he at last succeeded in licking them, as it were, into shape and symmetry, as the bear does her cubs.

‡ *Archæologia*, vol. xiii. p. 148.

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of popular education. In colleges, where the use of the learned languages prescribed by their statutes gave a species of authority and sanction to classical notation, we find that the Roman numerals were used in some instances as late as the year 1600, and in others as late as 1700; and the same feeling operates generally even to this day, to preserve their use in monumental and other inscriptions.

The use of them said to have been taught in the University of Paris in 1306.

(83.) A nearly contemporary author in the account of the life of Baldwin, Archbishop of Treves, states that he learned the use of these figures in the University of Paris, in the year 1306; * a fact, which would, if well authenticated, show that these figures had become the subject of common and popular knowledge at an earlier period than in this country. Professor Leslie,† mentions a Tract in the German language, dated 1390, *De Algorismo*, in which the notation and the common operations of this Arithmetic are very distinctly explained. It is impossible, however, in cases where the date of the introduction of these numerals is to be determined by their appearance in a manuscript, to say that other manuscripts of greater antiquity may not have contained them, unless the external evidence which they afford is farther confirmed by the contents of the manuscript itself.

The work of Planudes

(84.) These numerals appear to have been known about the middle of the XIIIth century in the Greek Empire at Constantinople, if we may judge from the work of Planudes, of which manuscripts are to be found in the Bodleian, Vatican, and Royal library at Paris, of the last of which Delambre has given an analysis;‡ it is entitled *Τοῦ φιλοσοφωτάτου μοναχοῦ καλουμένου Μαξίμου τοῦ Πλανουδη ψηφοφορία κατ' Ἰνδους, ἡ λεγομένη μεγάλη*. Vossius, indeed, has placed Planudes in the middle of the XIVth century; but if it be true, that he dedicated other works to the Emperor Michael Palæologus, he must have flourished about the period which we have mentioned. The forms of the digits, which he says are Indian, bear a considerable resemblance to the Arabic; and, with respect to the zero, he observes, *τιθέασι δε ἕτερον τι σχῆμα ὅ καλοῦσι τζίφραν, κατ' Ἰνδοὺς σημαῖνον οὐδεν: ἡ δὲ τζίφρα γραφεται οὕτως ο*: the term *τζίφρα*, or *cifra*, is derived from the Arabic term *tsaphara*, (*quod vacuum aut inane est*,) and corresponds to the Sanskrit term for it, *sûnya*, which signifies *blank* or *void*. The use of the zero or cypher, so important and so essential to this species of Arithmetic, has led to a more comprehensive meaning of the term, all the digits being designated by the general term of *cypher*, and the verb *to cypher* having the same signification as to calculate or work with these figures. To return, however, from this digression, it is clear from an examination of this work of Planudes, that it must have been translated or collected from the Arabic writers, as the distribution of the subject, and the rules of operation, are nearly the same as are found in the arithmetical and algebraical works in that language.

Origin and meaning of the term cypher.

Different origins assigned to these numerals, by

(85.) The work which we have just noticed is another amongst the numerous testimonies which may be brought to establish the Indian origin of those numerals. It must not be imagined, however, that the opinions of learned men have at all times agreed in assigning to them such an origin, as there are few subjects concerning which a greater number of extravagant theories have been formed. One of the earliest and most popular of these is the one which was first propagated by Dasypodius, and afterwards maintained by Huet, Bishop of Avranches,* that these figures were formed from the Greek letters for the nine digits. Dr. Bernard, in his Tables,† acquiescing in this theory, has stated that these figures passed from the Greeks to the Indians in 710, from the Indians to the Arabians in 800, and from thence to the Spaniards in the year 1000, for all which periods he has given their forms, though it would be difficult to refer to his authorities, and still more difficult to confirm them. Dr. Ward, Rhetoric Professor at Gresham College, adopting the same views, has traced their course also from Greece to India, and from thence through Arabia to the Moors in Africa and in Spain. It is unnecessary to quote more examples of the names even of distinguished men who have written in favour of an hypothesis so entirely unsupported by facts. Gatterer‡ imagined that he had discovered in Egyptian manuscripts written in the enchoriac character, that the digits were denoted by nine letters, and that a tenth sign performed the office of zero; and, as if not contented with assigning to the Egyptians the knowledge of this Arithmetic, it must be known likewise to Cecrops and Pythagoras, and that it formed part of the mysterious science which was transmitted to his followers. It was, probably, a vestige of this mystical knowledge which showed itself in the manuscript of Boethius, which Weidler considered as old as the VIIIth century, and in which the Arabic numerals were used.§ Wachter, however, found no difficulty in tracing them to a natural origin, as well as the Roman numerals, in the different combinations of the fingers; thus unity is expressed by the outstretched finger, and by repeating and varying this character, we have got = for 2, ≡ 3 ♢, or ✧ for 4, ƽ for 5, &c. which have degenerated from long use, and for the greater convenience of writing, to their present forms.‖ A Dutchman of the name of Rudbeck, and Brinhorne, a Swede, with singular boldness and patriotism, have claimed the invention for the Celts, or Scythians of the North of Europe; whilst a Spaniard named Antonio Nassau has referred it to the Carthaginians in Africa, on the authority of some Tyrian inscriptions, in which characters somewhat resembling the Arabic figures have been discovered.¶ The last opinion which we shall notice is that of Calmet,** which was originated by Mabillon,†† that these figures were part of the signs or abbreviations of Tiro, the freedman of Cicero, which were so extensively used in short-hand writing by the ancients. It is a sufficient answer to such an hypothesis to state, that we find amongst those signs simple symbols for 11, 12, 13, &c. which could not have been the case if they had involved, in any way whatsoever, the

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Dasypodius, Huet, &c.

Gatterer,

Wachter,

Rudbeck and Brinhorne,

Nassau,

Mabillon and Calmet,

* *Chronicon Gottwicense*, p. 114.
† *Philosophy of Arithmetic*, p. 114.
‡ *Hist. de l'Astronomie Ancienne*, tom. I. p. 518.

* *Demonstratio Evangelica*, p. 647.
† See his Tables, or Plate, printed in 1689.
‡ *Weltgeschichte bis Cyrus*, p. 586; *Elementa rei Diplomaticæ*, p. 65.
§ *Philosophical Transactions for* 1744, No. 472, p. 81.
‖ *Naturæ et Scripturæ Concordia*, c. 4.
¶ *Chronicon Gottwicense*, p. 114.
** *Mémoires de Trevoux*, for 1753, p. 1630. *De re Diplomaticâ*, p. 215.
†† *Nouveau Traité de Diplomatique*, tom. III. p. 527.

Arithmetic. principle of arithmetical notation by nine digits and zero.

Explanation of figures in Plate III.

(86.) In Plate III. we have given the principal forms of these numerals amongst different nations of Asia, as well as at different periods since their first introduction into Europe. The following are the explanations of the several divisions of this Plate, to which references are made by means of the Roman numerals.

I. The Sanskrit numerals in the Devanagari characters, or *divine* characters of Nagari: from the *Sanskrit Grammar* of Dr. Wilkins.

II. The Bengali numerals, from Halhed's *Grammar*, p. 132; the forms of these numerals are slightly varied from the Sanskrit; the zero is a small circle, and not a simple dot, as was usually the case in the Sanskrit: it is probable that this symbol for zero was borrowed from the Arabs at a later period.

III. Ancient Persian numerals, from a manuscript of the *Zendo-Vesto*, in the Bodleian library.

IV. Mahratta numerals, from a manuscript of Mr. Perry, copied by Mr. Astle in his *Origin and Progress of Writing*, Plate 30.

V. The Thibet numerals, from another manuscript of Mr. Perry, also copied by Mr. Astle; it is obvious that all these numerals have a common origin.

VI. The Arabic numerals, which are specifically called Indian, from De Sacy's *Grammaire Arabe*, Plate 8.

VII. The Arabic numerals, called *Gobar*, in which there is no zero, the nine digits having a dot attached to them to denote *tens*, two *dots* to denote *hundreds*, and so on: likewise taken from De Sacy.

VIII. Numeral characters from the manuscripts of Maximus Planudes.

IX. Numeral characters used in manuscripts in the XVth century.

X. The wood-cut, with numerals, in the *Myrrour of the World*, by Caxton, A. D. 1480.

XI. Fac simile of the memorandum of Petrarch on the manuscript of St. Augustin: *Hoc immensum opus donavit mihi vir egregius Johannes Boccacius de Certaldo, poeta nostri temporis et Florentia Mediolanum ad me pervenit* 1355, *Aprilis* 10.

XII. The date 1245, from the manuscript in the Strozzi library, but which has every appearance of having been added in the XVth century.

XIII. *Anno Dom.* 1292, *ad meridiem civitatis Toleti.* This is the date found at the commencement of what is called Roger Bacon's Calendar, in the British Museum, which Ames erroneously imagined to be the date of the manuscript.

XIV. *Christi* 1334 *incompleto:* this date is written upon a manuscript of *Postillæ in Psalterium* by Nicholas de Gorron; upon which is written, *Accommodatus iste Liber Magistro Thome Durant sacre pagine Professori, post festum S. Petri Cathedre, Anno Christi* 1334 *incompleto.* There is every reason for supposing that this date is perfectly genuine.*

XV. *Psalterium, cum calendario auctiori, hymnis Ecclesiasticis, Litania et vigiliis mortuorum; in usum Johannæ, Regis Ricardi* 2 *matris scriptum*, A. D. 1380. This very curious calendar commences with an account of *Algorism*, as usual in that period, and contains a more than common quantity of purely astronomical knowledge, the tables being calculated to the meridian of Oxford.†

* Casley, *Catalogue of Manuscripts in the British Museum.*
† Ibid. Plate 16.

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XVI. *Liber olim de Claustro Roffensi, per Benedictum episcopum datus:* to which is added, *Iste liber ligatus erat Oxonii, in Catstrete, ad instantiam Reverendi Domini Thome Wybarum, in sacra Theologia Baccalarii monachi Roffensis, Anno Domini* 1467.* Dates, which before the middle of this century were very rarely expressed in Arabic numerals, became extremely common afterwards.

XVII. A series of dates, from 1445 to 1587, for the purpose of exhibiting the changes which these figures underwent during that period.†

Two distinct species of Arithmetic among the ancients.

(87.) There were two distinct species of Arithmetic which were cultivated by the ancients, one practical, and the other purely speculative. The history of the first species we have already entered into with sufficient detail; the second, however, merits a distinct and separate notice, not merely from the great and, in some degree, extravagant importance which was attached to it by the Pythagorean and Platonic philosophers of antiquity, but likewise from its influence upon the particular speculations of arithmetical and other writers in modern times; we shall, therefore, give a very brief abstract of this Arithmetic, as far as in the first place regards the several species of numbers, and their arithmetical relations with each other; and secondly, those properties of numbers, which were supposed to lead to the knowledge of nature, and the principles of the true philosophy.

Definitions of unity.

(88.) Euclid has defined *unity* to be that according to which every thing which exists is called one; and *number* to be *multitude*, composed of unities.‡ The first of these definitions is one amongst innumerable other proofs that the ancients mistook the province of definition in attempting to explain a term such as unity, expressing an idea which does not admit of resolution into others more simple than itself; the consequence of this practice has been, that these definitions of the same term have no accordance with each other, and are in many cases absurd or unintelligible: thus one author says, that the *monad* is the principle and element of number, which while multitude is diminished by subtraction, is deprived of all number, and remains fixed and unchanged, since division cannot proceed beyond it.§ Another asserts, that the *monad* is one multitude: some say that it is the confine of number and parts; others that it is the form of forms, as comprehending causally all the ratios which are in number.‖ Another favourite subject also of disputation in the schools, was, whether *unity* is a number, and which was treated in many cases without reference to the definition of number itself:¶ thus, according to the Euclidean definition of numbers, the question must be answered in the negative; but if we define number to be that quantity by which every thing is numbered, it would be answered in the affirmative; since unity is a quantity which may be numbered by itself. We do not propose these questions as deserving of grave and serious discussion, nor shall we attempt to reconcile and explain definitions which are apparently so con-

* Casley, Plate 16.
† *Nouveau Traité de Diplomatique*, tom. iii. Plate 60.
‡ Lib. vii. def. 1, 2.
§ Theonis Smyrnæi *Mathematica*, p. 23.
‖ Taylor's *Theoretic Arithmetic*, p. 4.
¶ Alstedii *Encyclopædia*, p. 804; *Arithmétique* de Simon Stevin, where the affirmative of this question is vigorously maintained.

tradictory, and which are so far removed above all simple comprehension; we merely mention them for the purpose of showing how readily even the most acute understanding may be bewildered in a labyrinth of absurdities, unless guided in all its reasonings by fixed and intelligible principles.

Different species of numbers.

Pariter pares.

Numbers are distributed into various species, such as odd and even, prime or compound, &c. Even numbers are separated into such as are *pariter pares*, comprehended in the series,

4, 8, 16, 32, 64,

Impariter pares.

all whose divisors are even: *impariter pares*, forming the series

6, 10, 14, 18, 22,

Pariter et impariter pares.

which being divided by an even number, have no odd number for their quotient: and lastly, *pariter* and *impariter pares*, comprehending all other even numbers, greater than 2, which are not included in the two other classes.

Perfect.

Amicable.

Diametral.

Again, numbers are *perfect*, when equal to the sum of their divisors, *deficient* when less, and *superabundant* when greater than that sum. Two numbers are *amicable*,* when equal to the sum of each others divisors; and *imperfectly amicable*† when the sum of their divisors is the same for both, though not equal to either of them. *Diametral numbers*, the sum of the squares of whose two factors is a square number, the square root of which is the diameter.‡

Triangular.

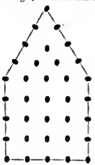
Polygonal numbers, which are of different species, such as *triangular* numbers, included in the series,

1, 3, 6, 10, 15, 21, 28,

Square.

Pentagonal.

which express the number of units which admit of being symmetrically arranged in an equilateral triangle, where there are 1, 2, 3, 4, 5, 6, 7, &c. units respectively corresponding to each side. *Square numbers*, the only species of polygonal number which Euclid has considered, as they alone strictly correspond in their properties, with the geometrical figures from which they derive their denomination. *Pentagonal numbers*, which only admit of symmetrical, or rather equidistant arrangement, in the equilateral, but not equiangular pentagon, which is formed by the junction of a square and equilateral triangle, as in the annexed figure:



These numbers are clearly, therefore, the sum of the triangular and square numbers corresponding to bases, which differ by unity, and are expressed by the series,

1, 5, 12, 22, 35, 51, 70, 92, &c.

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Since *triangular* numbers are formed by the addition of the terms of the series of natural numbers

1, 2, 3, 4, 5, &c.

square numbers, by the addition of the alternate terms of this series, or of the odd numbers,

1, 3, 5, 7, &c.

and *pentagonal* numbers, by the sums of every third term of that series, or of

1, 4, 7, 10, 13, &c.

Hexagonal.

So likewise *hexagonal* numbers are formed by the addition of every fourth term, beginning with the first, which are,

1, 5, 9, 13, &c.

and *heptagonal* numbers, by the addition of every fifth term, and so on for the polygonal numbers of higher orders: in those cases, therefore, there is evidently no reference whatever to geometrical figures; and it is quite clear, likewise, that the polygonal numbers of the ancients have no analogy to the figurate numbers of modern times.

If the terms of the series of triangular numbers be added together, we get the series

1, 4, 10, 20, 35, 56, &c.

Pyramidal.

the first in the series of solid pyramidal numbers, and which are called triangular pyramidal numbers. If the terms of the series of squares be otherwise added together, we get a series of *square pyramidal* numbers, which are

1, 5, 14, 30, &c.

In both cases these numbers would express the number of equal spheres, which can be placed in complete contact with each other, when those in the base form an equilateral triangle or square; and if the highest sphere, or the first number in the series be wanting, we get a *defective* pyramid; and hence the numbers 3, 9, 19, &c. are called *defective triangular pyramids*, and similarly for those cases where the base is a square or any other figure.

Cubes.

Parallelopipedons.

Lateruli.

Asseres.

If the series of squares be multiplied in succession by the natural numbers, we get the series of *cubes*. Numbers arising from three unequal factors, as 3, 4, 5, are called *parallelopipedons*, and sometimes *σκαληνοι σκαληνοι*, or in Latin *gradati*, and by others *βωμισκοι*, or *little altars*, from the resemblance which they bear to their analogous solids. When two equal numbers are multiplied into a less, as $3 \times 3 \times 2$, the result is denominated a *laterculus* or *tile*; but if two equal numbers are multiplied into a greater, the product is called *asser*, or a *plank*. All these distinctions were studiously multiplied, and their denominations founded upon some fancied analogy or resemblance to objects of sense.

Square numbers, oblong, &c.

Numbers were also distinguished into square, *ετερομηκεις*, and oblong; the second were the products of two numbers which differed by unity, and the last, the products of any other unequal numbers; and it was observed, that whilst square numbers are formed by the addition of the terms of the series of odd numbers, those of the second class are formed by the addition of the terms of the series of even numbers.

Different species of ratios.

Minute and trivial as many of the distinctions of the different species of numbers may appear to be, they are much less so than those which refer to ratios and proportions, and their different species; thus we

* Such are the numbers 284 and 220.

† The sum of the divisors of 27 and 35 respectively are equal to 13.

‡ The number $12 = 3.4$ is a diametral number, and 5 the diameter, since $3^2 + 4^2 = 25 = 5^2$.

Arithmetic. have ratios of equality, of greater and less inequality; of the second, there are five species, and as many corresponding to them of the third; thus we have multiple ratio, where the antecedent is a multiple of the consequent; superparticular,* where the antecedent is equal to the consequent and some part of it, which may become therefore, sesquialteran, sesquitertian, sesquiquintan, &c. as 3 to 2, 4 to 3, 5 to 4, &c. proceeding to infinite subdivisions; superpartient,† where the antecedent contains the consequent once, and some multiple of its parts, as 5 to 3, 7 to 4, 19 to 10, and so on; multiple superparticular,‡ where the antecedent contains a multiple of the consequent with some part of it, as 5 to 2, 23 to 11, &c.; and multiple superpartient,§ where the antecedent contains a multiple of the consequent and of some part of it, as 11 to 3, 17 to 7, and so on.

Different kinds of proportion. They distinguished three principal species of proportion, arithmetic, geometric, and harmonic, which were known to the more ancient philosophers; but later writers added three others, which were opposite to the former. Thus, if α, β, γ, are three numbers, arranged in the order of their magnitude, they will be related to each other, according to the conditions of the fourth *analogy*, or *μεσοτης*, if

$$\alpha : \gamma :: \beta - \gamma : \alpha - \gamma \qquad \|$$

of the fifth, if

$$\beta : \gamma :: \beta - \gamma : \alpha - \beta \qquad ¶$$

and of the sixth, if

$$\alpha : \beta :: \beta - \gamma : \alpha - \beta. \qquad **$$

To these six, four others were added by later writers, to complete the *decad*; these are as follows:

$$\alpha : \gamma :: \alpha - \gamma : \beta - \gamma \qquad \dagger\dagger$$
$$\alpha : \gamma :: \alpha - \gamma : \alpha - \beta \qquad \ddagger\ddagger$$
$$\beta : \gamma :: \alpha - \gamma : \beta - \gamma \qquad \S\S$$
$$\beta : \gamma :: \alpha - \gamma : \alpha - \beta \qquad \|\|$$

Many important properties of numbers discovered by the ancients. It is certain that many curious and valuable properties of numbers were discovered by the ancients, in their pursuit after these relations and analogies, which were supposed to lead to the knowledge of the true philosophy; every number became, as it were, possessed of a property, and all numbers possessed some relative analogy with each other, to which a name could be given; but it is a tedious and unprofitable task to wade through their multiplied refinements, their laborious demonstrations, that all equality proceeds from inequality; and again, that all inequality may be reduced to equality; that squares and cubes partake of the nature of identity or sameness, and a great variety of other questions equally trivial and unintelligible; and we have been chiefly induced to enlarge upon this subject to the extent which we have done, from observing the influence of these speculations, both upon the form and substance of nearly all works on Arithmetic which appeared before the end of the XVIth century.

Euclidean Arithmetic. The VIIth, VIIIth, IXth, and Xth Books of Euclid, form an abstract of the most rational portion of the science of numbers, as known to the ancients; many of the absurdities even of Plato himself, and still more of many of his commentators and successors, being excluded by the rigid form of his demonstrations, and the intimate dependence of his propositions on each other; a circumstance which drew a remark from Lord Bacon, that nothing had been added to this science since his time, which was worthy of the lapse of so many ages, the truth of which it would be difficult to dispute; in reality, the powers of the human mind were frittered away in the minute distinctions and endless refinements of the mystical philosophy of the schools of Pythagoras and Plato, which Bacon considered as *expatiatio quædam speculationis*; verifying, in short, the profound observation which he afterwards subjoins, that the human mind is naturally disposed to indulge itself in such speculations, when not occupied with the earnest pursuit of truth: *Hoc enim habet ingenium humanum, ut cum ad solida non sufficiat, in supervacaneis se atterat.*

Platonic Arithmetic. "Arithmetic," according to the opinion of the Platonic philosophers, "was not to be studied with gross and vulgar views, but in such a manner as might enable men to attain to the contemplation of the nature of numbers; not for the purpose of dealing with merchants and tavern-keepers, but for the improvement of the mind, considering it as the path which leads to the knowledge of truth and reality."* In order to show how perfectly removed their arithmetical speculations were from all practical objects, it is only necessary to refer to a few examples of the manner in which they philosophised about numbers. Thus, perfect numbers compared with those which are deficient or superabundant, are considered as the images of the virtues, considered as equally remote from excess and defect, and constituting a mean point between them; thus, true courage is a mean between audacity and cowardice, and liberality between profusion and avarice: in other respects also, this analogy is remarkable; as perfect numbers, like virtues, are few in number, and generated in a constant order; whilst superabundant and deficient numbers are like vices, infinite in number, disposable in no regular series, and generated according to no certain and invariable law.

Pythagorean Arithmetic. The tracing of analogies, like the preceding, accompanied as is generally the case with the most eloquent of the philosophical writers of antiquity, with moral illustrations of uncommon elegance and beauty, may be considered as furnishing, at least, a pleasing, if not a useful, exercise of the understanding; but, in other cases, analogies are taken for proofs, and assumed as the bases of the most absurd and inconsistent theories; and it would be difficult to refer any chapter in the history of the human mind which affords a more degrading picture of the perversion of the reasoning faculties, than that which would be furnished by the collection of the ravings of the Pythagorean philosophers on the subject of numbers and their properties; thus Pythagoras considered "number as the ruler of forms and ideas, and the cause of Gods and dæmons:" and again, that "to the most ancient and all-powerful creating Deity, number was the canon,

* *Λογος επιμοριος*, Theon. Smyrn. pars ii. cap. 24.
† *Λογος επιμερης*, Ibid. cap. 25.
‡ *Λογος πολλαπλασιεπιμοριος*, Ibid. cap. 26.
§ *Λογος πολλαπλασιεπιμερης*, Ibid. cap. 27.
‖ As, for instance, the numbers 6, 5, 3.
¶ As 2, 4, 5.
** As 1. 4. 6.
†† As 6, 8, 9.
‡‡ As 6, 7, 9.
§§ As 4, 6, 7.
‖‖ As 3, 5, 8.

* *Λογιστικὴ καὶ ἀριθμητικὴ, ὁλκὸς καὶ ἀγωγὸς πρὸς ἀλήθειαν ἅπτεσθαι δὲ λογιστικῆς, μὴ ἰδιωτικῶς, καὶ, ὡς ἐπὶ θέαν τῆς τῶν ἀριθμῶν, φύσεως ἀφίκωνται τῇ νοήσει. οὐδὲ πράσεως χάριν ἐμπόρων ἢ καπήλων ἀλλ' ἕνεκα ψυχῆς, τῆς ἐπ' ἀλήθειαν, καὶ οὐσίαν ὁδοῦ.* Theon. Smyrn. cap. i.

the efficient reason, the intellect also, and the most undeviating balance of the composition and generation of all things." Again, Philolaus declared, "that number was the governing and self-begotten bond of the eternal permanency of mundane natures." Another said, "that number was the judicial instrument of the Maker of the universe, and the first paradigm of mundane fabrication." And Taylor, their modern commentator and advocate, in order to add to this climax of absurdities, asks whether it is possible that these philosophers could have spoken thus sublimely of number, unless they had considered it as possessing an essence separate from sensibles, and a transcendency fabricative, and at the same time paradigmatic.*

Denarius Pythagoricus of Meursius.

The learned Meursius† has made a collection from the writings of Pythagorean philosophers, of the names and properties which they assigned to all numbers from 1 to 10, abounding, as might be imagined, with all kinds of absurdities and contradictions. We shall give a few specimens. Unity, or the *monad*, is termed by Nicomachus, "Intellect, male and female, God, and in a certain respect, matter; the recipient of all things (πανδοχευς,) chaos, confusion, commixture, obscurity, darkness, a chasm, Tartarus, Styx, and terror; the absence of mixture, a subterranean profundity, Lethe, a rigid virgin and Atlas; the axis, the sun, and Pyralios; Morpho, the tower of Jupiter, and the spermatic reason; Apollo, likewise, the prophet and soothsayer." The commentators on this passage say, that the monad is called *intellect*, as being the origin and fountain of all numbers, in the same manner as intellect is of all ideas: it is called *male* and *female*, as containing in itself *causally*, the odd and the even, the former corresponding to the *male*, and the latter to the *female*: as it is the cause of multitude, therefore it is called God, and matter in a *certain respect* only, as God is the first, and matter the last of things, and each subsists by negation of all things, and consequently matter is said to be dissimilarly similar to divinity: again, the *monad* was considered as the recipient of all things, from its analogy to the divinity, as all things are comprehended in the ineffable nature of divinity: it is called chaos, as being primæval and first born, from which all things have sprung, as all numbers from unity; but it is needless to pursue this tissue of absurdities to the conclusion, as we have given more than enough to satisfy our readers of the nature of their idle speculations.

The work of Bungus on the properties of numbers.

(89.) This passion for discovering the mystical properties of numbers descended from the ancients to the moderns, and numerous works have been written for the purpose of explaining them; amongst others we may mention that of Petrus Bungus,‡ whose work on this subject extends to 700 quarto pages; illustrating all the properties of numbers, whether mathematical, metaphysical, or theological; not content with collecting all the observations of the Pythagoreans concerning them, he has referred to every passage in the Bible in which numbers are mentioned, incorporating, in a certain sense, the whole system of Christian and Pagan theology: our limits will allow us to notice but one or two specimens of his reasoning; thus, the number 11 being the first

* Taylor's *Theoretic Arithmetic*, p. 165.

† *Denarius Pythagoricus*, Lugd. Batavorum, 1631.

‡ Petri Bungi Bergamotis, *Numerorum Mysteria*, 1618.

which *transgresses* the decad, denotes the wicked who transgress the *Decalogue*, whilst 12, the number of the Apostles, is the proper symbol of the good and the just; the number, however, upon which above all others he has dilated with peculiar industry and satisfaction, is 666, the number of the beast in the *Revelations*, the symbol of Antichrist; he shows that this is the number denoted by the words, τειταν, λαμπετις, λατεινος, and is particularly anxious to reduce the name of the impious and perfidious Hæresiarch Martin Luther, to a form which may express the same formidable number; for this purpose he transfers the numeral power of the Greek to the Latin alphabet, and after Italianizing and mispelling his name, he finds that

M (30) A (1) R (80) T (100) I (9) N (40)
L (20) V (200) T (100) E (5) R (80) A (1)

constitute the number 666. He seems conscious, however, of the liberties which he had been obliged to take in effecting his purpose, and consoles himself with his better success in its Hebraized form, לולתר, *Lulter*, which expresses the same number.*

Properties of the numbers three and seven.

(90.) The numbers 3 and 7 were, for very obvious reasons, the subject of particular speculation with the writers of that age; and every department of nature, science, literature, and art, was ransacked for the purpose of discovering ternary and septenary combinations. The excellent old monk Pacioli, or Fra. Lucas de Burgo Sancti Sepulchri, the author of the first printed Treatise on Arithmetic, has enlarged upon the first of these numbers in a manner which is rather amusing, from the quaint and incongruous mixture of the objects which he has selected for illustration.

Lucas de Burgo on ternary combinations.

"There are three principal sins," says he, "avarice, luxury, and pride; three sorts of satisfaction for sin, fasting, almsgiving, and prayer; three persons offended by sin, God, the sinner himself, and his neighbour; three witnesses in heaven, *Pater*, *verbum*, *spiritus sanctus*; three degrees of penitence, contrition, confession, and satisfaction, which Dante has represented as the three steps of the ladder that leads to purgatory, the first marble, the second black and rugged stone, the third red porphyry. There are three sacred orders in the church militant, *subdiaconati*, *diaconati*, and *presbyterati*: there are three parts, not without mystery, of the most sacred body made by the priest in the mass; and three

* It must be confessed, however, that these attacks on the great Reformer were not altogether unprovoked, either by himself or his followers; he himself interpreted this number to apply to the duration of Popery; and his friend and disciple, Stifel, the most acute and original of the early mathematicians of Germany, appears to have allowed himself to be seduced by these absurd speculations; he relates, in an appendix to his edition of Christopher Rudolph on *Algebra* in 1571, that whilst a monk at Esslingen in 1520, and when infected by the writings of Luther, he was reading in the library of his convent the 13th Chapter of *Revelations*, it struck his mind that the *Beast* must signify the Pope, Leo X.; he then proceeded in pious hope to make the calculation of the sum of the numeral letters in *Leo decimus*, which he found to be M, D, C, L, V, I; the sum which these formed was too great by M, and too little by X; but he bethought him again, that he had seen the name written *Leo X.*, and that there were ten letters in *Leo decimus*, from either of which he could obtain the deficient number, and by interpreting the M to mean *mysterium*, he found the number required, a discovery which gave him such unspeakable comfort, that he believed that his interpretation must have been an immediate inspiration of God.

This is not the only instance in which this excellent person gave way to these idle fancies; and he prophesied the downfall of the papacy on more than one occasion, and had the misfortune to live to see his own predictions falsified.

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times he says *Agnus Dei*, and three times, *Sanctus*; and if we well consider all the devont acts of Christian worship, they are found in a ternary combination; if we wish rigbtly to partake of the holy communion, we must three times express our contrition, *Domine non sum dignus*; but who can say more of the ternary number in a shorter compass, than what the prophet says, *tu signaculum sanctæ trinitatis*. There are three Furies in the infernal regions; three Fates, Atropos, Lacbesis, and Clotho. There are three theological virtues: *Fides, spes, et charitas. Tria sunt pericula mundi: Equum currere; navigare, et sub tyranno vivere*. There are three enemies of the soul: the Devil, the world, and the flesh. There are three things which are in no esteem: the strength of a porter, the advice of a poor man, and the beauty of a beautiful woman. There are three vows of the Minorite Friars: poverty, obedience, and chastity. There are three terms in a continued proportion. There are three ways in which we may commit sin: *corde, ore, ope*. Three principal things in Paradise: glory, riches, and justice. There are three things which are especially displeasing to God: an avaricious rich man, a proud poor man, and a luxurious old man. And all things, in short, are founded in three; that is, in number, in weight, and in measure."

Trinads of Bungus.

The collection of *trinads* which Bungus bas given would alone form a little volume, embracing, as they do, every species of knowledge, art, and science; he bas observed them in friendship, beauty, colours, eternity, in the first letter of the alphabet, in the monad, in music, in poetry, in a point, in a circle, in magnitude, in time, in primitive theology, and, in short, in almost every imaginable thing; so general, indeed, according to him, are these ternary combinations, that they make some approach to a general law of nature.

Septenary combinations

(91.) If the number 3 has been honoured with particular commemoration, the number 7 has received an equal, if not greater distinction. In the year 1502 there was printed at Leipsie a work entitled *Heptalogium Virgilii Salzburgensis*,* in honour of the number 7, and expressly composed for the use of students of the University of Leipsic; it consists of seven parts, each consisting of seven divisions. We think it unnecessary to detain the reader with the enumeration of the *septads* which this work contains, forming a collection, as might indeed be expected, of the most gross absurdities; our object being merely to show from this instance, as might be done from a multitude of others, how general was this passion for philosophising about the properties of numbers; so much so indeed, that vestiges of it may be discovered at a very late period, when the principles of just and philosophical reasoning were generally understood and practised.†

Ancient writers on Arithmetic.

(92.) The history of Arithmetic, down to the period of the introduction of the Arabic numerals, would be little benefitted by the analysis of the arithmetical writings of the Platonic school, presenting as they do no great variety in their form, and still less in their object; the chief of those whose works we possess, was Nicomachus of Gerasos, the author of a Treatise entitled *Isagoge Arithmetica*, and who flourished probably about the Christian era, though his date is uncertain; of this work, the Arithmetic of Boethius is in many respects a mere translation; it was honoured also with a commentary by Jamblichus, whose work, entitled Τὰ Θεολογούμενα τῆς Ἀριθμητικῆς, surpasses, in visionary speculations on the properties of numbers, the most absurd and extravagant of his predecessors. Martianus Capella, who flourished in the VIth century, wrote a Poem on the seven liberal arts, including Arithmetic and Geometry, entitled *De Nuptiis Mercurii et Philologiæ*. Theon of Smyrna, whose work we have had occasion to quote, and who flourished about the beginning of the IIId century, wrote expressly on those parts of Mathematics which were necessary for the understanding of the works of Plato. Porphyry, who flourished about the same time, aod who wrote on Arithmetic and the mysteries of numbers. Proclus, who in bis Commentary in the Ist Book of Euclid has furnished us with so much information on the history of the Mathematical sciences amongst the ancieots. Cassiodorus, a contemporary of Capella, Photius, Philo, Thymaridas, and a multitude of others, whose works have perished, but of whom we find notices in ancient writers, and who all, in common with other mathematicians of their age, appear to bave written for the purpose of expounding and amplifying the doctrines of Plato: so universal was the reverence in which they were held both by Christians and Pagans during the first seven centuries after the Christian era.

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Nicomachus

Jamblichus.

Martianus Capella.

Theon.

Porphyry.

Proclus

Cassiodorus, &c.

History of Arithmetic is the order of subjects.

(93.) In sketching the history of the progress of Arithmetic, after the introduction of the Arabic numerals, we shall follow the arrangement of subjects and not of authors, proceeding from numeration in a regular series through the common operations of Arithmetic. We are well aware of the many advantages regarding the history of scieoce, which arise from a regular analysis of the works of the principal writers, a plan which has been adopted so successfully by Delambre in his *History of Ancient Astronomy*, by Kæstner, and by other authors; but the extensive details in which such a plan must necessarily lead us, would be inconsistent with the limits within which we are confined by the nature of this work, and which indeed we have already very much exceeded; we shall include, however, in our present sketcb, occasional notices of the works which we shall have occasion to refer to, without attempting any further analysis of them than may be generally called for by the immediate subject of discussion.

Numeration in the *Lílávati*.

(94.) The process of numeratioo, as distinguished from notation, will of course vary with different nations, according to the genius of the language to wbich it is accommodated. Amongst the Hindoos, who possess distinct names for the first nineteen terms* of the

* Kæstner, *Geschichte der Mathematik*, vol. i. p. 204.

† As a final example, we will merely mention the following work, the contents of which, as might be expected, are quite worthy of the title: "*The Secrets of Numbers according to Theological, Arithmetical, Geometrical, and Harmonical computation. Drawn, for the better part, out of those Ancients, as well as Neoteriques. Pleasing to read, profitable to understande, opening themselves to the capacities of both learned and unlearned; being no other than a key to lead men to any doctrinal knowledge wha soever.* By William Ingpen, Gent., London, 1624." The first chapter is entitled, "*The excellencie of numbers: and how far they stretch towards the attaining of all manner of sciences.*"

* Ganésa, one of the Commentators on the *Lílávati*, says that these names were carried to a greater extent by Srid'hara, and other writers; but he does not notice them, in consequence of the little utility of such designations, as well as of the numerous contradictions which are found amongst them, from their not being always appropriated to the same numbers.

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decuple series of numbers beginning from unity, it assumes the form which is of all others the most simple in principle; it being merely necessary, in passing from notation to numeration, to repeat in succession the name of the digits with that of the corresponding term of the series; the practice, however, of such numeration is extremely tedious and embarrassing, from its preventing that distribution of large numbers into classes of superior units proceeding by thousands, myriads, or millions, &c. which is so useful in relieving the memory from the burden of so many independent terms, and in also assisting the mind in the conception and comparison of large numbers.

Amongst the Italians.

(95.) The Italians from an early period adopted the mode of numeration which is now in universal use, distributing the digits into periods of six, and, consequently, proceeding by millions; the units in the several classes thus formed being millions, billions, trillions, &c. (Art. 16.) Such is the process of numeration which is given by Lucas de Burgo.

Amongst the Spaniards.

(96.) The Spaniards adopted the term *cuento* to denote a million, and the following is the table of numeration which is given in the *Arithmetic* of Juan de Ortega* in 1536:

10.	Dezena.
100.	Centena.
1000.	Millar.
10000.	Dezena de millar.
100000.	Centena de millar.
1000000.	Cuento.
10000000.	Dezena de cuento.
100000000.	Centena de qo. (*cuento.*)
1000000000.	Centena de millar de qo.

Their numeration was thus limited to eleven places of figures, as the term *cuento* did not admit of composition with the terms for two, three, four, &c. in the same manner as the term *millione* in *billione, trillione, quadrillione, &c.* In an earlier Spanish author, however, on the same subject,† we find the term *millione* applied to designate 1000000000000, or a billion, whilst *cuento* is used in its ordinary sense for 1000000; thus the number

957 | 653 | 978 | 245 | 349 | 186 | 357 | 243

consisting of twenty-four places, and separated into periods of three, is expressed in Latin, adapted to the Spanish numeral language, by *nonaginti quinquaginta septem millia | sexcenti quinquaginta tres cuentos | nonaginti septuaginta octo | ducenti quadraginta quinque milliones | trecenti quadraginti novem millia | centum octoginta sex cuentos | trecenti quinquaginta septem millia | ducenti quadraginta tres;* the misapplication of the term *million*, which is found in this case, is a very curious example of a practice which we have already had occasion to remark on more than one occasion.

The term million: when first used in England.

(97.) We have no means of ascertaining the precise period at which this term was introduced into our

* *Tractado subtilisimo de Arismetica y de Geometria; compuesto y ordenado por el reverendo padre fray* Juan de Ortega, *de la orden de los predicadores.* This is a work of some merit, and we shall afterwards have occasion to notice a method which it contains for approximation to the square and cube root.

† *Arithmetice Practice seu Algorismi Tractatus* à Petro Sanchez Teruelo *noviter compilatus explicatusque. Impressus Parisiis per* Thomam Rees *in domo rubra post Carmelitas, Anno* 1513.

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own language. Bishop Tonstall,* who has discussed at great length the Latin nomenclature of numbers, speaks of the term *million* as in common use, but he rejects it as barbarous. *Quartus locus,* says he, *exhibet mille; septimus millena millia; vulgus millionem barbare vocat.* Again he says, *Decimus locus capit millies millena millia; vulgus milliones millionum vocat.* In this case, however, the combination of these terms is erroneous, as it would designate a million of millions, or a *billion*. It is not easy to say what class of persons were meant to be designated by the term *vulgus*; but, most probably, the arithmetical writers of this and other countries; at all events, the term appears in Recorde's *Arithmetic*, and in all subsequent writers on this subject.

When first used in Germany.

(98.) It appears to have been admitted into German at a much later period than into English and French. Kæstner says,† that he found it in no German author on Arithmetic in the first half of the XVIth century; and Clavius‡ is the first writer of that nation, who in a Latin Treatise on Arithmetic has noticed the term; in the chapter on numeration he says, *Si more Italorum millena millia appellare velimus milliones, paucioribus verbis et fortasse significantius numerum quemcunque propositum exprimemus.* He does not seem to have carried the innovation farther, as we afterwards find *billions* expressed by *milliones millionum*, which are the highest numbers which he has occasion to use.

Divisions of numbers into periods of three and six.

(99.) It has been from a very early period the custom of writers on Arithmetic to separate numbers into periods of three and of six, as the numeration in most European languages must proceed by thousands and millions; these periods are called *membres* by Stevinus,§ amongst whose *definitions* we find the following: *Chasques trois characteres d'un nombre s' oppellent membre, des quels le premier, sont les premiers trois characteres à la dextre: et le second membre, les trois characteres suivant, vers la sinestre: et ainsi par ordre du troisieme membre et autres suivants, tant qu'il y en aura au nombre proposé.* Instead of *million*, he says, *mille mille;* for a *thousand millions*, he uses *mille mille mille;* and for a *billion, mille mille mille mille*, and so on for higher numbers. If we might be allowed to judge from this practice and numeral phraseology of Stevinus, as well as from the observation of his contemporary Clavius, we might imagine that the term million was not yet in general use amongst mathematicians. A different system, however, began to prevail at no very distant period; for we find Albert Girard, in his *Commencemens de l'Arithmetique,*‖ in his account of numeration which he terms *Prolation des Nombres*, dividing the places into periods of six, which he terms *premiere*

* *De Arte Supputandi*, p. 4. In numeration he divides the places into periods of three, and calls 1000000 *millena millia*, or *millies millena;* 1000000000, *millies millena millia;* 1000000000000, *millies millena millia millies*, and so on, proceeding by analogy with the practice of classical authors in the construction of their expressions for high numbers.

† *Geschichte der Mathematik*, vol. i p. 145.

‡ Christophori Clavii Bambergensis, e S. J. *Epitome Arithmeticæ Practicæ*, Rom 1583.

§ *Arithmetique, Livre Premier Definitiones*, 1584 This work, which bears many marks of the acuteness and originality of its author, was published first in Flemish, and afterwards translated into barbarous French The whole works of Stevinus were collected together, and published at Leyden in 1634, the year after his death, by his friend and commentator Albert Girard.

‖ *Invention Nouvelle en Algebre*, par Albert Girard, Mathematicien, Amsterdam, 1629.

masse, seconde masse, troisieme masse, respectively, the first of which only is divided into two periods of three places.

Fundamental operations of Arithmetic. (100.) The fundamental operations of Arithmetic, as given in the *Lilávati*, are eight in number; namely, addition, subtraction, multiplication, division, square, square root, cube, cube root:[*] to the first four of these the Arabs added two, namely, *duplation* and *mediation* or *halving*, considering them as operations in some degree distinct from multiplication and division, in consequence of the readiness with which they were performed; and they appear as such in many of the books of Arithmetic of the XVIth century.[†]

Addition. (101.) With respect to the two first operations, whether in Sanscrit or other authors, we shall not find much to remark. The rule given in the *Lilávati*, in the first case, is as follows: "The sum of the figures, according to their places, is to be taken in the direct, or inverse order;" which is interpreted by the Scholiast to mean, "from the first on the right towards the left, or from the last on the left towards the right." In other words, that they commenced indifferently with the figures in the highest or lowest places, a practice which would not lead to much inconvenience when their mode of working addition is considered; thus to add 2, 5, 32, 193, 18, 10, 100, they proceed as follows:

Amongst the Hindoos.

Sum of the units. . . . 2, 5, 2, 3, 8, 0, 0. . . .	20
Sum of the tens 3, 9, 1, 1, 0	14
Sum of the hundreds 1, 0, 0, 1.	2
Sum of the sums	360

If they had commenced with the figures in the highest places, the process would have stood as follows:

Sum of the hundreds	2
Sum of the tens	14
Sum of the units	20
Sum of the sums. . . .	360

Subtraction. (102.) The process of subtraction was commenced likewise either from the right or from the left, but much more commonly from the latter; and it is a circumstance sufficiently remarkable, that this practice of commencing subtraction from the highest place, which is subject to considerable inconvenience, should have been so very general. It is found in Arabic writers, in Maximus Planudes, and in many European writers as late as the end of the XVIth century.

Planudes. In Planudes, numbers to be added or subtracted are placed underneath each other, as in modern books of Arithmetic; and the sum in one case, and the difference in the other, is placed above the whole. When the digits in the subtrahend are greater than those in the minuend, a unit is placed beneath them, as in this example:

1 8 7 6 9	remainder.
5 4 6 1 2	minuend.
3 5 8 4 3	subtrahend.
1 1 1 1	

In performing the operation, 3 is increased by the unit in the next place to the right, and similarly for 5, 8, 4; and the digits thus increased are subtracted from the digits above, when increased by 10, in order to get the remainder.

In other cases the process is arranged as follows:

0 6 7 7 9	remainder.
2 9 9 1	
3 0 0 2 4	minuend.
2 3 2 4 5	subtrahend.

The digits 3, 0, 0, 2 in the minuend are replaced by 2, 9, 9, 1; and then 5 is subtracted from 4, 4 from 1, 2 from 9, 3 from 9, and 2 from 2, in order to get the remainder. It is obvious, that when such a preparation is made, it is indifferent whether the operations proceed from right to left, or from left to right.

Tonstall. (103.) Bishop Tonstall attributes the invention of the modern practice of subtraction to an English Arithmetician of the name of Garth; a method by which any number, however great or however intricate, might be subtracted, *manentibus notis universis*. This method he has illustrated with great detail, and has added for the assistance of the learner a *subtraction table*, giving the successive remainders of the nine digits when subtracted from the series of natural numbers from 11 to 19 inclusive, the only cases which can occur in practice. In speaking of the methods of preceding writers he has given the following, which will be at once explained by the example by which he has illustrated it:

2	9	10	10
3	0	1	0
1	1	1	1
1	8	9	9

The digits in the minuend are replaced by the numbers, whether digits or not, from which the subtraction must be made.

Ramus. (104.) In the *Arithmetic* of Ramus, which was published in the year 1584, though written at an earlier period, we find the operation performed from *left* to right; and the same practice is followed by some other writers of his school.[*] Thus in subtracting 345 from

[*] The subject of the 12th Chapter of the *Brahma-sphuta-sidd'hanta* of Brahmegupta, written in the VIIth century, is Arithmetic; and it commences by defining the knowledge which constitutes a *ganaca*, or *calculator*: "He who distinctly and severally knows addition and the rest of the twenty logistics, and the eight determinations, including measurement by shadow, is a calculator." The Scholiast on this passage states those rules to be the eight fundamental operations, five rules of reduction of fractions, rule of three terms, (direct and inverse,) of five terms, seven terms, nine terms, eleven terms, and barter, which are twenty arithmetical operations. Mixture, progression, plane figure, excavation, stack, saw, mound and shadow, are eight determinations.

[†] This distinction was abandoned when the processes for multiplication became more general and uniform, as an absurd and unnecessary refinement; thus Gemma. Frisius, in his *Arithmetica Practicæ Methodus Facilis*, published in 1548, speaks of this practice with great contempt: *Solent nonnulli duplationem et mediationem assignare species distinctas a multiplicatione et divisione. Quid vero moverit stupidos illos nescio, cum et definitio et operatio eadem sit.* A different reason, however, is given by Lucas de Burgo for abandoning this distribution of the parts of Algorithm. "The ancient Philosophers," says he, "assign nine parts of algorism; but we will reduce them to seven, in reverence of the seven gifts of the Holy Spirit: namely, numeration, addition, subtraction, multiplication, division, progression, and extraction of roots."

[*] Ramus was obliged to quit his country, and take refuge in Germany, during the persecutions of the Protestants in France. He there established a school of philosophy and mathematics, chiefly distinguished for the introduction of more minute and logical subdivisions of the subjects of discussion, whether mathematical or not, than is to be found amongst preceding writers. He had many followers and most zealous admirers, who constituted his mathematical school, and who wrote upon Arithmetic

Arithmetic. History.

432, the sums to be subtracted, and the remainder are written as follows:

$$\begin{array}{ccc} & 8 & 7 \\ \cancel{4} & \cancel{3} & \cancel{2} \\ \cancel{3} & \cancel{4} & \cancel{5} \end{array}$$

When 3 is subtracted from 4, the remainder should be 1, bot it is replaced by zero, since the next digit in the subtrahend is greater then the one corresponding to it in the miouend: in this case also the remainder, which would be 9, is reduced to 8, since the next digit, 5 in the suhtrahend, is greater than 2 which is above it; the last remainder, 7, is not altered.

Orontius Fineus.

Ramus speaks with great respect of Orontius Fineus,* bis predecessor in the professorship of methematics at Paris, as heving revived, and in some measure iotroduced, the study of those sciences in France. He was also the author of e work on Arithmetic,† where the process of subtraction is taught under the same form in which it is found in modern books of Arithmetic. It is difficult to account for the adoptioo of this very incoovenient practice hy Ramus, when the other method must heve been familiar to him; and we cen only attribute it to that love of singularity which led him to aspire to the fouodation of e school of his own.

Multiplication. Methods in the *Lilávati*.

(105.) The author of the *Lilávati* has ooticed six different modes of multiplying numbers, and two others are mentioned hy his commentators; these will be best explained by their application to the following example, which is giveo io that work:

"Beautiful and dear Lilávati,‡ whose eyes are like e fawn's; tell me what are the oumbers resulting from one huodred and thirty-five taken into twelve? If thou be skilled in multiplication, hy whole or by parts, whether by division or separatioo of digits, tell me, euspicious woman, what is the quotient of the products, divided hy the same multiplier?"

Statemeot. Multiplicand, 135. Multiplier, 12.

(1.) Product. (Multiplying the digits of the multiplicand oecessarily by the multiplier.)

$$\begin{array}{ccc} 1 & 3 & 5 \\ 12 & 12 & 12 \\ \hline & 12 & 60 \\ & 3 & 6 \\ \hline & 16 & 20 \end{array}$$

(2.) Or subdividing the multiplier into parts, es 8 and 4, and severally multiplying the multiplicaod by them; thus

$$\begin{array}{ccc} 135 & 8 & 1080 \\ 135 & 4 & 540 \\ \hline & & 1620 \end{array}$$

(3.) Or the multiplier 12 being divided by 3, the quotient is 4; by which and by 3, successively multiplying, the last product is the result; thus

$$\begin{array}{cccccc} 135 & 4 & 20 & 540 & 3 & 120 \\ & & 12 & & & 15 \\ & & 4 & & & \\ \hline & & 540 & & & 1620 \end{array}$$

(4.) Or taking the digits as parts, viz. 1 end 2, the multiplicand being multiplied hy them severally, and the products added together, according to the places of figures; thus

$$\begin{array}{cc} 135 & 135 \\ 1 & 2 \\ \hline & 270 \\ & 135 \\ \hline & 1620 \end{array}$$

(5.) Or the moltiplicand being multiplied by the multiplier less 2, viz. 10, and added to twice the multiplicand; thus

$$\begin{array}{ccc} 135 & 10 & 1350 \\ 135 & 2 & 270 \\ \hline & & 1620 \end{array}$$

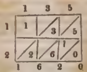
(6.) Or the multiplicand being multiplied by the multiplier increased by eight, viz. 20, and eight times the multiplier being subtracted; thus

$$\begin{array}{ccc} 135 & 20 & 2700 \\ 135 & 8 & 1080 \\ \hline & & 1620 \end{array}$$

The other two methods ere given in the commentary of Genésa:

Reticulated multiplication.

(1.) Form e series of equal squares, the number of



which in length is the same as the number of places in the multiplicand, and the number in depth the same as the oumber of places io the multiplier; divide these squares by diagonals, end write the multiplicand and multiplier on the adjacent sides of the rectaogle, each digit being placed opposite to e square, end the highest place io both being reckoned from the same angle. Multiply the several digits of the multiplicand and moltiplier together, placing the several products in the squares which ere commoo to the two digits which are multiplied successively together; the digit io the unit's place being put in the lower half, and that io the place of tens being put in the higher division of each square which is formed by its diagonel. The entire product is found by adding the digits between the same diagonals successively together.

and Algebra; of these was Bernard Salignac, of Bordeaux, his companion in exile, who wrote a work on Arithmetic, as well as *Tractatus Arithmetici Partium et Alligationis*, 1575; Christian Urstis or Urstisius, whose work entitled *Elementa Arithmeticæ, logicis legibus explicata*, was published in 1579; Joannes Freigius, Christopher Clavius, and many others. Ramus afterwards returned to Paris, and became one of the victims of the massacre of St. Bartholomew.

* In his Preface to his *Arithmetic*.

† Orontii Finei Delphinatis, Regii Mathematicarum Lutetiæ Professoris, *De Arithmetica Practica, libri quatuor*, 2d edit 1555.

‡ It was the daughter of Bhascara who is apostrophised in this very affectionate manner, whom he would not allow to marry, in consequence of having discovered, by an astrological scheme, that such an event would be fatal to his own life. It was by way of consolation that he dedicated this work to her, and called it by her name.

Arithmetic.

The shabacah of the Arabians.

This method of multiplication, which appears to have been very popular in the East, was adopted by the Arabs, who termed it *shabacah*, or *network*, from the *reticulated* appearance of the figure which it formed, and also by the Persians, under a slight alteration of form. It is found likewise amongst the early Italian writers on Algebra; and the same principle may be recognised in the process of multiplication by Napier's bones.

Cross multiplication.

The eighth and last method of multiplication is described by Ganésa in the following terms: "After setting the multiplier under the multiplicand, multiply unit by unit, and write the result underneath; then, as in cross multiplication, multiply unit by ten, and ten by unit, add together, and set down the sum in a line with the foregoing results; next multiply unit by hundred, and hundred by unit, and ten by ten; add together and set down the result as before, and so on with the rest of the digits; this being done, the sum of the results is the product of the multiplication." Thus,

$$\begin{array}{r} 135 \\ 12 \\ \hline 10 \\ 11 \\ 5 \\ 1 \\ \hline 1620 \end{array}$$

The Commentator, however, considers this method as difficult, and not to be learnt by dull scholars without oral instruction.

Multiplication considered as very difficult.

(106.) The number and variety of these methods would seem to show that the operation of multiplication was considered as one of considerable difficulty; and it is sufficiently remarkable, that the ordinary process of multiplying the multiplicand by the successive digits of the multiplier, and adding together the several results arranged in their proper places, should not be found amongst them. We find no notice of the multiplication table either amongst them or the Arabs; at all events it did not form a part of their elementary system of instruction, a circumstance which would account for some of the expedients which they appear to have made use of, for the purpose of relieving the memory from the labour of forming the products of the higher digits with each other.

Hindoo methods adopted generally by the Arabians.

(107.) The Arabs adopted most of the Hindoo methods of multiplication, and added some others of their own. They appear to have adopted the methods of Apollonius for the multiplication of articulate numbers, as far as the determination of the order of their product was concerned: we find amongst them many peculiar contrivances for the multiplication of numbers between 5 and 10, 10 and 20; of numbers between 1 and 10 into others between 10 and 20; of numbers between 10 and 20 into others between 20 and 100; and so on. They may be considered also as the authors of the method of *quarter squares*, or of finding the product of two factors by subtracting the square of half their difference from the square of half their sum.

The Arabians the inventors of the proof by casting out the 9s.

(108.) The Arabs were, most probably, the inventors of the proof of the accuracy of arithmetical operations by casting out the 9s, which is as yet unknown to the Hindoos; they called it *tarazu*, or the *balance*. In general, however, they contented themselves with the inheritance of the science transmitted to them from the Greeks, or with what they received from the East, with little or no attempt to add to them by native inventions.

History.

The substance of the work of Planudes derived from the Arabians.

(109.) It is one amongst many proofs that the work of Planudes was chiefly collected from Arabic writers, that he was acquainted with this method of casting out the 9s. In the operation of multiplicatioo itself, he has chiefly followed the method of multiplying *crosswise*, or *κατὰ τὸν χιασμὸν*, from the figure of χ, which is employed to connect the digits to be multiplied together; thus, in multiplying 24 into 35, the factors are written thus,

$$\begin{array}{c} 8\,4\,0 \\ 3\,5 \\ \chi \\ 2\,4 \end{array}$$

His methods of multiplication.

Multiply 4 into 5, (*μονάδες*,) write down 0 and retain 2 for the next place; multiply 4 into 3, and 2 into 5, the sum is 22, which added to 2 makes 24, (*δεκάδες*;) write down 4 and retain 2; lastly, multiply 2 into 3, add 2, which makes 8, (*ἑκατοντάδες*); we thus get the product 840.

This is not the only process of multiplication which he has given; there is another which he acknowledges to be very difficult to perform with ink upon paper, (*ἐπὶ χάρτου διὰ μέλανος*,) but very commodious on a board strewed with sand, where the digits may be readily effaced and replaced by others; thus, taking the same example,

$$\begin{array}{c} 8\,4\,0 \\ 7' \\ 6'\,2' \\ 3\,5 \\ 2\,4 \end{array}$$

we multiply 2 into 3, write 6 above 3; again, multiply 2 into 5, the result is 10; add 1 to 6, and replace it by 7, or write 7 above it; multiply 4 into 3, the product is 12; write 2 above 5, and add 1 to 7, which is replaced by 8, or 8 written above it; lastly, multiply 4 into 5, the result is 20; add 2 to 2, place 4 above it, and after it the cypher; the last figures, 840, or those which remain without *accents*, will express the product required.

Methods of multiplication in Italian writers on Arithmetic.

(110.) The Italians, who cultivated Arithmetic with so much zeal and success, from a very early period adopted from their oriental masters many of their processes for the multiplication and division of numbers; adding, however, many of their own, and particularly those which are practised at this day. In the *Summa de Arithmetica* of Lucas de Burgo we find eight different methods of multiplication, some of which are designated by names of a very quaint and fanciful nature. We shall mention them in their order:

1. *Multiplicatio: bericuocoli e schacherii.* The second of these names is derived from the resemblance of the written process to the squares of a chess-board; the first from its resemblance to the *chequers* on a species of sweetmeat or cake made chiefly from the paste of *bacochi* or apricots,* which were commonly used at festivals. The process is as follows:

* *Bericuocolo: spezie di confortino; si facevano prima quel confortini de pasta de bacochi, com' e da credere.*

Arithmetic.

				9	8	7	6
				6	7	8	9
			8	8	8	8	4
		7	9	0	0	8	
	6	9	1	3	2		
5	9	2	5	6			
6	7	0	4	8	1	6	4

This method of multiplication, denominated *schachero* at Venice, *bericuocolo* at Florence and Verona, and at Verona and some other cities of Italy *organetto*, is exhibited by Tartaglia,* and later Italian writers, without the squares, in the appearance of which these singular names originated. It thus became the method which is now universally used, and which was adopted from the beginning of the XVIth century by all writers on Arithmetic, nearly to the exclusion of every other method.

2. *Castelluccio; by the little castle.* It is difficult to discover the reason of this denomination.

```
  9876
  6789
  --------
  61101000
   5431200
    475230
     40734
  --------
  67048164
```

This was one of the methods much practised by the Florentines, by whom it is sometimes termed *all' indietro*, from the operation beginning with the highest places, *more Arabum*, according to the statement of Pacioli.

3. *Columna, o per taveletta; by the column, or by the tablets.* These were tables of multiplication commonly called *libretti*, or *librettine*, and at Florence *caselle*; they were arranged in columns, the first containing the squares of the digits, the second the products of 2 into all digits above 2; the third of 3, into all digits above 3; and so on, extending in some cases as far as the products of all numbers less than 100 into each other. Pacioli says, that these tablets were learned by the Florentines *etiam a cunabulis*; and their familiarity with them was considered by him as a principal cause of their superior dexterity in arithmetical operations, and he consequently seizes every opportunity of impressing on the mind of the student in Arithmetic the necessity of a perfect acquaintance with these tables. Tartaglia also, after giving some examples of their utility, earnestly entreats every amateur (*dilettante*) of the practice of Arithmetic to force himself to learn, if not the whole, at least the greater part of them, and in particular to make himself familiar with those numbers which are used in the division of the coins, weights, and measures of the city in which he resides; of this kind, in the *magnificent* city of Venice, are the numbers 12, 20, 24, 25, 32, and 36.

History.

This method is used in multiplying any number, however large, into another which is within the limits of the table. Thus, to multiply 4685 into 13, the digits of the multiplicand are multiplied successively into 13, and the results formed in the ordinary manner.

4. *Crocetta sive casella; by cross multiplication.* A method which is said to require more exertion of the understanding than any other,* particularly when many figures are to be combined together. The following examples will explain it sufficiently:

1 3 6 9 2 0 7 9 3 6

Pacioli, who rarely omits an opportunity of moralizing, after expressing his admiration of this method, as *una bella et sotil cosa*, but one which *vol cervello a casa e l'occhio a botega*, proceeds to enlarge on the great difficulty of attaining excellence, whether in morals or in science, and on the species of analogy which exists between them; that whilst with respect to one there is no virtue without labour, so in respect to the other the saying of the philosophers is equally just, *quod virtus est circa difficile:* that whilst the good and the wise are few, and of a rare occurrence, the wicked and foolish are met with everywhere, according to that other saying, *stultorum numerus est infinitus.*

5. *Quadrilatero; by the square.* A method which is characterised as elegant,† and as not requiring the operator to attend to the *places* of the figures when performing the multiplications:

	5	4	3	2	
	5	4	3	2	
1	0	8	6	4	
1	6	2	9	6	4
2	1	7	2	8	2
2	7	1	6	0	6
	2	9	5	0	6

* *Numerie Misure*, pars 1ma Venice, 1556. This is a work in three large volumes; the first of which contains the most elaborate system of practical and mercantile Arithmetic that was known in that age, and which we shall have very frequent occasion to refer to. The other volumes are divided into five parts, the subjects of which are geometry, mensuration, speculative arithmetic, and algebra. Its author, Nicolas Tartaglia, justly celebrated for his important discovery of the method of solution of cubic equations, derived his name, according to the testimony of Tiraboschi, from the following incident. He was born at Brescia in 1500, and at the sack of that city, by the French in 1512, was left for dead with three sword-cuts on his head and two on his face and lips; by the care of his mother, however, he recovered, but in consequence of the wound on his lips he lisped or stammered so much, that he was nicknamed by the boys Tartaglia, from the Italian word by which this imperfection is designated. In later life he retained a name which was not without interest as connected with the story of his misfortunes. In 1534 he settled at Venice, and became Professor of Arithmetic, a situation which he filled with extraordinary reputation for twenty-five years.

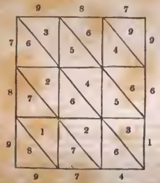
Some idea may be formed of the opinion entertained of Tartaglia and of his work, from the following title of an abridged translation of it: *L'Arithmetique de Nicolas Tartaglia, Brescian, grand mathematicien, et prince des praticiens. Recueillie et traduite de l'Italien en Francois, par* Guillaume Gosselin *de Caen. Dediée a tres illustre et vertueuse princesse Marguerite de France, reyne de Navarre,* 1578

* *Et quel modo vole alquanto piu fantasia e cervello che alcuno degli altri.*

† *Quale e bello e non bisogna tenere a mente le dicine.*

6. *Gelosia sive graticola; latticed multiplication.* "It is called by this name," says Pacioli,* "because the disposition of the operation resembles the form of a *lattice*, or *gelosia*, a term by which we designate the blinds or gratings which are placed in the windows of houses inhabited by ladies, so that they may not easily be seen, as well as by other *nuns*, in which the lofty city of Venice greatly abounds; and it is not surprising that the vulgar have found such names for this operation, inasmuch as astronomers, even in our days, have assumed the names and positions of many stars from animals and terrestrial material forms." The method in question will be understood from the subjoined example, and is clearly the same as one of those noticed above as in common use amongst the Hindoos, Arabians, and Persians.

To multiply 987 into 987:



7. *Ripiego;* multiplication by the *unfolding* or resolution of the multiplier into its component factors: thus to multiply 157 by 42, resolve 42 into its *ripieghi*, 6 and 7, and multiply successively by them:

$$\begin{array}{r} 157 \\ 6\,.\,42\,.\,7 \\ \hline 942 \\ 7 \\ \hline 6594 \end{array}$$

8. *Scapezzo;* multiplication by *cutting up*, or separating the multiplier into a number of parts, which compose it by their addition: thus, multiply 2093 by 17.

$$\begin{array}{r} 2093 \\ 17 = 10 + 7 \\ \hline 20930 \\ 14651 \\ \hline 35581 \end{array}$$

In some cases, both multiplicand and multiplier are separated into parts: thus, to multiply 15 into 12, we may separate 15 into 4, 5, 6, and 12 into 2, 4, 6, and proceed as follows:

$$\begin{array}{ccc|c} 4\,, & 5\,, & 6 & \\ 2\,, & 4\,, & 6 & \\ \hline 8 & 16 & 24 & 30 \\ 10 & 20 & 30 & 60 \\ 12 & 24 & 36 & 90 \\ \hline 30 & 60 & 90 & 180 \end{array}$$

Origin of the sign of multiplication.

(111.) In another Italian Treatise on Arithmetic published in 1567,* we find the same distinctions preserved, and the same names, or nearly so, attached to them; the method of cross multiplication is expressly attributed to Leonardus Pisanus, who derived it in common with Maximus Planudes from the Hindoos, through the Arabians; and it is not improbable that the St. Andrew's cross, which is the sign of multiplication, was derived from the custom of uniting the numbers to be multiplied together by lines which crossed each other, as in this example:

$$\begin{array}{c} 5\;9 \\ \times \\ 4\;7 \\ \hline 2773 \end{array}$$

Other methods of multiplication.

(112.) Both Lucas de Burgo and Tartaglia have mentioned the names of other methods of multiplication which were made use of in their time; such were the methods *per coppa* or *calice*, *per rombo*, *per triangolo*, *per diamante*. The first was most probably of the following kind, at least if we may judge from the very imperfect description of it which Pacioli has given:

$$\begin{array}{c} 234 \\ 234 \\ \hline 46116 \\ 682 \\ 12 \\ 9 \\ 8 \\ \hline 54756 \end{array}$$

Excessive fondness for novel methods of multiplication.

(113.) An extraordinary passion seems to have prevailed in that age for the invention of new forms of multiplication, and every professional practitioner of Arithmetic (and such were to be found in every mercantile city of Italy) considered it as an important triumph of his art if he could produce a figure more elegant and more refined in its composition and arrangement than those which were used by others. They are all of them, however, characterised by Pacioli as inconvenient, at least compared with those which he had given; and Tartaglia treats them as trifling and superfluous, such as any one may invent who is acquainted with the 2d Proposition in the IId Book of Euclid.

In performing multiplication *a bocca over per testa orally, or by the head,* that is, *senza penna*, says Tar-

* *Or sesto modo de moltiplicare e chiamato* gelosia: *ovvero per graticola: e chiamasi per questi nomi, perche la dispositione sua quando si pone in operatione torne a modo di graticola ovvero di gelosia. Gelosia intendiamo quelli graticelli chi si costumano mettere alle fenestre delle case dove habitano donne; acio non si possino facilmente vedere o altre religiose, di che molto abonda la excelsa cita di Venegia. E non e maraviglia che l' vulgo habe trovato questi vocaboli a tali operationi; per che ancora li astronomi hanno assumpti de molte stelle nomi e siti loro, da animali e forme terrestri materiali.*

* *Le Pratiche delle due Prime Mathematiche di* Pietro Cataneo Sienese. In Venetia, 1567.

Florentine indigitation.

taglia, the Florentines make use of a species of indigitation, working numbers by the inflections of the figures. The methods for this purpose which he describes are similar, though not identical with the methods of Bede and others which have been already described above; they furnish one amongst innumerable other proofs of the proficiency of that extraordinary people in Arithmetic, as well as in all the other arts of civilized life.

Multiplication table.

(114.) We have before mentioned that the Hindoos had no proper knowledge of the multiplication table; and though the Arabs used sexagesimal tables to aid them in their operations upon sexagesimals, they do not appear to have made use of the table of Pythagoras as the basis of their arithmetical education; the credit of introducing it, therefore, is due to the early Italian writers on this science, who probably found it in the writings of Boethius, and adopted it from thence. Familiar as the use of it, even on a very extended scale, appears to have been among the Italians, and particularly amongst the Florentines, yet many writers of other countries considered it important to relieve the memory from the labour of retaining it for the products of all digits exceeding 5, by giving rules for the formation of them; the principal rule for this purpose, called *regula ignavi*, or the *sluggard's rule*, was adapted from the Arabians, and is found in Orontius Fineus, Recorde, Laurenberg, Alstedius, and most other writers on Arithmetic between the middle of the XVIth and XVIIth centuries. It is as follows: *Subtract each digit from* 10, *and write down the difference; multiply these differences together, and add as many tens to their product as the first digit exceeds the second difference, or the second digit the first difference.* The following are examples:

Sluggard's rule.

7 3	8 2	8 2	9 1	9 1
×	×	×	×	×
6 4	7 3	8 2	8 2	9 1
4 2	5 6	6 4	7 2	8 1

The principle of this rule is too obvious to require demonstration, and we merely mention it as an instance of the disposition of the inferior writers of that as well as of other ages, to adhere to trifling and particular processes, when the same thing may be effected more rapidly by one which is general. The Arabians, as we have seen, not only made use of the rule in question, but likewise of others similar in principle, which included numbers of two places of figures; a practice which may be accounted for, and in some measure justified, by their very general use of sexagesimals, and the consequent importance of being able to form the products which are found in a sexagesimal table.

Other methods of multiplication.

(115.) Other expedients have been proposed to relieve the memory in the process of multiplication, from the labour of carrying the tens. The following is proposed by Laurenberg, an author who endeavoured to elevate the character of the common study of Arithmetic, by collecting all his examples from classical authors, and by making them illustrative of the geography, chronology, weights and measures of antiquity. It will be readily understood from his example:

```
   5142
     43
 ------
    106
   1532
    108
  2046
 ------
 221106
```

Rule for casting out the 9s.

(116.) "Multiplication," says an old author in the quaint pedantic language of his time, "observeth collocation, proceedeth to operation, and concludeth with probation." This probation, or the rule for proving the accuracy of this and other arithmetical operations by casting out the nines, one of the few additions which the Arabians made to the sciences which they derived from the Greeks and the Hindoos, is found in the earliest European writings on Arithmetic, beginning with Leonardus Pisanus. It is stated also with great detail by Lucas de Burgo, who has applied it to all the four fundamental rules; he has given likewise a method of proving the truth of these operations by casting out the sevens, a process much less rapid and commodious than the other, though founded upon the same principle: it was requisite, however, in this case, to get the remainders by actual division by 7, and not, as in the other case, by casting out the nines from the sum of the digits.

Process of division in the Lilavati.

(117.) The extreme brevity with which the rules of operation are stated in the *Lilavati*, renders it difficult for us to describe the Hindoo processes for division: we are directed *to abridge the dividend and divisor by an equal number, whenever that is practicable*, that is, to divide them both by any common measure; thus, instead of dividing 1620 by 12, we may divide 540 by 5, or 405 by 3. We find, however, in one of the commentators on this work, a description of the process of long division, which if exhibited in a scheme would exactly agree with the modern rule; taking the example just given, "*the highest places*," says Manoranjana, "*of the proposed dividend* 16 *being divided by* 12, *the quotient is* 1; *and* 4 *over. Then* 42 *becomes the highest remaining number, which divided by* 12 *gives the quotient* 3, *to be placed in a line with the preceding quotient: thus* 13 *remains* 60, *which divided by* 12 *gives* 5, *and this being carried to the same line as before, the entire quotient is exhibited.*"

Italian methods of division.

(118.) We shall pass over the processes of division which are given in Arabic writers and Planudes, which exhibit nothing which merits much remark, and shall proceed at once to the notice of the methods which are given in early Italian writers. There are four different methods given by Lucas de Burgo, which are as follow:

1. *Partire a regolo*
   a tavaletta
   a la dritta

is the same operation, and is sometimes also termed *partire per testa*, or *division by the head;* in this case the divisor is a single digit, or a number of two places, such as 12, 13, &c. included in the *librettine*, or Italian tables of multiplication.

2 divisor	6	16
9876 dividend	3478	12387
4938 quotient	$579\frac{4}{6}$	$774\frac{3}{16}$

"This method of division," says Lucas de Burgo, "is called by the *vulgar* the *rule*, from the similitude,

Arithmetic.

of the figure to the carpenter's rule, which is made use of in the making of dining-tables, boxes, and other articles, which rules are long and narrow. So likewise is the scheme which is formed by the scholar in this division, whose length consists of many figures, whilst it is only one in breadth: it looks therefore like a *rule*, since its *themologia*, or derivation, springs from thence." The second of these denominations arises from the process being founded on the *librettine*, or multiplication tables; and the last, from the operation beginning from the left, and proceeding to the right; it is called by Tartaglia *partire per colona over di testa, over per discorso, over per toletta.* The three first appellations are easily understood; the last is a popular corruption of *tavoletta.*

2. *Per ripiego*; by resolving the divisor into its simple factors, or *ripieghi*:

	63	divisor
7	250047	dividend
9	35721	
	3969	quotient.

3. *A danda; " the third method of division is called,"* says the author, *" by practicians a danda, it is thus called for reasons which they will easily discover in the operation itself."*

Divisor.	Provenieus.
9876	9876
9733	5376
88884	
86513	
79008	
75057	
69132	
59256	
59256	

This process is evidently the same as our common process for long division; as the reader, however, may not be able to find out the reason of its name from the operation so readily as Pacioli* supposes, the following explanation of a later author may remove the difficulty. "The method of division called *a danda*," says Cataneo, "is most necessary to every person who wishes to become an expert reckoner, and it is thus called *a danda*, or *by giving*, since after every subtraction made in the operation, we *give* or *add* one or more figures on the right hand, so that the remainder after subtraction, combined with the figure or figures given, may admit of being divided by the divisor."

This process, however, is, in the opinion of our author, much less pleasant than the following, which is termed,

4. *Galea vel galera vel batello*; from the form of the process resembling a *galley*, "the vessel of all others most feared on the sea by those who have good knowledge of it; the most secure and swiftest; the most rapid and lightest of the boats that pass on the water."† The following is the form of the process:

History.

To divide 97535399 by 9876,

$$\begin{array}{ccccccccl}
 & & 1 & \cancel{3} & & & & & \\
 & & \cancel{7} & \cancel{6} & \cancel{5} & & & & \\
 & & \cancel{8} & \cancel{7} & \cancel{9} & & & & \\
 & \cancel{1} & \cancel{4} & \cancel{5} & \cancel{4} & \cancel{4} & & & \\
 & \cancel{8} & \cancel{6} & \cancel{1} & \cancel{0} & \cancel{7} & 2 & & \\
 & \cancel{9} & \cancel{7} & \cancel{5} & \cancel{5} & \cancel{6} & \cancel{5} & & \\
\cancel{1} & \cancel{6} & \cancel{3} & \cancel{6} & \cancel{1} & \cancel{3} & \cancel{7} & 3 & \\
\cancel{9} & \cancel{7} & \cancel{5} & \cancel{3} & \cancel{5} & \cancel{3} & \cancel{9} & \cancel{9} & (9\ 8\ 7\ 6 \\
 & \cancel{9} & \cancel{8} & \cancel{7} & \cancel{6} & \cancel{6} & \cancel{6} & \cancel{6} & \\
 & & \cancel{9} & \cancel{8} & \cancel{7} & \cancel{7} & \cancel{7} & & \\
 & & & \cancel{9} & \cancel{8} & \cancel{8} & & & \\
 & & & & \cancel{9} & & & &
\end{array}$$

Such is the form of the *galley*, as given by Lucas de Burgo. In order to explain its construction, we will examine the several steps of the process: write down first the dividend, and underneath the divisor, commencing with the second figure of the dividend, as the number formed by the first four figures of it is less than the divisor; the result corresponding to the first figure in the quotient will stand as follows:

$$\begin{array}{cccccccccl}
 & & 8 & 6 & & & & & & \\
 & & \cancel{9} & \cancel{7} & 5 & & & & & \\
 & \cancel{1} & \cancel{6} & \cancel{3} & \cancel{6} & 1 & & & & \\
 & \cancel{9} & \cancel{7} & \cancel{5} & \cancel{3} & \cancel{5} & 3 & 9 & 9 & (9 \\
\cancel{9} & \cancel{8} & \cancel{7} & \cancel{6} & & & & & &
\end{array}$$

The product of 9 into 9 is 81, which subtracted from 97 leaves 16: cancel 97 in the dividend and 9 in the divisor: 9 times 8 is 72, which taken from 165 leaves 93: cancel 163 in the last remainder and 8 in the divisor: 9 times 7 is 63, which taken from 933 leaves 870: cancel 933 in the second remainder and 7 in the divisor: 9 times 6 is 54, which taken from 705 leaves 651: cancel 705 in the third remainder and 6 in the divisor: the remainder after the first division is

8651399.

We now proceed to the second figure in the quotient, which is 8, when the scheme, with the divisor in its proper place, appears as follows:

$$\begin{array}{ccccccccl}
 & 8 & 6 & & & & & & \\
 & \cancel{9} & \cancel{7} & 5 & & & & & \\
\cancel{1} & \cancel{6} & \cancel{3} & \cancel{6} & 1 & & & & \\
\cancel{9} & \cancel{7} & \cancel{5} & \cancel{3} & \cancel{5} & 3 & 9 & 9 & (98 \\
\cancel{9} & \cancel{8} & \cancel{7} & \cancel{6} & 6 & & & & \\
 & 9 & 8 & 7 & & & & &
\end{array}$$

which is equivalent (the quotient determined not being considered) to

8651399 (
9876

The scheme, after the second division is completed, will stand as follows:

$$\begin{array}{ccccccccl}
 & & 7 & & & & & & \\
 & & \cancel{8} & & & & & & \\
 & \cancel{1} & \cancel{4} & 5 & & & & & \\
 & \cancel{8} & \cancel{6} & \cancel{1} & 0 & & & & \\
 & \cancel{9} & \cancel{7} & \cancel{5} & \cancel{5} & & & & \\
\cancel{1} & \cancel{6} & \cancel{3} & \cancel{6} & \cancel{1} & 5 & & & \\
\cancel{9} & \cancel{7} & \cancel{5} & \cancel{3} & \cancel{5} & \cancel{3} & 9 & 9 & (98 \\
 & \cancel{9} & \cancel{8} & \cancel{7} & \cancel{6} & \cancel{6} & & & \\
 & & \cancel{9} & \cancel{8} & \cancel{7} & & & &
\end{array}$$

* *Il partire a danda e molto necessario a chi esperto ragioner esser desidera ed e chiamato a* danda *el detto modo perche ogni sottrattion fatta nel operare se li da uno o piu figure dal lato destro, talmente che la detta sottrattione con la figura o figure date si possi partir per il suo partitore.* Pietro Cataneo, *le Pratiche delle due prime Mathematiche.*

† *" El piu temuto nel mare da quelli chi ne hanno bona notitia, el piu sicuro, el piu veloce: el piu snello, el piu leggiadro legno che vada per aqua."* Cataneo speaks of this method as more beautiful and rapid, though less secure than the other. *" Il partir a galera,"* says he, *" e molto leggiadro e speditivo, ma non tanto sicuro per un principiante quanto il partir a danda et infra queste due modi ci corre gran disguaglianza nel operare, per le moltiplicatione et sottrattione, perche in questo di galera vanno per testa: come per esempio ti mostrerò."*

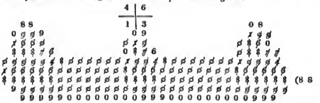
Arithmetic. The product of 8 and 9 is 72, which taken from 86 leaves 14: cancel 86 and 9 in the divisor: 8 times 8 is 64, which taken from 145 leaves 81: cancel 145 and 8 in the divisor: 8 times 7 is 56, which taken from 811 leaves 755: cancel 811 and 7 in the divisor: 8 times 6 is 48, which taken from 53 leaves 5: cancel 53 in the dividend and 6 in the divisor, and we get the remainder 750599 after the second division.

It is very amusing to observe the enthusiastic admiration of Lucas de Burgo for this method of division; when describing the preceding method, he seems little satisfied with it, and looks forward with a species of impatience to the method *a lo golea*, which possesses, both in the description of it and in the operation itself, a certain charm and solace. In truth, says he, it is a noble thing to see in any species and scheme of numbers a galley perfectly exhibited, so as to be able to observe in their disposition and arrangement its stem and stern, its mast, its sail, its yards, and its oars, launched in the spacious ocean of Arithmetic. Well, then, may he exhort the student before he ventures upon this last and most difficult of arithmetical labours, to invoke Apollo in the words of Dante, at the commencement of his *Paradiso*, History.

O buono Apollo, ad ultimo lavoro,

to inspire him with genius and resolution which may be equal to so important an undertaking.

It appears, according to the statement of Tartaglia, to have been the custom at Venice for masters to propose to their pupils, as the last proof of their proficiency in this process of division, examples which would produce the complete form of the galley, with its masts and pendant. The last addition was supplied by the scheme for the proof of the accuracy of the operation by casting out the 9s. The following is the example which he gives:



The galley method of division superseded all others.

(119.) The analysis which we have given of this operation will be quite sufficient to enable the reader to pursue it to its conclusion, and to understand its application to other cases; after making every allowance, however, for the influence of our own practice in division, and for the facility which long familiarity with this method must necessarily have given, we must still feel some degree of surprise at this preference given to it, on account of the superior expedition with which the operation was performed. In this opinion, however, nearly every writer on Arithmetic appears to have agreed, as late as the end of the XVIIth century. It was adopted by the Spaniards, French, Germans, and English, and it is the only method which they have generally thought it necessary to notice: it is found almost universally in the works of Tonstall, Recorde, Stifelius, Ramus, Stevinus, and Wallis; and it was only at the beginning of the XVIIIth century that this method of division, called by English Arithmeticians the *scratch* way of division, from the *scratches* used in cancelling the figures, was superseded by the method which is now in common use, which was specifically called Italian division, from the country from which it was derived.

Division considered very difficult.

(120.) The older writers on Arithmetic appear to have considered division as an operation of considerable difficulty, and one which required a very close and steady application of the understanding. *Duro coso e lo partita*, says Lucas de Burgo; and Ramus, after working through an example of long division, in itself sufficiently complicated and operose, subjoins the following remarks: *Hoc exemplum et similia divisoris plurium notarum mathematicum illam et Platonis μεταστροφὴν et Aristotelis ἀφαιρεσιν in auditorum animis imprimis suscitabant et confirmabant. Pes bonus, oculus bonus ait tyronibus lanista: mens bona, memoria bona, manus bona in quotidiana divisionis experientiâ dicat hic arithmeticus discipulo. Varietas enim tam multiplicis in unâ numeratione numerationis erectam mentem et constantem memoriam, fidelemque manum maxime omnium requirunt. Ac jam nemo sibi arithmeticæ discipulus verèque studiosus videatur, nisi singulis arithmetici studii diebus divisionem vel quam maximam potuit, effecerit.* A modern school-boy might well exult in his superior dexterity in this operation, when the greatest masters of arithmetical knowledge of former times felt so strongly impressed with a sense of its difficulty, and thought it necessary to enforce its constant and daily practice as the only means of acquiring an adequate knowledge of it.

(121.) Recorde, indeed, has noticed the Italian method of division, "which," he says, "I first learned of, and is practised by my ancient and especial loving friend, Master Henry Bridges, wherein not any one figure is cancelled or defaced." The *modus operandi*, as given by him, may be seen from the following example: Italian method noticed by Recorde.

```
33) 7890 (239 3/33
    66
    129
     99
    ---
     300
     297
     ---
       3
```

(122.) The Arabians and Persians perform division by a process which is in some degree analogous to their methods of multiplication, as every step in the process is put down at length. It will be best explained by an example: let it be required to divide 197685 by 573: Arabian and Persian methods of division.

	3	4	5		
			3	4	5
1 1	9 5	7	6	8	5
	4 2	7 1			
	2	6	6 9		
	2 2	5 0	7		
		5 2	7 8		
		2	9 1	8 2	
		2 2	8 5	6	
			3 3	6 5	
				1 1	5 5
	5	5 7	5 7 3	7 3	3

Method of forming squares in the *Lilávati*.

(123.) The author of the *Lilávati* has given rules for the formation of squares and cubes, as well as for the extraction of the corresponding roots. The rule for the formation of the square, which is very ingenious, is as follows: "Place the square of the last digit over the number; and the rest of the digits doubled and multiplied by the last are to be placed above them respectively; then repeating the number with the omission of the last digit, perform the same operation: thus to find the square of 297,

```
    4
   36
    81
    28
    126
      49
   -----
     297
   -----
   88209
```

Hindoo process for extracting the square root.

(124.) In performing the converse operation, every uneven place is marked by a vertical line, and the intermediate even digits by a horizontal one; but if the place be even it is joined with the contiguous odd digit. If we take for an example,

88209

We subtract from the last uneven place 8, the square 4, and there remains 48209. Double the root 2, and divide by that (4,) the subsequent even digit 48; quotient 9 a higher one cannot be taken; for the root of the foregoing digit would become greater than 2: the remainder is 12209. From the uneven place (with the residue) 122, subtract the square of the quotient 9, viz. 81, the remainder is 4109. The double of the quotient 18 is to be placed in a line with the former double number 4. By this divide the even place 410; the quotient is 7 and the remainder 49: to which uneven digit the square of the quotient 49 answers without residue. The double of the quotient 14 is put in a line with the preceding double number 58, making 594: the half of which is the root sought, 297.

The preceding account of the Hindoo operation for extracting the square root is taken from the Commentators on the *Lilávati*; and making allowances for some little obscurity of expression, which most probably arises from the difficulty of conveying from Sanscrit to a modern language the full force of its idioms and phrases, we shall find little which differs from the rule which is given in our books of Arithmetic. The same observation may be extended to the rule for the extraction of the cube root, which exhibits very little that is peculiar, if we except the difference which is found in their methods of multiplication and division, from those which are now adopted by European authors.

Arabian process for extracting the square root.

(125.) The method of extracting the square root made use of by the Arabians, resembled their method of division, as far at least as the difference between the two operations will allow; and a little examination would serve to show that they are both founded upon the Greek methods of performing these operations upon sexagesimals. An example (to extract the square root of 301401) will be quite sufficient to explain the process which they followed:

	5		4		9
3 2	0 5	1	4	0	1
	5 4	1 0			
	1	1 1	4 6		
		9 9	8 7	0 2	
				8 8	1 1
				0	8
	1	0	1		
	5		4		

Different modes of extracting the square root.

(126.) There is not much room for variety in the rule for performing this operation, and the varieties which are found in the form of the process itself are generally varieties only in the process of performing division. The earlier European writers on Arithmetic constantly refer to the 4th Proposition of the IId Book of Euclid

for the proof of the rule which they followed, with the exception of the method of pointing, which is a very obvious consequence of the principle of notation by local value. We shall give one or two examples of the form of the process in two or three authors, which will be quite sufficient for our purpose.

Pelletier. The first is from the *Arithmetic* of Pelletier,* the first edition of which was published in 1550. It is required to extract the square root of 92416.

$$\begin{array}{l} 0 \\ \cancel{9}\dot{2}41\dot{6} \\ \quad 604 \\ \qquad 4 \quad (304 \\ \hline \quad 2416. \end{array}$$

It may be as well to give the statement of the process in the quaint and rude language of the author himself: *En somme, tout l'affaire des extractions quarrées se pourra retenir par cinque mots, savoir est, chercher, doubler, diviser, multiplier et soultraire. Premierement, faut chercher la racine du nombre compris en la dernier point, et d'ici lui nombre oter le nombre quarré de telle racine. Secondement, faut doubler ce qui est la et mettre le double entre les points. Tiercement, faut diviser par la double, c'est adire, savoir combien de fois il est contenu au nombre superieur. Quartement, faut multiplier le diviseur joint avec la figure nouvelle mise derniere le demicercle, par la figure meme. Finablement, faut soultraire du nombre superieur, ce qui proviendra de telle multiplication et susscrire le residu, s'aucun on y a.*

Lucas de Burgo. The second example is from the work of Lucas de Burgo, and is the form of the process which was most commonly adopted: it is required to extract the square root of 99980001.

$$\begin{array}{c} 0\ 0 \\ \cancel{0}\ \cancel{1}\ \cancel{8} \\ \cancel{1}\ \cancel{8}\ \cancel{7}\ \cancel{0} \\ 2\ \cancel{0}\ \cancel{8}\ \cancel{8}\ \cancel{0} \\ \cancel{0}\ \cancel{9}\ \cancel{9}\ \cancel{6}\ \cancel{9}\ \cancel{9}\ \cancel{8}\ \cancel{0} \\ \cancel{1}\ \cancel{8}\ \cancel{7}\ \cancel{7}\ \cancel{8}\ \cancel{9}\ \cancel{8}\ \cancel{0} \\ \cancel{9}\ \cancel{9}\ \cancel{9}\ \cancel{8}\ \cancel{0}\ \cancel{0}\ \cancel{0}\ \cancel{1} \quad (9\ 9\ 9\ 9 \\ \cancel{9}\ \cancel{8}\ \cancel{9}\ \cancel{8}\ \cancel{9}\ \cancel{8}\ \cancel{9} \\ \cancel{1}\ \cancel{1}\ \cancel{9}\ \cancel{9}\ \cancel{9} \\ \cancel{1} \end{array}$$

This scheme will require no explanation to one who is acquainted with the *galley* form of division.

Stifelius. Stifelius, who usually sought to generalize the methods of his predecessors, has considered the process for extracting the square root in connection with those for higher powers; by observing the formation of the powers themselves, he discovered the following schemes or *pictures* (as he calls them) for extracting the square, cube, biquadrate, &c. roots. Calling, for greater clearness, the terms of a binomial root a and b, we shall have for the square root

$$\begin{array}{lcl} a - 20 - b & & \\ & & b^2 \end{array}$$

For the cube root,

$$\begin{array}{rcrcl} a^2 & - & 300 & - & b \\ a & - & 30 & - & b^2 \\ & & & & b^3 \end{array}$$

For the fourth root,

$$\begin{array}{rcrcl} a^3 & - & 4000 & - & b \\ a^2 & - & 600 & - & b^2 \\ a & - & 40 & - & b^3 \\ & & & & b^4 \end{array}$$

For the seventh root,

$$\begin{array}{rcrcl} a^6 & - & 7000000 & - & b \\ a^5 & - & 2100000 & - & b^2 \\ a^4 & - & 350000 & - & b^3 \\ a^3 & - & 35000 & - & b^4 \\ a^2 & - & 2100 & - & b^5 \\ a & - & 70 & - & b^6 \\ & & & & b^7 \end{array}$$

These schemes require little explanation; a is the greatest integer in the root of the first period; in extracting the square root it must be multiplied by 20 to get the divisor, and from thence we determine b; after which the sum of the product of a, 20 and b and b^2, must be subtracted from the first remainder. We will propose, as an example, to find the square root of 6765201

$$\begin{array}{l} \quad \cancel{7} \\ \quad \cancel{6}\ \cancel{7}\ \cancel{6}\ \cancel{5}\ \cancel{2}\ \cancel{0}\ \cancel{1}\ (2601 \\ 2 \quad - 20 - 6. \ 276. \\ \qquad\qquad\ \ 36. \\ 26 \ - 20 - 0.\ 0. \\ 260 - 20 - 1 \ 5201 \\ \qquad\qquad\quad 1 \end{array}$$

(127.) The invention of rules for approximating to the square and other roots of numbers, where those roots were surds, was a favourite speculation with the earlier writers on Arithmetic and Algebra. In order to state these rules with greater brevity, and to estimate more readily their relative accuracy, we shall express them in algebraical language. Rules for approximating to surd roots.

1. The rule given by the Arabs is expressed by the formula, By the Arabs.

$$\sqrt{(a^2 + x)} = a + \frac{x}{2a}.$$

This approximation gives the root in excess; but to increase its accuracy, we may repeat the process, making use of the root thus obtained. Thus the first approximation to the square root of 7 is $2\frac{3}{4}$; its square is $7\frac{9}{16}$. divide $\frac{9}{16}$ by twice $2\frac{3}{4}$, and the quotient is $\frac{9}{88}$, which taken from $2\frac{3}{4}$ gives $\frac{233}{88}$ for the second approximation.

This is the rule which is given by Lucas de Burgo,* and subsequently by Tartaglia,† who derived it, in common with the rest of his countrymen, from Leonard of Pisa.

2. The rule given by Juan de Ortega‡ is expressed by the formula, Ortega.

$$\sqrt{(a^2 + x)} = a + \frac{x}{2a + 1}.$$

This approximation is in defect, but, generally speaking, more accurate than the former.

3. The third method of approximation was proposed by Oronce Finé, or Orontius Fineus, Professor of Mathematics in the university of Paris, and who long enjoyed an uncommon reputation, in consequence of his having introduced the knowledge of the Mathema- Orontius Fineus.

* *L'Arithmétique de Jaques Pelletier de Mans departie en quatre livres; revue et corrigée.* A Lion, 1554, p. 136.

* *Summa de Arithmetica*, &c., p. 46.

† *Numerie misure, parte* iii.

‡ *Tratado subtilissimo de Arithmetica*, &c., 1534.

Arithmetic. tics of Italy amongst his countrymen, notwithstanding his absurd attempts to effect the quadrature of the circle. It consisted in adding 2, 4, 6, or any even number of cyphers to the number whose root was required, and then reducing the number expressed by the additional digits of the root, which were thus introduced to sexagesimal parts of an integer: thus, to extract the square root of 10 add six cyphers, thus,

10000000 (	3	162
		60
	9	720
		60
	43	200
		60
	12	000

The root of 10, as thus determined, when expressed in sexagesimals, is $3 \,.\, 9' \,.\, 43'' \,.\, 12'''$.

Near approach to decimals. The example which we have given is the most remarkable approximation to the invention of *decimals* which preceded the age of Stevinus. If the author had stopped short at the first separation of the digits in the root, it would have expressed the square root of 10 to 3 places of decimals; but the influence of the use of sexagesimals, so familiar to the mathematicians of that age, diverted him from this very natural extension of the decimal notation, and retarded for more than half a century this great improvement in the science of calculation.

Methods founded upon it. This very considerable improvement upon the ordinary method of approximating to the values of surd roots, as might be expected, excited the attention of contemporary mathematicians. They did not, however, follow the example of its author in proceeding to sexagesimals, but merely subscribed as a denominator to the whole root considered as integral, uniting with half as many cyphers as had been added in the first instance; thus $\sqrt{10} = \frac{3142}{1000}$. It is under this form that it is noticed by Tartaglia, who contends, however, that his own method, which we have noticed above, was capable of giving results to greater accuracy; by Recorde, in his *Whetstone of Wit*; by Buckley,* who has described the method in the following verses:

Quadrato numero senas præfigito ciphras,
Producti quadri radix per mille secatur.
Integra dat quotiens, et pars ita recta manebit
Radici ut veræ, ne pars millesima desit.

* Buckley was a native of Litchfield, and Fellow of King's College, Cambridge. He was also mathematical tutor to King Edward VI. His *Arithmetica Memorativa* was published in 1550, and subsequently reprinted at Cambridge at the end of Seton's *Logic*, in 1631. It consists of about 200 verses, describing, with great perspicuity, the principal rules of Arithmetic. He has also noticed the second of the methods of approximation which we have mentioned above as follows:

Modus colligendi minutias ex residuo
Duplo radicis numerus superadditur unus
Producto numerum mox supra scribe relictum
Lineolâ adjecta numeros quæ separet ambos.

The practice of expressing the principles and rules of algorithm in verse was very common before the invention of printing, and many examples of such treatises may still be found in manuscript libraries. They were usually confined, however, to the most simple and elementary rules of the science, and cannot be considered as exhibiting, like the work of Buckley, its most improved state at the time they were written.

History. Pelletier also, the pupil of Orontius Fineus, when speaking *de la manière de justifier les racines des nombres non quarrés*, after noticing the second of the preceding methods of approximation, has described this, which he considers as more accurate and much less tedious than any other.

Methods for extracting the cube root. (128.) We do not consider it necessary to notice in detail the methods of extracting the cube root of numbers which are found amongst the Hindoos, Arabians, and earlier European writers, as they present no variations from the methods which are now in use, which may not be inferred at once from the corresponding methods for the extraction of the square root. It may be sufficient for us to observe, that we find no trace of its existence amongst the Greeks, though it is not very probable that it was altogether unknown to them; and though it formed an essential part of all treatises on Arithmetic, whether Sanscrit, Arabian, Persian, or European, we may consider that their authors were generally ignorant of the principles upon which the rule was founded, and in some cases were incapable even of applying it in practice.*

Methods of approximation to surd cube roots. (129.) Under such circumstances, it is not surprising that mistakes should have been made in their methods of approximating to surd cube roots; that of Lucas de Burgo may be seen from the formula,

$$\sqrt[3]{(a^3 + x)} = a + \frac{x}{(3\,a)^2}$$

which Tartaglia says he got from Leonard of Pisa, who had it from the Arabians; and he expresses his surprise that he should have committed so grievous an error, unless he had done so without consideration. The method of Orontius Fineus errs as much in excess as that of Pacioli in defect, as will be at once seen from the formula which expresses it,

$$\sqrt[3]{(a^3 + x)} = a + \frac{x}{3\,a}.$$

Tartaglia criticises the method of Cardan founded on the formula,

$$\sqrt[3]{(a^3 + x)} = a + \frac{x}{3\,a^2}$$

with great bitterness, as might naturally be expected from one who had been so treacherously defrauded by him of an important discovery; his own method, more accurate than the former, but erring in defect, whilst the other erred in excess, is given by the formula,

$$\sqrt[3]{(a^3 + x)} = a + \frac{x}{3\,a^2 + 3\,a}.\dagger$$

In later times, methods of approximation have been proposed, whether founded upon rational or irrational formulæ, which give results much more accurate‡ than any of the preceding; as the discussion of such formulæ, however, belongs more properly to the history of Algebra than Arithmetic, we think it unnecessary for us to notice them in this place.

Notation of fractions in the Lilâvati. (130.) Fractions in the *Lilâvati* are denoted by writing the numerator above and the denominator below, without any line between them. The introduction of this line of separation is due to the Arabs, and we find it amongst the earliest European manuscripts on Arith-

* Such at least was the accusation advanced by Tartaglia against Jean Buteon, or Buteo, the author of a Treatise on *Arithmetic*.

† It is likewise given in the *Arithmetic* of Juan de Ortega.

‡ Of this kind are the rational and irrational formulæ of Halley.

Arithmetic.

metic. To denote fractions of fractions, such as $\frac{2}{3}$ of $\frac{4}{5}$, they are written consecutively, thus,

$$\begin{array}{cc} 2 & 4 \\ 3 & 5 \end{array}$$

To represent a number increased by a fraction, the fraction is written beneath the number; and when the fraction is to be subtracted from the number, a dot is prefixed to it; thus $2\frac{1}{4}$ is denoted by

$$\begin{array}{c} 2 \\ 1 \\ 4 \end{array}$$

and $3 - \frac{1}{4}$ by $\begin{array}{c} 3 \\ \cdot 1 \\ 4 \end{array}$

Its imperfections.

In other cases, their notation is not intelligible without verbal explanation; thus to denote "two thirds less one-eighth, and then diminished by three-sevenths of the residue," the fractions are written underneath each other, as follows:

$$\begin{array}{c} 2 \\ 3 \\ \cdot 1 \\ 8 \\ \cdot 3 \\ 7 \end{array}$$

In general, however, it may be remarked, that the invention of a distinct, expressive, and comprehensive notation, is the last step which is taken in the improvement of analytical and other sciences; and it is only when the complexity of the relations which are sought to be expressed in a problem is so great as to surpass the powers of language, that we find such expedients of notation resorted to, or their importance properly estimated. We, consequently, find the Hindoos, Arabians, and earlier European writers, singularly deficient in artifices of notation, and compelled therefore to express in words the relation of the numbers which appear in their problems, or to make use of the same notation for different relations. The following problems, given in the *Lilávati*, will serve more fully to explain our meaning.

(1.) "The quarter of a sixteenth of the fifth of three quarters of two-thirds of a moiety of a *dramma*, was given by a person from whom he asked alms: tell me how many cowry shells* the miser gave, if thou be conversant in Arithmetic, with the reduction called subdivision of fractions."

Statement.

$$\begin{array}{ccccccc} 1 & 1 & 2 & 3 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 & 16 & 4 \end{array}$$

Reduced to homogeneousness $\frac{6}{7680}$; in least terms $\frac{1}{1280}$, a single cowry shell.

(2.) "Tell me, dear woman, quickly, how much a fifth, a quarter, a third, a half, and a sixth make, when added together."

Statement.

$$\begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & \quad 29 \\ 5 & 4 & 3 & 2 & 6 & \quad 20 \end{array}$$

(3.) "Tell me what is the residue of three, subtracting these fractions."

* A *dramma* is equivalent to 1280 cowry shells.

History.

Statement.

$$\begin{array}{ccccccc} 3 & 1 & 1 & 1 & 1 & 1 & \quad 31 \\ 1 & 5 & 4 & 3 & 2 & 6 & \quad 20 \end{array}$$

In all these problems the statement or notation employed is the same, though the operations to be performed are essentially different.

Rules for the reduction of fractions.

(131.) The *Lilávati* contains four rules for the reduction and assimilation of fractions, as well as the application of the eight fundamental rules of Arithmetic to them; the rules themselves are generally sufficiently simple and clear, and differ so little from those which are used in modern practice, that any detailed notice of them is unnecessary. The author, however, in the enunciation of the following problem, would seem to intimate that operations with fractions were not without their difficulty, and that it required all the confidence of long practice to avoid making mistakes.

"Tell me the result of dividing five by two and a third, and a sixth by a third, if thy understanding, sharpened into confidence, be competent to the division of fractions."

Meaning of the term algorithm.

(132.) The term algorithm, which originally meant the notation by nine figures and zero, subsequently received a much more extensive signification, and was applied to denote any species of notation whatever for the purpose of expressing the assigned relations of numbers or quantities to each other: thus we find Stifelius speaking of the algorithm of fractions and of fractions of fractions, of the algorithm of proportions, of binomial surds, of cossic numbers, &c.; and an equally extended use of the term is sometimes made in modern times.*

Algorithm of fractions of fractions.

The algorithm of fractions of fractions, if we may be allowed to use this term, varied with different authors; thus with Lucas de Burgo

$$\begin{array}{ccc} & v^a & \\ 2 & \overline{\quad} & 4 \\ \overline{3} & \underline{\quad} & \overline{5} \\ & v^a & \end{array}$$ was equivalent to $\frac{2}{3}$ of $\frac{4}{5}$ or to $\frac{2}{3} \times \frac{4}{5}$,

where v^a denoted *via*, or *times*; with Stifelius, *three-fourths* of *two-thirds* of *one-seventh* was denoted by

$$\begin{array}{ccc} & & \frac{3}{4} \\ & \frac{2}{3} & \\ \frac{1}{7} & & \end{array}$$

and the same quantity was represented by Gemma Frisius by

$$\frac{3}{4} \;\Big|\; \frac{2}{3} \;\Big|\; \frac{1}{7},$$

a notation extremely simple and convenient.

Pacioli denotes that two fractions are to be multiplied together by writing them thus,

$$\begin{array}{ccc} 2 & \overline{\quad} & 3 \\ \overline{3} & \underline{\quad} & \overline{4} \end{array}$$

* It is amusing to observe the very general ignorance of the earlier writers on the origin and meaning of the Arabic terms which were made use of in the sciences; it was quite common with the Italian and German writers on Algebra to speak of Geber as its inventor; and Gosselin, who in 1667 translated and abridged the work of Tartaglia, says that Algorithm was derived from Algus, the inventor of the notation by nine figures and zero.

connecting the numerators and denominators which are to be multiplied together by a line. When two fractions are to be added together, or subtracted from each other, the operations to be performed are indicated as follows,

$$\begin{array}{ccc} 8 & & 9 \\ \frac{2}{3} & \times & \frac{3}{4} \\ & \overline{\ \ } & \\ & 12. & \end{array} \ fa\ \frac{17}{12}\ vel.\ \frac{1}{12},$$

where those quantities are to be multiplied together which are coonected by the lines.

Utrum multiplicatio fractorum augeat?

(133.) The good old monk seems extremely embarrassed by the usage and meaning of the term *multiplication* in the case of fractions where the product is less than the *multiplicand*, and he proposes the question, *Utrum multiplicatio fractorum augeat?* In order to show that this question must be answered in the affirmative, he refers to the passage in Genesis, "*Increase and multiply, and replenish the earth*;" and, again, to the promise to Abraham, "*I will multiply thy seed like the stars of the firmament, or the sand on the sea shore*," to show from the authority of God himself, that *to multiply* means *to increase*; but in what manner is this to be reconciled with the numerical result in those cases? namely, by supposing that the units of the product are of greater *virtue* and *significancy* than those of the factors; thus if $\frac{1}{2}$ and $\frac{1}{2}$ represent adjacent sides of a square, their product $\frac{1}{4}$ will represent the area of the square itself.

The same difficulty appears to have occurred to most other writers of his own and the subsequent age, who were not all of them equally satisfied with the correctness of his explanation. Tartaglia says, that the meaning of the term *multiplication* is different when the multiplier is an integer or a fraction, denoting increase in one case and diminution in the other. Bishop Tonstall, however, in discussing the example $\frac{2}{3} \times \frac{3}{4} = \frac{6}{12}$, has explained the result in this case with singular clearness and good sense: "*Cur id autem ita fiat*," says he, "*si rationem poscis, illa est; quod si numeratores in se soli ducerentur, viderentur integra inter se multiplicari atque ita numerator nimium cresceret. Veluti in exemplo dato, dum duo in tria ducuntur, fiunt 6, quæ, si nihil præterea fieret, viderentur integra; cæterum quia non duo integra per tria: sed duæ tertiæ unius integri per tres ejus quartus multiplicandæ sunt: similiter partium denominatores in se ducuntur: ut postea divisione quæ per denominatoris multiplicationem fit, (quanto enim magis denominator crescit, tanto magis partes comminuuntur) numeratoris augmentatio tantum corrigitur, quantum plus justo creverat, atque ea ratione ad æqualitatem redigitur.*"* The whole dispute furnishes a curious example of the embarrassing effect produced by the use of a term to which a specific and restricted meaning is attached, to denote a general operation, the meaning and interpretation of which must vary with the nature of the quantities to which it is applied.

(134.) There is so little difference between the operations in fractions, as they appear in ancient and modern books of Arithmetic, that we feel it to be altogether unnecessary to detain the reader by any further details on the subject. In the works of Lucas de Burgo and Tartaglia we find the number of cases and their subdivisions unnecessarily multiplied; and the reader upon

* *De Arte Supputandi.*

this, as well as upon other parts of Arithmetic, is frequently more embarrassed than instructed by the minuteness of their explanations. The charge of prolixity, indeed, has been made against Italian writers on this as well as other subjects of every age, and it is quite impossible to deny the truth of its application to the works of which we are speaking. It would be unjust, however, not to attribute much of this to the want of generality and comprehensiveness of the rules and operations which is characteristic of the early state of every science; and the same defect, though in a less degree, is observable in most of the writers of other countries who flourished at that period, with one memorable exception, however, in the case of Stifelius, whose brevity, and consequent obscurity, is as embarrassing to the reader as the tediousness of his predecessors and contemporaries.

Prolixity of Italian writers.

Invention of decimal fractions: circumstances which lead to it.

(135.) We have noticed above a method of approximating to the square and cube roots of numbers, which makes a near approach to the invention of *decimal fractions*, though it will not be found to have in any way contributed to that most important improvement in Arithmetic, at least if we may judge from the form under which it was first exhibited by its author.

It would seem rather to have been suggested by the convenience which was felt in the sexagesimal Arithmetic in the treatment of fractions, and by observing the connection between the series of natural numbers and a geometrical series, whether continued upwards or downwards. Archimedes had observed how the order of the term, formed by the product of any two terms of such a series, might be determined from the sum of their *exponents*, or the terms in the series of natural numbers corresponding to them; and Stifelius extended this remark by continuing the Arithmetic as well as the Geometric series downwards; thus,

-4	-3	-2	-1	0	1	2	3	4
$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16

The same distinguished author observed also, that the proposition would be equally true if the arithmetical series was reversed, and the positive terms made the exponents of the descending terms of the geometric series which were less than 1; thus,

$$0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,$$

might be considered as the exponents of the sexagesimal or *astronomical* series:

$$\frac{1}{60}\quad \frac{1}{3600}\quad \frac{1}{216000}\quad \frac{1}{12960000}\quad \frac{1}{777600000}$$

Improved notation of sexagesimals.

It was with reference to this principle that Stifelius ventured to simplify the sexagesimal notation by writing the numbers 2, 3, 4, &c., accentuated, above the places of the *minuta secunda, tertia, quarta*, &c.; thus,

$$\begin{array}{ccccc} \text{Grad.} & \text{Min.} & \tilde{2} & \tilde{3} & \tilde{4} \\ 2, & 3, & 7, & 20, & 44, \end{array}$$

means $2^{\circ}, 3^{\mathrm{I}}, 7^{\mathrm{II}}, 20^{\mathrm{III}}, 44^{\mathrm{IV}}$, and

$$\begin{array}{cccc} \text{Hor.} & \text{Min.} & \tilde{2} & \tilde{3} \\ 6, & 20. & 40, & 59, \end{array}$$

means 6 hours, 20 minutes, 40 seconds, $59'''$; and similarly in other cases. It is sufficiently curious that

Stifelius, after thus viewing the theory of sexagesimals under this very general form, should not have extended it to decimal fractions; more particularly as the following remark shows that he was sensible that they depended upon the same principle. "*Facile enim vides, ut numerus ille* 60, *id est, sexagenorius, limes sit totius negotii hujusmodi fractionum, quemadmodum* 10, *id est denorius, limes est calculationum vulgarium:*" in other words, that 60 in one case and 10 in the other were the *roots* of the geometrical series, to which the same series of exponents corresponded.

La Disme of Stevinus.

(136.) Stevinus, in his *Arithmétique*, adopted the views of Stifelius with respect to the exponents of terms in a geometrical series, and applied them to correct the barbarous mode of designating roots and powers of quantities which had been prevalent before his time; thus making a very near approach to the very important theory of indices, as they are now used. We find no traces, however, of decimal Arithmetic in this work; and the first notice of *decimal*, properly so called, is to be found in a short tract, which is put at the end of his *Arithmétique* in the collection of his works by Albert Girard, entitled *La Disme*. It was first published in Flemish about the year 1590, and afterwards translated into barbarous French by Simon of Bruges. The ludicro-serious dedication is addressed *Aux ostrologues, arpenteurs, mesureurs, de Tapisserie, goviers, stereometriens en general, maistres de monnoye et a tous marchands*; and describes in very express and ample terms the advantages to be derived from this new arithmetic: decimals are called *nombres de disme*; and those in the first place whose sign is (1) are called *primes*, those in the second place whose sign is (2) are called *secondes*, and so on; whilst all integers are characterised by the sign (0), which is put after or above the last digit. We will subjoin a few of his examples of arithmetical operations by means of these decimals.

1. Addition.

$$\begin{array}{rrrrr} & 0\,(1) & (2) & (3) & \\ 2 & 7 & 8 & 4 & 7 \\ 3 & 7 & 6 & 7 & 5 \\ 8\;7 & 5 & 7 & 8 & 2 \\ \hline 9\;4 & 1 & 3 & 0 & 4 \end{array}$$

or, 941 (0) 3 (1) 0 (2) 4 (3)

2. Multiplication.

$$\begin{array}{rrrrrrrr} & & & & (0) & (1) & (2) & \\ & & & & 3 & 2 & 5 & 7 \\ & & & & 8 & 9 & 4 & 6 \\ \hline & & & & 1 & 9 & 5 & 4 & 2 \\ & & & 1 & 3 & 0 & 2 & 8 \\ & & 2 & 9 & 3 & 1 & 3 \\ & 2 & 6 & 0 & 5 & 6 \\ \hline & 2 & 9 & 1 & 3 & 7 & 1 & 2 & 2 \end{array}$$

or, 2913 (0) 7 (1) 1 (2) 2 (3) 2 (4)

3. Division.

$$\begin{array}{cccccc} (0) & (1) & (2) & (3) & (4) & (5) \\ 3 & 4 & 4 & 3 & 5 & 2 \end{array} \text{ by } \begin{array}{cc} (1) & (2) \\ 9 & 6 \end{array}$$

$$\begin{array}{cccccc} & & & \cancel{1} & & \\ & & \cancel{1} & \cancel{8} & \cancel{6} & \\ & \cancel{5} & \cancel{1} & \cancel{1} & \cancel{4} & \\ & \cancel{7} & \cancel{6} & \cancel{3} & \cancel{7} & \\ \cancel{3} & \cancel{4} & \cancel{4} & \cancel{3} & \cancel{5} & \cancel{2} \\ & \cancel{9} & \cancel{6} & \cancel{6} & \cancel{6} & \\ & & \cancel{9} & \cancel{9} & \cancel{9} & \end{array} \qquad \begin{array}{cccc} (0) & (1) & (2) & (3) \\ (3 & 5 & 8 & 7 \end{array}$$

4. Indefinite division.

$$\frac{40}{3} = \begin{array}{cccc} 0 & (1) & (2) & \\ 1 & 3 & 3 & 3 \end{array}$$

Stevinus afterwards proceeds to enumerate the advantages which would result from the decimal subdivision of the units of length, area, and capacity, of money, and lastly of a degree of the quadrant; in the increased uniformity of notation, and increased facilities in performing all arithmetical operations in which fractions of such units were involved.

Translation of this work into English.

(137.) Whatever advantages, however, this admirable invention, combined as it still was with the addition of the exponents, possessed above the ordinary methods of calculation in the case of abstract or concrete fractions, it does not appear that they were readily perceived or adopted by his contemporaries. We can discover no notice whatever of the improvement before the beginning of the following century. In 1608 the tract in question was translated into English by Richard Norton, Gentleman, under the following title: *Disme, The arte of tenths, or decimal Arithmetike, teaching how to perform all computations whatsoever, by whole numbers without fractions, by the four principles of common Arithmetike: namely, addition, subtraction, multiplication, and division, invented by the excellent mathematician*, Simon Stevin.

The art of tens by Henry Lyte.

(138.) This publication does not appear to have exeited any very general or immediate notice. In the year 1619, however, we find its contents embodied in an English work, of which the following is the title: *The art of Tens, or decimal Arithmetike, wherein the art of Arithmetike is tought in a more exact and perfect method, avoyding the intricacies of fractions. Exercised by* Henry Lyte, *Gentleman, and by him set forth for his countries good. London*, 1619. It is dedicated to Charles, Prince of Wales; and in his advertisement he says, that he had been requested for ten years to publish his exercises in decimal Arithmetike. After enlarging upon the advantages which attend the knowledge of this Arithmetike to landlords and tenants, merchants and tradesmen, surveyors, guagers, farmers, &c., and in all men's affairs, whether by sea or land, he adds, "if God spare me life, I will spend some time in most cities in this land for my countries good to teach this art. I hold the lively voice of a meane speculator somewhat practised, furthereth tenfold more in my judgement than the finest writer that is." It is not necessary to proceed further with an analysis of the contents of this volume, as it contains nothing, either in notation or otherwise, which is essentially different from what was given by Stevinus.

Improvement of notation introduced by Napier.

(139.) The last and final improvement in this *decimal Arithmetic*, of assimilating the notation of integers and decimal fractions, by placing a *point*, or *comma*, between them, and omitting the exponents altogether, is unquestionably due to the illustrious Napier, and is not one of the least of the many precious benefits which he conferred on the science of calculation. No notice whatever is taken of them in the *Mirifici Logarithmorum canonis descriptio*, nor in its accompanying tables, which was published in 1614. In a short abstract, however, of the theory of these logarithms, with a short table of the logarithms of natural numbers, which was published by Wright, in London, 1616, we find a few examples of decimals, expressed with reference to the decimal point; but they are first distinctly noticed in the *Rabdologia*, which was published in 1617. In an *Admonitio*

Rabdologia.

Arithmetic.

pro decimali Arithmetica he mentions in terms of the highest praise the invention of Stevinus, and explains his notation; and without noticing his own simplification of it, he exhibits it in the following example, in which it is required to divide 861094 by 432.

```
          64
         136
         316
      118,000
      141
     402
    429
    861094,000 (1993,273
    432
    3888
     3888
      1296
      ----
        864
        3024
         1296
```

The quotient is 1993,273, or
1993,2′7″3‴

the form under which he afterwards writes it, in partial conformity with the practice of Stevinus.

The same form is adopted in an example of abbreviated multiplication, which subsequently occurs in the solution of the following question.

Abbreviated multiplication.

If 31416 be the approximate value of the circumference of a circle whose diameter is 10000, what is the numerical value of the circumference of a circle whose diameter is 635.

```
Complete.              Abbreviated *
 31416                  31416
   635                    635
 ---------              --------
 1884|96                1884|9 .
   94|248                 94|2 . .
   15|7080                15|7 . . .
 ---------------        --------
 1994 9' 1' 6" 0''''    1991 8
```

Decimals not necessary for logarithmic tables.

(140.) The poblicatinn of tables of logarithms, to whatever base they might be calculated, was by no means necessarily connected with the knowledge and use of the decimal Arithmetic. The theory of *absolute* indices, in its general form, at least, was at that time unknown; and logarithms were not considered as the indices of the *base*, but as measures of ratios merely. Under this view of their theory, it was clearly a matter of indifference whether we assumed the measure of the ratio of 10 to 1, to be 1, 10, 100, 100000000, or 1,00000,00000, the number assumed by Briggs in his *Arithmetica Logarithmica*. Thus the *absolute* logarithms of 15, 55, and 155, to ten places, are

Absolute and relative logarithms.

1,1760912591
1,7403626895
2,1903316982

whilst their *relative* logarithms, that of 10 being 1,00000,00000, are

1,17609,12591
1,74036,26895
2,19033,16982

History.

In one case the logarithms are expressed by decimals, in the other by whole numbers; they have the same *characteristics*, and it is obvious that their use in calculations is exactly the same. It is under the latter form that the logarithms are given in the earlier tables, such as those of Napier, Briggs, Vlacq, Kepler and Bartsch.

Noticed very particularly in the *Logarithmical Arithmetike*

(141.) The preceding statement will sufficiently explain the reason why no notice is taken of *decimals*, in the elaborate explanations which are given by Napier, Briggs, and Kepler, of the theory and construction of logarithms; and indeed we find no mention of them in any English author between 1619 and 1631. In that year the *Logarithmical Arithmetike* was published by Gellibrand, and other friends of Briggs, who died the year before, with a much more detailed and popular explanation of the doctrine of logarithms than was to be found in the *Arithmetica Logarithmica*. It is there said that the logarithms of 19695, of $1969\frac{5}{10}$, $19\frac{695}{1000}$ are

4,29435,59851
3,29435,59851
1,29435,59851

differing merely in their characteristic; and $\frac{5}{10}$, $\frac{695}{1000}$ are called decimal fractions. Rules are also given for the reduction of vulgar to decimal fractions by a simple proportion; and lastly a table for the reduction of shillings, pence, and farthings, to decimals of a pound sterling, of which the following is a specimen:

s.		*p.*		*f.*	
19	95000	11	045833	3	0031248
17	85000	5	020833	1	0010416

Different notations of decimals.

(142.) From this period we may consider the decimal Arithmetic as fully established, inasmuch as the explanation of it began to form an essential part of all books of practical Arithmetic. The simple method of marking the separation of the decimals and integers by a *comma*, of which Napier had given a solitary example, was not however generally adopted. The following are different modes of writing them, which are found amongst English and foreign authors:

34 . 1′ . 4″ . 2‴ . 6⁗
(1) (2) (3) (4)
34 . 1 . 4 . 2 . 6
34 . 1 4 2 6⁗
34 . 1 4 2 6 (4)
34 . 1 . 4 . 2 . 6
34 $\underline{1426}$
34 | $\underline{1426}$
34'1426
34,1426

Oughtrede's *Clavis*.

(143.) Amongst the authors who contributed most to the propagation of this Arithmetic we must mention the celebrated Oughtrede.[8] His *Clavis Mathematica* was first published in 1631, in the first chapter of which, *De*

---

* This is the first example which we have discovered of this abbreviated multiplication: the use of it, however, became very popular in a short time afterwards, as furnishing some relief in the management of the large numbers which were made use of in the construction of tables of sines, &c. Many examples of this species of multiplication and division may be found in the work of Kepler, on *Logarithms*, in Oughtrede's *Clavis*, in Wallis's *Algebra*, &c.

8 William Oughtrede was a fellow of King's College, Cambridge, and he always writes *Ætonensis* after his name. In those days the members of those royal foundations had not yet begun to consider the pursuits of literature and science as incompatible with each other. His works enjoyed a well deserved reputation in his day, and he is spoken of in his old age with singular reverence by Wallis. He died in 1660, in his 87th year, from excess of joy on hearing of the restoration of the monarchy.

Arithmetic. *Notatione*, we find the following table, with its accompanying explanation.

Integri.						*Partes.*
9	8 7 6	5 4 3	2 1 0 $\bar{1}$ $\bar{2}$	$\bar{3}$ $\bar{4}$ $\bar{5}$	$\bar{6}$ $\bar{7}$ $\bar{8}$	$\bar{9}$
M	M M M	M M M	C X I | X C	M M M	M M M	M
M	M M M	C X I		I X C	M M M	M
M	C X I				I X C	M

In hac tabellâ numeri superiores sunt indices sivè exponentes terminorum utrinque ab unitate continuo proportionalium; affirmativi in integris, negativi in partibus. Estque progressio in decuplâ ratione versus sinistram, et in subdecuplâ versus dextram; sicut literæ subscriptæ ostendunt. Estque igitur progressio ab unitate in integris 1, 10, 100, 1000, 10000. *Et in partibus,* $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$, $\frac{1}{10000}$. *Et sic in infinitum.*

The integers he separates from the *decimals* or *parts*, by a mark ⌊__ which he calls the *separatrix*, as in the examples 0⌊56, 48⌊5, for .56 and 48.5; and in giving examples of the common operations of Arithmetic he unites them under common rules.

Other authors. (144.) The view of the theory of decimals which was given by Oughtrede was generally adopted, and in some cases his notation also, by English writers on Arithmetic for more than thirty years after this period. Amongst others may be mentioned Nicholas Hunt, whose *Handmaid to Arithmetick* was published in 1633: John Johnson, whose *Arithmetic* was published in 1657. Jonas Moore, Professor of the Mathematics in the city of Durham, whose *Arithmetic* was published in 1660, with a dedication to James, Duke of York, a work which long enjoyed a considerable reputation. Samuel Jeake, merchant, whose *Compleat Body of Arithmetick* was written in 1671, though not published before 1701; a work of considerable learning and research on every subject connected with practical Arithmetic, and particularly in weights and measures: besides many others, whose works we have had no opportunity of examining.

Logistica Decimalis of Beyer. (145.) In the year 1619 there appeared at Frankfort a work with the following title: *Logistica Decimalis, dasist: Kunstrechnung mit Zehentheilichen Bruchen, denen Geometris, Astronomis, Landmessern, Ingenieurn, Wisiren, und insgemein allen Mechanicis und Arithmeticis in unglaublicher Leichterung ihrer muhsamen Rechnungen, Extractionen der Wurzeln, sonderlich aus Irrationalzahlen, auch zur construction einer neuen Tabulæsinuum, und andrer vielerhand nutzlicher canonum etc, uber die maass dienstlich und nothwendig, beschrieben durch* Johann Hartman Beyern, *D. Med.* The author states, that he first thought upon the subject of this decimal Arithmetic in the year 1597, but that he was prevented from pursuing it for many years by the little leisure afforded him from his professional pursuits. He makes no mention of Stevinus, and assumes throughout the invention as his own. The decimal places are indicated by the superscription of the Roman numerals, though the exponent corresponding to every digit in the decimal places is not always put down: thus 34.1426 is written $34^{\circ}.1^{\text{I}}4^{\text{II}}2^{\text{III}}6^{\text{IV}}$, or $34^{\circ}.14^{\text{II}}26^{\text{IV}}$, or $34^{\circ}.1426^{\text{IV}}$.

Acquainted with the works of Stevinus and Napier. (146.) The author must have been acquainted with the *Rabdologia* of Napier, as the thirty-ninth chapter of his book is devoted to the explanation of the construction and use of these rods, which enjoyed a most extraordinary popularity at that period: under such circumstances he could not have been ignorant either of Napier's notation or of the work of Stevinus, and we may very reasonably doubt, therefore, the truth of his pretensions to originality, or that he should so long have concealed an invention of such immense importance to the science of calculation.

History.

Albert Girard's notation. (147.) The works of Stevinus were published in 1625 by his friend and pupil Albert Girard, whose own work, entitled *Invention nouvelle en Algebre*, appeared in 1629. It contains the exposition of the principles of Arithmetic and Algebra, and we may naturally expect to find, therefore, examples of the use of decimals under their most improved form. In the solution of the equation,

$$1\ (3)\ \text{esgale}\ 3\ (1) - 1$$

$$\text{or, } x^3 = 3x - 1$$

by a table of sines, of which method he was the author, we find the three roots written as follows:

$$\left.\begin{array}{r} 1,532 \\ 347 \\ -\ 1,879 \end{array}\right\}$$

"*Ce qui est exprimè*," says he, "*en disme jusques en trines.*" On another occasion he denotes the separation of the integers and decimals by a vertical line: "*Divisez* 3218 *par* 10," says he, "*il viendra* $321\frac{8}{10}$, *le nombre est ainsi tracè* 321 | 8; *si par* 100, *ainsi* 32 | 18; *et si par* 1000, *ainsi* 3 | 218." He does not always, however, adhere to this simple notation, as we afterwards find the square root of 4½ expressed by 20816 (4); and on another occasion we find similar vestiges of the original notation of Stevinus.

Slow progress of improvements in notation. (148.) Whoever has studied the history of the progress of the mathematical sciences must have remarked the extreme slowness with which improvements in notation have been admitted into general use. In the infancy of those sciences more attention is paid to the *modus operandi*, to the actual rule for performing the operation, than to the form under which it is exhibited; and in many cases improvements in notation, the most important in their consequences, have originated as much in accident as design, or at all events their authors have had little notion of the effect of the change which they were making. When Napier disencumbered the decimal notation of the numeral exponents of Stevinus, the improvement in point of simplicity and practical usefulness which was thus produced, was apparently so obvious as to have at once recommended it to universal adoption; yet we find it timidly proposed, and not always followed even by its author; and though the work which contained it was very generally circulated and read, yet the notation was not admitted in principle for fifteen years after its first publication, even in our own country, at a period when the discussions connected with the theory of logarithms and the construction of tables, were calculated to bring decimal numbers and their notation into particular notice. On the continent of Europe this notation was not adopted generally before the middle of the century; and even in the year 1656 we find the Jesuit Andrew Tacquet, in his *Arithmetic*,* giving an account of the theory of decimals, and uniting them with Roman numerals as exponents, as if no improvement had taken place since the original publication of Stevinus.

* *Arithmeticæ Theoria et Praxis, auctore* Andrea Tacquet, *Antwerpiæ, e Societate Jesu Lovanii*, 1656.

Arithmetic of concrete quantities.

(149.) We shall now proceed to the history of the Arithmetic of *concrete* or *denominate* numbers, which forms the second and last division of our subject, and we shall commence with a few introductory remarks on the divisions of the primary units of weights and measures of different countries, and on the ultimate units in which they are made to terminate.

Observation of Tartaglia on the primary units of weights and measures.

(150.) It is a remark of Tartaglia, that mankind have generally attempted in the selection of the *ultimate* units of concrete quantities to imitate the indivisible abstract unit of number, or the mathematical point of geometers; in other words, that those units have been assumed to be quantities of their species so small, or of a nature so invariable, as to be considered as in some measure indivisible as to *sense*. By way of illustration, he refers to examples derived from the coins, weights, and measures, which were used in Italy. Thus the *ultimate* unit of money is in Venice termed a *piccolo*, or *bagatino*, terms used to express their extreme minuteness; and in other cities of Italy a *dinaro*; of the weight of medicines, gold, and precious articles, a *grain* of *barley*; of other valuable goods, though less precious than the former, a *caratto*,* equal to four grains; for common merchandise an *onza*, or ounce: in all these cases, the minuteness of the ultimate division being proportioned to the value of the articles which were required to be estimated. For measures of length, this unit was a *grain* of *barley* in breadth, and similarly in other cases.

To what extent derived from natural sources.

(151.) This observation is sufficiently curious, and quite worthy of the very acute and philosophical genius of its author, though we may not feel disposed to admit its truth to the extent asserted, or in the precise terms in which it is expressed; it directs us, however, to an inquiry of some interest respecting the nature of these ultimate units, and to the extent to which they, in common with other measures of length, weight, and capacity, are derived from natural sources, and therefore generally adopted by different people independently of each other. We shall commence with measures of length.

Measures of the Hindoos.

(152.) Amongst the Hindoos, 8 breadths of a *barley corn*, or 3 grains of rice in length, make a *finger*; 4 times six fingers make the cubit, or fore arm; 4 cubits make a *staff*, which is usually the height of a man's body; and 20 cubits make the *bambu pole*, which is used in measuring land and considerable distances.

Of the Hebrews

(153.) Amongst the Hebrews, 6 *barley* corns in their greatest thickness, or 2 in *length*, make the *etsbang*, or finger's breadth; 4 of these make the *tophach*, *palm*, or *hand*; 3 of which were equal to the *zereth*, or *span*, the distance between the ends of the thumb and the little finger when stretched out to their greatest extent; the double of the *span* made the *ammah*, or *ordinary cubit*, the length from the elbow to the extremity of the fingers. To these may be added the *paynam*, or *foot*, and the *tsugad*, or *pace*, derived in common with most of their other measures from the parts of the human body.

Of the Greeks.

(154.) There are many reasons which should make us expect to find a resemblance between the Greek

* This term is derived from the Greek κερατιον, the carob seed, or sweet bean, which in the Greek *physical* weights was considered as equivalent to 3½ grains of wheat.

measures and those which were used by the Hebrews and Phœnicians; we consequently have the δακτυλος, or *finger*, the σπιθαμη, or *span*, the πους, or *foot*, the πηχυς, or *cubit*, the οργυια, or *fathom*, the distance of the out-stretched hands,* with other intermediate measures derived from the same natural source.

Of the Romans.

(155.) Amongst the Romans we find the *digitus*, the *pollex*, or *thumb's breadth*, equal to an inch; the *palmus minor*, or *common palm* of 4 digits; the *palmus major*, corresponding to the σπιθαμη of the Greeks; the *pes*, or *foot*; the *gressus*, *gradus*, or *step*; and the *passus* of 5 feet, which was double of the step; the *ulna*, or *ell*, which corresponded to the *cubit*, is a term used in later authors, and is the origin of one of the most common and most variable of the measures of modern Europe.

Of the Venetians.

(156.) Amongst the Greeks and Romans we find no trace of the ultimate unit of length, the barley corn, either in length or breadth, which was referred to by the Hindoos and Hebrews as making some approach to an invariable standard: it reappeared, however, in modern Europe. Thus the Venetian measures commence with the *grano de orgio*,† or *barley corn*, 4 of which make a *dedo*, (a corruption of *digitus*,) and 4 *dedi a palmo*. Other measures are Roman, such as *passo*, consisting of 5 feet, and each foot of 12 *onze*, or inches. In our own country we assume 3 barley corns, taken from the middle of the ear, and placed end to end, as the standard of an inch. But it is not necessary to pursue this inquiry further, as the examples which we have already produced are sufficient to show that the ordinary measures of length have been *generally* derived from the dimensions of the human body, or of spaces included in our ordinary motions; and likewise that some other ultimate unit (generally a barley corn) has been assumed, as a space so small as to call for no further subdivision, at least in the ordinary cases where measures of length are required, and also of a nature so constant, or at least esteemed to be so, as to serve as a corrective to the extreme diversity of the other and greater measures when derived from their natural sources.

English inch.

Measures commonly derived from the human body

Longer measures of length.

(157.) For longer measures of length, where the parts of the human body could no longer be referred to, we must expect still less uniformity in the selection of superior units. There is a general resemblance, both in name and use, between the *bambu pole* of the Hindoos, the *kaneh* or *reed* of six cubits of the Hebrews, the ακαινα of the Greeks, the *decempes* of the Romans, the Spanish *stadale* of 11 feet, the French *perche*, and

* An old English author says that a pair of compasses with one leg in the navel would graze with the other the top of the head, the sole of the foot, and the extremities of the out-stretched arms; without resorting to the confirmation of such an experiment, we may assume this measure to be equal to the ordinary height of a man. The term *fathom* is used in nautical measures as being the portion of the sounding or other line which can be grasped between the hands at one time.

† Tartaglia considers the *grain of barley* as constituting the most correct *fundamento*, or basis of measures of length. It is much less variable in breadth than the grain of wheat. He allows, however, that it may be more *corpulent* in one country than another; a fact which he ascertained from comparing the *verga*, or yard of England, with the number of grains which are allowed for it, with the measures of Italy, and which he attributes to the coldness of our climate. He was furnished with the means of making this comparison by his friend and pupil Richard Wentworth, to whom he dedicates the *first* part of his work.

Arithmetic.

the English *pole*, *rod*, or *perch*, whose lengths were taken from that of the reed or rod which was used in the measurement of land and large distances.

Day's journey.

(158.) In the East, and even in modern Europe, distances were reckoned by the hour or day's journey. Thus, in Hebrew, the *cibrach haarets*, or half day's journey, was the distance which could be travelled from meal to meal. The unit of space of the Chinese is the *lih*, the distance which can be attained by a man's voice, thrust forth with all his force in a calm season, upon a clear plain; for greater or lesser distances they proceed to the multiplication or subdivision of this distance by 10, presenting thus an unique example of an uniform scale of measures of length. In the days of archery a *bow shot* presented a measure of a similar character of very general and popular usage. The Greek *σταδιον* was probably derived from the particular length of the *course* of their chariots in their public games; whilst the origin of our own *furlong*, a measure of nearly corresponding length, is sufficiently obvious in its derivation (*quasi furrow long*.) The *parasang* of ancient Persia consisted of 30 stadia, and is of unknown origin; and the same observation may be made of the *σχοινος* of double its length, a measure of the ancient Egyptians, which is mentioned by Herodotus.

Chinese lih.

Bow shot

Stadium.

Furlong.

Mile.

(159.) The *milliare*, *milliarium*, or *mille passus* of the Romans is the origin of the modern *mile*, varying in different countries of Europe from its extreme length in the German mile of 22,500 feet to the Italian of 5000; a circumstance which clearly shows that the classical name was *borrowed* to designate a large distance, without any reference to its precise signification. The term *league* has been supposed to be derived from the German *lugen* to see, and that it originally expressed the distance which could readily be seen by the eye on a plain surface; and it certainly would require all the vagueness and uncertainty which would attend the assignation of such a space, to account for its different lengths in the leagues of Germany, Spain, and Sweden, in the four leagues of France under the old monarchy, and in the common and nautical league of England.

League.

Measures of weight.

(160.) As there are no natural, or very obvious standards, from which we can readily derive our measures of weight, we may therefore expect to find them of a much more arbitrary character, in their designations at least, than the measures of length. It is very curious, however, to find how often a *grain of barley* has been taken as their basis. Thus, amongst the Hindoos the weights are derived from the *barley corn* and *gunja*, or seed of the *abrus precatorius*, which is considered as equivalent to two of them. The Greeks make two *σιτάρια*, or grains of barley, equivalent to the *χαλκοι*, their most minute piece of copper money, 4 of these equal to the *κερατιον*, or *carob seed*, and 8 to the *θερμος*, or *lupine*. The Romans made their weights, however, terminate in the *siliqua*, or *κερατιον*, deriving them directly from the Greeks, and, therefore, not proceeding lower than such weights as were in actual use. Amongst the Italians and all other European nations the *grain of barley* and the *carat*, which is equivalent to four of them, have been assumed as the basis of all existing weights.

Divisions of the Greek litra.

(161.) It is not surprising that the divisions of the Grecian *litra*, or pound, which were made use of in the division of their medicines, should have been adopted in modern Europe, when the influence of the writings of their physicians is considered; with them, 24 grains made the *gramma*, 3 *grammata* the *drachm*, or *dram*, 8 *drams* the *ουγγια*, or *ounce*, and 12 *ounces* the *litra*, or *pound*. The Romans translated *gramma* into *scriptulum*, *scripulum*, or *scrupulum*, which we have retained. The same divisions are continued in the Apothecaries' pound, and, therefore, in medical prescriptions in almost every country in Europe. The Greeks had a second pound of 16 *physical* ounces, called the *mna*, or *mina*, a term derived from the Hebrew *maneh*, a weight of nearly the same magnitude. The pound of Cairo* is divided in 12 ounces, each ounce into 12 *dirhems*, each *dirhem* into 12 *carats*, and each *carat* into 4 *grains*: though these divisions of the pound differ from the Grecian, there is no doubt that *dirhem* and *drachm* are the same word, and most probably derived from some common Phœnician root.

History.

Of the Egyptian pound.

Venetian lira of weight.

(162.) The Venetian *libra*, or *lira*, of weight is divided into 12 *oncie*, each *oncia* into 6 *sazzi*, each *sazzo* into 24 *caratti*, and each *caratto* into 4 *grani d'orgio*. In this case, as well as in that of the Egyptian ounce, we find a departure from the Grecian subdivisions, though in all three of them the ounce is made to consist of the same number (576) of grains. The modern Romans, however, have adhered to the divisions of the *pound* which prevailed amongst the ancients; it being divided into 12 *oncie*, each *oncia* into 8 *dramme*, each *dramma* into 3 *scrupoli*, each *scrupolo* into 2 *oboli*, each *obolo* into 4 *silique*, and each *siliqua* into 12 *grani*. In this case, the number of grains bears no relation to the weight which they represent; a circumstance which can only be accounted for by their being of perfectly abitrary value.

Divisions of the three English pounds.

(163.) The following are the divisions of the three pounds which are made use of in this country:

Troy.

24 grains make a pennyweight.
20 pennyweights make an ounce.
12 ounces make a pound.

Apothecaries.

20 grains make a scruple.
3 scruples make a dram
8 drams make an ounce.
12 ounces make a pound.

Avoirdupois.

20 grains make a scruple.
3 scruples make a dram.
8 drams make an ounce.
16 ounces make a pound.

The two first *pounds* are the same weight, but differently subdivided. The ultimate subdivisions of the *pound avoirdupois* coincide with those of the Apothecaries' pound, though they are never resorted to in practice.

Origin of the terms Troy and Avoirdupois

(164.) The pound troy is said to have derived its name from the town of Troyes, where a celebrated fair was formerly held, and where this weight was used. Whatever opinion, however, may be entertained of this derivation of the name, which is not very satisfactory, it is certain that it was never used in any public document before

* Bishop Hooper, in his *Inquiry into the state of ancient Measures*, is disposed to consider the pound of Cairo as exactly corresponding to our pound troy, which he supposes to have been derived from it.

Arithmetic. the statute of the 12th of Henry VII., where its subdivisions are given, and where it is said that every *gallon* shall consist of 8lbs *troy* of wheat. The origin of the term *avoir du pois*, as applied to a specific weight, is still more difficult to trace. It is first used in this sense in the statute of the 24th of Henry VIII., which fixes the *maximum* prices of provisions during a time of scarcity, and orders that carcasses, beef, pork, victuals, &c. shall be sold by the lawful weight called *haber-de-pois*.* In former times, this term appears to have designated *commodities*; thus the statute of the 9th of Edward III., made at York, speaks of damage done to the king and his subjects by people of cities, &c. not suffering merchants, strangers which do bring and carry by sea or land, *vins, aver-du-pois et autres victuailles, et autres choses vendables*. Again, in the statute of staple of the 27th of the same king, it is said, *Itempur ces que nous avons entendu que auscuns marchauntz achatent avoir de pois leynz et autres marchandises per un pois et vendent par un autre.* The most natural inference to be drawn from these passages is, that the term which was originally made use of to designate every description of heavy merchandise, was afterwards transferred to the *weight* itself, by which they were most commonly estimated.

The pound Troy the legal and statutable weight of this kingdom.

(165.) The pound troy must be considered as the original legal and statutable weight of this kingdom, though the *libra mercatoria*, corresponding nearly with the pound avoirdupois, was the weight which was in most common usage. In the statute of the 31st of Edward I. it is said, that "by the consent of the whole realm of England, the king's measure was made, so that an English *penny* which is called the *sterling*, round without clipping, shall weigh 32 grains of wheat, well dryed and gathered out of the middle of the ear; and 20 pence make an ounce, and 12 ounces a pound, and 8 pounds a gallon of wine, and 8 gallons of wine a bushel of London, which is the eighth of a quarter." The same division of the pound and gallon are mentioned likewise in the statute of the 12th of Henry VII., and in all the numerous statutes which were made from time to time for securing uniformity of weights, it is the pound troy which is considered as the standard and legal weight.

The libra mercatoria.

(166.) Whether this was the *legitimum pondus*, which was recognised in the time of Henry II., it is impossible now to ascertain; at all events, though this weight was the favourite of the legislature, there was another pound, one-fourth greater, which was in more general use; it is mentioned in the *Fleta*, in the time of Edward I., in an account of the possessions of the abbey of Bewley in Hampshire,† and also in a *Tractatus de Ponderibus* of the same age, where the two pounds are said to consist of 20 and 25 shillings respectively: in the statute of the 54th of Henry III., where the composition of the gallon and pound troy are given, there is mentioned also *una libra, pondus viginti quinque solidorum legatium sterlingorum*. On many other occasions this *libra mercatoria* is referred to, and we may consider its use, indeed, in mercantile transactions and ordinary sales as nearly universal. History.

Laws for securing uniformity of weights.

(167.) It was one of the articles of the great charter, that there should be one weight and one measure throughout the realm; and the repeated efforts of the legislature to secure this object appear to have been thwarted by the prevalence of the *customary* pound, as well as by local variations in other measures. In the 14th of Edward III., standard ells, bushels, gallons, and pounds, sealed with the kings's iron seal, were sent to the sheriffs of the different counties, and directed to be kept and adhered to, under severe penalties. In the 27th of the same king, however, we find that the complaint was general, that merchants bought by one weight and sold by another. In the 16th of Richard II. these standards are directed to be kept by the clerks of the market. Enactments on the same subject were made in almost every subsequent reign; but whether it arose from the multitude of statutes, many of which were inconsistent with each other, from the rapidity with which many of them were repealed, or from the imperfections of the standards themselves, (made by rude artists, and tried by methods which were equally rude,) it is certain that the uniformity at which the legislature aimed was never attained; repeated complaints were made of the frauds which were practised by false and unjust measures, and particularly in the case of the purveyors in the reigns of Elizabeth, James, and his unfortunate successor.

Committee of weights and measures in 1758.

(168.) In the year 1758 a committee was appointed to inquire into the original standards of weights and measures of this kingdom, and to examine the standards which were preserved in the Exchequer, Guildhall, and elsewhere. The report, which was drawn up by Lord Carysfort, and read on the 28th of May of the same year, is very learned and elaborate, referring to all the statutes which bear upon the subject, and containing the results of the examination of most of the existing standards, made chiefly under the direction of the celebrated instrument-maker Bird. The standard bushel (Winchester) of 1601 was found to contain 2124 cubic inches, though it was defined by the statute of the 1st of William and Mary that it should contain 2150. The gallon, quart, and pint, of the same date, contained 271, 70, 34$\frac{3}{4}$ cubic inches respectively, and similar and even greater variations were found to exist in the standards of weights and measures of length; under these circumstances, it was recommended that a *new yard* and a *pound troy*, made by Bird from a mean of those which were preserved in the Exchequer, or rather copies of those which were made with great care and accuracy by Graham for the Royal Society in 1742, should be the standard yard and pound troy, by which all other weights and measures should be regulated; and that the wine gallon, beer gallon, and bushel, should contain 224, 282, and 2150 cubic inches respectively. A second report was made in the following year, chiefly consisting of recommendations for the general adoption and enforcement of these standards; but as the bills which were founded upon them, and which were proposed in 1765, never passed into a law, it is not necessary for us to particularize them further.

Committee of 1818.

(169.) In the year 1818, Sir Joseph Banks, P. R. S., Sir George Clerk, Mr. Davies Gilbert, Dr. W. H. Wollaston, Dr. Thomas Young, and Captain Kater, were appointed commissioners under the privy seal, for the purpose of forming new standards of weights

* The same statute is reenacted for the following year, but was repealed altogether in the 33d year of this reign, upon the petition of the butchers, who declared that they should be ruined if this custom of selling provisions by weight, which had never been the case before, should continue to be enforced.

† Lord Carysfort's *Report of a Committee to ascertain the original standards of Weights and Measures of this Kingdom.* 26th May, 1758.

Arithmetic.

Their report.

and measures, or of determining the relations of those already in use to some invariable standard existing in nature. Their report, which was founded partly upon the report of a committee for the same objects in 1814, and upon inquiries which had been conducted chiefly by Captain Kater since that time, into the lengths of the seconds pendulum expressed in terms of existing standards, is of uncommon importance, from the authority and accuracy of its determinations, and still more so from its chief recommendations having passed into a law. It commences by deprecating any great or violent changes in the standards already in use, as well from the great derangement which such alterations would produce in the ordinary transactions of commerce and trade, as from there being no peculiar advantage in having such standards commensurable with any invariable quantity existing in nature; that it would not be expedient to alter the subdivision of those measures already in use, proceeding as they do in most cases according to the duodecimal scale, or by numbers admitting of two or three successive bisections, and which were, therefore, better accommodated to practical uses, than if the subdivisions had been adapted to the decimal scale. That the parliamentary standard yard, made by Bird in 1760, should be considered as the imperial standard yard of Great Britain;* that the length of the pendulum vibrating seconds in the latitude of London, according to the determination of Captain Kater, was equal to 39.13929 inches of this yard; a relation of lengths which would always furnish the means of recovering this standard in case it should be lost or injured; that though it is apparently more philosophical to determine the measures of capacity immediately from those of length, yet in practice they are much more easily deduced from measures of weight. That one-half of the double pound troy which was made by Bird, upon the recommendation of the committee of 1758, should be considered as the Imperial standard pound troy, containing 5760 grains, whilst the avoirdupois pound should contain 7000 grains; that in case this standard should be lost or injured, it might be recovered from the knowledge of the fact, that a cubic inch of distilled water, of the temperature of 32° of Fahrenheit, weighs 252.724 of this pound when the barometer is at 30°. That it was found upon examination, that the legal standards of capacity were at variance with each other, and that the ale gallon contained $4\frac{1}{2}$ per cent. more than the corn gallon, though it did not appear that this difference was sanctioned by the legislature; that the Winchester gallon, according to the definition in the statute of the 1st of William and Mary, should contain 269 cubic inches, whilst in other acts it was fixed at $272\frac{1}{4}$. That the ale gallon of the Exchequer contained 282 cubic inches, whilst the wine gallon was fixed by the statute of the 5th of Queen Anne at 231; that as it appeared that 10 pounds avoirdupois of distilled water at the temperature of 62°, weighed in air when the barometer is at 30°, was equal to 277.2 cubic inches, it was expedient to assume this capacity as the the Imperial gallon, eight of which should make the Imperial bushel; and that there should be but one common gallon for corn, ale, and wine.

History.

A bill embodying these recommendations, drawn up by Sir George Clerk, was passed in 1824, having been proposed, but rejected in the preceding session of parliament.

Inconveniences attending the introduction of the Imperial bushel.

(170.) The only important alteration which this bill proposed was in our measures of capacity; and it may very reasonably be doubted, whether this change was altogether consistent with one of the wisest recommendations of the committee: that the new gallon should contain exactly 10 pounds of distilled water, was not a necessary condition for recovering the standard hereafter in case it should be lost, though it might make the process for that purpose more easy,* and the accidental coincidence of this assumed weight with one of the standard pints of the Exchequer, which contained exactly 20 ounces of distilled water, was a circumstance altogether unworthy of notice. It is undoubtedly desirable that the same term, gallon, should indicate the same absolute measure of capacity for whatever articles it was used; though the inconvenience which arises from the double or triple meaning of a term is trifling and speculative, whilst that which is produced by the identification of its signification may be serious and real. It is true, indeed, that the alteration of the ale and wine gallon was easily and rapidly effected, as both the measures and the articles measured are under the simultaneous and universal control of the excise; and it was argued as a justification of the change of the corn gallon, that the bushel in ordinary use was almost universally greater than the Winchester bushel; but still it was a legal standard which was recognised by the legislature, and which long custom had rendered familiar to the farmers, a class of men who are generally adverse to all changes. It formed an essential part in all leases where the rent is regulated by the price of corn; and the departure from this standard, which local custom had in some cases sanctioned, was not generally very considerable, was always understood, and was rapidly disappearing. Under such circumstances, we may be almost justified in characterising this act as an example of rash and inconsiderate legislation, which enforced a tax of £150,000. upon a class of men for a merely speculative object, which altered the conditions of so many thousand leases, and which afforded, by the penalties by which its adoption was enforced, endless opportunities for fraud and litigation.

New French measures.

(171.) If ever an opportunity presented itself for the establishment of a system of weights and measures upon perfectly philosophical principles, it undoubtedly occurred in the early part of the French revolution, when the entire subversion of all the old establishments, and the hatred of all associations connected

* The report itself recommended as the standard yard the one which was used by General Roy, in the measurement of the base on Hounslow Heath for the great trigonometrical survey; it was found, however, upon further examination, before the bill was passed into a law, that it agreed less with the average of the other standards than that made by Bird, which was preserved in the Tower, which was then adopted in preference.

* It is provided in the Act, that in case any dispute should arise concerning the accuracy of any of these measures of capacity, whether gallon or bushel, where reference cannot readily be made to a standard, the parties must proceed before a justice of the peace, who is required to verify the measure by weighing its content of rain water of the temperature of 62° of Fahrenheit against the statutable weights. With every respect for the unpaid magistrates of this country, we should like to know how many of them would be either disposed or able to undertake the investigation when appealed to, and what would be the average degree of confidence to which their determination would be entitled; we may venture to say, that no measures, however just and accurate, could stand the test of such an inquiry.

Arithmetic. History.

with them, had created a passion for universal change. The extreme diversity also of the old French weights and measures in different provinces of the kingdom, whether of the same or different denominations, was productive of the greatest inconvenience in the transactions of commerce and trade; and philosophers as well as others had long been anxious for the introduction of some more uniform system, founded, if possible, upon some invariable quantity existing in nature.

Proposal of Picart.

The celebrated Picart, who first measured a degree of the meridian in France, proposed, in accordance with a suggestion of Huygens in his *Horologium Oscillatorium*, that the length of the pendulum vibrating seconds should be adopted as the unit of length, and that it should be called *le rayon astronomique*. The discovery of Richer, however, at Cayenne, in 1671, that pendulum was not of the same length for different latitudes, deprived it of that absolute and invariable character which was considered essential to such a standard.

Of Cassini.

At a subsequent period, Cassini proposed that this standard should be derived from the magnitude of the earth,* and that $\frac{1}{6000}$th part of a minute of a degree should be considered as the *pied géométrique*, and that a *toise* should be considered as the $\frac{1}{60000}$th part of a degree.

Of Mouton.

The same idea was adopted by an astronomer of the name of Mouton, who recommended that a *minute* of a degree should be considered as the superior unit of length under the name of *mille*, whilst the other measures, proceeding in the subdecuple series, should be called respectively, *centuria*, *decuria*, *virga*, *virgula*, *decima*, *centesima*, *millesima*, or otherwise *stadium*, *funiculus*, *virga*, *virgula*, *digitus*, *granum*, *punctum*.

Of De la Condamine.

In the year 1748 M. de la Condamine, who had recently returned from measuring a degree at the equator in Peru, in a Memoir read to the Academy of Sciences, resumed the idea of the pendulum as the unit of length, and recommended as the best means of quieting the feelings of national jealousy which would attend its selection for the latitude of London, Paris, or even of the parallel of 45°, which passes through France, that it should be taken on the equator: under such circumstances he felt persuaded that a sense of its advantages would insure its immediate adoption by all the scientific bodies of Europe, and that it would speedily be received into general use.

Commission of 1790.

(172.) In the year 1788, when the ferment of the revolution was beginning partially to show itself, the same subject was resumed; and in 1790 it was proposed by Talleyrand to the Constituent Assembly, that a commission should be appointed to report on the measures which were proper to be taken; and in consequence, Borda, Lagrange, Laplace, Monge, and Condorcet were appointed commissioners.

Their report.

Their report, which was made in the following year, after noticing the proposals which had been made to make the length of the seconds pendulum at the equator, and at 45°, the unit of measures, considers them in one respect as deficient in the character of a perfect standard, inasmuch as their determination would involve the heterogeneous element of time; that no such objection applies to an unit which shall be a definite portion of the length of a quadrant of a meridian of the earth. They therefore propose that the 10000000th part of the quadrant shall be called the *mètre*, and considered as the primary standard of measures of length, weight, and capacity; that the quadrant shall be divided into 100 degrees, the degree into 100 minutes, and the minute into 100 seconds; that the subdivisions of all measures should be adapted to the decimal scale; that in order to determine the *mètre*, an arc of the meridian, extending from Dunkirk to Barcelona, 6⅓ degrees to the north and 3 degrees to the south of the mean parallel of 45°, should be measured; and that subsequently the weight of a *décimètre* cubed of distilled water at the temperature of melting ice should be determined, as the unit of measures of weight.

Proceedings for the determination of the bases of the new metrical system.

(178.) Immediate steps were now taken for the execution of this great undertaking, under the direction of a committee of the most celebrated men of science in France. The measurement of the northern part of the arc from Dunkirk to Rodez was assigned to Delambre, and of the south to Mechain. The account* given by the former, of the difficulties which he encountered in the course of his operations, from the jealousy and alarm of the country-people, is extremely interesting. His first commission ran in the name of the king; and his labours began when the name of the king was a signal for outrage and violence. When his work was half done, he received the alarming intelligence, that his name, as well as that of Borda, Laplace, Lavoisier, Coulomb, and Brisson, had been struck out of the commission of weights and measures by the committee of Public Safety,† who assigned as their reason for this proceeding, that they required for the public service those only who were worthy of confidence, from their republican virtues and their hatred to kings. Fortunately, however, he was enabled to continue his observations, though with great difficulty and some danger,‡ until the termination of the reign of this sanguinary faction, when the names of the displaced members were restored to the commission, and the measurement of the whole arc completed. The length of the *mètre* which resulted was found to be 443.296 *lignes* less then the *mètre provisoire*, which had been adopted *provisionally*§ in 1794,

* It was contended by Paucton, in his *Métrologie*, that the side of the great pyramid was the exact $\frac{1}{480}$th part of a degree of the meridian, and that the founders of that mighty monument designed it as an imperishable standard of measures of length. Absurd as this notion apparently is, it was patronized by the celebrated Bailli, with his usual fondness for extravagant hypotheses, and who conjectured that both in it and in the *coudée nilométrique*, or *cubit of the nilometer*, was to be found the invariable standard of measures derived from the magnitude of the earth: it was somewhat unfortunate for both these suppositions, that the length of the side of the great pyramid was found to be 716½ French feet, instead of 684¼, and the cubit of the nilometer 20.54 inches instead of 19.892, as it should have been.

* The decree is signed by Barrere, Robespierre, Billaud Varenne, Couthon, and Collot d'Herbois.

† *Base du système métrique. Discours préliminaire.*

‡ The letter which he received in answer to an application which he made to be allowed to complete a certain series of triangles, in order that much of his previous labours might not be rendered useless, is an admirable specimen of the style which was fashionable at that period.

Citoyen,

La commission des poids et mesures a chargé l'un de ses membres de se rendre auprès de toi pour te remettre l'arrêté du comité de salut public qui te concerne et pour concerter avec toi les moyens de clore tes operations de manière que les signaux restent inutiles: elle t'invite à terminer la redaction de tes calculs et la copie de tes observations ainsi que tu les proposes.

18 Nivôse, an. 2.

§ By order of the Committee of Public Safety, who were determined to avail themselves of the *impulsion révolutionnaire* to effect this change, before the conclusion of the labours of the commission.

Arithmetic. before the completion of the operation, by $\frac{1}{1000}$ of a *ligne*.*

The determination of the *unity of weight*, an operation of great delicacy and difficulty, was specially confided to Lefevre Gineau, who assigned to the *kilogramme*, or *décimètre* cubed of distilled water at its *greatest density*, and not at the temperature of melting ice as at first proposed, a weight of 18827.15 grains, *poids de marc*.

The whole of these operations were conducted under the general superintendence of a numerous commission of members of the Institute, as well as of commissioners from Italy, Spain, Holland, and Switzerland; and all the instruments made use of, the journals of observations, and the calculations founded upon them, were submitted to their examination.

Report of the commissioners in 1798.

(174.) The report of the commissioners was made on the 10th of Prairial, 1798, and on the 4th of Messidor, the *original mètre* and *kilogramme* (*les étalons prototypes*) were presented, with a pompous address, to the two councils of the Legislative Body. In speaking of the *mètre* it is said, *Cette unité, tirée du plus grand et des plus invariables des corps que l'homme puisse mesurer, a l'avantage de ne pas différer considérablement de la demi-toise et des plusieurs autres mesures usitées dans les différens pays: elle ne choque point l'opinion commune. Elle offre un aspect qui n'est pas sans intérêt. Il y a quelque plaisir pour un père de famille à pouvoir se dire: "Le champ qui fait subsister mes enfans est une telle portion du globe. Je suis dans cette proportion conpropriétaire du monde."* After mentioning the extraordinary precautions which had been taken by the commission, and enumerating in imposing language the names of the *Savans étrangers et nationaux* who composed it, it is announced that these prototypes shall be deposed amongst the national archives, to be preserved with a religious care, from whence *jamais l'ignorance et la férocité des peuples barbares ne les enlèveront; à la vaillance, au patriotisme, aux vertus d'une nation éclairée sur ses intérêts, sur son honneur, sur ses droits. Mais si un tremblement de terre engloutissoit, s'il étoit possible qu'un affreux coup de foudre mît en fusion le métal conservateur de cette mesure, il n'en résulteroit pas, citoyens législateurs, que le fruit de tant de travaux, que le type général des mesures put être perdu pour la gloire nationale, ni pour l'utilité publique.* By way of provision against such a catastrophe, as *un moyen conservateur du mètre*, it is added, that Borda had determined with great accuracy the length of the seconds pendulum at Paris, and the repetition of the experiments at any future period would furnish the means of recovering the original relation of its length to that of the *mètre*, and consequently of determining the length of the *mètre* itself.

New nomenclature of weights and measures.

(175.) The nomenclature of the new weights and measures underwent various changes. It was proposed by the first commission, that the old names should be preserved as much as possible, with significations adapted to the new system. The law of the 18th of Germinal, 1794, which established the provisional *mètre* and *kilogramme*, altered the old names entirely; whilst the law of the 13th Brumaire, 1798, which succeeded the report of the commission, reverted in a great measure to the system of names which were first proposed. They are as follows: History.

Measures of length.	Synonyms.	Mètre.
Lieue	Myriamètre	10000
Mille	Kilomètre	1000
Hectomètre		100
Perche	Décamètre	10
Mètre		1
Palme	Décimètre	$\frac{1}{10}$
Doight		$\frac{1}{100}$
Trait		$\frac{1}{1000}$

Measures of weight.	Synonyms.	Kilogramme.
Millier		1000
Quintal		100
Myriogramme		10
Livre	Kilogramme	1
Once	Hectogramme	$\frac{1}{10}$
Gros	Décagramme	$\frac{1}{100}$
Dénier	Gramme	$\frac{1}{1000}$
Grain	Décigramme	$\frac{1}{10000}$

Measures of capacity: the unit is the *mètre* cubed.

Muid	Stère	1
Setier	Décistère	$\frac{1}{10}$
Boisseau		$\frac{1}{100}$
Pinte		$\frac{1}{1000}$
Verre		$\frac{1}{10000}$

Measures of area: the unit is the *mètre* squared.

Arpent	Hectare	1000
Décare		100
Perche	Are	10
Mètre carré	Déciare	1

(176.) The establishment of the French system of weights and measures was an event of considerable importance to the scientific world, from the imperishable nature of its bases, and from the confidence to which their determination is entitled. The power and influence of popular and national prejudices must for ever prevent the universal adoption of this or any other system, however perfect; but it is of comparatively little consequence whether they are actually adopted by any nation or nations, so long as they furnish a standard of reference by which those in use may be estimated, and by that means their value become universally known. It is only by such means that the fluctuating and variable standards of different nations may be made to speak the same language. The decimal subdivision of these measures possessed many advantages on the score of uniformity, and was calculated to simplify in a very extraordinary degree the Arithmetic of concrete quantities. It was attended, however, by the sacrifice of all the practical advantages which attend subdivisions by a scale admitting of more than one bisection, which was the case with those previously in use; and it may well be doubted, whether the loss in this respect was not more than a compensation for every other gain. Advantages and disadvantages of the new system.

(177.) The centesimal division of the quadrant was not called for by any principle of uniformity, and it at once sacrificed all the conveniences which attend its trisection, which is so important for artists in the division of circular instruments. It at once also made useless all the trigonometrical tables which were already calculated, at least without previous and troublesome reductions. If the change had been confined to the cen- The centesimal division of the quadrant less convenient than the sexagesimal.

* The length of the provisional *mètre* was determined from the data furnished by Lacaille in 1758, who had assigned to a degree of the meridian in latitude 45° a length of 57027 toises.

tesimal division of the minute, second, &c. it might have been generally adopted, and others would have very readily abandoned the use of sexagesimals, proceeding as they do by too high a scale to be conveniently used: as it was, the new division of the quadrant was never generally used even in France, notwithstanding the great authority of Laplace, and was abandoned in later life, even by Delambre himself, by whom it was once so zealously recommended.*

Difficulties attending the introduction of the new system.

(178.) The reception which the new measures experienced in France furnishes a curious proof of the extreme difficulty of counteracting the prejudices, or altering the habits of a whole people; and an instructive lesson of the danger and inefficacy of any legislative interference with them, unless called for by great and manifest advantages, and capable of being readily and universally enforced. In no other nation was the grievance of variable and uncertain weights and measures so intolerable; in no other nation was the occasion for their reformation so favourable, when the current of popular opinions and habits had been diverted from its ordinary channel by the violent concussion of the revolution; in no other country could the change proposed have been recommended by a greater or more imposing authority; yet we find that the people obstinately adhered to their ancient measures and their ancient names. The *mètre* was a new and unintelligible name, associated in their minds with no former or natural measure, and by no means recommended by its enabling them to ascertain the definite portion of the earth's surface which their farms occupied. In other cases, the union of old names with new measures made their introduction more easy; but it required the influence of many years, and all the authority of the government, to effect even their partial adoption; and even at this time, we find their *mètre* and its third part, the *foot*, with the *duodecimal* as well as the decimal division, in almost universal use.

Reductions of weights and measures: Their complexity.

(179.) The reduction of weights, measures, and coins, from greater to lower denominations, and the contrary, forms an important article in all books of Arithmetic, and requires, of course, a perfect knowledge of their several subdivisions. In Italian books of Arithmetic, these reductions become extremely complicated, from their generally extending to the weights and measures of other cities, besides those in which the authors lived, where they varied extremely, both in denomination and value. As an example, we shall give from Tartaglia the measures of length and area which were used in many of the cities of northern Italy, though he declares that the list which he gives does not include the hundredth part of the cities of Italy, in which such variations are found.

Italian measures of area.

Verona.

Measures of length.		Measures of area.	
Pertica,	6 piedi.	Campo,	24 vanezze.
Piede,	12 oncie.	Vanezza,	30 tavole.
Oncia,	12 ponti.	Tavola,	36 piedi.
		Piede,	12 oncie.
		Oncia,	12 ponti.
		Ponto,	12 athomi.
		Athomo,	12 menicoli.

Padua.

Measures of length.		Measures of area.	
Pertica,	6 piedi.	Campo,	4 quarteri.
		Quartero,	210 tavole.
		Tavola,	36 piedi.

Treviso.

Pertica,	5 piedi.	Campo,	1250 tavole.
		Tavola,	25 piedi.

Milano.

Zucata,	12 braccia.	Pertica,	24 tavole.
Brazzo,	12 oncie.	Tavola,	12 piedi.

The *tavola* is the square of the *zucata*, which is divided into 12 piedi.

Bergamo.

Cavezzo,	6 braccia.	Pertica,	24 tavole.
Brazzo,	12 oncie.	Tavola,	12 piedi.

The *tavola* is the square of the double *cavezzo*.

Mantua.

Cavezzo,	6 braccia.	Bioleo,	100 tavole.
Brazzo,	12 oncie.	Tavola,	12 piedi.

The *tavola* is the square of the double *cavezzo*.

Brescia.

Cavezzo,	6 braccia.	Pio,	100 tavole.

Firenza.

Brazzo,	12 oncie.	Staiora,	12 panore.
		Panora,	12 pugnore.
		Pugnora,	12 braccia.
		Brazzo,	12 oncie.

Meaning of oncia.

Throughout modern Italy, *oncia* has the same meaning with the *uncia* of the ancient Romans, designating a twelfth part of the next superior integer, whatever that integer may be.

General prevalence of duodecimal division.

(180.) The prevalence, likewise, of the duodecimal division in all these cases is sufficiently remarkable. The subdivisions of the *oncia* never extended in practice beyond the *ponto*.* The other terms *athomo* and *menicolo* are introduced by Tartaglia himself, to express the more minute terms in the duodecimal multiplication of length into length. The misapplication of the names of measures of length to designate the area of the squares described upon them is common in all languages; but in some of the cases above-mentioned the *foot* is taken as the first of the duodecimal subdivisions of the *tavola*, without any reference to the measures of length which it commonly designates.

Interchange of terms.

The origin of this interchange of terms, and their misapplication to denote things essentially different from each other, is to be ascribed in part to the poverty of language, and partly, likewise, to the ignorance of most men of the proper force and meaning of the terms which they use. In the case of terms applied to designate the successive subdivisions of any class of concrete quantities, it is a natural and easy process of the mind to consider them as more connected with the relative magnitude of the next superior unit than with the peculiar nature of the magnitude itself. In illustration of the truth of this observation, we may refer to the very general meaning given to the terms which were originally confined to denote the subdivisions of the Roman *as*.

* *Base du système métrique*, tom. iii. p. 309.

* In some cases, the common people called the *ponto de terra*, or any smaller subdivision of the *oncia*, *danaro de terra*, borrowing the name of the imaginary coin so called.

Arithmetic.

Multiplication of denominate numbers into each other.

(181.) The Arithmetic of compound or denominate quantities, their addition, subtraction, multiplication, and division, as well as their reduction, presents not much room for variety, and they will be found, upon examination of the arithmetical works of the last three centuries, nearly under the same form. A question, however, appears to have arisen, whether it was possible to multiply denominate numbers together, or to divide them by each other.

Opinion of Stifelius:

It was remarked by Stifelius,* that *numerus vulgariter denominatus, non potest multiplicari per alium numerum vulgariter denominatum nisi alter eorum denominationem suam deponat et fiat abstractus*: but again, that *alter per alterum dividi potest, modo ambo eandem habeant denominationem*. In the latter case, the numbers are reduced to the same lowest denomination, and their relation to each other is identical with the relation of the resulting numbers, considered as abstract, and, consequently, their quotient may be considered as an abstract number.

Of Tartaglia.

Tartaglia† has quoted this remark of Stifelius with disapprobation, and seems to speak of the possibility of multiplying money by money, and weights by weights; but as such an operation might appear to many people *cosa nuova e forsi strania*, he defers the further discussion of such peculiarities, or, at all events, of the exceptions to such an opinion, to another occasion.

Duodecimal multiplication of length into length.

(182.) The exceptions which probably suggested themselves to the mind of Tartaglia were those in which the product of length into length produces area, or where the product of area into length, or of length into length into length, produces capacity. It was not considered, that in these cases the multiplication took place as if the numbers were abstract, the inferior subdivisions forming a series of duodecimals of the primary unit; and that the relation between the product and the component factors (sides or edges of the rectangle or parallelopipedon) was merely numerical, the concrete units being essentially different from each other: in other words, that it was an extension of the meaning of the term *multiplication* to apply it to such cases; as the analogy of which the terms in their order are the product, the multiplicand, the multiplier, and unity, which existed in one case, no longer existed in the other, at least in its proper and strict sense.

Tartaglia has given many examples of these duodecimal multiplications, as well as of the inverse operation of division; and we have seen before, that he extended the nomenclature of the duodecimal subdivisions, so as to include all the terms which resulted from them: beyond these cases, however, he has not ventured to proceed, and we may consider the boast that he would produce numerous instances in reprobation of the opinion of Stifelius, as a proof of the envious and contentious spirit with which he criticized the writings of his contemporaries, of which he has been accused in severe terms by Bombelli.‡

Rule of Three:

(183.) The Rule of Three, emphatically called from its great usefulness the Golden Rule, both by ancient and modern writers on Arithmetic, is so simple in principle, that we can expect to find very few essential variations in the form in which it is stated.

In the Lilávati.

In the *Lilávati* we find the ordinary divisions of the rule into direct and inverse, simple and compound, with statements for performing the requisite operations, which are sufficiently clear and definite, due allowance being made for the ordinary obscurity of Sanskrit phraseology on scientific subjects. The terms of the proportion are written consecutively, without any marks of separation between them: the first of them is termed the *measure*, or *argument*; the second is its *fruit*, or *produce*; the third, which is of the same species with the first, is the *demand*, *requisition*, *desire*, or *question*. When the *fruit* increases with the increase of the *requisition*, as in the direct rule, the second and third terms must be multiplied together, and divided by the first; when the *fruit* diminishes with the increase of the *requisition*, as in the inverse rule, the first and second must be multiplied together, and divided by the third.

No proof of the rule is given, and no reference to the doctrine of proportion upon which it is founded. Proofs, indeed, are never given in the *Lilávati*, and on this occasion are hardly required; the proposition is so readily deduced by the common sense of mankind, when its terms are once understood, that it acquires very little additional evidence from a formal demonstration.

Compound proportion.

Under compound proportion are included the rule of five, seven, nine, or more terms. The terms are in these cases divided into two sets, the first belonging to the argument, and the second to the requisition: the fruit in the first set is called the produce of the argument; that in the second is called the divisor of the set: they are to be transposed, or reciprocally to be brought from one set to the other; that is, put the fruit in the second set, and the divisor in the first; in other words, transpose the fruits in both sets. This rule, which is sufficiently obscure, will be further explained in some of the examples which follow.

Examples.

Example 1. If two and a half *palas** of saffron be obtained for three-sevenths of a *nishca*,† say instantly, best of merchants, how much is got for nine *nishcas*? Statement:

3 5 9
7 2 1 Answer, 52 *palas* and 2 *carshas*.

Rule of three inverse.

Example 1. If a female slave, 16 years of age, bring 32 *nishcas*, what will one aged 20 cost? If an ox, which has been worked a second year, sell for 4 *nishcas*, what will one which has been worked 6 years cost?

1st question.

Statement: 16 32 20. Answer, $25\frac{3}{5}$ *nishcas*.

2nd question.

Statement: 2 4 6. Answer, $1\frac{1}{3}$ *nishcas*.

Value of slaves and oxen.

The value of living beings is supposed to be regulated by their age, the maximum of value of female slaves being fixed at 16 years of age, and of oxen after 2 years' work; and their relative value in the present case being 8 to 1. So important was this traffic considered, and so fixed were the principles by which it was regulated, that in the *Arithmetic* of Srid'hara it is made the subject of a distinct chapter: this is not the only instance in which the examples given in books of Arithmetic will convey important information concerning civil institutions and the trade or commerce of nations.

* *Arithmetica Integra*, p. 81.
† *Numeri e misure; in fine libri tertii*, pars i.
‡ *Algebra*, Preface.

* A *pala* = 4 *carshas*; a *carsha* = 16 *mashas*; and a *masha* = 5 *gunjas*, or 10 *grains of barley*.
† A *nishca* = 16 *drammas*; a *dramma* = 16 *panas*; a *pana* = 4 *cacinis*; and a *cacini* = 20 *cowry shells*.

Arithmetic.

Example 2. If a *gadyana* of gold of the touch of ten may be had for one *nishca* of silver, what weight of gold of fifteen touch may be bought for the same price?

Statement: 10 1 15. Answer, $\frac{2}{3}$.

Touch of gold.

The fineness of gold in the East is usually determined by its colour on the touchstone, which long experience makes a sufficiently delicate test of the quantity of alloy. European goldsmiths, from a very early period, have been accustomed to divide an unit of gold in 24 parts, called *caratti*,* or *carats*, and to estimate its fineness, or degree of purity, by the number of carats of pure gold which it contained.

Rule of five terms.

Rule of five terms.

Example 1. If the interest of a hundred for a month be five, what is the interest of sixteen for a year?

Statement:	1	12,	
	100	16	
	5		or, transposing the fruit,
	1	12	
	100	16	
		5	

product of the larger set 960, of the lesser 100. Quotient $\frac{960}{100}$, or $\frac{48}{5}$, which is the answer.

Example 2. Forty is the interest of a hundred for ten months; a hundred has been gained in eight months: of what sum is it the interest?

Statement:	10	8, or, transposing,		10	8
	100			100	
	40	100		100	40

whence the answer $\frac{800}{4}$ is obtained.

Interest of money in India.

The interest of money, if we may judge from the examples in Brahmegupta and *Lilávati*, varied from $3\frac{1}{2}$ to 5 per cent. per month, exceeding greatly the enormous interest paid in ancient Rome. The case is similar, though not to the same degree, in modern India, where it is not uncommon for native merchants or tradesmen to give 30 per cent. per annum.

Of seven terms.

Rule of seven terms.

Example. If three cloths, two wide and five long, cost six *panas*, tell me how many cloths, three wide and six long, should be had for six times six?

Statement:	2	3, or, transposing the fruits,		2	3
	5	6		5	6
	3			3	
	6	36		36	6

The answer is 10.

Of nine terms.

Rule of nine terms.

Example. The price of a hundred bricks, of which the length, the base, and breadth, are respectively sixteen, eight, and ten, is settled at six *dinâras*. We have received a hundred thousand of other bricks, a quarter less in every dimension: say what we ought to pay?

Statement:	16	12
	8	6
	10	$\frac{30}{4}$
	100	100000
	6	

or, transposing the fruit and denominator, it becomes,

16	12
8	6
10	30
100	100000
4	6

The answer is $2531\frac{1}{4}$.

* Tartaglia, *Numeri e misure*, para i.

History.

Rule of eleven terms.

Of eleven terms.

Example. Two elephants, which are ten in length, nine in breadth, thirty-six in girt, and seven in height, consume one *drona* of grain; how much will be the rations of ten other elephants, which are a quarter more in height and other dimensions?

Statement:	2	10
	10	$\frac{50}{4}$
	9	$\frac{45}{4}$
	36	45
	7	$\frac{35}{4}$
	1	

the fruit and denominator being transposed, the answer is $\frac{3125}{256}$.

The principle of this very curious example would be rather alarming, if extended to other living beings besides elephants.

Barter.

The last example which we shall give is one of barter, included by Brahmegupta and Bhascara under this very comprehensive rule.

If a hundred of mangoes be purchased for ten *panas*, and of pomegranates for eight; how many pomegranates for twenty mangoes?

Statement:	10	8, or, transposing,	8		10
	100	100		100	100
	20				20

The answer is 25.

Rules used in Italy.

(184.) It was usual, according to Lucas de Burgo, for students in Arithmetic, who wished to learn the practice of *la regola del tre*, (or *la regola delle tre cose*, as it was designated with manifest impropriety by the *grossi* or ignorant,) to commit to memory one or other of the two following rules:

1. *La regola del tre vol che si moltiplichi la cosa che l'huomo vol saper per quella che non e simigliante e partir per l'altra che e simigliante a quel che ne vene e de la natura de quella che non e simigliante e sia la valuta de la cosa che volemo inquirere.*

2. *La regola del tre vol che si guardi la cosa mentovata doi volte delle quale la prima e partitore. E la seconda si moltiplichi per la cosa mentovata una volta. E quella tal moltiplicatione si parta per detto partitore. E quello che ne viene de detto partimento sia de la natura de la cosa mentovata una volta; si deve mettere in lo mezzo quando si opera.*

Tartaglia has mentioned the first of these rules nearly in the same terms. He has given also a third rule, differing in expression only from the preceding; it is as follows:

La regola dal tre sono tre cose, la prima che si mette debbe esser sempre simile a quella che sta di drio e di drio debbe star la cosa che si vol saper e multiplicar la contra quella che sta de mezzo e quel produtto partirlo per la prima e sara fatta la ragione: e nota che quella che venira sara sempre simile alla cosa, che sta di mezzo.

This rule, expressed in popular or vernacular language, formed part of a system of instruction of the practice of this rule, adapted to those who had not sufficient time to acquire, genius to comprehend, or memory to retain the rules for the reduction and incorporation of fractions; a system reprobated by Tartaglia, and attributed by him partly to the ignorance of the ancient teachers of Arithmetic at Venice, and partly to the stinginess and avarice of their pupils, who grudged the time and expense requisite for attaining a perfect understanding of the peculiarities of fractions.

(185.) An arithmetician of Verona, named Francesco

Arithmetic.

Amended rules of Feliciano da Lazesio.

Feliciano da Lazesio, the author of a work on Arithmetic, entitled *Scala Grimaldelli*, objects to the memorial rules of De Burgo as being too general, as it is very possible that the three quantities may all be of the same or all of different species or denominations. Thus, in the following question, *If 3 ducats produce me 4, what will 6 ducats produce?* the three quantities are of the same species and denomination; whilst in this, *If 8 ducats purchase 4 braccia of cloth, how much may be got for 24 lire?* the quantities are all of different denominations. For these reasons, he wishes to distinguish the quantities into *agents* and *patients*, and those again into *actual* or *present* and *future*. The first term of the proportion is the *agent a presente*, and its corresponding *patient* is the second; the third term is formed by the *agent de futuro*, and its *patient* is the quantity to be determined. With respect to these objections, it may be observed, that the first is true though merely verbal; that the second is merely imaginary, inasmuch as the first and third terms are reducible to the same denominations; and granting the accuracy of the distinctions made in the proposed alterations, yet they injure the simplicity of the old rules, by introducing metaphysical considerations, which are not very readily apprehended by students whose minds are not disciplined to the labour of systematic thought.

Example from Lucas de Burgo:

(186.) The following example will show the manner in which the operation was stated and performed by De Burgo.

If a hundred pounds of fine sugar cost 24 ducats, what will be the cost of 975 pounds?

Partitore ouchari.	*Cosa mentovata una volta: multiplicatore: ducati.*	*Coso che volemo sapere: l'altro moltiplicatore.*

$$\frac{100}{1} \overset{via}{\times} \frac{24}{1} \;\overset{vt.}{\underset{vt.}{=\!=\!=}}\; \frac{975}{1}$$

$$\begin{array}{r}975\\24\\\hline 3900\\1950\\\hline 23400\end{array}\qquad\begin{array}{ccccc} & & \cancel{0} & & \\ & \cancel{0} & \cancel{4} & \cancel{0} & \\ \cancel{0} & \cancel{3} & \cancel{4} & \cancel{0} & \cancel{0}\\ \cancel{2} & \cancel{3} & \cancel{4} & \cancel{0} & \cancel{0}\\ \cancel{1} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0}\\ & \cancel{1} & \cancel{0} & \cancel{0} & \\ & & \cancel{1} & & \end{array}\;(234\ \textit{ducati}$$

The quantities are exhibited under a fractional form, for the purpose of making the process more general, being equally applicable to fractions and whole numbers. It is sufficiently curious, that he should have considered it necessary to construct the *galea* for the division by 100.

From Tartaglia.

(187.) The following example of the same process, with fractions in every term, is given by Tartaglia.

If $3\frac{1}{2}$ pounds of rhubarb cost $2\frac{1}{3}$ ducats, what will be the cost of $23\frac{3}{4}$ pounds?

$$\overset{lire.}{\frac{7}{2}} \times \;\|\; \overset{ducati.}{\frac{7}{3}} \;\overline{\underline{\;\|\;}}\; \overset{lire.}{\frac{95}{4}}$$

$$\begin{array}{rr} & 3\\ & 4\\\cline{2-2} & 12\\ & 7\\\cline{2-2} \textit{partitor} & 84\\\cline{2-2} & 95\\ & 7\\\cline{2-2} & 665\\ & 2\\\cline{2-2} \textit{da partir} & 1330\end{array}$$

$$\begin{array}{cccc} & 0 & 7 & \\ & \cancel{4} & \cancel{9} & \\ 0 & \cancel{5} & \cancel{9} & 0\\ \cancel{1} & \cancel{3} & \cancel{3} & 0\\ & \cancel{8} & \cancel{4} & \cancel{4}\\ & & \cancel{8} & \end{array}\;(15\ \textit{ducati.}$$

$$\begin{array}{cccc} 0 & 0 & 0 & \\ \cancel{1} & \cancel{6} & \cancel{8} & 0\\ & \cancel{8} & \cancel{4} & 4\\ & & 8 & \end{array}\;(20\ \textit{grossi.}$$

History.

Different modes of writing the terms of the proportion.

(188.) Different methods have been adopted by different authors, for representing the terms of the proportion in this rule. We will state a few of them with reference to the following question.

If 2 apples cost 3 soldi, what will 13 cost?

Statement of Tartaglia:

Se pomi 2 ‖ val soldi 3 ‖ che valera pomi 13.

Other Italian writers write the numbers consecutively with mere spaces, and no distinctive marks between them; thus,

Pomi.	Soldi.	Pomi.
2	3	13

or thus,

1ma.	2a.	3tia.
2	3	13

In Recorde and the older English writers, they are written as follows:

Apples.	Pence.
2	3
13	$19\frac{1}{2}$ answer.

A later English author,* whose work was first published in 1562, and afterwards in 1594, writes them thus:

2 3 13

The custom, which prevailed generally during the XVIIth century, was to separate the numbers by a horizontal line,† as follows:

* *The Well-spring of the Sciences, which teacheth the perfect worke and practice of Arithmetike, set forth both in whole numbers and in fractions; set forth by* Humfrey Baker, Londoner, 1562. In speaking of this rule, he says, "The rule of three is the chiefest, and the most profitable, and most excellent rule of all Arithmetike. For all other rules have neede of it, and it passeth all other: for the which cause, it is sayde the philosophers did name it the Golden Rule but now, in these later days, it is called by us the Rule of Three, because it requireth three numbers in the operation"

† *Vulgar Arithmetike, explaining the secrets of that art by* Noah Bridges, 1653, p. 127: "It was the custom in that age for every writer on the most trivial subjects to have his work recommended by the poetical contributions of his friends. The fame of Mr. Noah Bridges is celebrated in every form of versification, and with every variety of extravagant eulogy. A Mr. Lovelace addresses *the incomparable Mr. Bridges on his (by his favour) Ne Vulgar Arithmetic.*

Behold yon sacred bard, that doth uncharm
All your learned knots, and your strong rules disarm.
Your priest, that with delight doth sacrifice
Past errors, doubts, mistakes, unto the wise;
And with this smooth, bright, polished, easie key,
He opes a dore unto a rosie way,
So far from vexing the distracted brain,
That sculls of lead his *golden rules* contain.
Here he the chaste Arithmetick addresses,
Left nak'd as truth, unbrades her very tresses
With such religious skilful lust, that we
The very secrets of her secrets see.

Apples.		Pence.		Apples
2	——	3	——	13

The whole algorithm of proportions appears to have received the particular attention of Oughtrede,* from whom the sign : :, to denote the equality of ratios, was derived. He states the rule of three as follows:

2. 3 :: 13.

In still later times, the simple dot which separated the terms of the ratios was replaced by two, as in the form which is now used:

2 : 3 :: 13.

Classification of questions in Lucas de Burgo and Tartaglia.

(189.) Both Lucas de Burgo and Tartaglia have sought to include, in the numerous examples which they have given, every possible case of mercantile practice; the first of these authors has classified his examples with reference to the manner in which particular goods are sold, whether by the hundred pounds weight, as was the case with the sugars of Palermo, Syria, Madeira, or Candy, the finer species of wool, wax, gums, medicines, &c.; or by the thousand pounds, as was the case with heavier merchandise, such as metals, vitriol, galls, rice, oils, &c., or by measures of capacity and by number. The latter has adapted his classification partly to the occurrence or non-occurrence of fractions in the different terms of the proportion, and partly to the peculiar difficulties which attend the statement or solution of the questions which are proposed. The questions themselves are in immense variety, and are stated and solved with great minuteness of detail.

Different species of practice.

(190.) Amongst other abbreviations of the process for the solution of the rule of three questions, of which the Italians were the inventors, and which were adapted to the purposes of their extensive commerce, may be mentioned the rules of practice. Tartaglia has divided them into four species, *la practica naturale*, *artificiale*, *Venetiana*, and *Firentina*; the three first of which form, severally, the subjects of the IVth, Vth, and VIth Books of the 1st part of his work. The first consists in multiplying the several parts of the quantity whose value is demanded by the several terms of the price of its primary unit, and reducing the terms of the several results in the manner which may be required; as in the following example:

Practica naturale.

What is the price of 23 braccia, 2 quarte, at 8 lire, 13 soldi the brazzo?

	l.	*s.*			
23 braccia for 8 lire. .	184	0		23 braccia, 13 soldi	23
23 braccia for 13 soldi	14	19			13
	198	19			69
2 quarte	4	5	6 piccoli.		23
Total amount	203	6	6		299
					*l*14. 19*s*.

The *natural practice* merely requires the knowledge of the four fundamental rules of Arithmetic, and the methods for the reduction of weights and measures from higher to lower denominations, and, conversely, without resorting to any of the artifices made use of in the other methods of practice, which require the instructions of a refined arithmetician. The rules of the first of these, as well as of the two others, as they appear in our author, differ very slightly from the rules of practice which appear in our books of Arithmetic, excepting only that they are not reduced to the same definite and systematic form, and are worked out as usual with a tedious particularity and diffuseness. They are chiefly founded upon the assignation of the aliquot parts of their coins, weights, and measures, and more especially those of the *ducat* and *lira*, or pound, of which an extended list is given. The following is an example:

Practica artificiale.

What is the value of 624 *stara*, 2 *quarte*, and 3 *quartoroli* of wheat, at 9 lire, 16 soldi, 8 pizzoli, the *staro*?*

Stara. . . .	624	2	3		
Lire	9	15	8		
At 9 lire	5616	0	0		
10 soldi.	312	0	0		
6 pizzoli	15	12	0		
2 pizzoli	5	4	0		
	6104	16	0		
For 2 quarte. . . .	4	17	10		
2 quartoroli	1	4	5	6	men[l].
1 quartorolo	0	12	2	9	
	6111	10	6	3	

Practica Venetiana.

The third species of practice, denominated Venetian, differs from the preceding only in the mode of solving questions, where the price is fixed at so much per hundred or thousand, whether pounds or of other denominations. The following are examples:

If a hundred pounds of sugar of Madeira cost 9 ducats, 18 grossi, what is the price of 3855 pounds?

The reason of the disproportion between fools and wise men is very satisfactorily explained:

> Why are *wise* few, *fools* num'rous in the exceme?
> 'Cause, wanting *number*, they are *numberlesse*.

Amongst other names in this list, we may be surprised to find those of Thomas Shirley and Elias Ashmole. Another poet, whose initials are T. D., is shocked and amazed at the title of his book:

> I stood amazed, when first I saw,
> Ev'n in thy title, such a flaw,
> As made me (though engaged) withdraw,
> Why should arithmetique now be
> Accounted vulgar? when we see
> It thus ennobled by thee.
> Alas! that title's now too poore,
> Since that thy cyphers stand for more
> Than all their decimals did before.
>
> Melisides, who ne're could thrive
> In computation beyond five,
> May now in's noddle millions hive.

Every student in Arithmetic may join in the poet's last wish, and in thinking that the author's recompence would be well merited.

> Nay more, dear friend, free from control
> I'll chant thy praise from pole to pole,
> Doe but once make our fractions whole.

The reader may be somewhat surprised to be informed, that the work which is thus blazoned into notice is at least a very trifling, if not a very vulgar production.

* *Clavis Mathematicæ*, p. 17.

* The *stara*, like all other primary units of weights and measures, varied extremely in different cities of Italy; at Venice it weighed 132, at Parma 110, and at Piacenza 90 lire; and 2 stara of Mantua were equal to $5\frac{1}{3}$ stara of Bergamo.

	3855		
	9 ducats, 18 grossi per hundred.		
At 9 ducats	34695		
12 grossi..	1927	12	
6 grossi..	963	18	
Ducats..	375	86	6
Grossi..	20	70	
Piccoli..	22	40	

Answer, 375 ducats, 20 grossi, 22 piccoli.

If one thousand pounds of Spanish wool cost 27 ducats, 16 grossi, what is the price of 9756 pounds?

	9756	
	27 ducats, 16 grossi.	
	68292	
	19512	
At 27 ducats..	263412	
12 grossi..	4878	
4 grossi..	1626	
Ducats..	269	916
		24
Grossi....	21	984
		32
Piccoli..	31	488

Answer, 269 ducats, 21 grossi, 31 piccoli.

Practica Firentina.

The Florentine practice differed in no essential point from the Venetian, adopting merely a somewhat different and more artificial distribution of the aliquot parts. The following is an example:

If one hundred pounds of mastic cost 25 lire, 12 soldi, what is the cost of 18 pounds, $4\frac{1}{2}$ ounces?

25 lire, 12 soldi che val, 18 lire, $4\frac{1}{2}$ oncie.

	2	2	8	12
	1	1	4	2
	360	0	0	
	90	0	0	
	9	0	0	
	1	16	0	
	8	0	0	
	0	8	0	
	0	2	8	
	1	1	4	
Lire.. 4	70	8	0	
Soldi 14	08			
Pizzoli 0	$\frac{3}{10}\frac{2}{0}$			

In this case, the 25 lire 12 soldi are divided by 12, to get the value of an ounce, and by 2, to get the value of $\frac{1}{2}$ an ounce. The rest of the process requires no explanation.

Praxis Italica of Stifelius.

(191.) *Praxis illa quam ab Italis ad nos devolutam esse arbitramur, est ingeniosa quædam inventio, quarti termini regulæ de Tri, ex tribus terminis, mediante distractione varia eorundem terminorum, distractarum particularum proportionatione atque denominationum vulgarium translatione.* This is the language of Stifelius,[e] and the following example will show that the *praxis*, which was known to or used by him, was Florentine.

Ulna venditur pro 15 grossis et 10 denariolis et uno obulo; quanti venduntur 48 ulnæ de panno eodem.

ulna.	fl.	gro.	den.	ulnæ.
1	0	15	$10\frac{1}{2}$	48
		7	6	*locus distractarum*
		7	3	*particularum.*
		1	$1\frac{1}{2}$	
	16			
	16			
	2	6		*locus productorum.*
	1	3		
		12		
		6		
Facit	36 *flor.*	6 *gro.*		*summa productorum.*

In order to understand this scheme it must be observed, that 21 grossen make a florin, and, therefore, 7×48 grossen is equal to 16 florins, the reason which suggested the distribution of the 15 grossen into 7, 7 and 1. The author has given six different dispositions of the same example, to show the variety of ways in which such solutions may be presented.

Rules of practice in English authors.

(192.) The great convenience of these rules for performing the calculations which were continually occurring, both in trade and commerce, made them a favourite study with practical arithmeticians, and they consequently assumed from time to time an increased neatness and distinctness of form. Stevinus, indeed, speaks of them with some contempt, as forming "a vulgar compendium of the rule of three, sufficiently commodious in countries where they reckon by *livres, sous,* and *deniers;*"* but such is his usual manner. Amongst the additions made to Recorde's *Arithmetic* by John Mellis of short Southwarke, Schoolmaster, in 1586, is one on certain "briefe rules, called rules of practise, of rare, pleasant and commodious effect, abridged into a briefer method than hath hitherto been published," where they are exhibited under a very simple and complete form. Later works gave them still greater compactness and brevity, such as that of Wingrave, as edited by Kersey; and in Cocker's *Arithmetic,*† and others printed towards the end of the XVIIth century, they assumed the form which they retain at present.

Questions on tare and tret, &c.

(193.) Amongst the questions proposed by Tartaglia, in illustration of the rules of different species of practice, are many, which he terms *ragioni doppie, treppie, qua-*

e *Arithmetica Integra*, p. 83.

* *La Pratique d'Arithmetique*, p. 721.

† The work of Edward Cocker, late practitioner in the arts of writing, arithmetic, and engraving, whose name for near a century enjoyed a species of proverbial celebrity, as synonimous with the science of numbers and accounts, was published after his death in 1677, by John Hawkins, a brother writing-master in Southwark: it is entitled, and very justly, "a plain and familiar method, suitable to the meanest capacity, for the full understanding of that incomparable art." In his preface, he speaks with great contempt of many of the pretended arithmeticians of his time: "For you, the pretended numerists," says he, "of this vapouring age, who are more disingeniously witty to propound unnecessary questions, than ingeniously judicious to resolve such as are necessary; for you was this book composed, if you will deny yourselves so much as to invert the streams of your ingenuity, and by studiously conferring with the notes, names, orders, progress, species, properties, proportions, powers, affections, and applications of numbers, delivered herein, become such artists, indeed, as you now only seem to be." It may be worth while observing, that this modest and useful book is not honoured with poetical recommendations.

Arithmetic. *druppie*, &c.; questions, in short, which should properly require, two, three, four, or more proportions for their solution. They are chiefly those in which deductions, whether per centage or otherwise, are to be made from the gross weight of the articles bought or sold; or a tax to be deducted from the gross produce of the sale. It may be proper to explain the terms made use of, which have had their origin in the local or general customs of commerce and trade, or from taxes imposed for the particular benefit of the state or city where the transactions took place.

Tarra. *Tarra*, the original of our word *tare*, is derived from the verb *tarare*, to *abate* or *diminish*, was an allowance or deduction of so much per cent. or otherwise, upon the gross weight of the goods sold, to make up for package, dust, waste, or other losses; it varied, according to the nature of the merchandise, from 2 to 10 per cent.; the weight *netto de tarra*, or *clear from tare*, becomes our *nett* or *neat weight*. The term *sottile*, or *subtle*, is used in the same sense.

Tret and cloff. The terms *tret* and *cloff* are of unknown, but probably of Dutch, origin. The first is an allowance of 4 pound in every 104 sold, for waste; *tare*, with the English merchants, being the variable allowance for boxes, package, &c. The term *cloff* has a peculiar as well as a general sense; in one case it denotes an allowance of 2 pound to the citizens of London on every draught of certain descriptions of goods which exceeded 3 cwt.; whilst in general it denotes a small allowance made on goods sold in *gross*, to make up for deficiencies in weight when they are sold in retail.

Messetaria. *Messetaria*, a Venetian term, sometimes expressed by the more general word *datio*, or *dazio*, *tax* or *impost*, was a double tax, varying generally from 1 to 3 per cent., which was paid both by the buyer and seller of different species of goods. It was a law of Venice, that when one of the parties was a *terrero*, or inhabitant of *terra firma*, the other might retain this tax in his hands, as he was responsible to the *datiari* for its payment.

Other terms. The following question of Lucas de Burgo introduces other terms to our notice:

El migliaro de ramo roso val ducati 96 : *el migliaro del stagno in verga val ducate* 90 : *el migliaro del piombo impiastre val ducati* 24 : *che varranno lire* 9876 *de bronzo, che tengano per migliaro* 250 *de stagno e di rame* 643 : *abbattendo* dono *del stagno* 4 *per cento* : *e* tara *del rame* 10 *per* 1000 : *e* callo *del piombo* 12 *per* 1000 : *e di gabella* pesa, senseria, e *bastagi in tutto* 6 *per cento*.

Of these terms, *dono* may be interpreted a *gift* or *voluntary deduction*, where no waste took place; and *callo*, a *descent* or *allowance*, for diminution of bulk and weight which took place in the process of mixture. Tartaglia has confined the application of this term to denote the waste, in bulk and weight, which took place in *new* oil, as distinguished from *old*. Of the other terms which require interpreting, *pesa* was the allowance for weighing; and *sensaria* or *senseria*, was the *fee* to the *sensale*, or agent, by whose means the bargain was made.

Question proposed by De Burgo. (194.) The same author, in speaking *di viaggiis mercatoriis*, has given in an example an account of the various charges to which a mercantile adventure was subject, which is not without its interest, and particularly in connection with the subject of our present discussion. "I buy," says he, "for 1440 ducats at Venice 2400 sugar loaves, whose nett weight is 7200 lire; I pay for *sensaria* 2 per cent., to the *weighers* and *porters* (*bastagi*) on the whole, 2 ducats; I afterwards spend in *boxes*, *cords*, *canvass*, and in fees to the ordinary *packers* (*legatori*) on the whole, 8 ducats; for *messetaria* on the first amount, 1 ducat per cent.; afterwards for *duty* and *tax* (*datio e gabella*) at the office of exports, 3 ducats per cent.; for writing directions on the boxes and booking their passage, 1 ducat; for the bark to Rimini, 13 ducats; in compliments to the captains, and in drink for the crews of armed barks on several occasions, 2 ducats; in expenses for provisions for myself and servant for one month, 6 ducats; for expenses for several short journeys or *trajects* over land here and there, for barbers, for washing of linen and of boots, for myself and servant, 1 ducat; upon my arrival at Rimini, I pay to the captain of the port for port dues, in the money of that city, 3 lire; for porters, disembarkation on land, and carriage to the magazine (*magazen*,) 5 lire; as a tax upon entrance, 4 soldi per *load* (*callo*,) which are in number 32, such being the custom; *per fontecaggia a malatesta di marsilio*, *soldi* 4 *per callo*; upon my arrival at the fair, I find that 140 lire of weight are there equivalent to 100 at Venice, and that 4 *lire* of their silver coinage are equal to a ducat of gold. I ask, therefore, at how much I must sell a hundred lire Rimini, in order that I may gain 10 per cent. upon my whole adventure, and what is the sum which I must receive in Venetian money?"

History. The author may well say, that a merchant ought to have *his wits at home* (*cervello a casa*) who undertakes all the reductions and calculations which an adventure of this kind would require.

Particular imposts on pepper, cloves, &c. (195.) Particular species of goods appear to have been liable to other imposts. Pepper, which was sold by the *cargo*, (a weight of 400 lire,) as well as some other articles, paid a small tax for the support of a hospital for the poor. *Cloves*, (*garofali*,) which constituted a most important article of a traffic at that period confined to Venice, and which were of great value, (16 *grossi*, or ⅔ of a ducat *per lira*,) were subject to some peculiar regulations, which appear to have been very embarrassing to Venetian arithmeticians; they were usually mixed up with *fusti*, or *stalks*, of much less value than the seed itself, and 3 *sazzi*, out of the 72 which every pound contained, were allowed by law. As the quantity of them was commonly much greater, it became a question of some complexity to determine the proper deduction to be made. Tartaglia has mentioned, on more than one occasion, the error which existed in the common process for this purpose.

Questions on loss and gain. (196.) The IXth Book of the 1st part of the work of Tartaglia is chiefly occupied with ordinary questions on loss and gain per cent.; and on the conversion of the coins, weights, and measures, of one state or city into those corresponding to them in another, either directly or with certain limitations as to gain per cent. They contain nothing which is worthy of any particular notice.

The inverse rule of three. (197.) The rule of three *alla riversa*, or as it is called, in the older English writers on Arithmetic, the *backer* rule of three, consists in making the third term the divisor, which under the direct rule was the multiplier, and conversely. In one case, if the second term be *doubled*, the result, which is of the same species, is *doubled*; in this case, if the second term be doubled, the result is halved. This is a test of very easy application, and

will at once ascertain in any particular case which of the two rules must be used.

In what cases used.

Of this kind are all questions where a rectangular space is given, and the length and breadth are variable, or those in which the number of measures in a given heap of corn, or of any other quantity, is required; or where the time is required, in which different sums will produce a given interest; or questions relating to the assize of bread,* where the weight obtained for a given sum is required, the price of the same whole being variable.

Rule of five terms.

(198.) There are different methods of solving questions included under the rule of *five* or *more* terms, whether by successive statements, or by the combination of all the conditions into one. The following example is given by Tartaglia:

If 9 porters drink in 8 days 12 casks of wine, how many casks will serve 24 porters for 30 days?

Let us first suppose the time the same, and state the question as follows:

If 9 porters drink 12 casks of wine in 8 days, how many casks will serve 24 porters for the same time?

The answer is 32; and the second question will stand as follows:

If 24 porters drink 32 casks of wine in 8 days, how many casks will serve them for 30 days?

The answer is 120; which is, likewise, clearly that corresponding to the first question proposed.

The general principle of the other rules which are made use of by Tartaglia, may be stated as follows:

The quantity mentioned once is of the same nature with that which is sought, and is put in the second place. Of the other pairs of quantities, two are put in the first and last places, and two in the third and fourth, in the same order in which they occur in the question. Multiply the fifth, the fourth, and the second together, and divide their product by the product of the first and third: the quotient is the quantity required.

In those cases, where the inverse rule would apply to the simple statement of three terms, omitting all the other quantities mentioned *twice*, the second of the quantities must become a factor of the divisor, and the other a factor of the dividend.

Tartaglia usually puts the quantity mentioned once only in the last place but one, instead of the second. The rule may be very easily accommodated to suit this change in the arrangement.

The statement of the question proposed above, according to the principle of this rule, is as follows:

9 —— 12 —— 8 —— 30 —— 24

Divisor 9×8. Dividend $12 \times 30 \times 24$.

$$\text{Quotient } \frac{8640}{72} = 120.$$

* In all countries, the price of bread has been under the control of the magistrates, as it was always considered necessary to protect the people against the combinations or impositions of the bakers; for this purpose, however, it was necessary that the persons who formed the *tariffa*, or *table of prices*, should be good arithmeticians, as Tartaglia has shown that all his predecessors, including Lucas de Burgo, were mistaken in making the price of the *soldo* or *penny* loaf vary inversely as the price of the *staro* of wheat, without taking into consideration the constant expense which attended the process of making the bread: his attention was called to this subject, when requested by the magistrates of Verona to extend the *tariff*, in order that it might meet the high prices occasioned during a period of great scarcity.

History. Examples.

We shall subjoin a few of the most interesting questions which Tartaglia has given in illustration of this rule.

Twenty braccia of Brescia are equal to 24 braccia of Mantua, and 28 of Mantua to 30 of Rimini; what number of braccia of Brescia corresponds to 39 of Rimini?

Rimini. Mantua. Mantua. Brescia. Rimini.

30 —— 28 —— 26 —— 20 —— 39

21840   780:   Answer, 28.

The *lira* of Pisa is equivalent to 11 *oncie* of that of Florence, and the *lira* of Florence to 13 *oncie* of that of Perugia; what is the relation between the *lira* of Pisa and that of Perugia?

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Perugia. Fiorenza. Fiorenza. Pisa. Perugia.

13 —— 12 —— 11 —— 12 —— 12

1728

Answer: they are equal to each other.

Eight *soldi* of Venice are equal to 13 of Ferrara, and 15 of Ferrara are equal to 9 of Bologna, and 12 of Bologna are equal to 16 of Pisa, and 24 of Pisa are equal to 32 of Genoa; it is required to find what number of Venetian *soldi* correspond to 300 of Genoa.

Gen. Ven. Pis. Pis. Bol. Bol. Fer. Fer. Gen.

32 — 8 — 24 — 16 — 12 — 9 — 15 — 13 — 300

10368000   59904   Answer, $173\frac{1}{13}$.

Six eggs are worth 10 *danari*, and 12 *danari* are worth 4 thrushes, and 5 thrushes are worth 3 quails, and 8 quails are worth 4 pigeons, and 9 pigeons are worth 2 capons, and 6 capons are worth a *staro* of wheat: how many eggs are worth 4 *stara* of wheat?

960.

1 — 6 — 10 — 12 — 4 — 5 — 3 — 8 — 4 — 9 — 2 — 6 — 4

622080.   Answer, 648.

Other questions cannot be resolved by one statement; of this kind are the two following:

Ten excavators, (*guastatori*,) such as are usually employed in digging iron ore, can dig out 12 *carra* or *loads* of earth in 16 hours, whilst 12 other common excavators, less powerful than the former, dig out only 9 *loads* of earth in 15 hours; it is required to find in what time they will conjointly dig out 100 *loads* of earth?

The first question is, what quantity would the second set excavate in 16 hours, the time in which the first are engaged, which will be found to be $9\frac{3}{5}$ *loads*; the question is then reduced to the following:

If 22 excavators dig out $21\frac{3}{5}$ loads in 16 hours, in what time will they dig out 100?

$$21\tfrac{3}{5} — 16 — 100.$$

A gentleman going to the wars pays 360 ducats for 12 *carrette*, or waggons, with a pair of oxen each, whilst 5 other waggons without oxen cost him 40; it is re-

Arithmetic. quired to find the sum which he must pay for 60 oxen alone. The first question to be solved is this, if 5 waggons cost 40 ducats, what will 24 cost? The result, which is 96, being subtracted from 360, will give the charge for the oxen, when the remainder of the question is easily solved.

Different species of Italian partnerships and compagnies.

(199.) There is nothing more remarkable in the ancient commercial system of Italy, than the number, variety, and, in some cases, complexity of their compagnies or partnerships. The associations of different individuals for conducting mercantile concerns, which are too extensive for the superintendence of one person, or which require a larger capital than one individual can furnish, must take place in all commercial countries; but in Italy, others appear to have been formed for purposes merely temporary, for a particular adventure, with two or three persons, who contributed money, goods, or labour, sometimes one, and sometimes the other, and who divided the profits in the proportion of the capital advanced, the value of the goods furnished, or the wages of the labour employed in conducting the concern. Even in the most common affairs of life, they appear to have delighted in such associations; and the partnerships which were formed between landlord and farmer throughout Italy, have given a very peculiar character, not only to their relation to each other, but likewise to the whole of their agricultural system.

(200.) We shall mention a few only of the vast variety of questions on this subject which are given by Lucas de Burgo and Tartaglia, with such remarks as they may appear to require.

Examples.

Three persons form a company, the first of whom contributes 235 ducats, the second 430, and the third 520; and at the end of a certain time, they find that their capital and gain amount to 1732 ducats; what portion belongs to each?

Principle of solution.

This, and all similar questions are solved upon the common principle, that the sum of the capitals contributed by A, B, C, is to A's capital as the amount of capital and gain together is to the sum due to A.

A person has four creditors, to the first of whom he owes 624 ducats, to the second 546, to the third 492, and to the fourth 368: he fails and runs away, and his creditors find the amount of his whole property to be only 830 ducats; in what portions ought it to be divided amongst them? The principle of this question is the same as the last.

Three persons form a company, the first of whom contributes 300 *fiorini*, the second 600 *canne* of cloth, and the third 1200 *lire* of saffron; they gain 900 *fiorini*, of which the first receives 60, the second 360, and the third 380: what was the value of the *canna* of cloth and of the *lira* of *saffron*?

The *fiorino* was the primary coin of Florence, and under the name of *florin* became the general coin of the south of Germany; a circumstance easily accounted for, by the political connection between them.

Three companions are in a ship, one of whom has a butt of *malvasia*, which holds 36 barrels, (*barile*,) another one of Greek wine, which holds 24, and the third one of wine of Romania, which holds 40. By a violent movement of the ship the butts are upset, and the wine is spilt in the hold. The butts are afterwards replaced and filled with the mixture: what portion of each wine do they severally hold?

This question is solved on the general principle of the *regula societatis*.

History.

Three soldiers, or *adventurers*, form a company for the division of the spoil they shall gain in the wars; the first, being more practised than the second, says that he shall claim twice as much as the second, and the second, being more expert than the third, claims three times as much as the third, who submits to the terms; they gain 120 ducats; what is the share of each?

In the solution of this and similar questions, it is convenient to take numbers, such as 6, 3, 1, in the proportion of the respective shares.

Aventurieri.

In many other examples *de rebus militaribus* which are given both by Lucas de Burgo and Tartaglia, the term *soldier* and *aventuriere* are used as synonimous; the fact is, that in that age a *national* army was nearly unknown in Italy, the wars being chiefly carried on by *aventurieri*, who hired themselves to any party for a limited service. Tartaglia had good reason to know how much the horrors of war were increased, when carried on by men who looked for their reward in the plunder which arose from the sacking of towns, and the wasting of a country.

Four persons, a gentleman, an artisan, a barber, and a friar, make a pilgrimage in company, and spend 60 ducats; the barber agrees to pay 4 times as much as the friar, and 4 soldi more, the artisan 3 times as much as the barber, and 16 soldi more, and the gentleman twice as much as the artisan, and 10 soldi more; what portion was paid by each?

The ducats are converted into soldi, and the sum of 4, 28, and 66, are subtracted from 1200, leaving 1102; this is divided, as in the last example, in the proportion of the numbers 1, 4, 12, and 24; after which the actual portions are easily assigned.

A celebrated case of a will.

A man lying on his death-bed bequeathed his goods, which were worth 1200 ducats, in this sort: because his wife was great with child, and he yet uncertain whether the child were a male or female, he made his bequest conditionally, that if his wife bare a daughter, then should his wife have two-thirds of his goods, and his daughter one-third; but if she were delivered of a son, then should his wife have one-third, and his son two-thirds. Now it chanced her to bring forth both a son and a daughter, the question is, how shall they part the goods agreeably to the testator's will?

We have given Recorde's statement, with a few alterations, of a question which has become unusually celebrated from the time of Lucas de Burgo. The scholar in the dialogue is made to remark, that if some cunning lawyers had this matter in scanning, they would determine the testament to be void, as being insufficient. The master, however, "proceeds to try the work, not by the force of law, but by proportion geometrical, seeing the testator did minde to provide for each of them;" and as the intention was, that the son should have double of the mother, and the mother double of the daughter, the property must be distributed amongst them in the order of the numbers 4, 2, and 1.

Other cases.

Tartaglia has proposed many other similar questions where the intention can be only inferred. Amongst others, those in which a testator, from ignorance of the nature of fractions, directs a distribution of his property in fractional parts, the sum of which is greater or less than unity; thus, one gives $\frac{1}{2}$ of his property to his son, $\frac{1}{3}$ to his nephew, and $\frac{1}{4}$ to his niece. In such cases the property must be divided in the proportion of such fractions.

A person furnishes a shop with different goods by

Arithmetic. Fellowship with time.

means of a capital of 300 ducats, on the 1st of January, 1550; six months afterwards, one of his friends comes and offers, upon condition of being taken into partnership, to add 500 ducats to the capital, the division of the gain to be made in the conjoint proportion of the capital and time; at the end of December, 1550, they find the whole gain 260 ducats, what portion is due to each?

This is an example of fellowship with time.

Many other examples of compagnies are given by Tartaglia, which, properly speaking, require the aid of algebra for their solution; of this kind is the following:

Two persons form a company, on condition that the first should contribute 3000 *lire*, and the second 600 with his personal services, and that the first should receive $\frac{5}{8}$, and the second $\frac{3}{8}$ of the whole gain; the first, however, adds 400 *fiorini* to his first capital, and in consequence receives $\frac{2}{3}$ of the gain, whilst the second gets only $\frac{1}{3}$; what is the relative value of the *florin* and the *lira*?

Alleged error of Lucas de Burgo.

Lucas de Burgo solves this question on the principle, that the consideration for personal services should be the same in both cases, or, in other words, that they should be considered as equivalent to the same capital; and that, consequently, the value of the *florin*, as determined from the question, should be $1\frac{1}{2}$ *lire*. Tartaglia considers this principle as erroneous, and contrary to the spirit of the agreement, by which the value of the personal services should increase in proportion to the increase of the joint capital; if the question be solved with this view of its meaning, the result would give the *florin* equal to $3\frac{1}{2}$ *lire*.

Many other questions of a similar nature had been solved by Lucas de Burgo, Piero Borgin of Venice, and particularly by Giovanni Sfortunati of Sienna, upon the first principle, and the error whether real or alleged, is pointed out with great detail by Tartaglia; he seems, indeed, to have experienced a peculiar satisfaction in finding out the faults of his predecessors, and he rarely omits an occasion of doing so, particularly in the case of his predecessor, Pacioli, who attempted the solution of many questions upon erroneous principles, or by methods which were insufficient for the purpose. The phrase, *mai falla*, which he so often uses with reference to his processes, must be admitted with extreme caution, being most frequently used when he is most liable to be deceived.

Frequent errors of Lucas de Burgo.

Soccide de bestiami.

(201.) There was another class of compagnies, termed in the provincial language of the north of Italy, *sozzidi*, or *soccide di bestiami*, which were so common, and which lead to so many very complicated questions, that they always formed the subject of a distinct chapter in Italian books of Arithmetic. They arose from the poverty of the farmers, who would not stock their farms from their own funds, and, in many cases, could not even buy the corn which was necessary for seed; the consequence was, that the landlords generally, and in some cases other persons, provided the whole or the greatest part of the stock, and entered into an engagement with the farmer to divide with him its whole produce at the end of 3, 4, or 5 years, or to divide in certain proportions with him the profits which occurred in the mean time, and the whole stock which remained at the conclusion of the *sozzido*.

"Whoever wishes to support himself in this world of misery, must govern and guide his life in the path of sweating industry; this man employs his money in merchandise, that man in trade; and amongst other laudable species of industry which we every day witness, we find some men who provide the means of life by the aid of brute animals; and this not by violence, provided it be exercised in the proper mode and in charity, according to the injunctions of the holy Scriptures, which say *In caritate sudate et radicate.*" Such is the preface with which Lucas de Burgo introduces the notice of these associations of the rich and the poor, which he says were peculiarly liable to imposition and fraud, and that, in consequence, it was highly dangerous to extend such agreements beyond 3, 4, or, at most, 5 years; and, in every case, he recommends them to be formed under the inspection and control of the bishop of the diocese, since *con tale consiglio salutifero raro si erra*; though we might very reasonably doubt, whether the prelates of Italy, or of any other country, either in that age or the present,* were the persons best calculated to regulate the terms, or to enforce the fulfilment of such bargains. We will subjoin a few examples of questions, which frequently arose out of the formation of such compagnies.

History. Observations of Lucas de Burgo.

Examples.

A person gives a shepherd *in sozzido* 720 sheep, to keep them and their produce for 5 years, and at the end of that period to divide equally with him the profit and the capital; at the end of 3 years and 8 months the shepherd dies, and his wife, who has no confidential person to manage the concern, (her son not being of sufficient age,) is compelled, with the consent of the principal, to terminate the *sozzido*: the number of sheep is found to be 1060; what number will each party receive?

If the contract had been completed, the widow would have claimed 530; the number now due to her will be to 530 in the ratio of $3\frac{2}{3}$ to 5.

A person gives *in sozzido* 24 cows, and the herdsman adds 6 to the number, to keep them for five years, and then to divide the capital and profits equally; at the end of 3 years and 4 months they agree to terminate their contract, when they find 80 head of cattle; what portion belongs to each?

A citizen gives *in sozzido* 18 sheep to a shepherd, who agrees to add 6 to their number, upon condition of dividing the whole equally at the end of four years; the contract being made, the shepherd returns home, and finds that the wolves have eaten two of his sheep, and he has, therefore, only 4 to add to the number which he receives from the citizen; at the end of three years they agree to divide the *sozzido*, and find that they have 66 sheep; what number must each receive?

Different species of barter.

(202.) The Italians distinguished three distinct species of *barratti*, or barters. The first, *simple*, where goods were exchanged against each other at their *ready money*, or *barter* price; the second, *compound*, where the exchange was partly in goods and partly in ready money; and the third, *barter with time*, where the barter price is affected by the time at which the payments, whether real or imaginary, are to be made; in this, as well as in every other department of their commerce, they appear to have been fond of engagements involving the

* In modern Tuscany, the landlord furnishes stock, seed, and implements of husbandry, and divides the produce equally with the tenant: the case is somewhat different in Lombardy, where the farms are large, and where, in consequence, the agricultural population is in the possession of much greater wealth.

Arithmetic.

most complex relations, trusting to their own dexterity in the management of such bargains, and relying upon the skill of their professional Arithmeticians for the resolution of questions, to which the majority of them must have been altogether unequal.

Frequency of frauds.

(203.) So frequent were the frauds which occurred in these transactions, either in the articles not corresponding to their samples, or in fixing the difference in the barter price and the price *a danari contadi*, or for ready money, or between the price for ready money and for time, that it became a proverbial saying, that one of the parties in a *barratto* was *imbratto*, *cheated*, or, more literally, *dirtied*. It was the custom, also, according to Lucas de Burgo, when the *sensaro*, or *agent*, showed bad articles for barter, to ask him if he gave a dowry with them, in allusion, says he, to the manner in which marriages are contracted in those days; for whilst beautiful and accomplished ladies are taken from their fathers houses almost pennyless, the ugly and ill-favoured are recommended by large dowries, a quality which never fails to procure a husband in this age of avarice, in defiance of the proverb, which says, *Ne per bo ne per vacca non taglia donna matta, la robba va e vene e chi a la moglia matta se la tene.*

Examples.

(204.) We will add a few examples of the different species of barter, which frequently lead to questions of a very difficult and embarrassing nature.

Two persons wish to barter, the one wax, which sells at $8\frac{1}{2}$ ducats per hundred pounds, whilst the other has wool, of which the same quantity sells for $39\frac{1}{2}$ ducats; how much wax must be given for 756 pounds of wool?

Two persons barter ginger and soap; the hundred pounds weight of the first is worth 16 ducats for ready money, and 18 for barter; the second is worth 22 ducats for the thousand pounds, for ready money; if the first pays for one-half of what he gets in ready money, what must be give in money and ginger for 7890 pounds of soap, so that the terms of the barter may be equal on both sides?

Two persons barter, the one wool, the other pepper and ginger; the hundred weight of pepper is estimated for ready money at 30 ducats, and for barter at 35; the hundred weight of ginger is estimated at 27 ducats for ready money, and for barter at 33; the hundred weight of wool is worth 10 ducats: at what price must the wool be estimated at barter, to receive an equal quantity of pepper and of ginger, and to gain 10 per cent. upon the capital?

A merchant sells to another a quantity of scarlet cloth at 6 ducats the *braccio*, if paid for at the end of 8 months, but the price for ready money is only $4\frac{1}{2}$ ducats; afterwards the first buys of the second a quantity of ginger for 15 ducats the hundred weight, payable in 10 months; the excess of the time above the ready money price, in proportion to the time, being the same as in the case of the cloth; what is the ready money price of the ginger?

Interest simple and compound.

(205.) The importance of the knowledge of the principles of simple and compound interest, discount, annuities, &c., with the proper rules for their calculation in mercantile and other transactions, is so great, that we may naturally expect to find the discussion of them occupy a considerable portion of all books of Arithmetic. The rules for such calculations, however, are founded upon algebraical formulæ, and for the most part, involve relations of quantities much too complicated for any merely arithmetical investigation; under such circumstances, the questions proposed rarely extend beyond the more common cases, such as simple interest, discount, the ordinary cases of compound interest and discount, and the determination of the value of temporary annuities.

History.

Usury considered disgraceful.

(206.) When the excessive interest which was charged for the use of money, in those countries where commerce had not accumulated capital, is considered, it is not very surprising that the popular indignation and prejudice should be directed against usurers. Under the Mosaic Law, this prejudice received a much higher sanction, and domestic usury was not merely discouraged, but forbidden: and in modern Europe, it was long before the same law, which was obligatory upon Jews towards Jews, ceased to be considered as not extending to the members of the new covenant; at all events, religious feelings and the denunciations of the church came partially in aid of those which were natural and hereditary. The practice of usury, indeed, during the middle ages was so universally odious, that it was confined to that race of men, who by a singular revolution had succeeded to the exclusive exercise of a traffic which had been forbidden to their forefathers; nor did this feeling cease to exist even in countries and cities where the conveniences of an extensive commerce rendered it, in some measure, necessary. It was, of course, recognised in the transactions of merchants with each other; and money, time, and the consideration for the delayed and anticipated payment of money, formed an important element in all purchases and sales: but when money was directly borrowed, not in the course of trade, it was commonly from a Jew; and our own Shakspeare has correctly represented the feelings with which such transactions were regarded: nay, even as late as in 1567, Cataneo, an arithmetican who resided in Venice, prefaces the chapter in his work which relates to interest and discount with the following terms: *Se quelli che alla poltronescha usura si danno di tal mestiero non si vergognano, manco mi debbo vergognare io d'insignare quanto debbi pagare quel pover disperato, che a tali diabolichi patti s'obbliga.*

Interest of money at Venice, in the 16th century.

(207.) Interest in Venice at the beginning of the XVIth century varied from 5 to 12 per cent. per annum: in commercial transactions a much higher interest was calculated upon, or rather a much greater consideration paid in the difference which existed between the ready money and time price of goods; but when money was lent or borrowed, upon good security, whether from Jew or Christian, it rarely exceeded the last sum which we have mentioned; it appears to have been estimated in very different ways; sometimes at so much per cent. by the year or the month, sometimes at so many *danari*, on each *lira per mensem*, and sometimes at so many on the 100 *lire per diem*: it is evident that these different customs must have materially increased the complexity of the rules for the calculation of interest.

Definitions of simple and compound interest.

(208.) Simple interest is that in which no interest springs from the interest, and compound interest, or *merito a capo d'anno*,* that in which the interest is reckoned upon the arrears of interest; it was the second species only which was properly called *usura*, and was rarely practised in the transactions of merchants with each other. Stevinus terms compound interest, *interest prouffitable*, or *celuy qu'on ajouste au capital*, whilst the corresponding discount is termed *interest dommageable*, or *celuy qu'on soubstrait du capital*.

* *Meritasi simplicemente quando dal merito merito non nasce.*

Different days fixed for the commencement of the mercantile year.

(209.) The solution of questions of simple interest and discount readily reduce themselves to the ordinary cases of the rule of the three; and there is nothing in the methods which are used for this purpose by Tartaglia, or his predecessors, which is particularly worthy of notice. In calculating the interest of a sum from one day to another, whether of the same or different years, the determination of the number of mooths or days in the interval was in some degree embarrassing; and Tartaglia is proud of a rule which he has given for this purpose. In passing from one city of Italy to another, an additional source of embarrassment presented itself, in the different days on which the year was supposed to commence: being reckoned at Venice from the 1st of March; at Florence, from the annunciation of the Virgin; and in most other cities of Italy, in obedieuce to the orders of the church, from Christmas day.

Balancing accounts and equation of payments.

(210.) In a running account between two merchants, involving sums borrowed and paid at different times, upon which simple interest, for the most part 12 per cent., was allowed, it was important on particular days to *balance* their accouots, a process which was denominated *saldare una ragione:* such adjustmeuts appear to have been very frequently repeated, in perfect consistency with those habits of formal punctuality for which the Italian merchants were so remarkable. In such cases, the interest upon the several sums, on the debtor and creditor side of each account, was calculated up to the giveo day, and the difference of the sum on each side, if any remained, was passed in one sum to the proper side of the ledger. Another process, also of very frequent occurrence, was to calculate the equated time of payment of sums due at different periods, a process called *recare (li pagamenti) a un dì.* It consisted in multiplying each sum by the time before it was due, and dividing their sum by the sum of the several payments; this rule, which is the one commonly used at this time, confouods interest with discouot, and excludes, of course, all consideration of compouod interest. Tartaglia was fully aware, that the priociple of this rule was erroneous; but the principles of algebra were in that age too imperfect to give the correct solution, or, at all events, to give the correct interpretation to it.

Questions on interest proposed to Tartaglia.

(211.) Tartaglia has given some examples of cases, chiefly of annuities, which were proposed to him professionally; the first, which is the following, was proposed by a Jew at Venice, on the 14th of April, 1550.

A person owes me 450 ducats, payable by 9 ducats a month for 50 months, and wishes to pay the whole at once to another person, who undertakes to discharge the debt; what sum must he pay, supposing ioterest be allowed at the rate of $9\frac{1}{2}$ per cent?

He finds the equated time of payment by the ordinary rule, which is $25\frac{1}{2}$ months, and then discounts 450 ducats, payable at the expiration of that time: the answer is 374 ducats, 9 *grossi*, and 30[illegible] *piccoli*.

A certaio *maestro da Barri* proposed the following question:

I lend a certain *university* 2814 ducats, on condition of receiving an annuity of 618 ducats for 9 years; what interest do I gain upon my money, the ducat being estimated at 10 *carlini*, and the *carlino* at 10 *grani*?

The answer, determined upon the same priociples, is 19 ducats, 5 *carlini*, and 3[illegible] *grani*.

Nothing can be more unjust and erroneous io principle than this mode of calculating annuities, particularly for a long term. Such questions were considered, indeed, as peculiarly difficult and embarrassing; and Tartaglia has mentioned several others of a similar nature at the conclusioo of his algehra.

Rules for calculating compouod interest.

(212.) Tartaglia has noticed five methods of finding the amount of a sum of mooey at compound interest. Suppose the question to be, to find the amount of *L*300. for 4 years at 10 per cent. *a capo d'anno*; the first is by the following four statements:

$$
\begin{array}{lllllllll}
100 & : & 300 & :: & 110 & : & 330 \\
100 & : & 330 & :: & 110 & : & 363 \\
100 & : & 363 & :: & 110 & : & 399\frac{3}{10} \\
100 & : & 399\frac{3}{10} & : & 110 & : & 439\frac{23}{100}
\end{array}
$$

The second merely replaces 100 and 110 by 10 and 11 in the proportion: the third, which is his own method, multiplies 300 four times successively by 11, and divides the last product by 10000: the fourth consists in adding four successive teoths to the principal: the last, in calculating the amount for *L*100., and then finding the amount for *L*300., or any other proposed sum by a simple proportioo. The last four methods are obvious consequences of the first, and with the exception of the last, are not readily applicable, unless the ioterest per cent. be an aliquot part of 100.

Discount and annuities.

(213.) With the exception of discount at compound interest, (*sconto a capo d'anno*,) and its application to correct in part the conclusions respecting the values of annuities, there are few, if any, other questions of compound interest which Tartaglia and his contemporaries can be said to have resolved.

A disputed case.

A very natural difficulty arose in the solution of questioos of this kind: "what is the interest of 100 for 6 months, interest being reckoned at the rate of 20 per ceot. per annum." Lucas de Burgo, Giovanni Sfortunati, and others, made out that this would be 10: io other words, they calculated that simple interest only being allowed, it was a matter of indifference into how many portions of time the whole period was divided, whether into months or half years; the conclusion, under such a view of the case, is correct, and merely proves the injustice of the very principle of simple interest in all cases which are prospective at least, if not in those which are past.

Tables of interest used in Italy.

(214.) Lucas de Burgo has an article entitled *Del modo a sapere componere le tavole del merito*; and he enlarges upon the great utility of such tables for saving the trouble of calculation, and says, that they usually embraced a period of 20 years, commencing with 5 per ceot., the lowest interest which could be imagined to be taken. This statement is sufficient to prove the existence of such tables among the Italians, though we are not aware of any work in which they are given.

Tables of Stevinus.

The first compound interest tables with which we are acquainted, are those which are given by Stevinus in his *Arithmetic*; they give the present worth of 10000000 from one to thirty years, in 16 tables, the ioterest being reckoned successively from 1 to 16 per cent., and in 8 other tables, where the interest is differently reckoned, according to the custom of Flanders, as one *denier* in 15, 16, 17, 18, 19, 20 (5 per cent.), 21, and 22. There are two colomns in each table, one giving the present worths above mentioned, and the other the values of annuities of 10000000 for the same period, which are, therefore, the sums of the numbers in the first column. The idea of the research of proportional numbers, for the solution of questions of interest nod annuities, was suggested by the tables of

Arithmetic. sines, &c. commencing from a radius of 10000000, and was one of many happy extensions of a common principle, which were made by this singularly acute and original author.

Bills of exchange. (215.) It is extremely difficult to establish from historical documents the absolnte antiquity of the use of bills of exchange, or to ascertain the country where, or rather the places between which, they first circulated. They are themselves documents of n very perishable nature; and the only methods by which we are likely to be able to trace their existeace, must be from their connection in some cases with historical transactions, or from their appearance in legal records of disputes which arose out of them; of the first kind is the very curious account given by Matthew Paris, which Macpherson has quoted,* of the attempt made by the Pope, in 1255, to depose Manfred, King of Sicily, and to place upon bis throne Edmund, the second son of our Henry III., upon condition of being remunerated for the expenses which he incurred; upon the faith of this promise, large sums of money were advanced to the Pope by merchants of Florence and Sienna, who were repaid upon the failure of the enterprise by bille drawn, at the suggestion of Henry himself, upon the prelates of England, who were compelled to pey them with *interest*, notwithstanding their protests, from apprehension of being subjected to a sentence of excommunication.

Used in the time of Henry III.

Different origins assigned to them. (216.) This very remarkable transaction would appear to prove that the use of bills of exchange was perfectly well known to the Italian merchants of that age, though it is probable that the date of their origin is much earlier. Savary, in his *Negociant Parfait*, and in his *Dictionnaire du Commerce*, says, that they were invented by the Jews who were expelled from France at different periods nnder Dagobert in 640, Philip the Long in 1180, and Philip Augustus in 1316, and who evailed themselves of bills of exchange to withdraw their property from France. At another period, also, when the Gebelins were expelled by the Guelphs, some Lombards took refuge in Amsterdam, and recovered their property by the same means. These facts, however, are not supported by any very satisfactory historical evidence; it is certain, indeed, that the Lombards, for the purpose of the very extensive commerce of Italy, were dispersed over every country in Europe, where they established themselves as merchants, money-changers, and baokers. Our own Lombard-street, which still retains its appropriate traffic, is a proof of their presence in nur own country; and the Exchange of Amsterdam was long known by the name of the *Place Lombarde*, from similar associations.

Essential to international commerce. (217.) It is not very easy, indeed, to imagine in what manner a very extensive international commerce could be carried on without the assistance of bills of exchange. Though the balance of trade might disappear in the intercourse of nations with each other, this could rarely be the case in the transactions of individual merchants; we may suppose, therefore, two merchants, A and B, at Venice, and two others, C and D, at Alexandria; A owes C the same sum that D owes B; instead of A sending specie to C, and D again to B, it would save all parties both risk and expense if A should pay the money immediately to B, and receive in return an order, or bill of exchange, which he would transmit to C, to enable him to receive the money from D, by which the accounts of the two parties would be cleared; such a process as this would be pointed out by the common sense of mankind, and the whole theory of exchanges does not require a much broader basis for its foundation.

Their principle.

History.

Not noticed in the Roman law. (218.) There is no notice of bills of exchange, or of any thing equivalent to them in the Code of Justinian, and it has been inferred from thence that they were unknown to the Romans, inasmuch as transactions conducted by means of them, are those which of all others require the most frequent control and regulation of the law, and they could not, therefore, have existed, at least to any extent, without its notice and interference. We must allow this circumstance great force as a negative argument, notwithstanding the authority of the passage of one of the letters of Cicero to Atticus, (xii. 24,) when making inqniries concerning his son's journey to Athens, and the supply of money which would be requisite for him; *permutari ne possit an ipsi ferendum?* The *permutatio* alluded to must have been equivalent in substance at least, if not in form, with a bill or letter of exchange, and it appears from e subsequent letter, (xii. 27,) that such was the expedient which was adopted.

Their use not altogether unknown.

The use of them considered by some as usury. (219.) Lucas de Burgo, who was duly impressed with a sense of the great importance of commerce to the wealth and power of a state, complains, that in his time it was the custom of many persons to murmur against those who dealt in bills of exchange, calling them *usurers* and worse than Jews; in his opinion, however, the inventors of them deserve a blessing with a hundred hands, as without them the very foundation of all that beneficial commerce would be destroyed, which was essential to the support of the Repnblic. It is true, indeed, that exchanges were sometimes practised in a manner which was neither commendahle with God nor man; but this observation could never be applied to their legitimate use in the general transactions of commerce.

Different species of cambio or exchanges. (220.) *Cambio*, according to the same author, might be explained generally by the popular phrase *to e da qua;* that is, *togli da me questo e da me tu questo altro, take this from me and give me that in return.* Four species of exchange ere noticed by Italian writers, which are *cambio menuto o commune, reale, secco e fittitio;* the first of them, *minute*, or *ordinary* exchange, is that in which gold or silver coin is given in exchange for other coins of different species or denominations, where the banker or money dealer retalns a small consideration for his trouble; an allowance, so far from being usurious and improper, that it is approved of by the most celebrated theologians and doctors of the church, and amongst them by Remond Raimondo, Thomas Aquinas, and, above all, by the most sacred doctor of our own order, says Pacioli, Ricardus Mediavillensis.

Cambio menuto.

Cambio reale. The second, or *real exchange*, is of all others the most important, being the very "water upon which the vessel of commerce floats," and is carried on by means of letters or bills of exchange, which have preserved very nearly the same form for four centuries at least, if not for a much longer period. We will give specimens of such letters of exchange as were drawn in the years 1404, 1494, and 1553.

Specimens of bills of exchange. 1. *Francisco da Prato et comp. a Barselona. Al nome di Dio, Amen, a. d' XXIII Aprile,* 1404.

Pagate per questa prima de camb. a usanza a Piero Gilberto e Piero Olivo scuti mille a sold. X. Barselonesi per scuto, e quali scuti mille sono per cambio che con

* *Annals of Commerce*, vol. i. p. 405.

Arithmetic. *Giovanni Columbo a grossi XXII. di g. scuto: et pag. a nostro conto et Christo vi guardi.*

Antonio Quarti Sali de Bruggias.

This is a bill of exchange which is given by Capmany* in his history of the town and commerce of Barcelona; it was found amongst the records of a reference made by magistrates of Bruges to those of Barcelona, respecting the practice which they followed in the case of bills of exchange which had been *protested*, and upon which undue charges had been made in its transit from the *drawer* to the *drawee*, and which in consequence the drawer refused to pay.

2. *Domino Alphano de Alphanis e compagni in Perosecia.*

1494. *a. d[i] 9 Agosto.*

Pagate per questa prima nostra a Ludovico de Francesco da Fabriano e compagni unce cento d'oro Napolitane in su la proxima fiera di Fuligni per la valuta d'altretante recevute qui dal magnifico homo meser Donato da Leggi quondum meser Priamo. E pone le per noi. Iddio da mal vi guardi.

Vostro Paganino da Paganini da Brescia.

This is a form given by Lucas de Burgo.

3. *A meser Ricardo Ventworth gentilhuomo Inglese in Londra.*

1553 *a. d[i] 4 Ottobrio in Venetia.*

A uso pagareti per questa prima, a meser Giovan da Mora delle presente latore lire vinticinque e soldi sedici de sterlini per la valuta de altri tanti per lui medesimo qua consignata e poneteli a vostro conto che Christo vi conservi secondo il desiderio vostro.

Andrea Dolphino dal bancho vostro servitor.

This form is given by Tartaglia, and is addressed to his friend, pupil, and patron, to whom the first part of his work is dedicated.

Observations upon them.

With respect to the form and wording of these bills, very few remarks are necessary. The debtor and creditor side of an account are always designated by *per* and *a*. *Uso*, or *usanza*, means the customary time in different cities between the acceptance and payment of the bill, varying from ten days to three or four months, according to their distance or the facility of communication; the first of these bills is remarkable, as furnishing an example of a bill drawn in Italian at Bruges for acceptance in Spain, a proof that it had become the universal language of commerce. The laws of all commercial towns gave extraordinary power to the holder of a *protested* bill, which had been refused acceptance, or payment, by the *drawee*, upon the person and goods of the drawer; and the consequence was, that such bills were considered the best of all securities for a debt which was not real; this circumstance, and the wish to evade the denunciations of the church against the practice of usury, will account for the origin of the other two species of exchange, which we shall now proceed to notice.

Cambio secco.

(221.) A wishes to borrow 300 ducats of B; B selects a place, Lyons for instance, where the exchange, from the balance of trade at that period of the year, is very low, say 60 ducats for a mark of gold; B receives a bill of exchange directed to an *imaginary* person at Lyons, directing him to pay 5 marks to the holder, at the rate of exchange at the fair of All Saints, when he knows that it is the highest, say 75 ducats; the bill is of course protested, returned, and A must pay B 345 ducats, with all the charges incurred; such exchange was called *cambio secco*, and was clearly a method of avoiding the penalties and discredit of usury.

History.

Practised in a certain sense by the farmers of Italy.

Tartaglia has illustrated this species of exchange by a practice which was very common in Italy, and which places the poverty of the farmers in a very striking light; the interval between harvest and seed-time in that country is very considerable, and it generally happened that the farmers were compelled by their poverty to dispose of the whole of their produce before that period arrived; the consequence of this forced market was, that the price of corn was very low immediately after, and very high immediately before the harvest; under these circumstances they were compelled to borrow the seed-corn upon condition of replacing an equal quantity, or paying the price of it in the month of May. The cases are clearly analogous, and show, in a very remarkable manner, the inconveniences occasioned by any interference with the regular trade in money, and the extraordinary expedients which were commonly resorted to for the purpose of gaining an exorbitant interest, which could not have been the case had moderate usury been sanctioned by custom or by law.

Cambio fittitio.

(222.) *Cambio fittitio*, called by the French *change feint*, or *adulterin*, when A sells goods to B for time on this condition, that in case the payment is not made when due, he shall be repaid by a bill of exchange, as in *cambio secco*, reserving to himself the choice of place and time. It is hardly necessary to observe, that such practices were of the kind which Pacioli characterises as commendable in the sight of neither God nor man.

The more rapid and secure communication which takes place between different places in modern times, and the many channels through which bullion may be transmitted, have materially lessened those extreme fluctuations in the course of exchange, which were formerly so common and so certain, and in which these *fictitious* exchanges originated.

The preceding account of the terms used in exchanges, which occur so frequently in Italian and other books of Arithmetic, is all that is requisite for our present history; we dare not venture upon their modern use, history, and, still less, theory, a subject of vast extent and difficulty; and we shall proceed, therefore, to the notice of another subject of purely Italian origin, the method of Book-keeping by Double Entry.

Italian book-keeping.

(223.) This method of book-keeping has been explained in great detail, in a distinct chapter by Lucas de Burgo, and is certainly one of the most refined inventions which could be devised to prevent the confusion which would otherwise arise in the registering of complicated mercantile transactions; and though some improvements have been introduced in later times, as far as regards brevity and compactness, yet in all essential points the system remains unchanged. A few words will be sufficient to explain the general principle of this method, particularly as distinguished from the more obvious method of recording accounts, which is called Book-keeping by Single Entry.

Principle of book-keeping by single entry.

(224.) In the latter of these methods, there is merely required a memorial of occurrences in the order of time, with a ledger in which the names of all the parties between whom transactions take place are entered, with an alphabetical index of reference; the debtor and creditor accounts of each party being arranged on the two

* Beckmann's *History of Inventions*; the same work contains a custom-house *tariff* for 1221, and also a decree of the council of Barcelona, dated 1394, ordering all bills of exchange to be accepted within 24 hours of their being presented.

opposite pages, which are presented at one opening, the first on the right hand and the second on the left; there is only one entry of each transaction, which is either debtor or creditor; such a method enables us to balance the accounts of each party, but presents no register by which the state of the stock in trade and the balances of capital and cash can be at once ascertained, without a separate and independent investigation.

Principle of book-keeping by double entry.

(225.) In book-keeping by double entry, three books are required, the *waste book* or *memorial*, the *journal*, and the *ledger*. This method differs from the former chiefly in making cash, stock, goods, &c., parties as well as persons, and in making a *debtor and creditor account* in every transaction; thus, if cloth is sold to A, A is made debtor to cloth, and cloth creditor to A; if cash is received from B, cash is made debtor to B, and B creditor to cash; and in every case the party, whether animate or inanimate, which receives is debtor to that which pays, and conversely. A *double entry* is, therefore, requisite in every transaction, and a balance may at any time be struck between things as well as persons; and in order to avoid the confusion which would arise in a direct transfer of accounts from the memorial to the ledger, before the proper relation of debtor and creditor in each transaction are distinctly ascertained and recorded, they are first entered in the order of time in the *journal*, in the same form in which they must appear in the ledger.

Qualifications of a merchant, according to De Burgo.

(226.) Lucas de Burgo prefaces his account of Italian book-keeping by an enumeration of the proper qualifications and qualities of a merchant. As he had passed the greatest part of his life in a city of noble merchants, and saw at the head of the government of his own country a family which had risen by commerce, it is very natural that he should have entertained the highest respect for a character and profession which not only led to wealth but to public honours; so high, indeed, was the general estimation of the merchants of Italy for honour and integrity, that the simple affirmation *a la fè d'un real mercatante*, or *by the faith of a true merchant*, was considered one of the most solemn that could be made; and so numerous were the accomplishments which were deemed necessary for him to possess, that it became a common and proverbial saying, "that it required more points to make a good merchant than to make a doctor of laws." Considering, indeed, the various accidents and dangers to which he is exposed by sea and land, in times of peace and plenty, of war and scarcity, of pestilence and disease, and on so many other occasions, if he possessed, like Argus, a hundred eyes, they would not be sufficient. His proper emblem is the cock, that watcheth by night and by day, in summer and in winter; so watchful and so constant ought his vigilance to be, always remembering the maxim of the laws, *vigilantibus et non dormientibus subveniunt jura*, as well as the declarations of the holy church and of Scriptures, that the *crown is promised to him that watcheth*. He should fear no fatigue, uniting with his labour the practice of piety and charity, trusting to the truth of the adage, *nec caritas opes, nec missa minuit iter*; to all these moral qualifications, on which the good old monk enlarges with such apparent delight, it is requisite that he should unite others of a more worldly nature; he must possess a sufficient capital in money or in goods; be a ready and expert reckoner; and possess the power of registering all his transactions in a clear and beautiful order, so that he may at once become acquainted with them by reference to his books; for the proverb which says, *ubi non est ordo ibi est confusio*, which is true on all other occasions, is more particularly so in the case of mercantile affairs.

Inventario.

(227.) Of the books which are requisite for a merchant, the first is the *inventario*, or *inventory* of all his possessions and goods of every description. The following is a specimen of the mode in which it was headed: *In the name of God, on the 8th of November, 1494, at Venice. Here follows the inventory of me* M. N. *of the street of the Holy Apostle, written with my own hand, of all my goods, moveable or immoveable, debts, credits, &c. which I possess in the world on this present day.* It then proceeds to enumerate, with the utmost minuteness, all his money in gold and silver, in coins of different descriptions, lands, houses, gardens, orchards, *sozzide de bestiami*, stock of all kinds, debts, credits, bills of exchange, &c. It was sometimes usual to copy the heads of this inventory into other books, which were used in the conduct of mercantile affairs, which we shall now proceed to notice.

Memoriale.

(228.) There are three books which were necessary for this purpose, the *memoriale*, *giornale*, and *quaderno*; the first, called sometimes *vacchetta*, *squartafoglio*, or *squartafaccia*, *little cow*, *crooked leaf*, or *crooked face*, from its rumpled appearance when old, corresponds to the *waste book* of our merchants, and contained an account of all transactions in the order of time, particularizing *el chi*, *el che*, *el quando*, *el dove*, *the whom*, *the what*, *the when*, *the where*, in the most minute manner, so that not an *iota* of the transaction may be omitted which may be requisite to make it fully understood; inasmuch as *al mercante le chiarezze mai furon troppo*, *a merchant cannot have too many explanations which tend to give greater clearness.*

Giornale.

(229.) The second book was the *giornale*, where the transactions are entered from the *memoriale* in the order of time, and arranged in the form of debtor and creditor, preparatory to their being copied into the *quaderno*; debtor is signified by *per*, and creditor by *A*; and the two entries with reference to them are separated by two lines, thus ||. There are two terms which are of frequent occurrence in these entries, *cassa* and *cavedale*, which it may be requisite to explain; the first, which was transferred from designating the *money box* to its contents, corresponds to our own term *cash*, and denotes the stock of money in hand; the second must be translated *stock*, and denotes the whole stock in trade, (*monte e corpo di faculta o di tutto il trafico*.) The first in Italian book-keeping, properly so called, was never made *creditor*, the second never *debtor*, contrary to the usage of modern times.

Cassa and cavedale.

Quaderno.

(230.) The last and most important book was the *quaderno*, or *ledger*, into which the entries of the *giornale* were transferred in the names or designations of the several parties, whether animate or inanimate, there being always two entries for each transaction, one *per* and the other *A*. It commenced with the *alfabeto*, *repertorio* or *trovarello*, called in Tuscan *stratto*, and was ruled with as many vertical lines as were requisite to contain the different denominations of money or goods which were required to be registered; the first page contained the *cash account*; when *stock* was debtor, the general term *cavedale* was used; when creditor, the entry took place under the head of the particular goods which were concerned in the transaction; the *milesimo*, or date of

the year, was put at the top of each page; the month and day in each separate entry.

The same transactions were recorded in the same *memoriale*, *giornale*, and *quaderno*; and to denote their connection with each other, they were all signed with the same letters, A, B, &c. The first set of these books, however, were generally marked with the sign of the cross; *that glorious sign from which all our spiritual enemies fly, and at the sight of which the whole host of hell most justly trembles*; and were called *memoriale croci*, *giornale croci*, and *quaderno croci*. In some places it was customary to authenticate the *memoriale* before proper officers appointed for that purpose; a most laudable and excellent practice, well calculated to prevent disputes and frauds, as the authenticity of the other books must be determined from it.

Modes of entering accounts of different classes of transactions.

(231.) The author then proceeds to explain the mode of recording and entering the accounts of different transactions, whether *baratti*, of all their different species, whether simple, compound or for time; *compagne*, whether personal, or what the French call *en commandite*, where money alone or goods are contributed; *conti di botega*, or accounts of traffic in detail, whether conducted in person or intrusted to another; *accounts with banks*, which were then established in Venice, Genoa, Bruges, Antwerp, and Barcelona; of *mercantile journeys* or *voyages*, where separate books must be kept, the principal ones being left at home; of *bills of exchange and transactions connected with them*, with the notice of the expense incurred in the salaries of factors and servants, in the ordinary maintenance of the household, as well as of extraordinary expenses incurred for gaming, pastimes, amusements, and pleasures of different kinds, which are not properly included under any kind of ordinary expenditure.

Striking a balance.

(232.) Various directions are likewise given about the mode of striking *a balance*, whether general or particular, and of transferring the accounts from one ledger to another; as also of extracting a balance sheet containing the *summa summarum*. Every merchant is likewise recommended to keep *un libro de pagamenti*, or *book of payments*; *un libro de recordanze*, or *memorandum book*; and likewise to copy into a separate book all letters, whether received or sent, which notice any circumstance, the particulars of which the regular books cannot register; the necessity also of making no change in the books is repeatedly and strongly enforced, and if an error is detected it must be entered as a distinct item in the ledger; in short, no precaution is omitted which is requisite to give perfect distinctness to the recording of mercantile transactions, however complicated they may be.

Other books.

Italian book-keeping in general use.

(233.) If we consider the extent and influence of Italian commerce, extending to every country in Europe, Asia, and Africa, which was at that time known, in most of which Italian agents, factors, bankers, and money changers were established, it is natural to suppose, that this system of book-keeping should be generally adopted, recommended as it was by those whose experience and superior progress in the arts of life gave authority to their opinions and practice; we, consequently, find this method described in a work written by a merchant of Nuremberg, named Gottlieb, in 1531. In 1543, Hugh Oldcastle, a schoolmaster of London, wrote a work on the subject, which was afterwards published in an improved form by James Peale in 1569, with the following title: *A Briefe Instruction how to keep Books of Accounts, after the order of Debtor and Creditor, and as well for proper Accounts, Partible, &c. by three Books, named the Memoriall, Journal, and Ledger.*

Work of Stevinus on book-keeping.

(234.) Beckmann has given an account of a work of Stevinus on Italian book-keeping, written, in 1606, for his patron Maurice, Prince of Orange, and dedicated to the great Duke de Sully, who had introduced it in the accounts of the finances of France under Henry IV. It was translated into Latin by Willebrod Snell, who has latinized the modern terms with considerable elegance and ingenuity. Book-keeping is called *Apologistica*, or *Apologismus*; the book-keeper, *Apologista*; the memorial, or waste book, is *liber deletitius*; the ledger, *codex accepti impensique*; the cash book, *arcarii liber*; book of expenses, *impensarum liber*; the profit and loss account, *lucri damnique ratiocinium*, *contentio seu comparatio sortium*; the final balance, *epilogismus*; and the counting-house, *logisterium*. In connection with the subject of the names which are commonly given to those books, we may observe, that the Italian term *quaderno* is of unknown derivation; and the remark may be extended to our own word *ledger*, so variously written at different periods of our language, though many derivations have been given; it is called by the French *grande livre*, and by the Germans *hauptbuch*, or *head book*, to express its great importance. The existence of so many independent names proves that ledgers were used for registering accounts in those countries long before the Italian method was known; as it would otherwise have been hardly possible to have adopted the system without also borrowing its entire nomenclature.

Different names for ledgers.

Modern works on the subject.

(235.) It is not our intention to proceed farther with the notice of the books on this subject, which have been written in such great numbers by merchants and others, and by whom the method itself has been modified, from time to time, to suit the wants and purposes of modern commerce. Amongst the best of these we may mention the system published by Malcolm at Edinburgh, in 1728, and by John Mair of Perth, in 1737. In the year 1796, an accountant of Bristol, of the name of Jones, published a work, by subscription, on book-keeping by single entry, with double money columns, for the purpose of showing that it might be made, by certain modifications, equally efficient with the system of double entry, and that it was essentially more simple. This attempted innovation, however, was the cause of a considerable controversy, and was closed by a pamphlet of Mr. Mill, who showed by reducing the waste book of Mr. Jones to a journal and ledger, according to the old method, that his system was essentially and unavoidably defective.

Alligation in the *Lilávati*.

(236.) The rule for Alligation, as well as that of Position, is of eastern origin, and appears in the *Lilávati*, though under a somewhat limited form; it is there called *suverna-ganita*, or *computation of gold*, and is applied generally to the determination of the fineness or *touch* of the mass resulting from the union of different masses of gold of different degrees of fineness. The questions mostly belong to alligation medial, and are of the following kind:

"Parcels of gold weighing severally ten, four, and two *máshas*, and of the fineness of thirteen, twelve, eleven, and ten respectively, being melted together, tell me quickly, merchant, who art conversant with the computation of gold, what is the fineness of the mass? If the twenty *máshas* above described be reduced to six-

teen by refining, tell me instantly the touch of the purified mass? Or, if its purity when refined be sixteen, prithee, what is the number to which the twenty *máshas* are reduced?"

Statement:

Touch	13	12	11	10
Weight	10	4	2	4
Products	130	48	22	40

The sum of the products, 240, divided by the sum of the weights, 20, gives the fineness after melting, which is 12.

After refining, the weight being 16, the touch is 15; the touch being 16, weight is 15.

"Eight *máshas* of ten, and two of eleven by the touch, and six of unknown fineness, being mixed together, the mass of gold, my friend, became of the fineness of twelve; tell the degree of unknown fineness?"

Statement: 10 11
8 2 6. Fineness of the mixture, 12.

From 12 × 16, or 168, subtract 8 × 10 and 2 × 11, the remainder, 90, divided by 6 gives 15 for the degree of the unknown fineness.

Example in alligation alternate.

(237.) The following is the only question given in illustration of the rule called *Alligation alternate:*

"Two ingots of gold, of the touch of 16 and 10 respectively, being mixed together, the weight became of the fineness of 12; tell me, friend, the weight of gold in both lumps?"

Rule.

The following is the rule which is given: "Subtract the effected fineness from that of the gold of a higher degree of touch, and that of the one of the lower degree of touch from the effected fineness; tell me, friend, the weight of gold in both lumps? The differences, multiplied by an arbitrarily assumed number, will be the weights of gold of the lower and higher degrees of purity respectively."

Statement: 16 10. Fineness resulting, 12.

If the assumed multiplier be 1, the weights are 2 and 4 *máshas* respectively; if 2, they are 4 and 8; if $\frac{1}{2}$, they are 1 and 2: thus, manifold answers are obtained by varying the assumption.

Extent to which the rule was known to the Hindoos

(238.) This rule, though perfectly distinct and clear, is formed for the case of two quantities only, and there is no appearance of its ever having been applied to a greater number; it involves, however, the principle of the rule which is now used, recognises the problem as unlimited, and shows in what manner an indefinite number of answers may be obtained. The extension of the rule to any number of quantities, though not an easy step, in a state of the mathematical sciences when the generalization of principles and methods were little sought after and rarely practised, was yet incomparably more so than the invention of the rule itself, even under its most limited form; it is for this reason that we feel compelled to ascribe the chief honour of this rule to the arithmeticians of Hindostan.

Its Arabic name.

(239.) It was this latter rule, under a more general form, that was denominated *Sekis* by the Arabians, a term meaning *adulterous*, inasmuch as it is not content with a single, and, as it were, *legitimate* solution of the question. It was sometimes called *Cecca* by the Italians, who appear to have known nothing further of the word than its Arabic origin; and it constitutes the *alligation alternate* of modern books of Arithmetic. It may be as well, for greater clearness, to state algebraically the nature of the problems which are proposed for solution by means of it, and also to prove the truth of the process.

Algebraical statement of the problem proposed to be solved.

(240.) Let a, b, c represent the several prices, degrees of fineness, or other common quality of the several ingredients; it is required to find quantities x, y, and z of each, so that the common quality of the compound may be denoted by d.

The equations which represent the conditions of the problem, are

$$\left.\begin{array}{r} a\,x + b\,y + c\,z = m\,d \\ x + y + z = m \end{array}\right\} \begin{array}{l} (1) \\ (2) \end{array}$$

or, eliminating m,

$$(a - d)\,x + (b - d)\,y + (c - d)\,z = o, \quad (3)$$

which is an equation of condition, which must be satisfied in all cases.

The value of m, therefore, makes no alteration in the relative values of x, y, and z, which must be assigned from equation (3); and the assignation of it can only, therefore, in a certain sense, be said to limit the indetermination of the problem.

If the quantity of one of the ingredients be assigned, if $z = k$, for instance, then the equation (3) becomes

$$(a - d)\,x + (b - d)\,y + (c - d)\,k = o. \quad (4)$$

In this case, the values of x and y must be determined absolutely, so as to satisfy this equation; and those values must satisfy another equation of condition, which is,

$$x + y = m - k. \quad (4)$$

If m be also assigned, the determination of x and y is complete, when there are only three ingredients.

The problem becomes more limited if x, y, and z are concrete quantities, negative values of which would admit of no meaning; and still more so, if, in addition, those values are likewise required to be integral; under such circumstances there may be no answer to the question, or at most but a limited number of them.

Different species of alligation.

In *alligation alternate*, the only limitation is in the price of the compound: in *alligation total*, there is a limitation both of the price and quantity of the compound: in *alligation partial* there is a limitation of the quantity of one of the ingredients, and of the price of the compound: in *alligation medial*, the prices and quantities of all the ingredients are given to find the price of the compound, and the problem is, of course, determinate.

Proof of the arithmetical rule.

(241.) The arithmetical rule for *alligating* the quantities in the three first cases is the same, and the accuracy of the result may be readily shown by exhibiting the process and the result in algebraical symbols.

Let the prices or quality of the several ingredients be denoted by $u + a$, $u + b$, $u - a'$, $u - b'$, and that of the mixture by u; to find the quantities of each, which are requisite to produce a compound of this price or quality?

We will unite them in three different ways:

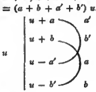
1.
$$u \left| \begin{array}{ll} u + a & b \\ u + b & a' \\ u - a' & b \\ u - b' & a \end{array} \right.$$

The quantities of each ingredient in their order being b', a', b, a, it is clear that the sum of the products of these quantities into their prices, ought to be equal to the product of the quantities into their mean price; thus,

Arithmetic.

$$b'(u+a) + a'(u+b) + b(u-a') + a(u-b')$$
$$= (a+b+a'+b')\,u + a\,b' + a'\,b - a'\,b - a\,b'$$
$$= (a+b+a'+b')\,u.$$

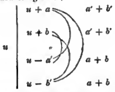
2.



In this case, also,

$$a'(u+a) + b'(u+b) + a(u-a') + b(u-b')$$
$$= (a+b+a'+b')\,u + a\,a' + b\,b' - a\,a' - b\,b'$$
$$= (a+b+a'+b')\,u$$

3. With double ligatures,



In this case, the several ingredients are respectively the sums of those which were determined by the single ligatures, and, of course, therefore answer the conditions of the question; or it may be shown as follows:

$$(a'+b')(u+a) + (a'+b')(u+b) + (a+b)(u-a')$$
$$+ (a+b)(u-b')$$
$$= 2\,.\,(a+b+a'+b')\,u + (a'+b')(a+b)$$
$$- (a+b)(a'+b')$$
$$= 2\,\{a'+b'+a+b\}\,u.$$

For alligation total. In *alligation total*, the quantity of each ingredient thus determined must be increased or diminished in the proportion of the sum of the ingredients determined to the sum required: in *alligation partial*, they must be altered in the proportion of the quantity of the ingredient determined to that which is required. For alligation partial.

In no case, does the rule attempt to determine all the answers of the question, aod in the two last cases, it only gives as many as can arise from variation of the ligatures.

Meaning of the term *consolare*. (242.) The earlier Italian writers on Arithmetic, in imitation of the practiee of their Arabian masters, have confined the applicatloo of this rule almost entirely to questions connected with the mixture of gold, silver, and other metals, with each other. This ooion was designated by the term *consolare*, which probably originated in the dreams of astrologers and alchemists: *Secondo che vogliono*, says de Burgo, *li astronomi, dei sonno li pianeti celestiali detti: per la virtu e ordinatione che da Dio ricevano hanno li detti dei metalli a generare e producere. Pero che la luna produce e genera argento morto: e lo sole genera l'oro. Delli altri metalli se taci.* It appears from hence, that it was considered the peculiar province of the suo to produce and generate gold; and as the process of the alchemists in transmuting the baser metals into gold was supposed to be under the influence of the sun, this gradual refinement, which they in common tended to produce, was designated by the common term *consolare*. Io later times it was applied to silver as well as gold, and still more generally to the common union of these metals with copper. History.

Mode of estimating the fineness of gold and silver. (243.) The fineness of gold was estimated by so many *carats*, or parts of 24, whilst that of silver was estimated by so many *lighe*, or parts of 12. The metals used in composition with them in coins were silver and copper, in the case of gold; and copper only, in that of silver: the baser metal in both cases being esteemed of no value with reference to the other. The noble metals were called *Fixed*, inasmuch as they did not waste during the process of refinement. We shall give a few examples connected with this subject.

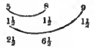
Examples. A person mixes 9 ounces of gold of 18 carats fine, 10 of 20 carats fine, and 11 of 22 carats fine; to find the fioeoess of the mixture?

A person mixes 9 marks of silver of 9 *lighe* of fineness, 13 of 8 *lighe*, and 14 of 10 *lighe*; of what *ligha* is the mixture?

I subject 82 ounces of gold of 18 carats fine to the fire for refinement, and draw out only 72 ounces; of what degree of fineness is it?

This is a common Inverse rule of Three question.

Given different species of silver of 9, 8, 5 *lighe*, respectively; io what proportions must we mix (*consolaremo*) 60 lbs., so that the compound may be of $6\frac{1}{2}$ *lighe*?



The answer:

34 lbs. 3 oz. 10 gr. + of 5 *lighe*.
12 lbs. 10 oz. 10 gr. + of 8 *lighe*.
12 lbs. 10 oz. 10 gr. + of 9 *lighe*.

A parish (*communità*) wish to *found* (*gittare*) a bell, composed of 5 metals, aod the hundred pounds weight of the basis cost 16 *lire*, of the second 18, of the third 20, of the fourth 27, and of the fifth 31. The whole weight of the bell is 2325 lbs., and it costs 488 *lire*, 5 *soldi*. What portions of each metal did they use?

The following is the form under which the ligatures are made by Tartaglia:

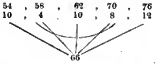
The price of the mixture is 21 *lire* the hundred pounds; and the quantities of each are, or may be, in the proportion of the oumbers 10, 6, 6, 3, 2.

A person has five kinds of wheat, worth 54, 58, 62, 70, 76 *lire* the *staro* respectively; what portion of each must be taken, so that the sum may be 100 *stara*, and the price of the mixture 66 *lire* the *staro*?

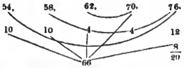
The following are different solutions of this question:

1st, In the proportion of the numbers 10, 4, 10, 8, 16.

Arithmetic.



2dly, In the proportion of the numbers 10, 10, 4, 4, 20.



3dly, In the proportion of the numbers 14, 14, 14, 24, 24.

54	58	62	70	76
10	10	10	12	12
4	4	4	8	8
			4	4
14	14	14	24	24

Tartaglia has given two other solutions of this example, arising from a different arrangement of the ligatures.

Examples on the mixture of medicines.

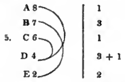
Suppose there are five simples, A, B, C, D, E, whose qualities are as followeth, viz. A is hot in 3°, B is hot in 2°, C is hot in 1°, D is cold in 1°, E is cold in 3°; it is required to mix four ounces of B with such quantities of the rest, so that the qoality of the medicine may be temperate?

It was the custom of the older physicians and pharmacopolists to classify medicines according to their degrees of *heat* or *coldness*, *moisture* or *dryness*. The scale for this purpose was adapted to the scale of the nine digits, the middle of which was *temperature*, as follows:*

History.

Indices.	Degrees.	
9	4	Qualities hot and dry.
8	3	
7	2	
6	1	
5	0	Temperature.
4	1	Qualities cold and moist.
3	2	
2	3	
1	4	

The solution of the question will stand therefore as follows:



The numbers 1, 3, 1, 4, 2, will, therefore, answer the question. The number 5 is sometimes called the *emergent* of the composition.

The following is a more elaborate example of the composition of a medicine called *Dianthus*, taken from Parkinson's *Herbal*, which is declared to be of a fine temperature or temperament, that is, somewhat more than a degree in heat, and somewhat less than a degree in dryness; in this case a *zero* is taken as the representative of *temperature*.

Ingredients.	Quantities.		Qualities. hot. cold. moist. dry.	Prodocts.
Rosemary flowers......	24	×	2 — 0 — 0 — 2	48 — 0 — 0 — 48
Red roses	18	×	0 — 1 — 0 — 1	0 — 18 — 0 — 18
Violets	18	×	0 — 1 — 2 — 0	0 — 18 — 36 — 0
Licorish	18	×	1 — 0 — 1 — 0	18 — 0 — 18 — 0
Cloves	4	×	3 — 0 — 0 — 3	12 — 0 — 0 — 12
Indian spikenard	4	×	1 — 0 — 0 — 2	4 — 0 — 0 — 8
Nutmegs	4	×	2 — 0 — 0 — 2	8 — 0 — 0 — 8
Galonga	4	×	3 — 0 — 0 — 3	12 — 0 — 0 — 12
Cinnamon	4	×	2 — 0 — 0 — 2	8 — 0 — 0 — 8
Ginger	4	×	3 — 0 — 0 — 3	12 — 0 — 0 — 12
Zedoary..............	4	×	2 — 0 — 0 — 2	8 — 0 — 0 — 8
Mace................	4	×	2 — 0 — 0 — 2	8 — 0 — 0 — 8
Wood of aloes	4	×	2 — 0 — 0 — 2	8 — 0 — 0 — 8
Cardamoms	4	×	3 — 0 — 0 — 3	12 — 0 — 0 — 12
Aniseeds	4	×	2 — 0 — 0 — 1	8 — 0 — 0 — 4
Dillseeds	4	×	2 — 0 — 0 — 3	8 — 0 — 0 — 12
	126			174 36 54 178

Hot. Cold.
$174 - 36 = \frac{138}{126} = 1\frac{2}{21}$. Hot

Dry. Moist.
$178 - 54 = \frac{124}{126} = \frac{62}{63}$. Dry.

Rules of Single and Double Position.

(244.) The rules of Single and Double Position are amongst the most celebrated in Arithmetie, and were generally discussed by the older writers with great diffuseness, in consequence of their furnishing the solutions of a vast number of questions, which would otherwise have required the aid of Algebra. It may conduce somewhat to the clearness of some of the details which follow, if we first state, in an algebraical form, the principles upon which these rules are founded.

* John Dee's *Mathematical Preface to Euclid*.

Arithmetic. Algebraical statement of their principles. Single position.

(245.) Single position includes those questions, in which there is a result which is increased or diminished in the same proportion with an unknown quantity which is proposed to be determined: of this kind are all questions which at once resolve themselves into the equation

$$a x = m. \qquad (1)$$

Rule. The process is as follows: assume a value of x, such as x', and let the result corresponding to it be m', or, in other words, let

$$a x' = m';$$

we from hence get,

$$\frac{x}{x'} = \frac{m}{m'}, \text{ or } x = \frac{m x}{m'}$$

or we must multiply the first result by the position, and divide by the new result corresponding to it.

Double Position. If, however, the question is proposed in such a manner, that the result, which is a function of the unknown quantity, does not increase in the same proportion with the increase of that quantity, or if it resolves itself into an equation of the form,

$$a x + b = m, \qquad (2)$$

we must then make two positions, or hypotheses, for the unknown quantity; let these be x' and x'', and let the corresponding *errors* be e' and e'', or, in other words, let

$$a x' + b = m + e'$$
$$a x'' + b = m + e'',$$

we from hence get

$$a (x' - x) = e'$$
$$a (x' - x'') = e' - e'',$$

which gives

$$\frac{e' - e''}{x' - x''} = \frac{e'}{x' - x}, \qquad (3)$$

and also

$$x = \frac{e' x'' - e'' x'}{e' - e''} \qquad (4)$$

Rules. The first of these results (3) being translated from algebraical into common language, shows that *the difference of the errors is to the difference of the positions, as the first error is to the difference of the first position and the quantity required*, a rule which is frequently used by De Burgo and Tartaglia.

The second (4) gives the common rule, that *the product of the first error into the second position, diminished by the product of the second error into the first position, and the result divided by the difference of the errors, gives the quantity whose value is required.*

Of course this rule must be modified according to the signs of the errors, whether both positive or both negative, or one positive and the other negative, and conversely: the sums being taken in the latter case, where the differences are taken before.

Applied to equations with two or more unknown quantities. It is not necessary that the question should at once resolve itself into an equation of the form (2), in order that it may come within the operation of this rule; if by means of any simple or obvious reduction, or by the solution of the intermediate equations, where there are more unknown quantities than one, it can be brought to a form in which the value of a function of the unknown quantity of the form $a x + b$ is given, it is equally resolvable by means of it.

History.

In the system of equations,

$$a x + b y = m \qquad (5)$$
$$\alpha x + \beta y = \mu. \qquad (6)$$

If we assume x' for the value of x, and determine the value of y corresponding to it from equation (5), we shall find,

$$a x' + b y' = m$$
$$\alpha x' + \beta y' = \mu + e'.$$

When the error e' is clearly the same as if we had first solved equation (5) with respect to y, and substituted the value thus found in equation (6): in other words, the error is the same as if we had commenced by reducing the system of equations to a single equation of the form

$$A x + B = \mu.$$

The same reasoning is clearly applicable to any system of equations containing more than two unknown quantities, where the error resulting from an erroneous assumption of the value of one of them necessarily shows itself in the result of one only of the equations: of this kind are the equations,

$$a x + b y = m$$
$$b' y + c z = n$$
$$c' z + d u = r$$
$$d' u + a' x = s.$$

If we assume x' as the value of x, determine successively the corresponding values of y, z, and u, from the three first equations, the error of the hypothesis appears only in the last equation, which becomes

$$d' u' + a' x' = e + e'.$$

A second hypothesis gives a second error, which combined with the first and the two positions, gives the true value of x, precisely in the same manner as if we had begun by reducing the four equations to one of the form

$$A x + B = s.$$

The preceding investigations include every rule which has ever been used for the solution of such questions in books of Arithmetic. It would be easy to form rules for the solution of systems of equations, by making distinct hypotheses for all the unknown quantities: thus, in the two equations, — Other rules easily formed

$$a x + b y = m$$
$$\alpha x + \beta y = \mu.$$

If we assume x' and y' for x and y, we shall get

$$a x' + b y' = m + e$$
$$\alpha x' + \beta y' = \mu + f,$$

from whence we readily find

$$x' - x = \frac{\beta e - b f}{a \beta - \alpha b}$$

$$y' - y = \frac{\alpha e - a f}{\alpha b - a \beta}$$

It is very easy to reduce these results into Arithmetical rules; but as the rules which are thus formed would be less simple than those which arise from the formulæ for the direct algebraical solution of the equations, it is clearly unnecessary to notice them further,

Arithmetic. particularly as they would find no application in the illustration of the methods which are found in books of Arithmetic.

Rule of single position in the *Lilávati*.

(246.) The rule of single position is the only one which is found in the *Lilávati*, where it is called *Ishtacarman*, or *operation with an assumed number*; we shall give a few examples from it, which, however, present nothing very remarkable beyond the peculiarities in the mode in which they are expressed.

Examples

Out of a heap of pure lotus flowers, a third part, a fifth, a sixth, were offered respectively to the gods Siva, Vishnu, and the Sun, and a quarter was presented to Bhavani; the remaining 6 were given to the venerable preceptor. Tell me quickly the whole numbers of flowers?

Statement: $\frac{1}{3}\ \frac{1}{5}\ \frac{1}{6}\ \frac{1}{4}$; known, 6.

Put 1 for the assumed number; the sum of the fractions $\frac{1}{3}, \frac{1}{5}, \frac{1}{6}, \frac{1}{4}$, subtracted from one, leaves $\frac{1}{20}$; divide 6 by this, and the result is 120, the number required.

Out of a swarm of bees, one-fifth part of them settled on the blossom of the *cadamba*, and one-third on the flower of a *silind'hri*; three times the difference of those numbers flew to the bloom of a *cutaja*. One bee, which remained, hovered and flew about in the air, allured at the same moment by the pleasing fragrance of a jasmin and pandanus. Tell me, charming woman, the number of bees?

Statement: $\frac{1}{5}, \frac{1}{3}, \frac{3}{1}$; known quantity, 1; assumed, 30.

A fifth part of the assumed number is 6, a third is 10, difference 4; multiplied by 3 gives 12, and the remainder is 2. Then the product of the known quantity by the assumed one, being divided by the remainder, shows the number of bees 15.

The following question is from the *Manoranjana:*

The third part of a necklace of pearls, broken in amorous struggle, fell to the ground; its fifth part rested on the couch, the sixth part was saved by the wench, and the tenth part was taken up by the lover; six pearls remained strung. Say of how many pearls the necklace was composed?

Statement: $\frac{1}{3}, \frac{1}{5}, \frac{1}{6}, \frac{1}{10}$; remained, 6. Answer, 30.

Derived by the Italian writers on Arithmetic from the Arabs.

(247.) The Italian writers on Arithmetic derived the knowledge of these rules immediately from the Arabians, designating them by the Arabic name *El Cataym*, or *Helcataym.*. *Costumasi*, says Lucas de Burgo, in *la practica de Arithmetica solversi molte e varie questioni per certa regola ditta el cataym. Quale (secondo alcuni) è vocabulo Arabo. E in nostra lingua sona quanto che a dire regola delle due false positioni.* The questions, proposed by him and by Tartaglia, are in immense variety, including every case of single and double position; and the rules which are given for this purpose, are such as would immediately result from the algebraical formulæ given above. A few examples will be sufficient to illustrate the form of the process which they followed:

Examples from De Burgo.

A person buys a jewel for a certain number of *fiorini*, I know not how many, and sells it again for 50. Upon making his calculation, he finds that he gains $3\frac{1}{3}$ *soldi* in each *fiorino*, which contains 100 *soldi*. I ask what is the prime cost?

Suppose it to cost any sum you choose; assume 30 *fiorini*, the gain upon which will amount to 100 *soldi*, or 1 *fiorino:* 1 added to 30 makes 31; and you say that it makes 50 between capital and gain; the position is therefore false, and the truth will be obtained by saying, if 31 in capital and gain arises from a mere capital of 30, from what sum will 50 arise. Multiply 30 by 50, the product is 1500; divide it by 31, the result is $48\frac{12}{31}$, and so much I make the prime cost of the jewel.

History.

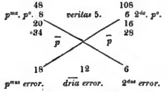
The above is a translation of the account given by De Burgo, of the first question which he has proposed on this subject.

Three persons have coins of the same kind and value; the second has twice as many as the first, and 4 more: the third as many as the first and second together, and 6 more; and the whole number is 44; how many had the first?

Suppose the number 8, then the second has 20, and the third 34; their sum is 62, and the error is 18, which is *plus*, or *piu*. Again, the first had 6, then the second has 16, and the third 28; the sum is 50, and the second error is 6, which is also *plus* or *piu*.

Consequently, $\dfrac{18 \times 6 - 6 \times 8}{18 - 6} = 5$, which is the true answer.

The following is the scheme which is given by De Burgo, which will require no explanation after the preceding statement.



Onde levate, says the author, *che sono le differentie, dice el commun proverbio, le parte stanno in pace. Siche tu vedi per dei falsità como siamo pervenuti a la verità. E questo è quello che diceva A. R. ex falsis verum: ex veris nil nisi verum.*

The following question admits not of translation: *Una matta de grue volano per airi e passan sopra un lago: dove una sta sottaqua: e sente quelle gridare Grugru; lei disse sete voi la su. La guida respose. Noi siamo tante che con altre tante e con la mita de tante e con teco in conto, siamo cento di ponto. Domando quasi te erano quelli che volevano?*

Questions proposed for amusement and pastime.

This is one of a multitude of questions which were proposed for amusement and pastime, and which were calculated to attract notice by the singularity of the terms in which they were expressed, or by presenting something remarkable in the conditions which they involved. Tartaglia says, that such questions were frequently proposed as puzzles, by way of dessert at entertainments, and has mixed up with his other questions on single position a large collection of such answers commonly proposed for this purpose. This practice, however, does not appear to have originated in Italy, as there are some circumstances which would make us refer them to the Greek arithmeticians of the IVth and Vth centuries, and probably even to an earlier period.

If the half of 5 were 3, of what number would 5 be the quarter? or if 4 were 6, what would 10 be? De Burgo has noticed other questions of this kind,

which are only remarkable for the violation of the propriety of language. The remark which he subjoins, shows that there were *bucks* in Italy as well as in other countries, and that the old monk felt a malicious pleasure in *posing* them, by the proposition of such questions. *In simil cosa*, says he, *siamo stati in disputatione in piu luoghi di Italia con molti che si tengono cervi e al bisogno non saltan troppo.*

The following questions are taken from the same author and Tartaglia, and will show the extent to which these rules were applied by them.

Questions from Tartaglia and De Burgo.

Two persons go to a fair, and the first says to the second, how many ducats have you? the second answered and said, if I had 30 of yours, I should have as many as you; and the second answered and said, if I had 30 of yours, I should have twice as many as you; how many ducats had each of them?

Tartaglia has given upwards of twenty questions, which are similar in principle to the preceding.

A schoolmaster, speaking of his scholars, says, if I had as many more, and half as many, and one quarter as many, and one-fifth as many, and 4 more, I should have 240; what number had he?

Two persons wish to buy a Turkish horse worth 120 ducats, but neither of them has sufficient money to pay for him. If I had $\frac{1}{3}$ of your money in addition to my own, I could just pay for him; upon which the second answered, if I had $\frac{1}{4}$ of your money besides my own, I could also pay for the horse; how much money had each?

A fisherman sold a sturgeon which weighed 60 pounds to three persons, the head to one, the tail to the second, and the body to the third; the head weighed $\frac{1}{5}$, the tail $\frac{1}{4}$ of the whole: what was the weight of the body?

A gentleman sends his servant to the garden of a lord, and tells him to go to the gardener and buy as many apples, that he may bring back one to his lady: he goes to the garden, which has four gates with four guards; nothing is paid on entering, but on quitting the gardens you must pay $\frac{1}{2}$ of the whole and 1 more at the first gate, $\frac{1}{2}$ of the remainder and 2 more at the second, $\frac{1}{2}$ of the remainder and 3 more at the third, and $\frac{1}{2}$ of the remainder and 4 more at the fourth: how many must he buy, so as to bring back one to his lady, (*la qual è gravida*)?

Indeterminate problems.

A gentleman asks a shepherd, what number of sheep he had, who answered, that when he numbered them 2 and 2, 3 and 3, 4 and 4, 5 and 5, 6 and 6, there remained 1 in each case, but if he numbered them by 7 and 7 there remained 0; what number of sheep had he?

This is an indeterminate problem, which Tartaglia solves by finding the least common multiple of 2, 3, 4, 5, 6, which is 60, and finding amongst the multiples of this number, increased by 1, those which were divisible by 7: of this kind are the numbers 301, 721, &c.

The following question is of a similar kind: *Un gentil' huomo incontrandosi con un contadino, che conduceva duoi sportoni di ovi sopra una cavalla a una città a vendere, e un cavallo di questo gentil' huomo si misse dietro a questa cavalla, talmente che gli fece rompere tutti quelli ovi: il gentil' huomo non volendo la rovina di quel contadino per voler gli pagar li delli ovi gli adimanda quanti erano, lui gli rispose che non sapeva quanti fossero, ma che sapeva ben a numerar li a 2 a 2 gli ne avanzava 1: similmente numerandoli a 3 a 3 gli ne avanzava 1 e cosi a 4 a 4 gli ne avanzava 1 e cosi a 5 a 5 gli ne avanzava 1: il medesimo faceva a 6 a 6, e a 7 a 7, e a 8 a 8, e a 9 a 9, e 10 a 10; ma numerandoli poi a 11 a 11 mi avanzava 0: si adimanda quanti erano li delli ovi.* The least number which answers the question is 25201.

A workman undertook to finish a piece of work in 16 days, and another workman undertook to do it in 20 days; in what time will they do it together?

If a person ask you how many Angels there are in Paradise, answer that there are three hierarchies, each consisting of 3 orders, and each order of 6666 legions, and in each legion there are 6666 Angels.

The answer is 399920004, which *secondo la opinion di theologhi sta bene.*

A person has 100 *stara* of wheat, and a miller has 3 mills, one of which would grind it in 10 days, the other in 5, and the third in 4; in what time will they grind it, all working together?

Four apples less a *danaro* are equal to 7 *danari* less one apple; what is the value of one apple?

A labourer undertakes a piece of work, upon condition of receiving 10 *soldi* for each day that he works, and of paying 15 *soldi* for every day that he is idle; at the end of 20 days the work is finished, and he receives only 15 *soldi*; how many days did he work, and how many was he idle?

Questions from the Greek *Anthologia.*

(248.) In the Greek *Anthologia* we find a collection of arithmetical problems, the greater part of which are attributed to one Metrodorus,* most of which are of the nature of those questions which are usually resolved by the rules of position; it is impossible, however, in consequence of the loss of all the Greek arithmetical writers subsequent to the age of Diophantus, to discover any traces of the methods which they made use of for their solution; whether these methods were merely tentative or identical with the rules of position. We will give a few instances:

1. *Δός μοι δύο μνᾶς, και διπλους σοῦ γίνομαι,*
Κἀγω λαβὼν σοῦ τὰς ἴσας, σοῦ τετράπλουν

Other examples are given of problems which are similar in principle.

2. The following refers to a bronze lion in a fountain, from the mouth, eyes, and heel of which the water flowed:

Χαλκέος εἰμι λέων, κρόυνοι δ'ἔμοι ὄμματα δοιὰ
Καὶ στόμα, σὺν δὲ θέναρ δεξιτεροῖο ποδος
Πλήθει δὲ κρητῆρα δυ' ἤμασι δεξιὸν ὄμμα
Καὶ λαιὸν τρισσαῖς καὶ πισύρεσσι θέναρ:
Αρκιον ἕξ ωραις πλῆσαι στόμα, σὺν ἅμα πάντα
Καὶ στόμα, καὶ γλῆναι καὶ θέναρ είπὸ πόσαις.

3.

Εγρεσθ' Ἠριγένεια παρέδραμε· πέμπτον ἔριθοι,
Λειπομένης τρισσῶν οἴχεται ογδοάτων.

4. The following possesses some interest, from its connection with the name of Diophantus:

Οὗτος τοι Διόφαντον ἔχει τάφον, ὃ μέγα θαῦμα
Καὶ τάφον ἐκ τέχνης μέτρα βίοιο λέγει:
Ἕκτην κουρίζειν βιότου θεὸς ὤπασε μοίρην
Δωδεκάτην επιθεὶς μῆλα πόρεν χλοάειν
Τῇ δ'ἄρ εφ'ἑβδομάτη τὸ γαμήλιον ἥψατο φέγγος
Ἐκ δὲ γάμων πέμπτω, παῖδ' ἐπενεύσεν ἔτει,
Αἴ αἴ τηλύγετον δειλὸν τέκος, ἥμισυ πατρὸς

* Brunck, *Anthologia*, vol. ii. p. 477.

Μέτρον τῶ κρυερὸν Μοῖρ' ὤφελεν βιότου.
Πενθας δ' ἂν πισύρεσσι παρηγορέων ἐνιαυτοῖς
Τῇδε ποσοῦ σαφὲη τέρμ' ἐπέρησε βίου.

The Epigram on the burdens of the mule and the ass, which has been so frequently quoted by writers on Arithmetic, we shall present to our readers in the translation of Philip Melancthon:

Mula asinusque duas imponit servulus utres
Impletas vino, segnemque ut vidit asellam
Pondere defessam vestigia ponere tarda
Mula rogat: Quid chara parens cunctare gemisque
Unam ex utre tuo mensuram si mihi reddas
Duplum oneris tunc ipsa feram: sed si tibi tradam,
Unam mensuram fiunt æqualia utrique
Pondera: mensuras dic docte geometra istas.

Rules of position attributed by some authors to Diophantus.

(249.) Some authors* have attributed the invention of the rules of position to Diophantus, though it is impossible to discover upon what grounds. It is most probable that the Greeks were in possession of some method of analyzing and solving such questions, otherwise it is hardly possible to conceive that they should have been proposed in such number and variety; and when we consider the nature and difficulty of the problems solved by Diophantus, in those parts of his works which remain to us, we should be fully justified in supposing that such methods were known.

Known to the Arabs, and whence derived by them.

(250.) The Arabs were in possession both of the rules for double and single position, with all their applications, and in this instance had advanced far beyond their Indian masters; and when we consider how small were the additions which they generally made to the sciences which passed through their hands, we might very naturally be inclined to suppose that their superior knowledge of these rules was derived from the Greek arithmeticians. There is, however, a vast gap in the history of the sciences after the time of Theon, and it is quite impossible to trace with certainty their transmission to the Arabs, or to ascertain through what channels some portions of Greek Astronomy at least, if not of other sciences, were transmitted to the Hindoos: under such circumstances we must rest contented with the rare and obscure hints which can be gathered from the writings of authors who flourished between the VIIth and the XIIth centuries, who had access to many arithmetical and other writings which have perished since that time.

Arithmetical writings attributed to Bede most probably spurious.

(251.) Amongst the earliest and most remarkable of these is our illustrious countryman Bede, amongst whose works there is a large collection of treatises on different arithmetical subjects, as well as many others *De computo ecclesiastico*, and on several points of astrology and astronomy: amongst the former is a collection of a great number of arithmetical problems and puzzles, which are extremely interesting under any circumstances, as the apparent originals of many of those which appear in the writings of the Italian arithmeticians, and which have been transmitted regularly downwards as stock questions to the authors of modern times. We once felt inclined to assign them a much earlier origin, and to suppose that they had been copied by Bede from the works of the Greek arithmeticians, particularly when we observe the resemblance between many of those questions and such as are found in the Greek *Anthologia*. A further examination, however, has given us good reasons for thinking that all these treatises are the production of a much later age; amongst others which are attributed to him, is one *de numerorum divisione*, which we found to be the identical treatise of Gerbert, with his prefatory letter to Constantine, which we have had particular occasion to notice above, from its importance in the controversy about the first introduction of Arabic numerals. An extended table of Pythagoras, which succeeds, is clearly the production of the same author, from its connection with the methods mentioned in the treatise in question for the multiplication of articulate numbers. In the *ratio cyclorum* which follows, he speaks of the present year 774, though he died in 735; and subsequently in an astrological treatise, *De Præcognitione copiæ et paupertatis futuræ*, he notices certain conjunctions and configurations of the planets which threaten ruin to Serugaga or Seville and Corduba, and famine to the Saracens, at least a century and a half before those names were known, and the whole is merely an extract from a Spanish calendar of the XIIth or XIIIth century. The whole treatise, *De computo ecclesiastico*, as well as those on other astronomical subjects, is clearly the production of a much later age; in short, there is so great a part of these treatises to which he clearly has no claim, that it is quite impossible for us not to look upon the whole as either spurious, or at least as of very doubtful authority.

The fact is, that the formation of calendars, and the composition of treatises *De computo ecclesiastico*, was a favourite employment of the more learned monks in the XIIth, XIIIth, and XIVth centuries, and it was a common practice to attribute the latter to some celebrated name: we have calendars of Roger Bacon without number, as well as a treatise of this nature, though it is nearly certain that he had nothing to do with the one or the other. In that age such impostures were easy, and were, indeed, considered meritorious, when their object was to give additional honour to a name such as that of Bede, so intimately connected with the glory of the order to which he belonged.

The first question in the collection would alone be sufficient to throw considerable doubt upon their authenticity.

Limax fuit ab hirudine invitatus ad prandium infra leucam unam: in die autem non potuit plusquam unam unciam pedis ambulare. Dicat qui velit in quot annos aut dies ad idem prandium ipse limax perambulavit.

In the answer to the question, it is said, that the *leuca*, or *league*, consists of 1500 *passus*, and each *passus* of 5 feet. Now it is very doubtful whether the *leuca*, or league, had yet become a recognised measure in France, and it is still more doubtful that a Saxon monk, residing in his monastery of Lindisfarn, should have taken such a measure in preference to one which was sanctioned by classical authority, or, at all events, familiar to the persons to whom his writings were chiefly addressed.

Though, for the reasons above-mentioned, we feel compelled to deny these questions the interest and importance which they would possess from the antiquity assigned to them, yet they are not without interest, as proving the general circulation, and even the antiquity, of a set of very curious questions, many of which have been familiar to us from our earlier years. We shall mention some of them as they occur, without any particular reference to the subject which we are immediately discussing, with such remarks as may naturally arise in connection with them.

* Gemma. Frisius, *Arithmeticæ Practicæ Methodus Facilis*, 1581.

Two men drive oxen on the same road: give me two of yours, says the first, and I shall have as many oxen as you; the other says, give me two, and I shall have twice as many as you; how many oxen had each?

The same, or nearly the same question is given above from the *Anthologia*.

Quidam senior salutavit puerum cui dixit. Vivas fili, vivas, inquit, quantum vixisti et aliud tantum et ter tantum addatque tibi Deus unum de annis meis et impleas annos centum.

The same question is frequently repeated with slight variations in its terms.

Quidam episcopus jussit 12 panes in clero dividi; Præcepit enim sic, ut singuli presbyteri binos acciperent panes, diaconi dimidium, lector quartam partem, ita tamen ut clericorum et panum idem sit numerus.

Tartaglia has proposed several questions which are resolved upon the same principle as this. Of this kind is the following:

Eighteen persons, men, women, and children, eat 18 pigeons; the men two each, the women 1, and the children $\frac{1}{2}$ of one; what number of men, women, and children were there respectively?

A father, on his death bed, leaves his 3 sons 30 vessels, 10 of which are full of wine, 10 of them half full, and 10 of them empty; in what manner must they be distributed, so that each may receive an equal quantity of wine and an equal number of vessels?

If we reduce the conditions of this question to equations, we shall find

$$x + y + z = 10 \qquad (1)$$
$$x' + y' + z' = 10 \qquad (2)$$
$$x'' + y'' + z'' = 10 \qquad (3)$$
$$z + x' + x'' = 10 \qquad (4)$$
$$y + y' + y'' = 10 \qquad (5)$$
$$z + z' + z'' = 10 \qquad (6)$$

$$x + \frac{y}{z} = 5 \qquad (7)$$

$$x' + \frac{y'}{z} = 5 \qquad (8)$$

$$x + \frac{y''}{z} = 5 \qquad (9)$$

The combination of equations (5) (6) (7) with (1) (2) (3), gives

$$z + \frac{y}{z} = 5$$

$$z + \frac{y'}{z} = 5$$

$$z'' + \frac{y''}{z} = 5$$

and consequently shows, that $z = x$, $z' = x'$, $z'' = x''$, or that each must have as many empty bottles as full ones; but it is evident, as well from the nature of the question as from the equations themselves, that the values of x, x', and x'', of y, y', and y'', and of z, z', and z'', are interchangeable, and that the equations are not independent of each other, and not sufficient therefore for the absolute determination of the unknown quantities.

There are two sets of values which will answer the conditions of the question.

$$\text{1st.} \left\{ \begin{array}{l} x = 5,\ y = 0,\ z = 5 \\ x' = 1,\ y' = 8,\ z' = 1 \\ x'' = 4,\ y'' = 2,\ z'' = 4 \end{array} \right.$$

$$\text{2d.} \left\{ \begin{array}{l} x = 2,\ y = 6,\ z = 2 \\ x' = 4,\ y' = 2,\ z' = 4 \\ x'' = 4,\ y'' = 2,\ z'' = 4 \end{array} \right.$$

The following three questions, given by Tartaglia, are of a similar character:

A citizen dying leaves 27 vessels, 9 of which are full of wine, 9 half full, and 9 empty, to be divided in equal number and quantity between three monasteries; namely, of Santa Maria dei Carmini, of Santa Maria della Pace, and of Santa Maria della Consolatione; how must they be distributed?

Two persons robbed a gentleman of a vessel of balsam containing 8 ounces, and whilst running away they met with a glassman, of whom they purchased in a great hurry two vessels, one containing 5 ounces, and the other 3; they at last reach a place of security, and wish to divide their spoil; how must this be done, so that each may have an equal portion?

Three persons have stolen a vessel of balsam containing 24 ounces, and have three vessels containing 5, 11, and 13 ounces respectively; in what manner must they proceed to effect the distribution, so that each may get an equal portion?

The difficulty of questions of this kind consists in their not being reducible to any regular analysis; the conditions to which they are subject not being expressible in algebraical language. The following representation will show one of the sets of successive steps which must be taken, in order to get an answer to the question.

Vessels..........	24 .	13 .	11 .	5
Successive contents	8 .	0 .	11 .	5
	0 .	8 .	11 .	5
	16 .	8 .	0 .	0
	16 .	0 .	8 .	0
	3 .	13 .	8 .	0
	3 .	8 .	8 .	5
	8 .	8 .	8 .	0

The following question is taken from Tartaglia: it is also found amongst those attributed to Bede, brothers and sisters being substituted for husbands and wives.

There are three men, young, handsome, and gallant, who have three beautiful ladies for wives, who are all jealous, as well the husbands of the wives as the wives of the husbands: being neigbours, they go in company to visit a shrine where indulgences are granted, and it happened that on their journey they have to pass a broad river, with neither a bridge nor passage boat; by good fortune, however, they find on the bank a very small boat, which can take no more than two at a time; in what manner must they pass, so as to give rise to no suspicion of jealousy?

If A, B, C represent the husbands, and a, b, c their respective wives, then a and b pass first, b returns and takes over c, c returns and remains with C, when A and B go over to a and b, A returns with a, and A and C pass over to their wives, c returns and brings back a, B returns and brings back b: they then, says Tartaglia, *attaccano il navello alla ripa e se ne vanno tutti a braccio a braccio con le sue donne al suo viaggio tutti allegri e gelosi.*

Tartaglia proposes the same question with 4 husbands and 4 wives, and the same method may clearly be adopted for passing any number of them, without violating the conditions, if it be allowed that the husband can protect his wife, or the wife her husband.

The following are questions, similar in principle though not in form, which appear in Bede, and which have likewise been frequently copied by other authors, probably from some common work.

A person is carrying a wolf, a goat, and a bundle of vetches, and meets with a river, which he can only pass in a small boat, and which will only hold himself and one of the other three; how must he contrive, so that the wolf may be kept from the goat, and the goat from the vetches?

A man, his wife, each a waggon load, and their two children, whose joint weight is equal to that of the father or mother, have to pass a river in a boat which can only bear the weight of a waggon load; how must their passage be effected?

Other questions are of a very trifling kind, being little more than a play upon words.

Bos qui tota die aratur, quot vestigia faciat in ultima rigâ?

Of the same kind are the two following questions from Tartaglia:

Uno cittadino ha un solo capretto e se ne vuol donar uno per uno al padre e uno al figliuolo; dimando come farà?

Uno cittadino ha 3 fasani, li quali vorria donar a duoi padri e duoi figliuoli e dargline uno per uno; dimando come lui farà?

Other questions relate to the degrees of relationship which result from the issue of extraordinary marriages.

Si duo homines ad invicem alter alterius sororem in conjugium sumpserent: dic (rogo) qua propinquitate filii eorum sibi pertineant?

Si relictam vel viduam et filiam illius in conjugium ducant pater et filius, sic tamen ut filius accipiat matrem et pater filiam: filii qui ex his fuerint procreati dic (quæso) quali cognationi subjugantur?

Divinations of numbers from certain data.

(252.) There are many questions proposed about the divination of a number, when the result is given, which arises from its being subjected to certain modifications, from additions, multiplications, &c.

Quomodo divinandum sit, qua feria septimanæ quilibet homo quamlibet rem fecisset.

A is directed to double the number, to add 5 to it, to multiply the sum by 5, and then by 10, and to give the result: B, who is informed of the operations to which it has been subjected, subtracts 250 from it, and the number of hundreds which remain, is the number required: in other words, if x be the number, $2x$, $2x + 5$, $10x + 25$, and $100x + 250$, will denote the successive results of the operations performed upon it; and, therefore, $(100x + 250) - 250 = 100x$, from whence the answer is obtained.

(253.) Such divinations were a source of a very popular species of pastime, and were in some measure equivalent to the solution of an equation, when the connection between the unknown quantity and the result, which arose from certain conditions, was previously known. The following are amongst the most common of those which are found in Tartaglia and later writers:

Game of the ring.

"If in any company," says Mellis, "you are disposed to make them merry by manner of divining, in delivering a ring unto any one of them, which after you have delivered it unto them, that you absent yourself from them, and they to devise after you are gone, which of them shall have the keeping thereof, and that you, at your returne, will tell them what person hath it, upon what hand, upon what finger, and what joint. Which to doe, cause the persons to sit downe all on a rowe, and to keepe likewise an order of their fingers; now after you are gone out from them to some other place, say unto one of the lookers on, that be double the number of him that hath the ring, and unto the double bid him add 5, and then cause him to multiplie that addition by 5, and unto the product bid him add the number of the finger of the person that hath the ring; and, lastly, to end the work, beyond that number towards his right hand, let him set downe a figure, signifying upon which of the joints he hath the ring, as if it be upon the second joint, let him put downe 2, then demand of him what number he keepeth, from the which you shall abate 250; and you shall have three figures remaining at least. The first towards your left hand shall signifie the number of the person which hath the ring, the second, or middle number, shall declare the number of the finger, and the last figure towards your right hand shall betoken the number of the joint."

If x be the number of the person, y of the finger, and z of the ring, then the course of the process gives successively $2x$, $2x + 5$, $10x + 25$, $10x + 25 + y$, $100x + 250 + 10y + z$, which, diminished by 250, gives the number expressed by the three digits x, y, z.

Other games.

Three persons play at the following game: one of them must form a wish which should be chosen emperor, which king of France, and which king of Naples; and the object of the game is, that a fourth person should be enabled from certain data to divine upon whom his choice had fallen. For this purpose, give to the first (say Hannibal) the number 1, to the second (Scipio) the number 2, and to the third (Pompey) 3, and tell him to double the number of him whom he wishes to be chosen emperor; add 5 to it; multiply the sum by 5, add to the product the number of the person whom he wishes to be king of France, add 10 to the result, multiply by 10, and then add the number of the person whom he wishes to be king of Naples; if 350 be subtracted from the last sum, the remaining digits will indicate the emperor and the two kings in their proper order.

In this case, if x be the number of the emperor, y of the king of France, and z of the king of Naples, then the process gives, successively, x, $2x$, $2x + 5$, $10x + 25$, $10x + 25 + y$, $10x + 35 + y$, $100x + 350 + 10y$, and $100x + 350 + 10y + z$.

A similar question would be amongst three persons who have secreted three articles, such as a glove, a purse, and a ring, to determine by whom the first has been taken, by whom the second, and by whom the third.

Another pastime described by Tartaglia was as follows:

Three persons seated round a table, upon which there are 18 balls, and also a piece of gold, a piece of silver, and a piece of copper: in the absence of a fourth, each person takes a coin; if the first takes the piece of gold, he also takes one ball, if the second 2 balls, and if the third 3; if the first takes the piece of silver, he takes 2 balls, if the second 4, and if the third 6; if the first takes the piece of copper, he takes also 4 balls, if the second 8, and if the third 12; the absentee upon his return is required, from the number of balls which remain,

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to assign the persons who have respectively taken the three coins.

If the three vowels, *a*, *e*, *o*, correspond to gold, silver, and copper, respectively, the persons will be indicated by the order of their occurrence in the following words, according as 1, 2, 3, 4, 5, or 6 balls remain on the table.

1	2	3	4	5	6
Absequor.	Belandes.	Latrones.	Dochmate.	Ocreas.	Reportant.
1 4 12	2 2 12	1 8 6	4 2 6	4 4 3	2 8 3

(254.) The principles upon which these puzzles and pastimes are founded, will show how easily they may be varied; and considering how much they were employed for the purposes of popular amusement, and how admirably they were calculated to excite the surprise and admiration of those who were ignorant of the mode in which they were formed and answered, we may naturally expect to find them modified in a vast variety of forms. Bachet de Meziriac, the commentator on Diophantus, was the author of a work on the subject of such problems,* containing a collection of all that were known in his time, accompanied by demonstrations and remarks, which in many cases show uncommon ingenuity; and a still greater number of them may be found in the *Mathematical Recreations* of Ozanam, as enlarged by Montucla. Referring our readers to their works for further information on this very entertaining subject, we shall conclude our observations relating to it, with a notice of the problem of the Turks and Christians, which has become unusually celebrated.

Problem of the Turks and Christians.

A ship, on board of which there are 15 Turks and 15 Christians, encounters a storm, and the pilot declares, that in order to save the ship one-half of the crew must be thrown into the sea: the men are placed in a circle, and it is agreed that every ninth man must be cast overboard, reckoning from a certain point. In what manner must the men be arranged, so that the lot may fall exclusively upon the Turks?

If the five vowels, *a*, *e*, *i*, *o*, *u*, represent the numbers 1, 2, 3, 4, 5, respectively, the rule for the arrangement of the men will be expressed by the occurrence of these vowels in the following distich or rubric:

From numbers' aid and art
Never will fame depart.

The vowel o indicates 4 Christians.
u ...... 5 Turks.
e ...... 2 Christians.
a ...... 1 Turk.
i ...... 3 Christians.
a ...... 1 Turk.
a ...... 1 Christian.
e ...... 2 Turks.
e ...... 2 Christians.
i ...... 3 Turks.
a ...... 1 Christian.
e ...... 2 Turks.
e ...... 2 Christians.
a ...... 1 Turk.

Bachet de Meziriac gives the following rubric:

Mort tu ne falliras pas
En me livrant le trespas.

The same purpose is answered by the Latin hexameter,

Populeam virgam mater regina ferebat.

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Tartaglia has given a series of nonsense verses, which will answer, respectively, for the cases where the lot falls on every third, fourth, fifth, sixth, seventh, eighth, ninth, tenth, eleventh, or twelfth person: those which correspond to the 9th are,

Documenta est decima perfecta,

or,

O brunetta rizza ale ferita Elena

or,

O puella irata est fetida effecta;

and for every 10th,

Rex Anglicus certo bona flamina dederat.

Legend of Josephus, related by Hegesippus.

(255.) If any reliance could be placed upon the truth of the following story, related by Hegesippus,* it would appear that the principles of such arrangements were understood and practised even in ancient times; after the storming of Jotapata by Vespasian, of which Flavius Josephus, the historian, was governor, he escaped with 40 of his companions to a lake or cavern; despairing of better fortune for their country, they determined on destroying themselves, notwithstanding the earnest exhortations of their commander, who was anxious that they should commit themselves to the clemency of Vespasian: finding all his entreaties vain, he at last hit upon the expedient of placing himself in such a position in the circle in which they were arranged, that every third man, reckoning from a certain point, being put to death, he should be one of the two which remained. The eloquence which had failed in persuading the whole body, was successful with his sole surviving companion; they agreed to live, and at once surrendered themselves to the mercy of their conquerors.

Theory of Stifelius of the invention of the rules of position.

(256.) Stifelius has given a very elegant theory of the steps which most probably led to the invention of the rules of position, which we shall give in his own words: *Inventurus author regulam falsi, dissimulabat se scire numerum illum, a quo 2 subtracta relinqueret* 3. *Recepit ergo primo* 4 *loco numeri illius: quem cum examinaret subtrahendo* 2, *vidit (loco* 3) *relinqui solummodo* 2. *Itaque deficere vidit unitatem et hunc numerum, cum defectu illo, separatim annotavit. Deinde recepit* 6, *quem numerum cum examinaret subtrahendo, vidit (loco* 3) *relinqui* 4. *Itaque superfluere vidit unitatem; et sic senarium cum superfluente unitate etiam separatim annotavit. Et sic postea exploravit qua ratione ex adnotatis numeris produceretur quinarius, qui videlicet subtractis a se* 2, *relinqueret* 3. *Facile enim videre fuit, qua ratione hoc fieret, scilicet ex aggregatione numerorum receptorum (id est* 4 *et* 6) *fiebant* 10. *Et ex aggregatione falsitatum, (id est* 1 *et* 1) *fiebant* 2. *Itaque ex divisione* 10 *per* 2, *proveniebatur* 5, *id est, numerus qui quærebatur.*

Figura positionum prædictarum

	4	10	6	
Minus	1		$\bar{1}$	Plus
		2		

Postea recepit 4 *et* 7, *et per eos simili modo tentavit invenire quinarium. Et cum videret figuram hujus inventionis sic stare (ut sequitur.)*

* *Problèmes plaisans et delectables qui se font par les nombres,* 1612.

* *De Bello Judaico et urbis Hierosolymitanæ excidio,* lib. iii. cap. 15.

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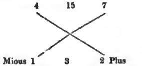
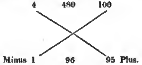
satis videbat, quod simplex aggregatio non responderet utrobique inventioni priori. Tentavit igitur omnes inveniendi modos possibiles, donec inveniret aggregationem mediante multiplicatione in cruce respondere: scilicet bis 4 et semel 7 (id est 8 et 7) faciunt 15, *quæ divisa per* 3, *faciunt* 5.

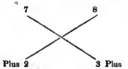
Figura inventionis prædictæ.



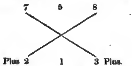
Postea, ut posset concludere, tentavit ejusdem numeri inventionem per 4 *et* 100: *et exibat figura inventionis, prædicto modo, hæc, respondens rei.*



Conclusit ergo inventiones hujusmodi esse ratas constanter, ubi falsitatum altera deficit, seu minus est, altera superfluente, seu plus existente. Deinde convertit se ad dexteram, tentans invenire hujusmodi inventiones per falsitates utrobique superfluentes. Recepit ergo pro experimento 7 *et* 8, *quibus numeris voluit invenire quinarium, modo prædicto: unde figura inventionis sic exibat.*



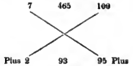
Sed hic cum videret aggregationem nihil fieri, tentavit rem per subtractionem. Et sic vidit operationem esse bonam et respondere rei.



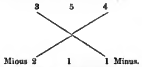
Hoc est, 2 *de* 3 *relinquunt,* 1 *divisorem: et* 3 *in* 7 *multiplicata faciunt* 21: *et* 2 *in* 8 *faciunt* 16. *At* 16 *de* 21 *relinquunt* 5 *dividenda per* 1 *divisorem.*

Postea, ut de inventione a dextris etiam concluderet, recepit 7 *et* 100, *quibus numeris quinarium produceret, modo prædicto: et exivit figura inventionis sic, ut sequitur.*

History.



Postea videns successum se habuisse talem a dextris, vertit se, ut idem experiretur etiam a sinistris. Recepit ergo 3 *et* 4, *id est, numeros quos sciebat allaturos esse falsitates deficientes utrinque. Per eos itaque quæsivit quinarium producere sicut prius, et invenit hanc figuram.*



Post tantos successus in questionibus ludicris, cœpit autor negotium illarum operationum transferre ad obscuras quæstiones, numerorum abstractorum et contractorum. Sentiens ergo immensam latitudinem negotii illius, magnifice lætabatur, reputans se reperisse thesaurum artis incomparabilem.

(257.) An addition was made to the Role of False by Gemma Frisius, which Stifelius characterises as *inventum valde egregium*: it consisted in applying it to the solution of such equations as

Extension of the rules of position by Gemma Frisius.

$$a x^2 = m, \quad \text{or,} \quad a x^2 + b = m,$$
$$a x^3 = m, \quad \text{or,} \quad a x^3 + b = m,$$

involving the squares, cubes, or higher powers of the unknown quantity; and the principle of it was merely that of considering x^2, x'^2, x''^2 (where x' and x'' are the positions) as simple quantities, such as X, X', X'', and treating them according to the ordinary rule; the determination of the value of X immediately leads to that of x. The following is an example:

To find two numbers in the proportion of the numbers 2 and 3, whose product shall be equal to 864.

		10368			
	1st Position.	Square.	Square.	2d Position.	
	2	4	16	4	
Product 6					24 Product
	3	9	19	6	
	Error 858			840 Error	
		18			

Assume 2 and 3, their product is 6, the error 858; again, assume 4 and 6, their product is 24, the error 840; the difference of the errors is 18: multiply 858 into 16, (the square of 4,) and from the product subtract the product of 4 × 840, which is 3360; the difference, 10368, divided by 18, gives 576, the square of 26, the first of the two numbers required.

(258.) We shall conclude our observations on this rule with an extract from Recorde, who, after remarking, that in other parts of Arithmetic the numbers are taken in just proportion, whilst in this rule they are not found by orderly work, but taken at all adventures, proceeds to say, "that sometimes being merie with my friends, and talking of such questions, I have caused them that proposed such questions, to call unto them such children and ideots as happened to be in the place, and

Statement of these rules by Recorde.

Arithmetic.

to take their answere, declaring that I would make them solve those questions that seemed so doubtful; and, indeede, I did answere to the question, and works the triall thereof also by those answeres which they happened at all adventures to make, which numbers seeing that they be taken as maketh false, therefore, is this rule for triflenesse, called the Rule of Falsehood, which rule, for readinesse of remembrance, I have comprised in these few verses following, in form of an obscure riddle.

> Gesse at this worke as hap doth leade,
> By chance to truth you may proceed.
> And first worke by the question,
> Although no truth therein be done.
> Such falsehoode is so goode a ground
> That truth by it will soon be founde.
> From many bate to many moe,
> From too fewe take too fewe also.
> With too much joyne too fewe againe,
> To too fewe adde too many plaine.
> In crosse waies multiplie contrarie kind,
> All truth by falsehoode for to finde.

Whatever other merits the composition of this riddle may possess, it is impossible to deny it the essential one of obscurity. —

Arithmetical and geometrical progressions.

(259.) The different species of Progressions, whether Arithmetical, geometrical, or musical, as well as the subject of combinations and permutations, whether we consider their theory, or a great portion of the problems which they lead to, more properly belong to Algebra than to Arithmetic, though they have generally been included in books on the latter subject, as well as the former. The great extent, however, to which this article has proceeded, compels us to pass them over without any notice beyond a few remarks; and we feel the less regret at the omission of more elaborate details, however interesting they might be, as they involve the developement of no principle which is essentially connected with the progress of Arithmetical science.

Particularly noticed by the Pythagorean arithmeticians.

(260.) The different progressions of numbers were the object of the particular attention of the Pythagorean and Platonic arithmeticians, who enlarged upon their most trivial properties with the most tedious minuteness. Their speculations, however, were directed to the elucidation of the mysterious harmonies of the physical and intellectual world, and had, therefore, no concern with the business of real life; and they, consequently, passed over, as altogether unworthy of notice, the solution of those questions which naturally arise from these progressions, and which appear in such numbers in Hindoo, Arabic, and modern European books on Arithmetic.

Questions on Arithmetic progression.

(261.) Amongst the questions attributed to Bede is the following:

There is a ladder with a hundred steps; on the first step is seated one pigeon, on the second 2, on the third 3, and so on, increasing by one from each step. Tell, who can, how many pigeons were placed upon the ladder?

Of the two following questions, which appear in all modern books of Arithmetic, the first originated with the Venetian arithmeticians, as might be conjectured from its subject; the second, of whose real origin we are ignorant, is the subject of a very common and popular wager.

How many strokes do the clocks of Venice strike in 24 hours?

If a hundred stones be placed in a right line, one yard from a basket, what length of ground must a person go who gathers them up singly, returning with them one by one to the basket?

History.

Geometric progression.

(262.) The extraordinary magnitude of the numbers which result from the summation of a geometrical series, is well calculated to excite the surprise and admiration of persons who are not fully aware of the principle upon which the increase of its terms depends; and examples are not wanting, where the rash and the ignorant have in consequence been seduced into ruinous or impossible engagements.

Celebrated question.

The most celebrated of these questions is the one which tradition has represented as the terms of the reward demanded of an Indian prince by the inventor of the game at chess; which was a grain of wheat for the first square on the chess board, two for the second, four for the third, and so on, doubling continually to 64, the whole number of squares.

Lucas de Burgo, who has solved this question, makes the number of grains

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which he proceeds to reduce to quantities of a superior denomination as follows:

6912	grains make a lira of Perugia.		
133	lire	....	mina.
3	mine	......	soma.
4	some	......	corba.
20	corbe	......	archa.
40	arche	......	barca.
100	barce	......	magazeno.
100	magazeni	..	castello.

The amount, expressed in *castles* of corn, would be 209022 with a fraction; he then recommends his reader to attend to this result, as he would then have a ready answer to many of these *babioni ignari de la Arithmetica*, who have made wagers on such questions, and have lost their money.

The case is similar to that of the ignorant and unfortunate host who undertook, on certain conditions, to give as many dinners to 10 persons as they could place themselves in different arrangements at the table.

In cases, indeed, of the formation of the terms of a geometric series, or in problems on permutations, where the result arises from the continued multiplication of the same or different factors, we speedily arrive at numbers which surpass the powers of the imagination to conceive; and arithmeticians have delighted in the proposition of questions which lead to such surprising conclusions. The amount of a *penny* put out to interest at five per cent. per annum, at the birth of our Saviour, would require more than 40 places of figures to express it; and many attempts have been made to exhibit this result in a form which may come within the grasp of the human mind. Political economists have appealed to the same principle to account for the rapidity with which population increases, when its progress is not checked by famine and disease; whilst the speculator on languages finds an unlimited supply of words in those permutations of the letters of the same or different alphabets, which form sounds within the compass of human utterance.

Conclusion.

(263.) We cannot conclude this history of Arithmetic without making some observations on the difficulty of the undertaking, and upon the many necessary defects under which it must labour. With the exception of the very able and interesting work of Professor Leslie on the *Philoso-*

Arithmetic. *phy of Arithmetic*, to whom we are under great obligations for having sketched an outline which we have endeavoured to fill up, the attempt may be considered as altogether new. The subject is hardly noticed in the work of Muntucla, which is otherwise so admirable in the early history of the mathematics; and the meagre sketch which Kæstner has given of some insulated works on the subject, generally contrives to omit almost every particular which is essentially connected with the history of the progress of the science; in short, there does not exist any source of information on this subject which can be deemed trust-worthy and authentic, except in the original authors themselves.

In writing the history of a science, the facts are generally distinct and positive, and the adjudication of the honour of different inventions and improvements, and of the just claims of different authors to them, may for the most part be made with certainty, from the examination of the original works taken in the order of time. On such subjects there is rarely any conflicting testimony, and it is seldom necessary to proceed to the nice weighing of probabilities, which is so frequently requisite in the history of events; but there are other difficulties, almost as considerable, which a scientific historian must encounter: he must not only perfectly understand the subject upon which he writes, but he must also understand it under the form in which it appears in the work which he examines: he must not only be able fully to appreciate the importance of a discovery or improvement, but likewise to determine how far a hint, or partial anticipation of it, may have contributed to its full developement: he must weigh the relative merits of the inventor and the expositor, of him who discovers a new region in science, and of him who, by subsequent and more minute examination, ascertains its full extent and boundaries, and makes its productions generally known.

In the history of Arithmetic, however, these difficulties present themselves under their least formidable aspect; the subject is easy under all its forms, and there can be little doubt or controversy about an improvement when made, though some might arise on the different steps which lead to it. Again, the number of original authors on this subject, since the invention of printing, at least, is very small; and when we have mentioned the great names of Lucas de Burgo, Stifelius, Tartaglia, Stevinus, and Napier, the additions made to the science by other authors are, generally speaking, of a very trifling importance; for on all subjects, where the difficulty of acquisition does not necessarily limit the number of authors, the great majority of writers are mere copiers of their predecessors, and are generally contented with some little alteration in form rather than in matter; and this is particularly the case with Arithmetic, a subject which so many must learn, and so many must teach; where the great number of readers has a natural tendency to make a great number of authors; and where the simplicity of form under which the rules of the science are exhibited, and the ease with which they may be learnt and practised, must always be considered of more importance than the originality of the matter. History

But though the number of authors whose works must be consulted is small, when we are in search of great and essential improvements in this science, yet there are other occasions where it is requisite to consult *all* those which belong to a particular period. This is the case when we wish to examine the progress of an improvement, and to ascertain the rapidity with which it came into general use, and the variations of form which it underwent between its first discovery and its final developement. Of this kind is the history of decimal fractions, from the first publication of Stevinus to the middle of the XVIIth century. In all cases of this kind we are sensible that this history must labour under great deficiencies, as there are no libraries in this country which contain *all or nearly all the books* which are requisite for this purpose, and there are no *classed catalogues* by which we can ascertain, without great labour, all the treasures which they contain.*

* We are glad to learn, that in one case, at least, this deficiency is speedily to be supplied, and that a *classed catalogue* of the library of the British Museum, and also of the magnificent gift of the King, is in active preparation. It is to be greatly lamented, however, that the national bounty should be distributed in such scanty sums to the support and increase of this great and important establishment; and that instead of a paltry allowance of *eight hundred pounds per annum*, for the purchase of books for the library, it should not be increased to at least as many thousands.

APPENDIX.

Work of the Abbé Hervas containing important information on South American numerals.

(264.) Since the first part of this article was written and printed, we have procured a copy of the work of the Abbé Hervas, entitled *Aritmética di quasi tutte le nazioni conosciute;* it contains the numerals in 175 languages, including those of more than thirty South American tribes, which he obtained chiefly from the ex-Jesuit missionaries who resided at Rome, after they had been obliged to quit their missions in South America, upon the extinction of their order; amongst those he particularly mentions Clavigero, the learned historian of Mexico, his native country, Gilii, the historian of the missions on the Orinoco, Camano and Velasco, the authors of important works on the languages and customs of several South American tribes; the information which he procured was chiefly from personal communication with them, and his inquiries were specifically directed to the construction of their numeral language, and to their practical methods of numeration. The materials which this work contains are particularly valuable, not only from their not existing in any other works, but likewise from their relating to tribes, many of which are in the lowest state of civilisation, amongst whom we must look for the most certain indications of the influence of practical methods of numeration upon the formation of their numerals.

(265.) Of the following four sets of numerals, which

Arithmetic. Numerals of the Peruvians, Araucani, Aimarri, and Sapibocones.

possess some points of resemblance, the first belongs to the Qquichnan, or ancient Peruvian language of the Incas, which was spoken anciently in Peru, and the influence of which extended for more than 40 degrees of latitude along the western coast of America. The second is from the language of the Araucani, the inhabitants of Chili, who were likewise included in the great empire of the Incas. The third is that of the Aimarri, a tribe in the north-eastern parts of Peru; and the last, of the Sapibocones, a neighbouring tribe.

Qquichua.	Araucana.	Aimarra.	Sapibocona.
1. Huc,	Kiñe,	Mai,	Pebbl.
2. Iscai,	Epu,	Paya,	Bbeta.
3. Kimsa,	Kula,	Kimsa,	Kimisa.
4. Tahua,	Meli,	Pusi,	Pusi.
5. Piebca,	Kechu,	Pisca,	Pissica.
6. Socta,	Kayu,	Sogta,	Snccute.
7. Canchis,	Relghi,	Pacalco,	Pacalucu.
8. Passac,	Pura,	Kimsacalco,	Kimisacalucu
9. Iscon,	Ailla,	Pusicalco,	Pusucalucu.
10. Chunca,	Mari,	Tunca,	Tunca.
11. Chunca hue niyoc,	Marikiña,	Tuncamayani,	Tuncapeepebbi.
12. Chunca iscai niyoc,	Mariepu,	Tuncapayani,	Tuncapeabbeta.
20. Iscaichunca,	Epumari,	Payatunca,	Bbetattunca.
30. Kimsachunca,	Kulamari,	Kimsatunca,	Kimisatunca.
40. Tahuachunca,	Melimari,	Pusitunca,	Pusitunca.
100. Pachac,	Pataca,	Pataca,	Tuncatunca.
1000. Huaranca,	Huaranca,	Huaranca,	Tuncatuncatunca.
1000000. Hnou.			

The two first systems are equally perfect, and similar in construction, though all the terms below 100 are essentially different from each other. The expressions for 11 and 12 in the first, mean *ten one with, ten two with*,—the signification of the *postposition yoc* being *with*, the particle *ni* being merely interposed for the sake of euphony: in the second, the expressions for the same numbers mean *ten one, ten two*. In the three first systems we find the same terms for 100 and 1000, affording an additional illustration of the truth of the observations made in Art. 17 and 21, on the transmission and adoption of the names of the higher orders of superior units.

The second and third of these systems are curious examples of the partial borrowing of numerals, by one people from another more advanced in civilisation; the names for 1 and 2 are most probably native in both, and that for 4 in one of them; whilst the names for 3, 5, and 6 are clearly Peruvian; the names for 7, 8, and 9 are clearly compound, meaning *two five, three five, four five*; *calco*, in one, and *lucu*, in the other, meaning *five*, or *hand*; showing that the influence of a natural method of numeration manifested itself even in a case where part of the numerals were borrowed from a nation who had altogether abandoned this manual Arithmetic. The duplication and triplication of the name for 10, in order to denote 100 and 1000, a simple and natural artifice for the expression of such numbers, will receive an additional illustration in the following system of numerals of the Cayubabi, a tribe inhabiting the banks of the Mamorè, which runs into the Marañon,

History. Numerals of the Cayubabi.

1. Carata.	10. Bururuche.
2. Mitia.	11. Bururuche-caratorogienê.
3. Curapa.	12. Bururuche-mitiarogienê.
4. Chadda.	19. Bururuche-chaddarirobogienê.
5. Maidarù.	20. Mitiaburuche.
6. Caratarirobo.	30. Curapahururuche.
7. Mitiarirobo.	100. Buruche buruche.
8. Curaparirobo.	1000. Bururuche peuêbururuche.
9. Chaddarirobo.	

Hervas says, that the name foe hand is *arue*, and that the names 6, 7, 8, 9, respectively, mean *one hand with, two hand with, three hand with, four hand with*. The name for 10, *or bururuche*, is probably derived from the *reduplication* of *arue*, quasi *aruearue*, or *hand hand*. If this derivation be well-founded, the name for 100 would be equivalent to *hand hand hand hand*, a very remarkable result of the composition of a simple term.

Numerals of Cora, Yucatan, and Mexico.

(266.) A still more remarkable example of the same fact will be found amongst the numerals of the Coran language, which is spoken in New Galicia, which we now subjoin, in conjunction with those of Mexico and Yucatan, with which they are intimately allied.

Azteck.	Yucatan.	Coran.
1. Ce,	Hunppel, or yax,	Ceiiut.
2. Ome,	Cappel, or ca,	Hualpoa.
3. Yei,	Oxppel, or yox,	Huacia.
4. Nahui,	Cammpel, or cantzel,	Moâcoa.
5. Macuill,	Hoppel, or ho,	Amxuoi.
6. Chicuace,	Uacppel, or uac,	Acevi.
7. Chicome,	Uucppel, or uuc,	Ahuapoa.
8. Chicuei,	Uaxacppel, or uaxac,	Ahuacica.
9. Chicunahui,	Bolonppel, or bolon,	Amoacua.
10. Matlactli,	Lahunppel, or lahun,	Tamoâmata.
11. Matlactli-occe,	Huncahunppel,	Tamoâmata-apon-ceiiut.
12. Matlactli-omome,	Lahca,	Tamoâmata-apon-hualpa.
15. Chaxtòli,	Holhunte,	
16. Chaxtòli-occe,		
20. Cempohuàli,	Kal, or hunkal,	Ceitevi.
30. Cempohuàli-i-pan-matlactli.		Ceitevi-poan-tamoâmata.
40. Ompohuàli,	Cakal,	Huahcatevi.
60. Epohuàli,	Oxkal,	Huacicatevi.
100. Macuilpohuàli,	Hokal,	Anxiitevi.
200. Matlacpohuàli,	Lahunkal,	Tamoâmatatevi.
400. Cen-tzontli,		Ceitevitevi.
800. Ontzontli.		
8000. Ce-xikipili,	Hunpic, or pic.	

Remarks on Mexican numerals.

In the list of Mexican numerals which is given in Art. 28, there are both deficiencies and inaccuracies: the name for 15 is *chaxtoli*, and the numeration recommences from it; the expression for 16 being *fifteen one*, for 17 *fifteen two*, and so on, precisely in the same manner as in the Welsh numerals, (Art. 22.) The name for 5, *macuili*, is derived from *maitl*, or *hand*; and the composition of the terms for 6, 7, 8 and 9, shows that *chicu* possessed a similar meaning, which appears again in the term for 15. The name *tzontli*, for 400, signifies, also, *hairs of the head*; and, probably, in ancient times was equivalent to *innumerable*, having subsequently acquired a definite signification, in the same manner as μυρια among the Greeks, when

Arithmetic. their numeration became more systematic.* Every circumstance which tends to illustrate the composition of the Mexican numerals possesses more than common interest, as they constitute the most perfect example of the vicenary scale, with the quinary and denary scales equally subordinate to it.

Remarks on the numerals of Yucatan. In the Yucatan, or Mayan, numerals, there are two sets of names for the digits, which are both used, and whose chief difference consists in the addition of the final *ppel*. The expression for 11 means *one ten*, for 12 *two ten*, for 15 *five ten*; a species of composition which might be ambiguous, if the system were denary and not vicenary. The term *pic*, or *hunpic*, *eight thousand*, or *one eight thousand*, is the termination of the Yucatan numerals. When the Yucatani speak of persons, they add the final *tul*, instead of *ppel*; thus, *huntul* means *one person*, *catul*, *two persons*, *lahcatul*, *twelve persons*. The Coran numerals which are given above, are those which are used for inanimate things; for living beings they postpone the particle *man*. Such instances of *imperfect abstraction* in the formation of numerals are not uncommon in South American languages.

Remarks on the Coran numerals. In the last of these systems the terms for 6, 7, 8, 9 are clearly compound. The general term for *hand* in the Coran language is *moamati*, which is clearly the basis of the name for 10; the expression for 11 means *ten above one*, that for 12, *ten above two*; the name for 20 is componnded of *ceitul*, one, and *tevit*, which is equivalent to the generic term *homo*, or *persona*, whilst that for 400 is *one twenty twenty*, or more literally *one person person*.

Numerals of the Otomiti. (267.) The following numerals of the *Otomiti*, a tribe allied to these above-mentioned, both in geographical situation and language, presents an example so common amongst Celtic nations, of the vicenary scale proceeding as far as 100 and then merging in the decimal.

1. Na.	12. Detta-ma-yoho.
2. Yoho.	19. Detta-ma-gueto.
3. Hiu.	20. Dotê.
4. Goho.	30. Dotê maretta.
5. Kueta.	40. Yotê.
6. Rato.	50. Yotê maretta.
7. Yoto.	60. Hiâte.
8. Hiato.	80. Huête.
9. Gueto.	100. Nato.
10. Detta.	1000. Namao.
11. Detta-ma-na.	

In this system, the names for 6, 7, 8, 9 are analogous to those for 1, 2, 3, 4, a clear indication of the quinary scale. The name *dotê*, for 20, is probably derived from *yohè*, *man*, which is its meaning in so many South American languages.

Numerals of the Guaranies. (268.) The numeral systems given above are those which have received the most complete developement; those which follow are not only extremely limited in extent, but may be considered as the expression of the practical methods of numeration, which are required for all numbers which exceed the radix of the natural scales.

Numerals of the Guaranies. (See Art. 30.)

1. Petey.
2. Mocôi.
3. Mbohapi.

* The term *cembati*, in the Coran language, signifies the *hairs of the head*, and also *innumerable*. See Art. 30.

History.

4. Irundi.
5. Irundi hae nirâi, *four and another*, or ace popetei, or the *one hand*, where *po* is hand, and *ace* the determinate article.
6. Ace popetei hae petêi abe, *the one hand and one besides*.
9. Ace popetei hae irundi abe, *the one hand and four besides*.
10. Ace pomocoi, *the two hands*.
20. Mbo-mbi-abe, *hands feet besides*.
30. Mbo-mbi hae pomocoi abe, *hands feet and two hands besides*.

The missionaries never heard a Guarani count beyond 30.

Of the Omoguas. (269.) The Omoguas, a tribe living in the kingdom of Quito, and speaking a dialect of the Guarani language, notwithstanding their immense distance from each other, have only five numerals, the last of which, *upapua*, signifies *hand*. By the combination of these, however, with the expressions for the hands and feet, they can proceed as far as a hundred.

Of the Zamucoes. (270.) The following are the numerals of the Zamucoes, one of the numerous tribes of Paraguay:

1. Chomara.
2. Gar.
3. Gaddive.
4. Gahagani.
5. Chuena yiminaete, *finished hand*.
6. Chomarahi, *one of the other*
7. Garihi, *two of the other*.
10. Chuena yimanaddie, *finished two hands*.
11. Chomara yiritie, *one of a foot*.
20. Chuena yiriddie, *finished feet*.

Their mode of expressing numbers beyond 20. The missionaries never heard a Zamuco express in words a number greater than 20: any number greater than 20 is designated by the term *unaha*, *many*: if the number greatly exceeds 20, they say *unahapuz*, *very many*; and to express in terms of increasing intensity their opinion of the magnitude of very large numbers, they say *unaahapuz*, *unaaahapuz*, *unaaaahapuz*, reduplicating continually the sound of the letter *a*. In common cases, however, in speaking of numbers within the compass of their methods of numeration, they take in their hand grains of rice, little stones, or seeds, and count them out until they have reached the number required, and then point to them, saying *choetie*, *like this*.

Of the Luli. (271.) The numerals of the Luli, another tribe of Paraguay, present an example of a very singular construction, where the mere poverty of words has caused an appearance of the quaternary scale.

1. Alapea.
2. Tamop.
3. Tamlip.
4. Lokep.
5. Lokep moilè alapea, *four with one*, or is-alapea, *hand one*.
6. Lokep moilè tamop, *four with two*.
7. Lokep moilè tamlip, *four with three*.
8. Lokep moilè lokep, *four with four*.
9. Lokep moilè lokep alapea, *four with four one*.
10. Is-yaoum, *all the fingers of hand*.
11. Is-yaoum moilè alapea, *all the fingers of hand with one*.
20. Is-elu-yaoum, *all the fingers of hand and foot*.
30. Is-elu yaoum moilè is-yaoum, *all the fingers of hand and foot with all the fingers of hand*.

Arithmetic. Their mode of expressing numbers beyond 30.

It is a rare thing for a Lulo to attempt the expression of a number beyond 30; when driven to it by necessity, they avail themselves of actions for the purpose. Thus, to express 40 he raises his open hands to his shoulders, and bending his head towards his feet, he says *tamop*, which means *twice of all that I show you:* with the same action, accompanied by the word *tamlip*, he expresses 60, and by saying *lokep moilè alapea*, he expresses 100.

Numerals of the Vileli.

(272.) The same expedients are made use of by the Vileli, a neighbouring tribe, to express such numbers, and it will be at once seen that their numerals, though essentially different, are formed upon the same principle.

1. Yaaguit, or agùit.
2. Ukè.
3. Nipetuei.
4. Yepcatalèt.
5. Isig-nislè yaaguit, *fingers of hand one*, meaning *all the fingers of hand one.*
6. Isig-teèt yaaguit, *hand with one.*
7. Isig-teèt ukè, *hand with two.*
10. Isig-ukè-nislè, *of hands two the fingers.*
11. Isig-ukè-nislè teèt yaaguit, *of hands two the fingers with one.*
20. Isig-ape nislè cavel, *fingers of hands and feet.*

Of the Mocobi.

(273.) The Mocobi are a tribe on the Paranà, in the neighbourhood of Buenos Ayres, the formation of whose numerals resembles that of the Luli, but which are still more remarkable for their extreme poverty.

1. Iniatedà.
2. Inabaca.
3. Inabacao-cainil, *two above.*
4. Inibacao-cainibà, *two above two*, or natolatata.
5. Inibacao-cainibà Iniatedà, *two above two one*, or natolatata ioiatedà, *four one.*
6. Natolatatata inibaca, *four two.*
7. Natolata-inibacao-caini, *four two above.*
8. Natolata-natolata, *four four.*

It ought to be observed, however, that the Mocobi possess practical methods of numeration as well as other tribes, and that the preceding numerals are never used, unless in cases where they wish to make an effort to dispense with the use of their hands and feet.

Of the Guaicurus.

(274.) The Mbayi, or Guaicurus, who live on the western bank of the river of Paraguay, are unable to express any number beyond 5, without the assistance of manual action.

1. Uninitegui.
2. Iniguata.
3. Iniguata dugani, *two over.*
4. Iniguata-driniguata, *two two.*
5. Oguidi, a word equivalent to *many*, and applied equally to all numbers above *four.*

Of the Betoi.

(275.) The Betoi are a nation who live on the banks of the Casanare, which runs into the Orinoco, who speak a language whose syntax and construction is singularly complex and artificial: their numeral language, properly speaking however, possesses only one, or at most two, independent names.

1. Edojojoi.
2. Edoi, *another.*
3. Ibutu, *beyond.*
4. Ibutu edojojni, *beyond one.*
5. Rumocoso, *hand.*

It must be kept in mind, that these people, as well as those last mentioned, possess practical methods of numeration which are equally extensive with those of other American tribes.

History. Of the Maipuri.

(276.) The Maipuri, the Tamonaki, and the Yaruroes, are considerable tribes who live on the banks of the Orinoco, who agree in their general methods of numeration, and who all give the name of *man*, or *Indian*, to the number 20.

Numerals of the Maipuri:

1. Papita.
2. Avanùme.
3. Apekivà.
4. Apekipaki, *three one.*
5. Papìtaerri capiti, *one only hand.*
6. Papìta yanà pauria capiti purenà, *one of the other hand we take.*
10. Apanumerri capiti, *two hands.*
11. Papìta yanà kiti purenà, *one of the toes we take.*
20. Papita camonèe, one *Indian or man.*
40. Avanùme camonèe, *two men.*
60. Apekivà camonèe, *three men.*

The preceding numerals are used when counting human beings: in speaking of other living beings, one is termed *pariata*, and two *arinùme.* In the case of inanimate objects, one is *pakiàta*, and two *okinùme;* and in reckoning time, the first is *mapukiò*, and the second *apucùnume.* We know of no other instance of variations equally numerous, with the exception of those of Japan, where the numerals are different, according as they are applied to measures, men, animals, inanimate things, days, nights, years, and the changes of the moon.

Of the Tamonaki.

(277.) Numerals of the Tamonaki:

1. Tevinitpe.
2. Acchiacke.
3. Acchialmòve.
4. Acchiackemneve, or acchiackere-penè.
5. Amnaitòne, *hand entire.*
6. Itaconò amnponà tevinitpe, *of the other hand one.*
10. Amna-acheponàre, *hands two.*
11. Puitta-ponà tevinitpe, *of the foot one.*
15. Iptaitone, *foot two hands.*
16. Itacono-puitta-ponà tevinitpe, *of the other foot one.*
20. Tevin-itòtn, *one Indian*, or *one man.*
21. Itatono itòto yamnar-ponà tevioitpe, *of the other Indian at the hand one.*
30. Itatono itòto-ponà amna-ache ponà, *of the other Indian hands two.*
40. Acchiakè itòto, *two Indians.*
100. Amnaitòne-itòto, *hand Indians*, or *five Indians.*

• There are only two numerals *tevin* and *occhia*, for one and *two*, which can properly be considered as independent, those for 3 and 4 being clearly compound. In no case, says the Abbé Gilii, does an Indian mention a number without a corresponding action: if he asks for a fruit he raises a finger; if he mentions five, he shows his whole hand; if ten, both his hands; and if twenty, he points the fingers of his hands to the toes of his feet. The Tamonaki call the *thumb* the *father* of the fingers; the *index* is termed the finger for pointing; and the ring finger is called the finger by the side of the little one.

Of the Yaruroes.

(278.) Numerals of the Yaruroes:

1. Caneame.
2. Noeni.
3. Tarani.
4. Kevvinè.
5. Caniicchimo, *coni, one, iecki, hand, mo, alone.*
10. Yoaicchibo, *all the hands.*

Arithmetic.

11. Taonepe-caneame, to the foot (*tao*) *one.*
12. Taonepe-noeni, *to the foot two.*
15. Canitaomo, *one foot alone.*
16. Caneamotaonepè-caneame, *one foot alone one.*
20. Canipumè, *one man.*
40. Noenipumè, *two men.*

In general, however, when they count beyond 20, they take grains of sand or stones of fruit, and make them into heaps of 20 each.

Important facts established by these numerals

(279.) We have to apologize to our readers for entering at so much length into the discussion of these South American numerals, but we must plead as our apology, the uncommon interest which they possess, as illustrating nearly all the remarks which we have had occasion to make, in connection with this subject in the first part of this Article; and as proving, almost to a demonstration, the general truth of the propositions which we have there stated, with respect to the origin and universality of the natural scales of numeration. It is extremely curious, likewise, to observe with what extreme difficulty these rude children of nature abstract words from things, and how little language, in many cases, at least, is able to keep pace even with the expression of the most common of their wants.

Contrast between the perfection of the syntax and the rudeness of the numerals of many of these languages.

(280.) Philologers have spoken with admiration of the wonderful syntax and construction of many of these languages, presenting so many examples of extreme refinement and complexity; and this has been observed in the languages even of those tribes whose numeral systems are the most imperfect: it is in vain to attempt to account for such facts upon ordinary principles, and the solution of our author is, of all others, the most rational, and the most becoming a Christian philosopher, who seeks for the origin of these languages, and the laws of their construction, not in the efforts of men for the mutual communication of their wants, but in the ordinance and institution of God himself.

Difficulties in ascertaining the correct grammatical construction of many of these numerals.

(281.) A person who examines minutely the analysis which is given above of the grammatical construction of many of these systems of numerals, will find reason to suspect the existence of very considerable inaccuracies in them. We have before remarked the extreme difficulty of writing down accurately the words of any language where the ear is the only guide; and the information which Hervas obtained from many of these Missionaries was derived from mere recollection, twenty years after they had been compelled to quit their stations, when old age and calamity had impaired the activity of their memory, as well as other faculties; besides, there were many other circumstances which combined to diminish the value of the information derived from such sources: the greater part of the Jesuits who were sent to South America were Spaniards possessing few of the advantages of education, which gave such celebrity to many others of their order; who, by living amongst savages, were compelled to adopt many of their habits; who had no opportunities of literary intercourse; who saw their few books and papers perishing from the damp and insects which infest the mighty forests which characterise that vast continent; and who were compelled to submit to privations, of which a lively image is given in the reply of the poor monk to Humboldt, when asked how long he had resided in his Reduction, "On such a day I shall have completed my twenty years of mosquitoes."

Numerals of Georgia.

(282.) Among the 100 systems of Asiatic numerals which Hervas has given, we find few which suggest any particular remarks, in addition to those which we have ourselves had occasion to make, if we except the numerals of different dialects of Georgia, which are adapted to the vicenary scale, and which present the only genuine example of it in any Asiatic language. The numerals of Georgia Proper are as follow:

History.

1. Erti.
2. Ori.
3. Sami.
4. Otchi.
5. Chuti.
6. Echsi.
7. Sciuiti.
8. Rua.
9. Zchara.
10. Athi.
11. Athierti, *ten one.*
12. Athiori, *ten two.*
15. Athichuti, *ten five.*
16. Athiechsi, *ten six.*
20. Ozierti, *one twenty.*
30. Samarti, *three ten*, or, more commonly in other dialects, *twenty ten.*
40. Ormazi, *two twenty.*
50. Ormazathi, *two twenty ten.*
60. Samotzi, *three twenty.*
70. Samotzathi, *three twenty ten.*
80. Otmozi, *four twenty.*
90. Otmozathi, *four twenty ten.*
100. Assi.
1000. Athachsi.

Basque numerals.

The author attempts to prove that these dialects are analogous to the Basque, and that their vicenary Arithmetic, as well as that of the Celtic nations, were derived from a common source. The following is a list of Basque numerals, which, though similar in construction, possess no other points of resemblance.

1. Bat.
2. Bi.
3. Iru.
4. Lau.
5. Bost.
6. Sei.
7. Zospi.
8. Zortzi.
9. Bederatz.
10. Amar.
11. Amaicu, *ten one.*
12. Amabi, *ten two.*
13. Amairu, *ten three.*
14. Amalau, *ten four.*
15. Amabost, *ten five.*
16. Amasei, *ten six.*
17. Amazospi, *ten seven.*
18. Amazortzi, *ten eight.*
19. Ameretzi, *ten nine.*
20. Oguei.
21. Oguei tabat, *twenty with one.*
30. Oguei taamar, *twenty with ten.*
40. Berroguei, *two twenty.*
50. Berroguei taamar, *two twenty with ten.*
60. Iruroguei, *three twenty.*
80. Lauroguei, *four twenty.*
100. Eun.
1000. Millà.

(283.) We have examined the other parts of the work of Hervas with considerable interest, as he has travelled

Observations on other portions of the work of Herras.

over a great part of the same ground with ourselves: he attempts to prove, that the quinary Arithmetic, or rather numeration by the fingers of one hand, was practised in the infancy of the world, and discovers vestiges of it in the very general resemblance of the name for *hand* and for *five*, or, at least, of the roots of those terms. It must be confessed, however, that, in his search after such analogies, he has ventured to travel further into the very dangerous regions of etymology than can be considered either prudent or judicious. He considers, however, the almost universal prevalence of the decimal scale as a proof, that it had superseded the quinary Arithmetic long before the dispersion of nations, and appeals, in confirmation of this opinion, to the affinity of the names for 6 and 7, which both possess the characteristic letter *s* in so many languages, and which, therefore, were most probably derived from some common source; he possessed not, however, the key which more modern philologists have found out, for the classification of European and Asiatic languages, and particularly of that great class of Indo Pelasgic languages, occupying a zone of more than two-thirds of the circumference of the globe, extending from the north-western extremity of Europe, through Persia and Hindostan, to the islands of the South Sea, and which will be found to comprehend the greatest part of the nations to whose numerals he has referred, in confirmation of this part of his theory.

The author has likewise discussed, with considerable learning, the alphabetical and symbolical Arithmetic of different nations, as well as the question so often agitated, of the origin of the notation by nine figures and zero, and the date and circumstances of its introduction into Europe. The opinions which he has advanced on these subjects are not materially different from our own, and though some of the facts which he has collected are new and important, we feel compelled to leave them unnoticed, as we have already trespassed too much upon the patience of our readers, to venture upon the addition of any further extracts to those we have already given.

ERRATA.

Page.	*Col.*	*Line.*	*Error.*	*Correction.*
397,	2,	14 from top,	[illegible],	[illegible].
do	do.	19 from bottom,	Ms,	Mo.
428,	1,	26 from bottom,	necessarily,	successively
429,	2,	2 from bottom	tacevano,	facevano.
433,	do.	25 from top,	9733 5376,	97535376.
445,	1,	3 from top,	wheat,	wine.
447,	2,	21 from bottom, after *lignes* put a comma.		
448,	1,	29 from bottom, after *enlèverent* omit semicolon.		

ARITHMETIC.

PART I.

Arithmetic. Part I.

Arithmetic of abstract numbers.

(284.) There are two great divisions of the Science of Arithmetic, to which we shall adhere generally in the following treatise.

The first comprehends the fundamental rules, Notation, Addition, Subtraction, Multiplication, and Division, which will vary according to the nature of the quantities which are considered, whether integers, ordinary or decimal fractions, or concrete or compound quantities; to which may likewise be added, the rules for the extraction of the square, cube, and other roots.

Of concrete numbers.

The second comprehends the application of these rules to the solution of such classes of questions as arise in the ordinary business of life; such as questions on the rule of three, practice, interest, and annuities, &c.; a division of oor subject which we shall treat with great brevity, as sufficisut information may be obtained upon it in our ordinary books of Arithmetic.

Explanation of signs.

(285.) As the following signs are very generally used, and cootribute greatly to the distinctness of notation in many cases, and to the abbreviation of language, it may be expedient to premise an explanation of them.

(1.) + *plus*, or *more*, the sign of addition; its signification in Arithmetie being, that the numbers between which it is placed are to be added together.

Thus $7 + 3$ denotes that 7 is to be added to 3: $\frac{1}{3} + \frac{1}{4}$ means that $\frac{1}{3}$ is to be added to $\frac{1}{4}$.

(2.) − *minus*, or *less*, the sign of subtraction; its arithmetical signification being, that the second of the numbers between which it is placed is to be subtracted from the other.

Thus $7 - 3$ means that 3 is to be subtracted from 7: $\frac{1}{2} - \frac{1}{3}$ means that $\frac{1}{3}$ is to be subtracted from $\frac{1}{2}$.

(3.) × *into*, the sign of multiplication, signifying that the numbers between which it is placed are to be multiplied together.

Thus 7×3 means that 7 is to be multiplied into 3.

(4.) ÷ *by*, the sign of division, signifying that the former of the two numbers between which it is placed is to be divided by the latter.

Thus $12 \div 3$ signifies that 12 is to be divided by 3.

This last sign is not very generally used, the more common practice being to write the divisor underneath the dividend, in the form of a fraction. Thus $12 \div 3$ is equivalent to $\frac{12}{3}$.

(5.) = *equal to*, signifies that the numbers between which it is placed are equal to one anuther.

Thus $7 + 3 = 10$.

There are other signs which we shall have occasion sometimes to make use of, but their explanation may be deferred until we come to the discussion of the operations for which they are required.

Numeration and Notation.

Definitions.

(286.) Arithmetical notation may be defined to be, the expression of any number in symbols which is already expressed in words; whilst the term numeration is generally applied to the converse process, of expressing in words a number which is already expressed in symbols.

We must, of course, suppose the learner to be acquainted with the meaning of all ordinary numerical terms, such as the names of the digits, tens, hundreds, thousaods, millions, &c., as also with the full import of the phrases for the expression of compound numbers, such as three hundred and sixty-five, one thousand eight hundred and twenty-six; ten millions, three hundred and ninety-five thousand, seven hundred and eighty-four; and so on. Unless possessed of such elementary and fundamental knowledge, it would be extremely difficult to make him comprehend the notation of numbers.

Notation of digits.

(287.) The nine digits, one, two, three, four, five, six, seven, eight, nine, are denoted by the nine figures,

1, 2, 3, 4, 5, 6, 7, 8, 9.

Zero, or nothing, is denoted by 0, which is also called a *cypher*.

Articulate numbers.

The articulate numbers of the first order, or ten, twenty, thirty, forty, fifty, sixty, seventy, eighty, ninety, are denoted by

10, 20, 30, 40, 50, 60, 70, 80, 90,

a cypher being written after the digital number, which must be multiplied ioto ten, in order to produce the corresponding articulate number.

The articulate numbers of the second order, one hundred, two hundred, three hundred, &c., are denoted by

100, 200, 300, 400, 500, 600, 700, 800, 900,

two cyphers being written after the respective digital numbers.

Articulate numbers of the third, fourth, or any other order, are denoted by writing three, four, or as many cyphers after the digital number as may be equal to the number which determines the order. Thus one thousand is denoted by 1000, twenty thousand by 20000, five hundred thousand by 500000, one million by 1000000, and similarly in other cases.

The zeros, or cyphers, therefore, though without value themselves, serve to mark the values of the digits which they succeed; those digits being supposed to be multiplied into ten, a hundred, thousand, &c., according as one, two, three, &c. cyphers or places succeed them.

Examples of the notation of compound numbers.

(288.) An example or two will best explain the principle of denoting compound numbers.

Let it be required to denote by figures the number seven thousand, six hundred, and ninety-five.

Write underneath each the digital and several articulate numbers of which this compound number is composed:

Five	5
Ninety	90
Six hundred..........	600
Seven thousand	7000

The number which is the sum of these several parts is denoted by 7695, where 5 is in the place of units, 9 of tens, 6 of hundreds, and 7 of thousands: in this case, therefore, the values of the several digits, 9, 6, 7, are determined by their position with respect to the place of units; and the number denoted, by writing those digits in succession in their several places, is equal to the sum of the numbers which they would express if written separately, with the same number of cyphers as of places after each.

Let it be required to write down the number twenty-three millions, sixty-nine thousands, one hundred and seven.

Seven	7
One hundred	100
Nine thousand	9000
Sixty thousand......	60000
Three millions....	3000000
Twenty millions ..	20000000

The number which is the sum of all these parts is written 23069107 in one line; the digits being written in succession, the zeros being written in those places to which no digit corresponds.

The principle of this notation, which is sufficiently illustrated by these examples, may be stated as follows: the values of the digits increase in a tenfold proportion, in passing from the place of units from the right hand to the left, being supposed to be multiplied by ten in the second place, by a hundred in the third, by a thousand in the fourth, by ten thousand in the fifth, and so on; and the number denoted by those digits written in succession, is the sum of the numbers which they severally denote, when their values are considered with reference to the place of units.

Thus, the number denoted by 2345 is equivalent to the sum of 2000, 300, 40, and 5; or, if we pass from symbols to numeral language, it is equal to two thousand, three hundred, and forty-five.

The number denoted by 8400018 is equivalent to 8000000, 400000, 10, and 8, or to eight millions, four hundred thousand, and eighteen.

The number denoted by 111000111 is equivalent to $100000000 + 10000000 + 1000000 + 100 + 10 + 1$, or to one hundred and eleven millions, one hundred, and eleven.

(289.) The following table of numeral terms, with the expressions in figures for the equivalent numbers, will materially assist the learner in the notation of any number when given in words, or in its numeration when expressed in symbols.

Table of numeral terms with their notation.

Unit	1
Ten......................................	10
One hundred	100
Thousand	1000
Ten thousand	10000
Hundred thousand	100000
Million.................................	1000000
Ten millions	10000000
Hundred millions....................	100000000
Thousand millions.................	1000000000
Ten thousand millions..............	10000000000
Hundred thousand millions	100000000000
Billion	1000000000000
Trillion..................	1000000000000000000
Quatrillion	1000000000000000000000000

(290.) Our language possesses no simple names for numbers in the decuple series, 1, 10, 100, &c., except for the 1st, 2d, 3d, 4th, 7th, 13th, 19th, &c.; and it has therefore been usual to separate numerical expressions into members or periods of six, the first embracing all numbers below a million, the second millions, the third billions, and so on, thus affording an aid to the eye, by which their numeration is more easily effected. Thus, the number denoted by

Numbers separated into periods of six places.

2340,064039,672107,

is two thousand three hundred and forty billions, sixty-four thousand and thirty-nine millions, six hundred and seventy-two thousand, one hundred and seven; and the number denoted by

10076,432897,158204,000621,

is ten thousand and seventy-six trillions, four hundred and thirty-two thousand eight hundred and ninety-seven billions, one hundred and fifty-eight thousand two hundred and four millions, six hundred, and twenty-one. The numeration of each period is the same as for the first six places, being only, instead of units, millions for the second period, billions for the third, trillions for the fourth, and so on.

(291.) As the values of the digits increase in a decuple proportion, in passing from the place of units from the right to the left, it is a very natural extension of this principle to consider digits on the right of the place of units as decreasing in a decuple proportion from left to right: thus

Principle of the notation of decimals.

32.124

would denote $3 \times 10 + 2 + \frac{1}{10} + \frac{2}{100} + \frac{4}{1000}$, a dot being placed after the place of units, to determine its position with respect to the other digits: and again,

7634.0345

is equivalent to $7 \times 1000 + 6 \times 100 + 3 \times 10 + 4 + \frac{0}{10} + \frac{3}{100} + \frac{4}{1000} + \frac{5}{10000}$. The digits in the 1st, 2d, 3d, 4th, &c. place to the right of the place of units, being divided by 10, 100, 1000, 10000, &c. respectively, whilst those in corresponding positions to the left are multiplied by the corresponding numbers in the same series. In the numeration of such numbers, the digits to the right of the place of units are *tenths, hundredths, thousandths, ten thousandths*, &c., corresponding to the places of *tens, hundreds, thousands, ten thousands*, amongst integral numbers. Thus

3.245

is read three, two-tenths, four-hundredths, five thousandths. Also,

.006934

is read six thousandths, nine ten-thousandths, three hundred-thousandths, four millionths, and similarly in other cases.

(292.) Such fractions are called *decimal fractions*, or *decimals*, to distinguish them from integers, though they are all equally subject to the same decimal scale, or classification of values. The dot also is termed the *decimal place*; all the digits to the right of it being considered as decimals, though in this respect no great regard is had to propriety of language. It would be more proper to place the dot beneath the digit in the place of units, which is the point of departure; the digits to the right and to the left, possessing value from their position with respect to it; but this would

Decimal point.

lead to some inconvenience, when there were no integral numbers in the expression. Thus

75.036

would be conveniently denoted by

75036;

but there would be some degree of awkwardness, though no ambiguity, in denoting

.00062

by

00062

Decimals and integers may be included under common rules.

(293.) It has been usual in books of Arithmetic to separate the rules for operation with integral numbers from those in which decimals are also involved; and though the notation of such quantities is reducible to a common principle in both cases, and though it would not be difficult to frame the rules for addition, subtraction, multiplication, and division, so as to include them both, we shall adhere to the common practice, as better adapted for the purposes of elementary instruction. A student in Arithmetic is not likely to possess much power of generalization, and it seems expedient that he should first be familiarized with the common operations with whole numbers only, without having additional difficulties thrown in his way, by the greater complexity of the rules which would be necessary, in order to embrace decimals as well as integers.

ADDITION.

Rule.

(294.) *To add is to collect several numbers into one sum.*

For this purpose, the numbers must be written underneath each other, so that units may stand under units, tens under tens, hundreds under hundreds; and we then proceed to add the digits in each column into one sum, and write the result underneath. Thus, if we have to add 321 to 237, they must be written thus,

$$\begin{array}{r} 321 \\ 2\cancel{3}7 \\ \hline 558 \end{array}$$

And the sum 558 is found by adding successively 7 to 1, 3 to 2, and 2 to 3.

Method of carrying tens.

But if the sum of the digits in the same column exceed 10, we must write down the excess, and *carry* 1 to the next column. Thus the sum of 27 and 56, or

$$\begin{array}{r} 27 \\ 56 \\ \hline 83 \end{array}$$

is found, by first adding 6 and 7 together, whose sum is 13: we write down 3, and carry 1 to the sum of the digits of the next column, which thus becomes 8.

Let it be required to add together 303, 727, 1069, and 35:

$$\begin{array}{r} 303 \\ 727 \\ 1069 \\ 35 \\ \hline 2134 \end{array}$$

The sum of the digits in the first column is 24, write down 4, and carry 2. The sum of $\cancel{2}$, 3, 6, 2, in the second column, is 13: write down 3, and carry 1: the sum of $\cancel{1}$, 7, 3, in the third column, is 11: write down 1, and carry 1; the sum of $\cancel{1}$ and 1 is 2, which, written down, gives the entire sum of the numbers required.

In this case, we have denoted the numbers which are *carried* from one column to another with *scratched* figures, to distinguish them from those which actually appear in the original sums to be added.

The principle of the rule for carrying the *tens* from one column to another, so important in the incorporation of numbers into one sum, whether in addition or multiplication, must be at once understood by any one who fully comprehends the principle of notation by nine figures and zero; and we should most probably create a difficulty where none existed before, by any attempt to explain it. Demonstrations become difficult and unsatisfactory, when the relation between the premises and conclusion is so simple that the mind at once perceives it; and in such cases, what is gained in form is generally lost in perspicuity.

Examples:

Examples.

1.	96341 25784 10001 7249 70000 ——— 209375	2.	12345 23456 34567 45678 56789 ——— 172835
3.	373737 363636 ——— 737373	4.	999999 101010 ——— 1101009

SUBTRACTION.

(295.) *To subtract one number from another, is to find their difference, or to find a number which added to the first will produce the second.*

Rule.

Place the number to be subtracted underneath the other, in the same manner as in addition, and then subtract the digits underneath successively from those above. Thus, to subtract 237 from 558, write them as follows:

$$\begin{array}{r} 558 \\ 237 \\ \hline 321 \end{array}$$

Subtract 7 from 8, the remainder is 1; 3 from 5, the remainder is 2; 2 from 5, the remainder is 3: we thus get the entire remainder, which is 321.

Process of borrowing ten.

In this example, the digits in the subtrahend are severally less than those above them; in case one or more of them are greater, we must add 10 to the upper digit, and increase the lower digit in the next column by 1; in other words, the 10 which we *borrow*, to increase the upper digit in the first column, we must *repay* by increasing the lower digit by 1 in the next: thus, in the example,

$$\begin{array}{r} 32 \\ 27 \\ \hline 5 \end{array}$$

we increase 2 by 10, which makes 12, from which we subtract 7, which leaves 5; we increase the digit 2, in the next column, by 1, which becomes 3, and being subtracted from 3 leaves no remainder.

Again, in the example,

34503
15376
19127

we add 10 to 3 and subtract 6 from 13, which leaves 7; we then increase 7 by 1, and 0 by 10, and, therefore, subtract 8 from 10, which leaves 2; we increase 3 by 1, and, therefore, subtract 4 from 5, which leaves 1; we increase 4 by 10, and then subtract 5 from 14, which leaves 9; we then increase 1 by 1, and subtract 2 from 3, which leaves 1, and thus get the entire remainder, which is 19127.

Examples. Examples.

1.	32104 7963 24141	2.	232323 41414 190909
3.	101101101 90909090 10192011	4.	987654321 123456789 864197532

MULTIPLICATION.

Definitions. (296.) *To multiply one number by another, is to add the first as often as the second denotes, or conversely.*

The first of these numbers is called the *multiplicand*, the second *the multiplier*, the *result* of their multiplication is called the *product*.

The definition which we have given of multiplication rather indicates what the operation is equivalent to, than guides us to the mode in which it may be performed: the product is the sum of the multiplicand repeated as often as there are units in the multiplier, but the object of *multiplication* is to enable us to find this sum, or product, by a short and simple process, which supersedes the necessity of these repeated additions.

Multiplication table. (297.) For this purpose, it is absolutely necessary to commit to memory the products of all numbers as far as 10 into 10, into each other. The following table extends as far as 12 into 12, and it is expedient and usual, though not necessary, to learn it under this extended form.

Multiplication Table.

1	2	3	4	5	6	7	8	9	10	11	12
2	4	6	8	10	12	14	16	18	20	22	24
3	6	9	12	15	18	21	24	27	30	33	36
4	8	12	16	20	24	28	32	36	40	44	48
5	10	15	20	25	30	35	40	45	50	55	60
6	12	18	24	30	36	42	48	54	60	66	72
7	14	21	28	35	42	49	56	63	70	77	84
8	16	24	32	40	48	56	64	72	80	88	96
9	18	27	36	45	54	63	72	81	90	99	108
10	20	30	40	50	60	70	80	90	100	110	120
11	22	33	44	55	66	77	88	99	110	121	132
12	24	36	48	60	72	84	96	100	120	132	144

Part I. Remarks connected with it. Many of the remarks, which an examination of the numbers in this table would suggest, however interesting, would be useless here, as having no connection with its immediate object and application: there are some others, however, which, though extremely simple and obvious, it may be worth while to point out.

Square numbers. The numbers included in the black squares, which form the diagonal of the great square, are the squares of the numbers from 1 to 12, or the products of 2 into 2, 3 into 3, 4 into 4, &c.

Two similar portions of the table. The portions of the table on each side of this diagonal are identical with each other, as will be immediately seen from an examination of the numbers in the squares on each side, whose diagonals are in the same straight line.

The same product from different factors. Those numbers which occur more than once on the same side of the diagonal, may arise from the product of different combinations of numbers between 1 and 12: thus 18 may arise from 6 and 3, or from 9 and 2; 48 from 6 and 8, or from 4 and 12; 72 from 8 and 9, or from 6 and 12; and, similarly, for the numbers 12, 20, 24, 30, 40, and 60. The only square numbers which occur in this portion of the square are 16 and 36, which are the squares of 4 and 6, or the products 8 and 2, and of 9 and 4, or 12 and 3.

Other remarks. The series of square numbers exceed by unity the number in the adjoining square in the same diagonal, which are 4 and 3, 9 and 8, 16 and 15, 25 and 24, 36 and 35, and so on, as far as 121 and 120: in other words, the square of a number exceeds by unity the product of the two numbers, which differ from it by 1, one in excess and the other in defect.

Pursuing the examination of numbers in the same diagonal, we find those in the second square from the centre differing from the square number placed therein by 4; those in the third by 9, in the fourth by 16, and so on: in other words, the product of two numbers, differing in excess and defect by 1, 2, 3, 4, &c. from any number, will be less than its square by the squares of that difference.

Conclusions like these may be generalized, and applied to any numbers whatsoever; but such generalizations must be made with the greatest caution and distrust, and never admitted as proved, unless it can be shown that the conclusion does not depend upon the particular magnitude of the numbers which are used.

Formation of squares from 12 to 50. (298.) There are some cases in which it is expedient to learn by heart the products of numbers beyond the limits of this table. Of this kind, are the squares of all numbers as far as 25 or 30, and even farther, the knowledge of which is frequently useful, and particularly so for enabling us to form very readily the products of numbers equidistant from them, by a method founded on the preceding observations.

The square of 13,	169.	22,	484.
14,	196.	23,	529.
15,	225.	24,	576.
16,	256.	25,	625.
17,	289.	26,	676.
18,	324.	27,	729.
19,	361.	28,	784.
20,	400.	29,	841.
21,	441.	30,	900.

It is a very amusing and instructive practice to observe the analogies which may exist amongst these, or any other connected series of numbers, and to notice such

Arithmetic. points of resemblance or diversity, as may serve the purposes of a technical memory. Thus the squares of 13 and 14 are 169 and 196, the two last digits being the same in each, but in an inverted order: the two last figures in the square of 15 are the two first in that of 16: the squares of 24 and 26, each differing from 25 by 1, differ from each other by 100: those of 23 and 27, differing from 25 by 2, differ from each other by 200: those of 22 and 28, differing from 25 by 3, differ from each other by 300: those of 21 and 29, differ by 400; of 20 and 30, by 500. If we extend the conclusion, the squares of 19 and 31 should differ by 600, or, in other words, the square of 31 should be 961; whilst, in the same manner, we should find the square of 32 to be 1024; that of 33 to be 1089, and similarly for other numbers as far as 50; the general rule being as follows: "if two numbers are equidistant from 25, the square of the greater exceeds the square of the less, by as many hundreds as the number itself exceeds 25."

Rule.

Formation of squares from 50 to 100.

(299.) If we wished to form the squares of all numbers above 50 from those below 50, it might be easily done by the following rule: if two numbers be equidistant from 50, the square of the greater exceeds that of the less by twice as many hundreds as the number itself exceeds 50. The truth of this rule would be readily ascertained from the actual formation and examination of any number of the squares themselves.

Importance of such observations

(300.) Observations like these are easily made, and save the memory from much useless labour; and it is impossible for a student to habituate himself too soon to the practice of such examinations as are the foundation of them. It is true, that the rules of Arithmetic are formed generally for the use of those who have not arrived at an age when the reflective and reasoning faculties are sufficiently exercised and strengthened to enable them to understand fully the principles of the rules which they follow: but it may justly be doubted, whether the acquiescence in this principle of education, is not much too general, and whether habits of investigation and inquiry are not checked, at least, if not destroyed, by teaching the student to follow merely mechanical rules, in which the understanding takes no part.

(301.) But it is proper to return from this digression to the immediate uses of the multiplication table, as exemplified in the process of multiplication of numbers, one or both of which are beyond the limits of the table.

Rule for multiplication where one factor exceeds the limits of the table.

Let one of the numbers only be within the limits of the multiplication table. In this case the greater number must be made the multiplicand, and the less number the multiplier. Multiply successively every digit of the multiplicand by the multiplier; the several products are known from the table, and in forming the whole product, we must carry the *tens* in the product of the first digit to the product of the second, and so on to the end. An example will explain our meaning more clearly.

Let it be required to multiply 237 by 9.

Write them as follows:

$$\begin{array}{r} 237 \\ 9 \\ \hline 2133 \end{array}$$

The product of 9 and 7 is 63; write down 3 and *carry* 6, or retain it in the mind as a number to be added to the next product: the product of 9 and 3 is 27; to this add 6, which makes 33; write down 3 and carry 3: the product of 9 and 2 is 18; to this add 3, and the sum is 21, which, written down, gives 2132, the entire product required. Part I.

The same result would be obtained by the addition of 237 nine times to itself, as follows:

$$\begin{array}{r} 237 \\ 237 \\ 237 \\ 237 \\ 237 \\ 237 \\ 237 \\ 237 \\ 237 \\ \hline 2133 \end{array}$$

The multiplication of the successive digits 7, 3, 2, by 9, is equivalent to the addition of these digits 9 times repeated, and the numbers carried in each case are obviously the same.

Let it be required to multiply 9876 by 12.

$$\begin{array}{r} 9876 \\ 12 \\ \hline 118512 \end{array}$$

The product of 12 and 6 is 72; write down 2 and carry 7: the product of 12 and 7 is 84, add 7, and the result is 91; write down 1, and carry 9: the product of 12 and 8 is 96, add 9, the result is 105; write down 5, and carry 10: the product of 12 and 9 is 108, add 10, the result is 118, which written down gives the entire product.

(302.) The next case to be considered is that in which both the numbers to be multiplied together exceed the limits of the table.

Rule where both the factors are beyond the limits of the table.

In this case it is most convenient to make that number the multiplier which possesses the smallest number of *digits*; we then multiply the multiplicand successively by the digits of the multiplier, placing the several products underneath each other, the digit in the units' place in the second under the digit in the tens' place in the first product, and so on throughout: we then add these results together, in order to get the entire product. Thus, suppose it were required to multiply 2349 by 876, the form of the process is as follows:

$$\begin{array}{r} 2349 \\ 876 \\ \hline 14094 \\ 16443 \\ 18792 \\ \hline 2057724 \end{array}$$

We first multiply 2349 by 6, the result is 14094; we next multiply 2349 by 7, the result is 16443, which is written underneath the first result, so that the last figure of one may be under the last but one of the other; we lastly multiply 2349 by 8, and the result is 18792, which is placed in a similar manner: the digits in the several columns are added together, and the final product is obtained.

If the several results had been written down at full length, the scheme of the process would have appeared as follows:

```
   2349
    876
-------
  14094
 164430
1879200
-------
2057724
```

The fact is, that the digits of the multiplier denote 800, 70, and 6, respectively, and we, properly speaking, multiply by 70 and 800, and not by 7 and 8. The result, however, of the multiplication of a number by 70 differs from its product by 7, merely in having an additional cypher after the significant digits; whilst the product produced by multiplying by 800 differs from that with 8, merely in having 2 additional cyphers after it: it is quite clear, however, that the final result which is obtained by following the directions of the rule, and omitting the cyphers, is the same as if they were inserted in full; and it is an important principle in all arithmetical rules, to dispense with the writing down of all figures which are superfluous in practice, however much they may otherwise contribute to make the operation better understood.

Products of numbers terminated by cyphers.

(303.) The product of 10 into 10 is 100, or, expressed in the abbreviated form which the use of signs enables us to give it,

$$10 \times 10 = 100.$$

Again,

$$10 \times 100 = 1000,$$
$$10 \times 1000 = 10000,$$
$$100 \times 100 = 10000,$$
$$100 \times 1000 = 100000,$$
$$1000 \times 1000 = 1000000,$$
$$100 \times 10000 = 1000000,$$
$$10 \times 100000 = 1000000,$$

It appears from hence, and these results are an immediate consequence of the decimal notation, that the product of two numbers, expressed by 1 and any number of cyphers after it, is the number denoted by 1 with as many cyphers as are equal to the sum of those in the two factors: and the same rule applies, as far at least as the number in the product is concerned, when any other numbers terminated by cyphers are concerned; thus the product of 30 and 300, or

$$30 \times 300 = 9000,$$
$$70 \times 800 = 56000,$$
$$1200 \times 1300 = 1560000,$$
$$16000 \times 16000 = 256000000.$$

Rule.

The rule, therefore, for such cases, may be stated as follows: "Multiply the significant digits as if there were no cyphers after them, and append to their product as many cyphers as are equal to the sum of the number of those in the multiplicand and multiplier." The following is an example:

```
      461200
      273000
------------
       13836
      32284
      9224
------------
125907600000
```

(304.) In case cyphers occur between the significant digits of the multiplier, they are, of course, passed over in the process of multiplication, and the first place of the product formed by the next significant digit is removed as many places to the left as there are cyphers passed over. We will take the following example:

Where cyphers occur between the significant digits.

```
      207392
      504003
------------
      622176
   829568
1036960
------------
104526190176
```

The reason of this rule will be manifest, if we should perform the multiplications with the cyphers as well as with the significant digits, in which case the process would produce the following scheme:

```
      207392
      504003
------------
      622176
     000000
    000000
   829568
  000000
1036960
------------
104526190176
```

Definition of multiplication must be modified when the multiplier is not an abstract whole number.

(305.) The definition which we have given of multiplication, considering it as equivalent to the addition of the multiplicand, repeated as often as unity is contained in the multiplier, is strictly applicable to those cases only where the multiplier is an abstract *whole* number: in all other cases, its meaning must be modified to suit the particular nature of the case, and at the same time to coincide strictly with the preceding, which is its *primitive* definition, in all those points which they possess in common. We shall have occasion to notice this subject again, when we come to the discussion of the multiplication of fractions.

(306.) Examples.

Examples.

```
1.   1826            2.    111
      365                  111
   ------                -----
     9130                  111
    10956                 111
    5478                 111
   ------                -----
   666490                12321

3.    12321          4.    7390460
      12321                5440300
  ---------             -----------
      12321                2217138
     24642               2956184
    36963               2956184
   24642               3695230
  12321                -----------
  ---------            40106319538
  151807041
```

DIVISION.

Definition.

(307.) *To divide one number by another, is to find how often the second is contained in the first; or, in other words, to find how often the second may be subtracted continually from the first, until nothing remains, or, at least, until the number which remains is less than the second.*

The *first* of these numbers is called the *dividend*, the *second* the *divisor*, and the number which results from the operation is called the *quotient*.

The *quotient* is perfect or complete when there is no *remainder*; *imperfect* when there is. In the first case, the product of the *quotient* and *divisor* produces the *dividend*; in the second case, this product differs from the *dividend* by the *remainder*.

The operation of Division is the inverse of that of Multiplication, and the rule is founded upon a retracing the steps of the process of multiplication. The different cases also depend entirely upon the divisor, in the same manner as the cases of multiplication depend upon the multiplier.

Rule when the divisor is within the limits of the multiplication table.

(308.) The first of these cases is, where the divisor is a number within the limits of the multiplication table: we write the divisor and dividend consecutively in the same line, merely separating them by a small curved line; we then inquire, how often the divisor is contained in the first one, two, or three figures of the dividend: we write the quotient below, and to the remainder we annex the next figure of the dividend; we find the quotient of this number, and repeat the same operation continually, until all the figures in the dividend are exhausted, and the quotient, whether perfect or not, is obtained.

Examples.

(309.) The following are examples:

(1.) 7) 168
      24

We find that 7 is contained twice in 16; the figure in the quotient is 2, and the remainder 2, to which we annex 8, which gives us 28 for the next number to be divided, of which the quotient is 4; and there is no remainder.

(2.) 12) 112496340
      9374695

In this case, it is necessary, at first, to take three places of the dividend, before we get a number which is greater than the divisor.

(3.) 4) 315
      $78\frac{3}{4}$

In this case, there is a remainder 3 after the operation, and it is usual to distinguish it from the integral part 78 of the quotient, by writing the divisor underneath it, with a line between: 78 is the *imperfect quotient* of 315 divided by 4; the complete quotient would require the remainder to be appended to it in the manner represented above.

Fractions—their origin and meaning.

(310.) The quantity represented by $\frac{3}{4}$ is termed a *fraction*, and originates in the process of division: it might be termed the *quotient* of 3 divided by 4; under such a view of its origin and meaning, it must be a quantity of such a kind, that when multiplied by 4 the product is 3; for the operation of division being the inverse of that of multiplication, it follows, that the number 3 being first *divided* by 4, and the *quotient* $\frac{3}{4}$ again *multiplied* by the same number 4, the final result must coincide with the original number 3.

We are enabled, in all cases, to make the quotient *complete* by appending the remainder, with the divisor underneath it, in the form of a *fraction*: and it must always be understood, when such a *fraction* is written after an integral number, without any sign being interposed, that it is to be added to the number which precedes it; thus $78\frac{3}{4}$ is equivalent to $78 + \frac{3}{4}$.

It is clear, likewise, that the same notation may be applied to denote the *quotient* of the *division* of any number by another; thus, the quotient of 315, divided by 4, may be denoted by $\frac{315}{4}$; for it answers the condition which the *quotient* must satisfy; that is, if multiplied by 4, it produces 315.

The term *fraction*, or *broken numbers*, which is generally applied to such quantities as $\frac{3}{4}$, originates in a view of their origin, which is different from the preceding, though it leads to the same conclusion, as we shall see when we come to the express discussion of such quantities.

When the divisor is made up of factors within the limits of the multiplication table.

(311.) There are many cases where the divisor is not within the limits of the table, but where it is the product of two or more numbers which are so, which may be known from trial, or otherwise; in such cases, the quotient may be obtained by successive division by the factors of the divisor, as in the following examples:

(1.) To divide 20390216 by 56:

8) 20390216
7) 2548777
   364111

As we obtain the same product 20390216, whether we multiply 364111 at once by 56, or first by 7, and then by 8; so, likewise, we produce the same quotient, whether we divide 20390216 at once by 56, or successively by 8 and 7.

(2.) To divide 7014596 by 72:

6) 7014596
12) 1169099 − 2
    97424 − 11

or, $97424\frac{68}{72}$.

The remainder.

The first remainder is 2; the second is 11; if this be reduced to the form of a quotient, it is equivalent to $\frac{11}{12}$, or $\frac{66}{72}$, multiplying and dividing by the same number 6: to this must be added the first remainder 2, which is equivalent to $\frac{2}{72}$; we thus get the whole additional part of the quotient, which is $\frac{68}{72}$.

Or the same result may be obtained as follows: every unit in the quotient of the division by 6, may be considered as corresponding to 6 units of the dividend: the remainder 11 of the second quotient is, therefore, equivalent to 66 units of the dividend, to which if 2 be added, the sum is 68, which, if reduced to the form of a quotient from the division by 72, gives the fraction $\frac{68}{72}$.

Long division.

(312.) When the divisor is not at once resolvable into factors within the limits of the table, or when its composition is unknown, we must resort to the process termed *long division*, which is applicable to all cases. It is as follows:

Rule.

Write the divisor and dividend consecutively, separating them by a curved line, as in *short* division; the quotient is written after the dividend, and separated from it in the same way as the divisor: inquire how often the divisor is contained in as many of the highest places of the dividend as there are places in the divisor; but if the number thus formed be less than the divisor, an additional place of the dividend must be taken; place the digit thus found in the quotient, and

multiply the divisor by it, and place the result beneath the assumed portion of the dividend, and subtract the one from the other; to the remainder append the next figure in the dividend, and repeat the process until all the places in the dividend are exhausted.

Examples. The process will be better understood from its application to a few examples.

Let it be required to divide 42075 by 275:

275) 42075 (153
275
1457
1375
825
825

The first three places of the dividend make a number which is greater than the divisor, which is contained once in it: the figure in the quotient is 1; subtract 275 from 420, the remainder is 145; append to this 7, the next figure in the dividend, when the number to be next divided becomes 1457; the number 275 by trial is found to be contained 5 times in it; write down 5 in the quotient, and multiply 275 by 5, the product is 1375, which subtracted from 1457 leaves 82; to this append 5, and the next number becomes 825, which contains the divisor thrice; write 3 in the quotient, and multiply 275 by 3, and the result is 825, which subtracted leaves no remainder.

If the process were written at full length, it would appear as follows:

275) 42075 (100 + 50 + 3 = 153
27500
14575
13750
825
825

We first multiply 275 by 100, and subtract the result, which leaves 14575: we next multiply 275 by 50, and subtract the result, which leaves 825: we then multiply 275 by 3, and subtract the result, which leaves no remainder. We have thus subtracted 100 + 50 + 3 or 153 times 275 from the dividend, and there is no remainder: in other words, 153 is the *perfect* quotient of the division under this form of the process: the cyphers are superfluous, and 5 is written once more than necessary. The other form of it, which is a *skeleton* of the complete one, is the best adapted to practice, inasmuch as it omits all unnecessary writing.

Let it be required to divide 29137062 by 5317:

5317) 29137062 (5479 $\frac{5219}{5317}$
26585
25520
21268
42526
37219
53072
47853
5219

In this example, we take 5 places of the dividend for the first division, though there are only 4 places in the divisor; the last remainder is 5219, and the quotient corresponding to it is the fraction $\frac{5219}{5317}$.

Let it be required to divide 31086917 by 71000.

Cyphers in the divisor. When there are cyphers after the significant digits in the divisor, we mark off as many places from the dividend as there are cyphers in the divisor, and then proceed to divide the remaining places of the dividend by that divisor, which arises from the omission of the cyphers.

71,000) 31086,917 (437 $\frac{30917}{71000}$
284
268
213
556
497
30917

The reason of this process will be at once seen if we write it at full length.

71000) 31086917 (400 + 30 + 7
28400000
2686917
2130000
556917
497000
59917

Cyphers in the divisor and dividend. If there are cyphers terminating both the dividend and divisor, we may obliterate altogether as many as are common to each of them.

Let it be required to divide 239406000 by 12100000:

121,00~~000~~) 2394,06~~000~~ (19 $\frac{9506}{12100}$
121
1184
1089
9506

Other examples. (313.) Other examples:

(1.) To divide 10000 by 3:

3) 10000
3333⅓

(2.) To divide 83016572 by 240:

8) 8301657,2
3) 1037707 − 1
345902 − 1

or, 345902 $\frac{92}{240}$.

(3.) To divide 29383945593000 by 84050000:

8405,0~~000~~) 2938394559,3~~000~~ (349600 $\frac{65593}{84050}$
25215
41689
33620
80694
75645
50495
50430
65593

Proof of Addition, Subtraction, Multiplication, and Division.

(314.) *On the methods of verifying the correctness of the operations in the Addition, Subtraction, Multiplication, and Division of whole numbers.*

Different methods have been proposed for verifying the correctness of the results obtained in the fundamental operations of Arithmetic. Thus, in Addition, we are directed to add the digits in the several columns downwards, and see whether the result thus obtained, agrees with that obtained by adding them in the contrary direction. In Subtraction, we must add the *remainder* to the *subtrahend*, and observe whether the sum equals the *minuend*. In Division, we are directed to multiply the divisor into the *complete quotient*, the result of which ought to equal the *dividend*; but the method, which of all others is the most popular, and we may add, likewise, the most general, is that which is founded upon casting out the 9's, the principle of which we shall now proceed to explain.

The remainder from dividing a number by 9 is the same as from dividing the sum of its digits by 9.

(315.) If we divide the series of articulate numbers 10, 20, 30, 40, 50, 60, 70, 80, 90, by 9, the remainders are 1, 2, 3, 4, 5, 6, 7, 8, respectively; and the same is the case by whatever number of cyphers these digits are succeeded: in other words, the remainder from the division of any number, such as 8 3 4 5 6 7 2 3, is the same, whether we consider it as equivalent to 80000000 + 3000000 + 400000 + 50000 + 6000 + 700 + 20 + 3, or as simply equal to the sum of its digits, or 8 + 3 + 4 + 5 + 6 + 7 + 2 + 3: that is, the remainder from dividing any number by 9, is the same as that which arises from dividing the sum of its digits by 9. It is this theorem which is the foundation of the rule in all cases.

In addition.

(316.) In the first place, for addition, the rule is as follows: *Cast the 9's out of the digits of the several sums to be added, and also out of the sum of the several remainders; the last remainder thus obtained is equal to the remainder which results from casting out the 9's from the sum of the sums.* The following is an example:

$$\begin{array}{r}
78403 - 4 \\
20465 - 8 \\
79639 - 7 \\
57341 - 2 \\
\hline
235848 - 3
\end{array}$$

The sum of 7 and 8 is 15, omit 9, there remains 6: the sum of 6 and 4 is 10, omit 9, there remains 1: the sum of 1 and 3 is 4, which is the remainder from *casting out the* 9's from 7, 6, 4, and 3, and also the remainder from dividing 78403 by 9: the remainders obtained by the same process from the other numbers, and from the sum of the sums, are 8, 7, 2, and 3, respectively; and the remainder from casting out the 9's from the sum of the remainders corresponding to the several numbers, is 3, the same as that from the sum of the numbers, as it ought to be, if the addition is correct, for the following reasons.

The numbers 4, 8, 7, 2 are the remainders, from dividing the several sums by 9: if we divide their sum by 9, the remainder must be the same as that which arises from the division of the sum of the remainders by 9, which is 3.

This proof, however, cannot be considered as complete, inasmuch as this agreement of the remainders may take place even when the addition is not correct: thus, the remainder from the division of the sum would be 3, if we should from mistake have written down 234048 for the sum, and not 235848; but if the remainders are not the same, the result is certainly wrong; and a mistake generally so considerable, as to produce a difference of 9 in the sum of the digits, can hardly be considered as within the limits of ordinary errors.

(317.) The rule for proving the correctness of the result of the multiplication of two numbers is as follows:

For multiplication.

Cast out the 9's from the digits of the multiplicand, multiplier, and product; multiply the remainders from the two first together, and cast out the 9's from their product: if the remainder which thus results is the same as that from the product of the two numbers, the operation most probably is correct; if not, it is certainly wrong.

The following is an example of its application:

$$\begin{array}{lr}
\text{Multiplicand} \ldots\ldots & 3748 - 4 \\
\text{Multiplier} \ldots\ldots & 6236 - 8 \\
 & \overline{} \\
\text{Product} \ldots\ldots & 23372528 - 5
\end{array}
\qquad
\begin{array}{ccc}
 & 5 & \\
4 & \times & 8 \\
 & 5 &
\end{array}$$

The remainders, 4 and 8, from the multiplicand and multiplier, are placed in the opposite angles of a St. Andrew's cross; in one of the remaining angles is placed the remainder, 5, from their product; in the last, is placed the remainder from the product, which is likewise 5, which shows that the multiplication is correctly performed.

Proof of the rule.

(318.) The proof of this rule may be readily derived from the general theorem mentioned above, and the consideration of the nature of multiplication. The multiplicand consists of a *multiple* of 9, and a remainder; and the same is the case with the multiplier. In forming the product, we add the multiplicand to itself as often as unity is contained in the multiplier: in the first place, we add the multiplicand as often as unity is contained in the portion of the multiplier, which is a multiple of 9; the sum is clearly a multiple of 9. Again, we add the multiplicand as often as unity is contained in the remainder from the multiplier; the sum will consist of a multiple of 9, arising from the repeated addition of the multiple of 9 in the multiplicand, and another part, which arises from the addition of the remainder of the multiplicand as often as unity is contained in the remainder of the multiplier, which is clearly equivalent to the product of these remainders, which is the only part of the entire product which is not necessarily a multiple of 9. If, therefore, we reject the 9's from this product of the remainders, the remainder which results, must clearly be the same as the remainder from casting out the 9's from the product of the multiplicand and the multiplier.

For division.

(319.) The process of the rule for proving the truth of division, must clearly be founded upon that for multiplication; the dividend corresponding to the product, and the divisor and quotient corresponding to the multiplicand and multiplier. We must cast out the 9's from the divisor and quotient, and from the product of the remainders, and the resulting remainder must be equal to that which arises from casting out the 9's from the dividend. In case the quotient is not complete, the remainder after the last division, must be subtracted from the dividend, before this rule is applicable.

FRACTIONS.

Fractions.

(320.) We have before spoken of the origin of fractions, in connection with the process of division, where they are

Arithmetic.

considered as representing the quotient of the division of the numerator by the denominator.

Thus $\frac{1}{4}$ represents the quotient of 1 divided by 4; $\frac{3}{7}$ the quotient of 3 divided by 7; and similarly in other cases.

Second interpretation of their meaning.

We are thus lead to another view of their meaning, which is very simple and intelligible. The *denominator* is said to denote the number of parts into which unity is divided, and the *numerator* denotes the number of those parts which are taken.

Thus, if 1 represented any concrete quantity whatever, and, therefore, divisible into parts: and if the number of equal parts was four, one of them would be denoted by $\frac{1}{4}$; two of them by $\frac{2}{4}$, three of them by $\frac{3}{4}$, and the whole by $\frac{4}{4}$, or 1: if another equal portion of another similar unit was added, the sum would be denoted by $\frac{5}{4}$; if two were added, the sum would be denoted by $\frac{6}{4}$; and, in a similar manner, we should be enabled to interpret the meaning of any fraction whose denominator is four, whether *proper* or *improper*.

In a similar manner, $\frac{3}{7}$ would denote 3 of the seven equal portions into which unity was divided, and $\frac{10}{7}$ would denote 10 of the same portions of which unity contained 7.

In conceiving the meaning of such quantities, the mind naturally resorts to actual objects, which are divisible into parts, of whatever nature they may be, depriving them of that abstract quality, which in their representation they possess equally with whole numbers.

What fractions are equivalent to each other?

(321.) The fractions $\frac{1}{2}$, $\frac{2}{4}$, $\frac{3}{6}$, $\frac{4}{8}$ are equivalent to each other, it being clearly indifferent, whether we divide unity into 2 parts, and take 1; into 4 parts, and take 2; into 6 parts, and take 3; or into 8 parts, and take 4: in short, all fractions are equivalent to each other, which may be derived from each other, by multiplying or dividing their numerators and denominators by the same number: thus $\frac{2}{3}$ and $\frac{8}{12}$ are equivalent to each other, it being the same thing whether we divide unity into 3 parts and take 2, or divide it into 4 times as many parts, and take 4 times as many of them. The same reasoning would apply to all other fractions which are thus related to each other.

Proposition.

It is an important proposition, which is founded upon the principle just mentioned, *that fractions are not altered in value by multiplying or dividing both their numerators and denominators by the same number.*

Reduction of fractions to their lowest terms.

(322.) It is frequently requisite, however, to reduce a fraction to its lowest terms, when its numerator and denominator admit of a common divisor, or *measure*; and the discovery of this *common measure* becomes an inquiry of importance. In some cases, it is discoverable by inspection: thus, 2 is a common measure of all even numbers; and fractions, such as $\frac{6}{14}$ and $\frac{22}{36}$, are at once reducible to $\frac{3}{7}$ and $\frac{11}{18}$; in other cases, the common measures are masked in the products in such a manner as not to be discernable, without some further knowledge of the composition of numbers: thus, $\frac{63}{108}$ is reducible to $\frac{7}{12}$, from our knowledge of the multiplication table, and the same means furnish us at once with the reductions of $\frac{45}{72}$, $\frac{88}{121}$, and $\frac{36}{144}$, to $\frac{5}{8}$, $\frac{8}{11}$, and $\frac{1}{4}$. The composition of the numerators and denominators of fractions, such as $\frac{91}{147}$, $\frac{105}{378}$, $\frac{1102}{1836}$, is not discoverable by such simple means, nor indeed by any methods which are not those of successive trials. There is a general method, however, of discovering the *greatest common measure* of any two numbers, the rule for which is as follows:

Part II.

Rule for finding the greatest common measure of two numbers.

Divide the greater number by the less, and the last remainder by the last divisor continually, until there is no remainder; the last divisor is the greatest common measure required.

Example.

Thus, let it be required to find the greatest common measure of 91 and 147: or, in other words, to reduce the fraction $\frac{91}{147}$ to its lowest terms.

```
91) 147 (1
    91
    --
    56) 91 (1
        56
        --
        35) 56 (1
            35
            --
            21) 35 (1
                21
                --
                14) 21 (1
                    14
                    --
                     7) 14 (2
                        14
                        --
```

The greatest common measure is 7, and the reduced fraction is, therefore, $\frac{13}{21}$.

Proof that 7 is a measure of 91 and 147.

It does not require a very difficult analysis of this operation to prove the truth of the conclusion which is thus deduced; it being merely requisite, for this purpose, to trace the steps in an inverse order: thus, 7 is a divisor of 14, and, therefore, of 14 + 7, or 21: it is a divisor of 21 and 14, and, therefore, of their sum, which is 35: it is a divisor of 35 and 21, and, therefore, of their sum, which is 56: it is a divisor of 56 and 35, and, therefore, of their sum, which is 91: it is a divisor of 91 and 56, and, therefore, of 147. It is thus shown to be a divisor or measure, both of 91 and 147: the only principle, involved in this proof, being the very simple one, that if a number divide two others, it will divide their sum.

And also the greatest of all such measures.

It only remains to show that 7 is the greatest number which divides 91 and 147, a conclusion which will be established, if it be shown that every divisor of 91 and 147 is necessarily a divisor of 7: for, let us suppose that some number greater than 7 is a divisor of 91 and 147; if so, it must divide their difference, which is 56; and since it divides 91 and 56, it must divide also their difference, which is 35; and if it divide 56 and 35, it must divide their difference, which is 21; and if it divide 35 and 21, it must divide their difference, which is 14; and if it divide 21 and 14, it must divide their difference also, which is 7: but no number greater than 7 can divide it; therefore 7 is the greatest of all the numbers which can divide 91 and 147.

Second example.

We will take another example. Let it be required to reduce the fraction $\frac{75}{405}$ to its lowest terms.

```
75) 405 (5
    375
    ---
    30) 75 (2
        60
        --
        15) 30 (2
            30
            --
```

Arithmetic. Part I.

The reduced fraction is $\frac{5}{27}$.

Proof.

It is very easy to show that 15 must he a divisor of 75 and 405: 15 is a divisor of 30, and, therefore, of twice 30 or 60; it is, therefore, a divisor of 60 and 15, and, therefore, of their sum, which is 75: it is a divisor of 75, and, therefore, of 5 times 75, which is 375: it is a divisor of 375 and of 30, and, therefore, of their sum, which is 405.

It is very easy, by a reversion of these steps, to show that every divisor of 405 and 75, is also a divisor of 15; and that, therefore, 15 is the greatest measure of these numbers; for every number which divides 405 and 75, divides also 405 and 375, and, therefore, their difference, which is 30; and if it divides 75 and 30, it must also divide 75 and 60, and, therefore, their difference, which is 15.

Its principle general.

The principle of this proof is independent of the particular numbers involved in the preceding examples, and, therefore, equally applicable to every other case. We may, therefore, consider the rule as universally true, and that it will in all cases lead to the detection of the greatest common measure, whenever such measures exist which are greater than nnity.

Other examples.

(323.) The following are other examples:

(1.) To reduce [illegible] to its lowest terms. Answer, [illegible].

(2.) To reduce [illegible] to its lowest terms. Answer, $\frac{3}{4}$.

(3.) To reduce [illegible] to its lowest terms. Answer, [illegible].

(4.) To reduce [illegible] to its lowest terms. Answer, [illegible].

Mode of comparing fractions with each other.

(324.) In order to compare fractions with each other, it is requisite to reduce them to a common denominator, when the relation between them will be that of their numerators: thus, $\frac{2}{3}$ and $\frac{3}{4}$ being redoced to equivalent fractions, with a common denominator 12, become $\frac{8}{12}$ and $\frac{9}{12}$, and the relation between is that of the numbers 8 and 9. But it is not in these cases only, in which we wish to compare the magnitudes of fractions with each other, that such reductions are requisite, inasmuch as they are reqnired whenever fractions are to be added together, or subtracted from each other. The following is the general rule by which it is effected:

Rule for reducing fractions to a common denominator

When any number of fractions are to be reduced to a common denominator, each numerator must be multiplied into all the denominators except its own, for a new numerator, and all the denominators must be multiplied together for a new common denominator.

The least consideration of this rule will show, that the numerator and denominetor of each fraction are multiplied by the same number; namely, by the product of all the denominators except its own, and, consequently, that its value is not altered. A few examples will show this more elearly.

Examples.

(1.) To reduce $\frac{3}{4}$ and $\frac{7}{9}$ to equivalent fractions having a common denominator.

To form the new numerators, $3 \times 9 = 27$.
$7 \times 4 = 28$.

To form the common denominator, $4 \times 9 = 36$.

The new fractions are $\frac{27}{36}$ and $\frac{28}{36}$, are formed hy multiplying the numerator and denominator of $\frac{3}{4}$ by 9, and the numerator and denominator of $\frac{7}{9}$ by 4.

(2.) To reduce $\frac{2}{3}$, $\frac{5}{11}$, $\frac{11}{14}$ to equivalent fractions having a common denominator.

For the numerators,

$$2 \times 11 \times 14 = 308$$
$$5 \times 3 \times 14 = 210$$
$$11 \times 3 \times 11 = 363$$

For the common denominator,

$$3 \times 11 \times 14 = 462$$

The new fractions are $\frac{308}{462}$, $\frac{210}{462}$, $\frac{363}{462}$, which are clearly equivalent to the original fractions, inasmuch as the numerator and denominator of $\frac{2}{3}$ are multiplied by the same number 11×14, those of $\frac{5}{11}$ by 3×14, and those of $\frac{11}{14}$ by 3×11.

(3.) To reduce $\frac{5}{6}$ and $\frac{7}{8}$ to equivalent fractions having a common denuminator.

These fractions, determined by the rule, would be $\frac{40}{48}$ and $\frac{42}{48}$, which are clearly reducible to two others, $\frac{20}{24}$ and $\frac{21}{24}$, which are equivalent to the former, but in lower terms.

Reduction of fractions to their least common denominator.

(325.) In this case, the rule gives a common denominator, which is not the least of those which cao be found, and it is always expedient, and sometimes important, to exhibit the fractions under their most simple form. It is on this account requisite to modify the rule, so that the common denominator which results from it may be the least possible.

It is obvious, that any denominator which is a mnltiple of all the denominators will answer for a common denominator, and the conditions of the question will be fulfilled, therefore, hy that denominator which is the least common multiple of the denominators.

Thus, in the last example, the denominators are 6 and 8, or 2×3, and 2×4, where 2 is their greatest common measure. It is clear, therefore, that $2 \times 3 \times 4$ is a multiple of 6 and 8, and it is their least common multiple: the two fractions, therefore, become $\frac{5 \times 4}{6 \times 4}$ and $\frac{7 \times 3}{8 \times 3}$, or $\frac{20}{24}$ and $\frac{21}{24}$.

Least common multiple of two or more numbers. Rule.

(326.) The solution of this qnestion requires the determination of the least common multiple of the denominators, which may be found upon the following principle: *Find out all the simple factors of the several numbers; the numbers formed by the multiplication of the simple factors, omitting one of them, as long as it occurs in any two of the numbers, is the least common multiple required.*

Thus, suppose it were required to find the least common multiple of 14 and 63; the numbers resolved into their factors are 7×2, and $7 \times 3 \times 3$. The least common multiple is, therefore, $7 \times 2 \times 9$, or 126

Let it be required to find the least common multiple of the numbers 8, 12, and 18.

The numbers resolved into their simple factors are $2 \times 2 \times 2$, $2 \times 2 \times 3$, $2 \times 3 \times 3$; and the least common multiple is, therefore, $2 \times 2 \times 2 \times 3 \times 3$, or 72.

Arithmetical rule.

(327.) There is a common arithmetical rule which leads to the same conelusion, and which is more convenient in practice than the one just given; it is as follows:

Write down in one line the numbers whose least common multiple is required: divide those which have a common measure by that common measure, and repeat these divisions as long as any common measure exists between two or more of them: the least common multiple is the continued product of the divisors, and of the quotients of the several divisions.

Thus, in the case of the example just given, we proceed as follows:

$$\begin{array}{r|l} 2 & 8,\ 12,\ 18 \\ \hline 2 & 4,\ 6,\ 9 \\ \hline 3 & 2,\ 3,\ 9 \\ \hline & 2,\ 1,\ 3 \end{array}$$

Then $2 \times 2 \times 3 \times 2 \times 1 \times 3 = 72$,

is the least common multiple required.

Let it be required to find the least common multiple of the nine digits:

$$\begin{array}{r|l} 2 & 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9 \\ \hline 2 & 1,\ 1,\ 3,\ 2,\ 5,\ 3,\ 7,\ 4,\ 9 \\ \hline 3 & 1,\ 1,\ 3,\ 1,\ 5,\ 3,\ 7,\ 2,\ 9 \\ \hline & 1,\ 1,\ 1,\ 1,\ 5,\ 1,\ 7,\ 2,\ 3 \end{array}$$

and $2 \times 2 \times 3 \times 5 \times 7 \times 2 \times 3 = 5040$

is the least common multiple required.

The least consideration of this process will show, that by means of it, when the same common factor occurs in two or more numbers, it is obliterated in all of them, but is preserved *singly* in the divisor, which becomes a factor of the least common multiple: it is, therefore, clearly identical with the rule first given, but is exhibited in a form which is better adapted to arithmetical practice.

Examples of the reduction of fractions to their least common denominator.

(328.) We will now resume the subject of the redoetioo of fractions to their least commoo deoominator, which introduced the process for fiodiog the least common multiple; and, suppose it was required to reduce the fractions $\frac{7}{8}$, $\frac{11}{12}$, and $\frac{17}{18}$ to their least common denominator.

The least commoo multiple of the denomioator is 72, as we have found above; divide 72 by 8, 12 and 18 respectively, and we shall get 9, 6, aod 4 for the respective moltipliers of the numerators; the fractions become then $\frac{7 \times 9}{72}$, $\frac{11 \times 6}{72}$, and $\frac{17 \times 4}{72}$; or $\frac{63}{72}$, $\frac{66}{72}$, and $\frac{68}{72}$ respectively.

Let it be required to reduce the fractions $\frac{1}{7}$, $\frac{3}{8}$, $\frac{4}{9}$, $\frac{5}{24}$, and $\frac{7}{72}$, to their least common denominator.

The least common multiple of the denominators is $7 \times 8 \times 9$, or 506. The multipliers of the numerators are 72, 63, 56, 21, 7.

The frections are

$$\tfrac{72}{504},\ \tfrac{189}{504},\ \tfrac{224}{504},\ \tfrac{105}{504},\ \tfrac{49}{504}.$$

Let it be required to rednce the fractione

$$\tfrac{3}{10},\ \tfrac{7}{100},\ \tfrac{8}{10000},\ \text{and}\ \tfrac{9}{1000000}$$

to a common denominator.

The least common multiple ie 1000000. The multipliers of the numerators are

100000, 10000, 100, and 1.

The fractions are

$$\tfrac{300000}{1000000},\ \tfrac{70000}{1000000},\ \tfrac{800}{1000000},\ \text{and}\ \tfrac{9}{1000000}.$$

Mixed numbers: their reduction.

(329.) *Mixed numbers* ere those which consist partly of whole numbers, and partly of fractions, of which we have already had examples in the quotieote from the division of a number by another, which ie not contained a certain number of times exactly io it; of thie kind are $2\frac{1}{3}$, $7\frac{2}{9}$, $23\frac{4}{5}$, $1059\frac{112}{144}$, &c. Such quantities are easily reducible to a fractional form; thue $2\frac{1}{3}$ is equivalent to $2 + \frac{1}{3}$, or to $\frac{2}{1} + \frac{1}{3}$; or, reducing them to a common denominator, to $\frac{6}{3} + \frac{1}{3}$, and incorporating them by adding the numerators, and subecribing the common denominator, to $\frac{7}{3}$.

In a similar manner, $7\frac{2}{9} = 7 + \frac{2}{9} = \frac{7}{1} + \frac{2}{9} = \frac{63}{9} + \frac{2}{9} = \frac{65}{9}$.

Again, $1059\frac{112}{144} = \frac{1059}{1} + \frac{112}{144} = \frac{1059 \times 144}{144} + \frac{112}{144} = \frac{152496}{144} + \frac{112}{144} = \frac{152608}{144}$.

Improper fractions.

(330.) Again, *improper* fractions (where the numerator exceeds the denomioator) are reducible to mixed numbers, by simply dividing the numerator by the denominator, according to the ordinary rule:

Thus,

$$\tfrac{7}{3} = 2\tfrac{1}{3}.$$
$$\tfrac{23}{14} = 1\tfrac{9}{14}.$$
$$\tfrac{100}{9} = 11\tfrac{1}{9}.$$
$$\tfrac{5907}{25} = 236\tfrac{7}{25}.$$
$$\tfrac{174213}{144} = 1209\tfrac{117}{144}.$$

Addition of fractions.

(331.) The addition of fractions to each other is effected by reducing them to a common denominator, addiog their numerators, and subscribing the common denominator.

Thus, let it be required to find the sum of $\frac{2}{3}$ and $\frac{3}{4}$.

Reduced to a common denominator, they become $\frac{8}{12}$ and $\frac{9}{12}$, and their sum, therefore, $\frac{17}{12}$.

Or, more formally, thus:

$$\begin{array}{rr} 2 \times 4 = & 8 \\ 3 \times 3 = & 9 \\ \hline & 17 \end{array} \qquad 3 \times 4 = 12.$$

and the sum required $\frac{17}{12}$.

To find the sum of $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$:

$$\begin{array}{rr} 1 \times 3 \times 4 = & 12 \\ 1 \times 2 \times 4 = & 8 \\ 1 \times 2 \times 3 = & 6 \\ \hline & 26 \end{array} \qquad 2 \times 3 \times 4 = 24.$$

and the sum required $\frac{26}{24}$ or $\frac{13}{12}$.

To find the sum of 3, $\frac{2}{3}$, $\frac{7}{5}$:

$$\begin{array}{rr} 3 \times 3 \times 5 = & 45 \\ 2 \times 1 \times 5 = & 10 \\ 7 \times 1 \times 3 = & 21 \\ \hline & 76 \end{array} \qquad 1 \times 3 \times 5 = 15.$$

and the sum is $\frac{76}{15}$, or $5\frac{1}{15}$.

To find the sum of $\frac{3}{10}$, $\frac{7}{1000}$, and $\frac{11}{1000000}$:

$$\begin{array}{lcr} 3 \times 10000 & = & 30000 \\ 7 \times 100 & = & 700 \\ 11 \times 1 & = & 11 \\ \hline & & 30711 \end{array}$$

and the fraction is $\frac{30711}{1000000}$.

Subtraction of fractions.

(332.) Fractions are subtracted from each other by reducing them to a commoo denominator, subtracting their numerators, and under the remainder subscribing the common deoominator.

Let it be required to eobtract $\frac{3}{4}$ from $\frac{4}{5}$.

The fractione reduced to a common denominator are $\frac{15}{20}$ and $\frac{16}{20}$, and their difference $\frac{1}{20}$.

To subtract $\frac{8}{13}$ from $\frac{7}{11}$:

$$\begin{array}{rr} 7 \times 13 = & 91 \\ 8 \times 11 = & 88 \\ \hline & 3 \end{array} \qquad 13 \times 11 = 143.$$

and the remaiodor is $\frac{3}{143}$.

To subtract $\frac{3}{100}$ from $\frac{9}{10}$:

$$\begin{aligned} 9 \times 10 &= 90 \\ 3 \times 1 &= 3 \\ &\overline{\ \ 87} \end{aligned}$$

and the remainder is $\frac{87}{100}$.

To subtract $2\frac{3}{4}$ from $7\frac{7}{9}$:

The mixed numbers are reduced to the fractions $\frac{11}{4}$ and $\frac{70}{9}$ respectively:

$$\begin{aligned} 70 \times 4 &= 280 \\ 11 \times 9 &= 99 \\ &\overline{\ 181} \end{aligned} \qquad 4 \times 9 = 36.$$

and the remainder is $\frac{181}{36} = 5\frac{1}{36}$.

Multiplication of fractions. (333.) Before we proceed to state the rule for the *multiplication* of fractions, it is proper, in the first place, to ascertain its *meaning* when applied to such quantities, and to show in what manner it is connected with the *definition* of the term in the case of whole numbers.

The product of a fraction multiplied by a whole number is derived at once from that definition without any modification of its meaning; thus, the product of $\frac{3}{5}$ multiplied by 4 is $\frac{12}{5}$, being the result of the addition of $\frac{3}{5}$ to itself, repeated 4 times.

But 4 is equal to $\frac{4 \times 7}{7}$, or its value is not altered by being multiplied and divided by the same number 7: therefore the fraction $\frac{3}{5}$ being multiplied by 4, the product divided by 7, and the result again multiplied by 7, its value is not altered. Let us take the operations in their order:

Meaning of the operation. Multiply $\frac{3}{5}$ by 4, the result is $\frac{3 \times 4}{5}$. Divide $\frac{3 \times 4}{5}$ by 7, the result is $\frac{3 \times 4}{5 \times 7}$, which must be the case, inasmuch as $\frac{3 \times 4}{5 \times 7}$, being multiplied by 7, the result $\frac{3 \times 4}{5 \times 7} \times 7$ is equivalent to $\frac{3 \times 4}{5}$; but if we stop before this last operation, the result $\frac{3 \times 4}{5 \times 7}$, which arises from multiplying by 4 and dividing by 7, may be considered as equivalent to the product of the fraction $\frac{3}{5}$ by the fraction $\frac{4}{7}$. When we multiply, therefore, by a fraction, we mean, that we multiply by its numerator, and divide by its denominator; the only signification which it can admit of, so as to be consistent with the definition of multiplication in the case of whole numbers.

Rule. The rule for the multiplication of fractions is founded upon this view of the meaning of the operation. We must multiply the multiplicand by the numerator, and divide by the denominator; or, in other words, we must multiply the numerators of the two fractions together for a new numerator, and the two denominators together for a new denominator.

Examples. Thus, the product of $\frac{17}{19}$ multiplied by $\frac{13}{14}$ is $\frac{17 \times 13}{19 \times 14} = \frac{221}{266}$.

The product of $\frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}$ is $\frac{2 \times 3 \times 4 \times 5}{3 \times 4 \times 5 \times 6} = \frac{2}{6} = \frac{1}{3}$.

The product of $9\frac{1}{10}$, $7\frac{1}{100}$, $1\frac{1}{1000}$, or, of $\frac{91}{10}, \frac{701}{100}, \frac{1001}{1000}$, is $\frac{91 \times 701 \times 1001}{1000000} = \frac{63856791}{1000000} = 63\frac{856791}{1000000}$.

Fractions of fractions. (334.) We frequently have occasion to make use of compound fractions, or fractions of fractions, such as two-thirds of three-fourths, $\frac{2}{3}$ of $\frac{3}{4}$ of $\frac{7}{8}$, and so on. A very little examination will show that the equivalent simple fractions are formed by multiplying the several fractions of the compound fraction together.

Their meaning. Thus, when we say two-thirds of three-fourths, we mean by it two-thirds of that portion of unity which three-fourths denotes; thus, if unity be divided into 4 equal parts, and three of these be taken, and if each of these be again divided into 3 equal parts, and 2 of each of them be taken, then each of these parts will be one-twelfth of the original unit, and the number of them taken will be 2×3, or 6; the result is, therefore, equivalent to $\frac{6}{12}$, or $\frac{2 \times 3}{3 \times 4}$, or $\frac{1}{2}$, the product of the multiplication of $\frac{2}{3}$ into $\frac{3}{4}$. The same reasoning will apply to all other cases of such compound fractions:

Examples. Thus, $\frac{1}{2}$ of $\frac{2}{3}$ of $\frac{7}{8} = \frac{1}{2} \times \frac{2}{3} \times \frac{7}{8} = \frac{14}{48}$.

Again, $\frac{3}{100}$ of $12\frac{6}{7} = \frac{3}{100} \times 12\frac{6}{7} = \frac{3}{100} \times \frac{90}{7} = \frac{270}{700} = \frac{27}{70}$.

Also, $\frac{4}{5}$ of $\frac{7}{9}$ of $10\frac{3}{4} = \frac{4}{5} \times \frac{7}{9} \times 10\frac{3}{4} = \frac{4}{5} \times \frac{7}{9} \times \frac{43}{4} = \frac{301}{45}$.

Rule for division of fractions. (335.) The rule for the division of fractions is founded upon that for multiplication, the operations being the *inverse* of each other; in other words, if we multiply and divide by the same fraction, the value of the multiplicand must remain unchanged: thus, if we multiply and divide $\frac{2}{3}$ by $\frac{5}{7}$, the first operation gives $\frac{2 \times 5}{3 \times 7}$; the second must give $\frac{2 \times 5 \times 7}{3 \times 7 \times 5}$, otherwise the result would not be equivalent to $\frac{2}{3}$.

Rule. When we divide, therefore, one fraction by another, we obtain the quotient by multiplying the numerator of the dividend into the denominator of the divisor for its numerator, and the denominator of the dividend into the numerator of the divisor for its denominator; and it is clear, that the same result would be obtained by *inverting* the terms of the divisor, and then proceeding as in multiplication.

Examples. The quotient of $\frac{3}{4}$ divided by $\frac{5}{7}$ is equal to $\frac{3}{4} \times \frac{7}{5} = \frac{21}{20}$.

The quotient of $\frac{9}{100}$ divided by $\frac{3}{10} = \frac{9}{100} \times \frac{10}{3} = \frac{3}{10}$.

The quotient of $3\frac{1}{6}$ divided by $9\frac{1}{2} = \frac{19}{6} \times \frac{2}{19} = \frac{2}{6} = \frac{1}{3}$.

The quotient of $\frac{3}{4}$ of $\frac{7}{8}$ divided by $\frac{2}{3} = \frac{3}{4} \times \frac{7}{8} \times \frac{3}{2} = \frac{63}{64}$.

The quotient of $\frac{1}{10}$ of $\frac{9}{100}$ divided by $\frac{3}{1000}$ of $\frac{7}{100}$ $= \frac{1}{10} \times \frac{9}{100} \times \frac{1000}{3} \times \frac{100}{7} = \frac{900}{21} = \frac{300}{7}$.

Interpretation of the meaning of some peculiar fractional forms. (336.) There are some consequences of the notation of fractions, and of the meaning attached to them, which, though legitimate and even necessary deductions from them, it may be requisite to explain; thus, let it be required to assign the proper meaning of the fraction $\frac{1}{\frac{1}{3}}$.

This is merely the mode of denoting the quotient of the division of 1 by $\frac{1}{3}$, which, if reduced according to the general rule, is equivalent to $1 \times \frac{3}{1} = 3$.

In the same manner, $\frac{\frac{2}{3}}{\frac{4}{11}}$ is the quotient of $\frac{2}{3}$ divided by $\frac{4}{11}$, and is therefore equivalent to $\frac{2}{3} \times \frac{11}{4} = \frac{22}{12}$

Arithmetic.

Extending this conclusion, the fraction $\frac{\frac{1}{\frac{2}{3}}}{\frac{6}{11}}$ is equivalent to $\frac{\frac{3}{2}}{\frac{6}{11}}$; and, again, to $1 \times \frac{15}{12}$, or $\frac{15}{12}$.

In the same manner, $\frac{\frac{1}{1}}{3} = \frac{1}{3}$, and $\frac{\frac{1}{1}}{\frac{1}{3}} = 3$, and there is, clearly, no limit to these different notations for the same quantity.

Continued fractions.

(337.) Again, fractions such as the following

$$\frac{1}{1+\frac{1}{2}}$$

must be reduced, by first incorporating the fractions in the denominator, and then proceeding by the general rule: it thus becomes

$$\frac{1}{\frac{3}{2}} = \frac{2}{3}.$$

Another example is the fraction

$$\frac{2}{3+\frac{4}{5+\frac{6}{7}}}$$

which must be reduced by successive operations: we shall thus find,

$$\frac{2}{3+\frac{4}{5+\frac{6}{7}}} = \frac{2}{3+\frac{4}{\frac{41}{7}}} = \frac{2}{3+\frac{28}{41}}$$
$$= \frac{2}{\frac{151}{41}} = \frac{82}{151}$$

Such fractione ere called *continued* fractions, and the reductions become very complicated, when the number of terms is great, unless simplified by rules founded upon algebraical formulæ.

Examples.

(338.) The following examples will furnish instancee of most of the preceding reductions.

(1.) Reduce the fractions $\frac{161}{391}$, $\frac{20323}{35333}$, and $\frac{44441}{42861}$ to their lowest terms.

(2.) Reduce the fractions $\frac{3}{4}$, $\frac{11}{15}$, and $\frac{19}{25}$ to equivalent fractions having a common denominator.

(3.) Find the least common multiple of 3, 7, 21, 27 and 63.

(4.) Reduce the fractions $\frac{1}{3}$, $\frac{7}{15}$, $\frac{12}{25}$, and $\frac{14}{35}$ to equivalent fractions having the least common denominator.

(5.) Reduce the mixed number $117\frac{1}{7}$ to an improper fraction.

(6.) Reduce the improper fraction $\frac{289}{47}$ to a mixed number.

(7.) Add together the fractione $\frac{3}{4}$ and $\frac{2}{3}$: $\frac{17}{18}$, $\frac{15}{8}$, and $\frac{19}{23}$: and $3\frac{1}{3}$, $7\frac{1}{2}$, and $10\frac{1}{10}$.

(8.) Subtract $\frac{8}{11}$ from $\frac{11}{13}$: and $3\frac{1}{3}$ from $7\frac{1}{2}$.

(9.) Multiply $\frac{23}{30}$ by $\frac{41}{46}$: $\frac{7}{3}$ into $\frac{5}{4}$ into $\frac{2}{3}$: $3\frac{1}{3}$ into $7\frac{1}{2}$, into $10\frac{1}{10}$.

(10.) What is the value of $\frac{1}{4}$ of $\frac{1}{3}$ of $\frac{2}{5}$?

(11.) Divide $\frac{7}{10}$ by $\frac{77}{100}$, and $\frac{2}{3}$ of $\frac{4}{7}$ by $\frac{3}{5}$ of $\frac{16}{23}$.

(12.) Reduce the fractions

$$\frac{\frac{2}{3}}{\frac{4}{5}},\ \frac{7\frac{1}{3}}{8\frac{1}{5}},\ \frac{\frac{1}{7}}{\frac{42}{5}},\ \text{and}\ \frac{\frac{2}{\frac{3}{4}}}{\frac{5}{6}}$$

to their most simple forme.

(13.) Reduce the continued fraction

$$\frac{3}{4+\frac{7}{10+\frac{13}{16+\frac{19}{22}}}}.$$

DECIMALS.

Decimals.

(339.) We have before explained the neture end origin of decimals, as connected with the notation by nine figures and zero; the digits on the right of the place of units being supposed to be divided by 10, 100, 1000, 10000, &c., in the same manner as those which are respectively equidistant on the left, are multiplied by the same numbers: thus,

$$78324.2454$$

is equivalent to

$$7 \times 10000 + 8 \times 1000 + 3 \times 100 + 2 \times 10 + 4 +$$
$$\frac{2}{10} + \frac{4}{100} + \frac{5}{1000} + \frac{4}{10000}$$

and any other number involving decimals is resolvable into its component parts in a similar manner.

The decimal

$$.14159$$

is equivalent to the sum of the fractions

$$\frac{1}{10} + \frac{4}{100} + \frac{1}{1000} + \frac{5}{10000} + \frac{9}{100000}$$

which, if reduced to their least common denominator, become

$$\frac{10000}{100000} + \frac{4000}{100000} + \frac{100}{100000} + \frac{50}{100000} + \frac{9}{100000};$$

and if we add them together they become

$$\frac{14159}{100000}.$$

In a similar manner, the decimal expression

$$3.003714$$

ie equivalent to

$$3 + \frac{3}{1000} + \frac{7}{10000} + \frac{1}{100000} + \frac{4}{1000000}$$

which, if reduced in a similar manner as the expression last given, becomes

$$\frac{3003714}{1000000}.$$

Conversion of decimals into equivalent fractions.

(340.) It appears from these examples (and the method which is made use of to effect these transformations is equally applicable to all cases,) that a decimal expression may be converted into an equivalent fraction by omitting the decimal point, and subscribing for a denominator 1, with as many cyphers as there are decimal places.

Thus, 90.090909

is equivalent to

$$\frac{90090909}{1000000}.$$

Again, $.023$

is equivalent to

$$\frac{23}{1000}$$

and $.0000301$

ie equivalent to

$$\frac{301}{10000000}.$$

Converse operation.

(341.) Conversely, any fraction whose denominator is 1, with cyphers *only* following it, may be converted into an equivalent decimal, by omitting the denominator, and striking off as many decimal places in the numerator as there are cyphers in the denominator.

Thus, $\frac{23}{100}$

is equivalent to

$$.33.$$

Again,

$$\frac{141432}{10000}$$

is equivalent to

$$14.1432,$$

and

$$\frac{4087}{10000000}$$

is equivalent to

$$.0004087.$$

It is very important to attend to this transition from decimals to equivalent fractions, and its converse, as it forms the foundation of the proofs of the rules for the multiplication and division of decimals.

Addition and subtraction of decimals.

(342.) The rules for the addition and subtraction of decimals are the same as those for whole numbers, care being taken to place the corresponding places under each other.

Let it be required to add 72.031 and 4.20123 together:

$$\begin{array}{r} 72.031 \\ 4.20123 \\ \hline 76.23223 \end{array}$$

Let it be required to add together 345.012, .02468, 7692.75, and 7.4000693:

$$\begin{array}{l} 345.012 \\ .02468 \\ 7692.75 \\ 7.4000693 \\ \hline 8045.1867493 \end{array}$$

Let it be required to subtract 3.04096 from 10.345072:

$$\begin{array}{r} 10.345072 \\ 3.04096 \\ \hline 7.304112 \end{array}$$

Let it be required to subtract 113.694 from 114:

$$\begin{array}{l} 114 \\ 113.694 \\ \hline .306 \end{array}$$

The process in this case might, perhaps, be more readily understood, if the decimals were written as follows:

$$\begin{array}{r} 114.000 \\ 113.694 \end{array}$$

It is obvious, that the addition of cyphers, after the significant digits in decimals, makes no alteration of their value. Thus, 114 is equivalent to 114.000, 07 is equivalent to .070000, and similarly in all other cases.

Multiplication of decimals.

(343.) The following is the rule for the multiplication of decimals:

Multiply the decimals as if they were whole numbers, and strike off from the product as many decimal places as are equal to the sum of the numbers of decimal places in the multiplicand and the multiplier.

Let it be required to multiply together 72.037 and 3.59:

$$\begin{array}{r} 72.037 \\ 3.59 \\ \hline 648333 \\ 360185 \\ 216111 \\ \hline 258.61283 \end{array}$$

The sum of the numbers of decimal places in the multiplicand and multiplier is 5, which is the number of decimal places which must be struck off from the product of the decimals, considered as integers.

Proof of the rule.

The reason of this rule will be obvious, if we convert the decimals into equivalent fractions: they thus become

$$\frac{72037}{1000} \text{ and } \frac{359}{100}$$

and their product is,

$$\frac{72037 \times 359}{100000} = \frac{25861283}{100000};$$

and if we pass from the fraction, which is the result of the multiplication, to the equivalent decimals, it becomes

$$258.61283.$$

The same reasoning will apply in all other cases; the numerators of the fractions equivalent to the decimals, are the integral numbers which result from removing the decimal point: their denominators are 1, with as many cyphers following as there are decimal places in each; the product of the fractions is the product of the numerators, which the operation performed according to the rule always gives, divided by the product of the denominators, which is clearly 1, with as many cyphers as are found in the denominators of both the fractions; and in the transition from the fraction to the equivalent decimal, we omit the denominator, and strike off as many decimal places from the numerator as there are cyphers in it.

Let it be required to multiply .00037 into .04145:

$$\begin{array}{r} .04145 \\ .00037 \\ \hline 29015 \\ 12435 \\ \hline .0000153365 \end{array}$$

In this case, it is requisite to place cyphers to the right of the integral product, in order to get the requisite number of decimal places.

Let it be required to multiply 310000 into .375.

$$\begin{array}{r} .375 \\ 310000 \\ \hline 375 \\ 1125 \\ \hline 116250.000 \end{array}$$

In this case the product is integral.

Division of decimals.

(344.) The following is the rule for the division of decimals:

Find the quotient in the same manner as if the decimals were whole numbers; then if the number of decimal places in the divisor be equal to the number in the dividend, the quotient obtained is correct: if the number of decimal places in the divisor be less than the number in the dividend, as many decimal places must be struck off from the integral quotient, as is equal to the excess of the number in one above the number in the other; and if the number of decimal places

Arithmetic. *in the divisor be greater than the number in the dividend, as many cyphers must be written after the figures in the quotient, (the whole being integral,) as is equal to the excess of the number of decimal places in the divisor above the number in the dividend.*

In the last case, it is usual, before the division is begun, to add cyphers to the dividend, until it has as many decimal places as the divisor.

Examples. Let it be required to divide 24.075 by 7.5 :

```
7.5) 24.075 (3.21
     22.5
     ----
       157
       150
       ---
         75
         75
```

The quotient of the numbers considered as integers is 321 : but there are 3 decimal places in the dividend, and only 1 in the divisor : we must strike off, therefore, 3 − 1, or 2 decimal places from the quotient, which thus becomes 3.21.

(2.) If the divisor had been 75, the quotient would have been .321.

(3.) If the divisor had been 7500, the quotient would have been .00321.

(4.) If the divisor had been .75, the quotient would have been 32.1.

(5.) If the divisor had been .075, the quotient would have been 321.

(6.) If the divisor had been .00075, the quotient would have been 32100.

The correctness of these results may be immediately shown by passing from the decimals to their equivalent fractions, which are $\frac{24075}{1000}$ and $\frac{75}{10}$: their quotient is $\frac{24075}{1000} \times \frac{10}{75} = \frac{24075}{75} \times \frac{10}{1000} = 321 \times \frac{1}{100} = \frac{321}{100} =$ 3.21.

In case (2), the quotient is $\frac{24075}{1000} \times \frac{1}{75} = \frac{321}{1000} =$ .321.

In case (3), the quotient is $\frac{24075}{1000} \times \frac{1}{7500} = \frac{321}{1000} \times \frac{1}{100} = \frac{321}{100000} =$ .00321.

In case (4), the quotient is $\frac{24075}{1000} \times \frac{100}{75} = \frac{321}{10} =$ 32.1.

In case (5), the quotient is $\frac{24075}{1000} \times \frac{1000}{75} =$ 321.

In case (6), the quotient is $\frac{24075}{1000} \times \frac{100000}{75} = 321 \times 100 = 32100$.

The same method of proof is applicable to all other cases, and will show very distinctly the principle upon which the rule is founded.

Let it be required to divide 298.89 by .1107 :

```
.1107) 298.8900 (2700
       2214
       ----
        7749
        7749
```

In this case, the number of decimal places in the dividend is made equal to the number of decimal places in the divisor.

Let it be required to divide 14 by .7854

```
.7854) 14.0000,0000000 (17.8253119
        7854
        ----
        61460
        54978
        -----
         64820
         62832
         -----
          19880
          15708
          -----
           41720
           39270
           -----
            24500
            23562
            -----
              9380
              7854
              ----
              15260
               7854
              -----
              74060
              69686
              -----
               4374
```

In this example, the operation does not terminate; and in order to continue it, we have added cyphers arbitrarily, in order to get a nearer approximation to the true value of the quotient; the last value obtained is $\frac{178253119}{10000000}$, and differs from its true value by $\frac{4374}{78540000000}$: and it is obvious, that by continuing the process we may obtain a decimal value of the quotient, differing from the true quotient by a quantity less than any that may be assigned.

(345.) The conversion of fractions into decimals, whether they terminate or not, is the most important use of these quantities, as it brings them under a uniform notation. The following are examples: Conversion of fractions into decimals.

(1.) $\frac{3}{4} = \frac{3.00}{4} =$ .75.

(2.) $\frac{1}{25} = \frac{1.000}{25} =$ .04.

(3.) $\frac{7}{16} = \frac{7.0000}{16} =$ .4375.

(4.) $\frac{11}{125} = \frac{11.000}{125} =$ .088.

(5.) $\frac{33}{250} = \frac{33.000}{250} =$ .132.

In all these cases, the factors of the denominators are either 2 or 5, and the decimals terminate. In example (1), the denominator is 2 × 2; in (2), it is 5 × 5; in (3), it is 2 × 2 × 2 × 2; in (4), it is 5 × 5 × 5; in (5), it is 5 × 5 × 5 × 2; and the number of decimal places in each case, never exceeds the greatest number of times that one or other of these factors are repeated.

The fact is, that 2 and 5 are the only divisors of 10, and, therefore, 2 × 2 and 5 × 5 are divisors of 100; 2 × 2 × 2 and 5 × 5 × 5 are divisors of 1000; 2 × 2 × 2 × 2 and 5 × 5 × 5 × 5 are divisors of 10000; and as the process of adding cyphers to the dividend in the division of decimals, is equivalent, as far as the division is concerned, to its multiplication by 10, 100, 1000, 10000, &c. respectively, it clearly follows, that when one, two, three, four, &c. of these factors 2 and What fractions produce finite decimals.

Arithmetic.

5, whether singly or conjointly, compose the divisor, that the division must terminate after one, two, three, four, &c. operations: it is for this reason, that the quotient cannot involve more decimal places than the greatest number of times that one or other of these factors is involved in the denominator.

What fractions produce interminable decimals.

But if the fraction in its *lowest* terms involves a factor in its denominator, not resolvable into the products of 2 or 5, such as 3, 6, 7, 9, 11, 12, &c., then the division can never terminate, and the equivalent decimal is interminable: for a number which is not a factor of 10, is not a factor of 100, or of 1000, or of 10000, and, consequently, the continuance of the operation brings us no nearer its termination. The following are examples:

(1.) $\frac{1}{3} = \frac{1.000}{3} = .333.....$

Circulating decimals.

The same figure is repeated continually, there being always the same remainder; and, therefore, the same quantity 10 to be divided. The decimal is, of course, indefinite, and is called a circulating decimal.

(2.) $\frac{1}{6} = .16666$.

The repetition begins in the second place, and the decimal is a circulating decimal like the former.

(3.) $\frac{1}{7} = .142857142857.....$

Whenever a remainder occurs, when cyphers only are brought down, which produces a quantity to be divided identical with any one preceding it, the same series of quotients and remainders must occur in the same order; the number of remainders different from each other which can occur in succession can therefore never exceed the divisor: in this case it is 6, and the *repeating* period in the circulating decimal produced is 142857.

(4.) $\frac{1}{9} = .11111....$

(5.) $\frac{1}{11} = .090909....$

(6.) $\frac{1}{12} = .08333...$

(7.) $\frac{1}{13} = .076923076923.....$

The repeating period is 076923.

(8.) $\frac{1}{15} = .06666...$

(9.) $\frac{1}{17} = .05882352941352941...$

In this case, the repeating period is 352941, and commences after the first five places.

(10.) $\frac{1}{19} = .052631578947368421052 6.$

The repeating period consists of 18 places

(11.) $\frac{16}{27} = .5925925.....$

(12.) $\frac{83}{9768} = .008497133497133....$

(13.) $\frac{7916}{1665} = 4.7543543....$

(14.) $\frac{355}{113} = 3.14159329203...$

Though in every case, when fractions are reduced to indefinite decimals, a repeating period may be found, yet, as the determination of it may, in an extreme case, require a number of divisions equal to the divisor itself, it may become too laborious to be practicable.

Conversion of circulating decimals into equivalent fractions.

(346.) The preceding examples will show in what manner circulating decimals are produced: it is frequently important, however, to reverse the process, and to pass from the circulating decimal to the equivalent fraction.

The rule for this purpose is as follows:

Multiply the circulating decimal by 1, with as many cyphers after it as there are decimal places before the second repeating period; and again multiply the circulating decimal by 1, with as many cyphers as there are places before the first repeating period: the products being subtracted from each other, and the remainder divided by the difference of the multipliers, will give the fraction which is equivalent to the circulating decimal.

Part I.

Let it be required to find the fraction which produces the circulating decimal

.0171717....

Multiply by 1000: the result is

17.1717....

Multiply by 10: the result is

.1717....

Subtract these results from each other, the remainder is

17:

which divided by 1000 − 10, or 990 gives $\frac{17}{990}$, the fraction required.

Let it be required to assign the fraction which produced the circulating decimal

.34500970097....

Multiply by 10000000, the result is

3450097.0097....

Multiply by 1000, the result is

345.0097....

Subtract the results from each other, and the remainder is

3449752:

which, divided by 10000000 − 1000, or 9999000, gives

$$\frac{3449752}{9999000}$$

for the fraction required, which, reduced to its lowest terms, becomes

$$\frac{145778}{410445}.$$

Infinite series.

(347.) Circulating decimals present the most familiar examples of the origin and meaning of infinite series: thus

.33333....

is equivalent to

$$\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + \text{\&c.}$$

where the terms are supposed to be continued indefinitely; the *sum* of the series is, likewise, the value of the circulating decimal, and the process which determines the one determines the other likewise.

Examples.

(348.) The following examples will illustrate most of the operations in decimals.

(1.) Add together .0345, 757.069, and 2.9168504.

(2.) Subtract 3.47965 from 5.111324.

(3.) Multiply .000395 into 27.0456.

(4.) Divide 9.6195 by 1.21.

(5.) Divide 233.91 by .345.

(6.) Reduce the fractions $\frac{11}{19}$, $\frac{100}{121}$, $\frac{8}{10019}$, and $\frac{99}{75425}$, to decimals.

(7.) Find the value of the circulating decimal

.003406969.

(8.) Find the sum of the infinite series $\frac{18}{1000} + \frac{18}{1000000} + \frac{18}{1000000000} + \text{\&c.}$

EXTRACTION OF ROOTS.

Square Root.

Extraction of the square root.

(349.) The process for extracting the square root must be founded upon the rule for the formation of the square, in the same manner as the rules for other inverse opera-

tions are founded upon those for the direct operation: the arithmetical process, however, for the formation of the square, leaves no traces of the root which are readily discoverable, in consequence of the incorporation of the parts which takes place in all arithmetical processes: we must divest the root, therefore, of its arithmetical character, at least, as far as notation is concerned, in order to detect the composition of its square.

Formation of the square.

(350.) Let it be required to form the square of 74: We will write it in the form

$$70 + 4,$$

and consider in what manner the result arising from multiplying this into $70 + 4$ is composed.

In the first place, there is 70 times 70, which is

$$4900.$$

In the second place, there is 70 times 4, which is

$$4 \times 70.$$

In the third place, there is 4 times 70, which is

$$4 \times 70.$$

In the fourth and last place, there is 4 times 4, which is

$$4 \times 4, \text{ or } 16.$$

If all these parts be added together, so as to form one sum, we shall get

$$4900 + \text{twice } 4 \times 70 + 16;$$

or the square of the number which is the sum of the parts 70 and 4, is the square of the first part + twice the product of the two parts + the square of the second part.

The same conclusion would be deduced, if the parts were 700 and 40, 7000 and 400, or any other numbers whatsoever.

Process for extracting the square root.

(351.) We shall now proceed to the inverse process, and let it be required to find the square root of

$$\begin{array}{rl}
 & 4900 + 560 + 16 \;(70 + 4 \\
 & 4900 \\
\hline
140 + 4) & + 560 + 16 \\
 & + 560 + 16
\end{array}$$

We first find the square root of 4900, which is 70, and subtract its square, which leaves $560 + 16$: we double 70, which gives 140, and divide 560 by it, in order to get 4, the second part of the root: we then add 4 to 140, and multiply the sum by 4, which gives $560 + 16$, the remaining part of the square.

We will now exhibit the same process under a somewhat more arithmetical form; let it be required to extract the square root of 5476:

$$\begin{array}{rl}
 & 5476 \;(70 + 4 \\
 & 4900 \\
\hline
140 + 4) & 576 \\
\hline
 & 560 \\
 & 16 \\
\hline
 & 576
\end{array}$$

Find the greatest multiple of 10, whose square is less than the given number; this is 70: subtract its square 4900 from 5476, the remainder is 576: double 70, which is 140, and divide the remainder by it, in order to find the second part of the root: the nearest whole number is 4: add 4 to 140: multiply $140 + 4$ by 4: the product of 4 and 140 is 560, and that of 4 and 4 is 16: their sum is 576, which subtracted leaves no remainder.

It remains to give the process a purely arithmetical form.

$$\begin{array}{rl}
 & 5476 \;(74 \\
 & 49 \\
\hline
144) & 576 \\
 & 576
\end{array}$$

Divide the square into periods of two, commencing from the place of units, by placing a dot over 6 and 4: find the greatest number whose square is less than the first period 54, which is 7: put 7 in the root, and underneath the first period write its square 49, which being subtracted, there remains 5: bring down the next period 76, and write it after the last remainder: double the root 7, which is 14, and divide 57 (omitting the last digit 6) by 14: the nearest number is 4, which must be placed after 7 in the root, and after 14 in the divisor: multiply the divisor 144 by 4: the product is 576, which subtracted, leaves no remainder: 74 is therefore the complete square root of 5476.

Second example.

(352.) We will proceed to another example, where there are 3 places in the root: let it be required to find the square root of 459684.

$$\begin{array}{rl}
 & 459684 \;(600 + 70 + 8 \\
 & 360000 \\
\hline
1200 + 70) & 99684 \\
\hline
 & 84000 \\
 & 4900 \\
\hline
 & 88900 \\
\hline
1340 + 8) & 10784 \\
\hline
 & 10720 \\
 & 64 \\
\hline
 & 10784
\end{array}$$

Or, more arithmetically, thus:

$$\begin{array}{rl}
 & 459684 \;(678 \\
 & 36 \\
\hline
127) & 996 \\
 & 889 \\
\hline
1348) & 10784 \\
 & 10784
\end{array}$$

The comparison of the two schemes of the process will show the reason of the abbreviations in the second: the square is first divided into periods of two by marking the first, the third, and the fifth digits: the greatest square less than the first period 45 is 36, which subtracted leaves 9: bring down the next period 96: double 6, the figure in the root, and divide 99 (omitting 6) by 12: the result (taken in defect) is 7: write 7 after 12, and multiply 7 into 127, and subtract the product 889 from 996: the remainder is 107: bring down the next period 84, and double 67, making 134: divide 1078 (omitting 4) by 134, the result is 8: write 8 in the root, and also after 134, and multiply 8 into 1348: the result 10784 being subtracted, leaves no remainder, and 678 is the complete root required.

The second scheme is the skeleton of the first, and is founded upon the general principle of all arithmetical rules, of avoiding all superfluous writing: the reason of the pointing every second figure of the square,

reckoning from the place of units, will be very obvious, when we consider that the number of cyphers after the significant digits in the square will be even, whether the number of cyphers in the root be odd or even.

When there are decimal places in the square.

(353.) If there are decimal places in the root, there will be double the number of them in the square, and, therefore, the number of decimal places in the square must always be even. In pointing, therefore, a square, which contains both integral and decimal places, we must begin from the place of units, and proceed both to the right and the left. The following is an example:

```
       1369.7401 (37.01
       9
      ----
   67) 469
       469
      -----
 7401) 7401
       7401
```

Indefinite approximation to the root.

(354.) When the number whose square root is required is not a complete square, we may *approximate* continually to the true value of the root, by adding pairs of cyphers to the root on the right of the decimal point as often as we choose. As an example, let it be proposed to extract the square root of 10.

```
        10.0000.... (3.162
         9
        ----
    61) 100
         61
        -----
   626) 3900
        3756
        ------
  6322) 14400
        12644
```

The square root of .1 is .3162...., the square root of .01 is .1, and that of .001 is .03162.

Square root of a fraction.

(355.) Let it be required to extract the square root of $\frac{3}{8}$. The fraction reduced to an equivalent decimal becomes

```
          .3750.... (.61237
           36
          -----
     121)  150
           121
          ------
    1222)  2900
           2444
          -------
   12243)  45600
           36729
          --------
  122467)  887100
           857269
          --------
            29831
```

Examples.

(356.) The following are examples of the different cases which occur in the extraction of the square root.

(1.) Extract the square root of 152399025.
(2.) Extract the square root of 119550.669121.
(3.) Extract the square root of .0000032754.
(4.) Extract the square root of 2.
(5.) Extract the square root of $\frac{1}{4}$.
(6.) Extract the square root of $795\frac{1}{3}$.

EXTRACTION OF THE CUBE ROOT.

Formation of the cube.

(357.) The formation of the cube, upon which the rule for the extraction of the corresponding root is founded, is more complicated than that of the square, and it is difficult to exhibit it clearly without the aid of algebraical symbols. We shall assume, however, for this purpose, 74, or $70 + 4$, for the root, of which the square is

$$4900 + \text{twice } 4 \times 70 + 16;$$

and in order to form its cube, it is requisite to multiply this result by $70 + 4$, which being done, the several results are as follows:

First, the product of 70 into 4900, which produces 343000, the cube of 70.

Secondly, the product of 70 into *twice* 4×70, which is equal to *twice* 4×4900.

Thirdly, the product of 70 into 16, which produces 16×70.

Fourthly, the product of 4 into 4900, which produces 4×4900.

Fifthly, the product of 4 into *twice* 4×70, which produces *twice* $4 \times 4 \times 70$, or *twice* 16×70.

Sixthly, the product of 4 into 16, which produces 64, the cube of 4.

If we combine all these results together, we shall find that the cube $70 + 4$, consists of

(1.) The cube of 70, or 343000.
(2.) Three times 4 into the square of 70, or *thrice* 4×4900.
(3.) Three times the square of 4 into 70, or *thrice* 16×70.
(4.) The cube of 4, or 64.

Inverse process.

(358.) Assuming the sum of these expressions for the cube, the steps in the reverse process are very obvious.

```
343000 + thrice 4 × 4900 + thrice 16 × 70 + 64 (70 + 4
343000
-----------
Thrice 4900 | thrice 4 × 4900 + thrice 16 × 70 + 64
            | thrice 4 × 4900 + thrice 16 × 70 + 64
```

We first subtract the cube of 70, (the cube of the highest multiple of 10, which is less than the cube:) we then take *thrice* the square of 70, or 3×4900 for a divisor of the first term of the remainder, by which means we determine 4, the second figure in the root: we then subtract $3 \times 4900 \times 4$, $3 \times 70 \times 16$, and 64 successively, in order to take away the complete cube of $70 + 4$.

We shall now put the same example under a more arithmetical form, and suppose that it is required to extract the cube root of 405224.

```
         405224 (70 + 4
         343000
        -------
 14700)  62224
        -------
         58800    3 × 4900 × 4
          3360    3 × 70 × 4 × 4
            64    4 × 4 × 4
        -------
         62224
```

We find the greatest multiple of 10 (70), whose cube is less than 405224, and subtract it, leaving the remainder 62224: we find the square of 70, which is 4900, and multiply it by 3, which is 14700, which we employ as a divisor of 62224, in order to find 4, the second figure in the root: we then add together thrice 4 into the square of 70, which is 58800,

thrice 70 into the square of 4, which is 3360, and the cube of 4, which is 64, and subtracting their sum, there is no remainder: therefore 70 + 4, or 74, is the cube root required.

It now remains to put the process under the most simple form which it admits of, omitting every figure and cypher which is not necessary in obtaining the result.

```
        405224 (74
        343
   147) 62224
        588       3 × 4 × 49
         336  .   3 × 16 × 7
          64      4 × 4 × 4
        62224
```

Rule. The cube is divided into periods of three places, beginning from the place of units; inasmuch as there are 3 cyphers in the cube of 70, 6 in that of 700, 9 in that of 7000, and similarly for higher orders of articulate numbers: 7 is the greatest number whose cube is less than the first period; the remainder is 62, to which the next period is annexed. In the divisor we put three times the square of seven, which is 147, and divide 622 (omitting the two last places) to get 4, the next figure in the root: we then form the products of 3 × 4 × 49, 3 × 16 × 7, and 4 × 4 × 4, and place them underneath each other, so that the second may advance one place beyond the first, and the third one place beyond the second: they are then added together, and their sum subtracted from the dividend, and, as there is no remainder, 74 is the cube required.

Second example. (359.) We will now proceed to a second example. Let it be required to find the cube root of 48228544:

```
                    48228544 (300 + 60 + 4
                    27000000
3 × 300 × 300=  }   21228544
  270000)       }
                    16200000  3 × 300 × 300 × 60
                     3240000  3 × 300 × 60 × 60
                      216000  60 × 60 × 60
                    19656000
3 × 360 × 360=  }    1572544
  388800        }
                     1555200  3 × 360 × 360 × 4
                       17280  3 × 360 × 4 × 4
                          64  4 × 4 × 4
                     1572544
```

Or, merely preserving the skeleton of this process, and conforming to the arithmetical rule, the scheme will appear as follows:

```
          48228544 (364
          27
     27)  21228    Dividend
          162      3 × 3 × 3 × 6
           324     3 × 3 × 6 × 6
            216    6 × 6 × 6
          19656    Subtrahend
   3888)   1572544 Dividend
           15552   3 × 36 × 36 × 4
             1728  3 × 36 × 4 × 4
               64  4 × 4 × 4
           1572544
```


Cube roots of decimals. (360.) Let it be required to extract the cube root of 27054.036008:

```
             27054.036008 (30.02
             27
       27)     54
     2700)     54036
   270000)     54036008
               540000    3 × 300 × 300 × 2
                 3600    3 × 300 × 2 × 2
                    8    2 × 2 × 2
               54036008
```

Indefinite cube roots. (361.) Let it be required to extract the cube root of 10:

```
            10.000000... (21.54...
             8
      12)    2000         Dividend
             12           3 × 2 × 2 × 1
               6          3 × 2 × 1 × 1
                1         1 × 1 × 1
             1261         Subtrahend
    1323)    739000       Dividend
             6615         3 × 21 × 21 × 5
              1575        3 × 21 × 5 × 5
                125       5 × 5 × 5
             677375       Subtrahend
  138675)    61625000     Dividend
               554700     3 × 215 × 215 × 4
                 10320    3 × 215 × 4 × 4
                    64    4 × 4 × 4
               55573264   Sobtrahend
                6051736
```

It is quite clear that the operation can never terminate, and that by continuing it we may obtain an approximate value of the cube root of 10 within any required limits of accuracy.

Examples. (362.) The following are other examples of the various cases which can occur in the application of this rule:

(1.) Let it be required to extract the cube root of 3430529217010729.

(2.) Let it be required to find the cube root of 1.879080904.

(3.) Let it be required to extract the cube root of .000000042875.

(4.) Let it be required to find the cube root of 3.

(5.) Let it be required to find the cube root of $\frac{1}{2}$.

Rules for extraction of higher roots. (363.) The invention of rules for the extraction of the fourth, fifth, and higher roots, depends upon the formation of the fourth, fifth, and higher powers, and is effected upon the same principles as those for the square and cube root, though they are not easily discovered without the aid of algebraical formulæ. The rules are also extremely complicated, and their application difficult and embarrassing, when they extend beyond two places of figures in the root; under such circumstances, therefore, it is expedient to defer the consideration of them until we can avail ourselves of algebraical formulæ, by

which the rules may be simplified, or other methods investigated, which may give approximate values of the roots.

Example of the extraction of the fourth root.

(364.) The extraction of the fourth root is equivalent to a double extraction of the square root, and such is the arithmetical method which is most convenient to follow. The following is an example:

Let it be required to find the fourth root of

29986576.

```
        29986576 (5476
        25
104)     498
         416
1087)    8265
         7609
10946)   65676
         65676
```



```
        5476 (74
        49
144)    576
        576
```

Consequently, 74 is the fourth root required.

Conclusion.

(365.) There are some other subjects which might be included in a Treatise on abstract Arithmetic, such as the notation of numbers proceeding according to scales different from the decimal, whether binary, quaternary, quinary, doodenary, &c., the formation and reduction of continued fractions, and some of the more obvious properties of numbers, all of which are more properly included under the Theory of Numbers: whilst the consideration of others, such as arithmetical and geometric progressions, combinations and permutations, which are commonly found in treatises on this subject, may with more propriety be deferred until we are enabled to investigate algebraically the formulæ upon which the rules are founded: we shall, therefore, close at this point our Treatise on the Arithmetic of Abstract Numbers.

ARITHMETIC.

PART II.

Arithmetic. Concrete numbers. (366.) NUMBERS are *concrete* when the units, of which they are composed, represent magnitudes to which a *denomination* is given: such as 17 *shillings*, 143 *yards*, 74 *pounds*, 23 *minutes*, 67 *gallons*.

Difference in the arithmetic of abstract and concrete numbers. The arithmetic of such numbers would be nearly identical with that of numbers which are abstract, if the concrete units of the same species of quantity were always of the same magnitude, not admitting of subdivision into others, which are multiples or submultiples of the first; in other words, if *shillings* were the only units of money, *yards* of length, *pounds* of weight, *minutes* of time, and *gallons* of capacity. Under such circumstances, such numbers would be subject to all the common operations of Arithmetic, whether of addition, subtraction, multiplication, or division, without any reference to the particular nature of the quantities which they denoted.

Again, supposing those subdivisions were in all cases adapted to the decimal scale, the operations on such quantities would be in every respect identical with those which are required in the arithmetic of decimals. The fact, however, is, that those subdivisions are rarely adapted to any regular scale; the duodecimal is most prevalent; in some cases they proceed by continued bisections; but most commonly the successive units are not the *same* submultiples or multiples of those which precede or follow them. It is this want of uniformity which renders it necessary for the student in the first instance to commit to memory tables of the subdivisions of coins, of the different units of weights, of measures of length, area, and capacity, of time, and of such specific quantities as are frequent subjects of consideration, but whose subdivisions do not conform to the general custom.

These successive units, though they neither follow the decimal or any other scale, may be brought within the rules of the Arithmetic of abstract numbers, by reducing the inferior units to a *vulgar* or *decimal* fraction of one of higher denomination. Such a mode of proceeding is not always the most convenient or expeditious; but in many questions it is absolutely necessary, and in every case it is more *general* than any other process which can be followed.

We shall now put down some of the more useful of these tables, accompanied with examples of the different species of reductions which will be required in the solution of questions, in which such quantities are involved.

Table of divisions of English money. (367.) *Table of Money.*

2 farthings make 1 halfpenny.
4 farthings .... 1 penny, (*d.*)
12 pence ...... 1 shilling, (*s.*)
20 shillings .... 1 pound, or sovereign, (*l*)
21 shillings .... 1 guinea.

Part II. Or, expressing each superior unit, not merely in terms of the next below it, but also of all others which are inferior to it, it will stand as follows:

qrs.		d.		s		l.
4	=	1				
48	=	12	=	1		
960	=	240	=	20	=	1

Various reductions. (368.) One of the most common species of reduction, is to express numbers of superior denominations in units of a lower denomination, and conversely. Thus, suppose it was required to find how many farthings there are in 4*s*. 3*d*.:

$$\begin{array}{l} s.\ \ d. \\ 4\ ,\ 3 \\ 12 \\ \hline 51 = 48 + 3 \\ 4 \\ \hline \end{array}$$

Answer, 204

We first reduce the shillings to pence, by multiplying the number of shillings 4 by 12: to the product 48, we add 3, and thus get 51, the whole number of pence in 4*s*. 3*d*.: if this number be multiplied by 4, the last result 204 is obviously the number of farthings required.

Let it be required to reduce £17. 13*s*. 3¾*d*. to farthings.

This sum might be written thus,

£17. 13*s*. 3*d*. 3*qrs*.

but it is more usual to express 3 *qrs*., or 3 farthings, by the equivalent fraction ¾*d*., or three-fourths of a penny.

$$\begin{array}{l} £.\ \ s.\ \ d. \\ 17\ ,\ 13\ ,\ 3\tfrac{3}{4} \\ 20 \\ \hline 353 = 20 \times 17 + 13 = \text{number of shillings.} \\ 12 \\ \hline 4239 = 12 \times 353 + 3 = \text{number of pence.} \\ 4 \\ \hline 16959 = 4 \times 4239 + 3 = \text{number of farthings.} \end{array}$$

The general rule for such reductions, whether of money or other classes of concrete units of the same species, is to multiply the superior units by the number which connects them with the unit next succeeding in the table, and to add to the result whatever units of the same order may appear in the sum to be reduced; and the process must be continued until we arrive at the units of the denomination required.

The following question is the converse of those just given.

Let it be required to find how mony pounds, shillings and pence there are in 17347 farthings.

$$\begin{array}{r} 4)\ 17347 \\ 12)\ \overline{4336} - 3 \\ 2{,}0)\ \overline{36{,}1} - 4 \\ \overline{18,\ 1,\ 4\tfrac{3}{4}} \end{array}$$

We first reduce the farthings to pence, by dividing by 4; we next reduce the pence to shillings, by dividing by 12; and we lastly reduce the shillings to pounds, by dividing by 20: the final result is £18. 1*s*. $4\frac{3}{4}$*d*.

The steps of this process, of passing from inferior to superior units, are clearly the inverse of those which are followed in passing from superior to inferior units.

The following are examples of the reduction of a compound expression to a simple fractional or mixed number.

What fraction of a pound is 2*s*. 7*d*.?

$$\begin{array}{lcl} \begin{array}{r} £. \\ 1 \\ 20 \\ \hline 20 \\ 12 \\ \hline 240 \text{ denominator.} \end{array} & \qquad & \begin{array}{r} s.\ \ d. \\ 2,\ 7 \\ 12 \\ \hline 31 \text{ numerator.} \\ \\ \\ \end{array} \end{array}$$

The fraction is $\frac{31}{240}$.

There are 31*d*. in 2*s*. 7*d*., and 240 in a pound; and, consequently, if unity be divided into 240 equal parts, and 31 of them be taken, the portion of unity, or of 1£., which they denote, is $\frac{31}{240}$.

What fraction of £3. 10*s*. is £2. 5*s*. $6\frac{1}{4}$*d*.?

$$\begin{array}{lcl} \begin{array}{r} £.\ \ s. \\ 3,\ 10 \\ 20 \\ \hline 70 \\ 12 \\ \hline 840 \\ 4 \\ \hline 3360 \end{array} & \qquad & \begin{array}{r} £.\ \ s.\ \ d. \\ 2,\ 5,\ 6\tfrac{1}{4} \\ 20 \\ \hline 45 \\ 12 \\ \hline 546 \\ 4 \\ \hline 2285 \end{array} \end{array}$$

The fraction is $\frac{2285}{3360}$, or, in lower terms, $\frac{457}{672}$.

The following questions are the converse of the preceding.

What is the value of $\frac{5}{7}$ of a pound?

$$\begin{array}{rl} & £. \\ & 2 \\ & 20 \\ \hline 7) & 40\ (5 \\ & 35 \\ \hline & 5 \\ & 12 \\ \hline 7) & 60\ (8 \\ & 56 \\ \hline & 4 \\ & 4 \\ \hline 7) & 16\ (2 \\ & 14 \\ \hline & 2 \end{array}$$

In £2. there are 40*s*., and, therefore, $\frac{5}{7}$£. is equivalent to $\frac{40}{7}$*s*.; but if we reduce this fraction to a mixed number, it becomes $5\frac{5}{7}$*s*.: but 5*s*. are equal to 60*d*., and, therefore, $\frac{5}{7}$*s*. is equivalent to $\frac{60}{7}$*d*., which, reduced to a mixed number, is $8\frac{4}{7}$*d*. Again, 4*d*. are equal to 16 farthings, and, therefore, $\frac{4}{7}$*d*. is equivalent to $\frac{16}{7}$ *qrs*. or, $2\frac{2}{7}$ *qrs*., or to $\frac{1}{2}$*d*. $\frac{2}{7}$ *qrs*.: the final answer, therefore, is 5*s*. $8\frac{1}{2}$*d*. $\frac{2}{7}$ *qrs*., or, as it is commonly written, 5*s*. $8\frac{1}{2}$*d*.$\frac{2}{7}$.

What is the value of $\frac{17}{113}$ of £2. 12*s*.?

$$\begin{array}{lcl} \begin{array}{r} £.\ \ s. \\ 2,\ 12 \\ 20 \\ \hline 52 \end{array} & \qquad & \begin{array}{rl} & 17 \\ & 52 \\ \hline & 34 \\ & 85 \\ \hline 113) & 884\ (7 \\ & 791 \\ \hline & 93 \\ & 12 \\ \hline 113) & 1116\ (9 \\ & 1017 \\ \hline & 99 \\ & 4 \\ \hline 113) & 396\ (3 \\ & 339 \\ \hline & 57 \end{array} \end{array}$$

The answer is 7*s*. $9\frac{3}{4}$*d*. $\frac{57}{113}$.

The reduction of shillings, pence, &c. to decimals of a pound, or any other superior unit, is extremely important, being the reduction which, of all others, is most frequently required. The following are examples:

What decimal of a pound is 2*s*. 6*d*.?

$$\begin{array}{r} 12)\ 6 \\ 20)\ \overline{2.5} \\ \overline{.125} \end{array}$$

In the first place, 6*d*. is equivalent to $\frac{6}{12}$*s*., which reduced to a decimal is .5: consequently, 2*s*. 6*d*. is equivalent to 2.5*s*.; but 2.5*s*. is equivalent to $\frac{2.5}{20}$£., or .125£.

The same result would be obtained by first reducing 2*s*. 6*d*. to a fraction of a pound, and then converting the fraction, which is $\frac{30}{240}$, or $\frac{1}{8}$, to an equivalent decimal.

What decimal of a pound is 19*s*. $11\frac{3}{4}$*d*.?

$$\begin{array}{r} 4)\ 3 \\ 12)\ \overline{11.75} \\ 20)\ \overline{19.9891666\ldots} \\ \overline{.99945833\ldots.} \end{array}$$

What decimal of 13*s*. is 12*s*. 7*d*.?

$$\begin{array}{r} 12)\ 7 \\ 13)\ \overline{12.5833\ldots} \\ \overline{.96794871\dot{7}9487\dot{1}\ldots} \end{array}$$

The following questions are the converse of those just given.

What is the value of .375£.?

£.
.375
20
7.500
12
6.00

The answer is 7*s.* 6*d.*

What is the value of ,0552084£.?

£.
.0552084
20
1.1041680
12
1.2500160
4
1.0000640

The answer is 1*s.* $1\frac{1}{4}d.$ $\frac{1}{15625}$.

What is the value of .0425 of 100£.?

.0425
100
4.25
20
5.00

The answer is £4. 5*s.*

Tables of the subdivisions of weights.

(369.) The following three tables contain the subdivisions of the weights which are used in this country.

Troy Weight.

24 grains make	1 pennyweight,	*dwt.*
20 pennyweights	1 ounce,	*oz.*
12 ounces......	1 pound Troy,	*lb.*

Or thus,

gr.	*dwt.*		
24 =	1	*oz.*	
480 =	20 =	1	*lb.*
5760 =	240 =	12 =	1

This weight is used in weighing gold, silver, jewels, and other articles of a costly nature.

Apothecaries' Weight.

20 grains make	1 scruple,	*sc.* or ℈
3 scruples....	1 dram,	*dr.* or ʒ
8 drams	1 ounce,	*oz.* or ℥
12 ounces	1 pound,	*lb.* or ℔

Or thus,

gr.	*sc.*			
20 =	1	*dr.*		
60 =	3 =	1	*oz.*	
480 =	24 =	8 =	1	*lb.*
5760 =	288 =	96 =	12 =	1

The apothecaries' pound is identical with the pound Troy, differing merely in its subdivisions. It is used by apothecaries in the composition of medicines.

Avoirdupois Weight.

16 drams make	1 ounce,	*oz.*
16 ounces......	1 pound,	*lb.*
28 pounds	1 quarter,	*qr*
4 quarters	1 hundred-weight,	*cwt.*
20 hundred-weight	1 ton,	*ton.*

Or thus,

dr.	*oz.*				
16 =	1	*lb.*			
256 =	16 =	1	*qr.*		
7168 =	448 =	28 =	1	*cwt.*	
28672 =	1792 =	112 =	4 =	1	*ton.*
573440 =	35840 =	2240 =	80 =	20 =	1

This weight is used in weighing all heavy articles, such as grocery goods, butter, cheese, meat, bread, corn, &c. and all metals, except gold and silver.

The pound avoirdupois is equal to 7000 grains Troy, and the relation of the ounce avoirdupois to the ounce Troy is that of $437\frac{1}{2}$: 480, which is nearly that of 11 to 12; in some cases, the dram avoirdupois is subdivided into 3 scruples, and each scruple into 10 grains: under these circumstances, the grain Troy is equal to 1.097 grains avoirdupois.

(370.) The following are examples of reductions connected with these tables. Reductions.

In 3 *lb.* 10 *oz.* 7 *dwt.* 5 *gr.*, how many grains?

lb. oz. dwt. gr.
3 , 10 , 7 , 5
12
46 *oz.*
20
927 *dwt.*
24
22253 *gr.* The answer.

In 1 *ton* 7 *cwt.* 2 *qr.* 17 *lb.*, how many pounds?

ton cwt. qr. lb.
1 , 7 , 2 , 17
20
27
4
110
28
3097 *lb.* The answer.

In 27 *lb.* 7 ℥. 2 ʒ. 1 ℈. 2 *gr.*, how many grains?

lb. ℥ ʒ ℈ *gr.*
27 , 7 , 2 , 1 , 2
12
331
8
2650
3
7951
20
159022 The answer.

What fraction of a pound Troy is 3 *oz.* 15 *dwt.* 12 *gr.*?

lb.	*oz. dwt. gr.*
1	3 , 15 , 12
12	20
12	75
20	24
240	312
24	150
5760	1812

Arithmetic.

The fraction is [illegible] = [illegible].

What decimal of a ton is 7 *cwt.* 3 *qr.* 27 *lb.* ?

```
28 = 7 × 4      7) 27
                4) 3.8571428
4 lb. = 1 qr.   4) 3.9642897
               20) 7.9910714
      The answer,   .39955357
```

What is the value of .12345 ℔. ?

```
   ℔.
 .12345
     12
 1.48140
      8
 3.85120
      3
 2.55360
     20
11.07200
```

The answer is 1 *oz.* 3 *dr.* 2 *sc.* 11 *gr.* [illegible].

Table of measures of length.

(371.) *Tables of Measures of Length.*

3 barleycorns (in length) make	1 inch,	*in.*
12 inches	1 foot,	*ft.*
3 feet	1 yard,	*yd*
6 feet	1 fathom,	*fth*
5½ yards	1 pole, or rod,	*po.*
40 poles	1 furlong,	*fur.*
8 furlongs	1 mile,	*mi.*
3 miles	1 league,	*lea.*
69½ miles	1 degree,	*deg.* or °.

Or thus,

```
  bar.    inch.
    3 =     1     foot.
   36 =    12 =    1     yard.
  108 =    36 =    3  =    1     pole.
  594 =   198 =   16½ =    5½ =  1 furlong.
23760 =  7920 =  660  =  220  = 40 = 1  mile.
190080 = 63360 = 5280 = 1760 = 320 = 8 = 1
```

Reductions.

(372.) In 3 *miles*, 2 *furlongs*, 7 *poles*, 3 *yards*, and 2 *feet*, how many inches?

```
mi. fur. po. yd. ft.
 3 , 2 , 7 , 3 , 2
     8
    26
    40
  1047
     5½
  5238
   523½
  5761½
     3
 17286½
    12
The answer, 207438
```

What decimal of a mile is 17 *yards*, 1 *foot*, 6 *inches*? Part II.

```
                 12) 6
                  3) 1.5
220 = 11 × 20    20) 17.5
                 11) .875
                  8) .0795454
                     .00994318
```

Required the value of .67 of a league?

```
 .67
   3
2.01
   8
 .08
  40
3.20
  5½
 100
  10
1.10
   3
 .30
  12
3.60
   3
1.80
```

The answer is 2 *mi.* 0 *fur.* 3 *pol.* 1 *yd.* 0 *ft.* 3 *in.*

Table of measures of area.

(373.) *Table of Measures of Area.*

144 square inches make	1 square foot.
9 square feet......	1 square yard.
30¼ square yards ..	1 square pole.
40 square poles	1 rood.
4 roods..........	1 acre.

Or thus,

```
 inches.    foot.
    144 =     1       yard.
   1296 =     9  =     1      pole.
  39204 =   272¼ =    30¼ =    1    rood.
1568160 = 10890  =  1210  =   40 =  1   area.
6272640 = 43560  =  4840  =  160 =  4 = 1
```

The names of the inferior units of area are identical with the names of those units of length which are the sides of the squares; and, in general, the distinguishing epithet (*square*) is altogether omitted, unless in those cases where the meaning is not clearly defined by the context.

Reductions.

(374.) What decimal of an acre is 1 *rood*, 17 *poles*, 12 *yards*?

```
      30¼) 12.
or,   121) 48.
       40) 17.39669
        4) 1.434917
The answer, .358729
```

What is the value of .12345 of an acre?

Arithmetic.

$$\begin{array}{r} .12345 \\ 4 \\ \hline .49380 \\ 40 \\ \hline 19.75200 \\ 30\frac{1}{4} \\ \hline 22.56000 \\ 18800 \\ \hline 22.74800 \\ 9 \\ \hline 6.73200 \end{array}$$

The answer is 19 *poles*, 22 *yards*, 6 *feet*.

Table of Measures of Capacity.

Table of measures capacity.

(375.) (1.) For wine, ale, and other liquids

2	pints make	1 quart.
4	quarts....	1 gallon.
42	gallons ..	1 tierce.
2	tierces....	1 puncheon.
63	gallons ..	1 hogshead.
2	hogsheads	1 pipe, or butt.
2	pipes	1 tun.

(2.) Dry measure, for corn, seeds, &c.

2	pints make	1 quart,	*qt.*
2	quarts....	1 pottle,	*pot.*
2	pottles ..	1 gallon,	*gal.*
2	gallons ..	1 peck,	*pec.*
4	pecks	1 bushel,	*bu.*
4	bushels ..	1 coom,	*coom.*
2	cooms....	1 quarter,	*qr.*
5	quarters ..	1 wey, or load,	*wey.*
2	weys	1 last,	*last.*

Or thus,

$$\begin{array}{l} \textit{pints.}\quad \textit{gallon.} \\ 8 = 1 \quad \textit{peck.} \\ 16 = 2 = 1 \quad \textit{bushel.} \\ 64 = 8 = 4 = 1 \quad \textit{coom.} \\ 256 = 32 = 16 = 4 = 1 \quad \textit{quarter.} \\ 512 = 64 = 32 = 8 = 2 = 1 \quad \textit{wey.} \\ 2560 = 320 = 160 = 40 = 10 = 5 = 1 \quad \textit{last.} \\ 5120 = 640 = 320 = 80 = 20 = 10 = 2 = 1 \end{array}$$

Imperial gallon.

(376.) The wine gallon formerly differed from the beer gallon, and both of them from the corn gallon; the first being 231 cubic inches, the second 282, and the third 271. In the Imperial measures of capacity, established by act of Parliament in 1824, there is only one gallon for wine, beer, and corn, or for liquid and dry measures, which is equal to 277.274 cubic inches.

The Imperial gallon is nearly $\frac{1}{5}$th larger than the old wine gallon, $\frac{1}{32}$th greater than the old corn gallon, and $\frac{1}{60}$th less than the old beer gallon. At least, these reductions are sufficiently accurate for ordinary reductions of the ancient to the modern measures.

Reductions.

(377.) What number of Imperial gallons are there in 3 pipes, 1 hogshead, 12 gallons, of the old wine measure?

$$\begin{array}{r} \textit{pipe hhd. gal.} \\ 3\ ,\ 1\ ,\ 12 \\ 2 \\ \hline 7 \\ 63 \\ \hline 5)\ 453 \\ 90\frac{3}{5} \\ \hline 362\frac{2}{5} \end{array}$$ The answer.

What number of Imperial bushels are there in 7 lasts, 7 quarters of the old measure?

$$\begin{array}{r} \textit{last}\ \ \textit{qr.} \\ 7\ ,\ 7 \\ 10 \\ \hline 77 \\ 8 \\ \hline 50)\ 616 \\ 12\frac{8}{25} \\ \hline 604\frac{17}{25} \end{array}$$ The answer.

What decimal of a hogshead are 3 gallons and 3 pints?

$$\begin{array}{rr} & 8)\ 3. \\ 63 = 7 \times 9 & 7)\ \overline{3.375} \\ & 9)\ \overline{.482142857} \\ & \overline{.053571428\ldots} \end{array}$$ The answer.

Table of the divisions of time.

(378.) *Table of Measures of Time.*

60	seconds make	1 minute,	*m.* or ′.
60	minutes	1 hour,	*hr.*
24	hours	1 day,	*day.*
7	days	1 week,	*wk.*
4	weeks......	1 month,	*mo.*

Or thus,

$$\begin{array}{l} \textit{seconds.}\quad \textit{minute.} \\ 60 = 1 \quad \textit{hour.} \\ 3600 = 60 = 1 \quad \textit{day.} \\ 86400 = 1440 = 24 = 1 \quad \textit{week.} \\ 604800 = 10080 = 168 = 7 = 1 \quad \textit{month.} \\ 2419200 = 40320 = 672 = 28 = 4 = 1 \end{array}$$

Different years.

(379.) The civil year, taking an average of four years, is $365\frac{1}{4}$; but if we take an average of 400 years, its length is 365.2425 days: this is different from the mean tropical year, upon which the recurrence of the seasons depends, whose length is 365.242264 days, differing from the former by .000136 day, or by about $11\frac{3}{4}$ seconds.

And months.

It is necessary, likewise, to distinguish between a *month*, as defined by the preceding table, a *calendar month*, which varies from 28 to 31 days, and an *astronomical month*, which is a synodical period of the moon, the mean length of which is 29.5505885 days. It is the second of these which is most commonly understood in arithmetical questions; and when the particular month is not specified, its length is assumed to be 30 days.

(380.) What decimal of a week is 1 hour, 27 minutes, and 14 seconds?

60) 14
60) 27.233
24) 1.45388
7) .0605787
The answer, .0086541

What is the value of .00693 of a year?

.000693
365¼
3465
4158
2079
173,25
.25311825
24
101247300
50623650
6.074838
60
4.490280
60
29.416800

The answer is 6 hours, 4 minutes, and 29½ seconds.

Addition and subtraction of concrete quantities.

(381.) The reductions which we have mentinned above, in connection with the several tables of weights and measures, are those which are most commonly required in erithmetical operatinns with concrete quautities, and particularly fur hringing them within the provinee of decimal arithmetic. It is not always expedient, however, to effect such reductions, and the addition and subtraction of such quantities, and their multiplication and division by abstract numbers, take place without any previous preparation.

In the addition and suhtraction of conerete quantities, it is requisite that they should be of the same kind, otherwise nu incorporation can take place in the results: and in performing the nperation, quantities of the *same denomination* must be pleced underneath each other: under such circumstances, the numbers in the same column are added together, or subtracted from eaeh other; end when the sum exceeds the number of units of that denomination, whieh constitutes an unit of the oext superior order, it must be divided by that number, and the *remainder* from the division left in the expression for the sum, and the *quotient* carried to the next column. A few examples will make this rule sufficiently clear.

Examples.

(1.)

£.	s.	d.
77	13	5½
15	19	11¾
107	7	2¼
327	16	8
528	17	3½

The nnmber of farthings is 6, which, being divided by 4, (4 *far.* = 1*d.*) gives e *quotient* 1, with a remainder 2: the number of penee, adding 1, is 27, which, divided by 12, (12*d.* = 1*s.*) gives a *quotient* 2, and a remainder 3: the number of shillings, adding 2, is 57, which, divided by 20, (20*s.* = £1.) gives a *quotient* 2, and a remainder 17: the number of pounds, adding 2, is 528: we thus get the eutire sum, which is £528. 17*s.* 3½*d.*

(2.)

	oz.	dr.	sc.	gr.
	8	5	1	8
	7	6	2	13
	11	7	0	0
	10	0	0	16
	1	2	2	8
	0	7	1	19
3*lb.*	4	5	2	19

The sum in the first column is 59, which, divided by 20, gives a quotient 2, with a remainder 19: the sum in the second colnmn, adding 2, is 8, which, divided by 3, gives a quotient 2, with a remainder 2: the sum in the third eolumn, adding 2, is 29, which, divided by 8, gives a quotient 3, with a remainder 5: the sum in the fourth column, adding 3, is 40, which, divided by 12, gives a qnotient 3, with a remainder 4.

(3.) Let it be required to subtract 12 ton, 7 cwt. 1 qr. 12 lb. 7 nz. from 15 ton, 11 cwt. 0 qr. 1 lb. 5 oz.

ton	cwt.	qr.	lb.	oz.
15	11	0	1	5
12	7	1	12	7
3	3	2	16	14

In the first eolumn, we borrow 1 lb. or 16 nz., and add it to 5; and from their sum, 21, we subtract 7, which leaves a remainder 14: we add 1 to 12, and borrow 1 qr., or 28 lh., from the third columa: we subtract, therefore, 13 from 29, and the remainder is 16: we add 1 to 1 in the third column, and borrow 1 cwt., or 4 qr., from the fuurth column, and, therefore, subtract 2 from 4, which leaves a remainder 2: we add 1 to 7 in the fourth column, and subtract, therefore, 8 from 11, which leaves a remainder 3: we subtract 12 from 15 in the fifth column, and the remainder is 3.

Multiplication and division of concrete quantities by abstract numbers.

(382.) In multiplying eoncrete quantities of different denominatinns by au abstract number, we multiply the terms in succession, beginning from the lowest, divide the results successively by the number which connects each term with the next superior, carry the quotients successively to the next product, and leave the remainders. The following are examples:

lea.	mi.	fur.	po.	yd.
20	2	7	38	4
				5
104	2	7	33	3½

We multiply 5 into 4, the produet is 20, which, divided by 5½, gives a quotient 3, and a remainder 3½: we multiply 5 into 38, add 3 tn the product, and divide the result, 193, by 40, which gives e quotient 4, and e remainder 33: we multiply 5 into 7, add 4 to the product, and divide the result 39 by 8, which gives a quotient 4, and a remainder 7: we multiply 5 into 2, and add 4 to the product, and divide the result 14 by 3, which gives a quotient 4, and a remainder 2: we lastly multiply 5 into 20, and add 4 to the result, which is 104.

In the division of concrete quantities of different denominations by abstract numbers, we commence with the highest, and proceed to the lowest, putting down the quotients, and carrying the remainders multiplied by the number which connects the several denominations with each other, and adding their products to the corresponding terms of the dividend. The following is an example:

What is the fifth part of 214 quarts, 7 bushels, and 3 pecks?

$$\begin{array}{r} qr. \;\; bus. \; pec. \\ 5)\; \underline{214 \;,\; 7 \;,\; 3} \\ 42 \;,\; 7 \;,\; 3 \;,\; 1 \; gal. \; 2 \; quarts \; \frac{2}{5}. \end{array}$$

The quotient of 214 is 42, and the remainder 4, which, multiplied by 8, and the product added to 7, makes the next number to be divided 39: the quotient of 39 is 7, with a remainder 4, which, multiplied by 4, and the result added to 3, makes the next number to be divided 19: the quotient of 19 is 3, and the remainder 4, which, multiplied by 2, is 8: the quotient of 8 (gallons) is 1, and the remainder 3, which, multiplied by 4, the result is 12, of which the quotient is 2, with a remainder 2.

In those cases in which the divisor is a mixed number, it is necessary to multiply both the dividend and divisor by the denominator of the fractional part, so that the divisor may become an integral number. The following is an example:

$$\begin{array}{r} wk. \; day. \; hrs. \; min. \\ 5\frac{1}{4})\; 3 \;,\; 6 \;,\; 14 \;,\; 53 \\ \underline{4} \\ 21)\; \underline{15 \;,\; 5 \;,\; 11 \;,\; 32} \\ 5 \;,\; 6 \;,\; 15 \;,\; 48'' \frac{4}{7} \end{array}$$

Duodecimal multiplication of length into length.

(383.) In some cases, concrete quantities are multiplied together, and a result is obtained which admits of interpretation: thus, length being multiplied into length produces area, and area into length produces capacity; the units in the products are different from those in the factors, and the meaning of the term multiplication must be modified, so as to suit this extended application of it: for this purpose, we must consider in what manner the result is obtained, and also what is the meaning of the units of which it is composed.

A rectangular area whose adjacent sides are 5 feet, and 3 feet respectively, may be separated into 3×5, or 15 equal squares, by dividing the opposite sides into 5 and 3 equal parts respectively, and drawing lines through the points of division: in this case, the rectangle is said to be the product of the two adjacent sides, represented by numbers, whilst the units in the numerical product are no longer lines, but squares described upon an unit of length: it is easy to extend this conclusion to the rectangle under two lines, which are denoted by 5.4, and 3.7, respectively, whose product is 19.98, which is 19 units or squares, and that portion of one of those squares, which .98, or $\frac{98}{100}$, represents.

In the same manner, the solid parallelopipedon, whose adjacent edges are 5 feet, 3 feet, and 4 feet, respectively, is equivalent to $5 \times 3 \times 4$, or 60 equal cubes, one whose edges is 1 foot; and it is in this sense, that the continued product of the numbers, whether whole or fractional, by which three lines are denoted, gives a numerical product, of which the units denote solids and not lines.

The subdivisions of feet proceed according to the duodecimal scale, and artisans, in estimating rectangular areas, or rectangular solids terminated by rectangular surfaces, are accustomed to multiply feet and inches into each other, for the purpose of obtaining the units of area (squares) or of capacity (cubes), which they contain: such quantities are called *duodecimals*, from the scale according to which they decrease,—and the process which is made use of for this purpose is strictly analogous to the multiplication of *decimals*, though requiring a different notation. The following are examples:

Examples.

1. Multiply 5 feet, 4 inches by 6 feet. 8 inches.

$$\begin{array}{r} ft. \;\; in. \\ 5 \;,\; 7 \\ \underline{6 \;,\; 8} \\ 33 \;,\; 6 \\ \underline{3 \;,\; 8 \;,\; 8} \\ 37 \;,\; 2 \;,\; 8 \end{array}$$

Or thus,

$$\begin{array}{r} 5\frac{7}{12} \\ \underline{6\frac{8}{12}} \\ 33\frac{6}{12} \\ \underline{3\frac{8}{12} \; \frac{8}{144}} \\ 37\frac{2}{12} \; \frac{8}{144} \end{array}$$

The reason of the first operation will be sufficiently obvious from the second form of the process: 5 ft. 7 in. is equivalent to $5\frac{7}{12}$ feet, and 6 ft. 8 in. to $6\frac{8}{12}$ feet: their product is found by multiplying these mixed numbers together, which is effected as follows: multiplying first by 6, we get $6 \times \frac{7}{12}$, which is $\frac{42}{12}$, or $3\frac{6}{12}$, and 6×5 is 30, which, added to the former, makes $33\frac{6}{12}$: we next multiply $\frac{8}{12}$ into $\frac{7}{12}$, the result is $\frac{56}{144}$, or $\frac{4}{12}\;\frac{8}{144}$, and, again, $\frac{8}{12}$ into 5, which is $\frac{40}{12}$, or $3\frac{4}{12}$, which, added to the former, makes $3\frac{8}{12}\;\frac{8}{144}$: the sum of these two products is $37\frac{2}{12}\;\frac{8}{144}$: if instead of retaining the denominators 12 and 144, we suppose their existence understood from the position of the numerator with respect to the place of units, we shall arrive at the precise process which is followed in duodecimal multiplication.

2. What is the number of cubic feet, inches, &c. in a piece of masonry, 9 feet, 3 inches long, 11 feet, 5 inches high, and 3 feet, 2 inches thick?

$$\begin{array}{r} ft. \;\; in. \\ 9 \;,\; 3 \\ \underline{11 \;,\; 5} \\ 101 \;,\; 9 \\ \underline{3 \;,\; 10 \;,\; 3} \\ 105 \;,\; 7 \;,\; 3 \\ \underline{3 \;,\; 2} \\ 316 \;,\; 9 \;,\; 9 \\ \underline{17 \;,\; 7 \;,\; 2 \;,\; 6} \\ 334 \;,\; 4 \;,\; 11 \;,\; 6 \end{array} \qquad \begin{array}{r} 9\frac{3}{12} \\ \underline{11\frac{5}{12}} \\ 101\frac{9}{12} \\ \underline{3\frac{10}{12}\;\frac{3}{144}} \\ 105\frac{7}{12}\;\frac{3}{144} \\ \underline{3\frac{2}{12}} \\ 316\frac{9}{12}\;\frac{9}{144} \\ \underline{17\frac{7}{12}\;\frac{2}{144}\;\frac{6}{1728}} \\ 334\frac{4}{12}\;\frac{11}{144}\;\frac{6}{1728} \end{array}$$

PROPORTION, THE RULE OF THREE, &c.

(384.) Before we proceed to the statement and explanation of the Rule of Three, the most important of all arithmetical rules, it appears to be requisite to give some account of the doctrine of ratios and proportion upon which it is founded.

Ratios.

Ratio exists between two numbers, or any quantities which are of the same kind, and admit of comparison in respect of magnitude: thus, we speak of the ratio

Arithmetic. of 3 to 5, of 7 days to 10 days, of 11 cwt. to 14 cwt., and so on: but it can have no existence between quantities which are dissimilar, such as £3. and 5 horses, 7 bushels and 9 feet, and so on, such quantities admitting of no comparison with each other.

How denoted. A ratio is denoted by placing two dots (:), one above the other, betweeo its terms: thus the ratio of 13 to 17 is denoted by 13 : 17: that of 3 feet to 7 feet by *ft. in.* 3 : 7; and similarly in all other cases; the first term being called the *antecedent*, and the secood the *consequent*.

Their meaning. (385.) The term ratio, however, does not convey at once to the mind a distinct idea of the nature of the comparison which is designated by it, or of the principles upon which the magnitude uf different ratios may be estimated: in order to *define* its meaning, the antecedeot is made the numerator, and the consequent the denominator of a fraction, and the magnitude of the fraction ascertains the value of the ratio: thus the ratio of 3 to 5 is denoted by $\frac{3}{5}$: by this means ratios are brought within the province of commoo arithmetic, and this assumption respecting the mode of denoting them, and thence of comparing them with each other, in reality constitotes the true arithmetical definition of the meaning of the term.

Proportion how denoted (386.) Proportion consists in the equality of ratios: thus the four qoantities 3, 5, 9, and 15, constitute a proportion, or are said to be proportionals, and are denoted usually in the following manner:

$$3 : 5 :: 9 : 15 :.$$

The sign (::) placed between the ratios of 3 : 5, and of 9 : 15, denotes the equality of the ratios; the whole expression is equivaleot to

$$\tfrac{3}{5} = \tfrac{9}{15},$$

the most convenient form of denoting it, inasmuch as the equality of these fractioos is the *test* of the proportionality of the terms.

If we reduce the two fractions to a common denominator, we shall find

$$\tfrac{3 \times 15}{5 \times 15} = \tfrac{5 \times 9}{5 \times 15}$$

and, therefore,

$$3 \times 15 = 5 \times 9$$

Product of the means equal the product of the extremes or, in other words, *the product of the two extreme terms of the proportion is equal to the product of the means*, a conclusion which is clearly general, inasmuch as the process which leads to it has no connection with the particular numbers above given.

It is an immediate corollary from this propositioo, that if the product of the *means* be divided by one of the *extremes*, the quotient is the other *extreme*; or if the product of the *extremes* be divided by one of the *means*, the quotient is the other *mean*.

It will readily follow from hence, that if three terms of a proportion are given, the fourth may be found, by multiplying the second and third together, and dividing by the first: thus, if it was required to find a fourth proportional to 8, 9, and 24, we should find $\frac{9 \times 24}{8} = 27$, for the number required.

The preceding propositions are all that are required in the solution of questions in the Rule of Three, which we shall now proceed to consider.

Part II. Rule of Three. (387.) The rule itself, and the principles upon which it is founded, will be best understood from its application to an example.

If 7 hats cost £9. 10*s*., what is the cost of 13?

Example. In this questioo, two of the three quaotities are of the same kind; the third is of the same nature with the quantity which is required to be determined.

Cunsidering this unknown quantity as the fourth term in a proportion, of which 7 *hats*, 13 *hats*, and £9. 10*s*. are the three first terms, they will stand as fullows:

hats		*hats*		£.		*s.*	
7	:	13	::	9	,	10	:

Or, reducing £9. 10*s*. to shillings:

$$\begin{array}{rrrrr} \textit{hats} & & \textit{hats} & & s. \\ 7 & : & 13 & :: & 190 : \\ & & & & 13 \\ \hline & & & & 570 \\ & & & & 190 \\ \hline & & & 7) & 2470 \\ \hline & & & & 352\frac{6}{7} \end{array}$$

We multiply the second aod third terms together, aod divide by the first, wheo we get $352\frac{6}{7}$, or £17. $12\frac{6}{7}$*s*., or £17. 12*s*. $10\frac{2}{7}$*d*. for the cost required.

The quantities which form the terms of the two ratios, of which the complete proportion is composed, are of the same kind; and these rates are, therefore, independent of the specific *denomination* of their terms: thus the ratio of 7 hats to 13 hats is identical with that of the abstract oumbers 7 aod 13, whilst the ratio of 190*s*. to $352\frac{6}{7}$*s*. is the same as that of 192 to $352\frac{6}{7}$: it is for this reason that we are allowed to multiply the mean terms, and divide by the extreme, precisely as in the case of whole numbers.

Names to distinguish the terms of the proportion (388.) It is convenient in the statement of this rule, to distinguish the two knowo terms which are of the same kind, by the names of the *argument* and the *demand*, and to designate the third koowo term as the *fruit* or *produce* of the *argument*, the unknown term being, therefore, the *fruit*, or *produce*, of the *demand*.

Thus, in the questioo proposed, the 7 *hats* are the *argument*, the 13 *hats* are the *demand*; and, consequently, £9. 10*s*. is the *fruit* of the *argument*, and £17. $12\frac{6}{7}$*s*. is the *fruit* of the *demand*, which is the answer to the question.

Rule of Three direct. (389.) If the *fruit* increase with the increase of the *argument*, the terms must be arranged in the following order;

The argument : *the demand* :: *the fruit of the argument* : *the answer*.

Inverse. If the *fruit* of the *argument* decrease with the increase of the *argument*, the order of the two first terms is inverted, and becomes as follows;

The demand : *the argument* :: *the fruit of the argument* : *the answer*.

Questions which come under the first arrangement belong to the *direct* Rule of Three; those which come under the second arrangement, belong to the *inverse* Rule of Three.

Examples. (390.) The following are examples:

(1.) What is the value of a cwt. of sugar, at 1*s*. $1\frac{1}{2}$*d*. per lb.?

```
lb.   lb.     s.  d.
1  :  112  ::  1 , 1½ :
                 12
                 ——
                 13½
                 112
                 ———
                 336
                112
                 56
                ————
           12)  1512
                ————
          2,0)  12,6
                ————
               £6 , 6s.  The answer.
```

In this case, 1 lb. is the *argument*, and 1s. 1½d. its *fruit*, whilst 112 lb. is the *demand*, and £6. 6s. is its *fruit*.

(2.) If the rents of a parish amount to £2340.17s. 6d., and a rate be granted of £137. 10s. 8d., what portion of it must be paid by an estate whose rental is £143. 9s. 10d.?

```
  £.    s.  d.     £.   s.  d.      £.   s.  d.
 2340 , 17 , 6  :  137 , 10 , 8  ::  143 , 9 , 10 :
   20               20                20
 ——————           ————              ————
 46817             2750              2869
    12               12                12
 ——————           —————             —————
 561810           33008             34438
                                    33008
                                   ——————
                                   275504
                                 10331400
                                103314
                               —————————— 12)
                   56181,0)  113672950,4 (2023
                             112362        ————
                             ——————  2,0) 16,8 , 7
                               131095     ————————
                               112362      8 , 8 , 7
                               ——————
                                187330
                                168543
                                ——————
                                 187874
                                      4
                                ———————
                                 751496 (1
                                 561810
                                 ——————
                                 189686
```

The answer is £8. 8s. 7¼d. $\frac{189686}{561810}$.

In this case, the terms are all of the same nature, though distinguished as the *argument*, its *fruit*, and the *demand*; they involve units of different denominations, and must all of them be reduced to the lowest. The statement, after these reductions are effected, would be

```
   d.          d.            d.
561810  :   33008   ::   1136729504 :
```

The answer, or fourth term, is of the same denomination with the third, inasmuch as the two first terms might be considered as abstract whole numbers.

(3.) How many quarters of wheat can I buy for 80 guineas at 8s. 6d. per bushel?

```
s.  d      guin.     bush.
8 , 6  :   80   ::   1   :
2          21
——        ————
17        1680
             2
          ———— 8)
     17)  3360 (197
          17    ———
          ——     24 , 5
          166
          153
          ———
           130
           119
           ———
            11
```

The answer is 24 *qrs.* 5 $\frac{11}{17}$ *bush.*

In this case, the two first terms are reduced to sixpences, instead of pence, by which means the result is more readily deduced.

(4.) If 12 men can reap a field of wheat in 3 days, in what time can the same work be performed by 25 men?

The *argument* is 12, and its *fruit* 3, and the *demand* is 25: it is obvious, that the increase of the *demand* must diminish the *fruit*, and, consequently, the statement must stand as follows:

```
men     men      days
25  :   12   ::   3  :
         3
        ——
   25)  36  (1 day.
        25
        ——
        11
        24
        ———
        264  (10 hours.
        25
        ———
         14
         60
        ———
        840  (33 minutes.
        75
        ——
         90
         75
         ——
         15
```

The answer is 1 *day*, 10 *hours*, 33⅗ *minutes*.

A very slight examination will show, that the proportion is correctly assumed in this case: if the number of reapers be doubled, the work will be done in half the time; if tripled, in one-third of the time; if quadrupled, in one-fourth of the time; and it is presumed, and indeed implied, that in all other cases, the time in which the same work may be done will be diminished or increased at the same *rate* with which the number of workmen is increased or diminished; and, consequently, the *argument* and *demand* must occupy a position in the terms of the proportion which is the inverse of that which they occupied in the Rule of Three Direct.

(5.) How much in length, that is 13½ poles in breadth, must be taken to contain an acre, which is 4 poles long and 4 poles broad?

```
poles   poles   poles
13½  :  40  ::  4  :
         4
       ----
13½)   160
 2       2
       ----
 27)   320 (11
       27
       ----
        50
        27
       ----
        23
         5½
       ----
       115
        11½
       ----
 27)   126½ (4
       108
       ----
        18½
         3
       ----
 27)    55½ (2
        54
       ----
         1½
        12
       ----
 27)    18 (0⅔
```

The answer is 11 *po.* 4 *yds.* 2 *ft.* 0⅔ *in.*

The greater the *breadth*, the less the *length*, the area remaining the same: the *demand*, 13½ poles, must, therefore, be put in the first place, and the *argument* 40 in the second.

(6.) If a certain number of men can throw up an entrenchment in 10 days, when the day is 6 hours long, in what time would they do it when the day is 8 hours long?

If the number of hours in each day be increased, the number of days will be diminished, the number of labourers and the work to be done remaining the same.

```
hours  hours    days
  8  :  6  ::   10  :
                 6
               ----
            8)  60
               ----
                7½ days.
```

Compound proportion. (391.) In many questions there are more *arguments* than one, with their corresponding *demands*. The following are examples:

(1.) If a family of 9 people spend £120. in 8 months, how much will serve a family of 24 people 16 months, at the same rate of living?

Arguments; 9 men and 8 months.
Their *fruit;* £120.
Demands; 24 men and 16 months.
The statement is as follows:

```
men    men      £.
 9  :  24  ::  120
 8  :  16
----  ----
72    144
       24
      ----
      384
      120
      ----
72)  46080 (640£. the answer.
     432
     ----
      288
      288
     ----
        0
```

The reason of this process will be evident, if we resolve it into two distinct statements: in the first place, suppose the time in both cases to be 8 months; then we should have

```
men    men    £.
 9  :  24  :  120  :
```

The fruit of the demand would be $\frac{24 \times 120}{9} = 320$.

Let us now suppose the number of men 24 in both cases, and the time different, when £320. will become the fruit of the argument, which is 8 months: we thus get

```
months  months
  8  :  16  ::  320  :  640
```

where the fourth term $640 = \frac{320 \times 16}{8} = \frac{24 \times 16 \times 120}{9 \times 8}$.

(2.) If a barrel of beer be sufficient to last a family of 7 persons 12 days, how many will be sufficient for a family of 14 persons for a year?

Arguments; 7 persons, 12 days.
Their *fruit;* 1 barrel.
Demands; 14 persons, 365 days.

```
 7  :  14  ::  1
12  : 365
---   ----
84   1460
      365
     ----
84)  5110 (60 70/84 barrels. Answer.
     504
     ----
       70
```

(3.) If 248 men, in 5 days of 11 hours each, can dig a trench 230 yards long, 3 wide, and 2 deep, in how many days, 9 hours long, can 24 men dig a trench of 420 yards, 5 wide, and 3 deep.

Arguments *direct;* 230 yds.: 3 yds.: 2 yds.
inverse; 248 men: 11 hours.
Their *fruit;* 5 days.
Demands *direct;* 420 yds.: 5 yds.: 3 yds.
inverse; 24 men: 9 hours.

$$248 \times 3 \times 2 \ : \ 420 \times 5 \times 3 \ :: \ 5 \ :$$
$$24 \times 9 \ : \ 248 \times 11$$

Arithmetic.

248	420
3	5
744	2100
2	3
1488	6300
9	11
13392	69300
24	248
53568	554400
26784	277200
321408	138600
	17186400
	5

```
321408) 85932000 (267[illegible] days.
        642816
        2165040
        1928448
         2365920
         2219856
          116064
```

This question would require five successive simple statements for its solution, three of them *direct*, and two of them *inverse*. In combining them into one statement or compound proportion, it is merely necessary to separate the *arguments* and *demands* into *direct* and *inverse*, and to multiply the *arguments* in the first into the *demands* in the second, for the first term; and the *demands* in the first into the *arguments* in the second, for the second term of the proportion.

Chain rule. (392.) The consideration of the preceding examples, and of the modes of solving them, would lead to a rule for their solution, in which it would be altogether unnecessary to arrange the terms in the form of a proportion: it would be as follows:

Write underneath each other the *direct* arguments and the *inverse* demands, and, in another column, write the *direct* demands and the *inverse* arguments, and underneath them the fruit: divide the product of the numbers in the second column by the product of the numbers in the first column; the quotient is the *fruit* demanded.

It is, of course, understood, that the corresponding quantities of the same *species* in each column are reduced to units (if necessary) of the same denomination.

It is this rule, which is denominated the Chain rule, which is extensively used in exchange operations, particularly by foreign merchants: the reason of its name will be understood from a particular mode of solving such questions of which examples may be seen in Art. 198, as well as from the modern practice. The following are examples of the use of this rule:

Examples. (1.) If 3 lb. of tea be worth 4 lb. of coffee, and 6 lb. of coffee be worth 20 lb. of sugar, how many pounds of sugar may be had for 9 lb. of tea?

9 lb. tea.
3 lb. tea = 4 lb. coffee.
6 lb. coffee = 20 lb. sugar.

$$\therefore \frac{20 \times 4 \times 9}{6 \times 3} = \frac{720}{18} = 40 \text{ lb. sugar.}$$

Part II.

If the *chain*, connecting the corresponding quantities, be added, it will stand as follows:

9 lb. tea.
3 lb. tea — 4 lb. coffee.
6 lb. coffee — 20 lb. sugar.

(2.) Required the value of the mètre of France in terms of the foot of Cremona, if 48 feet of Cremona = 56 English feet, and the mètre be = 39.371 English inches.

1 foot of Cremona.
48 feet of Cremona = 56 feet English
1 foot English = 12 inches.
39.371 inches = 1 mètre.

The result is $\frac{14}{39.371}$ mètres = 1 foot of Cremona, or 1 mètre = 2.812 feet.

(3.) Find the value of a kilogramme of gold, weighing 15434 Troy grains, $\frac{9}{10}$ fine, at £4. per ounce Troy. $\frac{11}{12}$ fine.

1 kilogramme.
1 kilogramme = 15434 Troy grains.
480 Troy grains = 1 ounce.
10 ounces French standard = 9 ounces fine.
11 ounces fine = 12 ounces English standard.
1 ounce English standard = £4.

$$\frac{15434 \times 9 \times 12 \times 4}{480 \times 10 \times 11} = £126.\ 5s.\ 6d.$$

(4.) What is the course of exchange between London and Paris resulting from the price of gold: the premium on the Paris mint price being 8 in the 1000, and the price itself being 78s. per ounce, English standard, which is $\frac{11}{12}$ ounce fine.

The course of exchange is expressed by the number of francs in a pound sterling. The mint price in France of a kilogramme of gold of 32.154 ounces, or 15434 grains Troy, being 3434.44 francs.

1 pound sterling.
1 pound sterling = 20 shillings.
78 shillings = 1 ounce standard gold.
12 ounces standard = 11 ounces fine.
32.154 ounces fine = 3434.44 francs, mint price.
1000 francs mint pr. = 1008 current.

$$\frac{20 \times 11 \times 3434.44 \times 1008}{78 \times 12 \times 32.154 \times 1000} = 25.3 \text{ francs per pound sterling.}$$

In making calculations for a variable premium and price of gold, it is usual first to determine the fixed number

$$\frac{20 \times 11 \times 3434.44}{12 \times 32.154 \times 1000} = 1.95823,$$

which, in the case before us, is multiplied by 1008, and divided by 78.

The same rule is applicable to the solution of all questions connected with the arbitration of exchange and other operations of commerce; numerous examples of which may be found in the second volume of Kelly's *Universal Cambist*.

Practice.

PRACTICE.

(393.) Practice is a compendious mode of solving Rule of Three questions, when the first term, or *argument*, is an unit, or 1; in this case, it is merely requisite to multiply the second term, considered as an abstract number, into the third term, in order to get the result.

Questions of this kind arise in the transactions of ordinary trade, where the price is required of a certain quantity of any species of goods generally estimated by weight: it is the particular nature of the questions proposed for the application of the rules of Practice, that makes it necessary for the student to make himself familiar with tables of the *aliquot* parts of a shilling and a pound sterling, and also with those of a cwt., quarter, and lb.

Table of aliquot parts.

(394.) *Aliquot parts of a shilling.*

6*d*. is $\frac{1}{2}$, a half.
4*d*. is $\frac{1}{3}$, a third.
3*d*. is $\frac{1}{4}$, a fourth.
2*d*. is $\frac{1}{6}$, a sixth.
$1\frac{1}{2}$*d*. is $\frac{1}{8}$, an eighth.
1*d*. is $\frac{1}{12}$, a twelfth.
$\frac{3}{4}$*d*. is $\frac{1}{16}$, a sixteenth.
$\frac{1}{2}$*d*. is $\frac{1}{24}$, or $\frac{1}{2}$ of a penny.
$\frac{1}{4}$*d*. is $\frac{1}{48}$, or $\frac{1}{4}$ of a penny.

Aliquot parts of a pound sterling.

10*s*. is $\frac{1}{2}$.
6*s*. 8*d*. is $\frac{1}{3}$.
5*s*. is $\frac{1}{4}$.
3*s*. 4*d*. is $\frac{1}{6}$.
2*s*. 6*d*. is $\frac{1}{8}$.
2*s*. is $\frac{1}{10}$.
1*s*. 8*d*. is $\frac{1}{12}$.
1*s*. is $\frac{1}{20}$.

Aliquot parts of a cwt.

2 *qrs*. is $\frac{1}{2}$.
1 *qr*. is $\frac{1}{4}$.
14 *lb*. is $\frac{1}{8}$.
8 *lb*. is $\frac{1}{14}$.
7 *lb*. is $\frac{1}{16}$.

Aliquot parts of a qr.

14 *lb*. is $\frac{1}{2}$.
7 *lb*. is $\frac{1}{4}$.
4 *lb*. is $\frac{1}{7}$.
$3\frac{1}{2}$ *lb*. is $\frac{1}{8}$.

Aliquot parts of a lb.

8 *oz*. is $\frac{1}{2}$.
4 *oz*. is $\frac{1}{4}$.
2 *oz*. is $\frac{1}{8}$.
1 *oz*. is $\frac{1}{16}$.

Examples of the different cases of practice.

(395.) The following examples will illustrate most of the cases which can arise, and which hardly merit a more formal classification.

(1.) Find the value of 733 (lb., oz., or units of any other species or denomination) at $\frac{3}{4}$*d*. each.

$\frac{1}{2}$*d*. is $\frac{1}{2}$	733
$\frac{1}{4}$*d*. is $\frac{1}{2}$	$366\frac{1}{2}$
	$183\frac{1}{4}$
12)	$549\frac{3}{4}$
2,0)	4,5 , $9\frac{3}{4}$
	£2 , 5*s* , $9\frac{3}{4}$*d*. The answer.

(2.) Find the value of 6771 at $8\frac{1}{2}$*d*.

6*d*. is $\frac{1}{2}$	6771
2*d*. is $\frac{1}{3}$	3385 , 6*d*.
$\frac{1}{2}$*d*. is $\frac{1}{4}$	1128 , 6*d*.
	282 , $1\frac{1}{2}$*d*.
2,0)	479,4 , $1\frac{1}{2}$*d*.
	£239 , 14*s* , $1\frac{1}{2}$*d*. The answer.

(3.) Find the value of 969 at 19*s*. 11*d*.

10*s*. is $\frac{1}{2}$£.	969
5*s*. is $\frac{1}{4}$£.	484 , 10
	242 , 5
4*s*. is $\frac{1}{5}$£.	193 , 16
6*d*. is $\frac{1}{8}$ of 4*s*.	24 , 4 , 6
3*d*. is $\frac{1}{2}$ of 6*d*.	12 , 2 , 3
2*d*. is $\frac{1}{3}$ of 6*d*.	8 , 1 , 6
The answer,	964 , 19 , 3

It would be very easy to select other aliquot parts which would equally make up 19*s*. 11*d*.

(4.) Find the value of 457 at £14. 17*s*. $9\frac{1}{2}$*d*.

	457
	14
	1828
	457
	6398
10*s*. is $\frac{1}{2}$£.	228 , 10
5*s*. is $\frac{1}{2}$ of 10*s*.	114 , 5
2*s*. 6*d*. is $\frac{1}{2}$ of 5*s*.	57 , 2 , 6
3*d*. is $\frac{1}{10}$ of 2*s*. 6*d*.	5 , 14 , 3
$\frac{1}{2}$*d*. is $\frac{1}{6}$ of 3*d*.	0 , 19 , $0\frac{1}{2}$
The answer,	£6804 , 10*s* , $9\frac{1}{2}$*d*.

(5.) Find the value of 17 cwt. 1 qr. 12 lb. at £1.19*s*.8*d*. per cwt.

	£. *s*. *d*.
	1 , 19 , 8
	17
1 *qr*. is $\frac{1}{4}$ *cwt*.	33 , 14 , 4
7 *lb*. is $\frac{1}{4}$ *qr*.	0 , 9 , 11
4 *lb*. is $\frac{1}{7}$ *qr*.	0 , 2 , $5\frac{3}{4}$
1 *lb*. is $\frac{1}{4}$ of 4 *lb*.	0 , 1 , 5
	0 , 0 , $4\frac{1}{4}$
The answer,	34 , 8 , 6

The preceding examples include most of the cases which are really different from each other, and are quite sufficient to exemplify the process to be followed in all those questions which are usually proposed for solution by the rules of Practice.

Tare and Trett.

(396.) Questions, in which it is required to determine the *neat*, or *nett*, weight, where the *gross* weight is to be diminished by allowances for *tare*, *trett*, &c., are commonly resolved by a similar method. The following are examples:

(1.) Find the *nett* weight where the *gross* weight is 173 cwt. 3 qrs. 17 lb. and the *tare* 16 lb. per cwt.

	cwt. qrs. lb.
14 *lb.* is $\frac{1}{8}$ *cwt.*	173 , 3 , 17
2 *lb.* is $\frac{1}{7}$	21 , 2 , 26
	3 , 0 , 11
	24 , 3 , 9
The answer,	149 , 0 , 8

(2.) What is the *nett* weight of 152 cwt. 1 qr. 3 lb. *gross*, *tare* 10 lb. per cwt., and *trett* being, as usual, 4 lb. in 104 lb, or $\frac{1}{26}$th part of the whole?

	cwt. qrs. lb.	
8 *lb.* is $\frac{1}{14}$ *cwt.*	152 , 1 , 3	
2 *lb.* is $\frac{1}{4}$ of 8 *lb.*	10 , 3 , 14	
	2 , 2 , 24	
	13 , 2 , 10	Tare.
26)	138 , 2 , 21	Suttle.
	5 , 1 , 9	Trett.
The answer,	133 , 1 , 12	Nett.

(3.) What is the gross weight of 27 cwt. 3 qrs. 16 lb., tare being 8 lb. per cwt., trett and cloff as usual, the last being 2 lb. in every 3 cwt.?

	cwt. qrs. lb.	
8 *lb.* is $\frac{1}{14}$ *cwt.*	27 , 3 , 16	
	1 , 3 , 27	Tare.
26)	25 , 3 , 17	Suttle.
	0 , 3 , 27	Trett.
168)	24 , 3 , 18	Suttle.
	0 , 0 , 17	Cloff.
The answer,	24 , 3 , 1	Nett.

INTEREST, DISCOUNT, BROKERAGE, AND OTHER QUESTIONS CONNECTED WITH THE PER CENTAGE RECEIVED OR PAID ON THE LENDING, BORROWING, INVESTMENT, TRANSFER, AND OTHER USES OF MONEY.

Interest.

(397.) Interest is the consideration due for the use of money, whether advanced as a loan, or due as a debt: it is generally estimated by the *per centage*, or sum allowed for £100. for 1 year.

The amount of this allowance will vary under different circumstances, being regulated by the nature of the security for the debt, and the abundance or scarcity of money: in this country it is limited by the law to five per cent., though a much higher interest is sometimes paid, under different forms, by which the provisions of the law may be evaded.

Interest simple and compound.

(398.) The interest is usually paid at the end of each year: if the payment be forborne for a longer period, the amount due will be different, according as it is estimated by *simple* or by *compound* interest. In the first case, no interest is paid on the amount of interest due and unpaid: in the second, the interest, when due, is supposed to be added to the principal, and the interest is subsequently calculated upon the whole amount.

Compound interest not legal: in what sense.

(399.) The law allows *simple* interest only; in other words, when the payment of the interest has been deferred for any number of years, such as 10, the person to whom it is due can only *demand* 10 times the interest due in one year, without any allowance of interest upon the amount of interest due: as the law, however, could enforce the payment of the interest at the end of each year, it may always be received, added to the principal, or otherwise invested, and thus compound interest may be legally secured, though not legally demanded.

Rule for finding simple interest.

(400.) The rule for finding the simple interest of any sum for any number of years is as follows: in order to determine, in the first place, the interest for one year, we must multiply the principal by the rate per cent., and divide the result by 100: this quotient, multiplied by the number of years, will give the interest required. The following are examples:

Example.

Required the simple interest of £237. 5s. 6d. for 3 years at 5 per cent.?

	£. s. d.
	237 , 5 , 6
	5
100)	11,86 , 7 , 6
	20
	17,27
	12
	3,30
	4
	1,20
	£. s. d.
	11 , 17 , $3\frac{1}{4}$
	3
The answer,	35 , 11 , $9\frac{3}{4}$

The first part of this process, for finding the interest for one year, is identical with the following Rule of Three statement:

£.		£. s. d.		£.	
100	:	237 , 5 , 6	::	5	:

Where £100. is the *argument*, £5. its *fruit*, and £237. 5s. 6d. the *demand*.

The whole process is equivalent to the following Double Rule of Three statement:

£.		£. s. d.		£.	
100	:	237 , 5 , 6	::	5	:
1	:	3			

Where £100. and 1 year are the *arguments*, £5. their *fruit*, and £237. 5s. 6d. and 3 years, the *demands*.

Second method of decimals.

(401.) Another method of solving such questions, is to reduce the shillings and pence to decimals of a pound sterling, and to find, from the rate of interest per cent.,

the rate for £1.: if the number of pounds sterling be multiplied by the interest of £1. for one year, it will give the whole interest for 1 year; and if the interest for 1 year be multiplied by the number of years, it will give the whole interest required.

Thus, in the last question, we should proceed as follows:

100) 5
.05

12) 6
20) 5.5
237.275 Number of pounds.
.05 Interest of £1.
11.86375
3 Number of years.
35.59125
20
11.82500
12
9.90000
4
3.60000

The answer is £35. 11*s*. 9¾*d*.

(2.) Let it be required to find the interest of £1229. 7*s*. 11½*d*. for 7½ years at 4½ per cent.?

By the first method:

£.	s.	d.
1229,	7,	11½
		4½
4917,	11,	10
614,	13,	11¾
100) 55,32,	5,	9¾
20		
6.45		
12		
5.49		
4		
1.99		

	£.	s.	d.
Interest for 1 year	55,	6,	5½
			7½
	387,	5,	2½
	27,	13,	2¾
The answer,	414,	18,	5¼

By the second method:

4) 2
12) 11.5
20) 7.9583
1229.39791
$.045. = \frac{4\frac{1}{2}}{100}$
614698955.
491759164
55.32290595
$7.5 = 7\frac{1}{2}.$
27661452975
38726034165
414.921794625
20
18.435892500
12
5.230910000
4
.923640000

The answer is £414. 18*s*. 5¼*d*. nearly.

The process might be shortened considerably by omitting all decimals after the fourth place, increasing the last figure by unity, when the next digit is equal to, or greater, than 5.

(3.) What is the interest due upon £450. at 3¾ per cent. per annum for 2¾ years and 67 days?

4) 3
100) 3.75
.0375

£.
450
.0375
18750
1500
$16.875\cancel{0}$
2.9335
84375
50625
50625
151875
33750
$49.5028\cancel{1}\cancel{2}\cancel{5}$
20
$10.056\cancel{2}$
12
.672
4
2.688

365) 67.
.1835
2.75
2.9335

The answer is £49. 10*s*. 0¾*d*. taking the nearest integral values.

(402.) The following questions are equivalent in principle to those in which the interest is required of a sum of money for one year only.

Other questions solved upon the same principle. Commission.

(1.) What is the *commission* on £769. 3*s*. 6*d*. at 2½ per cent.?

```
  £.    s.  d.
 769 ,  3 , 6
           2½
 ------------
1538 ,  7 , 0
 384 , 11 , 9
 ------------
100) 19.22 , 18 , 9
        20
      ----
      4.58
        12
      ----
      7.05
```

The answer is £19. 4s. 7d.

Brokerage. (2.) What is the *brokerage* on £7999. 11s. 4d. at ¼ per cent.?

```
   4) 7999 , 11 ,  4
      --------------
100) 19.99 , 17 , 10
        20
      -----
      19.97
         12
      -----
      11.74
          4
      -----
       2.96
```

The answer is £19. 19s. 11¾d.

Insurance (3.) What is the *insurance* upon £24034. 14s. 2d. at 11½ per cent.?

```
      £.     s.  d.
   24034 , 14 ,  2
                11½
  ----------------
  264381 , 15 , 10
   12017 ,  7 ,  1
  ----------------
100) 2763.99 , 2 , 11
          20
       -----
       19.92
          12
       -----
       11.15
```

The answer is £2763. 19s. 11d.

Sale of stock (4.) What is the value of £8334., 3 per cent. stock, at 81⅞ per cent.?

```
    8) 7             8334
       ---          .81875
100) 81.875        -------
       ---          327500
    .81875         245625
                  245625
                 655000
                 ----------
                 6823.46250
                         20
                     ------
                     9.2500
                         12
                     ------
                       3.00
```

The answer is £6823. 9s. 3d.

(5.) What is the value of £2170. 5s. 6d. Bank stock at 217¼ per cent.?

```
12) 6            2170.275
    ---              217¼
20) 5.5          ---------
    ----         15191925
2170.275         2170275
                4340550
                  54256875
                ------------
                4714.9224|375
                       20
                  -------
                  18.4480
                       12
                  -------
                   5.3760
                        4
                   ------
                   1.5040
```

The answer is £4714. 18s. 5½d.

It is unnecessary to give other questions connected with the purchase or sale of other species of stock, whose value is estimated by the rate per cent. at which it is saleable for ordinary money, as they are all of them solved upon the same principle with those above given.

Discount. (403.) Discount is the deduction made in consideration of the payment of money before it is due.

Present worth. The *present worth* of a *principal* sum due hereafter, is the sum which, if paid *immediately*, will *amount*, at simple interest, to the *principal* when that principal is due.

The *discount* is, therefore, the difference between the *present worth* and *principal*.

Rule. In questions respecting discount, the *principal* must be considered as the *amount* of the *present worth* put out to interest at a certain rate per cent. for the time which elapses before the principal is due; and in reducing such questions to a statement, we must consider the *amount* of 100 for that time as the *argument*, its *interest* as the *fruit*, and the *principal* as the *demand*.

(1.) What is the discount of £400. due 2 years hence at 5 per cent.?

The interest of £100. for 1 year is £5.
2 years is £10.

```
 £.     £.
110  :  10  ::  400
                 10
          ---------
          11,0) 400,0
```

The answer is £36 , 7s , 3¼d.

The present worth is, therefore,

£400. – £36. 7s. 3¼d. = £363. 12s. 8¾d.

Or it may be found at once by the following statement,

```
 £.     £.
110  :  100  ::  400
                 100
          ----------
          11,0) 4000,0
                -------
          £363 , 12s , 8¾d.
```

(2.) What is the present worth of £273. 4s. 6d. due at the end of 3 months, discounting at 4½ per cent.?

The amount of £100. in 1 year is £104. 10s.
¼ year is £101. 2s. 6d.

Arithmetic.

```
  £.  s. d.     £.        £.  s. d.
101 , 2 , 6  :  100  ::  273 , 4 , 6
     20                        20
   ----                      ----
   2022                      5464
     12                        12
  -----                     -----
  24270           2427,0)  655740,0 (270£.
                           4854
                           -----
                           17034
                           16989
                           -----
                             450
                              20
                            ----
                            9000 (3
                            7281
                            ----
                            1719
                              12
                           -----
                           20628 (8
                           19416
                           -----
                            1212
                               4
                            ----
                            4848 (2
                            4854
```

Answer, £270. 3s. 8½d.

(3.) What ready money will discharge a debt of £1377. 13s. 4d., due 2 years, 3 quarters, and 25 days hence, discounting at $4\frac{3}{8}$ per cent. per annum?

```
365.) 25              8) 3
     -----               -----
      .0685              4.275
      2.75
     ------
      2.8185
      4.375
     -------
      140925
     197295
     84555
   112740
   ----------
   12.3309375, or 12.33£. nearly.

112.33  :  100  ::  1377.6666
                     100
                ----------
        112.33) 137766.66 (1226
                11233
                -----
                 25436
                 22466
                 -----
                  29706
                  22466
                  -----
                   72406
                   67398
                   -----
                    5008
                      20
                  ------
        112.33)   100160 (8
                   89864
                  ------
                   10296
                      12
                  ------
        112.33)   123552 (11
                  123563
```

The present worth is £1226. 8s. 11d. nearly.

Part II.

Compound interest.

(404.) We have before mentioned the essential distinction between simple and compound interest: it remains to consider the principles upon which it may be calculated.

Rule.

The most simple and obvious method is to calculate the interest for 1 year, to add it to the principal, and thus to find the amount at the end of the first year: this amount becomes the principal upon which the interest for the second year must be calculated, and thus the whole amount at the end of it may be determined: the second amount becomes the principal for the third year, and by the same process we may find the amount at the end of the third year: by continuing this process, we may find the amount during any number of years during which the interest is supposed to accumulate: the difference between the first principal, and the last amount, is the compound interest required.

(1.) Required to find the compound interest of £320. for 3 years at 4 per cent. per annum?

$\frac{4}{100} = \frac{1}{25}$

```
          £.
     25) 320              Principal.
          12 , 16
     -------------
     25) 332 , 16         Principal for 2d year
          13 ,  6 ,  3      nearly.
     -------------
     25) 346 ,  2 ,  3    Principal for 3d year.
          13 , 16 , 10
         ------------
         359 , 19 ,  1    Last amount.
         320 ,  0 ,  0
         ------------
The answer, 39 , 19 ,  1  Interest.
```

(2.) Required the amount of £760. 10s. forborne 3 years at $4\frac{1}{2}$ per cent.?

$\frac{4\frac{1}{2}}{100} = .045$

```
  £.   s.
 760 , 10 = 760.5     Principal.
            1.045
           ------
            38025
           30420
          76050
          --------
          794.7225    2d Principal.
           1.045
         ---------
          39736125
         31788900
        79472250
        ------------
        830.485|ø/25  3d Principal.
        1.045
       ---------
        4152425
       3321940
       8304850
      ------------
      867.8568 25     Final amount.
            20
      -------
      17.1360
           12
      -------
       1.6320
            4
      -------
       2.5280
```

The amount of £1. in 1 year = 1.045.

The answer is £867. 17s. 1½d.

Arithmetic.

In this case we determine the amount of £1. in one year, and multiply the principal by it, in order to determine its amount also: the same process is applied to the several principals in succession.

It would clearly lead to the same conclusion, if we first multiplied the decimal expressing the amount of £1. in one year into itself once, twice, thrice, &c., according as the interest or amount is to be calculated for 2, 3, 4, or a greater number of years; and, lastly, multiply the last product by the first principal.

(3.) Let it be required to find the amount of £1057. 2*s*. 6*d*. for 5 years at 4 per cent.

```
The amount of £1. for 1 year is 1.04     (1.)
                                1.04     (2.)
                                ----
                                 416
                                1040
                              ------
                              1.0816
                                1.04     (3.)
                              ------
                               43264
                              108160
                            --------
                            1.124864
                                1.04     (4.)
                            --------
                               44892
                              112480
                            --------
                            1.169794
                                1.04     (5.)
                            --------
                               46788
                              116970
                            --------
                            1.216529
                            1057.125
                          ----------
                               60825
                              24330
                             12165
                            85155
                           60825
                          121650
                        ------------
                        1285.9925|625
                                20
                            --------
                            19.8500
                                 12
                            -------
                              10.20
```

The answer is £1285. 19*s*. 10*d*

When the number of years is considerable, the calculation of compound interest becomes extremely laborious: in such cases it is generally necessary to have recourse to logarithms.

We shall not proceed to the consideration of questions on the amounts of annuities, accumulating at simple or compound interest, the present worth of annuities, whether perpetual or limited, equation of payments, &c. the rules for which are founded upon algebraical formulæ, without the aid of which they admit not of explanation or proof.

BARTER.

Barter.

(405.) Questions in Barter usually resolve themselves, with very slight modifications, into ordinary cases of the Rule of Three.

Part II.

(1.) How much sugar, at 9*d*. per lb., must be given in barter for 17 cwt. of tobacco, at £3. 10*s*. per cwt.?

```
         £.  s.
          3, 10
             17
         ------
         59, 10
 d.      £.  s.      lb.
 9  :    59, 10  .:  1 .
         20
         ----
         1190
           12
      -------
   9) 14280
      -------
  28) 1586
      -------
    4) 56, 18
      -------
       14,  0
```

The answer is 14 cwt. 0 qr. 18⅔ lb.

(2.) A merchant barters 1200 lb. of pepper, at 13*d*. per lb., for equal quantities of two species of cotton at 7*d*. and 11*d*. per lb., and for ⅓*d*. in money; how many lbs. of each sort must he receive, and how much in money?

```
          lb.
         1200
           13
       ------
   12) 15600
       ------
   20) 130,0
       ------
    3) 65
       ------
       £21, 13, 4  sum paid in money.
        43,  6, 8  the amount bartered
                     in goods.
```

Now it is evident, that 7 + 11, or 18*d*., is expended for every lb. of each species of cotton which is given in exchange: consequently,

```
 d.     £. s. d.     lb.
 18  :  43, 6, 8  ::  1
        20
       ----
        866
         12
      ------
  18) 10400 (577 7/9 lb.   The answer.
        90
       ---
       140
       126
       ---
        140
        126
        ---
         14
```

PROFIT AND LOSS.

Profit and Loss.

(406.) Questions connected with the gain or loss per cent. upon goods bought in gross and sold in detail, or conversely, and, in short, under any other circumstances, are resolved by one or more statements by the Rule of

Three, combined in some cases with reductions, which are suggested by the nature of the questions proposed.

(1.) At 1*s*. 3½*d*. in the pound profit how much is gained per cent. ?

1 : 100 :: 1, 3½
12
15
4
4) 6200
12) 1550
2,0) 12,9, 2
£6, 9*s*, 2*d*. Answer.

(2.) Bought goods at 7¾*d*. per lb., and sold them at £4. 15*s*. per cwt., what is the gain or loss per cwt. ?

First statement:

lb. : *lb.* :: *d.*
1 : 112 :: 7¾ :
4
31
112
112
336
4) 3472
12) 868
2,0) 7,2, 4 per cwt.

Cost price, £3, 12*s*, 4*d*.

Second statement:

£. s. d. : £. s. :: £.
3, 12, 4 : 4, 15 :: 100
20 ... 20
72 ... 95
12 ... 12
868 ... 868) 114000 (131
868
2720
2604
1160
868
292
20
5840 (6
5208
632
12
7584 (8
6944
640
4
2560 (2
1736
724

131, 6, 8½
100, 0, 0
£31, 6, 8½ The answer.

(3.) If I buy tobacco at 10 guineas per cwt., at what rate must I sell it per lb. so as to gain 12 per cent. ?

First statement:

lb. : *lb.* :: £. *s.*
112 : 1 :: 10, 10 :
20
112) 210 (1*s*.
112
98
12
1176 (10*d*.
112
56
4
224 (2*q*.
224

The cost price per lb. is 1*s*. 10½*d*.

Second statement:

£. : £. :: *s. d.*
100 : 112 :: 1, 10½ :
12
22
4
90
112
100) 10080
4) 100.$\frac{8}{10}$
12) 25.$\frac{1}{5}$
2*s*, 1⅕*d*. The answer.

(4.) If when I sell cloth at 10*s*. per yard, I lose 5 per cent., how much shall I gain or lose per cent. by selling it at 12*s*. 6*d*. per yard ?

Statement:

s. : *s. d.* :: £.
10 : 12, 6 :: 95 :
12 ... 12 ... 150
120 ... 150 ... 4750
95
120) 14250 (118$\frac{9}{12}$
120
225
120
1050
960
90

The gain per cent. is £18¾.

(5.) Sold goods for £75., and by so doing I lost 10 per cent., whereas in the regular course of trade I should have gained 30 per cent.: how much were they sold under their proper value ?

Statement:

```
 £.     £.      £.
 90  .  130 ::  75 :
                13
               ----
               225
               75
               ----
            9) 975
               ----
               108⅓
```

The goods were, therefore, sold at £108⅓ − £75., or £33. 6s. 8d. under their just value.

FELLOWSHIP.

Fellowship.

(407.) This is the rule by which the individual shares are assigned in joint stock transactions with two or more partners.

Single and Double.

The questions will be different according as the several stocks, or their equivalents, are invested for the same or different periods of time: questions of the first kind belong to Single, and those of the second kind to Double Fellowship.

Single Fellowship.

(408.) In Single Fellowship, the accumulated capital, or the gain or loss upon it, is divided in the proportion of the several capitals, or their equivalents, which are invested in the concern.

(1.) A and B gain by trading £750.; A's original stock was £500., and B's £850.; in what ratio must they divide the profits?

First statement:

```
               500
               850
               ----
              £1350   Joint stock.
 £.            £.
1350   :      500  ::  750
                       500
                     ------
              1350) 375000 (277
                    2700
                    -----
                    10500
                     9450
                    -----
                    10500
                     9450
                    -----
                     1050
                       20
                    -----
              1350) 21000 (15
                    1350
                    -----
                     7500
                     6750
                    -----
                      750
                       12
                    -----
              1350)  9000 (6
                     8100
                    -----
                      900
                        4
                    -----
              1350)  3600 (2
                     2700
                    -----
                      900
```


Second statement:

```
 £.      £.      £.
1350  :  850 ::  750 :
                 850
                -----
                37500
               6000
              ------
      1350) 637500 (472
            5400
            -----
             9750
             9450
            -----
             3000
             2700
            -----
              300
               20
            -----
      1350)  6000 (4
             5400
            -----
              600
               12
            -----
       350)  7200 (5
             6750
            -----
              450
                4
            -----
      1350)  1800 (1
             1350
            -----
              450
```

Consequently, £277. 15s. $6\frac{1}{2}d.\ \frac{900}{1350}$. A's portion.
£472. 4s. $5\frac{1}{4}d.\ \frac{450}{1350}$. B's portion.
the sum of which is £750.

In practice it is not necessary to work out the two statements, inasmuch as A's portion subtracted from £750. will give B's portion.

(2.) Three persons, A, B, C, invest £134. 10s., £340. 5s., and £425. 5s., respectively, in a partnership; at the end of 3 years they find the value of their capital reduced by losses to £500., what portion of the loss must they severally sustain?

```
 £.   s.
134 , 10   A's capital.
340 ,  5   B's capital.
425 ,  5   C's capital.
---------
900 ,  0
500
---
400   The loss.
```

The following are the three statements:

```
       £.       £.       £.   s.
(1.)  900  :  400  ::  134 , 10  :  A's loss.
(2.)  900  :  400  ::  340 ,  5  :  B's loss.
(3.)  900  :  400  ::  425 ,  5  :  C's loss.
```

	£.	s.	d.
Consequently, A's loss =	59	15	$6\frac{1}{2}\ \frac{6}{9}$.
B's loss =	151	4	$5\frac{1}{4}\ \frac{3}{9}$.
C's loss =	189	0	0
Their sum =	400	0	0

(409.) In Double Fellowship we must multiply each separate capital, or its equivalent, into the time of its employment, and proceed with the products in the same manner as with the simple capitals in questions in Single Fellowship.

Double Fellowship.

Arithmetic.

(1.) A employs a capital of £500. in trade, and at the end of 3 years takes B into partnership, who advances a capital of £800.: at the end of 6 years from this time, they have gained £600.: in what ratio must the profits be divided?

$500 \times 9 = 4500$, the product of A's capital and time.
$800 \times 6 = 4800$, the product of B's capital and time.
9300

```
9300  :  4500  :: 600
                   45
             93) 27000 (290
                 186
                 ---
                  840
                  837
                  ---
                    30
                    20
             93)   600 (6
                   558
                   ---
                    42
                    12
             93)   504 (5
                   465
                   ---
                    39
                     4
             93)   156 (1
                    93
                   ---
                    63

9300  :  4800  :: 600  :
                   48
             93) 28800 (309
                 279
                 ---
                   900
                   837
                   ---
                    63
                    20
             93)  1260 (13
                   93
                   ---
                   330
                   279
                   ---
                    51
                    12
             93)   612 (6
                   558
                   ---
                    54
                     4
             93)   216 (2
                   186
                   ---
                    30
```

	£.	s.	d.
Consequently, A's share is	290 ,	6 ,	$5\frac{1}{4}\frac{63}{93}$.
B's share is	309 ,	13 ,	$6\frac{1}{2}\frac{18}{93}$.

Part II.

(2.) A ship's company take a prize of £4000., which is to be divided amongst them in proportion to their pay, and to the time they have been on board. There are 6 officers, who have 120*s*. a month, and have been on board six months; 12 midshipmen, who have each 40*s*. a month, who have been on board 4 months; and 110 sailors, who have 30*s*. a month, and have been on board 3 months: what sum must each receive?

We must first determine the sum due to officers, midshipmen, and sailors, considered as each constituting one body, and then divide the respective sums by the number of officers, midshipmen, and sailors.

$$6 \times 6 \times 120 = 4320$$
$$12 \times 4 \times 40 = 1920$$
$$110 \times 3 \times 30 = 9900$$
$$16140$$

The following are the statements:

				£.		
16140	:	4320	::	4000	:	Officers' portion.
16140	:	1920	::	4000	:	Midshipmen's portion.
16140	:	9900	::	4000	:	Sailors' portion.

Consequently,

	£.	s.	d.
The 6 officers receive	1070 ,	12 ,	$7\frac{1}{2}\frac{1116}{1614}$.
The 13 midshipmen	475 ,	16 ,	$8\frac{1}{2}\frac{1572}{1614}$.
The 110 sailors....	2453 ,	10 ,	$7\frac{1}{2}\frac{540}{1614}$.
Each officer receives	178 ,	8 ,	$9\frac{1}{4}$.
Each midshipman..	39 ,	13 ,	$0\frac{3}{4}$.
Each sailor	22 ,	6 ,	1.

The reader is referred to the historical notice of the Rule of Three, Practice, Tare and Trett, Interest, Discount, Barter, Loss and Gain, and Fellowship, for other examples in illustration of these rules.

The ample notice which is given in the history of Arithmetic of the rules of Alligation and of Single and Double Position, supersedes the necessity of the more formal statement of these rules, which is given in ordinary books of Arithmetic: such rules, indeed, possess very little practical interest or importance, as the questions to which they apply are more generally, if not more readily, solved by algebraical processes.

ALGEBRA.

Relation between Arithmetic and Algebra.

(1.) ARITHMETIC and ALGEBRA are Sciences the object of which is to trace the relations and properties of NUMBER. Through the medium of Number, quantity in general is brought under their dominion; but they reject the consideration of those properties which are peculiar to particular species of quantity, being strictly confined to those which appertain to quantity in the abstract. Number may be properly said to be a means for expressing the abstract relation of one quantity to another of the *same kind*; that is to say, the relation which they have independently of the species to which they belong. Thus, a certain length called a *foot* has a relation to another length called an *inch*. Again, a certain portion of time called a *year* has a relation to another part of time called a *month*. Now, although the quantities between which these relations subsist be different in species, the one being *space* and the other *time*, yet, notwithstanding this, the relations are the same, and are both expressed by the *number* 12. In this respect then, as being independent of any particular species of quantity, Arithmetic and Algebra agree, and are, so far, equally abstract.

But although Arithmetic is abstract as to the species of quantity, yet the *relations* which it contemplates, and whose properties it investigates, are *particular*. In other words, its objects are particular numbers, and their properties and its notation, at least that of modern Arithmetic, are the nine Arabic digits, and 0, or cypher. It teaches the method of expressing, by various combination of these, all particular numbers whatever; and it investigates the properties of particular numbers, and the methods of performing the various Arithmetical operations on them, and the solutions of problems respecting them.

In the process of generalization, Algebra however advances further than Arithmetic. The Algebraist, not confining himself to the properties and relations of particular numbers, takes a wider range, and investigates the relations which may be considered common to all number, and so departs one step farther from specific quantity. While the Arithmetician is abstract as to the quantity, but particular as to the relation, the Algebraist is abstract as to both. An example will illustrate this:

The problem, "To divide the number 10 into two parts, one of which is double the other," is Arithmetical. It is abstract, as to the quantity expressed by the number 10; but the relations of the parts, into which it is proposed that this number be divided, to each other, and to the whole, are particular. Let the problem be modified, so that the *relations*, as well as the quantities, shall become abstract, and it ceases to be an Arithmetical question, and becomes Algebraical; in which case it is expressed thus, "To divide any given number into two parts which shall bear to each other a given ratio."

The former problem is, evidently, only one individual of an extensive class which is comprised under the latter. When the former has been solved, the result is merely the calculation of one particular numerical question; on the other hand, the solution of the other furnishes a general method for the calculation of any of the class of the questions of which the former is only an example.

Algebra called Universal Arithmetic.

(2.) It is from this circumstance that Newton called Algebra *Universal Arithmetic*. If this were the only respect in which the powers of Algebra exceed those of common Arithmetic, the propriety of the title could scarcely be disputed. But the student will not have penetrated very deeply into the science before he will perceive, that the title *Universal Arithmetic* very inadequately expresses the nature, objects, and extent of this department of Analysis.

Difference of notation.

(3.) From the more abstract nature of the objects of Algebra, it follows that the notation of Arithmetic is insufficient for its processes. The numerical symbols are essentially particular, and are therefore incapable of expressing the general relations which are here contemplated. Instead, therefore, of the Arabic digits, and their combinations, the letters of the alphabet have been by universal consent adopted to express numbers in Algebra. Thus the example already given would be thus expressed, "To divide a given number a into two parts, such that one should bear to the other the given ratio of $m : n$." It would be, evidently, impossible to express this problem by the symbols of Arithmetic; for the moment particular numbers should be introduced to express the different data, the problem would lose its *general* character, and become an ordinary Arithmetical question.

The change in the nature of the symbols used to express the numbers which are contemplated in Algebra, renders a change in the manner also of expressing the relations and operations on these numbers necessary. In Arithmetic, the operations on numbers are actually performed, and the results actually obtained; but in Algebra, the operations and results are not actually effected, but only expressed. Thus, if in Arithmetic it be proposed to add 5 to 7, the process is *effected*, and the result is 12. In Algebra, if it be required to add the number a to the number b, the process of addition is *indicated* by the sign + called *plus*, and the result, or the *sum* of the numbers a and b, is expressed by $a + b$.

Advantages of Algebraic notation

In examining these two processes it is remarkable, that in the Arithmetical result no trace whatever is left of the process by which it was obtained. The sum 12 might have been obtained by the addition of 8 and 4, or 9 and 3, or various other numbers, for any thing which can be inferred from the mere result. But in the Algebraical result the process is quite apparent, and is in effect actually expressed by $a + b$; for although two other numbers, as c and d, might have the same sum as a and b, yet that sum would be expressed by $c + d$, and not by $a + b$. This remark, which will be found of some importance, is equally applicable to the result of every Algebraical investigation, as compared with an Arithmetical process.

(4.) It must be apparent from these observations,

Algebra.

that Arithmetic and Algebra are so closely connected, that it is difficult to treat of either without, in some degree, encroaching on the province of the other. In the natural order of ideas in the human mind, the particulars precede the generals; and, therefore, although all the particular properties and theorems of numbers which form the subject of Arithmetic, are included in the more general results of Algebra, yet we have given the former priority in our series of Mathematical papers.

In the following Treatise on Algebra we shall avoid, as far as possible, a repetition of the demonstration of principles already established in our Treatise on Arithmetic, yet some repetition will be unavoidable, to give that connection to the chain of reasoning without which our investigations would be, in a great degree, unintelligible.

For history, see History of Analysis.

(5.) We do not propose in this place to enter into any historical account of the origin and progressive improvement of Algebra. This and the other departments of analytical science are so intimately connected, and, consequently, every great step in the improvement of any one of them has produced such important effects on the others, that it has been thought advisable, instead of introducing each part of analysis by a historical notice, to conclude our Mathematical papers with one comprehensive History of Analysis. This, together with the historical notices of Geometry and Arithmetic already given, will, it is hoped, form a very complete History of the Mathematical Sciences.

We shall devote the present article to an elementary Treatise on Algebra in the most improved state to which it has been brought in modern times. For an account of the principal works on Algebra, we refer to the general catalogue of Mathematical works at the conclusion of the History of Analysis.

SECTION I.

Notation.

Symbols twofold.

(6.) In Algebra, numbers and the operations to which they are conceived to be submitted are represented by arbitrary symbols.

Hence there are necessarily two systems of symbols; one to express the numbers themselves, and the other to represent the operations to be effected on them.

Numbers are, by universal consent, expressed by the letters of the alphabet, except in certain cases in which *particular* numbers are used, in which case the symbols and notation of common Arithmetic are preserved.

In Algebra, and indeed in Mathematical science generally, there are two distinct species of questions:

Theorem.

1. A *Theorem*, the object of which is to establish certain known or given properties of numbers.

Problem.

2. A *Problem*, the object of which is to determine certain numbers, certain other numbers being known or given, which have, with the numbers required, known or given relations.

In every Problem, therefore, there are two distinct sets of numbers to be expressed, the known or given, and the unknown or sought.

(7.) It is an universal custom to express the known or given numbers by the first letters of the alphabet, a, b, c, &c., and the unknown or sought numbers by the last, x, y, &c.

Notation.

Each of the operations to which numbers may be submitted, is expressed by a peculiar symbol. The four elementary operations, *Addition*, *Subtraction*, *Multiplication*, and *Division*, are expressed as follows:

Addition.

(8.) 1. *Addition*. When two numbers are added together, the process is signified by the sign $+$, called *plus*, placed between the symbols which express the numbers; and the whole combination, the symbols with the sign between them, is understood to express the result of the process, or the sum of the numbers.

Thus $7 + 5$ represents 12, $a + b$ represents the sum of the numbers represented by a and b. It may, and frequently does happen, that more than two numbers are to be added. This is expressed by the interposition of the sign $+$ between every successive pair of them. Thus, if 7, 5 and 3 are to be added, their sum is expressed by $7 + 5 + 3$, which arithmetically would be 15. If a, b, and c, are to be added, their sum is expressed by $a + b + c$.

In this case, it is evidently indifferent in what order the operations may be performed. Thus, the sum will be the same if b be first added to a, and then c added to their sum, as if c were added to a, and b added to their sum; that is, $a + b + c$ is equal to $a + c + b$. And, in the same manner, it is equal to $b + c + a$, and to $b + a + c$. In a word, the sum will be the same in whatever order the letters may be written.

It may happen, that the letters which are added together are equal to each other. Thus, if a, b, and c, were equal, their sum would be $a + a + a$. It is not usual, however, to express it in this way. The sum in this case is expressed by the single letter a with a number prefixed to it thus, $3\,a$, signifying the number of times the same letter would occur in the sum were it expressed in the manner it would be expressed had the letters been different. Thus, $a + a + a + a + a$ is expressed by $5\,a$. This number is called the *coefficient* of the letter; thus, in $5\,a$, 5 is the coefficient of a. Coefficient.

When a letter having a coefficient is to be added to another, the sign of addition precedes the coefficient. Thus, if $5\,b$ be to be added to a, the sum is expressed by $a + 5\,b$.

(9.) 2. *Subtraction*. When one number is to be subtracted from another, the operation is expressed by the sign $-$, called *minus*, placed after the *minuend* and before the *subtrahend*, and the whole combination of symbols expresses the remainder. Subtraction

Thus, if 5 be to be subtracted from 7, the process is expressed by $7 - 5$, which represents the remainder 2. If a be the minuend, and b the subtrahend, $a - b$ represents the remainder.

It may happen, that a number is to be subtracted from the sum of several others, $a + b + c$. In this case this sum may be treated as a single quantity, in which case it is usual to enclose it in a parenthesis, thus, $(a + b + c)$, or to draw a line over the letters, called a *vinculum*, thus, $\overline{a + b + c}$, in which case the remainder will be expressed thus, $(a + b + c) - d$, or $\overline{a + b + c} - d$. These combinations of symbols signify, that a, b, and c, are to be first added together, and then the number d subtracted from the result.

To express the remainder in this case it is not, however, necessary to resort to a parenthesis or *vinculum*. It is evident, that the number d will be subtracted from the sum $a + b + c$, if it be subtracted from any one o

Algebra.

its component parts, a, b, c. Hence it follows, that the remainder, which we have expressed above by $(a + b + s) - d$, may also be expressed by $a + b + s - d$, without the parenthesis. The expression in this case may be understood to mean the sum of a, b, and the remainder $s - d$ found by subtracting d from s. In the same way, the remainder might be expressed by $a + b + s - d$, or $a - d + b + c$, or by the same four letters placed in any order whatever, provided the same sign $+$ or $-$ precede the same letters.

Here we should observe, that if the first letter a be transposed, so as to be preceded by any other letter, the sign $+$ must be prefixed to it. This is obvious, since $a + b$ is necessarily equivalent to $b + a$. But further it is necessary, that if the quantity d, to which the sign $-$ is prefixed, be placed first, it will not be correct to place it without any sign prefixed, for in that case the meaning of the whole combination would be changed. Thus, $d + a + b + c$ would signify the sum of the four numbers a, b, c, and d, instead of the remainder when d is subtracted from the sum of a, b, and c. If d, therefore, be placed first, it will be necessary to prefix to it the sign $-$; indicating, that the manner in which it is to be combined with the other quantities is by subtraction. In the same sense, therefore, when no sign is prefixed to the first quantity, the sign $+$ is to be understood.

These symbols $+$ and $-$ are called the *signs* of the quantities to which they are prefixed; their true and only meaning is, as already explained, to indicate the manner in which the quantities which follow them are to be united with the other quantities with which they may happen to be combined, i. e. whether they are to be added or subtracted. In this sense, the quantities might with great propriety be denominated in reference to their signs, *additive* and *subtractive*; an *additive* quantity being one which has the sign $+$, and a *subtractive* quantity one which has the sign $-$. But long established usage has given to these signs the names *positive* or *affirmative*, and *negative*; that being called a *positive* or *affirmative quantity* which has the sign $+$, and that a *negative quantity* which has the sign $-$. These terms are apt to convey wrong ideas; but the student should endeavour to retain the notions of *additive* and *subtractive*, and annex them to the names *positive* and *negative*.

Positive and negative quantities.

By generalizing the preceding results, it will be easy to see, that if several quantities be united by different signs, the value of the whole combination will necessarily remain the same in whatever order they may be written, provided that the same signs are always prefixed to the same letters. Thus the following combinations,

$$
\begin{gathered}
a - b + c - d + e - f \\
a + c - b - d + e - f \\
a + c - d - b + e - f \\
a + e - d + c - b - f \\
a + c - d + e - f - b \\
- b + a - d + s - f + e \\
- b - d - f + a + c + e \\
\text{\&c. \&c.}
\end{gathered}
$$

all express the same result. In effect, in all these cases the same operations are performed with the same quantities, but they are performed in different orders, and this difference of orders produces no effect on the final result.

If several quantities be successively subtracted from the same quantity, the remainder is the same as if their sum were at once subtracted from it. Hence we perceive that the combinations

Notation.

$$
\begin{gathered}
a + s + e - b - d - f \\
a + c + e - (b + d + f)
\end{gathered}
$$

are equivalent. Now if d and f are each equal to b, we shall have the expression equivalent to

$$a + c + e - 3b.$$

Hence it appears, that if several negative quantities be equal, they may be replaced by a single letter with a coefficient, as explained in (8) with respect to positive quantities.

It should also be observed, that if several quantities enclosed in a parenthesis, or under a vinculum, be positive, and that the negative sign be prefixed to the parenthesis, the parenthesis may be removed by making all the quantities negative. This is evident from the preceding example.

(10.) 3. *Multiplication.* When two numbers are multiplied together, the process is represented by the sign $\times$ placed between them, and the whole combination represents their product. Thus 5×7 represents the product of 5 and 7; $a \times b$ represents the product of a and b. But when letters are used, which is generally the case, the product is signified by a point placed between them thus, $a \, . \, b$, or more usually by writing the letters like those of one word, thus ab. This notation could not be used with particular numbers, because there would then be no distinction between the notation for expressing 7 times 5, and the number seventy-five. Both would be written 75.

Multiplication.

The terms *multiplicand* and *multiplier* as used in Arithmetic are preserved in Algebra. There is, however, no difference between the relations which these numbers bear to the product, and it is better to call them by the common name *factors*. In other words, the product ab will be the same, whether a be multiplied by b or b by a, and it is indifferent whether it be written ab or ba. In fact, the product has a relation to its factors, which is called a *symmetrical relation*. It is such, that if the values and names of the factors be interchanged, the product remains unaltered.

Factors.

It may happen, that three or more numbers are multiplied continually into one another. In this case, the process, if the factors be particular numbers, is expressed by the interposition of the sign $\times$ between every successive pair of factors; and if the factors be letters, the product is expressed by writing them as in one word. Thus, $7 \times 5 \times 3$ signifies the product of 7 and 5 multiplied by 3, or the continued product of 7, 5 and 3, or 105. Also, $abcd$ expresses the continued product of the numbers, a, b, s, and d.

If the several factors of a product be equal, it is called a *power*, and said to be the *second*, *third*, &c. power, according to the number of equal factors it contains. Thus, aa is the *second power* of a, aaa the *third power* of a, $aaaa$ the *fourth power*, &c.

Powers.

This, however, is not the way in which powers are usually expressed. The number of times the same letter occurs as a factor, is expressed by placing the particular number above the letter, thus a^2, a^3, a^4, &c., which expresses aa, aaa, $aaaa$, &c.; and if a occurred m times as a factor, the power would be expressed a^m.

The second power is usually called the *square*, and the third power the *cube*. For the reasons of these denominations, see GEOMETRY, pp. 330, 352, 353, also the Definitions, pp. 314, 350.

Algebra. Exponent. The number which thus denotes the number of equal factors in the power is called the *exponent*, and sometimes the *index* of the power.

If it be necessary to express 10 times the continued product of the 5th power of a, the 4th power of b, the 3d power of c, the 2nd power of d and e, it is done by this very concise notation $10a^5b^4c^3d^2e$.

Division. (11.) 4. *Division*. When one number is to be divided by another, the process is signified by placing the dividend above a line, and the divisor below it. If a be the dividend and b the divisor, the quote is expressed by $\frac{a}{b}$. Division is also sometimes expressed by placing the sign : or $\div$ between the dividend and the divisor, thus $a : b$, or $a \div b$, either of which signify the quote of a divided by b.

Monome. Polynome. (12.) A *simple* quantity is one in which the letters of which it is composed are not connected by addition or subtraction, or by the signs $+$ or $-$. Thus, all quantities expressed by a single letter are necessarily simple. The quantities $a\,b$, $\frac{a}{b}$, &c. are simple, but $a + b$, $a - b$, &c. are compound. Simple quantities are called *monomes*, and sometimes *terms*. Compound quantities, consisting of two parts, are called *binomes*, and all others *polynomes*.

Like monomes (13.) Simple quantities are said to be *like* when they are composed of the same letters combined in the same manner. *Like* quantities may, therefore, differ both in their signs and coefficients. The quantities $+ 3\,a$ and $- 5\,a$ are *like*; also $+ 3\,a\,b$ and $- 10\,a\,b$. The quantities $+ \frac{7\,a}{b}$ and $- \frac{6\,a}{b}$ are *like*, but $+ 3\,a\,b$ and $+ \frac{3\,a}{b}$ are *unlike*, because although they are expressed by the same letters, those letters are not combined in the same manner.

(14.) The sign $=$ interposed between two quantities, whether simple or compound, expresses their equality. Thus,

$$a + b = c + d,$$

means that the sum of a and b is equal to the sum of c and d.

(15.) The sign $>$ means *greater than*, and $<$ *less than*. Thus, $a > b$ means that a is greater than b; and $a < b$, that a is less than b.

Dimensions. (16.) Each of the literal factors of a monome or term, is called a *dimension* of the term. The degree of a term is its number of dimensions. Thus $a\,b$ is of the second degree, $a\,b\,c$ of the third degree. But in estimating the dimensions of a quote, those of the divisor must be subtracted from those of the dividend. Thus $\frac{a\,b}{c}$ is of the first degree, because there are two dimensions in the dividend, and one in the divisor. Again, $\frac{a\,b\,c}{d}$ is of the second degree, &c. The reason of this will appear hereafter.

Homogeneous monomes. (17.) Monomes are said to be *homogeneous* when they are of the same degree, and a polynome is said to be homogeneous when all its terms are of the same degree.

It should be observed, that the numeral coefficient is not reckoned as a dimension.

Addition.

SECTION II.

Addition.

Addition defined. (18.) Several Algebraical quantities are said to be added together, when they are arranged in a series, and connected by their proper signs. In some cases it happens, that the operations of addition or subtraction, indicated by the connecting signs $+$ or $-$, may be actually performed, and two or more of the quantities may be thus incorporated, and the result so far simplified. According to what has been already observed, when the same quantities are to be thus *added* together algebraically, the result will be the same in whatever order the operations may be performed.

(19.) When the quantities to be added are *unlike*, that is, expressed by different letters, they do not admit of being incorporated by the operations indicated by the signs by which they are connected. In this case, algebraical addition consists merely in arranging them in a series, the proper sign being prefixed to each, and the aggregate is called their *algebraical sum*. When it is considered that the numbers represented by different letters may be referred to different units, the impossibility of incorporating them will be at once perceived. In the compound quantity $a + b - c$, a may represent *miles*, b *furlongs*, and c *perches*; in which case, were the *numbers* represented by a and b to be actually added, and that represented by c subtracted from the result, the number thus obtained would neither represent the miles, the furlongs, nor the perches, in the proposed distance.

(20.) It is otherwise, however, if the quantities to be added, or any of them, be *like*, (13.) In this case, they are necessarily referred to the same unit, and may always be incorporated by the actual arithmetical operations indicated by the signs which connected them. Thus, if the quantities to be added be $+ 2\,a$ and $+ 3\,a$, it is evident that the sum

$$+ 2\,a + 3\,a$$

is equal to $5\,a$.

(21.) Also, if the quantities to be added be $+ a$, $- 2\,b$, and $- 3\,b$, the result is

$$+ a - 2\,b - 3\,b,$$

that is, twice b is to be subtracted from a, and from the remainder $3\,b$ is to be subtracted. The result will clearly be the same, if in the first instance five times b were subtracted from a. Thus, if a be a foot and b be an inch, two inches are first subtracted, which leave ten inches, and again three inches are subtracted from the remaining ten, and the remainder is seven inches. Had five inches been subtracted *at once*, the remainder would have been the same. Hence we infer the following equality,

$$a - 2\,b - 3\,b = a - 5\,b.$$

So that $- 2\,b - 3\,b$ is equal to $- 5\,b$; hence *negative* quantities when *like* are incorporated by addition in the same manner as *positive* quantities.

(22.) If the quantities to be incorporated be *like*, but have different signs, the process is effected by arithmetical subtraction. Let the quantities be $+ 5\,a$ and $- 3\,a$. Being connected with their proper signs, the result is

$$+ 5 \qquad a,$$

which means the actual remainder obtained by subtracting three times a from five times a. This is evidently twice a, so that

$$+5a - 3a = +2a.$$

In this case, therefore, the coefficient of the negative quantity is subtracted from that of the positive quantity, and the remainder is the coefficient of the result.

In the example just given, the coefficient of the positive quantity was greater than that of the negative, and the process was sufficiently obvious. There is, however, somewhat more difficulty in the case in which the coefficient of the negative quantity is greater than that of the positive quantity, and, therefore, cannot be subtracted from it. Let the quantities to be added be $+a$, $+2b$ and $-5b$. The result is

$$+a+2b-5b.$$

By what has been proved in (21) $-5b$ is equivalent to $-2b-3b$, and, therefore,

$$+a+2b-5b = +a+2b-2b-3b.$$

But it is evident that $2b - 2b = 0$, or neutralize each other, and may be altogether omitted, and we infer the following equality,

$$+a+2b-5b = +a-3b,$$

and hence

$$+2b-5b = -3b.$$

Thus, *when the coefficient of the negative quantity is greater than that of the positive quantity, the latter must be subtracted from the former, and the remainder will be the coefficient of the result, the sign of which will be negative.*

Rules for addition.

By generalizing the results of the preceding observations, we shall obtain the following rules for algebraical addition.

Rule I.

To add like quantities with like signs.

Add their coefficients, and to the sum affix the common letter or letters, and prefix the common sign.

Rule II.

To add like quantities with unlike signs.

Add the coefficients of the positive quantities, and likewise those of the negative quantities, and subtract the lesser sum from the greater. To the remainder affix the common letter or letters, and prefix the sign of those quantities of which the sum of the coefficients is the greater.

Rule III.

To add unlike quantities.

Let them be arranged in a series in any order, and connected by their proper signs.

Rule IV.

To add mixed quantities, like and unlike.

Add the like quantities by the first and second rules, and the results may be added by the third rule.

Term addition used in an extended sense.

(23.) It will be observed, that the term *addition* in Algebra is used in a very extended sense, the process being as often arithmetical subtraction as arithmetical addition. Were it not for the difficulty and inconvenience arising from any change in the nomenclature of a science, it would be desirable that the algebraical operation called *addition* should be otherwise denominated. But the same reasons which suggest the expediency of this change would equally apply to subtraction, multiplication, division, and numerous other terms. It is therefore, perhaps on the whole, better to retain old terms in new and extended senses, than to invent new ones at the risk of obscurity to students, and to the manifest inconvenience of adepts in the science. Algebraical addition is nothing more than the incorporation of a number of simple quantities by the arithmetical processes of addition and subtraction indicated by their signs, as far as that incorporation is rendered possible by the nature of the quantities.

Examples.

$$\begin{array}{r} +x \\ +2x \\ +3x \\ +4x \\ \hline +10x \end{array} \qquad \begin{array}{r} +5a \\ +4a \\ +3a \\ +2a \\ \hline +14a \end{array} \qquad \begin{array}{r} -2\dfrac{a}{b} \\ -\dfrac{a}{b} \\ -4\dfrac{a}{b} \\ \hline -7\dfrac{a}{b} \end{array} \qquad \begin{array}{r} -10ab \\ -4ab \\ -3ab \\ -ab \\ \hline -18ab \end{array}$$

$$\begin{array}{r} +2ay \\ -5ay \\ +4ay \\ -7ay \\ \hline -6ay \end{array} \qquad \begin{array}{r} +7x \\ -4y \\ +x \\ -8y \\ \hline +8x-12y \end{array} \qquad \begin{array}{r} a \\ -2x \\ 4a \\ -5x \\ \hline 5a-7x \end{array} \qquad \begin{array}{r} 2ay \\ -2by \\ 5ay \\ -6by \\ \hline 7ay-8by \end{array}$$

(24.) It sometimes happens, when a compound quantity is to be added to a simple or another compound quantity, that the operation is not actually performed but only signified. In this case, the compound quantity to be added is enclosed in a parenthesis, or placed under a *vinculum*, and connected by the sign $+$, with the quantity to which it is to be added. Thus, if $a - b$ is to be added to $10a$, the result may be expressed thus, Vinculum

$$10a + (a - b)$$

$$\text{or, } 10a + \overline{a - b}.$$

In this case, $a - b$ is considered as a single quantity, and the sign $+$, which precedes the parenthesis, or the vinculum, does not belong to the first quantity a, but to the result of the process indicated by $a - b$. Therefore the above complex quantity might also be expressed thus, without any change in its meaning, or in its value, (9,)

$$10a + (-b + a)$$

$$\text{or, } 10a + \overline{-b + a}.$$

SECTION III.

Subtraction.

(25.) Subtraction, in the popular or arithmetical sense of the word, implies *diminution*. When any quantity is said to be *subtracted* from another, that other is supposed to be diminished by the quantity so

Algebra. subtracted or taken away from it. In Algebra, however, the term acquires in its signification an extension analogous to that already given to the term addition. To explain the meaning of subtraction in Algebra, we shall define it with reference to addition. By addition we solve the problem, "Given two quantities to find their algebraical sum." By subtraction, then, we solve the problem, "Given one of two quantities, and their algebraical sum, to find the other." Thus, subtraction may be conceived to be nothing more than *undoing*, or destroying, the effect of a previous addition.

Let A represent any algebraical quantity, whether simple or compound, from which it is proposed to subtract another simple or compound quantity, which we shall call B. The quantity A may here be conceived to be the *algebraical sum* of B, and some other quantity which it is proposed to discover. Let this other quantity, whether simple or compound, be called x, (7.) Thus, by our hypothesis, $A = x + B$.

As A was obtained by annexing (18) the quantities expressed by B to x with their proper signs, the effect of this process will be destroyed by annexing to A the quantities represented by B *with their signs changed.* This process gives $A - B = x + B - B$.

But as $B - B$ is equal nothing, we have $A - B = x$.

If B were originally negative, the process would become $A = x - B$.

$$A + B = x - B + B, \text{ but } - B + B = 0$$
$$\therefore^{*} A + B = x.$$

Rule. (26.) Hence we may infer the following *General Rule:*

"To subtract one algebraical quantity from another, change the signs of the subtrahend, or conceive them changed, and add the quantities by the rules of addition."

EXAMPLES.

From	$5a - 2b$	$a - 2b + 3$
Subtract	$2a + 5b$	$4a + 9b - 5$
	$3a - 7b$	$-3a - 11b + 8$
From	$5ab - 18$	$8a - 2b - 5$
Subtract	$-ab + 12$	$-a + 3b + 2$
	$6ab - 30$	$9a - 5b - 7$

Signs of compound quantities. (27.) By what has been proved in the last section, respecting the incorporation of algebraical quantities, by actually effecting the operations indicated by the signs which connect them, it easily appears, that the sign of every compound quantity may be inferred from the signs and values of its component parts.

If the simple component parts of a compound quantity be all positive, it is evident that the whole quantity is positive; for if all the parts could be reduced to the same denomination, and, therefore, rendered *like*, upon incorporating them the result would be positive, (19.)

The same reasoning proves, that if the signs of all the component parts be negative, the sign of the whole is negative.

If a compound quantity be composed partly of positive and partly of negative quantities, the sign of the whole will be the same with that of those quantities, positive or negative, which have the greater sum. If the sum of the positive parts exceed the sum of the negative parts, the whole is positive; and if the sum of the negative parts exceed the sum of the positive parts, the whole is negative. Multiplication.

Hence it will easily appear, that by changing the signs of all the component parts of any compound quantity, the sign of the whole is changed.

(28.) Hence it follows, that if the signs of the several quantities within a parenthesis, or under a vinculum, be changed, and at the same time the sign which is prefixed to the parenthesis be changed, no real change in the compound quantity is produced, because the two effects counteract or compensate each other. Thus, if the quantity $+ (a - b)$ be changed to $+ (- a + b)$ its sign is changed. But if, again, this latter be changed to $(\quad a + b)$, its sign being again changed, the quantity becomes what it was before, so that

$$+ (a - b) = - (- a + b).$$

Hence we are always at liberty to change the sign of a parenthesis, provided the signs of the quantities enclosed be also changed.

(29.) If a complex quantity enclosed in a parenthesis, be connected with other quantities by the sign $+$, the parenthesis may be removed, the signs of the quantities enclosed in it being preserved. Thus, $a + (b - c)$ is equal to $a + b - c$; and $a + (\quad b + c)$ is equal to $a - b + c$.

This follows from the rules of addition; for the meaning of $a + (b - c)$ is that the compound quantity $b - c$ is to be added to a, which is done by connecting them by their proper signs. *Addition*, Rule III. (22.)

(30.) But if a compound quantity enclosed in a parenthesis, be connected with other quantities by the sign $-$, in order to remove the parenthesis it will be necessary to change the signs of all the simple quantities within it. For the sign $-$, which precedes the parenthesis, indicates that the complex quantity included within it, is to be subtracted from those quantities with which it is connected. This subtraction, as has been already shown, is effected by changing the signs of the quantities within the parenthesis.

SECTION IV.

Multiplication.

Multiplication defined. (31.) MULTIPLICATION, in the original sense of the term, means the continual addition of the same quantity as many times as there are units in the integer, which is called the *multiplier*. This term, however, like *addition* and *subtraction*, has acquired an extended signification, and the sense in which it was first used is only a particular case of its present more universal application.

It is observed by some writers, that the multiplicand may be any quantity, but that the multiplier must always be an abstract number. This, however, is by no means true. In algebraical calculations, heterogeneous quantities, or rather the numbers representing them, are constantly combined by multiplication, and it often happens, that the factors of the same product have different significations, and are referred to units of different species. Thus, if b represent the base of a parallelopiped in Geometry, and a its altitude, the product $a\,b$ represents its volume. Here the factors

* $\therefore$ signifies, *therefore*

Algebra. are not only heterogeneous, but each of them different in species from the product.

(32.) The difficulty of conceiving the multiplication of heterogeneous quantities will disappear by considering, that the letters in Algebra are not the *immediate* representatives of quantities but of *numbers*, and these numbers express the quantities in reference to their specific units. Thus, in the example just given, b is the number of *superficial units* in the base of the solid, a the number of *linear* units in its altitude, and $a\,b$ the number of *solid* units in its volume. Under this view, there is no more difficulty in conceiving the base b multiplied by the altitude a, than if the altitude were an abstract number.

(33.) Multiplication, in the most general sense of the term, is an operation by which a *fourth proportional** is found to the unit, and two numbers which are called *factors*, and the fourth proportional so found is called their product, (10.)

Thus, if a and b be the factors, and $a\,b$ the product, we have $1 : a :: b : a\,b$.

Since the transposition of the means does not disturb the proportion, it follows that there is no essential distinction between the factors, nor any grounds for giving them different denominations, such as *multiplicand* and *multiplier*, (10.)

Rule of signs. When the factors of a product are considered as having signs, as being *positive* or *negative*, a question arises as to what sign the product should receive. The rule commonly received is this, "When the two factors have the same sign, the sign of the product will be $+$, whether the common sign of the factors be $+$ or $-$; and when the two factors have different signs, the sign of the product is always $-$." This rule is generally briefly expressed thus, "In multiplication like signs produce $+$, and unlike signs produce $-$."

To give a general and unobjectionable demonstration of this rule has occasioned some embarrassment with elementary analytical writers. Before we enter upon any investigation respecting it, let us recur to the meaning of positive and negative quantities.

(34.) A positive quantity is one to which the sign $+$ is prefixed, and a negative quantity is one to which the sign $-$ is prefixed. The signs *in general* imply a connection with other quantities, the one signifying a connection by addition, and the other by subtraction. In this way, therefore, we are to consider positive and negative quantities as actually connected with some other quantities by the arithmetical operations of addition or subtraction.

(35.) The meaning of *positive* and *negative quantities* being thus explained, the following seems to be the most unobjectionable of the proofs usually given for the rule of signs.

1. Let it be required to multiply the number $A + a$ by the number B, A and B signifying absolute arithmetical numbers independently of any signs. If A be multiplied by B, the product is A B. But a greater number than A, viz. $A + a$ is to be multiplied by B; and, therefore, the product must be greater than A B by the product a B. Hence the whole product is A B $+ a$ B.

Thus it follows, that if a positive algebraical quantity $(+ a)$ and an absolute number B be multiplied together, the product will be a positive algebraical quantity, $+ a$ B.

Multiplication.

2. Let it be required to multiply $A + a$ and $+ b$ together. By the last case, the product of A and $+ b$ is $+ A\,b$. Now if $+ b$ be multiplied by a greater number than A, the product must be proportionally greater. Therefore the product of $A + a$ and $+ b$ must be greater than the product of A and $+ b$ by the product of $+ a$ and $+ b$. Hence the product of $A + a$ and $+ b$ is $A\,b + a\,b$. Hence the product of $+ a$ and $+ b$ is $+ a\,b$.

Hence, when the signs of both factors are $+$, the sign of the product is $+$.

3. Let it be required to multiply $A - a$ and $+ b$ together. By the first case, the product of A and $+ b$ is $+ A\,b$; but this is too great by the product of a and b. Hence the true product is $A\,b - a\,b$. Hence the product of $- a$ and $+ b$ is $- a\,b$. When the signs of the factors are unlike, the sign of the product is, therefore, $-$.

4. Let it be required to multiply $A - a$ by $B - b$. The product of $A - a$ and B is $A\,B - a\,B$. But this is too great by the product of b and $A - a$, or $A\,b - a\,b$. To obtain the true product it will, therefore, be necessary to subtract $A\,b - a\,b$ from $A\,B - a\,B$, the result of which is $A\,B - a\,B - A\,b + a\,b$, from which it follows that the multiplication of $- a$ and $- b$ gives the product $+ a\,b$. From this and the second case we infer that *like signs produce* $+$.

From the principles thus established, the following consequences may be deduced without difficulty.

(36.) The sign of the continued product of several factors is determined by the number of the negative factors. If there be an even number of negative factors or none, the product is positive.

(37.) If there be an odd number of negative factors, the product will be negative.

(38.) It is evident, that the same reasoning will apply if there be no positive factor; and. therefore, if all the factors be negative the product is positive or negative, according as the number of factors is even or odd.

(39.) If the signs of all the factors of a product be changed, the sign of the product will not be changed if the number of factors be even.

(40.) If the number of factors in the product be odd, by changing the signs of all the factors the sign of the product is changed.

(41.) When the factors of a product have numeral coefficients, these should evidently be multiplied together, and their product taken as the coefficient of the sought product. Thus, the product of $2\,a$ and $3\,b$ is $6\,a\,b$.

Multiplication of complex quantities. (42.) In all the preceding observations, the factors are supposed to be single quantities. We shall now consider the case in which one of the factors is a compound algebraical quantity. In this case, the product is found by multiplying the simple factor by each term of the complex factor, according to the rules already established, and adding the results. The sum thus obtained will be the true product. This may readily be proved by the definition of multiplication already given, and the properties of proportions established in the Treatise on GEOMETRY in this work. But we shall not here enter into any detail of the demonstration, the principle being sufficiently evident.

(43.) If both factors be complex quantities, the

* For the nature and properties of *ratios and proportions*, the reader is referred to the Article GEOMETRY, p. 319

Algebra. product is obtained by multiplying each term of the one factor by each term of the other, and adding together all the products by the ordinary rules of algebraical addition. The validity of this process may be easily inferred from the last.

SECTION V.

Division.

(44.) DIVISION bears the same relation to multiplication that subtraction bears to addition. It is the *undoing* of what has been done, or is conceived to have been done, by multiplication. As multiplication, therefore, is the compounding two factors together so as to form a product, so division, on the other hand, consists in the decomposition of a product into its factors. In multiplication and division there are three quantities concerned, the *two factors* and the *product*. Now any two of these three being given, the remaining one may be found. When the *two factors* are given to find the product, the process is called *multiplication*; and when the product and one of the factors are given to find the other factor, the process is called *division*. The product is in this case called the *dividend*, the given factor the *divisor*, and the required factor the *quote*.

Rule of the signs. (45.) The rule for deducing the sign of the quote from those of the dividend and divisor, is the same as the rule in multiplication, and may be derived from it. Let d be the divisor, D the dividend, and q the quote. Since the product of d and q is equal to D, it follows that when d and q have like signs D is $+$, and when d and q have unlike signs D is $-$. Hence it may easily be inferred, that in division, as in multiplication, "like signs give $+$, and unlike signs give $-$."

(46.) If the divisor and dividend be monomes, the division will not be capable of being executed exactly, unless all the factors of the divisor, both numeral and literal, be also factors of the dividend; for the dividend is considered as the product of the divisor and the number sought, or the quote. Let $a\,b$ be the dividend, and a the divisor, it is evident that in this case the quote is b.

Let the divisor be $5\,a^3$, and the dividend $15\,a^4\,b$. In the first instance the quote is expressed by (12) $\dfrac{15\,a^4\,b}{5\,a^3}$.

This must be such a quantity as multiplied by $5\,a^3$ will produce $15\,a^4\,b$. Let it be x, so that $x \times 5\,a^3 = 15\,a^4\,b$. But $15\,a^4\,b = 3 \times 5 \times a^3 \times a\,b$, $\therefore$ $x \times 5\,a^3 = 3\,a\,b \times 5\,a^3$.

Hence it is evident that $x = 3\,a\,b$.

In the same manner, if the dividend be $22\,a^5\,b^3\,c$, and the divisor $11\,a^3\,b$, the quote will be $2\,a^2\,b^2\,c$.

Division of monomes. By generalizing these results we shall obtain the following rule for the division of monomes:

"The sign of the quote being determined as in multiplication: 1. let the coefficient of the dividend be divided by the coefficient of the divisor, and let the result be taken as the coefficient of the quote; 2. let each of the literal factors, which are common to the dividend and divisor, be written after this coefficient with an exponent equal to the excess of the exponent in the dividend above the exponent of the same letter in the divisor. 3. Let each of the literal factors which occur in the dividend, but not in the divisor, be then written as factors of the quote." Division.

The following examples will illustrate this process:

$$\frac{75\,a^5\,b^4\,c^2\,d}{25\,a^3\,b\,c} = 3\,a^2\,b^3\,c\,d \quad \frac{93\,a^7\,x^3\,y}{31\,a\,x} = 3\,a^6\,x^2\,y.$$

(47.) There are certain circumstances under which the above process cannot be executed: 1. The coefficients may not be divisible one by another. 2. The exponent of some letter in the dividend may be less than that of the same letter in the divisor. 3. There may be literal factors in the divisor which are not contained in the dividend at all. In these cases, the division cannot be effected, and the quote must be expressed by the notation described in (12.)

Let the dividend be $16\,a^3\,b\,c$, and the divisor $10\,a^2\,b^2\,c^2$. In this case 16 is not divisible by 10, neither can the exponents of b and c in the divisor be subtracted from those of the same letters in the dividend. In this case, however, the expression $\dfrac{16\,a^3\,b\,c}{10a^2b^2c^2}$ for the quote may be simplified by removing the common factors 2, a^2, b, and c. The quote thus becomes $\dfrac{8\,a}{5\,b\,c}$.

When complete or exact division cannot be effected, the fractional expression for the quote may, therefore, be simplified by the following rule:

"1. Divide the coefficients of the dividend and divisor by their common factors. 2. If the same letter occur in both dividend and divisor with different exponents, subtract the lesser exponent from the greater, and in place of the greater exponent place the remainder, omitting the letter with the lesser exponent altogether. 3. Let the letters which are common, and have equal exponents, be altogether omitted. 4. Let the letters not common retain their places."

(48.) It is evident that the value of a quote depends on the *ratio* of the dividend to the divisor; and however they may be changed, provided their ratio be preserved, the quote will retain the same value. Since $q \times d = \mathrm{D}$, $\because$ by (33)

$$d : \mathrm{D} :: 1 : q.$$

The value of q must remain unchanged so long as the unit bears to it the same ratio, that is, so long as the divisor bears to the dividend the same ratio.

Hence it follows, that any common factors may always be removed from the divisor and dividend without affecting the quote.

(49.) If the rule for the subtraction of the exponents of the same letter in the divisor and dividend, when these exponents are unequal, be applied to the case in which they are equal, the result will assume a peculiar form. Let the dividend be $a\,b^2$ and the divisor b^2. The quote is $\dfrac{a\,b^2}{b^2}$; applying to this the rule for the case in which the exponent in the dividend is greater than in the divisor, the result is $a\,b^{2-2} = a\,b^0$. But the quote being evidently a, we have $a = a\,b^0$, $\because$ $b^0 = 1$. It is usual, therefore, to say that "any quantity having o for its exponent is $= 1$." This, however, is to be considered as a matter *purely conventional*, the symbol a^0 having no other meaning than *an expression of the result of the division of two powers of the same letter, having the same exponent, the process being conducted by the rule established for the case in which the* Value of a^0.

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exponent of the dividend is greater than that of the divisor.

The use of this notation is to preserve in the result the marks of the process by which it was obtained.

(50.) If the dividend be the continued product of several factors, it will be divided by any number by dividing any one of its factors by that number. Thus, $8 \times 9 = 72$, and $\frac{8}{2} \times 9 = 36 = \frac{72}{2}$; and the same, of course, applies to algebraical quantities.

Division of polynomes.

(51.) Hitherto we have supposed the divisor and dividend to be *monomes*. Let us now suppose the letter D a polynome. In this case, the quote q must be such a polynome as multiplied by the divisor d, (supposed a monome,) will give a product equal to D.

By (42) it appears that q is multiplied by d, by multiplying each of its terms by d, and the product will therefore be a polynome, whose terms are the products of d, and the several terms of q. But this polynome must be identical with D, and, therefore, each of the terms of D must be the product of d and the several terms of q. Hence the several terms of q must be the quote found by dividing the terms of D severally by d.

It follows from this, that in order that a polynome should be exactly divisible by a monome, each of the terms of the polynome must be divisible exactly by the monome; otherwise the quote will include terms of a fractional form.

Let $2\,a\,x^2$ be the divisor, and let the dividend be

$$10\,a^7\,x^5 + 20\,a^5\,x^4 - 12\,a^2\,x^3 + 6\,a^5\,x^2 - 2\,a\,x^2;$$

when the several terms have been divided by $2\,a\,x^2$ by the rules established for monomes, the quote will be

$$5\,a^6\,x^3 + 10\,a^4\,x^2 - 6\,a\,x + 3\,a^4 - 1$$

(52.) If the divisor be a polynome and the dividend a monome, the exact division is impossible, and the quote can only be expressed in the fractional form. For the quote cannot be a monome, since the product of a monome quote and a polynome divisor would give a polynome dividend, contrary to hypothesis. Neither can the quote be a polynome, since the product of the quote and divisor, both polynomes, could not give a monome dividend. In this case, therefore, the quote must be expressed as in (47,) and may be simplified if there be any factor of the dividend which is common to all the terms of the divisor. This factor may be removed, since both dividend and divisor may be divided by the same quantity without affecting the value of the quote.

The case in which the divisor is a monome, and the dividend a polynome, admits of a similar simplification when all the terms of the dividend contain a factor common with the divisor.

(53.) We shall now consider the case in which both the dividend D and the divisor d are polynomes. Each of the terms of the dividend D being the product of a term of the divisor d, and one of the quote q, it follows that if we find a term of the dividend which is divisible by a term of the divisor, this quote will be a term of the quote q. Having thus found any one term (A) of the quote q, this term being multiplied by the whole divisor d, gives a product A d, which is to be considered as that part of the dividend which has been divided by d. This being subtracted from the whole dividend D, the remainder is all that is now to be divided by d. As before a term of this remainder is selected, which is exactly divisible by some term of the divisor, and the quote being found, it is inserted with its proper sign as another term of the quote q; and so the process is continued until a term of the quote q is found, which, multiplied into the divisor d, will be equal to all that has remained of the dividend. In this case, the division is complete. But if in any of the remainders there is no term which is exactly divisible by a term of the divisor, the division cannot be effected, and we conclude that there is no polynome q, which, multiplied by d, will exactly give the product D.

Division.

In multiplying two polynomes together, it frequently happens that the partial products of the several terms of the factors destroy or modify each other, by those which are similar being incorporated by addition or subtraction. It may, therefore, happen that some of the terms of the product of two polynomes are the sum or difference of the product of two or more terms of the factors, and not the product itself of these terms. In the selection, therefore, of a term of the dividend, which is to be considered as the product of a term of the divisor, and one of the quote, it is necessary that this term should be one which cannot have proceeded from the combination of two partial products of d and q, by the addition or subtraction of similar terms; for if it were so, it is plain that we should not be justified in concluding, that by dividing it by the term of the dividend we should obtain a term of the quote.

When the same letter occurs in two polynomes, powers of that letter must occur in their product, and one at least of these powers must have a higher exponent in the product than in either of the factors. The term containing the highest power is that which proceeds from multiplying together the two terms of the factors which contain the same letter with the highest exponents. The exponent of the corresponding term of the product will be their sum, and no other term of the product can contain the same letter with so high an exponent. This term, therefore, can suffer no modification by addition or subtraction with any other term, and must always be actually the immediate product of the two terms of the factors which contain the highest exponent of the same letter.

Hence it follows, that if there be a letter which occurs with exponents in the divisor and dividend, its highest exponent in the latter being greater than its highest exponent in the former, that term of the dividend which contains this letter with the highest exponent, must be the immediate product of that term of the divisor which contains the same letter, with the highest exponent and a term of the quote, which term is therefore immediately found by dividing the one by the other. Then, by the means already explained, a new dividend is obtained; and in this, likewise, a term is to be found, in which the exponent of some letter is higher than in the other terms, and so on. It is generally convenient to select the highest power of the *same letter* in each partial dividend, as that which is to determine the term of the quote; this, however, is not absolutely necessary.

In writing down the dividend and divisor preparatory to division, it is not *necessary* to place the terms of either in any one particular order rather than another. But it is *convenient* to place first in each the two terms by the division of which the first term of the quote is to be determined. If, after the first subtraction the

 highest power of the same letter in the next dividend be selected, it will be also convenient that it should stand first in the remainder, and, therefore, that it should be placed as the second term in the original dividend. By continuing this reasoning we shall find, that the terms of the dividend should be arranged according to the *descending powers* of that letter, whose highest power is selected for determining the first term of the quote. By such an arrangement, the first term of each remainder will be that which contains the highest power of the same letter, and will, therefore, be that which is proper to determine a term of the quote. Since the first term of the quote is to be multiplied by the divisor, and the result to be placed under the dividend, preparatory to subtraction, it is evidently convenient that the terms of the divisor should also be arranged according to the descending powers of the same letter; for, in that case, the corresponding powers of the terms of the subtrahend will come under those of the dividend preparatory to the subtraction.

Rule. Hence we obtain the following rule for the division of polynomes:

"Arrange the terms of the divisor and dividend according to the descending powers of any letter which is common to them, placing in each the term containing this letter, with the highest exponent first, and each succeeding term having that letter with a higher exponent than that which follows it. Let the first term of the dividend be then divided by the first term of the divisor, and the result with its proper sign will be the first term of the quote. Let this term be then multiplied by the whole divisor, and the product subtracted from the dividend. Let the first term of the remainder be divided by the first term of the divisor, and the result, with its proper sign, will be the second term of the quote. Let this, in like manner, be multiplied by the whole divisor, and the product subtracted from the first remainder. The second remainder then, constituting a new dividend, must be treated as the former remainder, and the process must be continued in this way until the multiplication of some term of the quote gives a product exactly equal to the last remainder, in which case the quote is complete, and the division effected."

(54.) It appears from what has been already proved, that if the term of the dividend which contains the highest power of a letter common to the dividend and divisor, be not exactly divisible by the term of the divisor containing the highest power of the same letter, the exact division is impossible; for, in this case, the dividend cannot be the product of the divisor and any polynome.

(55.) It is plain, that if the divisor contain any letter which is not found in the dividend, the division is impossible. For a product must contain every letter which enters either of its factors, and the division is never possible, except when the divisor is a factor of the dividend. On the other hand, the dividend may contain a letter or letters which do not appear in the divisor, because a product may contain letters which do not appear in one of its factors, since they may be letters of the other factor. If the dividend D contain any letter a not contained in the divisor d, the division may be effected by arranging the dividend by the powers of the letter a. Since the divisor d, by hypothesis, does not contain the letter a, and yet the product of the quote q and the divisor d is identical with the dividend, it follows, that the quote must consist of a series of terms affected by the same powers of a as appear in the dividend. Let $A a^m$ be any term of the dividend, and $B a^m$ the corresponding term of the quote. It follows, that $d B a^m = A a^m$, or $d B = A$, or $B = \frac{A}{d}$; and since the same observation may be applied to each of the terms, we deduce the following rule for division, when the dividend contains any letter which does not appear in the divisor: "Let the dividend be arranged by the powers of this letter, and let each of the multipliers of the powers be divided by the divisor. The several quotes thus found, will be the multipliers of the corresponding powers of the same letter in the quote."

SECTION VI.

Of Simple Powers and Roots.

(56.) As powers of the same quantity would be multiplied by writing them down as one word, it is evident that the number of equal factors in their product would be the *sum* of the numbers of equal factors in each of the powers so multiplied. But as the exponents express the number of those factors, we may immediately infer, that "when powers of the same quantity are multiplied together, the sum of their exponents is the exponent of the product." Thus Multiplication of powers.

$$a^2 \times a^3 = a^5 \qquad a^3 a^4 = a^7$$
$$a^7 \times a^9 = a^{16} \qquad a^5 a^4 = a^9$$

and in general $\quad a^m a^n = a^{m+n}$,

m and n being any positive integers.

(57.) From (46) it appears, that if a power of any quantity be divided by a power of the same quantity having a lesser exponent, the quote will be found by subtracting the exponent of the divisor from that of the dividend; and it has been shown (49) how this rule has been conventionally extended to the case in which the dividend and divisor have equal exponents. It may also happen, that the exponent of the divisor is greater than that of the dividend; in which case, if the division were performed according to the rule established for the case in which the exponent of the dividend is greater than that of the divisor, the exponent of the quote would be negative. Thus we should have, for example, $\frac{a^3}{a^5} = a^{-2}$. But according to the established rules of division (46,) we should have $\frac{a^3}{a^5} = \frac{aaa}{aaaaa} = \frac{1}{aa} = \frac{1}{a^2}$. Nevertheless, in order to generalize, as far as possible, the processes in algebraical investigations, it is found expedient to extend the rule for the division of powers, established in (46,) to the cases in which the exponent of the divisor is equal to, or greater than, that of the dividend. What we have already observed in the case of equal exponents, should, however, be carefully attended to in the case of the exponent of the divisor being greater than that of the dividend. The negative exponent, which the quote acquires in this case, is to be understood only as indicating that the power which is affected by it has been obtained by applying the rule to a case to which it is not applicable, otherwise than by general consent. Division of powers. Negative exponents.

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By the quantities a^0, a^{-1}, a^{-2}, a^{-3}, &c., we are, therefore, to understand the quantities, 1, $\frac{1}{a}$, $\frac{1}{a^2}$, $\frac{1}{a^3}$, &c., and in general a^{-m} is only another way of expressing $\frac{1}{a^m}$, or a quantity with a negative exponent is

Reciprocals.

the *reciprocal* of the same quantity with the same positive exponent.

(One quantity is said to be the *reciprocal* of another when their product is equal to the unit.)

(58.) The student will find no difficulty in extending the rules for the multiplication and division of quantities with positive exponents to the case in which the exponents of the factors are one or both negative.

E. g. $a^{-m} \times a^{-n} = a^{-m-n}$, and $\frac{a^{-m}}{a^{-n}} = a^{-m+n}$.

Involution.

(59.) To find any required power of a quantity, or, as it is called, to *raise any quantity to a required power*, it is only necessary to form a product in which that quantity shall be repeated as a factor as often as there are units in the exponent of the power to which it is to be raised. Now if the quantity to be raised be a power of a simple quantity, as a^m, it is plain that the continual multiplication of this will give a product such as $a^{m+m+m\,\&c.}$, where m is contained in the exponent as often as there are units in the exponent of the power to which it is to be raised. Let this exponent be n; it is then evident, that the n^{th} power of a^m is a^{mn}. Thus, if it be required to find the third power of a^m, we have

$$a^m a^m a^m = a^{m+m+m} = a^{3m},$$

and this is true whether the exponent m be positive or negative.

The general rule, therefore, to raise any simple quantity to any required power, is to "Multiply the exponent of the quantity by the exponent of the power to which it is to be raised, and the product is the exponent of the power sought."

(60.) The terms *power* and *root* are correlatives; if a be the m^{th} power of b, b is called the m^{th} root of a, and *vice versâ*. The notation by which the root is expressed, is the mark $\sqrt{\ }$ called a radical, placed over the letter, with an exponent to the left indicating the order of the root. The quantity which is placed under the radical, is called its *suffix*. Thus $\sqrt[3]{a}$ means the third root of a, or that quantity of which the number a is the third power. When no exponent is expressed, the symbol means the square root, thus $\sqrt{a}$ is the square root of a, or the number whose square is a. The processes by which powers and roots are found, are called respectively, *Involution* and *Evolution*.

Evolution.

(61.) If the n^{th} root of a simple quantity, such as a^m, were required, it is evident that, if m were a multiple of n, such as rn, the quantity a^m or a^{rn} would be the n^{th} power of a^r, and, therefore, the n^{th} root would be found by dividing the exponent m or rn by the exponent n of the root required. If, however, m be not a multiple of n, the n^{th} root cannot be algebraically extracted. In this case, however, the same rule is extended analogically, and the root is signified by assuming the quota of the exponent m of the given quantity, by the exponent n of the required root, which is, therefore, expressed $a^{\frac{m}{n}}$. Thus the conventional notation derived from the extraction of the roots of powers, gives another method of expressing radical quantities; thus,

Of Simple Powers and Roots.

$\sqrt{a} = a^{\frac{1}{2}}$ $\sqrt[3]{a} = a^{\frac{1}{3}}$ $\sqrt[4]{a^3} = a^{\frac{3}{4}}$, &c.

Multiplication of fractional powers.

(62.) Fractional powers of the same quantity are multiplied by adding their exponents. That is,

$$a^{\frac{m}{n}} \cdot a^{\frac{m'}{n'}} = a^{\frac{m}{n}+\frac{m'}{n'}}\,*$$

To prove this, let $x = a^{\frac{m}{n}}$ $\quad x' = a^{\frac{m'}{n'}}$

Hence $\quad x^n = a^m \quad x'^{n'} = a^{m'}$

Let the one be raised to the n'^{th} power, and the other to the n^{th} power, and we have

$$x^{nn'} = a^{mn'} \qquad x'^{n'n} = a^{m'n}$$

These being multiplied, give

$$x^{nn'}\,x'^{nn'} = a^{mn'+m'n}$$

Taking the nn'^{th} root of these, we obtain

$$xx' = a^{\frac{mn'+m'n}{nn'}} = a^{\frac{m}{n}+\frac{m'}{n'}}$$

$$\therefore a^{\frac{m}{n}} \cdot a^{\frac{m'}{n'}} = a^{\frac{m}{n}+\frac{m'}{n'}}$$

The same demonstration will be applicable if $\frac{m}{n}$ and $\frac{m'}{n'}$, are one or both negative.

Division of fractional powers.

(63.) Since the product of two powers (whether fractional or negative, or both) is obtained by adding the exponents, it follows that the quote is obtained by subtracting the exponent of the divisor from that of the dividend. For the dividend being the product of the quote and divisor, its exponent must be the algebraical sum of the exponents of the quote and divisor, (62;) therefore the exponent of the quote must be the result of the subtraction of the exponent of the divisor from that of the dividend. Thus the rule for the division of powers, established in the case where the exponents are positive integers, is general.

Involution and evolution.

The rule for the multiplication of powers of the same quantity being generalized, the extension of that for their involution immediately follows. If $a^{\pm\frac{m}{n}}$ be continually multiplied into itself, until a product be found having a number of factors which we may call p, the exponent $\pm\frac{m}{n}$ will be added p times, and the new exponent will be $\pm p \cdot \frac{m}{n}$. Thus the p^{th} power of $a^{\pm\frac{m}{n}}$ is $a \pm p \cdot \frac{m}{n}$, that is, the power is obtained by multiplying the exponent of the root by that of the required power.

From the preceding rule is immediately derived the extension of the rule (61) for the evolution of powers of the same quantity.

(64.) We shall now take the most general possible case of the involution or evolution of powers. Let it be required to find the $\left(-\frac{r}{s}\right)^{th}$ power of $a^{\frac{m}{n}}$. The general rule applied to this case gives

$$\left(a^{\frac{m}{n}}\right)^{-\frac{r}{s}} = a^{\frac{m}{n} \times -\frac{r}{s}} = a^{-\frac{mr}{ns}}$$

* To avoid a multiplicity of different letters, and to give symmetry to the expressions, it is usual to express different quantities by the same letter, distinguishing it, however, by accents thus, m, m', m'', m''', &c.

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To prove this it is only necessary to retrace the symbols to their original signification. The student will find no difficulty in doing so.

The chief advantage which the extension of the properties established for positive integral exponents to exponents of all kinds is, that it saves the necessity of registering in the memory, and practising a different system of rules, and renders the results of algebraical investigations more simple and symmetrical. Besides this, it reduces all the operations on radicals to operations on fractions, with which every student is familiar.

Surds.

(65.) The rules established for positive integral exponents being extended to those which are fractional and negative, it may be asked how far the same rules may be applicable to the cases where the exponents are numbers *incommensurable with the unit*. Such numbers are called *irrational numbers* or *surds*. As, for example, the m^{th} root of an integer which is not itself the m^{th} power of an integer. Thus $\sqrt{3}$, $\sqrt[3]{5}$, &c. are surds. We shall see hereafter, that although no integer or fraction can exactly express the values of such numbers, yet we can always find a fractional number which differs from the value of any given irrational number by a quantity less than any assigned number; and in applying the rules already established for exponents to an irrational exponent, it is to be understood that they are applied to the fractional number, which represents the approximate value of the irrational exponent. In fact, in numerical applications we can form no distinct idea of an irrational number, otherwise than that which we may form of the exact fractional number which represents its value approximately.

With this limitation, therefore, and in this sense, the properties just established may be extended to all exponents, whether rational or irrational, positive or negative.

Signs of powers and roots.

(66.) We have not yet pointed out how the signs of powers and roots depend on the signs of the quantities themselves. If the quantity which is involved be positive, all its powers must be positive, for a product is positive if all its factors be so. But in general (36) a product is positive when it has either an even number of negative factors or none; and negative whenever it has an odd number of negative factors. Hence it follows, that *all* powers of a positive quantity are positive, and also all *even* powers of a negative quantity are positive. Thus the only powers which can be negative are the odd powers of negative quantities. The successive powers of $+a$ are $+a^2$, $+a^3$, $+a^4$, &c. and those of $-a$, are $-a$, $+a^2$, $-a^3$, $+a^4$, &c. being alternately negative and positive.

From this it appears, that the odd powers of $+a$ and $-a$ differ from each other in sign, each having the sign of its root; but that the even powers are the same, being in both cases positive. Thus $+a^2$ is at the same time the square of $+a$ and of $-a$; and, in like manner, $+a^4$, $+a^6$, &c. are the fourth, sixth, &c. powers of $+a$, and also of $-a$.

It follows, therefore, that $+a$ and $-a$ have equal claims to be considered as the square root of $+a^2$, the fourth root of $+a^4$, and the same of any positive even power of a. Thus it appears, that every positive quantity must have at least two even roots, which differ only in their signs. Hence it is usual to prefix the double sign $\pm$ to an even radical, thus $\pm\sqrt{a}$, indicating thereby that a has two square roots, one with the sign $+$, and the other with the sign $-$, but otherwise the same.

Of Simple Powers and Roots.

Impossible or imaginary quantities.

(67.) Since no quantity whether positive or negative raised to an even power can be negative, it follows that no negative quantity can have an even root. Nevertheless it frequently happens in algebraical investigations, that negative quantities are found to occur under even radicals, and although such a result is always the consequence of a falsehood or contradiction in the reasoning on which it is founded, yet it is found expedient to preserve the mode of expressing it. The symbol $\sqrt{-A}$, or $\sqrt[m]{-A}$, (where m is even) therefore expresses the result of an operation which cannot be performed, yet such expressions are submitted to the same rules, and subject to the same operations as similar radicals affecting positive quantities. They are said, though improperly, to express *impossible or imaginary quantities*. In effect, they do not represent any quantities whatever, and are merely indicative of an absurdity in the process from which they have been derived.

(68.) With reference to such symbols, algebraical quantities are said to be *real* or *imaginary*. An *imaginary* quantity is the even root of a negative quantity, and every other quantity is said to be *real*.

We shall reserve the further consideration of Imaginary expressions for a subsequent part of this section.

(69.) To obtain any proposed power of a product, it is necessary to raise each of its factors to that power. Thus if the third power of $2a^3b^2$ be required, we have $(2a^3b^2)^3 = 2a^3b^2 \times 2a^3b^2 \times 2a^3b^2$. But as the order of the factors is indifferent, this may be expressed $(2a^3b^2)^3 = 2 . 2 . 2 . a^3 . a^3 . a^3 . b^2 . b^2 . b^2$, or $(2a^3b^2)^3 = 2^3a^9b^6$. By generalizing this result, we obtain the following rule, "To raise a monome to any given power, raise the numeral coefficient to that power, and multiply each of the exponents by the exponent of the power to which it is to be raised."

(70.) Hence we may immediately infer the following rule for extracting any proposed root of a monome: "Extract the root of its coefficient, and divide each exponent by the exponent of the required root."

In order, therefore, that a monome should be a complete power of the m^{th} order, it is necessary that its coefficient should be a complete power of that order, and that the exponent of each of its literal factors should be a multiple of the exponent of the root required. Otherwise the result will be an *algebraical surd*.

Simplification of surds.

(71.) In cases in which the roots of monomes do not admit of absolute extraction, and are, therefore, algebraical surds, they are nevertheless frequently capable of considerable simplification. Some important reductions on such quantities are founded on the following theorem: "The m^{th} power of a product is equal to a product of the m^{th} powers of its factors, and the m^{th} root of a product is equal to the product of the m^{th} roots of its factors, the exponent of the m being any number whatever, integral or fractional, positive or negative, rational or irrational."

The first part of this theorem when m is a positive integer has been already proved. Let it then be a fraction $\frac{m}{n}$. The first part of the theorem announced

algebraically is $(a\,b\,c\,d)^{\frac{m}{n}} = a^{\frac{m}{n}}.\,b^{\frac{m}{n}}.\,c^{\frac{m}{n}}.\,d^{\frac{m}{n}}$. Let

$$a'^n = a \quad b'^n = b \quad c'^n = c \quad d'^n = d$$

$$\therefore a' = a^{\frac{1}{n}} \quad b' = b^{\frac{1}{n}} \quad c' = c^{\frac{1}{n}} \quad d' = d^{\frac{1}{n}}$$

and $a'^m = a^{\frac{m}{n}} \quad b'^m = b^{\frac{m}{n}} \quad c'^m = c^{\frac{m}{n}} \quad d'^m = d^{\frac{m}{n}}$

Hence we infer, that

$$a'^m b'^m c'^m d'^m = a^{\frac{m}{n}}.\,b^{\frac{m}{n}}.\,c^{\frac{m}{n}}.\,d^{\frac{m}{n}}$$

or $$(a' b' c' d')^m = a^{\frac{m}{n}} b^{\frac{m}{n}} c^{\frac{m}{n}} d^{\frac{m}{n}}.$$

But also $$a'^n b'^n c'^n d'^n = a\,b\,c\,d,$$

$$\therefore (a' b' c' d')^n = a\,b\,c\,d$$

$$\therefore a' b' c' d' = (a\,b\,c\,d)^{\frac{1}{n}}$$

$$\therefore (a' b' c' d')^m = (a\,b\,c\,d)^{\frac{m}{n}}$$

$$\therefore (a\,b\,c\,d)^{\frac{m}{n}} = a^{\frac{m}{n}}.\,b^{\frac{m}{n}}.\,c^{\frac{m}{n}}.\,d^{\frac{m}{n}}$$

In this reasoning we assume the first part of the theorem to be true, when the exponent is an integer, whether positive or negative; but these cases may be at once inferred from what has been already proved. As roots are only fractional powers otherwise expressed, the preceding demonstration establishes the second part of the theorem.

(72.) Since the divisor and dividend of a quote may be multiplied or divided by the same number, (48.) without changing the value of the quote, it follows that the exponent of a radical, and the exponent of its suffix, may be multiplied by the same quantity without changing its value. For

$$\sqrt[m]{a^n} = a^{\frac{n}{m}}$$

also $$\sqrt[pm]{a^{pn}} = a^{\frac{pn}{pm}} = a^{\frac{n}{m}}$$

$$\therefore \sqrt[m]{a^n} = \sqrt[pm]{a^{pn}}$$

Reduction of radicals.

(73.) By this principle, two radicals may be reduced to the same exponent. For this purpose, all that is necessary is to multiply the exponent of each radical, and that of its suffix, by the exponent of the other radical, and the product of the exponent will then be the common exponent. Let the two radicals be

$$\sqrt[m]{a^n} \text{ and } \sqrt[m']{a'^{n'}}$$

then $$\sqrt[m]{a^n} = \sqrt[mm']{a^{m'n}}$$

$$\sqrt[m']{a'^{n'}} = \sqrt[mm']{a'^{mn'}}$$

The radicals have thus mm' as their common exponent. It will be easily perceived that this is in effect reducing the fractional exponents of the equivalent fractional powers to a common denominator. (Section IX.)

The rule may be extended to several radicals. "Multiply the exponent of each radical, as well as that of its suffix, by the product of the exponents of all the other radicals; and the product of all the exponents will be the common exponent." It sometimes happens, that the radicals can be reduced to a lower common exponent than the product of their exponents. But as the whole doctrine of the reduction of radicals may be resolved to the reduction of the equivalent fractional powers, we refer the student to Section IX., the results of which will be immediately applicable to the fractional exponents, whose denominators are the exponents of the radicals, and whose numerators are the exponents of their suffixes. The lowest common exponent will then be the least common multiple of the exponents of the several radicals.

Addition of radicals.

(74.) The principles which have just been established form the foundation of the rules for effecting on radicals the elementary operations of *addition*, *subtraction*, &c.

Radicals are said to be *similar* when they have the same exponent and the same suffixed quantity.

Similar radicals are added or subtracted by adding or subtracting their coefficients, and prefixing the sum or difference as the coefficient of the result. Thus

$$3\sqrt[3]{b} + 2\sqrt[3]{b} = 5\sqrt[3]{b}$$

$$3\sqrt[3]{b} - 2\sqrt[3]{b} = \sqrt[3]{b}$$

$$3\,m\sqrt{a} - 4\,n\sqrt{a} = (3\,m - 4\,n)\sqrt{a}.$$

Radicals which are not similar, may sometimes become so by reduction. Let them first be reduced to a common exponent. If then the suffixes have a common factor, and the factors not common be complete powers of the order expressed by the common exponent, the radicals will be reduced to a common suffix, (the common factor,) by bringing the roots of the factors not common outside the radical. Thus, the radicals $\sqrt{48\,a\,b^2}$ and $\sqrt{75\,a}$ are reduced thus, $\sqrt{48\,a\,b^2} = \sqrt{3.16\,.\,a\,b^2}$: $\sqrt{75\,a} = \sqrt{3.25\,.\,a}$. $3\,a$ is a common factor. The factors not common are $16\,b^2$ and 25, which are the squares of $4\,b$ and 5. Hence $\sqrt{48\,a\,b^2} = 4\,b\sqrt{3\,a}$, $\sqrt{75\,a} = 5\sqrt{3\,a}$. Hence $\sqrt{48\,a\,b^2} \pm \sqrt{75\,a} = (4\,b \pm 5)\sqrt{3\,a}$. Again, $\sqrt[3]{8\,a^3\,b + 16\,a^4} = \sqrt[3]{8\,a^3\,(b + 2\,a)} = 2\,a\sqrt[3]{b + 2\,a}$.

$$\sqrt[3]{b^4 + 2\,a\,b^3} = \sqrt[3]{b^3\,(b + 2\,a)} = b\sqrt[3]{b + 2\,a}$$

$$\therefore \sqrt[3]{8\,a^3\,b + 16\,a^4} \pm \sqrt[3]{b^4 + 2\,a\,b^3} = (2\,a \pm b)\sqrt[3]{b + 2\,a}.$$

If the radicals be not similar, they do not admit of being incorporated, and their addition or subtraction can only be expressed by the usual signs, + or −, between them.

Multiplication of radicals.

(75.) To multiply radicals, reduce them to the same exponent, multiply their suffixes, and prefix the common exponent. Thus $\sqrt[m]{a} \times \sqrt[m]{b} = \sqrt[m]{a\,b}$.

For the product of the m^{th} roots is equal to the m^{th} root of the product (71.)

Division.

In like manner, to divide radicals reduce them to a common exponent, and divide their suffixes. Thus

$$\frac{\sqrt[m]{a}}{\sqrt[m]{b}} = \sqrt[m]{\frac{a}{b}}$$

Involution and evolution of radicals

(76.) The rules for the involution and evolution of radicals may be immediately obtained by converting them into powers with fractional exponents. If it be required to raise $\sqrt[m]{a^n}$ to the p^{th} power, we have $\sqrt[m]{a^n} = a^{\frac{n}{m}}$ $\because (\sqrt[m]{a^n})^p = a^{p\frac{n}{m}}$. Now it will be proved in Section IX., that a fraction is multiplied by

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any number, either by multiplying its numerator, or dividing its denominator. Hence

$$(\sqrt[m]{a^n})^p = a^{\frac{pn}{m}} = \sqrt[m]{a^{pn}}$$

or,

$$= a^{\frac{n}{m \div p}} = \sqrt[m \div p]{a^n}$$

Hence to raise a radical to the p^{th} power, it is necessary either to multiply the exponent of the suffix, or to divide that of the radical by p.

(77.) In the same manner we may infer, that to take the p^{th} root of a radical it is necessary to divide the exponent of the suffix, or multiply the exponent of the radical by p. For the process must be exactly the reverse of evolution.

(78.) The theorems which have been established in the preceding articles, respecting the calculation of radicals, are founded upon the supposition, that if the powers of the same degree of two qoantities be equal, the quantities themselves will be equal. So long as the theorems are applied to absolute nombers this is strictly true, but some modification will be necessary when applied to algebraical quantities. We have already seen that all the even powers of $+ a$ and $- a$ are the same; and we should, therefore, be wrong in concluding, that if $+ a^2$ be the square of two quantities, these two quantities must be algebraically equal, since one might be $+ a$, and the other $- a$. We shall see hereafter, that every quontity has as many roots of the m^{th} order algebraically different, as there are units in m, and, therefore, it would be wrong to conclude, that if the m^{th} powers of two quantities were equal, the quantities themselves would be equal, since there might be two different m^{th} roots of the same quantity. These observations are more especially to be attended to io cases where imaginary expressions are concerned.

(79.) Every simple imaginary quantity may be considered as the product of a real quantity, and a power of $- 1$, whose exponent has an odd numerator and even denominator. The terms of the series 0, 2, 4, 6, &c. are severally the doubles of those of the series 0, 1, 2, 3, &c. Hence if m represent ony term of the latter series, $2m$ will represent any even integer or any term of the former series. In like manner, the successive terms of the series of odd numbers, 1, 3, 5, 7, &c., may be found by adding one to each of the terms of the first series, so that any term of the last series may be represented by $2m + 1$, m, as before, being any term of the second series.

A simple imaginary expression has been defined to be a negative quantity raised to a fractional power, the denominator of whose exponent is even. The numerator must therefore be odd; for if both were even, the fraction might be reduced to lower terms. The numerator of the exponent may, therefore, be represented by $2m + 1$, and the denominator by $2n$. The quantity, therefore may be expressed in the form $(- A)^{\frac{2m+1}{2n}}$, A being a real quantity. But by (71) we hove

$$(- A)^{\frac{2m+1}{2n}} = (- 1)^{\frac{2m+1}{2n}} . (+ A)^{\frac{2m+1}{2n}}$$

But $(+ A)^{\frac{2m+1}{2n}}$ is a real quantity; let it be called B, and we have

$$(- A)^{\frac{2m+1}{2n}} . = (- 1)^{\frac{2m+1}{2n}} . B.$$

The results of all operations performed with imaginary monomes are, therefore, to be determined by considering the properties of those powers of $- 1$ which have fractional exponents with even denominators.

On Prime and Compound Integers.

(80.) To determine in general what powers of an imaginary quantity are real, it is only necessary to find what integral multipliers will render the exponent of $(- 1)$ in its coefficient an integer. Let

Real powers of imaginary quantities.

$$(- A)^{\frac{2m+1}{2n}} = (- 1)^{\frac{2m+1}{2n}} . B.$$

In order that the exponent of $- 1$ should become an integer, its numerator must be either $2n$, or some multiple of it. Hence the real powers of such an imaginary expression are $2n$, $4n$, $6n$, &c., and they are alternotely negative and positive. All other powers are imaginary.

The product of any two quadratic imagioary expressions is real and negative.

$$\sqrt{-a} . \sqrt{-b} = (-1)^{\frac{1}{2}} . \sqrt{a} . (-1)^{\frac{1}{2}} \sqrt{b}$$
$$= (-1)^{\frac{1}{2}} (-1)^{\frac{1}{2}} \sqrt{ab}$$
$$= (-1) \sqrt{ab} = - \sqrt{ab}$$

The prodnct of three such factors woold be imaginary.

$$\sqrt{-a} \sqrt{-b} \sqrt{-c} = - \sqrt{ab} . \sqrt{-c} = - \sqrt{-1} \sqrt{abc}$$

again, if a fourth imaginary factor be introduced, we have

$$\sqrt{-a} \sqrt{-b} \sqrt{-c} \sqrt{-d} = - \sqrt{-1} . \sqrt{-d} \sqrt{abc}$$
$$= - (-1)^{\frac{1}{2}} (-1)^{\frac{1}{2}} \sqrt{abcd}$$
$$= + \sqrt{abcd}$$

By continuing this reasoning it will be evident, that a product consisting of an even number of quadratic imagiory factors will be real, and will be positive or negative, according as half the number of factors is even or odd.

SECTION VII.

On Prime and Compound Integers.

(81.) *Number* is defined to be the abstract ratio of any quantity to another of the same kind, which is called the *unit*. As the terms of a ratio may be either commensurable or incommensurable, number is accordingly divided into two species. Number.

(82.) A *rational number* is thot which is commensurable with unity.

(83.) An *irrational number* is that which is incommensurable with unity. Irrational numbers are sometimes called surds.

(84.) *Rational numbers* are of two species, *integral* and *fractional*.

(85.) An *integer* is a multiple of unity.

(86.) A *fraction* is a *submultiple* of unity, or a multiple of a submultiple of unity.

Integers are divided into *prime* and *compound*.

(87.) A *prime integer* is one which is oot measured by any integer greater than unity, as 3, 5, 7, 11, &c. Prime integer.

(88.) A *compound integer* is one which is measured by an integer greater than unity, as 4, 6, 9, &c. Compound integer.

(89.) An integer which measures another is called a divisor of it; and if it be a prime integer, it is called a *prime divisor* or *prime factor.*

(90.) Every compound integer is the product of its prime divisors. Thus 10 is measured by 2 and 5, and $10 = 2 \times 5$. It should, however, be observed, that the same prime divisor may occur more than once as a factor. Thus the only prime divisors of 12 are 2 and 3. But 12 is not equal to 2×3, but $= 2 \times 2 \times 3$; the prime factor 2 occurring twice. In like manner, 2 is the only prime factor of 16, but $16 = 2 \times 2 \times 2 \times 2$.

(91.) Two integers are said to be prime to each other, when they have no integral common factor greater than unity. Thus 7 and 9 are *prime to each other*, although 9 is not a *prime integer*.

If either or both of two integers be *absolutely* prime, it is evident that they must be *relatively* prime, since one or both has no factor greater than unity. This is subject, however, to the exception of the case in which one being prime the other is a multiple of it.

The least integers in a given ratio.

(92.) *The least integers in a given ratio measure all other integers which are in the same ratio.*

Let m and n be the least integers, and let M, N be any others in the same ratio.

If m measure M, it is evident from the nature of proportion that n must measure N by the same number. Thus, if m be contained in M t times, without a remainder, n must also be contained in N t times without a remainder.

Also, from the nature of proportion it appears, that if m be contained in M t times with a remainder m' less than m, n must be also contained t times in N with a remainder n' less than n. Thus we have

$$M = m t + m'$$
$$N = n t + n'$$

Hence

$$m : n :: m t + m' : n t + n'$$

but also

$$m : n :: m t : n t$$
$$\because m t : n t :: m t + m' : n t + n'$$
$$\because m t : m t + m' :: n t . n t + n'$$
$$\because m t : m' :: n t : n'$$
$$\because m t : n t :: m' : n'$$
$$\because m \ : n \ :: m' : n'$$

Hence m' and n' are integers less than m and n, and in the same ratio with them, which is contrary to the hypothesis. It follows, therefore, that there can be no remainders, and that m and n must measure M and N the same number of times, so that

$$M = m t$$
$$N = n t$$

It is plain that M and N are divisible by t, which is, therefore, a common measure, and, therefore, M and N cannot be prime.

It appears also, that t is the greatest common measure of M and N; for if there were a greater, the quotes found by dividing M and N by it would be less than m and n, and yet would be in the same ratio with them, which contradicts the hypothesis.

Also it follows, that m and n are prime, for if they admitted of a common factor, let it be t'. Then $m = m' t'$, $n = n' t'$, and we should have

$$m : n :: m' : n'$$

m', n' being less integers than m and n, and in the same ratio.

It follows, also, that prime integers are the least in their own ratio; for if they were not, they would have a common measure, as has been already proved.

Thus prime integers are equisubmultiples of all other integers in their own ratio, and the primes in any given ratio are found by dividing any integers in that ratio by their greatest common measure.

(93.) If an integer a measure one of two prime integers m, it must be prime to the other n. For any common measure of a and n would also measure m, which is a multiple of a, and would therefore be a common measure of m and n, which contradicts the hypothesis.

(94.) If an integer m measure a product $a\,n$, and be prime to one factor n, it must measure the other a. For let it measure $a\,n$ by c, so that

$$m c = a n \because m : n :: a : c.$$

Since m is prime to n it measures a.

(95.) If an integer a be prime to two others, m, n, it will be prime to their product.

For if not, let c be a common measure of a and $m\,n$. Since c measures a it is prime to m, (93;) and since it measures $m\,n$ it must measure n, (94.) It, therefore, is a common measure of a and n, which are $\because$ not prime, which contradicts the hypothesis.

The same principle being extended, shows that if an integer be prime to any number of integers, it will be prime to their continued product, and that if any number of integers be severally prime to any number of others, the continued product of the former will be prime to the continued product of the latter.

Hence, if two integers be prime to each other, every power of the one will be prime to every power of the other.

For a more complete discussion of the properties of prime and composite integers, we refer to our Treatise on ARITHMETIC. We have confined ourselves here strictly to what is indispensably necessary to render the doctrine of fractions in Section IX. intelligible

SECTION VIII.

Of the Greatest Common Measure and the Least Common Multiple.

The measure of any two quantities

(96.) If a quantity a measure two others, b and c, it will also measure their sum $(b + c)$ and their difference $(b - c)$.

For let a measure b m times, and c n times, so that $b = m a$, $c = n a$, $\because b + c = m a + n a = (m + n) a$, $b - c = m a - n a = (m - n) a$. Since $m + n$ and $m - n$ are integers, a measures $b + c$ and $b - c$.

(97.) If the division of a greater quantity by a lesser be partially effected, and the integral part of the quote be obtained, any quantity which measures both the divisor and dividend must measure the remainder, and any quantity which measures both the divisor and the remainder must measure the dividend. Let d be the divisor, D the dividend, q the integral part of the quote, and r the remainder. Hence $D - q d = r$, any

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quantity which measures d must measure $q\,d$, and if it measure D also, it will measure $D - q\,d$, or r, (96.) Also, we have $D = q\,d + r$, and any quantity which measures d and r will also measure $q\,d + r$ or D.

The greatest common measure.

(98.) To determine the greatest common measure of two quantities.

Let the lesser be A, and the greater B.

Let B be divided by A, and if there be no remainder, A is the greatest common measure, since it measures itself and B.

But if there be a remainder, let it be R. It is necessarily $<$ A. Let A be divided by R, and let the remainder be R'. Again, let R be divided by R', and let the remainder be R'', and in this manner let the process be continued, dividing each remainder by that which immediately succeeds it, until some remainder be found which measures the preceding remainder. This remainder is the *greatest common measure*.

First, it is a *common measure*; for it measures itself and the last divisor, and, therefore, measures the last dividend. But this divisor and dividend were the remainder and divisor in the preceding division, and since it measures these, it must measure the preceding dividend; and by the same reasoning it may be proved, to measure every divisor and dividend until we arrive at the given quantities A and B, which are the first divisor and dividend. It is, therefore, a common measure of these.

Secondly, it is the *greatest common measure*; because every other common measure can be proved to measure it. Every common measure of A and B must measure the first remainder R. But R and A are divisor and dividend in the second process of division. Therefore the same common measure measures the second remainder, and so on, until we arrive at the last remainder, which it also measures. But this remainder has been proved to be a common measure, and since every other common measure measures it, it must be the greatest common measure.

In this process each successive remainder is less than that which precedes it, and the process may be continued *ad infinitum*, the remainders continually diminishing in magnitude, and none ever found which will measure that which precedes it. In this case, by continuing the process, a remainder may be found which is less than any assignable quantity. It is not difficult to perceive, that in this case the given quantities are incommensurable; for if they had a common measure, however small, the process above described might be continued, until a remainder be found smaller than this common measure; but this common measure would measure every remainder, and would, therefore, measure a quantity less than itself, which is absurd. Hence the two given quantities admit no common measure, or are incommensurable.

If two quantities be commensurable, all their common measures may be found by determining their greatest common measure. Let this be M. Every other common measure of the two given quantities measures this, and *vice versâ*, it is plain that every quantity which measures M must measure the given quantities. Now the greatest quantity which measures M is $\frac{1}{2}$ M. The next in magnitude is $\frac{1}{3}$ M, the next $\frac{1}{4}$ M, and so on; the common measures forming the series M, $\frac{M}{2}$, $\frac{M}{3}$, $\frac{M}{4}$, $\frac{M}{5}$ &c

If the two given quantities be integers which are prime to each other, the last remainder will be unity.

The Greatest Common Measure and the Least Common Multiple.

It is evident that the greatest common measure of two quantities, A and B, is also the greatest common measure of the lesser A, and the remainder resulting from the division of the greater B and the lesser A, and also the greatest common measure of every divisor and remainder to the end of the process.

The greatest common measure of three quantities.

(99.) To determine the greatest common measure of three quantities A, B, C, let the greatest common measure M of A and B be found, and next let the greatest common measure M' of M and C be found. This will be the greatest common measure of A, B, and C.

First, it is a common measure; for since M' measures M, it must measure A and B, which are multiples of M; and it also measures C, and is, therefore, a common measure.

Secondly, it is also the greatest common measure; for any other m, since it measures A and B, must measure their greatest common measure M, and since it measures M and C, must measure their greatest common measure M', and is, therefore, less than M'.

In the same manner, the greatest common measure of four or more quantities may be found, viz. by finding the greatest common measure of two, then the greatest common measure of that and the third, and so on.

The greatest common measure of four quantities being known, all other common measures may be found in the same manner as for two.

The least common multiple of two quantities.

(100.) *To determine the least common multiple of two quantities* A, B.

Let the common multiple sought be m A and n B, so that

$$m\,A = n\,B,$$

and that m and n be integers. The question then is to determine what are the least integral values of m and n, which are consistent with the equality of m A and n B. From this equality we deduce

$$m : n :: B : A.$$

Hence m and n must be the least integers in the ratio of B : A. Let e be the greatest common measure of B and A. By (92) we have

$$m = \frac{B}{e} \quad n = \frac{A}{e}$$

Hence we obtain

$$m\,A = \frac{B\,A}{e} = n\,B.$$

The least common multiple of two quantities is, therefore, their product divided by their greatest common measure.

If the quantities be not numbers, this result may be found more intelligible if announced thus, To find the least common multiple of two quantities, let either of them be multiplied by the number of times their greatest common measure is contained in the other.

The least common multiple measures every other common multiple.

(101.) The least common multiple of two quantities measures every other common multiple. For let m be the least common multiple of A and B, and let M be any other common multiple; if m do not measure M, let there be a remainder r less than m. Since A and B measure m and M, they must also measure r, which is therefore a common multiple of A and B, and less than m, which is the least common multiple, which is absurd.

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Hence if the least common multiple m of two quantities be known, every other common multiple may be found; for the least number which m measures is $2m$, and the next is $3m$, and so on. So that the succession of common multiples is m, $2m$, $3m$, $4m$, &c.

If the two quantities be prime integers, their least common multiple is their product. For their greatest common measure is unity.

We shall treat of the greatest common measure of Algebraic quantities hereafter.

SECTION IX.

On Fractions.

Fractions.

(102.) Any quantity being divided into any number of equal parts, one, or the aggregate of several of these parts, is called a *fraction* of that quantity. The quantity which is so divided may be itself a number; and as, in explaining the theory of fractions, it is convenient to suppose that all fractions arise from the division of the same *whole*, we shall consider this to be the *unit*.

The value of a fraction, therefore, depends on two things, first, on the number of equal parts into which the unit is divided, and secondly, on the number of these parts which constitute the fraction. Two integers are, therefore, necessary to express the value of a fraction; that which expresses the number of parts into which the unit is divided is called the *denominator*, and that which expresses the number of these parts in the fraction is called the *numerator*.

Denominator. Numerator

(103.) A fraction bears the same ratio to the unit as its numerator bears to its denominator. For the former expresses the number of equal parts in the fraction, and the latter expresses the number of the same parts in the unit.

(104.) Hence it appears, that a fraction is equivalent to the quote arising from the division of its numerator by its denominator. For the quote bears to the unit the same ratio as the dividend (or the numerator) bears to the divisor (or the denominator,) (48.) Since, then, the quote and fraction both bear the same ratio to the unit, they are equal.

(105.) Hence the notation used to express a fraction is the same as that used to express the division of the numerator by the denominator. If a be the numerator, and b the denominator, the fraction is $\frac{a}{b}$.

Ratios expressed by fractions.

(106.) If the numerators a, c of two fractions bear the same ratio to their denominators b, d, the fractions are equal, and *vice versâ*, (48.) That is, if

$$a : b :: c : d$$

$$\therefore \frac{a}{b} = \frac{c}{d}.$$

The latter may, therefore, be considered a more concise way of denoting proportion.

Terms of a fraction.

(107.) Since the value of a fraction depends on the relative, and not the absolute values of its terms, it follows that the same fraction may be expressed in an infinite variety of different terms. Any change may be made upon the terms of a fraction which does not affect their ratio, without changing its value.

Of Fractions.

It is, however, most frequently desirable that fractions should be expressed in their lowest possible terms, and these are evidently the least quantities in the ratio of the numerator to the denominator. Whether the fraction be arithmetical or algebraical, these terms are found by dividing the terms of the proposed fraction by their greatest common measure.

(108.) It is evident, also, from what has been established in the last section, that all terms in which a fraction can be expressed are equimultiples of its least terms.

(109.) Also it appears, that both terms of a fraction may be multiplied, or divided, by the same quantity without changing its value.

Reduction of a fraction to a given denominator.

(110.) It is sometimes necessary to change the *denomination* of a fraction, that is, to find an equivalent fraction having a given denominator. It is first to be observed, that this is only possible when the given denominator is a multiple of the least denominator of the proposed fraction; for it has been already proved, that all terms in which a fraction can be expressed are equimultiples of its least terms. The numerator sought will then be the same multiple of the least numerator, as the given denominator is of the least denominator. The practical process for determining the numerator is obvious. Let $\frac{a}{b}$ be the fraction, d the given denominator, and x the sought numerator. We have by hypothesis $\frac{x}{d} = \frac{a}{b}$, multiplying these equal quantities by d we obtain $x = \frac{a\,d}{b}$; in order that the problem be possible, it is necessary that b should measure $a\,d$.

Reduction of fractions to the same denominator.

(111.) If several fractions be required to be *reduced to the same denomination*, let them be first reduced to their lowest terms. The common denominator to which they are then to be reduced, must be a common multiple of their denominators, (110;) and they may be reduced to any common denominator which is a common multiple of their denominators. The several numerators are found by taking the same multiple of the numerator of each fraction, as the common denominator assumed is of the denominator or the fraction. Or, what is the same, let the common denominator be divided by each of the given denominators, and let the quotes be severally multiplied by the respective numerators. The several products will be the numerators of the fractions sought.

The least terms in which several fractions can be expressed, consistently with having a common denominator, is when the common denominator is the *least common multiple* of their denominators.

Relative magnitudes of fractions.

(112.) The relative magnitudes of two fractions may be known by reducing them to a common denominator. For then, since the parts of unity which compose them are the same, they are as their numerators. Let the fractions be $\frac{a}{b}$ and $\frac{c}{d}$. When reduced to a common denominator they become $\frac{a\,d}{b\,d}$, $\frac{b\,c}{b\,d}$ which are as $a\,d : b\,c$ $\therefore \frac{a}{b} : \frac{c}{d} :: a\,d : b\,c$. Hence fractions are as the alternate products of their numerators and denominators.

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Addition and subtraction.

(113.) Several fractions united with the signs + or − may be incorporated or reduced to one fraction by reducing them to a common denominator, and adding or subtracting their numerators according to the signs with which the fractions are connected; taking the result of this as the numerator, and subscribing the common denominator. For by reducing them to a common denominator, the parts of the unit which they severally contain are equal, (102;) by adding or subtracting the numerators, and subscribing the common denominator, these parts are added, and their magnitude preserved, thus $\frac{a}{b} \pm \frac{c}{d} = \frac{a\,d}{b\,d} \pm \frac{c\,b}{b\,d} = \frac{a\,d \pm c\,b}{b\,d}$.

Multiplication.

(114.) The multiplication of the numerator of a fraction by any number, has the same effect on its value as the division of the denominator by the same number. Let the fraction be $\frac{a}{b}$. If its numerator be multiplied by c, it becomes $\frac{a\,c}{b}$. If its denominator be divided by c, it becomes $\frac{a}{b \div c}$. To prove that these results are equal, let both numerator and denominator of the former be divided by c, and we have

$$\frac{a\,c}{b} = \frac{a\,c \div c}{b \div c} = \frac{a}{b \div c}.$$

In precisely the same manner it may be proved, that the division of the numerator of a fraction by any number, has the same effect upon its value as the multiplication of the denominator by the same number.

By an integer.

(115.) A fraction is multiplied by an integer, by multiplying its numerator by the integer. For this multiplies the number of parts in the fraction without changing the value of these parts. Also, the same effect is produced by dividing the denominator by the integer.

(116.) A fraction is divided by an integer, by dividing its numerator by the integer. For this divides the number of parts in the fraction without affecting the value of these parts. The same effect is produced by multiplying the denominator.

By a fraction.

(117.) The multiplication of any quantity by a fraction is an operation compounded of a multiplication by its numerator, and division by its denominator. If the multiplier be $\frac{a}{b}$, and the quantity be first multiplied by the numerator a, the product thus obtained will be b times the true product; because the multiplier a was b times the true multiplier $\frac{a}{b}$. Hence, to obtain the true product, it will be necessary to divide the product already obtained by b.

Four ways.

(118.) Hence, if one fraction $\frac{a}{b}$ be required to be multiplied by another $\frac{c}{d}$, it is necessary to multiply it by c, and to divide it by d. As each of these operations can be performed in two different ways (115,) it follows that there are four ways in which one fraction may be multiplied by another; these four ways are represented as follows:

1. $\frac{a\,c \div d}{b}$; 2. $\frac{a\,c}{b\,d}$; 3. $\frac{a}{b\,d \div c}$; 4. $\frac{a \div d}{b \div c}$.

Of fractions.

Of these the second is the most usual, because it is always possible to effect the operations. The others are only used when the divisions indicated can be effected without remainders. However, when this is the case, they are to be preferred to the second, because they give the product sought in lower terms.

Division.

(119.) The division of any quantity by a fraction $\left(\frac{a}{b}\right)$ is an operation compounded of a division by the numerator, and a multiplication by the denominator. If the quantity be divided by the numerator a, the quotient will be the b^{th} part of its true value, because the divisor a is b times the true divisor. Hence to obtain the true quotient it will be necessary to multiply the quotient already obtained by the denominator b. It appears, therefore, that dividing by the fraction $\frac{a}{b}$ is the same as multiplying by $\frac{b}{a}$. That is, to divide a quantity by a fraction, it is necessary to multiply it by the reciprocal of that fraction.

(120.) If one fraction $\frac{a}{b}$ be required to be divided by another $\frac{c}{d}$, it is the same as if required to multiply it by $\frac{d}{c}$. Hence there are four ways (118) of effecting the object. The quote may, therefore, be expressed in any of the following ways:

Four ways.

1. $\frac{a\,d \div c}{b}$; 2. $\frac{a\,d}{b\,c}$; 3. $\frac{a}{b\,c \div d}$; 4. $\frac{a \div c}{b \div d}$.

The second is the most usual, for the reasons assigned in (118,) but the others, when possible, are to be preferred.

Value of $\frac{0}{b}$

(121.) The denominator of a fraction being supposed to remain the same, if the numerator be diminished the fraction itself will evidently be proportionately diminished. If the numerator be indefinitely diminished, and ultimately be supposed to become $= 0$, the fraction will also become $= 0$. Let the fraction be

$$\frac{a - x}{b}.$$

While b and a remain unvaried, let the value of x continually approach to equality with a; the fraction will evidently be constantly diminished in value, and the ultimate value when $x = a$ is considered to be $= 0$.

Thus, when the numerator $= 0$, and the denominator is finite, the fraction $= 0$.

If, however, the denominator $= 0$, the case is otherwise. Let the fraction be

$$\frac{b}{a - x}.$$

Value of $\frac{b}{0}$

While a and b remain unvaried, let x continually approach to equality with a. The nearer x approaches to a, the greater ratio will b bear to $a - x$, and, therefore, the greater will be the value of the fraction. As x approaches to equality with a, there is no limit to the increase of the fraction. When $a = x$ the fraction is, therefore, considered infinitely great.

Hence a fraction whose denominator $= 0$, and whose numerator is finite, is infinite.

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Such a quantity is generally expressed by the symbol ∞; so that $\frac{b}{0} = \infty$.

Infinity.

If the numerator of a fraction be increased, the same effect is produced upon its value as if its denominator were proportionately diminished, and *vice versâ*. Hence, and from what has been just established, we may infer, that when the numerator is infinite, and the denominator finite, the fraction is infinite. That is

$$\frac{\infty}{b} = \infty;$$

and that when the denominator is infinite, and the numerator finite, the fraction is $= 0$, or

$$\frac{b}{\infty} = 0.$$

Strictly speaking, the symbols ∞ and 0, in these cases, should be considered as representing quantities indefinitely increased, and indefinitely diminished; and when they are called *infinity* and *zero* or *nothing*, it is for brevity, and to avoid a circumlocutory description of unlimited increase and diminution.

From the preceding observations, combined with the principles established in Section VI., it follows, that when an algebraical quantity becomes $= 0$, owing to particular values or relations being assigned to the letters of which it is composed, all its powers which have positive exponents will be $= 0$, and those which have negative exponents become infinite. Hence we infer generally, that

Powers of 0

$$0^m = 0,\ 0^{-m} = \infty,$$

where m represents any *positive* number, integral or fractional, rational or irrational.

In a fraction whose numerator and denominator are algebraical quantities, it sometimes happens, that when particular values or relations are assigned to the letters of which these quantities are composed, they will both become $= 0$, so that the fraction will assume the form $\frac{0}{0}$.

To determine what the true value of the fraction is in this case, we must examine under what circumstances a quantity becomes $= 0$. If it be the product of any number of factors, it will necessarily $= 0$, when any factor having a positive exponent $= 0$; and if such a factor occur with positive exponents in both numerator and denominator, the fraction will necessarily have the form $\frac{0}{0}$. Let the fraction be $\frac{A}{B}$, A and B representing any algebraical quantities, which both become $= 0$, when some particular values or relations are ascribed to the letters of which they are composed. Let the factor of A which $= 0$ be F, and let $F^m A' = A$; A' being not divisible by F, and m being a positive number.

Value of $\frac{0}{0}$

Also, let F^n be a factor of B, and let $F^n.\ B' = B$, B' not being divisible by F; in other words, let F^m and F^n be the highest powers of F which divide A and B. Hence we have

$$\frac{A}{B} = \frac{F^m.\ A'}{F^n.\ B'}.$$

If under the given conditions $F = 0$, the fraction will become $\frac{0}{0}$, but its true value will depend on the relation between m and n.

Of Fractions.

1. If $m = n$ ∴

$$\frac{A}{B} = \frac{A'}{B'},$$

under which form both terms are finite, and the value of the fraction is made evident.

2. If $m > n$

$$\frac{A}{B} = F^{m-n}.\ \frac{A'}{B'}.$$

Since $F = o$, and $(m - n)$ is positive, ∴ $F^{m-n} = o$, ∴ $F^{m-n} \times \frac{A'}{B'} = o$. Hence the fraction in this case $= o$.

3. If $m < n$ ∴

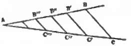
$$\frac{A}{B} = F^{-(n-m)} \times \frac{A'}{B'},$$

and since $n - m$ is positive, ∴ $F^{-(n-m)} = \infty$ ∴ $F^{-(n-m)} \times \frac{A'}{B'} = \infty$. The value of the fraction is, therefore, in this case infinite.

Hence, in general, if an algebraical fraction assumes the form $\frac{0}{0}$, by a factor common to both numerator and denominator becoming $= 0$, the value of the fraction is found by dividing both terms by the common factor, if it have the same exponent in both. It is $= 0$ if the exponent be greater in the numerator than in the denominator; and it is infinite, if it be greater in the denominator than in the numerator.

(122.) If, however, the numerator and denominator do not become $= 0$, by reason of a common factor being $= 0$, but are absolutely each $= 0$, the value of the fraction is indeterminate. The symbol 0 being indicative of a quantity infinitely diminished, it will be easily understood that a fraction whose numerator and denominator are infinitely diminished, (except under the circumstances already mentioned,) may have any value whatever. Let $\frac{a}{b}$ be any fraction, and let another $\frac{a'}{b'}$ be assumed, whose numerator and denominator are respectively the 10^{th} parts of a and b. Then by (106) $\frac{a}{b} = \frac{a'}{b'}$. The same would be true if $\frac{a'}{b'}$ were the 100^{th}, or 1000^{th}, or ten millionth parts of a and b. In this way, both the numerator and denominator may be infinitely diminished, and each tend to the limit 0, and yet the value of the fraction will remain what it originally was. And as its original value may have been that of any number, it follows, that $\frac{0}{0}$ may have any value whatever.

The same may be illustrated geometrically: thus, let parts A B, A C be taken on the legs of an angle,



and of such magnitudes that A B : A C :: a : b. Hence $\frac{A\,B}{A\,C} = \frac{a}{b}$, in this case the ratio of a to b may have any value whatever; and, therefore, the

fraction $\frac{a}{b}$ may have any value whatever. Let the base B C be moved parallel to itself towards the point A, so as successively to assume the positions B′ C′, B″ C″, B‴ C‴, &c. It is plain, that the ratios A B′ : A C′, A B″ : A C″, &c. remain the same; and, therefore,

$$\frac{A\,B'}{A\,C'} = \frac{A\,B''}{A\,C''} = \ldots = \frac{a}{b};$$

and these ratios continue the same until the line B C arrive at the point A, at which both terms of the fraction become $= 0$, and it assumes the form $\frac{0}{0}$. Throughout these changes it may have had any value whatever; and that value which it is supposed to have throughout the changes, whatever it be, is its value when it becomes $\frac{0}{0}$.

Value of $\frac{\infty}{\infty}$ Algebraical fractions also sometimes assume the form $\frac{\infty}{\infty}$. This is always in consequence of a *vanishing factor* with a *negative* exponent occurring in both numerator and denominator. This may always be reduced to an equivalent fraction of the form $\frac{0}{0}$, by removing the factor with the negative exponent from the numerator to the denominator, or *vice versâ*, changing the sign of its exponent. This is equivalent to multiplying or dividing both terms of the fraction by the same number.

SECTION X.

Of Equations.

(123.) When a problem is to be resolved by Algebra, the first step of the process is to translate its various conditions from the ordinary language in which they are usually announced, into the peculiar language of Algebra. The result of this is always an *equation*, and the *resolution* of this equation gives the solution of the proposed problem. Let us suppose, for example, that a certain number is required, such that if it be added to a given number a, the result will be equal to double another given number b. Now if the number sought be called x, when added to a the result would be $a + x$, and this by the proposed condition must be equal to $2\,b$, that is, $a + x = 2\,b$; such is the proposed problem when stated algebraically.

Equation defined. An equation is, then, a proposition stating that the result of certain operations performed on certain numbers, is equal to the result of other operations performed on other numbers, the numbers, the operations, and the equality being expressed by algebraical symbols.

(124.) Every equation consists, therefore, of two parts, connected together by the sign $=$, the part to the left of this sign being called the *first member*, and the other the *second member*. Thus the *first member* in the example already given is $a + x$, and the *second member* is $2\,b$. Members.

(125.) Every statement of the equality of arithmetical or algebraical quantities is not, however, called an equation. The statements

$$5 = 2 + 3 \qquad a - a = 0$$
$$10 = 2 \times 5 \qquad a - 2\,b + 2\,a = 3\,a - 2\,b$$

and, in general, all equalities which are such that the operations indicated by the signs *can* be performed, and when performed render both members of the equality identical, are called *identities*. Identities.

(126.) The *degree* of an equation is determined by the exponent of the highest power of the unknown quantity which occurs in it. Thus, an equation in which only the single dimension of the unknown quantity occurs, is called an equation of the *first degree*. Such is Degree

$$a + x = 2\,b.$$

One in which the highest dimension is the square of the unknown quantity, is called an equation of the second degree, or quadratic. Such is

$$3\,x^2 \quad 5 + 2\,x = 10\,x^2.$$

The equation

$$10\,x^3 - 2\,x^2 + 3\,x = 10$$

is *cubic*, or of the *third degree*, and so on.

It should be observed, that in determining the *degree* of an equation, it is supposed that no fractional power of the unknown quantity occurs in it, or that the unknown quantity is not contained under any radical, and also that the unknown quantity does not occur in the denominator of any fraction. A method will be hereafter explained, by which such equations may be converted into equivalent ones, in which the unknown quantity does not occur in this way. In fact, to determine the degree of an equation, it must be reduced to a series of monomes, in each of which a power of the unknown quantity occurs as a factor, the exponent of which is neither negative nor fractional.

(127.) Equations, therefore, with relation to the *exponent* of the unknown quantity, are classed in *degrees*. With respect to the nature of the *coefficients* of the unknown quantities, they are divided into *numerical* and *literal*. Numerical and literal.

A numerical equation is one in which the coefficients of the unknown quantity are all particular numbers. Such are the equations

$$3\,x + 4\,x = 10$$
$$2\,x - 5 = 8.$$

A literal equation is one in which the coefficients of the unknown quantity are expressed by letters, or by letters and numbers combined. Such are

$$x^2 + a\,x = b$$
$$2\,a\,x + b = 3\,c\,x.$$

It will be observed, that, as applied to equations, the term *coefficient* acquires an extended signification. In this case it signifies the factor, whether literal or numeral, or both, by which the power of the unknown quantity which enters any term of the equation is affected. Thus, the coefficients of

$$A\,x^3,\ 10\,b\,x^5,\ (a + b)\,x^2,\ 3\,(A - c)\,x^4,\ .$$

are respectively

$$A,\ 10\,b,\ a + b,\ 3\,(A - c).$$

Whenever the data of the problem are particular numbers, the equation to which it is reduced will be *numerical*. The problem in this case is always a *particular* one.

But if the problem be *general*, the data are expressed by letters, and the equation is literal.

(128.) The value of the unknown quantity in any equation, whether numerical or literal, is such a number or letter, or combination of letters, as being sub-

Algebra. stituted for the unknown quantity would convert the *equation* into an *identity*. (125.)

(129.) A value of the unknown quantity, which thus converts the equation into an identity, is said to *satisfy* the equation, and such a value is called a *root* of the equation. It will be seen hereafter, that an equation may have more roots than one.

Roots.

(130.) The root of an equation, or the value of an unknown quantity, would be determined if we could effect such changes as, without disturbing the equality of its members, would disengage the unknown quantity from those known quantities with which it is combined, and so dispose the several quantities, that the unknown quantity shall stand alone in the first member of the equation, while the second member consists of given quantities only combined by signs, indicating the operations to be effected on them. The second member will then be the value of the unknown quantity, or the *root* of the equation.

(131.) In determining, therefore, the root of an equation, it is of importance to be able to disengage the unknown quantity from those known quantities with which it may be combined; and the general principle by which we are enabled to effect this is, that "Any change may be made on the two members of an equation which does not disturb their equality." The *same change* may always, therefore, be effected on the two members of an equation.

Hence it follows, "That the same quantity or equal quantities may be added to or subtracted from both members of an equation."

(132.) It follows from this, that any term may be transferred from one member of an equation to the other by *changing its sign*; for this is equivalent to adding that quantity with an opposite sign to both members. Thus, if

$$x + a = b,$$

adding $-a$ to both members,

$$x + a - a = b - a$$
$$\therefore x = b - a,$$

which is equivalent to transferring a to the second member, changing its sign.

Again, if

$$x - a = b$$
$$x - a + a = b + a$$
$$\therefore x = b + a,$$

in which, as before, $-a$ is transferred to the second member, changing the sign.

(133.) The signs of all the terms of an equation may be changed. For this is equivalent to transferring all the terms of the first member to the second, and *vice versâ*, by (132;) it being evidently indifferent which member is written first.

(134.) Both members of an equation may be multiplied by the same quantity or equal quantities.

By this means, if the unknown quantity be divided by any known quantity, whether simple or complex, it may be disengaged from it by multiplying both members of the equation by the divisor. Thus, if

$$\frac{x}{a} + b = c.$$

By multiplying both members by a, we obtain

$$x + a\,b = a\,c.$$

(135.) Also, if the unknown quantity occur either singly or in combination with known quantities as a divisor, it may in like manner be disengaged by multiplying both members of the equation by such divisor. Of Equations.

This process is called "clearing the equation of fractions."

(136.) If several terms of an equation have different denominators, the equation may be cleared of fractions by multiplying both members by the least common multiple of the denominators.

(137.) Both members of the equation may be divided by the same quantity or equal quantities.

By these means, if the unknown quantity be affected by a known quantity, or several known quantities, as factors, it may be disengaged from them by dividing both members of the equation by them. Thus, if

$$a\,x + b = c,$$

by dividing both members by a we obtain

$$x + \frac{b}{a} = \frac{c}{a}.$$

(138.) Both members of an equation may be raised to the same power, or the same root of them may be extracted.

By this, when the unknown quantity, either singly or in combination with known quantities, is raised to any power, or affected by any radical, it may be disengaged.

(139.) In order to prepare an equation for solution, it is necessary to reduce it to that state in which the first member will be a series of monomes, each having a power of x, with a positive integer as its exponent, and the second member a known quantity or some combination of known quantities. To this state every algebraic equation may be reduced, by the several means which have been just explained.

1. To clear the equation of fractions, find the least common multiple of all the denominators which occur in the equation, and multiply both members by this. There will be no denominator, literal or numeral, in the resulting equation.

2. Bring the radicals or terms affected by fractional exponents, and involving the unknown quantity, successively to stand alone as one member, all the other quantities being transferred to the other member, and raise both members to that power expressed by the exponent of the radical, or the denominator of the fractional exponent. Each of these operations will remove a radical, and by their successive application all the radicals may be removed from the equation.

3. Reduce to a single term all the terms of which the same power of the unknown quantity is a factor. This may be done by enclosing all the coefficients of such terms with their proper signs in a parenthesis, incorporating by addition or subtraction such as admit of it, and multiplying the whole parenthesis by the power of the unknown quantity, which is the common factor. Thus, if the several terms be

$$a\,x^3 - b\,x^3 + 3\,x^3 - 5\,x^3$$

we have

$$(a - b + 3 - 5)\,x^3$$

or,

$$(a - b - 2)\,x^3.$$

4. These reductions being made, let the term in which the highest power of the unknown quantity occurs be placed first, and the others in the descending order of their exponents; the terms which are independent of the unknown quantity forming the second member. The form to which an equation of the third degree would be thus reduced, would be

$$A\,x^3 + B\,x^2 + C\,x = D.$$

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A, B, C being general representations of the coefficients, and D of the quantities independent of x.

5. The equation may be still further simplified, by dividing both members by any one of the coefficients. That which is usually chosen is the coefficient of the highest dimension. If this division were effected, an equation of the fourth order would assume the form,

$$x^4 + a x^3 + b x^2 + c x = d,$$

and in general an equation of the n^{th} order would have the form

$$x^n + a x^{n-1} + b x^{n-2} + c x^{n-3} + \text{\&c.} = K,$$

K representing the terms which are independent of x.

(140.) We have already stated, that equations are classed according to their *degrees*. It is evident that by the process we have just explained, an equation of the first degree would be reduced to the form $x = K$, which, without further investigation, would give the value of the unknown quantity.

We shall now proceed to the consideration of problems, the solutions of which depend on equations of the first degree.

SECTION XI.

Of Equations of the First Degree including one unknown quantity.

(141.) The algebraical solution of a problem consists of two very distinct parts. The first consists in the translation of the conditions of the problem from the common popular language in which it is usually proposed, into the peculiar analytical language of the science. This is what is called "reducing the problem to an equation." The other part consists in discovering the value of the unknown quantity from the equation, or "solving the equation." No general rules can be given for the reduction of a problem to an equation; experience alone, and the study of a number of well-selected examples, will attain this end. The following directions will be found, however, of considerable use: "Let the problem be considered as having been already solved, and the known quantities being represented either by particular numbers or by letters, and the unknown quantity always by a letter; indicate by algebraic signs the various relations and operations to which these quantities would be submitted, were the unknown quantities known." The result of such a process generally gives two different systems of operations on the data of the problem, and the unknown quantity, by which some one quantity may be obtained, and the two algebraical expressions of the results of these operations, in general, furnish the two members of the primary equation.

(142.) We shall now proceed to give a few examples of the investigation of problems which are reduced to equations of the first degree; offering such general observations as the peculiar circumstances of each problem may suggest.

Examples.

(143.) *A fox is started at sixty of his own paces from a hound, nine of his paces being made in the same time as six of the hound, but three paces of the hound being equal to seven of the fox. It is required to determine how many paces the hound will have made when he shall have overtaken the fox?*

Simple Equations.

Let H be the length of each pace of the hound. Since three of the hound's paces are equal to seven of the fox's, if 3 H be divided by seven, the result is the length of one pace of the fox, which is, therefore, $\frac{3\,H}{7}$. At setting out, the fox is sixty of his own paces distant from the hound. Hence this distance is

$$60 \times \frac{3\,H}{7} = \frac{180\,H}{7}.$$

Let the distance sought be x, that is, the number of paces the hound has made at the moment he overtakes the fox. The distance the fox will, therefore, have run will be

$$x - \frac{180\,H}{7};$$

that is, the distance gone over by the hound, diminished by the distance between them at the moment of departure. The spaces x and $x - \frac{180\,H}{7}$ being run over in the same time, must be in the same ratio as the speed of the two animals. It is granted that the fox makes nine paces while the hound makes six, or, what is the same, the fox makes three while the hound makes two. Thus, three times $\frac{3\,H}{7}$, which is the fox's pace, is made in the same time as 2 H. Hence, the spaces the animals move through in the same time are as $\frac{9\,H}{7} : 2\,H$, or as $9 : 14$. Hence we have

$$\frac{x - \frac{180\,H}{7}}{x} = \frac{9}{14}.$$

Which, being cleared of fractions, becomes

$$14\,x - 360\,H = 9\,x$$
$$\therefore\ 14\,x - 9\,x = 360\,H$$
$$\therefore\ 5\,x = 360\,H$$
$$\therefore\ x = 72\,H.$$

The hound will, therefore, have made 72 paces when he shall have overtaken the fox.

(144.) *To divide a line of 15 inches length into two such parts that one of them shall be three-fourths of the other.*

In this case, if one of the parts be called x, the other will be $15 - x$. The number represented by x is here understood to express inches. Now, by the conditions of the question, one of the parts is three-fourths of the other. Three-fourths of x is expressed $\frac{3\,x}{4}$; now this and the other part $15 - x$ must be equal. Thus we have the equation

$$15 - x = \frac{3\,x}{4}.$$

It may be useful to the student to compare this process of reduction with the observations in (141.)

Clearing this equation of fractions by multiplying both members by 4, we obtain $60 - 4\,x = 3\,x$. Transferring $-4\,x$ to the second member, changing the sign $60 = 7\,x$, or $7\,x = 60$. Dividing both members by the coefficient 7,

$$x = \frac{60}{7} = 8\tfrac{4}{7}$$

This in inches is the length of one part, and since the whole line is 15 inches, the other part must be $6\frac{3}{7}$ inches.

(145.) In this instance the question is particular, and the equation numerical. It would, perhaps, be better in every case where a particular problem is proposed, to generalize it in the first instance. The result will then be a literal equation, which, when solved, will give a general formula, by which not only the proposed question may be solved, but also every question of the same class. The preceding problem generalized would be as follows:

(146.) To divide a given line a into two such parts that one shall be m times the other, (m being any number, integral or fractional.)

The statement would now be thus: Let one part be x, and the other must be $a - x$. By the conditions of the problem, $a - x$ and $m\,x$ must be equal. Hence $m\,x = a - x$. Transposing $-x$, and changing its sign, we have $m\,x + x = a$. Collecting within a parenthesis the coefficients of x, we have $(m + 1)\,x = a$. Dividing by $(m + 1)$, we obtain

$$x = \frac{a}{m+1},$$

which is one of the parts. The other part will be $a - x$. Hence

$$a - x = a - \frac{a}{m+1}.$$

The second member of this equation may be considered as a mixed number, and, therefore, the first part a is to be multiplied by $m + 1$, and a subducted from the result. The process will be understood from the following steps:

$$a - x = \frac{a\,(m+1)}{m+1} - \frac{a}{m+1},$$

$$\therefore\ a - x = \frac{a\,(m+1) - a}{m+1},$$

$$= \frac{m\,a}{m+1}.$$

Hence we obtain the following general rule for the solution of all such questions. To find one part, divide the proposed line by the number which is given, expressing the proportion of the parts increased by unity, and the quote is one part. Multiply this quote by the same number, and the product is the other part.

It is, however, worse than useless to translate into popular language thus, the formulæ derived from general algebraical investigation; they are clearer and more compendious, and much more easily retained in the memory, wherever it is necessary to do so, when expressed in their algebraical form. We have in the present instance reduced the result to ordinary language, only to show that this result is really a general theorem or rule, and not merely the solution of a particular question or problem. In the particular instance given, at first we have $a = 15$ and $m = \frac{3}{4}$. Hence $m + 1 = \frac{7}{4}$. Hence we have

$$x = 15 \div \frac{7}{4} = \frac{4 \times 15}{7} = \frac{60}{7}$$

$$a - x = 15 - \frac{60}{7} = \frac{105 - 60}{7} = \frac{45}{7},$$

which are equivalent to the results first obtained.

(147.) *A labourer is engaged for 48 days on these conditions: for each day he works he is paid two shillings, but forfeits one shilling for every idle day; at the end of the 48 days he is entitled, under the terms of the agreement, to 21 shillings: it is required to calculate the number of days he worked, and the number he was idle?*

By the conditions of the problem, if the number of days on which he worked were multiplied by 2, we should have the wages of the entire of these days. The number of days on which he was idle will express the number of shillings which he forfeited. The latter subtracted from the former will leave a remainder equal to the sum to which he is entitled at the conclusion of the stipulated period. This sum is, however, given to be 21 shillings. If, therefore, an algebraical formulæ be adapted to represent the result of the several operations above mentioned, and be taken as the first member of the equation, 21 shillings will be the second member.

Instead, however, of stating the question in the first instance as a particular one, we shall generalize it.

Let a be the number of days for which the labourer is engaged.

Let x be the total number of working days, and, therefore, $a - x$ the number of idle ones.

Let m be the number of shillings he is paid for each working day, and n the number which he forfeits for each idle day.

Let S be the whole sum to which he is entitled at the end of the period by the terms of the agreement.

The total number of shillings earned on the x working days will be $m\,x$, and the total number forfeited on the $a - x$ idle days will be $n\,(a - x)$.

Hence, the total sum to which he will be entitled at the conclusion of the period a, will be $m\,x - n\,(a - x)$. But this, by the conditions of the question, is granted to be equal to S. Hence we obtain the following equation,

$$m\,x - n\,(a - x) = S$$

or,

$$m\,x - n\,a + n\,x = S.$$

Collecting within a parenthesis the coefficients of x, we obtain

$$(m + n)\,x - n\,a = S.$$

Transposing $n\,a$,

$$(m + n)\,x = S + n\,a.$$

Dividing both members by $(m + n)$

$$x = \frac{S + n\,a}{m + n},$$

which gives the number of working days.

To determine the number of idle days, we have

$$a - x = a - \frac{S + n\,a}{m + n}$$

$$= \frac{a\,(m + n) - S - n\,a}{m + n}$$

$$\therefore\ a - x = \frac{a\,m - S}{m + n},$$

which is the number of idle days.

In the particular question proposed, we have $a = 48$, $m = 2$, $n = 1$, and $S = 21$. Hence

$$x = \frac{21 + 48}{2 + 1} = \frac{69}{3} = 23$$

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$$a - x = \frac{96 - 21}{2+1} = \frac{75}{3} = 25.$$

Thus the number of working days was 23, and the idle ones 25, amounting together to the whole period of 48 days.

(148.) It is evident that the general values of x and $a - x$ would solve the problem with equal facility had any other rate of payment, or any other period, been made the subject of a similar agreement. In fact, the result of the general algebraical investigation is not so much an absolute solution of the problem, as an indication of a method by which similar problems may always be solved.

Negative roots.

If the particular numbers represented by a, m and S be such that the product $a\,m$ is less than S, it is evident that the formula expressing the number of idle days will represent a negative number. A question then arises, what is meant by the labourer having worked a negative number of days?

To explain this, we must refer to the meaning of the symbols. m is the number of shillings paid to the labourer for each day that he works. a is the total period agreed upon. $m\,a$ is, therefore, the sum which he should receive if he worked every day of the entire period, and spent no day idle. But, under the circumstances which we have supposed, he becomes entitled to a sum S greater than the sum $m\,a$, to which he would have been entitled had he worked every day of the stipulated period. The inference is, that instead of being idle on any of the stipulated days, he must have worked as many additional days as would entitle him to that sum by which S exceeds $m\ a$. Consequently, the formula for $a - x$, which, when positive, signifies the excess of the stipulated period over the working days, signifies, when it becomes negative, the excess of the working days above the stipulated period. Such a result as a negative number of days, considered merely by itself, is unmeaning, but when the circumstances which led to that result are examined, it leads to a modification of the original question. It shows that the conditions proposed are inconsistent with the data, and it indicates, that to render them consistent, either the data or the conditions must be modified; and, further, it points out what the necessary modifications are. In the present instance we find that the sum S, to which it is asserted, in the original question, that the labourer is entitled at the end of the stipulated time, is greater than he could have made in that time without any idle days at all; and, therefore, that if the question be modified, and rendered consistent by changing the data, it will be necessary to regulate the numbers represented by a, m, and S, so that S shall not exceed $a\,m$, which may evidently be effected by increasing a or m, or both, or by diminishing S, or by all these changes combined.

If, however, it be desired to modify the conditions of the original question, so as to render them consistent with the data, we must examine the original statement. This is $m\,x - n\,(a - x) = \text{S}$. Now if $a - x$ be negative, as is supposed in the present case, that is, if $x > a$ the quantity $-\,n\,(a - x)$ is positive, and the equation being written thus, $m\,x + n\,(x - a) = \text{S}$, expresses that the total sum S receivable by the labourer is composed of m shillings for each of the x working days, together with n additional shillings for each of the $(x - a)$ days which he works over and above the stipulated period of a days. Thus the n shillings, which in the case of idle days was a forfeit, becomes a premium in the case of supernumerary working days. The question, therefore, will be thus modified:

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A labourer is engaged for a *days at* m *shillings per day, on condition that he shall forfeit* n *shillings per day for as many days as his number of working days shall fall short of the stipulated period* a, *and that, in addition to* m *shillings per day, he shall receive a premium of* n *shillings a day for as many days as his working days shall exceed the stipulated period* a. *At the cessation of his labour he becomes entitled, under the terms of the agreement, to a sum of* S *shillings. It is required to assign the number of working days, and to determine the number of idle or supernumerary working days, as the case may be.*

(149.) A further advantage which general algebraical investigations possess over particular numerical questions is, that the same general formula may be the means of solving other problems, besides even the general one from which it results. In the problem just investigated, the formula

$$x = \frac{a\,m + \text{S}}{m + n}$$

expresses in general a relation between the numbers represented by x, a, m, n, and S. Now if any one of these five quantities be unknown, and all the others known, the value of the unknown quantity may always be determined.

Let us suppose, for example, that S is the unknown quantity; the question will then be, to determine the sum to which the labourer will be entitled at the cessation of his labour, the number of working days x, the daily wages m, the forfeit or premium n, and the stipulated period a, being all given. To solve this problem, it is only necessary to consider S as the unknown quantity, and solve the equation for it. Multiplying both members by $m + n$ we have

$$(m + n)\,x = a\,m + \text{S},$$

and transposing $a\,m$ we have

$$(m + n)\,x - a\,n = \text{S},$$

or, $$\text{S} = (m + n)\,x - a\,n.$$

This gives the sum to which the labourer is entitled.

In the first example, $m = 2$, $n = 1$, $a = 48$, and $x = 23$; hence

$$\text{S} = (2 + 1)\,.\,23 - 48\,. = 69 - 48$$

$$\therefore \text{S} = 21.$$

In this case, also, it might so happen, that the particular values assigned by the data to the quantities x, m, n, and a, would render the value of S negative. Let us consider the meaning of such a result.

By the equation

$$\text{S} = (m + n)\,x - a\,n$$

or, $$\text{S} = m\,a - (a - x)\,n,$$

it appears, that if S be negative we must have $a - x$ positive, or $a > x$, and $m\,x < (a - x)\,n$, that is, the number of working days x is less than the stipulated period a, and the entire wages $m\,x$ of the working days is less than the sum $(a - x)\,n$ forfeited for the idle days. Hence, on the whole, the labourer is a loser by the excess of the sum forfeited $(a - x)\,n$ over the wages $m\,x$, that is, by the positive value of the negative result S.

Thus it appears, that the sum supposed in the statement to be gained by the labourer becoming negative in the result, proves that this sum is not gained, but lost. The problem should therefore be modified, so

Algebra. as that the required quantity would be the balance for or against the labourer on closing the account.

(150.) These observations lead us to the consideration of the nature of negative quantities. When positive and negative quantities are considered merely as members of polynomes, and therefore connected by their proper signs with other quantities, their meaning is obvious, and they might more properly be called *additive* and *subtractive* quantities; as has been already explained. But we have seen that a negative quantity is frequently the result of a calculation, and, therefore, not considered as a member of a polynome. What then, it may be asked, can be its meaning in this case?

The most simple process from which a negative quantity can result is subtraction. Let the problem proposed be to find a number, which, when added to a given number b, will produce a given sum a. Thus, if x be the number, we have

$$b + x = a$$
$$\therefore x = a - b.$$

If we suppose $a = 30$ and $b = 20$, we have

$$a = 30 - 20 = 10;$$

in this case the result is positive, and is the true solution of the problem proposed. But suppose that $a = 20$ and $b = 30$, we should have

$$x = 20 - 30.$$

Putting this expression under the form

$$x = 20 - 20 - 10$$

we have $20 - 20 = 0 \therefore$

$$x = -10,$$

a negative solution.

To explain the meaning of this, let us recur to the original statement,

$$b + x = a$$

or, $$30 + x = 20,$$

which expressed in ordinary language is, "To determine the number which, added to thirty, will produce a sum equal to twenty;" a problem manifestly impossible, twenty being less than thirty.

But now let us replace x by the value which the algebraical process gives for it, and the statement becomes $30 - 10 = 20$. So that the absolute or arithmetical value of the result obtained is a number which, *subtracted* from thirty, will give a *remainder* equal to twenty.

If the original problem be considered arithmetically, the negative solution indicates an inconsistency between the data and the conditions, and the necessity of a modification of one or both. But if it be considered algebraically, no such inconsistency exists; because here the term addition is taken in a larger sense, and includes the addition of negative quantities, which is arithmetical subtraction.

To determine the modification which is necessary to remove the inconsistency of a problem which gives a negative solution, it is only necessary to change the sign of x in the equation to which this problem is reduced, and then to translate the new equation into ordinary language. The necessity of employing negative quantities in algebraic investigations, has introduced a phraseology respecting them which, understood literally, seems absurd. A negative quantity as $-a$ is said to be *less than nothing;* and one negative quantity a being *numerically greater* than another $-b$, is said to be less than it. Thus -1 is said to be less than 0, and -3 less than -2. This phraseology is, however, to be considered rather conventional, and derived, by analogy, from the effects of arithmetical operations on positive and absolute numbers. It has, however, been necessary to adopt it in Algebra, in order to generalize the investigations and their results.

Simple Equations.

It is a general principle, that when one absolute quantity is subtracted from another, that other is diminished by the operation. Thus the operations represented by $5 - 1$, $5 - 2$, $5 - 3$, &c. have the effect of producing a constant diminution of the number 5. Now let this process be continued, the successive results are $5 - 4$, $5 - 5$, $5 - 6$, $5 - 7$, $5 - 8$, &c. In an arithmetical view, all the operations represented here after $5 - 5$ cannot be performed. But in Algebra it is necessary to perform them as far as can be done, and to represent by a certain symbol that part which cannot. Thus six units cannot be taken from five units; but five of the six can, and the remaining unit which cannot is represented by placing the negative sign before it thus, -1. In the same manner, $5 - 7$, $5 - 8$, &c. are represented by -2, -3, &c. Now as in absolute numbers the remainder diminishes as the subtrahend increases, the same property is extended analogically to those imaginary remainders which are the results of subtractions which cannot be executed; and we consider $5 - 5$ to be greater than $5 - 6$, and $5 - 7$ greater than $5 - 8$, &c.; that is, 0 is greater than -1, and -2 greater than -3, and so on.

This phraseology is not so inconsistent with the language used in the most ordinary affairs of life as it may at first appear. If we estimate the property of any individual, we first compute his actual possessions and the debts due to him; from these we subtract the debts which he owes, and the remainder may be considered as the value of his property. Now if it so happen, that the amount of his debts exceed the amount of his possessions, and the debts owing to him, we say that he is worth *less than nothing*. In this case, the result of the above-mentioned subtraction would be a negative quantity, and one of precisely that amount by which, in popular language, the individual in question is said to be poorer than he who neither has, nor owes a shilling.

In like manner, if the debts of A exceed his effects by a, and the debts of B exceed his effects by $a + b$, we say that A is richer or less poor than B. Now, in this case, the results obtained by subtracting the debts from the value of the effects in both cases are negative; but the value in the case of A is numerically *less* than in the case of B, although A is said to be *more* wealthy than B.

From these considerations we derive a method of expressing algebraically, that a quantity as a is positive or negative. If we wish to express that a is positive, we write $a > 0$, and if it be negative, we write $a < 0$.

SECTION XII.

Of Equations of the First Degree containing two or more unknown quantities.

(151.) In some of the examples given in the last section, more than one quantity was unknown, but in

Algebra. all the instances which occurred, there was such an obvious connection between the unknown quantities, that one unknown symbol signifying one of them, was by proper combination with the data made to express the other unknown quantities. Thus, in the problem (144,) one part of the line being x, it is known that the other part, which *à priori* may be considered equally unknown, is $15 - x$. But if this problem were at once treated as one involving two unknown quantities, we should consider the two parts as characterised by x and y, and we should have the equation

$$x + y = 15$$

by one condition, and

$$y = \frac{3}{4} x$$

by the other.

The examination of the following problem will lead us to the general principles by which questions involving two unknown quantities may be solved.

(152.) *Given the sum (a) and the difference (b) of two numbers, to find the numbers themselves.*

Let x and y be the numbers, we have, by the conditions of the problem, $x + y = a$, $x - y = b$. Since equal quantities added to equal quantities give equal results, we obtain, by adding these equations,

$$2x = a + b,$$

an equation which is independent of the unknown quantity y. This being divided by two, gives

$$x = \frac{1}{2}(a + b).$$

In like manner, subtracting the one from the other, we obtain $2y = a - b$, $\therefore y = \frac{1}{2}(a - b)$, and thus the values of the two unknown quantities are found, and we have established the following theorem.

"Of two unequal quantities the greater is equal to half the sum of their sum and difference, and the less is equal to half the difference of the sum and difference."

Upon examining the preceding process it will be found, that the contrivance by which the values of the unknown quantities have been determined, has been that of obtaining from the two given equations, each containing two unknown quantities, a single equation containing but one unknown quantity, and from this equation obtaining the value of that. This, being done with respect to each of the unknown quantities, will determine their values.

(153.) By generalizing the results, we shall obtain methods of solving all questions where two equations containing two unknown quantities are given.

After the proposed equations are cleared of fractions and radicals, as they cannot include any powers of the unknown quantities higher than the simple dimensions, they must have the forms

$$\left.\begin{array}{l} a x + b y = c \\ a' x + b' y = c' \end{array}\right\} [1]$$

a, b, c, a', b', c', being general representatives of any numbers positive or negative, which may happen to be the results of the reduction of the equations by the process of clearing them of fractions and radicals, or powers of the unknown quantities with fractional exponents. It should, perhaps, be here observed, that if an equation of two unknown quantities contain a term of which the product (xy) of the unknown quantities is a factor, it is accounted an equation of the second degree; since, although it contains no term in which the second *power* of either unknown quantity occurs as a factor, yet it contains a term in which the unknown quantities *combined* occur in two dimensions.

Simple Equations.

The process by which a single equation [1,] containing only one unknown quantity, is obtained from the two equations, is called *elimination*; and the unknown quantity which is made to *disappear*, is said to be *eliminated*. There are three methods by which this end is attained.

Elimination —three methods.

1. The first is the *method of addition* or *subtraction*. This method consists in equalizing the coefficients of the same unknown quantity in the two equations, by multiplying both members of each by such a number as will render the coefficients of the same unknown quantity in each equal. This is done on the same principle as that by which fractions are reduced to a common denominator. Let the least common multiple of the coefficients of the same unknown quantity be found, and let this be divided by the coefficient of that unknown quantity in each equation; the quotes will be the numbers by which it will be necessary to multiply the two equations in order to equalize the coefficients. Thus, if the equations be those of [1] the least common multiple of the coefficients of y is $a a'$; consequently the multipliers sought are a' and a, and when these are respectively multiplied into the two members of each equation we obtain

Method of addition or subtraction.

$$\left.\begin{array}{l} a a' x + b a' y = c a' \\ a a' x + b' a y = c' a \end{array}\right\} [2]$$

in which x has the same coefficient.

Again, if the equation be

$$6x + 8y = 50$$
$$8x + 6y = 48.$$

The least common multiple of 6 and 8 is 24, which divided by 6 and 8 gives 4 and 3. These, being multiplied by both members of each equation, give

$$24x + 32y = 200$$
$$24x + 18y = 144,$$

in which x has the same coefficient.

(154.) The same unknown quantity being by these means reduced to the same coefficient in both equations, the next step of the process is to subtract the one equation from the other, if this common term have the same sign in both, and to add them together if the common term have a different sign in the one and the other. In either case, the result of the process will be an equation containing but one unknown quantity. In the first case, the two equations will be of the form [2,] which, being subtracted, the latter from the former, give $(b a' - b' a) y = (c a' - c' a)$. [3.]

In the second case, the equations will be of the form

$$a a' x + b a' y = c a'$$
$$- a a' x + b' a y = c' a,$$

which, being added, give

$$(b a' + b' a) y = (c a' + c' a). \qquad [4.]$$

In every case, therefore, in which the coefficients of the same unknown quantity have been equalized, that unknown quantity may be eliminated by addition or subtraction, and an equation obtained, including only

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the remaining unknown quantity, the value of which may be found by the methods explained in the last Section.

Method of comparison.

2. The second method of elimination is called the *method of comparison*, which consists in bringing the same unknown quantity to stand alone as the first member of each equation; and thus the second member of each equation would include only the remaining unknown quantity. These second members being necessarily equal, since the first member is common, may be assumed as the two members of a new equation, which will therefore contain but one unknown quantity, and therefore the other unknown quantity is by these means eliminated.

Thus, in the equations [1,] the first being divided by a, and the second by a', we have

$$\left.\begin{array}{l} x + \dfrac{b}{a}\,y = \dfrac{c}{a} \\[2ex] x + \dfrac{b'}{a'}\,y = \dfrac{c'}{a'} \end{array}\right\}$$

$$\left.\begin{array}{l} \therefore\ x = \dfrac{c}{a} - \dfrac{b}{a}\,y \\[2ex] x = \dfrac{c'}{a'} - \dfrac{b'}{a'}\,y \end{array}\right\}$$

The second members of this latter system being assumed as the two members of the same equation, give

$$\frac{c}{a} - \frac{b}{a}\,y = \frac{c'}{a'} - \frac{b'}{a'}\,y$$

which, being cleared of fractions, becomes

$$c a' - b a' y = c' a - b' a y,$$

and the known and unknown quantities being brought to opposite sides, we have

$$b' a y - b a' y = c' a - c a',$$

or

$$(a b' - a' b)\,y = c a' - c' a,$$

which is the same with [3,] and would, if the sign of a' were negative, be the same as [4.] Thus these two methods lead precisely to the same results.

Method of substitution

3. The third method of elimination is called the *method of substitution*, and in principle is the same as the *method of comparison*, differing from it only in appearance. The method of substitution consists in bringing one of the unknown quantities in one of the equations to stand alone as its first member. The second member will, therefore, include only the other unknown quantity. This member is then *substituted* in place of the other unknown quantity in the second equation; by which substitution the second equation will contain only one unknown quantity, and therefore the elimination will be effected.

To apply this to the equations [1,] we have by the first

$$x = \frac{c}{a} - \frac{b}{a}\,y.$$

The second member being substituted for x in the second equation, it becomes

$$a\left(\frac{c}{a} - \frac{b}{a}\,y\right) + b\,y = c',$$

which, being cleared of fractions and reduced, becomes

$$(b' a - a' b)\,y = (c a' - a' c)$$

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which is the same as [3,] and if a' were negative would be the same as [4.]

(155.) All equations whatever of the first degree between two unknown quantities can be reduced to the forms

$$\left.\begin{array}{l} a x + b y = c \\ a' x + b' y = c' \end{array}\right\}\ [1.]$$

Hence it follows, that the solution of the equations [1] will furnish general formulæ by which the values of the unknown quantities in any given equations of the first degree may be computed. By the investigations already given, it appears that the values of x and y, derivable from the equations [1,] are

$$y = \frac{c a' - c' a}{b a' - b' a}$$

$$x = \frac{c b' - c' b}{a b' - a' b}.$$

By substituting in these formulæ the particular values of a, b, c, a', b', and c' in any proposed equations, the values of the unknown quantities may be at once obtained without further investigation.

Examples

(156.) The following example will illustrate these principles:

Two couriers depart in the same direction from two places on the same road, the distance between which is a, *one* A *goes* m *miles, and the other*, B, n *miles per hour. It is required to determine at what distances from the points of departure the one will overtake the other.*

Let x and y be the two distances. As these distances are travelled in the same time, we have

$$n x = m y,$$

and also

$$x - y = a.$$

Hence, by elimination, we obtain

$$x = \frac{a m}{m - n} \qquad y = \frac{a n}{m - n}.$$

If $m < n$, and therefore $m - n < 0$, these values for x and y will be negative. This indicates that the courier A can never overtake the courier B in the proposed direction, but that if they travel in the opposite direction, the courier B will overtake the courier A at the distances indicated by the values of x and y determined above.

(157.) It might happen that the values obtained for the unknown quantities from two given equations would be fractions, whose denominators are $= 0$. In this case the roots are said to be *infinite*, (121.) But the origin of such a result is always an absurdity or inconsistency in the two given equations. It will be easy to show this by the general formulæ [1.]

The condition under which the values of x and y derived from these equations are infinite, is

$$a b' - a' b = 0.$$

This gives

$$\frac{b}{a} = \frac{b'}{a'}. \qquad [2.]$$

Now if both members of the first of the equations [1] be divided by a, and of the second by a', they become

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$$\left.\begin{array}{l} x+\frac{b}{a}y=\frac{c}{a} \\ x+\frac{b'}{a'}y=\frac{c'}{a'} \end{array}\right\} [3.]$$

By the condition [2,] the first members of these equations are equal, whatever values be ascribed to x and y; and, therefore, unless the data be so related that the second members are also equal, the equations are inconsistent and contradictory.

In the same example, if $m = n$ the results will be infinite. In this case, the rates of travelling of the two couriers would be the same, and consequently the one would never overtake the other, and the condition of the question would be inconsistent with the data.

There are instances, however, in which these infinite results do contain the true solution of the problem. The student will find them occur frequently in our Treatise on ANALYTIC GEOMETRY.

(158.) If the second members of these equations were equal, as well as the first, it is evident that the two equations would be identical. The conditions under which this would take place would then be

$$\frac{b}{a}=\frac{b'}{a'}, \qquad \frac{c}{a}=\frac{c'}{a'},$$

from which we infer

$$ab'-a'b=0, \quad ca'-c'a=0.$$

Also, by these last equations, we obtain

$$\frac{a}{a'}=\frac{b}{b'}, \qquad \frac{a}{a'}=\frac{c}{c'},$$

$$\therefore \frac{b}{b'}=\frac{c}{c'} \quad \therefore cb'-c'b=0.$$

It therefore follows, that under these circumstances the values of x and y would assume the form $\frac{0}{0}$.

In this case there would be in effect but *one* equation for the determination of *two* unknown quantities, and the data would then be evidently insufficient for the solution of the problem. This will be easily perceived by substituting particular numbers for the general symbols. Let the equation be

$$2y+3x=50.$$

In this equation, any number whatever being substituted for x, a *corresponding number* may always be determined, which substituted for y will satisfy the equation. Let y be brought to stand alone as the first member, and we obtain

$$y=50-\frac{3}{2}x.$$

Now suppose $x = 2 \therefore$

$$y=25-3=22.$$

These two values, 22 and 2, being substituted for y and x in the proposed equation, it becomes

$$44+6=50$$

which is an identity.

Again, let any other value be substituted for x, as 5, we find

$$y=25-\frac{3}{2}.5=\frac{50-15}{2}=\frac{35}{2},$$

these two values $\frac{35}{2}$ and 5 being substituted for y and x in the original equation, give $35 + 15 = 50$, which is an identity.

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In like manner, any other value being ascribed to x, a corresponding value of y would be found, which would satisfy the equation.

To generalize this principle, in the equation

$$ax+by=c,$$

let any value y' be ascribed to y, so that the equation becomes

$$x+by'=c$$

$$\therefore x=\frac{c-by'}{a}$$

Substituting this value of y for y in the first equation, it becomes

$$a.\frac{c-by'}{a}+by'=c,$$

or

$$c-by'+by'=c,$$

or

$$c=c,$$

which is an identity.

Thus it appears, that there may be an infinite number of systems of values of two unknown quantities, each of which will equally satisfy the proposed equation, which, therefore, leaves the values of the unknown quantities indeterminate.

It may, therefore, be assumed generally, that when the values of the unknown quantities which result from two equations assume the form $\frac{0}{0}$, the two equations differ only in appearance, but are really one and the same, at least they are such that one may be *inferred* from the other. In this case, therefore, there is but one equation in reality between the two unknown quantities, and their values are indeterminate.

In the example (156) if $a = 0$ and $m = n$, the values of x and y would assume the form $\frac{0}{0}$, and the problem would be indeterminate. In this case the distance between the places of departure being $a = 0$, they would necessarily be the same. Also, m and n being equal, the couriers would travel at the same rate, and since they are supposed to move in the same direction they would necessarily keep always together. Hence, as the object of the problem is to assign the place at which they will be found together, every part of their road has in this case an equal claim to be considered as the point required. Hence the *indeterminateness* indicated by the form of the roots $\frac{0}{0}$.

There is, however, an exception to this principle; for it might so happen that the root assumed the form $\frac{0}{0}$, from having in both its numerator and denominator a common factor of the form $a - a$. The true value of the root would then be found by dividing both numerator and denominator by this common factor, (121.)

(159.) In order to determine the values of two unknown quantities, it is therefore necessary that there should be two *independent equations* between them; that is, two equations such that one cannot be inferred from the other. Two independent equations necessary.

(160.) Three or more independent equations would be more than sufficient data for the determination of two unknown quantities, and the result would be, that different and inconsistent values of the same unknown

quantity would be obtained from each pair of equations.

(161.) It might happen that the values obtained for the unknown quantities would be $= 0$. To determine the circumstances under which this could happen, it is only necessary to consider under what circumstances the formulæ

$$y = \frac{cb' - bc'}{ab' - ba'} \quad x = \frac{ac' - ca'}{ab' - ba'}$$

shall become $= 0$. That this should happen, it is necessary that

$$cb' - bc' = 0 \qquad ac' - ca' = 0,$$

but that $ab' - ba'$ *should not* $= 0$, because in that case the values of x and y would assume the form $\frac{0}{0}$.

Let c be eliminated by the preceding equations, and the result is

$$\frac{c'}{b'}\,(ab' - ba') = 0.$$

Now since $ab' - bb'$ cannot $= 0$, we must have $c' = 0$, and in like manner it can be proved that $c = 0$. Hence the form of the equations must be

$$ax + by = 0$$
$$a'x + b'y = 0.$$

The number of equations should be equal to that of the unknown quantities.

(162.) The principle by which one unknown quantity is eliminated by two equations may be generalized. If several equations of the first degree be given, including several unknown quantities, any one of these unknown quantities may be made to stand as the first member of any one of the equations,—the other terms being all transferred to the other member, by the methods already explained. The second member may then be substituted for the unknown quantity which stands alone in the first member, in all the other equations. One equation, therefore, has served to eliminate one unknown quantity from all the other equations, and the number of equations, as well as that of unknown quantities, is thus diminished by one. The same process may be repeated with another equation and another unknown quantity, and the number of equations and of unknown quantities will then be diminished by two; and so the process may be continued. If the number of unknown quantities be equal to the number of independent equations, it is clear that by eliminating all the unknown quantities but one, we shall also have reduced the number of equations to a single one. This single equation will determine the value of the remaining unknown quantity.

Consequences if not.

But if the number of equations were less than the number of unknown quantities, after reducing the number of equations to a single one, the number of unknown quantities remaining in it would be two or more, and it would therefore be insufficient to determine their values, and the problem would be indeterminate. But, on the other hand, if the number of unknown quantities be less than the number of equations, after reducing their number by elimination to one, more than one equation would remain, and the results would be contradictory if the the given equations were independent.

Examples.

(163.) We shall give one or two examples:

1. *How many times do the hands of a watch coincide between noon and midnight, on the supposition that there is only an hour hand and a minute hand; and what are the exact moments of their coincidence. Also, what would be the number of coincidences of three hands moving on the same centre, an hour, minute, and second hand, and what would be the exact moment of their coincidence?*

2. *A number is composed of three digits, of which the sum is given. The digit in the unit's place is* m *times that in the hundred's place; and on adding a given number consisting of three digits to the sought number, the digits will be reversed. Investigate a general formula for the solution of this class of problem, and apply it to the case where the sum of the digits is* 11, m = 2, *and where the number added is* 297.

3. *A sum of* £100,000. *is placed at interest, one part at* 5 *per cent., another at* 4 *per cent. The total interest is* £4640.; *it is required to assign the proportions which are placed at each rate.*

4. *Three persons,* A, B, C, *have certain sums which they place at interest.* B *and* C *have each given numbers of pounds more than* A. *The rates of interest of* B *and* C *exceed that of* A *by given sums; and also the revenues of* B *and* C *exceed that of* A *by given sums. It is required to determine the capitals of* A, B, *and* C, *and also the rates of interest they respectively receive.*

Three unknown quantities

(164.) We have already proved that all equations of the first degree between two unknown quantities may be reduced to the general forms, (153,)

$$ax + by = c$$
$$a'x + b'y = c'.$$

The same reasoning by which this was established will likewise prove the equations between three unknown quantities may each be reduced to the form

$$ax + by + cz = d,$$

and as in every determinate problem there must be three of these equations, they may be represented thus:

$$ax + by + cz = d$$
$$a'x + b'y + c'z = d'$$
$$a''x + b''y + c''z = d''.$$

It is evident how these observations may be extended to any number of equations between the same number of unknown quantities.

It should be observed, that it is by no means necessary that *all* the unknown quantities engaged in the problem should occur in each equation; and although they appear to do so in the above general formulæ, yet, as it is supposed that any one or more of the general coefficients $a', a'' \ldots b', b'' \ldots$ &c. may be $= 0$, they are not so restricted. These general coefficients are, in fact, the aggregates of the coefficients of each unknown quantity, in any particular question, after the equations have been cleared of fractions and reduced, as explained in (139.)

(165.) Rules may be assigned and established by which, when any number of equations of the first degree between the same number of unknown quantities are given, the values of these unknown quantities may be severally obtained without the usual process of elimination, or any other preparatory investigation.

If there be but one unknown quantity, the equation may always be reduced to the form

$$ax = b,$$

a being the algebraic sum of all the coefficients of the unknown quantity, and b the algebraic sum of the

terms which have no unknown factor. The general formula for x in this case is obviously

$$x = \frac{b}{a}.$$

We have already shown that when there are two equations with two unknown quantities, the formula for their values are

$$x = \frac{c b' - b c'}{a b' - b a'}, \qquad y = \frac{a c' - c a'}{a b' - b a'}.$$

The rule by which these formulæ may always be found is as follows:

1. They have a common denominator. With the letters a and b, which express the coefficients of x and y, form the two arrangements $a\,b$ and $b\,a$, and place between them the sign $-$, and place an accent on the last factor of each term. Thus we first write

$$a b - b a,$$

and then placing the accents we have

$$a b' - b a',$$

which is the common denominator.

2. To determine the numerator of the value of each unknown quantity, substitute for the letter expressing the coefficient of that unknown quantity in the denominator (already found) the absolute quantity c, and preserve the accents as before. Thus, to determine the numerator of the value of x, we change a in the common denominator into c, and the result is

$$c b' - b c';$$

and to obtain the numerator of y we change b into c, and obtain $\quad a c' - c a'$.

(166.) Let us now consider the general formulæ for the values of three unknown quantities derived from the equations of (164.)

Let z be eliminated, by multiplying the first equation by c', and the second by c, and subtracting the one from the other. The result is

$$(a c' - c a') x + (b c' - c b') y = d c' - c d'.$$

In like manner, eliminating z by the second and third, we obtain

$$(a' c'' - c' a'') x + (b' c'' - c' b'') y = d' c'' - c' d'';$$

eliminating y by these two equations, by the usual methods, we obtain

$$[(a c' - c a')(b' c'' - c' b'') - (a' c'' - c' a'')(b c' - c b')] x =$$
$$(d c' - c d')(b' c'' - c' b'') - (d' c'' - c' d'')(b c' - c b')$$

Developing the several products, and dividing by the common factor c', and arranging the factors of each term in the order of the accents, the equation becomes

$$(a b' c'' - a c' b'' + c a' b'' - b a' c'' + b c' a'' - c b' a'') x =$$
$$d b' c'' - d c' b'' + c d' b'' - b d c'' + b c' d'' - c b' d''.$$

Whence we obtain

$$x = \frac{d b' c'' - d c' b'' + c d' b'' - b d' c'' + b c' d'' - c b' d''}{a b' c'' - a c' b'' + c a' b'' - b a' c'' + b c' a'' - c b' a''};$$

and by a similar process we obtain

$$y = \frac{a d' c'' - a c' d'' + c a' d'' - d a' c'' + d c' a'' - c d' a''}{a b' c'' - a c' b'' + c a' b'' - b a' c'' + b c' a'' - c b' a''};$$

$$z = \frac{a b' d'' - a d' b'' + d a' b'' - b a' d'' + b d' a'' - d b' a''}{a b' c'' - a c' b'' + c a' b'' - b a' c'' + b c' a'' - c b' a''}.$$

(167.) The last two formulæ might be deduced from the first by the *symmetrical* nature of the proposed equations. It is evident, if in the three original equations of (164) the letters x, a, a', a'' were changed into y, b, b', b'', or in z, c, c', c'', and *vice versâ*, the equations would remain unchanged. Hence we are authorized to make similar changes in the formulæ which are deduced from these equations. If, then, in the formula for x, the letters x, a, a', a'' be changed into y, b, b', b'', and *vice versâ*, we shall obtain the formula for y; and by changing x, a, a', a'' into z, c, c', c'', and *vice versâ*, we shall obtain the formula for z. This principle will be found of very extensive use in analysis.

(168.) The preceding formulæ for x, y, and z, like the former, have a common denominator, and may be found by the following rule:

1. To form the common denominator, write the denominator $(a b' - b a')$ in the case of two unknown quantities without the accents, thus

$$a b - b a;$$

introduce the letter c in all possible positions in each of the terms $a\,b$ and $b\,a$; that is, last, middle, and first; and write the successive results one after another, affecting them alternately with the signs $+$ and $-$. The result will be

$$a b c - a c b + c a b - b a c + b c a - c b a;$$

accenting the second factor of each term with $'$, and the third with $''$, the formula becomes

$$a b' c'' - a c' b'' + c a' b'' - b a' c'' + b c' a'' - c b' a'',$$

which is the common denominator.

2. To form the numerator of the formula for each unknown quantity, it is only necessary to substitute for the letter expressing its coefficient in the denominator the absolute term d, and to preserve the accents. Thus, to determine the numerator of the value of x, it is only necessary to change a into d, and the result is

$$d b' c'' - d c' b'' + c d' b'' - b d' c'' + b c' d'' - c b' d'',$$

and similarly for y and z.

(169.) The *law* by which the arrangement of the terms of these formulæ is governed appears upon inspection, and may be extended to the cases of four or more unknown quantities. A general demonstration of the law has been given by Laplace, in the proceedings of the Institute for the year 1772. It is, however, of too complicated a nature to be properly inserted here.

(170.) The values of the unknown quantities deduced from any system of equations must be either positive, negative, $= 0$, of the form $\frac{A}{0}$, or of the form $\frac{0}{0}$.

If the value we obtain for an unknown quantity be positive, it is *generally* a value which solves the problem which was reduced to the proposed equations. It is not, however, *always* so. The equation, or the system of equations, is not always the exact translation of the proposed problem into the language of Algebra. There are frequently some peculiar conditions in the proposed problem, which the analyst is obliged to omit from their not being of a nature to allow of being expressed in an equation. The problem which is expressed by the equations is therefore more *general* than the problem from which the equations are deduced;

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and the roots of it, from the peculiar values of the data, may happen to be of such a nature, that they are inconsistent with those conditions of the problem which are not expressed in its algebraical statement. Thus, suppose that the problem was such that the sought number must, from its nature, be an integer, but that the data were such that the result of the equation gave it a fractional value. This value is a true and full *solution of the equation*, but it is not a *solution of the problem* from which the equation was deduced. The cause of which is, that the condition that the root should be integral was not expressed in the equation, and the result indicates that the data of the proposed problem are inconsistent with that condition. Instances of this will be seen hereafter.

(171.) If the values obtained for any of the unknown quantities be *negative*, a modification of the original problem is suggested, as has been already explained in the case of a single unknown quantity. The modifications thus suggested may be determined by recurring to the original equations, and changing in them the signs of those unknown quantities which are negative. As this is determined on the same principles as in the case of a single unknown quantity, it will be unnecessary here to enter further upon the subject.

The observations already made on the other peculiar forms *scil.*, 0, $\frac{A}{0}$, and $\frac{0}{0}$, in the cases of one and two unknown quantities, are also applicable to the results of equations of several unknown quantities.

SECTION XIII.

Of Equations of the Second Degree.

(172.) After an equation which results from the conditions of a problem expressed algebraically has been reduced in the manner explained in (139,) the result, if it be an equation of the second degree, must have the form

$$A x^2 + B x = C.$$

As the coefficients A and B are respectively the algebraical sums of the several coefficients of x^2 and x, and C the algebraical sum of those terms not affected by x as a multiplier, it follows in general that A, B, and C may have any values positive, negative, or $= 0$. But it should be observed, that if $A = 0$ the equation is no longer of the second degree; this case we shall therefore omit in the consideration of these general equations. If the equation be divided by A, and that we suppose

$$\frac{B}{A} = p, \quad \frac{C}{A} = q,$$

it becomes

$$x^2 + p x = q,$$

where, as before, p and q may each be positive, negative, or $= 0$.

If $p = 0$, the form of the equation becomes

$$x^2 = q.$$

This form is sometimes called a *pure quadratic* equation, and by some authors an *incomplete quadratic* equation.

If p be not $= 0$, the equation is called a *complete* or *affected quadratic* equation.

Quadratic Equations.

The square roots of both members of the former being taken (138) we have

$$x = \pm \sqrt{q}.$$

If q be a number, this is done by the rules of ordinary arithmetic. If q be a simple algebraical quantity, its root, when it has one, may be obtained by the principles established in Section VI. If it be a complex algebraical quantity, the method of obtaining the root will be explained in a subsequent section.

It may be observed, generally, that if $q > 0$, there will be two values of x whose arithmetical value is the same, but whose algebraical values have different signs, (66.) If $q < 0$, there is no arithmetical value of x, and its algebraical values are *imaginary*, (68.)

(173.) The method of solving a complete equation of the second degree is deduced from a comparison of its first member with the form for the square of a binomial, the first term of which is x. Let

$$x + a = b;$$

squaring both members, we have

$$x^2 + 2 a x + a^2 = b^2. \qquad [1.]$$

This is evidently a complete equation of the second degree, and may be solved by taking the square roots of both members. Upon comparing it with the form

$$x^2 + p x = q, \qquad [2]$$

they are found to differ only in this, that there is an absolute term (a^2) in the first member of the former which does not appear in the latter. This term is the square of half the coefficient of x in the former. We are, however, allowed to add the same known quantity to both members of an equation without disturbing their equality. Hence, the first members of the two equations will be assimilated, as to their form, by adding to both members of the latter the square of half of the coefficient p; that is, $\frac{p^2}{4}$. By this change it becomes

$$x^2 + p x + \frac{p^2}{4} = \frac{p^2}{4} + q,$$

or

$$x^2 + 2 \cdot \frac{p}{2} x + \frac{p^2}{4} = \frac{p^2}{4} + q. \qquad [3.]$$

The first member here becomes identical with that of [1,] by changing $\frac{p}{2}$ into a. Hence it is easily seen that the first member of [3] is the square of

$$x + \frac{p}{2}.$$

General formulæ for solution.

Taking, then, the square roots of both members of [3,] we obtain

$$x + \frac{p}{2} = \pm \sqrt{\frac{p^2}{4} + q}$$

$$\therefore x = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} + q}.$$

Hence we derive a general rule by which the value of x in an equation of the second degree may at once be obtained.

"Let the equation be first reduced to the form [2,] (which if it be a quadratic equation it always can;) the value of x will be found by taking the coefficient of x (p), changing its sign, and dividing it by 2, and

adding to it, or subtracting from it, the square root of the quantity, which is the algebraic sum of the square of the half coefficient $\left(\frac{p^2}{4}\right)$ and the absolute quantity (q)."

Hence it will be observed, that a quadratic equation always has two roots, inasmuch as the radical is susceptible of two signs.

Properties of the roots.

(174.) We shall now proceed to consider some general properties of the roots of equations of the second degree.

After modifying the formula

$$x^2 + p\,x = q; \qquad [1]$$

by the addition of $\frac{p^2}{4}$ to both members, we obtained

$$\left(x + \frac{p}{2}\right)^2 = \frac{p^2}{4} + q.$$

Let the second member of this equation be called m^2, so that

$$\left(x + \frac{p}{2}\right)^2 = m^2,$$

or

$$\left(x + \frac{p}{2}\right)^2 - m^2 = 0,$$

or

$$\left(x + \frac{p}{2} + m\right)\left(x + \frac{p}{2} - m\right) = 0. \quad [2.]$$

The first member of this equation is the product of two factors, and the second member is 0. Now it is evident that a product will become equal 0 when either of its factors = 0. Hence the last equation will always be fulfilled by the condition expressed by either of the following equations,

$$x + \frac{p}{2} + m = 0,$$

$$x + \frac{p}{2} - m = 0,$$

or

$$x = -\frac{p}{2} - m,$$

$$x = -\frac{p}{2} + m;$$

or, if m be replaced by its value,

$$x = -\frac{p}{2} - \sqrt{\frac{p^2}{4} + q},$$

$$x = -\frac{p}{2} + \sqrt{\frac{p^2}{4} + q}.$$

Since then the equation [1,] or its equivalent [2,] can only be fulfilled by one or other of the factors of [2] being = 0, it follows, "That an equation of the second degree admits of two roots, but not of more."

(175.) If the equation [1] be reduced to the form

$$x^2 + p\,x - q = 0, \qquad [3]$$

its first member must be equivalent to that of [2.] Let x' x'' be the roots of this equation. It is evident that we have

$$x' = -\frac{p}{2} - m,$$

$$x'' = -\frac{p}{2} + m;$$

and by this [2] becomes

$$(x - x')\,(x - x'') = 0;$$

the first member of which being equivalent to that of [3] gives

$$x^2 + p\,x - q = (x - x')\,(x - x''). \qquad [4.]$$

The following identity

$$x^2 + p\,x - q = \left(x^2 + p\,x + \frac{p^2}{4}\right) - \left(\frac{p^2}{4} + q\right)$$

is equivalent to

$$x^2 + p\,x - q = \left(x + \frac{p}{2}\right)^2 - \left(\frac{p^2}{4} + q\right).$$

From which we immediately infer

$$x^2 + p\,x - q =$$

$$\left(x + \frac{p}{2} + \sqrt{\frac{p^2}{4} + q}\right)\left(x + \frac{p}{2} - \sqrt{\frac{p^2}{4} + q}\right),$$

or

$$x^2 + p\,x - q = (x - x')\,(x - x'').$$

Product of roots.

(176.) By developing the product which forms the second member of the identity [4,] we obtain

$$x^2 + p\,x - q = x^2 - (x' + x'')\,x + x'\,x''.$$

As this has been proved true, whatever value be ascribed to x, let x be supposed = 0. Hence we obtain

$$-\,q = x'\,x''.$$

Subtracting this from the former, we obtain

$$x^2 + p\,x = x^2 - (x' + x'')\,x;$$

and dividing by x, and omitting the common term, we have Sum of roots.

$$+\,p = -\,(x' + x'').$$

Hence we infer, that in a quadratic equation reduced to the form [1,] "The absolute quantity (q) with its sign changed is equal to the product of the roots; and the coefficient (p), with its sign changed, is equal to their sum."

It will be easy to verify these results by actual addition and multiplication.

(177.) The roots of a quadratic equation are rational or irrational, according as the quantity under the radical is an exact square or not. If it be not an exact square, and the equation be numerical, the values of x', x'' may be obtained with any degree of approximation which may be required in rational numbers by the arithmetical rules for the extraction of the square root. If the equation, however, be literal, there is no other way of signifying the root when the quantity under the radical is not an exact square than by the radical itself, or by the equivalent notation of fractional exponents already explained. When rational.

(178.) If the quantity under the radical be negative, the radical, and therefore the roots of each of which it is a part, will be imaginary, (68.) Of the two terms under the radical, one is always positive, being the square of $\frac{p}{2}$, a quantity supposed to be real. Hence, in order that the suffix of the radical be negative, two things are necessary: 1. that the absolute quantity (q) be negative, and, 2. that it be greater than the square of the half coefficient $\left(\frac{p^2}{4}\right)$. It is under these conditions only that the roots will be imaginary; and since the *same radical* enters both roots, they must always be both real or both imaginary together. When imaginary

From the signs and values of the coefficient and

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absolute quantity, it may, therefore, be always determined whether the roots be real or imaginary.

Signs determined.

(179.) The signs of the roots, when real, may be at once deduced from the properties already established; and from the principle that if a product of two factors be positive, its factors will have the same sign, and if it be negative, they will have different signs.

Hence, since the product of the roots has always a different sign from the absolute quantity (q.) (176.) it follows, that when the absolute quantity is negative in [1,] the roots have the same sign, and when it is positive they have different signs.

(180.) When two quantities have the same sign, their common sign is that of their algebraical sum; and when they have different signs, the sign of the greater is that of their algebraical sum. Hence, when the roots have the same sign, that sign will be different from the sign of the coefficient (p,) and when they have different signs, the sign of the lesser root will be that of the coefficient (p) (176.)

(181.) As p and q may be each positive or negative, the general formula [1] includes under it the four following cases: 1. $x^2 + p\,x = +q$; 2. $x^2 - p\,x = +q$; 3. $x^2 \quad p\,x = -q$; 4. $x^2 + p\,x = -q$. By what has been just established, it follows, that the roots in the first two formulæ, first, are always real; secondly, that they have different signs, the root whose arithmetical value is greater being negative in the first, and positive in the second. Also, that the roots in the last two formulæ, first, are real or imaginary, according as $\frac{p^2}{4}$ is greater or less than q; and, secondly, that when they are real they are both positive in the third formula, and both negative in the fourth.

(182.) When quantities have the same sign, their algebraical sum is also their arithmetical sum, and when they have different signs, their algebraical sum is their arithmetical difference. Hence it follows, that in the first two of the above formulæ, the coefficient p is the arithmetical difference of the roots, and in the last two, it is their arithmetical sum. The first two formulæ, therefore, if interpreted in ordinary language, become "Given the difference of two numbers, and their product, to determine the numbers themselves;" and the last two, "Given the sum of two numbers, and their product, to determine the numbers themselves." To one or other of these classes, every problem which produces a quadratic equation can, therefore, be ultimately resolved.

Difference of roots.

(183.) To obtain the formula for the algebraic difference of the roots of an equation of the second degree, let the values of x' be subtracted from that of x'':

$$x'' = -\frac{p}{2} + \sqrt{\frac{p^2}{4} + q}$$

$$x' = -\frac{p}{2} - \sqrt{\frac{p^2}{4} + q}$$

$$x'' - x' = 2\sqrt{\frac{p^2}{4} + q}$$

Twice the radical is, therefore, the difference of the roots, and is positive or negative, according to the manner in which the subtraction is performed.

In order that the roots may be equal, it is, therefore, necessary that the suffix of the radical $= 0$, and this can only happen when the absolute quantity (q) is negative, and equal to the square of the half coefficient. In that case, the value of each root will be the half coefficient with its sign changed. This may be easily verified.

Quadratic Equations.

(184.) If $q = o$, the expressions for the roots become

$$x' = -\frac{p}{2} - \frac{p}{2} = -p,$$

$$x'' = -\frac{p}{2} + \frac{p}{2} = 0;$$

one of the roots being equal to the coefficient with its sign changed, and the other being $= 0$. This might also be inferred from q being the product of the roots. If a product $= 0$, one of its factors must $= 0$, and therefore one of the roots must $= 0$. The sum of the roots $(-p)$ will then be equal to the other root. It will be seen, hereafter, that this is only a particular case of a much more general principle.

(185.) In considering the case of *pure*, or *incomplete*, equations of the second degree, we have already disposed of the case in which $p = 0$.

If $p = 0$, and also $q = 0$, both roots are $= 0$; for since their product $= 0$, one of them at least must $= 0$, but since their sum also $= 0$, the other must $= 0$.

(186.) There is a case which frequently occurs in algebraical investigations, to explain which we must recur to the original form in which we expressed (172) an equation of the second degree:

$$a\,x^2 + b\,x = c.$$

This equation being solved by the general rule gives

$$x = \frac{-b \pm \sqrt{b^2 + 4\,a\,c}}{2\,a}.$$

If we now suppose that $a = 0$, the values of x become

$$x = \frac{-b \pm b}{0}.$$

If the upper sign be taken, we have

$$x = \frac{-2\,b}{0}$$

and for the lower sign,

$$x = \frac{0}{0},$$

the one being a symbol of infinity, and the other indeterminate.

To trace the circumstance which gave rise to these results, it is only necessary to determine, what effect the hypothesis $a = 0$ would produce upon the primitive equation. It is evident that it would reduce it to the form $b\,x = c$. The division by a, which was effected preparatory to the solution as a quadratic equation, involved a distinct, though implicit, condition, that the value of a was not $= 0$. The condition that $a = 0$, *subsequently* introduced, contradicts this, and hence the absurdity of the results.

This process is what is called shifting the hypothesis, and is too often used by analytical writers, who attempt to account for the results obtained, and to give them a meaning, notwithstanding the evident sophistry and invalidity of the process by which they were obtained

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In the present instance it is evident, since the original equation becomes $b\,x = c$ when $a = 0$, that x has but one value, and that is

$$x = \frac{c}{b}.$$

If in this case $b = 0$, the equation becomes $0 = c$, which is absurd, if c be not $= 0$, and if $c = 0$, it becomes an useless identity.

It is, perhaps, worth observing, that if a, b, and c all $= 0$, the equation $a\,x^2 + b\,x = c$ will be necessarily true, whatever value may be ascribed to x. The problem is in this case indeterminate, and the equation is said to be "satisfied by its coefficients."

SECTION XIV.

Of Inequalities.

(187.) An *inequality* is a proposition which expresses algebraically, that one quantity is greater or less than another. Inequalities are therefore of two kinds, and must be expressed in either of the following forms,

$$A > B$$
$$A < B,$$

according as the first member is greater or less than the second.

In an equality it is a matter of indifference on which side of the sign $=$ either member is placed. It is otherwise with an inequality; for if it be necessarily true in one position, it will be evidently false when the members are transposed. If, however, at the same time that the members are transposed, the sign of inequality be reversed, the transposition is valid, and the statement continues true. Thus, if $A > B$, $\therefore B < A$; and if $A < B, \therefore B > A$, which is evident from the meaning of the symbols.

(188.) Several of the changes allowable on equalities are also allowable on inequalities. Thus, quantities which are algebraically equal, may be added to, or subtracted from both members of an inequality. It is evident, that if $A > B$, $\therefore A + C > B + C$, and $A - C > B - C$. In executing these transformations it should, however, be remembered, that of two negative quantities that which is *numerically* less is *algebraically* greater.

(189.) Hence a quantity may be transferred from one member of an inequality to the other, provided that its sign be changed; for this is the same as subtracting it algebraically from both members. Thus, if $A > B + C$, $\therefore A - C > B$; and if $A > B - C$, $\therefore A + C > B$.

(190.) Hence we may infer, that if the signs of both members of an inequality be changed, the species of the inequality must also be changed. For if $A > B$, $\therefore A - B > 0$ by (189,) $\therefore - B > - A$, by (189,) or $- A < - B$ by (187.)

(191.) Both members of an inequality may be multiplied by the same positive quantity; but if they be multiplied by the same negative quantity, the species of inequality will be changed.

For since products having a common factor are in the same ratio as the factors not common, the *numerical* inequality of both members will remain of the same species, whether the multiplier be positive or negative. If the multiplier be positive, the signs of both members remain unchanged, and therefore the species of inequality remains the same; but if the multiplier be negative, the signs of both members are changed, and therefore the species of inequality must be changed. Thus, if both members of $A > B$ be multiplied by $+ C$, we have $A\,C > B\,C$; but if they be both multiplied by $- C$, the effect is the same as if they were first multiplied by $+ C$, and the signs then changed. The first result would be $A\,C > B\,C$; and changing the signs we should have by (187) $- A\,C < - B\,C$.

Inequalities.

(192.) The same principles exactly, will authorize us to divide both members of an inequality by the same positive or negative quantity under similar restrictions.

(193.) The corresponding members of inequalities of the same species may be added one to another. Thus, by adding $A > B$, $A' > B'$, we obtain $A + A' > B + B'$.

The validity of this inference may be easily established. It is evident, that the quantity which it is necessary to add to the lesser member of an inequality, in order to convert it into an equality, must be positive. Hence m and m' will be positive quantities in the equalities

$$A = B + m,$$
$$A' = B' + m'.$$

These being added give

$$(A + A') = (B + B') + (m + m').$$

Since m and m' are both positive their sum is positive, hence $A + A' > B + B'$.

(194.) A similar principle, however, is not true as respects the subtraction of similar inequalities. It does not follow, that if $A > B$, $A' > B'$, that $A - A' > B - B'$. For, as before, let $A = B + m$, $A' = B' + m'$, $\therefore (A - A') = (B - B') + (m - m')$.

The quantity $m - m'$ may be either positive or negative. If it be positive, we have $A - A' > B - B'$, and if it be negative, $A - A' < B - B'$.

(195.) Both members of an inequality may be raised to the same power, or the same roots may be extracted, observing the condition, that if in the process of involution or evolution the signs of the members be preserved, the species of the inequality is also to be preserved; but if the signs be changed, the species of inequality is also to be changed.

(196.) It is evident, that the sign of the greater member of an inequality if negative may be made positive, and the lesser member if positive may be made negative, because by this process the former is algebraically increased, and the latter algebraically diminished.

(197.) For the same reason any positive quantity may be added to the greater member, or subtracted from the lesser, and any negative quantity may be added to the lesser member, or subtracted from the greater.

SECTION XV.

On the changes in sign of a rational and integral formula of the first or second degree, produced by changes in the value ascribed to the unknown or variable quantity in it.

(198.) When an algebraical formula contains a

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quantity which is unknown or indeterminate, combined by given operations with other quantities which are given, it is said to be a *rational formula* when the unknown or indeterminate quantity is not, either by itself or in combination with other quantities, affected by a radical or a fractional exponent. It is likewise said to be integral when the unknown quantity, either by itself or in combination with other quantities, is not found in the denominator of any fraction, or affected by a negative exponent. The *degree* of the formula, like that of an equation, is decided by the highest integral exponent. Every rational and integral formula of the first degree must, therefore, have the form $A x + B$, and every rational and integral formula of the second degree must have the form $A x^2 + B x + C$.

The general symbols A, B, C being supposed to represent given quantities, it follows that the values of these formulæ will entirely depend on the values which may be ascribed to the unknown or indeterminate quantity x. We propose in this Section to determine how the signs of the quantities represented by these formulæ, depend on the values which may be ascribed to x, and to distinguish what values of x will render them positive or negative.

This may be considered as a more general investigation than the solution of equations which is the determination of the values of x, which render these formulæ $= 0$.

The formula of the first degree presents no difficulty. It may obviously be expressed in the form $A\left(x + \frac{B}{A}\right)$ Let the value of x, which renders it $= 0$, be x'. We then have $x' = -\frac{B}{A}$, and the original formula by this substitution becomes $A(x - x')$. This being the product of two factors, its sign will be $+$ or $-$, according as its factors have like or unlike signs. Hence if $A > 0$,* all values of $x > x'$ render the formula > 0, and all values of $x < x'$ render it < 0. If $A < 0$, all values of $x > x'$ render the formula < 0, and all values of $x < x'$ render it > 0. Hence we find

$$A x + B > 0 \text{ if } \begin{cases} A > 0 \text{ and } x > -\frac{B}{A}. \\ A < 0 \text{ and } x < -\frac{B}{A}. \end{cases}$$

$$A x + B < 0 \text{ if } \begin{cases} A > 0 \text{ and } x < -\frac{B}{A}. \\ A < 0 \text{ and } x > -\frac{B}{A} \end{cases}$$

$$A x + B = 0 \text{ if } x = -\frac{B}{A}.$$

Hence, if x be supposed to assume all possible values from an unlimitedly great positive value decreasing to 0, and then to pass through all negative values from 0 to an unlimitedly great negative value, the formula $A x + B$ becoming $= 0$, when $x = -\frac{B}{A}$ will be positive for all values on the one side of this, and negative for all those on the other side of it. The formula may thus be conceived to change its sign in passing through zero, and constantly to maintain the same sign, while x is on the same side of the value which renders the formula $= 0$, so that throughout the whole variation of x the formula suffers but one change of sign.

Sign of an Integral Formula.

This, however, is not the case with any other rational and integral formula. In the formula

$$A x^2 + B x + C,$$

let the values of x which render this $= 0$, be x' and x''. We have (175)

$$A x^2 + B x + C = A(x - x')(x - x'').$$

The quantities x' x'' are subject to all the circumstances incident on the roots of an equation of the second degree: they may be, 1. real and unequal; 2. real and equal; 3. imaginary. We shall consider successively these cases.

(199.) 1°. If the quantities x' x'' be real and equal, the formula

$$A x^2 + B x + C,$$

or its equivalent

$$A(x - x')(x - x'')$$

is the product of three factors. If two of these have the same sign, the sign of the product will be that of the third factor; and if two have opposite signs, the sign of the product will be different from that of the third factor. Of the two roots x' and x'' (being unequal) let $x' > x''$. If a value be ascribed to x which is between the values of the roots, that is, greater than the lesser root and less than the greater root, the factors $x - x'$ and $x - x''$ will have different signs, and therefore the sign of the whole formula will be different from the sign of A; but if the value ascribed to x be beyond the limit of either root, that is, if it be greater than the greater root or less than the lesser root, the signs of the factors $x - x'$ and $x - x''$ will be the same, and the sign of the whole formula will be that of A.

Thus it appears, that while continually increasing values are ascribed to x, from negative infinity to positive infinity, the formula of the second degree suffers two changes of sign in passing twice through zero; that for the values of x between those which render it equal to zero, it is > 0 when $A < 0$, and < 0 when $A > 0$; and that for all values of x beyond the limits of the roots on either side, it is continually > 0 or < 0, according as $A > 0$ or < 0.

(200.) 2°. If the roots x', x'' be equal, the formulæ is reduced to $A(x - x')^2$, x' expressing the common value of the two roots. In this case the factor $(x - x')^2$ is essentially positive, whatever be the sign of $x - x'$, except when $x = x'$, when it $= 0$. Hence for all values of x whatever, except that particular value x', which renders the formula $= 0$, the sign of the formula will be that of A.

(201.) It may be observed, that in this case the formula is a perfect square. For the condition on which the equality of the roots x', x'' depends is, that the suffix of the radical should $= 0$. And this gives $B^2 - 4 A C = 0$,

$$\therefore B = 2\sqrt{A C},$$

which being substituted in the original formula it becomes

$$A x^2 + 2\sqrt{A C}\, x + C,$$

which is equivalent to

$$(\sqrt{A} \cdot x + \sqrt{C})^2.$$

* It should be carefully observed, that $>$ and $<$, and the terms greater and less, mean *algebraically* greater or less, and not *arithmetically*, see Sect. XIV.

(202.) It is evident also that this condition can only be fulfilled when A and C have the same sign. For if they had different signs, 4 A C would be essentially negative, and therefore $B^2 - 4AC$ would be the sum of two quantities essentially positive, and could not $= 0$.

(203.) 3°. If the roots x', x'' be imaginary, there are no real values of x which render the formula $= 0$. In this case the sign of the formula must be otherwise determined. Any real value being ascribed to x, let the corresponding value of the formula be y, so that

$$A x^2 + B x + C = y,$$

$$\therefore x^2 + \frac{B}{A} x + \frac{C}{A} = \frac{y}{A},$$

which, being solved for x, gives

$$x = -\frac{B}{2A} \pm \sqrt{\frac{B^2 - 4AC}{4A^2} + \frac{y}{A}},$$

$$\therefore x = \frac{-B \mp \sqrt{B^2 - 4AC + 4Ay}}{2A}.$$

Since, by hypothesis, in the present case, the values x', x'' are imaginary, it is necessary that $B^2 - 4AC < 0$. But also it is supposed that the values of x are real.

Hence $B^2 - 4AC + 4Ay > 0$;

and since $B^2 - 4AC < 0$, we have

$$4Ay > 0,$$

$$\therefore Ay > 0.$$

Hence it follows, that y must always have the sign of A, whatever be the value of x, provided it be real.

Thus it appears, that when the values of x which render a rational and integral formula of the second degree $= 0$ are imaginary, all real values of x whatever will render the same formula positive when $A > 0$, and negative when $A < 0$.

It appears, as in the case where $x' = x''$, that in this case A and C must have the same sign.

SECTION XVI.

Of Maxima and Minima.

(204.) The species of problems having for their object the determination of *maxima* and *minima*, belong more properly to the *Differential Calculus* than to pure Algebra. For the complete discussion of them we therefore refer the reader to that subject. A particular class of these questions may, however, be solved by the aid of the theory of equations of the second degree; and as they frequently occur in the more elementary parts of analysis, and particularly in the application of Algebra to Geometry, we shall here explain the methods of investigating them.

When certain operations are to be performed on given numbers, it may so happen that the magnitude of the result will depend on the manner in which these operations are performed. In such a case it may be required to determine how the proposed operations should be performed, in order that the resulting quantity should be of the greatest or the least values which it could have consistently with the proposed conditions. Such values are called *maxima* and *minima*.

This will, perhaps, be better understood by an example. Let it be required to divide a given number ($2a$) into two parts, whose product is a *maximum*; that is, whose product is greater than the product of any other two parts into which the number could be divided.

Let y be the sought maximum value, and x one of the sought parts, the other being $2a - x$ we have

$$x(2a - x) = y.$$

It is plain, that as there are an infinite variety of ways in which the proposed number may be divided into two parts, there is an infinite variety of values which may be ascribed to the part x. In fact, x may be conceived to express any number which is less than the given number $2a$. The value of the product y will altogether depend on the value ascribed to x. Under these circumstances x is called a *variable*, and y is said to be a *function* of x. The word *function* being a term implying a quantity or symbol, the value of which depends, by some given condition, on the value of another quantity called the *variable*; *function* and *variable* being therefore correlative terms.

In order to determine the value of x, which renders y a maximum, let the first member of the equality be developed, and the result is

$$2ax - x^2 = y,$$

$$\therefore x^2 - 2ax = -y.$$

Let this be solved as if y were a given quantity, and the result is

$$x = a \pm \sqrt{a^2 - y}.$$

By the primitive equation the value of y depends on that of x. If such a value were ascribed to x as would make $y > a^2$, that value would render the radical in the last equation imaginary. But as this radical is a part of the value of x by the last equation, that value of x will itself be imaginary. Hence no real value of x will render $y > a^2$. The greatest value which y can receive, consistently with the reality of x, is when $y = a^2$. This therefore is the maximum value sought. But it is still necessary to determine the parts into which the number is divided, in order that the product of its parts may have this value. This may be found by substituting a^2 for y in the last equation, the result of which is

$$x = a \pm \sqrt{a^2 - a^2} = a,$$

$$\therefore 2a - x = a.$$

The parts into which the number is divided are therefore equal. From which we deduce the following general theorem, "If a number be divided into any two unequal parts, their product is always less than the square of half that number."

This principle might also be established still more simply, by taking half the difference of the parts as the variable, instead of one of the parts themselves. As before, let the number be $2a$, and let one of the parts be $a + x$. The other will be $2a - (a + x) = a - x$; it is evident that $2x$ is the difference of the parts, and therefore x is half their difference. We have then

$$(a + x)(a - x) = y,$$

$$a^2 - x^2 = y,$$

$$\therefore x^2 = a^2 - y,$$

$$\therefore x = \sqrt{a^2 - y}.$$

As before, if $y > a^2$, x would be imaginary. Therefore

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y is a maximum when $= a^2$, $\therefore x = 0$. Since the difference of the parts $= 0$, the parts are equal.

Each step of this process involves a principle which merits attention. By the original statement it appears, that of two unequal quantities the greater is equal to half their sum increased by half their difference, and the less is equal to half their sum diminished by half their difference. The first equation shows that the product of two unequal quantities is equal to the square of half their sum diminished by the square of half their difference; and as the difference must always be less than the sum, it is apparent, even without having recourse to any reasoning on imaginary quantities, that the product of the unequal parts must be less than the square of half the sum, or than the product of the equal parts.

(205.) The general principle by which the property of imaginary roots of quadratic equations becomes instrumental in the solution of questions respecting *maxima* or *minima*, will now be easily comprehended. If the roots of either of the formulæ

$$x^2 - p x = -q,$$
$$x^2 + p x = -q,$$

be real, it has been already proved, that q cannot exceed $\frac{p^2}{4}$. Hence, if $\frac{p^2}{4}$ be supposed to be a given quantity, and q to be variable, and at the same time the values of x be supposed to be real, the greatest value which q can have is $\frac{p^2}{4}$, in which case

$$x = \pm \frac{p}{2},$$

the upper sign applying to the first, and the lower to the second formula.

Again, if q be supposed given, and p variable, the least value which p can have consistently with the reality of the roots, is when $\frac{p^2}{4} = q$, or $p = 2\sqrt{q}$. In this case p is a minimum, and the value of x is

$$x = \pm \frac{p}{2} = \pm \sqrt{q}.$$

(206.) The principle may, however, be stated still more generally. When the result of any problem is a quadratic equation, and that a quantity whose *maximum* or *minimum* value is to be determined, enters in combination with given quantities under the radical in the solution of the equation, all values of that quantity which render the suffix of the radical negative must be rejected, since they render the roots imaginary, but that value which renders the suffix of the radical $= 0$, and which stands between those which render it positive or negative, will be the maximum or minimum value sought. Whether this value be a maximum or minimum, must be decided by the peculiar circumstances of the question.

(207.) Let it be proposed to divide a given number ($2a$) into two parts, such that the sum of the squares of these parts shall be greater or less than the sum of the squares of any other parts into which the same number could be divided, or such that it shall be a *maximum* or *minimum*.

As before, let x be one of the parts, the other will be $2a - x$, and let the sum of the squares be y, so that

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$$x^2 + (2a - x)^2 = y,$$
$$\therefore 2x^2 - 4ax + 4a^2 = y,$$
$$\therefore x^2 - 2ax = \frac{y}{2} - 2a^2,$$
$$\therefore x = a \pm \sqrt{\frac{y}{2} - a^2}.$$

The value of y, which renders the suffix of the radical $= 0$, being found, $y = 2a^2$ is evidently the least value which it can have consistently with the reality of x. The sum of the squares is therefore a *minimum* when it is equal to twice the square of half the given number, and the corresponding values of the parts are $x = a$, $2a - x = a$. The number is therefore divided into equal parts.

There is no value of y greater than a^2 which will render the suffix negative; on the contrary, the value of the suffix is continually augmented as increasing values are ascribed to y. There is, however, notwithstanding this, a limit. It will be remembered, that the value of y depends on that of x; and the mere inspection of the original equation will show, that if x be increased without limit, y will be also increased without limit, and therefore no major limit to y can be inferred from the *algebraical* statement of the question. In the problem itself, however, the number $2a$ is supposed to be divided into two parts. Neither of these parts can then be greater than the whole, consequently x cannot exceed $2a$. If x were supposed $= 2a$, which is the extreme case, the other part $2a - x$ would $= 0$, and y would be greater than it could be under any other circumstances. Why, then, it may be asked, does not this result from the algebraic investigation? The difficulty will be removed by examining more closely the algebraic statement.

The equation

$$x^2 + (2a - x)^2 = y$$

means simply that the square of a number represented by x, added to the square of another number represented by $2a - x$ produces a result $= y$. Now there is nothing here which limits the magnitude of x, or makes it necessarily less than $2a$. The number $2a - x$ may be negative, and yet its square will be positive. In this case $2a$ will be the arithmetical *difference* of the numbers x and $2a - x$, and not their arithmetical sum as enunciated in the problem. So that, as frequently happens, the algebraical statement is more general than the original problem; and hence it arises, that although in the original problem there is a major limit to the value of y, there is no major limit to it in the more general algebraical statement, because the particular condition which produced the major limit is the very condition by whose omission the problem is generalized.

(208.) Let it be required *to divide a number* ($2a$) *into two parts, x, $2a - x$, such that the sum of the quotients of each part by the other shall be a maximum or minimum.*

Let y be the sum of the quotes. The statement after reduction becomes

$$x^2 - 2ax = -\frac{4a^2}{2 + y}$$
$$\therefore x = a \pm \sqrt{a^2 - \frac{4a^2}{2 + y}}.$$

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The suffix of the radical being equated with zero gives

$$a^2 - \frac{4a^2}{2+y} = 0,$$

$$\therefore 2 + y - 4 = 0,$$

$$\therefore y = 2,$$

$$\therefore x = a, \quad 2a - x = a.$$

The number therefore must be divided into equal parts, and the sum of the quotes is 2, each quote being I.

In this case the sum is evidently a *minimum*. For the increase of y produces a diminution in the negative part of the suffix of the radical; and it is obvious that no increase whatever beyond the value a will ever render the suffix negative; and as the diminution of y increases the negative part of the suffix, no diminution below a will ever render the suffix of the radical positive.

SECTION XVII.

Arithmetical Progression.

(209.) A SERIES of quantities so related that each term exceeds that which precedes it, or is exceeded by it by the same quantity, is called an *arithmetical series*, and its terms are said to be in *arithmetical progression*.

Thus, in the series a_0, a_1, a_2, a_3, &c.

if $\quad a_1 - a_0 = a_2 - a_1 = a_3 - a_2$, &c.

the quantities are in arithmetical progression.

Thus,	1,	4,	7,	10,	13, &c.
	1,	3,	5,	7,	9, &c.
	20,	18,	16,	14,	12, &c.

are severally arithmetical series.

The difference of every two consecutive terms in the series being the same, is called the *common difference*. The series may be conceived to be generated by the constant *addition* of this common difference to the first term; when the series increases the common difference being positive, and when it decreases being negative.

Thus, if a be the first term and x the common difference, the successive terms of the series will be

$$a,\ a + x,\ a + 2x,\ a + 3x, \text{ \&c.}$$

The coefficient of x in any term is evidently equal to the number of preceding terms, so that the m^{th} term T will be $T = a + (m - 1)x$. This general formula will determine each of the terms by substituting successively for m the numbers 1, 2, 3, &c.

(210.) The sum of any two terms equally distant from a given term in an arithmetical series is equal to twice the given term. Let the given term be $a + mx$. The preceding and succeeding terms are

$$a + (m-1)x,\ a + (m+1)x,$$

which added are $2(a + mx)$. In like manner the terms, *two* distant on each side, are

$$a + (m-2)x,\ a + (m+2)x,$$

which added give $2(a + mx)$, and in general the terms distant n terms on each side are

$$a + (m-n)x,\ a + (m+n)x,$$

which being added give $2(a + mx)$.

In the same manner it may be proved, that the sum of any two adjacent terms is equal to the sum of any two terms equally distant from them

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(211.) Hence if any number of quantities be in arithmetical progression, the sum of the first and last terms is equal to the sum of the second and penultimate, or of any two terms equally distant from the extremes; and if the number of terms be odd, there being one term equally distant from the extremes, the sum of the extreme terms is equal to twice this middle term.

(212.) If three quantities be in arithmetical progression, the mean is equal to half the sum of the extremes, and the common difference is equal to half the difference of the extremes. Let the quantities be a, b, c. Hence

$$2b = a + c$$

$$\therefore b = \frac{1}{2}(a + c)$$

$$\therefore a - b = a - \frac{1}{2}a - \frac{1}{2}c = \frac{1}{2}(a - c) = b - c.$$

(213.) Let it be required to determine the sum of n terms in arithmetical progression, of which the first a_1 and the last a_n are given. Let the common difference be x, and the sum S; $\therefore$

$$S = a_1 + (a_1 + x) + (a_1 + 2x) + (a_1 + 3x) + \ldots \ldots \ldots \{a_1 + (n-1)x\}.$$

But if we arrange the terms in the opposite direction, beginning with a_n, we shall have

$$S = a_n + (a_n - x) + (a_n - 2x) + (a_n - 3x) + \ldots \ldots \ldots \{a_n - (n-1)x\}.$$

Adding these series, and observing that there are n terms, we have

$$2S = (a_1 + a_n)n$$

$$\therefore S = (a_1 + a_n)\frac{n}{2};$$

that is, the sum of the series is equal to the sum of the first and last terms multiplied by half the number of terms.

(214.) When an arithmetical series with a determinate number of terms is given, there are five quantities, *viz.* the first and last terms a_1, a_n, the common difference x, the number of terms n, and the sum of the series S, between which there subsists a relation which is expressed by the two equations,

$$a_n = a_1 + (n - 1)x$$

$$2S = (a_1 + a_n)n.$$

Hence it follows, that if any three of these five quantities be given, the remaining two may be found, and thus there arises the ten following problems:

	Given.		Sought.
1.	a_1, x, n	....	a_n, S
2.	a_1, x, a_n	....	n, S
3.	a_1, x, S	....	n, a_n
4.	a_1, n, a_n	....	x, S
5.	a_1, n, S	....	x, a_n
6.	a_1, a_n, S	....	x, n
7.	x, n, a_n	....	a, S
8.	x, n, S	....	a, a_n
9.	x, a_n, S	....	a, n
10.	x, a_n, S	....	a, x.

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(215.) All these probleme are solved by equations of the first degree, except those in which a and n and a_n and n are unknown. These are resolved by equations of the eecond degree, and it should be observed, generally, that every value of n muet be rejected, except those which are positive integers; for, from its nature, n cannot be negative or fractional.

SECTION XVIII.

Geometrical Progression.

(216.) A series of quantities are said to be in geometrical progression, when they increase or decrease in a *common ratio*. Thus, geometrical progreseion is equivalent to *continued proportion*. A series in geometrical progression may always be cooceived to be generated by a *constant multiplier*. For let the constant ratio of each pair of successive terms be $1 : r$, and let a be the first term. It is evident that $a\,r$ will be the second term, since $a : a\,r :: 1 : r$, and, for a similar reason, the third, fourth, &c. terms are $a\,r^2$, $a\,r^3$, &c. Thus, the n^{th} term is $a\,r^{n-1}$.

If the common multiplier r be > 1 the series increases, and if it be < 1 it decreases.

(217.) The product of any two terms equally distant from a given term is equal to the square of the given term. Let the given term be $a\,r^m$, the preceding and following terme are $a\,r^{m-1}$, $a\,r^{m+1}$, of which the product is $a^2\,r^{2m} = (a\,r^m)^2$.

Those which are two terms distant on each side are

$$a\,r^{m-2}, \quad a\,r^{m+2}$$

the product of which is the same, *scil.* $= (a\,r^m)^2$.
And in general the terms which are n terms distant on each side are $a\,r^{m-n}$, $a\,r^{m+n}$, and their product is

$$a^2\,r^{2m} = (a\,r^m)^2.$$

In like manner it may be proved, thet the product of any two successive terms is equal to the product of any two terms equally dietant from them in the series. Let the two adjacent terms be

$$a\,r^m, \; a\,r^{m+1},$$

and the two terme distant on each side by n terms are

$$a\,r^{m-n}, \; a\,r^{m+n+1},$$

which multiplied give

$$a^2\,r^{2m+1} = a\,r^m \times a\,r^{m+1}$$

$$\therefore a\,r^{m-n} \times a\,r^{m+n+1} = a\,r^m \times a\,r^{m+1}.$$

(218.) Hence if any number of quantitiee be in geometrical progression, the product of the extreme terms is equal to tha product of any two terms equally distant from them; and if the number be odd, this product is equal to the square of the eingle term whieh is equally distaot from the extremes.

(219.) If three quantities be in geometrical progression, the square of the mean is equal to the product of the extremes, and, therefore, either extreme is found by dividing the square of the mean by the other. If a, b, c be the three quantities

$$a\,c = b^2 \therefore a = \frac{b^2}{c}$$

(220.) Let it be required to determine the sum (S) of any determinate nnmber (n) of terms in geometrical progreesion. Let the first term be a_1. Hence we have

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$$S = a_1 + a_1\,r + a_1\,r^2 + \ldots\ldots a_1\,r^{n-2} + a_1\,r^{n-1};$$

multiply both members of this equality by r, and we obtain

$$S\,r = a_1\,r + a_1\,r^2 + \ldots\ldots a_1\,r^{n-2} + a_1\,r^{n-1} + a_1\,r^n.$$

Subtracting this from the former we obtain

$$S\,(1 - r) = a_1 - a_1\,r^n$$

$$\therefore S = a_1 . \frac{1 - r^n}{1 - r}$$

$$\text{or } S = a_1 . \frac{r^n - 1}{r - 1}.$$

(221.) When the series is decreasing, it may be continued to an unlimited number of terms and yet have a finite eum. In this case the multiplier r ie < 1, and r^n undergoes unlimited diminution, ae its exponent n ie unlimitedly increased; aod if n be supposed infinite, r^n will become $= 0$. Hence the sum of the series will be

$$S = a\,\frac{1}{1 - r}.$$

(222.) If $r = 1$ the formula for S assumes the form $\frac{0}{0}$. Thie indicates, either that the problem to determine the sum of the series is then indetermioate, or that the formula for S has a common factor in both numerator and denominator which becomes $= 0$ when $r = 1$. This latter, in fact, takes place in the present instance. For if tha division indicated by the formula $\frac{1 - r^n}{1 - r}$ be actually performed, we ehall have

$$\frac{1 - r^n}{1 - r} = 1 + r + r^2 + r^3 + \ldots\ldots + r^{n-1}.$$

Now if in this $r = 1$ the second member becomes $= n$.

(223.) Between the five quantitiee a_1, r, n, a_n, S, there subsist two equations, *scil.*

$$S = a_1\,\frac{r^n - 1}{r - 1}$$

$$a_n = a\,r^{n-1};$$

which, as in arithmetical progression, eoahle us when any three of the five quantities are given to determine the other two. But the solution of the several problems present in this ease greater difficulties. The four cases in which the unknown quantities are a_n S, r S, a_1 S, and a_1, a_n, offer no particuler difficulties, being all reduced to equations of the first degree. Tha two cases in which $a_1\,r$ and $a_n\,r$ are sought, depend on the eolution of equatioua of the n^{th} degree. By the formulæ above mentioned we deduce

$$(S - a_n)\,r^n - S\,r^{n-1} + a_n = 0$$

$$a_n = a_1\,r^{n-1},$$

the solutioo of which for r ie necessary in the former case. The degree of the problem, therefore, in this case depends on the number of terme in the series. In the latter case the equation is

$$a_1\,r^n - S\,r + S - a_1 = 0.$$

(224.) The four other eases where n ie unknown, depend on the resolution of an eqnation in which the unknown quantity occurs as an exponent. The inves-

tigation of equations of this kind will be explained in a subsequent section.

SECTION XIX.

Of the Indeterminate Analysis.—One simple equation with two unknown quantities.

(225.) When the number of equations which result from the conditions of a problem is less than the number of unknown quantities, the data are insufficient for the solution, and the values of the sought quantities cannot be determined, or rather there are an infinite variety of values of the unknown quantities which will equally satisfy the conditions of the problem, and all of which, therefore, have an equal claim to be considered as its solution.

To take a very simple instance, suppose it be required to find two numbers which have a given ratio the one to the other; let the ratio be $m : 1$. If x and y be the sought numbers we have $x = my$, which expresses the condition of the problem. Here then there are two unknown quantities and but one equation. One of the unknown quantities y may be supposed to have any value whatever, and the equation will determine a corresponding value of the other, so that the two values will satisfy the proposed condition. Thus the variety of systems of values will be absolutely infinite.

(226.) The variety of values of the unknown quantities in an indeterminate problem may, however, be restricted by conditions which do not admit of being expressed in the equation to which it is reduced. Thus, suppose it be required that the values of the unknown quantities be integers, all the systems of fractional values which satisfy the equation must then be rejected, and only the integral values retained. In the problem already given, let $m = \frac{5}{6}$ $\therefore$

$$x = \frac{5}{6}\,y.$$

Any value whatever being assigned to y, a value of x may be found, which, together with the value so assigned to y, will satisfy this equation. But it is required by the problem that the values of the unknown quantities should be integers. Hence we infer, first, that no fractional value can be assigned to y, and, secondly, that no integral value can be assigned, except one which is divisible by 6. For the product of the assigned value and 5 must be exactly divisible by 6, since x must be an integer. But 6 is prime to 5, and therefore must measure the value of y. Hence the only values assignable to y are

6, 12, 18, 24, &c.

and the corresponding values of x are

5, 10, 15, 20, &c.

(227.) The object of the indeterminate analysis, as applied to equations of the first degree, is to assign the systems of *positive and integral values* of the unknown quantities which satisfy them, if there be any such.

The general equation of the first degree between two unknown quantities, is

$$A\,x + B\,y = C. \quad (1.)$$

In this case the quantities A, B, and C may be supposed to be integers, since if they were fractions they could be reduced to integers by multiplying the entire equation by any common multiple of their denominators. It may also be supposed, that A, B, and C, have no common measure; for if they had, the entire equation might be divided by it.

These reductions having been previously performed, if A and B be not prime, let their greatest common measure be M, and let the whole equation be divided by it.

$$\frac{A}{M}\cdot x + \frac{B}{M}\cdot y = \frac{C}{M} \quad (2)$$

M, by hypothesis, does not measure C, therefore $\frac{C}{M}$ is an irreducible fraction. But $\frac{A}{M}$ and $\frac{B}{M}$ being integers, since it is required that x and y should be integers, it is necessary that each term of the first member of (2) should be an integer. And as the sum or difference of two integers must be an integer, it is evident that the first member is an integer, whatever be the signs of its terms. The second member, however, is an irreducible fraction, which is absurd. Hence there are no integers, positive or negative, which will solve the equation (1) when the coefficients A, B are not prime.

(228.) Let the coefficients A, B be now supposed prime, and let $A < B$. By solving the equation for that unknown quantity which has the lesser coefficient we have

$$x = \frac{C}{A} - \frac{B}{A}\,y. \quad [1.]$$

If $C > A$ the division indicated by $\frac{C}{A}$ may be partially effected. Let the integral part of the quote be Q and the remainder R, and also let the integral part of the quote $\frac{B}{A}$ and the remainder be q and r, so that

$$\frac{C}{A} = Q + \frac{R}{A}$$

$$\frac{B}{A} = q + \frac{r}{A}$$

which substitutions being made change the equation to

$$x = Q + \frac{R}{A} - q\,y - \frac{r}{A}\,y$$

$$\therefore x - Q + q\,y = \frac{R}{A} - \frac{r}{A}\,y.$$

The first member of this being an integer, let it be t, so that

$$t = \frac{R}{A} - \frac{r}{A}\,y$$

$$\therefore\ y = \frac{R}{r} - \frac{A}{r}\,t. \quad [2.]$$

Let the integral parts of the quotes $\frac{R}{r}$, $\frac{A}{r}$ be Q', q', and the remainders be R', r', and the equation becomes

$$y = Q' + \frac{R'}{r} - q'\,t - \frac{r'}{r}\,t$$

$$\therefore y - Q' + q' t = \frac{R'}{r} - \frac{r'}{r} t.$$

In like manner, the first member of this being t', we have

$$t' = \frac{R'}{r} - \frac{r'}{r} t$$

$$\therefore t = \frac{R'}{r'} - \frac{r}{r'} t'. \quad [3.]$$

As before, let the integral parts of the quotes $\frac{R'}{r'}, \frac{r}{r'}$ be Q'', q'', and the remainders R'', r'', and we have

$$t = Q'' + \frac{R''}{r'} - q'' t' \quad \frac{r''}{r'} t'$$

$$\therefore t - Q'' + q'' t' = \frac{R''}{r'} - \frac{r''}{r'} t',$$

the first member of this being an integer, let it be t'', so that

$$t'' = \frac{R''}{r'} - \frac{r''}{r'} t'$$

$$\therefore t' = \frac{R''}{r''} \quad \frac{r'}{r''} t''. \quad [4.]$$

If this process be continued, the denominator of the fractions in the second member of some of the equations [2,] [3,] [4,] &c. must at length become $= 1$. For since the numbers r, r', r'', &c. are the several remainders from effecting the divisions indicated by $\frac{B}{A}, \frac{A}{r}, \frac{r}{r'}$, &c., the last remainder must be the greatest common measure of B and A, (98.) These numbers are by hypothesis prime, and therefore their greatest common measure is unity. Let us then suppose that the remainder which becomes the denominator of [5] is $= 1$. We have

$$t'' = R''' - r'' t'''. \quad [5.]$$

By the equation [5] t'' may be eliminated from [4,] and by the equation thus found, t' may be eliminated from [3,] and by the equation resulting from this last process t may be eliminated from [2,] so that we shall have y expressed as a function of t''' alone.

By this equation y may be eliminated from [1,] and x will be obtained as a function of t'''. The values of x and y being thus obtained as functions of t, we may obtain an unlimited number of pairs of values of x and y, by substituting for t in each of the values thus obtained the terms of the series,

$$0,\ 1,\ 2,\ 3,\ 4,\ \&c.$$

$$-1,\ -2,\ -3,\ \ \ 4,\ \&c.$$

(229.) We shall now illustrate these principles by applying them to some examples.

Let the given equation be

$$13x + 16y = 97$$

$$\therefore x = \frac{97}{13} - \frac{16}{13} y \quad [1]$$

$$\therefore x = 7 + \frac{6}{13} - y - \frac{3}{13} y.$$

Hence the second equation will be

$$t = \frac{6}{13} - \frac{3}{13} y$$

or

$$y = 2 - \frac{13}{3} t. \quad [2.]$$

The value of x may be obtained in terms of t, by substituting this value of y in [1,] which gives

$$x = \frac{97}{13} - \frac{32}{13} + \frac{16}{3} t = \frac{65}{13} + \frac{16}{3} t.$$

By effecting the division [2,] becomes

$$y = 2 \quad 4t - \tfrac{1}{3} t,$$

$$\therefore t' = -\tfrac{1}{3} t,$$

$$\therefore t = -3t'. \quad [3.]$$

Eliminating t between [2] and [3] we obtain

$$x = 5 - 16t'$$

$$y = 2 + 13t'.$$

It is evident that the elimination of t' by these equations would give the original equations, as should be the case, since they have been derived directly from it.

By substituting for t' successively in the above equations the values

$$0,\ 1,\ 2,\ 3,\ 4,\ \&c.$$

we obtain the following systems of values of x and y;

$$x = +5,\ -11,\ -27,\ -43,\ -59,\ \&c.$$

$$y = +2,\ +15,\ +28,\ +41,\ +53,\ \&c.$$

and by substituting successively for t' the values

$$-1,\ -2,\ -3,\ -4,\ \&c.$$

we obtain

$$x = +21,\ +37,\ +53,\ +69,\ \&c.$$

$$y = -11,\ -24,\ -37,\ -50,\ \&c.$$

Any of these systems of values substituted in the original equation, will be found to change it to an identity. Thus we have

$$13 \times 5 + 16 \times 2 = 65 + 32 = 97$$

$$13 \times 11 + 16 \times 15 = \quad 143 + 240 = 97$$

$$13 \times 21 - 16 \times 11 = 273 - 176 = 97$$

&c. &c. &c.

It appears that the equation admits but one solution in positive integers, which corresponds to $t' = 0$, and is $x = 5$, $y = 2$.

It does not always happen, however, that the number of integral and positive solutions is limited. Let us consider the equation

$$17x - 49y = -8,$$

$$\therefore x = -\frac{8}{17} + \frac{49}{17} y \quad [1]$$

$$\therefore x = -\frac{8}{17} + 2y + \frac{15}{17} y$$

$$\therefore t = -\frac{8}{17} + \frac{15}{17} y$$

$$\therefore y = \frac{8}{15} + \frac{17}{15} t \quad [2]$$

$$\therefore y = \frac{8}{15} + t + \frac{2}{15} t$$

$$\therefore t' = \frac{8}{15} + \frac{2}{15} t$$

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$$\because t = \frac{15}{2} t' - \frac{8}{2} \qquad [3.]$$

$$\because t = 7t' + \tfrac{1}{2} t' - 4$$

$$\because t'' = \tfrac{1}{2} t'. \qquad [4.]$$

By the equations [1,] [2,] [3,] [4,] the values of x and y being found in terms of t'', we have

$$x = 49\,t'' - 12$$
$$y = 17\,t'' - 4.$$

Substituting

$$1,\ 2,\ 3,\ 4,\ \&c.$$

for t'', we have

$$x = 37,\ 86,\ 135,\ 184,\ \&c.$$
$$y = 13,\ 30,\ \ 47,\ \ 64,\ \&c.$$

So that the number of positive and integral solutions is unlimited.

The number of negative integral solutions is also unlimited, as may be proved by substituting,

$$0,\ -1,\ -2,\ -3,\ \&c.$$

successively for t''.

(230.) It will be observed, that in the values of x and y, obtained in terms of the last indeterminate quantity which is introduced, the coefficients of the indeterminate in the value of x, is the same with that of y in the original equation; and the coefficient of the indeterminate in the value of y, is the same with that of x in the original equation. This may be easily demonstrated.

Let us suppose, as before, that the process stops at equation [5.] By substituting the value of t'' obtained in [5,] for t'' in [4,] the coefficient of t''' in the resulting equation will evidently be $\frac{r'}{r''} \times r'' = r'$. This again being substituted in [3] the coefficient of t''' will be $-\frac{r}{r'} \times \frac{r'}{r''} \times r'' = -r$. The process of substitution being continued to [2,] the coefficient of t''' will be $\frac{A}{r} \times \frac{r}{r'} \times \frac{r'}{r''} \times r'' = A$, and in [1] it will be $-\frac{B}{A} \times \frac{A}{r} \times \frac{r}{r'} \times \frac{r'}{r''} \times r'' = -B$. Hence, in this case, the coefficient of t''' in the value of x is $-B$, and that of t''' in the value of y is A; the former being the coefficient of y in the original equation, the sign being changed, and the latter the coefficient of x.

It will be easy to generalize this demonstration. Let the number of equations obtained before a remainder $= 1$ is found, be n. The last equation will then be

$$t^{(n-2)} = R^{(n-2)} - r^{(n-2)} \cdot t^{(n-1)},$$

the numbers within the parenthesis denoting the number of accents with which each letter is affected. After this substitution is made in the $(n-1)^{th}$ equation, the coefficient of $t^{(n-1)}$ in it will be

$$(-r^{(n-2)}) \times \left(-\frac{r^{(n-3)}}{r^{(n-2)}}\right) = +r^{n-3}$$

The substitution being continued to the $(n-2)^{th}$ equation, the coefficient of $t^{(n-1)}$ in it will be

$$\left(-r^{(n-2)}\right) \times \left(-\frac{r^{(n-4)}}{r^{(n-3)}}\right) \times \left(-\frac{r^{(n-3)}}{r^{(n-2)}}\right) = -r^{(n-4)}$$

In the $(n-3)^{th}$ equation, the coefficient of the indeterminate $t^{(n-1)}$ after substituting will be

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$$\left(-r^{(n-2)}\right) \times \left(-\frac{r^{(n-4)}}{r^{(n-3)}}\right) \times \left(-\frac{r^{(n-5)}}{r^{(n-4)}}\right) \times \left(-\frac{r^{(n-3)}}{r^{(n-2)}}\right)$$
$$= +r^{(n-5)}$$

It appears, therefore, that after substitution the coefficients of the indeterminate $t^{(n-1)}$ in each successive equation, beginning from the last, have signs alternately $-$ and $+$, and that the values of these coefficients are the successive remainders resulting from processes of division. The coefficient of $t^{(n-1)}$ in the third equation will be the first remainder, and in the second equation the coefficient A, and in the first the coefficient B. One of these last will have the sign $+$, and the other the sign $-$, according to whether the total number of equations be odd or even. If it be odd, the second equation will stand in an even order, counting from the last; and in this case the sign of A in the second equation will be $+$, or in general it will be the same with that which it has in the original equation; and that of B in the first equation will be $-$, or different from that which it holds in the original equation, and *vice versâ* when the number of equations is even.

SECTION XX.

On Continued Fractions.

(231.) When a fraction in its lowest terms is expressed by any high numbers, it is often desirable to obtain a fraction nearly equivalent to it in lower numbers, and also in this case to determine the limit of error to which we are subject in using this approximate value for the true.

Let it be proposed to find an approximate value for $\frac{159}{493}$ in lower terms. To effect this, let both terms be first divided by 159. Hence we obtain

$$\frac{159}{493} = \frac{1}{3 + \frac{16}{159}}$$

If the fraction $\frac{16}{159}$ be neglected, the value $\frac{1}{3}$ will be too great, since the denominator will be too small. But if $\frac{16}{159}$ be replaced by 1, the value $\frac{1}{4}$ will be too small, the denominator being too great. Hence the value is between $\frac{1}{3}$ and $\frac{1}{4}$.

A further approximation may be obtained by proceeding in the same manner with the fraction $\frac{16}{159}$, which gives

$$\frac{16}{159} = \frac{1}{9 + \frac{15}{16}}$$

$$\therefore \frac{159}{493} = \cfrac{1}{3 + \cfrac{1}{9 + \frac{15}{16}}}$$

If the fraction $\frac{15}{16}$ be neglected, $\frac{1}{9} > \frac{16}{159}$, and $\because \frac{1}{3 + \frac{1}{9}} < \frac{159}{493}$. But

$$\frac{1}{3 + \frac{1}{9}} = \frac{9}{28}$$

Hence the value of the proposed fraction is less than $\frac{1}{3}$ and greater than $\frac{9}{28}$.

Now the difference between these two limits is $\frac{1}{84}$.

and, therefore, either of these two values is within $\frac{1}{84}$ of the true value.

The approximation may be carried still further, by treating the fraction $\frac{16}{15}$ like the former, by which we obtain

$$\frac{159}{493} = \cfrac{1}{3 + \cfrac{1}{9 + \cfrac{1}{1 + \frac{1}{15}}}}$$

If the last fraction $\frac{1}{15}$ be neglected here, $\frac{1}{1} > \frac{15}{16}$. Hence the denominator $9 + \frac{1}{1}$ is too great, $\because$ the denominator $3 + \cfrac{1}{9 + \frac{1}{1}}$ is too small, and $\because$ the assumed value for the fraction would be too great. This value, when reduced, is $\frac{10}{31}$. Hence we infer

$$\frac{159}{493} < \frac{10}{31} \text{ and } > \frac{9}{28}.$$

The difference between these limits is $\frac{1}{868}$, which is, therefore, greater than the error to which we should be subject in using either of these for the true value of the fraction.

(232.) The meaning of the expression

$$a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + 1}}} \\ \&c.$$

must now be apparent. Such an expression is called a *continued fraction*.

(233.) From the example already given, we may derive the following rule for converting any ordinary fraction into a continued fraction: "Let the terms of the fraction be submitted to the process necessary for finding their greatest common measure, and let it be continued until a numerator is found which exactly measures its denominator, which, when the terms of the fraction are prime, will always be unity; the successive quotes obtained in this process will be the denominators of the fraction which constitute the successive members of the continued fraction."

Let $\frac{M}{N}$ be the fraction which is to be converted into a continued fraction, and let a be the integral part of the quote $\frac{M}{N}$, b the integral part of the quote of N by the first remainder, c that of the first remainder by the second remainder, and so on. Hence we have

$$\frac{M}{N} = a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \cfrac{1}{e +}}}} \\ \&c.\ \&c.$$

The value a is called the first approximation to $\frac{M}{N}$, $a + \frac{1}{b}$ the *second approximation*, $a + \cfrac{1}{b + \frac{1}{c}}$ the *third approximation*, and so on. Let these successive approximations be called x_1, x_2, x_3, &c., and if they be reduced to simple fractions, we have

$$x_1 = a$$

$$x_2 = \frac{a\,b + 1}{b}$$

$$x_3 = \frac{(a\,b + 1)\,c + a}{b\,c + 1}$$

$$x_4 = \frac{[(a\,b + 1)\,c + a]\,d + a\,b + 1}{(b\,c + 1)\,d + b}$$

&c. &c.

By inspecting these values, the *law* by which they may be derived from one another is very apparent. To find the third x_3, the numerator of x_2 is multiplied by the third qoote d, and the numerator of x_1, added to the result; and, in like manner, the denominator of x_3 is found by multiplying the denominator of x_2 by the third quote, and adding to the product the denominator of x_1. Also, the numerator and denominater of x_4 are found by multiplying the numerator and denominator of x_3 by the fourth quote, and adding to the results the numerator and denominators of x_2. And, in general, the numerator and denominator of x_n are found by multiplying the numerator and denominator of x_{n-1} by the n^{th} qnote, and adding to the result the numerator and denominator of x_{n-2}.

Hence, if the numerator and denominator of x_{n-2} be A_{n-2}, B_{n-2}, and those of x_{n-1} be A_{n-1}, B_{n-1}, and those of x_n be A_n, B_n, and that q be the n^{th} quote, we have

$$A_n = A_{n-1} \cdot q + A_{n-2}.$$

$$B_n = B_{n-1} \cdot q + B_{n-2}.$$

(234.) We shall now determine the difference between every two successive approximations. Let

$$x_{n-2} = \frac{A_{n-2}}{B_{n-2}}$$

$$x_{n-1} = \frac{A_{n-1}}{B_{n-1}}$$

$$\because x_n = \frac{A_{n-1} \cdot q + A_{n-2}}{B_{n-1} \cdot q + B_{n-2}}$$

Hence we find

$$x_{n-1} - x_{n-2} = \frac{A_{n-1}}{B_{n-1}} - \frac{A_{n-2}}{B_{n-2}}$$

$$= \frac{A_{n-1}\,B_{n-2} - A_{n-2}\,B_{n-1}}{B_{n-1}\,B_{n-2}}$$

$$x_n - x_{n-1} = \frac{A_{n-1} \cdot q + A_{n-2}}{B_{n-1} \cdot q + B_{n-2}} - \frac{A_{n-1}}{B_{n-1}}$$

$$= \frac{A_{n-2}\,B_{n-1} - A_{n-1} \cdot B_{n-2}}{(B_{n-1} \cdot q + B_{n-2})\,B_{n-1}}$$

Hence it appears, that the numerators of the differences between every two successive approximations are equal, but have different signs, and that the denominators are the products of the denominators of the approximations themselves.

To determine the constant value of the numerators of tha differences, it will be sufficient to determine any one of them. We have

$$x_1 = a \qquad x_2 = \frac{a\,b + 1}{b}$$

$$\because x_2 - x_1 = \frac{+1}{b}.$$

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Hence, the constant value of the oumerator of the differences is unity; and as the oumerator of the difference of the first and second is $+1$, that of the second and third is -1, and so on.

(235.) Since the denominators of these differences are essentially positive, it follows that the differences themselves are alternately negative and positive, that is

$$\begin{aligned} x_2 - x_1 &> 0 \\ x_3 - x_2 &< 0 \\ x_4 - x_3 &> 0 \\ x_5 - x_4 &< 0 \\ \&c. \quad &\&c. \end{aligned}$$

Hence we infer that

$$\begin{aligned} x_1 &< x_2 \\ x_2 &> x_3 \\ x_3 &< x_4 \\ x_4 &> x_5 \\ &\&c. \end{aligned}$$

Since the numerator of the difference between each successive pair of approximatinns is constantly the same, and the denominator constantly increasing, it follows that this difference is constantly diminishing. Hence we have

$$\begin{aligned} x_2 - x_1 &> x_3 - x_2 \therefore x_1 < x_3 \\ x_3 - x_2 &> x_4 - x_3 \therefore x_2 > x_4 \\ x_4 - x_3 &> x_5 - x_4 \therefore x_3 < x_5 \\ \&c. \qquad &\qquad \&c. \end{aligned}$$

Now since x_1 is evidently less than x, it follows that $x_2 > x$, $x_3 < x$, $x_4 > x$, &c., and in general the approximations of an *odd* order are $< x$, while those of an even order are $> x$. Also, since $x_1 < x_3 < x_5 < {}_7$, &c., and all of these are $< x$, it follnws that the further we continue the approximations the oearer will thuse of an odd order approsch to equality with x, *all*, however, being $< x$. And since $x_2 > x_4 > x_6$, &c., and all of these $> x$, it follows that the further we proceed with the approximations, the mnre nearly those of an even order will approach to x, all being $> x$. The limit of error caused by any approximation will be found by taking the difference between it and that which is oext above it, if it be of an odd order; and that below it, if it be of an even order.

But a still more exact limit may be determined. Let the value of all the remaioing part of the continued fractioo after the $(n-1)^{th}$ approximation be y; that is, if q be the n^{th} quote, and r, s, &c. the succeeding quotes, let

$$y = q + \cfrac{1}{r + \cfrac{1}{s + 1}} \quad \&c.$$

Now we have

$$x_n = \frac{A_{n-1} \cdot q + A_{n-2}}{B_{n-1} \cdot q + B_{n-2}}$$

But if we change q ioto y, this will become the exact value of x ∴

$$x = \frac{A_{n-1} \cdot y + A_{n-2}}{B_{n-1} \cdot y + B_{n-2}},$$

$$\therefore x - x_{n-2} = \frac{(A_{n-2} B_{n-1} - A_{n-1} B_{n-2})\, y}{(B_{n-1}\, y + B_{n-2})\, B_{n-2}},$$

$$x - x_{n-1} = \frac{A_{n-1} B_{n-2} - A_{n-2} B_{n-1}}{(B_{n-1} \cdot y + B_{n-2})\, B_{n-1}},$$

$$\therefore x - x_{n-2} = \frac{\pm y}{(B_{n-1} \cdot y + B_{n-2})\, B_{n-2}},$$

$$x - x_{n-1} = \frac{\mp 1}{(B_{n-1} \cdot y + B_{n-2})\, B_{n-1}}.$$

Siuce y cannot be less than 1, it follows that the difference between x_n and x cannot be greater than

$$\frac{1}{(B_{n-1} + B_{n-2})\, B_{n-1}} = \frac{1}{B^2_{n-1} + B_{n-1} \cdot B_{n-2}}.$$

This gives the limit of error still more nearly thao $\frac{1}{B_{n-1}\, B_{n-2}}$ before obtained. It also furnishes another limit, *scil.* $\frac{1}{B^2_{n-1}}$, though not so exact as that established above.

Thus we may infer that the n^{th} approximation differs from x by a quantity less than the fraction whose numerator is unity, and whose denominator is the square of the denominator of this approximation; or still more oearly by a fraction whose numerator is unity, and whose denominator is the product of the deoominator of the n^{th} approximation, and the sum of the denomioators of the n^{th} and $(n-1)^{th}$ approximatinos.

(236.) We shall nnw iovestigate, by means of a continued fraction, the value of the circumference of a circle whose diameter is unity. This is known to be nearly equivalent to 3,14159, or $\frac{314159}{100000}$. Converting this into a continued fractinn, we have

$$x = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{25 + \cfrac{1}{1 + \cfrac{1}{7 + \frac{1}{4}}}}}}}$$

Hence we find

$$x_1 = \tfrac{3}{1},\ x_2 = \tfrac{22}{7},\ x_3 = \tfrac{333}{106},\ x_4 = \tfrac{355}{113},\ x_5 = \tfrac{9208}{2931},$$
$$x_6 = \tfrac{9563}{3044},\ x_7 = \tfrac{76149}{24239},\ x_8 = \tfrac{314159}{100000}.$$

If wa assume $\frac{22}{7}$ as the true valoe, the error must be less than $\frac{1}{7\,(7+1)} = \frac{1}{56}$. But this is even oearer the true value, which is between $\frac{22}{7}$ and $\frac{333}{106}$, and is therefore nearer to it than the difference of these, which is $\frac{1}{742}$. In cases, therefore, where extreme accuracy is oot required, $\frac{22}{7} = 3\frac{1}{7}$ may be taken to represent the circumfereuce. This was the approximation of Archimedes.

If the fourth approximation be taken for x, the error must be less than the difference between the fourth and fifth approximations, which in this case is $\frac{1}{113 \times 2931}$ $< ,00001$. Thus then $\frac{355}{113}$ differs from the circumference by less thao the ten thousandth part of the diameter.

SECTION XXI.

Of Exponential Equations.

(237.) An exponential equation is one in which the unknown quantity is an exponent, as $a^x = b$.

To explain the method of solving such an equation as this, we shall take, in the first place, a particular case. Let the equation be $3^x = 243$. Substitute successively for x the integers 1, 2, 3, &c., and we find

$$3^1 = 3,\quad 3^2 = 9,\quad 3^3 = 27,$$
$$3^4 = 81,\quad 3^5 = 243,$$
$$\therefore x = 5.$$

In this case it happens, that the second member of the equation is an exact power of 3. But let us suppose that the equation is $2^x = 6$, we have

$$2^2 = 4$$
$$2^3 = 8.$$

Consequently x is > 2, and < 3. Let $x = 2 + \frac{1}{x'}$.

In this case $\frac{1}{x'}$ must be a proper fraction. We have

$$2^{2+\frac{1}{x'}} = 6,$$

$$\therefore 2^2 \times 2^{\frac{1}{x'}} = 6, \therefore 2^{\frac{1}{x'}} = \frac{6}{4} = \frac{3}{2}, \therefore 2 = \left(\frac{3}{2}\right)^{x'}$$

Let 1, 2, 3, &c. be successively substituted for x', and we have

$$\left(\frac{3}{2}\right)^1 = \frac{3}{2},\quad \left(\frac{3}{2}\right)^2 = \frac{9}{4}.$$

Now $\frac{3}{2} < 2$ and $\frac{9}{4} > 2$, $\therefore x' > 1$ and < 2. Let

$$x' = 1 + \frac{1}{x''},$$

$$\therefore \left(\frac{3}{2}\right)^{1+\frac{1}{x''}} = 2,\quad \therefore \frac{3}{2} \times \left(\frac{3}{2}\right)^{\frac{1}{x''}} = 2,$$

$$\therefore \frac{3}{2} = \left(\frac{4}{3}\right)^{x''}.$$

Again

$$\frac{4}{3} < \frac{3}{2},$$

$$\left(\frac{4}{3}\right)^2 = \frac{16}{9} > \frac{3}{2}.$$

Hence we infer that $x'' > 1$ and < 2. Let

$$x'' = 1 + \frac{1}{x'''},\quad \therefore \left(\frac{4}{3}\right)^{1+\frac{1}{x'''}} = \frac{3}{2},$$

$$\therefore \frac{4}{3}\left(\frac{4}{3}\right)^{\frac{1}{x'''}} = \frac{3}{2},$$

$$\therefore \frac{4}{3} = \left(\frac{9}{8}\right)^{x'''}.$$

Again, substituting 2 and 3 for x''' we find

$$\left(\frac{9}{8}\right)^2 = \frac{81}{64} < \frac{4}{3},$$

$$\left(\frac{9}{8}\right)^3 = \frac{729}{512} > \frac{4}{3}.$$

Hence it follows that $x''' > 2$ and < 3. Let

$$x''' = 2 + \frac{1}{x^{\text{iv}}},\ \therefore$$

$$\left(\frac{9}{8}\right)^{2+\frac{1}{x^{\text{iv}}}} = \frac{4}{3},\quad \therefore \frac{81}{64}\left(\frac{9}{8}\right)^{\frac{1}{x^{\text{iv}}}} = \frac{4}{3},$$

$$\therefore \frac{9}{8} = \left(\frac{256}{243}\right)^{x^{\text{iv}}}$$

By proceeding with this as before, we should find the two successive integers between which the value of x^{iv} lies, and so proceed another step; and thus the investigation might be continued as far as is desired.

We have, then,

$$x = 2 + \frac{1}{x'},\ x' = 1 + \frac{1}{x''},\ x'' = 1 + \frac{1}{x'''},\ x''' = 2 + \frac{1}{x^{\text{iv}}}$$

$$\therefore x = 2 + \cfrac{1}{1 + \cfrac{1}{x''}} \qquad = 2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{x'''}}}$$

$$= 2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{x^{\text{iv}}}}}}.$$

If we omit the fractions $\frac{1}{x''}$, $\frac{1}{x'''}$, &c., we have, as a first approximation,

$$x = 2 + \frac{1}{1} = 3.$$

If we include $\frac{1}{x''}$, omitting $\frac{1}{x'''}$, $\frac{1}{x^{\text{iv}}}$, &c., we have, as a second approximation,

$$x = 2 + \cfrac{1}{1 + \cfrac{1}{2}} = 2 + \frac{1}{2} = \frac{5}{3}.$$

Again, by including $\frac{1}{x'''}$, we have

$$x = 2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2}}} = 2 + \cfrac{1}{1 + \cfrac{2}{3}} = 2 + \frac{3}{5} = \frac{13}{5}$$

and by continuing the process we should approximate without limit to the value of x.

(238.) The general method then for resolving the equation $a^x = b$ by approximation, is to find the highest exact power of a which is contained in b. Let this be a^n, so that $a^n < b$ $a^{n+1} > b$. Hence the value of x must be between n and $n + 1$. Let

$$x = n + \frac{1}{x'}\ \therefore$$

$$a^{n+\frac{1}{x'}} = b\ \therefore a^n.a^{\frac{1}{x'}} = b$$

$$\therefore \left(\frac{b}{a^n}\right)^{x'} = a.$$

In the same manner x' is found to be between the limits n' and $n' + 1$. Let $x' = n' + \frac{1}{x''}$ and proceed in the same manner. Finally we shall have

$$x = n + \cfrac{1}{n' + \cfrac{1}{n'' + \cfrac{1}{n''' + \cfrac{1}{\&c.}}}}$$

By continuing the process the value of x may thus be obtained within any proposed degree of approximation.

By the results of Section XX. it appears, that $n + \frac{1}{n'}$ differs from x by a quantity less than $\frac{1}{(n + n)\, n'}$. Also, that the third approximation differs by a quantity less than $\frac{1}{(n' n'' + n' + 1)\,(n' n'' + 1)}$ and so on.

SECTION XXII.

Of Permutations and Combinations.

(239.) If there be any number of quantities or things which we shall represent by letters a, b, c, &c., the various *orders* in which it is possible to arrange these are called *permutations*. Thus, if there be two, a, b, they may be arranged in either of two ways $a\,b$ or $b\,a$, and they are said to be susceptible of but two *permutations*. If there be three, a, b, c, they may be arranged in six ways, $a\,b\,c$, $a\,c\,b$, $b\,a\,c$, $b\,c\,a$, $c\,a\,b$, $c\,b\,a$, and are said, therefore, to be susceptible of six *permutations*.

(240.) Let it be required to determine in general the number of permutations of which m letters, a, b, c, &c. are susceptible.

Let x be the number of permutations sought, and z be the number of permutations of which $m - 1$ of the given letters are susceptible. The remaining letter may be placed either *before* the first letter in any one of these permutations, or after the first or any succeeding letter. It may, therefore, have m different places, and for each of the z permutations of the $m - 1$ letters there are m permutations of the total number. Hence the total number of permutations is $m\,z$.

If $m = 2$ it is evident that $z = 1 \because x = 2$.

If $m = 3 \because z = 2$, and $x = 1\,.\,2\,.\,3$.

If $m = 4 \because z = 1\,.\,2\,.\,3$, and $x = 1\,.\,2\,.\,3\,.\,4$.

And in general we may infer, that by continuing the process we should have

$$x = 1\,.\,2\,.\,3\,.\,4 \ldots\ldots m - 1\,.\,m.$$

(241.) If there be any number of quantities or things represented, as before, by letters, a, b, c, &c., a group consisting of any number of these, without regard to their order, is called a *combination*; and whenever two such groups differ in a single letter, they are considered as different *combinations*. Thus, if the given quantities be a, b, c, d, $a\,b\,c$ and $a\,b\,d$ are different *combinations*; but $a\,b\,c$, $a\,c\,b$, $c\,a\,b$ are all the same *combination*.

Combinations are denominated combinations of two, three, four, &c., according to the number of letters of which each group is composed.

Each *combination* is susceptible of *permutation*. Combinations differing in the order of their letters may be called *permuted combinations*.

(242.) To determine the number of permuted combinations of n letters which can be formed from m letters, m being supposed greater than n.

Let the number of permuted combinations of $n - 1$ of the m letters be z, and the sought number be x.

Any one of the z permuted combinations of $n - 1$ letters being taken, and the remaining $m - (n - 1)$ of the m letters being successively annexed to it, will give a corresponding number of permuted combinations of n letters, and this being done with each of the z permuted combinations, we have $x = z\,(m \quad n + 1)$. If $n = 2 \because n - 1 = 1$. In this case it is evident that $z = m$. Hence $z = m\,(m - 1)$.

$$\text{If } n = 3 \because n - 1 = 2 \because z = m\,(m - 1)$$
$$\because x = m\,(m - 1)\,(m - 2).$$
$$\text{If } n = 4 \because n - 1 = 3 \because z = m\,(m - 1)\,(m - 2) \because$$
$$x = m\,(m - 1)\,(m \quad 2)\,(m - 3)$$

and in general

$$x = m\,(m - 1)\,(m - 2)\,(m - 3) \ldots\ldots (m - n + 1).$$

(243.) To determine the number of combinations of n letters which can be formed of m letters.

The number of permuted combinations was found in the preceding article. Thus, to determine the number of different combinations, it is only necessary to divide the number of permuted combinations by the number of permutations of which n letters are susceptible. Hence the number of different combinations sought is

$$\frac{m\,(m - 1)\,(m - 2) \ldots\ldots (m - n + 1)}{1\;.\;2\;.\;3 \;\ldots\ldots\ldots\ldots\ldots\;n}.$$

Since this number must, from its nature, be an integer, it appears that the continued product of all the integers from m to $m - (n - 1)$ inclusive, is divisible by the continued product of all the integers from 1 to n inclusive, n being less than m.

(244.) It is not difficult to prove that the number of combinations of n letters to be made from m, is equal to the number of combinations of $m - n$ to be made from m. Let $m - n = n'$. The number of combinations of n' letters is

$$\frac{m\,(m - 1)\,(m - 2) \ldots\ldots (m - n' + 1)}{1\;.\;2\;.\;3 \ldots\ldots\ldots\ldots\ldots\ldots n'}$$

Substitute $m - n$ for n' and we have

$$\frac{m\,(m - 1)\,(m - 2) \ldots\ldots (m + 1)}{1\;.\;2\;.\;3 \ldots\ldots\ldots\ldots (m - n)}. \qquad [1]$$

First, let $m - (n - 1)$ be greater than $n + 1$. Then the preceding number may be expressed

$$\frac{m\,(m - 1)\,(m - 2) \ldots (m - (n - 1))\,(m - n) \ldots (n + 1)}{1\;.\;2\;.\;3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (m - n)}$$

The factors from $(m - n)$ decreasing by unity to $n + 1$ inclusive, are here common to both numerator and denominator, and may, therefore, be omitted, and the result is

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$$\frac{m\,(m-1)\,(m-2)\ldots\ldots(m-(n-1))}{1\,.\,2\,.\,3\,\ldots\ldots\ldots\ldots\,n}$$

which is the number of combinations of n letters to be made from m letters.

Secondly, let $m-(n-1)$ be less than $n+1$, and therefore also $m-n$ is less than $n+1$. Let both numerator and denominator of [1] be multiplied by the successive integers from n to $m-(n-1)$ inclusive, and it will become

$$\frac{m\,(m-1)\,(m-2)\ldots\ldots(m-(n-1))}{1\,.\,2\,.\,3\,\ldots\ldots\ldots\ldots\,n}$$

which, as before, is the number of combinations of n letters to be made from m letters.

SECTION XXIII.

Of the Binomial Theorem.

(245.) If the square, the third, fourth, &c. powers of a binomial be obtained by actual multiplication, the results will be as follows:

1. power...... $x+a$.
2. power...... $x^2+2\,x\,a+a^2$.
3. power...... $x^3+3\,x^2a+3\,a\,x^2+a^3$.
4. power...... $x^4+4\,x^3a+6\,x^2a^2+4\,x\,a^3+a^4$.

&c. &c.

In cases, however, where high powers are required, the process of involution would be very laborious, and where the exponent of the required power is expressed by a letter, and not by a particular integer, we should not be able to express it at all, unless the *law* were known by which the exponents and coefficients of the successive terms of the series are derived from the exponent of the power.

The rule which determines the method of deriving the exponents and coefficients from the exponent of the required power in general, and independently of any particular value which that exponent may have, is called the *binomial Theorem;* and the series thus found, and which would also result from the continued multiplication by which the ordinary process of involution is conducted, is called the *developement* of the power.

Newton first assigned the law by which the binomial developement was governed, but did not give any demonstration of it. Since his time, however, the theorem has been submitted to rigorous proof.

(246.) We shall first consider the case in which the exponent of the power is a positive integer. The question then is, to obtain the developement of $(x+a)^m$, m being a positive integer.

If any number m of simple binomials of the forms $(x+a)$, $(x'+a')$, $(x''+a'')$, &c. be multiplied so as to form a continued product, it is evident that the developement of this product would consist of products formed of every possible combination of m quantities, which could be formed from the $2\,m$ simple quantities, $x, x', x'' \ldots.\ a, a', a'' \ldots..$ If the accents be all removed from letters x, and they be supposed to become equal, the product formed of their combination will be x^m. Those products in which but one letter a enters, will have $m-1$ factors of x. In these, therefore, x^{m-1} will be multiplied by each of the letters a, and the sum of all these terms will be represented by x^{m-1} multiplied by the sum of all the letters, $a, a', a'' \ldots.$ Let this be expressed by $S\,(a)_1$. Hence the first two terms of the developed product is $x^m+x^{m-1}\,S\,(a)_1$. Those terms which have $m-2$ factors of x will be multiplied by the letters a combined in pairs; and will be equivalent to x^{m-2} multiplied by the sum of every combination of two of the letters, a, a', a'', &c. Let this sum be represented by $S\,(a)_2$ and the first three terms of the product are $x^m+x^{m-1}.\,S\,(a)_1+x^{m-2}\,S\,(a)_2$. And by continuing the same reasoning, and preserving the same notation, the continued product of m factors of the form $(x+a)\,(x+a')\,(x+a'')\ldots\ldots$ is, when developed,

Binomial Theorem.

$$x^m+x^{m-1}\,S\,(a)_1+x^{m-2}\,S\,(a)_2+x^{m-3}\,S\,(a)_3+\text{\&c.}$$

By the preceding section it appears, that the number of terms in $S\,(a)_1$, $S\,(a)_2$, $S\,(a)_3$, &c. respectively are

$$m,\ \frac{m\,(m-1)}{1\,.\,2},\ \frac{m\,(m-1)\,(m-2)}{1\,.\,2\,.\,3}\ \text{\&c.}$$

Now if the accents be removed from the letters a, and they be supposed to become equal, we have evidently

$$S\,(a)_1=m\,a\quad S\,(a)_2=\frac{m\,(m-1)}{1\,.\,2}\,a^2$$

$$S\,(a)_3=\frac{m\,(m-1)\,(m-2)}{1\,.\,2\,.\,3}\,a^3$$

$$S\,(a)_4=\frac{m\,(m-1)\,m-2)\,(m-3)}{1\,.\,2\,.\,3\,.\,4}\,a^4$$

&c. &c.

Hence we obtain

$$(x+a)^m=x^m+m\,x^{m-1}a+\frac{m\,(m-1)}{1\,.\,2}\,x^{m-2}\,a^2$$

$$+\frac{m\,(m-1)\,(m-2)}{1\,.\,2\,.\,3}\,x^{m-3}\,a^3$$

$$+\frac{m\,(m-1)\,(m-2)\,(m-3)}{1\,.\,2\,.\,3\,.\,4}\,x^{m-4}\,a^4+\text{\&c.}\ [1]$$

which is the *binomial series.*

(247.) It is plain that the coefficient of the r^{th} term of this series is the number of combinations of r letters which can be formed from m letters, and that the exponent of a in each term is equal to the number of preceding terms, and in the r^{th} term it is therefore $r-1$; while the exponent of x is the given exponent m diminished by this number, and is therefore $m-(r-1)$. Thus the r^{th} term of the series is

$$\frac{m\,(m-1)\,(m-2)\,(m-3)\ldots.(m-r+1)}{1\,.\,2\,.\,3\,.\,4\,\ldots\ldots\,(r-1)}\,x^{m-r+1}\,a^{r-1}.$$

It appears that the sum of the exponents of x and a in every term is the same, and $=m+1$.

(248.) Each successive term of the series may be conceived to be produced by multiplying the preceding term by a fraction, one factor of which is $\frac{a}{x}$, and the other factor having for its numerator the given exponent m, diminished by one less than the number of preceding terms, and for its denominator the entire number of preceding terms. Thus the *third* term is

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found by multiplying the second by $\frac{m-1}{2} \cdot \frac{a}{x}$, the fourth by multiplying the third by $\frac{m-2}{3} \cdot \frac{x}{a}$, &c. In this case the numeral factor $\frac{m-1}{2}$, $\frac{m-2}{3}$, &c. by being multiplied into the preceding coefficient, produces the next coefficient, and the literal factor $\frac{a}{x}$, produces the literal part of the term. The exponent of x is thus continually diminished by one each step, while that of a is increased by one. This generating fraction for the r^{th} term is

$$\frac{m-r+2}{r-1} \cdot \frac{a}{x}.$$

The series terminates when the generating fraction becomes $= 0$. Let n be the number of terms; the generating fraction of the $(n+1)^{th}$ term must $= 0$. Hence by substituting $n+1$ for r in the numerator of the generating fraction, and putting it $= 0$, we have

$$m - n - 1 + 2 = 0$$
$$\therefore n = m + 1.$$

Hence the total number of terms in the series exceeds the given exponent by one.

(249.) If x be changed into a, and *vice versâ* in the equality [1] we shall have

$$(a+x)^m = a^m + m\,a^{m-1}\,x + \frac{m\,(m-1)}{1\;.\;2}\,a^{m-2}\,x^2$$
$$+ \frac{m\,(m-1)\,(m-2)}{1\;.\;2\;.\;3}\,a^{m-3}\,x^3$$
$$+ \frac{m\,(m-1)\,(m-2)\,(m-3)}{1\;.\;2\;.\;3\;.\;4}\,a^{m-4}\,x^4 + \ldots . \quad [2]$$

This series can only differ from [1] in having the terms in an opposite order. It appears, however, that the coefficients remain exactly the same, from whence we infer, that in the binomial series [1] *the coefficients of every pair of terms equally distant from the extreme terms are equal.* This might also be inferred from (244.)

(250.) The coefficients depending entirely on the exponent m will be the same, whatever values x and a be supposed to have. Let $x = a = 1$, and the series becomes

$$(1+1)^m = 2^m = 1 + m + \frac{m\,(m-1)}{1\;.\;2}$$
$$+ \frac{m\,(m-1)\,(m-2)}{1\;.\;2\;.\;3} + \&c.$$

In this case the second member of the equality is reduced to the sum of the coefficients. Thus it appears, that the sum of the coefficients of the binomial series is equal to that power of 2 whose exponent is equal to the exponent of the binomial.

(251.) If the second member a of the binomial be negative, it is sometimes called a *residual*. In this case the odd powers of a will be negative, and as these are factors of the alternate terms beginning from the second, these terms will be negative, and the binomial will assume the form

Binomial Theorem.

$$(x-a)^m = x^m - m\,x^{m-1}\,a + \frac{m\,(m-1)}{1\;.\;2}\,x^{m-2}\,.\,a^2$$
$$- \frac{m\,(m-1)\,(m-2)}{1\;.\;2\;.\;3}\,x^{m-3}\,a^3, \&c.,$$

(252.) We have hitherto considered the binomial series as representing the developement only where the exponent m is a positive integer, and the demonstration derived from the properties of combinations; and the continued multiplication of different binomials evidently proceeds on that hypothesis. It may, however, be proved, that the series will maintain the same form, and be governed by the same law, when the exponent is negative or fractional. The following demonstration, given by EULER, extends to the cases where m is any rational number, positive or negative.

(253.) Let $(x+a)$ be expressed in the form $x\left(1+\frac{a}{x}\right)$ and we have

$$(x+a)^m = x^m\left(1+\frac{a}{x}\right)^m$$

or if $z = \frac{a}{x}$,

$$(x+a)^m = x^m\,(1+z)^m$$

The question is then, to show that the developement

$$(1+z)^m = 1 + m\,z^{m-1} + \frac{m\,(m-1)}{1\;.\;2}\,z^{m-2}$$
$$+ \frac{m\,(m-1)\,(m-2)}{1\;.\;2\;.\;3}\,z^{m-3} + \&c.$$

is true whatever be the value of m.

Let the problem be converted, and let us inquire what algebraical expression has the preceding developement when m is a fraction. Let the sought expression be y, so that

$$y = 1 + m\,z + \frac{m\,(m-1)}{1\;.\;2}\,z^2$$
$$+ \frac{m\,(m-1)\,(m-2)}{1\;.\;2\;.\;3}\,z^3 + \&c. \qquad [1.]$$

Let m' be another fractional exponent, and y' the corresponding algebraical expression or equivalent for the series, $\therefore$

$$y' = 1 + m'\,z + \frac{m'\,(m'-1)}{1\;.\;2}\,z^2$$
$$+ \frac{m'\,(m'-1)\,(m'-2)}{1\;.\;2\;.\;3}\,z^3 + \&c. \qquad [2.]$$

If these two equalities be multiplied, the first member of the result will be $y\,y'$; but to ascertain by direct multiplication the form of the second member would be attended with some difficulty. It is evident, however, that the product of the second members of [1] and [2] will necessarily be the same in form, whatever m and m' be supposed to represent; and, therefore, whatever form that product will have when m and m' are supposed to be positive integers, will necessarily also have when they are fractions. But in the former case we have

$$(1+z)^m = 1 + m\,z + \frac{m\,(m-1)}{1\;.\;2}\,z^2 +$$
$$\frac{m\,(m-1)\,(m-2)}{1\;.\;2\;.\;3}\,z^3 + \&c.$$

$$(1+z)^{m'} = 1 + m'z + \frac{m'(m'-1)}{1 \,.\, 2}z^2 + \frac{m'(m'-1)(m'-2)}{1 \,.\, 2 \,.\, 3}z^3 + \&c.$$

$$\therefore (1+z)^{m+m'} = \left(1 + mz + \frac{m(m-1)}{1 \,.\, 2}z^2 + \frac{m(m-1)(m-2)}{1 \,.\, 2 \,.\, 3}z^3 + \&c.\right)$$
$$\times\left(1 + m + z\frac{m'(m'-1)}{1 \,.\, 2}z^2 + \frac{m'(m'-1)(m'-2)}{1 \,.\, 2 \,.\, 3}z^3 + \&c.\right)$$

But also

$$(1+z)^{m+m'} = 1 + (m+m')z + \frac{(m+m')(m+m'-1)}{1 \,.\, 2}z^2 + \&c.$$

Hence the second member of this last equality gives the form of the product of the second members of [1] and [2] when m and m' are positive integers, and the same form must continue when they are fractions. Thus we have

$$yy' = 1 + (m+m')z + \frac{(m+m')(m+m'-1)}{1 \,.\, 2}z^2 + \&c. \qquad [3.]$$

By continuing the same reasoning, if m be supposed successively to assume the values m', m'', m''', &c. all fractional, and y', y'', y''', &c. be the corresponding equivalents of the series, and $r = m + m' + m'' +$ &c., we shall have

$$yy'y''y''' \ldots = 1 + rz + \frac{r(r-1)}{1 \,.\, 2}z^2 + \frac{r(r-1)(r-2)}{1 \,.\, 2 \,.\, 3}z^3 + \&c. \qquad [4.]$$

Now suppose $m = m' = m'' =$ &c., and let q be the number of repetitions, so that $r = qm$, the equality [4] becomes

$$y^q = 1 + mqz + \frac{mq(mq-1)}{1 \,.\, 2}z^2 + \frac{mq(mq-1)(mq-2)}{1 \,.\, 2 \,.\, 3}z^3 + \&c.$$

or since m is supposed to be fractional let $m = \frac{p}{q}$ $\therefore$

$$y^q = 1 + pz + \frac{p(p-1)}{1 \,.\, 2}z^2 + \frac{p(p-1)(p-2)}{1 \,.\, 2 \,.\, 3}z^3, \&c.$$

But the second member of this, since p is a positive integer, is equal to $(1+z)^p$ $\therefore$

$$y^q = (1+z)^p$$

$$\therefore y = (1+z)^{\frac{p}{q}} = (1+z)^m$$

Hence the developement

$$(1+z)^m = 1 + mz + \frac{m \,.\, m-1}{1 \,.\, 2}z^2 + \frac{m(m-1)(m-2)}{1 \,.\, 2 \,.\, 3}z^3 + \&c.$$

holds good when m is a fraction and positive.

To extend the proof to negative exponents it is only necessary in [3] to suppose $m' = -m$ $\therefore m + m' = 0$

$$\therefore yy' = 1,$$

$$\therefore y = \frac{1}{y'} = y'^{-1}$$

But we have already proved that

$$y' = (1+z)^{m'}$$

$$\therefore y = \frac{1}{(1+z)^{m'}} = (1+z)^{-m'} = (1+z)^m$$

and, therefore,

$$(1+z)^m = 1 + mz + \frac{m(m-1)}{1 \,.\, 2}z^2 \;\&c.$$

is true when m is negative.

Thus, then, the binomial theorem is extended to all cases in which the exponent is a rational number, whether positive or negative.

(254.) This extension of the principle being made, there will be no difficulty in applying it to the approximation to the roots of numbers.

In the series

$$(x+a)^m = x^m\left\{1 + m \,.\, \frac{a}{x} + \frac{m \,.\, (m-1)}{1 \,.\, 2} \,.\, \frac{a^2}{x^2} + \&c.\right\}$$

Let $m = \frac{1}{n}$. Hence

$$\sqrt[n]{x+a} + \sqrt[n]{x}\left\{1 + \frac{1}{n} \,.\, \frac{a}{x} - \frac{1}{n} \,.\, \frac{n-1}{2n} \,.\, \frac{a^2}{x^2} + \frac{1}{n} \,.\, \frac{n-1}{2n} \,.\, \frac{2n-1}{3n} \,.\, \frac{a^3}{x^3} - \&c.\right\}$$

Let it be proposed to apply this series to the extraction of the cube root of 31. To effect this, it is necessary first to find the nearest complete cube to 31, which is $27 = 3^3$ $\therefore \sqrt[3]{27} = 3$. Hence

$$31 = (x+a) = 27 + 4$$

$$\therefore \sqrt[3]{27+4} = 3\left(1 + \frac{1}{3} \,.\, \frac{4}{27} - \frac{1}{3} \,.\, \frac{1}{3} \,.\, \frac{16}{729} + \frac{1}{3} \,.\, \frac{1}{3} \,.\, \frac{5}{9} \,.\, \frac{64}{19683} - \&c.\right)$$

$$\therefore \sqrt[3]{31} = 3 + \frac{4}{27} - \frac{16}{2187} + \frac{320}{531441} - \&c.$$

The first three positive terms of the series expressed in decimals, are

$$\left.\begin{array}{r} 3 = 3{,}00000 \\ \frac{4}{27} = 0{,}14815 \\ \frac{320}{531441} = 0{,}00060 \end{array}\right\} = 3{,}14875$$

and the first two negative terms are

$$\left.\begin{array}{r} -\frac{16}{2187} = -\,0{,}00731 \\ -\frac{2560}{43046721} = -\,0{,}00006 \end{array}\right\} = -\,0{,}00737$$

$$\therefore \sqrt[3]{31} = 3{,}14138$$

(255.) A series is said to *converge*, when the numerical value of each term is less than that of the preceding term; and is said to converge more or less rapidly, as the ratio of each term to that which succeeds it is a greater or lesser ratio. By means of such a series we can always approximate in rational numbers to the value of that quantity of which it is the developement. Such a developement may be considered to be numerically equivalent to the quantity from which it was obtained. It, however, frequently happens, that the successive terms of the developement, instead of decreasing, increase. In this case, no numerical equality exists between the series and the quantity whose developement it represents; and the sign $=$ placed between them, is to be understood only as indicating, that the one is obtained from the other by a certain process which has been instituted, and these observations are equally applicable, whether the developement be obtained by the binomial theorem or any other way.

If any number of terms of a converging series, beginning from the first, be taken to represent the value of the whole, there is a certain error introduced, the limits of which may in some cases be assigned. Let the series be

$$a - b + c - d + e - f + \&c.$$

the terms being supposed to decrease. Let it be required to assign the error to which we are subject in taking $a - b + c$ for the whole series. Let x be the quantity to which the whole series is equal. Since each positive quantity is greater than the negative quantity which immediately succeeds it, it follows, that the successive quantities $(a - b)$, $(c - d)$, $(e - f)$, &c. are all positive, and, consequently, the sum of any number of them, commencing from the first, will be less than x. Hence

$$x > a - b$$
$$x > a - b + c - d$$
$$x > a - b + c - d + e - f$$
$$\&c. \quad \&c.$$

Again, for the same reason, the successive quantities $(-b + c)$, $(-d + e)$, $(-f + g)$, &c. are negative, and, therefore, when the sum of any number of them be added to a, the result is greater than x, so that

$$x < a$$
$$x < a - b + c$$
$$x < a - b + c - d + e$$
$$x < a - b + c - d + e - f + g$$
$$\&c. \quad \&c.$$

Hence it follows, that the value of x is between the values of a and $a - b$; it is also between those of $a - b$ and $a - b + c$; also between those of $a - b + c$ and $a - b + c - d$, and so on. Therefore, if a be taken as equal to x, the error will be less than b; if $a - b$ be taken for x, the error will be less than c, and so on.

To apply this to the example already given, we have

$$\sqrt[3]{31} = 3 + \frac{4}{27} - \frac{16}{2187} + \frac{320}{531441} - \frac{2560}{43046721} + \&c.$$

If the first two terms be taken to represent $\sqrt[3]{31}$, the assumed value will be greater than the true value, by a fraction less than $\frac{16}{2187}$; if these terms be taken, the assumed value will be less than the true by a fraction less than $\frac{320}{531441}$, &c.

(256.) The general rule for the application of the binomial theorem to the approximation to the roots of numbers is as follows: Let the n^{th} root be sought: find the nearest complete n^{th} power to the proposed number, and let this be p^n, and let the difference between this and the proposed number be q, so that the proposed number being N, we shall have

$$N = p^n + q$$

when $p^n < N$, and

$$N = p^n - q$$

when $p^n > N$. In this series

$$(x + a)^{\frac{1}{n}} = x^{\frac{1}{n}} \left\{ 1 + \frac{1}{n} \cdot \frac{a}{x} - \frac{1}{n} \cdot \frac{n-1}{2n} \cdot \frac{a^2}{x^2} + \frac{1}{n} \cdot \frac{n-1}{2n} \cdot \frac{2n-1}{3n} \cdot \frac{a^3}{x^3} \; \&c. \right\}$$

substitute p^n for x, and q for a. The result will be a converging numerical series, provided a be less than x. The several terms being reduced to decimals, and combined by addition or subtraction, as indicated by the signs, will give the value of the root with any required degree of approximation.

(257.) Since the successive terms of the developements of $(x + a)^m$ and $(x - a)^m$ differ in nothing but the signs of the alternate terms, beginning from the second, it follows that if their developements be added, the result will be twice the sum of the alternate terms, beginning from the first; and if they be subtracted, their difference will be twice the sum of the alternate terms, beginning from the second.

(258.) Also, since the alternate terms of the series, beginning from the first, contain only even powers of a, and the alternate terms, beginning from the second, contain only odd powers of a, it follows that the developement of

$$(x + a)^m + (x - a)^m$$

contains no odd power of a, and that of

$$(x + a)^m - (x - a)^m$$

contains no even power of a.

(259.) If a be a quadratic surd, such as $\sqrt{b}$, $\sqrt{3}$, &c. the developement of

$$(x + a)^m + (x - a)^m$$

will be rational, since all the even powers of $\sqrt{b}$, $\sqrt{3}$, &c. are rational.

Also, if a be an imaginary quantity of the second order, such as $\sqrt{-b^2}$, $\sqrt{-3}$, &c. the developement of

$$(x + a)^m + (x - a)^m$$

will be real. Also, in this case, the developement of

$$\frac{(x + a)^m - (x - a)^m}{\sqrt{-1}}$$

will be real.

(260.) The developements obtained by the ordinary process of division, may also be obtained by the binomial theorem. Thus, by division we find

$$\frac{1}{a + b} = \frac{1}{a} - \frac{b}{a^2} + \frac{b^2}{a^3} - \frac{b^3}{a^4} + \&c.$$

The same may be obtained by the binomial theorem, by substituting for $\frac{1}{a + b}$ its equivalent $(a + b)^{-1}$.

SECTION XXIV.

Method of Indeterminate Coefficients.—Of Series.

(261.) If the *form* of the developement of any quantity be assumed, the determination of the developement is reduced to the investigation of the values of the coefficients of the powers of thet quantity by which the series is supposed to be arranged. These coefficients being supposed to be independent of the latter quantity, will be the same, whatever value be assigued to it; and on this fact is founded that method of developement called the *method of indeterminate coefficients*.

Let the formula to be developed be

$$\frac{a}{b+b'x},$$

and let the form of the required developement be

$$A_0 + A_1 x + A_2 x^2 + A_3 x^3 + A_4 x^4 + \&c.$$

A_0, A_1, A_2 .... being quantities indeterminate or unknown, but supposed to be independent of x. Equating the series with the formula it represents, we have

$$\frac{a}{b+b'x} = A_0 + A_1 x + A_2 x^2 + A_3 x^3 + A_4 x^4 + \&c.$$

Since the values of the several coefficients in each member of this equation are independent of x, they will be the same whatever value x be supposed to receive. If $x = 0$ the equction becomes

$$\frac{a}{b} = A_0,$$

which determines the value of the first coefficient.

Making this eubstitution, clearing the equation of fractions, bringing all the terms to the same side, end arranging them by the ascending powers of x, we have

$$\begin{array}{r|r|r|r|l} A_1 b & x + A_2 b' & x^2 + A_3 b & x^3 + A_4 b & x^4 + \ldots = 0 \\ + \dfrac{a b'}{b} & + A_1 b' & + A_2 b' & + A_3 b' & \end{array}$$

which being divided by x becomes

$$\begin{array}{r|r|r|r|l} A_1 b & + A_2 b & x + A_3 b & x^2 + A_4 b & x^3 + \ldots = 0 \\ + \dfrac{a b'}{b} & + A_1 b' & + A_2 b' & + A_3 b' & \end{array}$$

In this, if $x = 0$ we have

$$A_1 a' + \frac{a b'}{b} = 0, \quad \therefore A_1 = -\frac{a b'}{b^2}.$$

This condition being observed, and the equation again divided by x, it becomes

$$\begin{array}{r|r|r|l} A_2 b & + A_3 a' & x + A_4 a' & x^2 + \ldots = 0 \\ - \dfrac{a b'^2}{b^2} & + A_2 b' & + A_3 b' & \end{array}$$

and x being again eupposed $= 0$, we have

$$A_2 b - \frac{a b'^2}{b^2} = 0, \therefore A_2 = \frac{a b'^2}{b^3},$$

and by continuing the same process we should find

$$A_3 = -\frac{a b'^3}{b^4}, \quad A_4 = \frac{a b'^4}{b^5}, \quad A_5 = -\frac{a b'^5}{b^6}, \&c.$$

In effect, each succeeding coefficient is found by multiplying the preceding one by

$$-\frac{b'}{b},$$

and each term ie found by multiplying the preceding term by

$$-\frac{b'}{b}\,x.$$

(262.) The principle here used when generalized, proves that if an equation of the form

$$A + B x + C x^2 + D x^3 + \ldots\ldots = 0$$

be fulfilled independently of x, it is necessary that eech of its coefficients severally should $= 0$; and it is, in fact, equivalent to the several equations

$$A = 0,\ B = 0,\ C = 0,\ \&c.$$

(263.) From this principle we may immediately infer, that if an equation of the form

$$a + bx + ex^2 + dx^3 + \ldots = A + Bx + Cx^2 + Dx^3 +$$

be fulfilled independently of x, (that is, be true whatever valne be ascribed to x,) we shall bave

$$a = A,\ b = B,\ e = C,\ \&c.$$

For it mey be reduced to the form

$$(a - A) + (b - B)\,x + (c' - C)\,x^2 + \ldots. = 0.$$

Hence by (261) we have

$$a - A = 0,\ b - B = 0,\ e - C = 0,\ \&c.$$

$$\therefore a = A,\ b = B,\ e = C,\ \&c.$$

(264.) We bave before stated, that in the epplication of the method of indeterminate coefficients the *form* of the developement ie essumed. It may so bappen, thet the form assumed is one in which the given quantity cannot be developed. In this ease the process will lead to some manifest absurdity, indicative of the falsehood involved in the equality which was instituted between the given expression and the proposed form of developement.

As an example of this, let it be required to develope the fraction $\frac{1}{3x - x^2}$ in ascending integral and positive powers of x, so thet

$$\frac{1}{3x - x^2} = A + B x + C x^2 + D x^3 + E x^4 + \&c.$$

Clearing this of fractions, and bringing all the terms to the same side, we bave

$$\begin{array}{r|r|l} -1 + 3Ax + 3B & x^2 + 3C & x^3 + \ldots. = 0 \\ - A & - B & \end{array}$$

Now if $x = 0$, we have $-1 = 0$, which is absnrd, and shows that the expression cannot be developed in the form required.

If, however, the original expression be resolvad into its factors $\frac{1}{x}$ and $\frac{1}{3 - x}$, the latter may be developed in the required form, and we find

$$\frac{1}{3 - x} = \frac{1}{3} + \frac{x}{3^2} + \frac{x^2}{3^3} + \frac{x^3}{3^4} + \&c.$$

$$\therefore \frac{1}{3x - x^2} = \frac{1}{3x} + \frac{1}{3^2} + \frac{x}{3^3} + \frac{x^2}{3^4} + \&c.$$

$$\text{or } \frac{1}{3x - x^2} = \frac{x^{-1}}{3} + \frac{x^0}{3^2} + \frac{x^1}{3^3} + \frac{x^2}{3^4} + \&c.$$

(265.) Let the expressiono t be developed be

$$\frac{a + a'x}{b + b'x + b''x^2},$$

and the form of the developement being as before, we have

$$\frac{a + a'x}{b + b'x + b''x^2} = A_0 + A_1 x + A_2 x^2 + A_3 x^3 + \&c.$$

which when reduced becomes

$$\begin{array}{c|c|c|c|c|l} A_0 b & + A_1 b & x + A_2 b & x^2 + A_3 b & x^3 + A_4 b & x^4 + ..0 \\ - a & + A_0 b' & + A_1 b' & A_2 b' & A_3 b' & \\ & - a' & + A_0 b'' & A_1 b'' & A_2 b'' & \end{array}$$

sioce this is fulfilled, independently of x, it gives

$$A_0 b - a = 0$$
$$A_1 b + A_0 b' - a' = 0$$
$$A_2 b + A_1 b' + A_0 b'' = 0$$
$$A_3 b + A_2 b' + A_1 b'' = 0$$
$$A_4 b + A_3 b' + A_2 b'' = 0$$
&c. &c.

Hence we obtain

$$A_0 = \frac{a}{b}$$

$$A_1 = -\frac{b'}{b} A_0 + \frac{a'}{b} = \frac{-b'a + a'b}{b^2}$$

$$A_2 = -\frac{b'}{b} A_1 - \frac{b''}{b} A_0 = \frac{a b'^2 - a'b'b - a b b''}{b^3}$$

$$A_3 = -\frac{b'}{b} A_2 - \frac{b''}{b} A_1$$

$$A_4 = -\frac{b'}{b} A_3 - \frac{b''}{b} A_2$$
&c. &c.

and in general

$$A_n = -\frac{b'}{b} A_{n-1} - \frac{b''}{b} A_{n-2}.$$

Thus we obtain a general rule for determining each successive coefficient, viz. "Multiply the preceding coefficient by $-\frac{b'}{b}$, and the last but one by $-\frac{b''}{b}$, and the sum of the results is the coefficient sought."

This rule applies to all the coefficieots after the secood term. The first two terms, however, must be determined by the formulæ established for them in particular.

Had the expression to be developed been

$$\frac{a + a'x + a''x^2}{b + b'x + b''x^2 + b'''x^3}$$

we should have had

$$A_0 b - a = 0$$
$$A_1 b + A_0 b' - a' = 0$$
$$A_2 b + A_1 b' + A_0 b'' - a''$$
$$A_3 b + A_2 b' + A_1 b'' + A_0 b''' = 0$$
$$A_4 b + A_3 b' + A_2 b'' + A_1 b''' = 0$$
&c. &c.

and in general

$$A_n b + A_{n-1} . b' + A_{n-2} . b'' + A_{n-3} . b''' = 0.$$

Hence we fiod

$$A_0 = \frac{a}{b}$$

$$A_1 = -\frac{b'}{b} A_0 + \frac{a'}{b} = \frac{-a b' + b a'}{b^2}$$

$$A_2 = -\frac{b'}{b} A_1 - \frac{b''}{b} A_0 + \frac{a''}{b}$$
$$= \frac{a b'^2 - a' b b' - a b b'' + a'' b^2}{b^3},$$

and the remaining coefficients would be determined by

$$A_3 = -\frac{b'}{b} A_2 - \frac{b''}{b} A_1 - \frac{b'''}{b} A_0$$

$$A_4 = -\frac{b'}{b} A_3 - \frac{b''}{b} A_2 - \frac{b'''}{b} A_1$$

$$A_5 = -\frac{b'}{b} A_4 - \frac{b''}{b} A_3 - \frac{b'''}{b} A_2,$$
&c. &c.

and in geoeral

$$A_n = -\frac{b'}{b} A_{n-1} - \frac{b''}{b} A_{n-2} - \frac{b'''}{b} A_{n-3}$$

(266.) In the three examples which have been given, of the application of the method of indeterminate coefficients, it may be observed, that in the first, each term was derived from that which immediately preceded it by multiplying by a coostant factor; in the secood, each term was derived from the two which immediately preceded it by multiplying each of them by a constant factor, *scil.*, that which immediately preeded by $-\frac{b'}{b}x$, and the other by $-\frac{b''}{b}x^2$. In like manner, in the third example each term is derived from the three preceding terms by multiplying them respectively by the constant factors $-\frac{b'}{b}x$, $-\frac{b''}{b}x^2$, $-\frac{b'''}{b}x^3$, and adding the results.

Series formed or *generated* in this way are called *recurring series*, and the system of coostant multipliers is called the *scale of relation*. The *order* of the recurriog series is determined by the number of constant multipliers in the *scale* of relation. Thus, the first of the preceding examples presents a recurring series of the *first order*, the secood gives one of the *second order*, and the *third* one of the *third order*, the *scales of relation* being respectively

$$\left(-\frac{b'}{b}x\right), \left(-\frac{b'}{b}x, -\frac{b''}{b}x^2\right),$$
$$\left(-\frac{b'}{b}x, -\frac{b''}{b}x^2, -\frac{b'''}{b}x^3\right).$$

It is evident, that by continuing the same reasoning we should find io geoeral that the developement of

$$\frac{a + a'x + a''x^2 + \ldots . a^{(n-1)}x^{n-1}}{b + b'x + b''x^2 + \ldots . b^{(n)}x^n}$$

would be a recurring series of the n^{th} order, of which the scale of relatioo would be

$$\left(-\frac{b'}{b}x, -\frac{b''}{b}x^2, -\frac{b'''}{b}x^3 \ldots\ldots -\frac{b^{(n)}}{b}x^n\right)$$

It is evident, that a recurring series of the first order is a geometrical progressioo.

SECTION XXV.

Of Logarithms

(267.) In the indeterminate equation $y = a^x$ for every value, whether positive or negative, which is assigned to x, there will result a corresponding value of y, and *vice versâ*, any numerical value whatever being assigned to y, there will be a corresponding number, which, substituted for x, will verify the equation. This, however, is on the condition that a is not $= 1$, for if it were, y would also be $= 1$, whatever value should be given to x. Let N, N′, N″ .... be any values of y, and let the corresponding values of x, determined either exactly or approximately, be n, n', n'', &c. we have

$$N = a^n,\ N' = a^{n'},\ N'' = a^{n''}, \text{\&c.}$$

The value of a being arbitrarily chosen, subject to the exception already mentioned, and being supposed to remain the same, there will be a fixed and constant relation between the numbers expressed by y and x. This peculiar relation is expressed by calling x the *logarithm* of y. Thus, n is the *logarithm* of N, n' of N′, &c. The constant quantity a is called the *base* of the logarithms.

(268.) In the theory of logarithms, therefore, all numbers are considered as powers of some one number, which is called the *base*, and the exponents of the powers are called the logarithms of the numbers. The *logarithm* is usually expressed by *log*, or simply the letter l placed before the number; thus log y, or ly, signifies the logarithm of y.

(269.) If the base of the system be 10, we have evidently

$$l\,(100) = 2,\quad l\,(1000) = 3,\quad l\,(10000) = 4, \text{\&c.}$$

since 100 is the square of 10, 1000 the third power, 10000 the fourth power, &c.

(270.) If all numerical values whatever be supposed to be successively ascribed to y, and written in one column, and that the corresponding values of x in the equation $y = a^x$ be determined, and written in another column, the corresponding values being placed opposite to one another, we shall have what is called *a table of logarithms*, so that when any number is given, its logarithm will be found registered in this table, and *vice versâ*, when any logarithm is given, the corresponding number may also be found. The nature of such a table, and the method of constructing it, we shall more fully explain hereafter. We shall at present show how such a table would be instrumental in expediting several numerical operations.

Let y, y', y'' .... be several numbers, and a be the base of the logarithms, we have

$$y = a^{ly},\quad y' = a^{ly'},\quad y'' = a^{ly''}, \text{\&c.}$$

By multiplying these we obtain

$$y\,y'\,y'' \ldots\ldots = a^{ly+ly'+ly''+\ldots\ldots}$$

But also $$y\,y'\,y'' \ldots\ldots = a^{l(yy'y''\ldots\ldots)}$$

$$\therefore\ l\,(y\,y'\,y''\ldots) = ly + ly' + ly'' + \ldots\ldots$$

That is, the logarithm of the continued product of any numbers is equal to the sum of the logarithm of the factors.

Hence, if it be required to multiply several numbers together, it is only necessary to obtain their logarithms from the table; add these logarithms together, and then obtain the number of which the sum is the logarithm. Thus continued multiplication is reduced to continued addition.

(271.) Let the equations $y = a^{ly}$, $y' = a^{ly'}$ be divided one by the other, and we obtain $\frac{y}{y'} = a^{ly - ly'}$. But also $\frac{y}{y'} = a^{\,l\left(\frac{y}{y'}\right)}\ \therefore\ l\left(\frac{y}{y'}\right) = ly - ly'$. That is, "The logarithm of the quote is equal to the logarithm of the dividend, *minus* the logarithm of the divisor."

If then it be required to divide one number by another, let the logarithms of these numbers be taken from the tables, and that of the divisor subtracted from that of the dividend, and let the number be found in the tables whose logarithm is equal to the remainder, this number is the quote. Thus division is reduced to subtraction.

(272.) Let both members of

$$y = a^{ly}$$

be raised to the n[th] power, and it becomes

$$y^n = a^{nly}$$
$$\therefore\ l\,(y^n) = n\,ly.$$

That is, the logarithm of any power of a number is equal to the logarithm of the number multiplied by the exponent of the power.

Hence, to obtain any required power of a number, let the logarithm of the number be found in the tables, and let the product of that and the exponent of the power be found by the rule (270,) and this being obtained, let the number be found in the tables of which it is the logarithm. This will be the required power.

(273.) Let the n^{th} root of both members of

$$y = a^{ly}$$

be taken, and we have

$$\sqrt[n]{y} = a^{\frac{ly}{n}}$$
$$\therefore\ l\sqrt[n]{y} = \frac{ly}{n}.$$

That is, the logarithm of any proposed root of a number is obtained by dividing the logarithm of the number by the exponent of the root.

Hence, to obtain any proposed root of a number, let its logarithm be taken from the tables, and let the number be divided by this by the rule (271,) and then let the number be found in the tables whose logarithm is equal to this quote. This number in the required root.

(274.) Thus it appears, that by the aid of a table of logarithms we shall be able to reduce all calculations where products, quotes, powers, or roots, are required to simple addition and subtraction.

(275.) The number most commonly taken for the base of a system of logarithms is 10. However, if a system be computed with respect to any base a, it will be easy to obtain from it a system relatively to another base a'. Let y be any number, and let ly be its logarithm relative to the base a, and $l'y$ relative to the base a'. We have

$$y = a^{ly} \qquad y = a'^{\,l'y}$$
$$\therefore\ a^{\,ly} = a'^{\,l'y}.$$

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Taking the logarithms of these relatively to the base a, we have

$$ly = l'y\, la' \because l'y = \frac{ly}{la'}$$

(276.) Hence, when the logarithm of numbers relatively to any base is known, the logarithms of numbers relatively to any other base may be found by dividing the given logarithms by the logarithm of the new base in the given system.

(277.) In the computation of logarithmic tables, it is not necessary actually to calculate the logarithms of fractions, because they can always be found from those of their numerators and denominators by the rule in (271.) Neither is it necessary, in the first instance, to compute the logarithms of any but prime integers, for all others being products of these, their logarithms may be derived by adding those of their factors, (270.) Thus $l6 = l2 + l3$.

(278.) We shall proceed first to explain the method of using tables of logarithms, and then to show the methods by which these tables are computed.

Let us suppose that the base of our system is 10. The only numbers whose logarithms are rational, are 100, 1000, 10000, &c. All others must be expressed approximately, and we shall suppose the approximation carried to seven decimal places.

If in the equation $y = 10^x$, $x = 0$, we have $y = 1$. Therefore the logarithm of $1 = 0$. This is common to all systems. If $x = 1, 2, 3, 4$, &c., we have $y = 10$, $y = 100$, $y = 1000$, $y = 10000$, &c. Hence the logarithm of the base itself is $= 1$, which is also common to all systems.

If $x = -1, -2, -3$, &c.

$$y = \frac{1}{10}, y = \frac{1}{100}, y = \frac{1}{1000}, \text{\&c.}$$

The logarithms of all numbers less than 1 are negative, and the logarithm of 0 is $-\frac{1}{0} = -\infty$, while the logarithm of an infinitely great number is $+\frac{1}{0} = +\infty$.

The logarithm of an integer < 10 is < 1, and, therefore, there is no significant digit before the decimal point in the value of such a logarithm. In this case, 0 may be conceived to precede the point, which always happens, therefore, when the number is expressed by a single digit.

If the number consist of two digits, it is between 10 and 100. Its logarithm is therefore > 1 and < 2, and, therefore, the digit which precedes the point in the logarithm is 1.

If the number consist of three digits, it is between 100 and 1000, and its logarithm is between 2 and 3. Therefore 2 is the digit which precedes the point in the logarithm.

In general, if the number consist of n digits, it is between 10^{n-1} and 10^n, and its logarithm is between $n-1$ and n, and therefore $n - 1$ must be the digit which precedes the point in the logarithm.

The digit which precedes the point in the logarithm of a number is called the *characteristic* of the logarithm. Thus the *characteristic* is always that integer which is one less than the digits of the number.

Logarithms.

(279.) If a number end with any number of cyphers, they may be cut off, and the logarithm of the remaining part found, as many units being added to it when so found as there were cyphers cut off. For let the value of the number without the cyphers be N, and let n be the number of cyphers cut off. The original number is $N \times 10^n$, the logarithm of which is $lN + n$.

In like manner, if a number be divided by a power of 10, the logarithm of the quote may be found by subtracting from the logarithm of the number as many units as there are in the exponent of the power. For let the number be N,

$$l\frac{N}{10^n} = lN - n.$$

Thus, to obtain the logarithm of any number having n decimal places, let the logarithm of the number considered as an integer be first found, and then let n be subtracted from it.

(280.) It appears from the uses of logarithmic tables already explained in multiplication, division, &c. that two processes are required in every operation: 1. to find the logarithm of a given number; and 2. to find the number corresponding to a given logarithm.

1. *To determine the logarithm of a given number.*

The given number must be either integral or fractional. If it be a fraction, its logarithm is the difference between those of its numerator and denominator. If the fraction be expressed as a decimal, its logarithm being found as an integer, it is only necessary to subtract from the characteristic as many units as there are decimal places. If the proposed number be composed of an integer and a fraction, it can be reduced to a fraction. Thus the determination of the logarithm of any number whatever, is resolved to the determination of the logarithms of integers.

The tables are usually constructed so as to give the logarithms of all integers within a certain limit. If then the integers whose logarithms are required be within this limit, their logarithms will be immediately found annexed to them in the tables.

If, however, it be desired to determine the logarithm of an integer greater than any tabulated integer, let the characteristic be first determined by the number of places. Then let such a number of decimal places be pointed off as will reduce the number of integral places to the greatest number of places in the tabulated integers. Thus, if the number of integral places in the proposed number be 8, and the greatest tabulated integers have but 5 places, it will be necessary to cut off three integral places by the decimal point. This will evidently produce no other effect upon the logarithm of the number, than to diminish its characteristic by as many units as there are places cut off. So that if the logarithm of the number so modified be determined, that of the sought number may be immediately obtained, by adding as many units to the characteristic as there were places cut off.

After the integral places have been thus reduced, let the value of the number be N. Let p be the number of places cut off by the decimal point, so that the original number is $N \times 10^p$. Find in the tables the two integers between which the value of N lies. Let them be n and $n + 1$; so that $N > n$ and $< (n + 1)$, and let the logarithms of n and $n + 1$ be found. We shall show hereafter that when numbers so high as n,

N, and $(n+1)$, are supposed to differ by a number less than unity, we may assume, without any considerable error, that the numbers are proportional to their logarithms, so that we shall have

$$N-n : (n+1)-n :: lN - ln : l(n+1) - ln,$$

or, $\quad lN - ln = \{ l(n+1) - ln \} \times (N-n)$

$$\therefore lN = \{ l(n+1) - ln \} \times (N-n) + ln.$$

In fact, the error which this proportion entails upon the value of lN, does not affect any of the first seven decimal places, and beyond these we do not usually require to extend the calculation.

The value of lN being found, we may immediately determine that of the given number $N \times 10^p$.

$$l(N \times 10^p) = \{ l(n+1) - ln \} (N-n) + ln + p.$$

By this formula, the numbers n and $n+1$, and their logarithms being known, lN may be computed

Example. Let

$$N \times 10^p = 34735879.$$

If we suppose that integers as far as those consisting of five places are tabulated, it is necessary to point off three decimal places. Hence

$$N = 34735{,}879,\ p = 3,\ n = 34735$$
$$n+1 = 34736,$$
$$ln = 4{,}5407673,$$
$$l(n+1) - ln = {,}0000125,$$
$$N - n = {,}879,$$
$$l(N \times 10^p) = {,}879 \times {,}0000125 + 4{,}5407673 + 3,$$
$$\therefore l\,34735879 = 7{,}5407783.$$

2. *To determine a number when its logarithm is given.*

The given logarithm may be positive or negative.

First, Let it be positive.

If the given logarithm be found in the tables, the corresponding integer will be prefixed to it,

If not, let us suppose, in the first instance, that its characteristic is that of the highest number included in the tables. In this case, its value will be found to be between two successive tabulated logarithms, let these be ln and $l(n+1)$, and let the sought number be N. By the formula already established we have

$$N - n = \frac{lN - ln}{l(n+1) - ln}$$

$$\therefore N = \frac{lN - ln}{l(n+1) - ln} + n;$$

for in this case $p = 0$.

Example. Let the proposed logarithm be

$$4{,}7325679,$$

4 being the highest characteristic in the tables. We find by the tables

$$ln = 47325626,$$
$$\therefore lN - ln = {,}0000053,$$
$$n = 54021,$$
$$l(n+1) - ln = {,}0000081,$$
$$\therefore N = \frac{53}{81} + 54021 = 54021{,}65$$

If the characteristic be less than the greatest characteristic of the tables, the formula for approximating to N will not give sufficient accuracy. In this case, therefore, it will be necessary to add as many units to the characteristic, as will render it equal to the highest characteristic of the tables; and to compensate for this, it is only necessary to point off as many additional decimal places in the result as there were units added to the characteristic.

If the characteristic of the given logarithm be greater than the greatest characteristic of the tables, it is necessary to subtract as many units from it as will render it equal to the highest tabular characteristic, and it will be necessary to multiply the number found by that power of 10, whose exponent is equal to the number subtracted from the given characteristic.

Secondly, Let the given logarithm be negative.

Let as many units be added to it as will render it positive, and make its characteristic equal to the highest characteristic of the tables. This being done, let the corresponding number be found in the manner already explained, and let it be divided by that power of 10 whose exponent is equal to the number of units added to the given logarithm, or, what is the same, let the decimal be moved as many places to the left as there were units added.

Example. Let the given logarithm be

$$-2{,}4537875.$$

The highest characteristic of the tables being 4, let 7 be added to this, and the result is

$$4{,}4537875 = \log 35173{,}25.$$

The point must now be moved 7 places to the left, and we obtain

$$-2{,}4537875 = \log 0{,}003517325.$$

A negative logarithm is always the logarithm of a proper fraction. If the sign be changed, it will be the logarithm of the reciprocal of this fraction. Hence arises another method of determining the number corresponding to a given negative logarithm. Let the number corresponding to the positive value of the given logarithm be found, and the reciprocal of this number is the number required. This method is inferior in accuracy to the last, because two approximations are necessary in it. First, an approximation to the number corresponding to the positive value of the given logarithm, and, secondly, an approximation in decimals to the value of the reciprocal. In cases, therefore, where much exactness is required, the former method is to be preferred. In other cases, however, the latter has the advantage of greater expedition.

(281.) In logarithmic calculations it frequently happens, that a number of logarithms are to be added or subtracted. The process is somewhat abridged by the use of what are called *arithmetical complements.*

The arithmetical complement of a logarithm is that number which is found by subtracting it from 10. Thus $10 - x$ is the arithmetical complement of x. Two numbers whose sum is 10 are arithmetical complements of each other. Thus, to determine the arithmetical complement of 6,347218, we have

$$\begin{array}{r} 10{,}000000 \\ 6{,}347218 \\ \hline 3{,}652782 \end{array}$$

It is easy to see that the arithmetical complement of a logarithm may be found at once by subtracting the first digit on the right from 10, and each of the others from 9.

To show the application of this principle, let several

logarithms be united together by the signs + and −, thus $l - l' + l'' - l''' + l'''' -$ &c. Let cl', cl''', &c. be the complements of l', l''', &c., it is evident that

$$l' = 10 - cl' \qquad l''' = 10 - cl'''$$

$$\therefore l - l' + l'' - l''' + l'''' + l = cl' + l'' + cl''' + l'''' - 20$$

or in general let $\Sigma(l)$ signify the sum of all the positive logarithms, and $\Sigma(cl')$ the sum of the complements of all the negative logarithms, and let the number of negative logarithms be n. The whole series will then be reduced to $\Sigma(l) + \Sigma(cl') - 10\,n$. Thus, instead of first adding all the positive numbers, then adding all the negative numbers, and then subtracting the latter sum from the former, we have only to add together all the positive logarithms, and the complements of the negative ones, and subtract from the result the number of n followed by 0; a process comparatively expeditious and simple.

(282.) Exponential equations, of which we have given approximate methods of solution in Sect. XXI. may be immediately solved by logarithmic tables. Taking the logarithms of both members of the equation

$$a^x = b$$

we obtain

$$x\,la = lb \therefore x = \frac{la}{lb}.$$

The unknown quantity may occur as the exponent of the exponent, as in $a^{b^x} = c$. In this case let

$$b^x = y \therefore a^y = c \therefore y = \frac{lc}{la}.$$

Hence $ly = llc - lla$. But by taking the logarithms of both members of $b^x = y$, we have $ly = xlb$ ∴

$$x = \frac{llc - lla}{lb}.$$

(283.) The meaning of the notation llc, lla, is obvious. The logarithm of the number c being found, it becomes in its turn a number whose logarithm is sought. Thus, the logarithm of lc is expressed by llc. It is, however, expressed with more elegance and brevity by l^2c, the number 2 not expressing an exponent, but merely the number of ls which precede c, written as a product or power would be.

In like manner it may be necessary to express the logarithm of l^2c which is expressed l^3c, and so on, the meaning of l^nc being sufficiently obvious.

It is evident, that $l^{n-1}c$ signifies the number whose logarithm is l^nc. Now if we suppose $n = 1$, we find that l^0c signifies the number whose logarithm is lc, and therefore $l^0c = c$. Again, by extending the analogy, let $n = 0$, and $l^{-1}c$ signifies the number whose logarithm is l^0c or c.

If we call l^2c the second logarithm of c, l^3c the third logarithm of c, and in general l^nc the n^{th} logarithm of c, the same analogy suggests the extension of the notation to $l^{-n}c$, which signifies the number whose n^{th} logarithm is c.

When the student shall have advanced into the higher departments of analysis, he will perceive the extensive use of the principles of notation to which we have just alluded, and of which the ordinary notation of powers are the earliest and simplest instance.

(284.) The numbers whose logarithms we have hitherto considered are all positive, and such are the only numbers whose logarithms are ever required in numerical calculations.

If, however, logarithmic calculation be applied to an algebraical formula such as

$$a^2 - b^2$$

which gives

$$l(a^2 - b^2) = l(a + b) + l(a - b)$$

it may so happen, that upon substituting the particular values for a and b, that $a - b$ may be negative. In which case the logarithm of a negative number would be required.

But in fact negative numbers have no logarithms. For in a logarithmic system all numbers whatever are considered as the powers of some one number arbitrarily assumed, but never changing in the same system, and the exponents of these powers are the logarithms. Now this fixed number or *base* is supposed to be such, that by constantly increasing its exponent from 0 to an unlimitedly great positive number, the value of the power will continually increase from unity to an unlimitedly great number; and by constantly increasing the negative value of its exponent, it would continually diminish to an unlimitedly small number. This would not be the case if a negative number were assumed as the base. On the other hand the power would sometimes be a negative quantity, (*scil.*, when the exponent would become an odd integer,) and sometimes an imaginary quantity, (*scil.*, when the exponent would have an even denominator.) That continuity which constitutes a part of the definition of logarithms would in these cases be broken.

It sometimes happens, that computation by logarithms is introduced into a numerical or algebraical problem, merely as a matter of convenience to expedite the process. If in such a case it should occur, that the quantity to which logarithms are to be applied be negative, let its sign be changed, and after its value (considered positively) has been ascertained by logarithmic computation, let its former sign be restored.

Thus, if the quantity $a^2 - b^2$ is to be computed, a being less than b. Let $b^2 - a^2$ be computed, and when determined let it be taken with a negative sign.

If, however, the question be such, that the application of logarithms is absolutely necessary to resolve it, the occurrence of the logarithm of a negative quantity is a symbol of absurdity, and must be understood in the same manner as an imaginary quantity. Suppose, for example, a question terminated in the equation

$$10^x = -100$$

$$\therefore x\,l\,10 = l(-100).$$

This is evidently an absurd equation, since there is no power of 10, whether the exponent be positive or negative, which is $= -100$.

(285.) We shall now proceed to explain methods of computing tables of logarithms.

The method of resolving the equation $y = a^x$, already explained (282.) would be attended with great labour where the computation would be required to be extended far, and would be absolutely impracticable in cases where a very high degree of approximation is required. The methods of expressing logarithms by series furnish much more exact results, and are at the same time more expeditious.

Algebra.

Let y be any number whose logarithm is to be expressed in a series. Applying the method of indeterminate coefficients we have

$$ly = A_0 + A_1 y + A_2 y^2 + A_3 y^3 + \&c.$$

If $y = 0$ the first member becomes infinite, and the second is reduced to A_0. Hence it appears, that the developement of y cannot be effected under the required form. If, however, we assume the first member to be $l(1+y)$, this difficulty will disappear, and we shall have

$$l(1+y) = A_0 + A_1 y + A_2 y^2 + A_3 y^3 + \&c.$$

which when $y = 0$ gives

$$l(1) = A_0 = 0$$

$$\therefore l(1+y) = A_1 y + A_2 y^2 + A_3 y^3 + A_4 y^4 + \&c. \quad [1]$$

In like manner we should have

$$l(1+x) = A_1 x + A_2 x^2 + A_3 x^3 + A_4 x^4 + \&c. \quad [2]$$

By subtraction we have

$$l\left(\frac{1+y}{1+x}\right)$$

$$= A_1(y-x) + A_2(y^2-x^2) + A_3(y^3-x^3) + \&c. \quad [3]$$

But $\dfrac{1+y}{1+x} = 1 + \dfrac{y-x}{1+x} = 1 + u$, if we suppose $u = \dfrac{y-x}{1+x}$.

And since

$$l(1+u) = A_1 u + A_2 u^2 + A_3 u^3 \&c.$$

we have

$$A_1 \cdot \left(\frac{y-x}{1+x}\right) + A_2 \left(\frac{y-x}{1+x}\right)^2 + A_3 \left(\frac{y-x}{1+x}\right)^3 + \&c.$$

$$= A_1(y-x) + A_2(y^2-x^2) + A_3(y^3-x^3) + \&c.$$

Dividing both members of this by $y - x$ it becomes

$$A_1 \frac{1}{1+x} + A_2 \cdot \frac{y-x}{(1+x)^2} + A_3 \cdot \frac{(y-x)^2}{(1+x)^3} + \&c.$$

$$= A_1 + A_2(y+x) + A_3(y^2+yx+x^2) + \&c.$$

As the several series are independent of any relation between y and x, let $y = x$, and the preceding equality becomes

$$A_1 \frac{1}{1+x} = A_1 + 2A_2 x + 3A_3 x^2 + 4A_4 x^3 + \&c.$$

$$\therefore 0 = \begin{array}{c|c|c|c|c|l} A_1 & +2A_2 & x + 3A_3 & x^2 + 4A_4 & x^3 + 5A_5 & x^4 + \\ -A_1 & + A_1 & +2A_2 & +3A_3 & +4A_4 & \end{array}$$

This being independent of x we shall have (261)

$$A_1 - A_1 = 0 \quad 2A_2 + A_1 = 0 \quad 3A_3 + 2A_2 = 0$$

and in general

$$nA_n + (n-1)A_{n-1} = 0.$$

Hence we find

$$A_2 = -\tfrac{1}{2}A_1 \quad A_3 = -\tfrac{1}{3}A_1 \quad A_4 = -\tfrac{1}{4}A_1$$

and in general

$$A_n = -\frac{1}{n} A_1$$

Hence we find

$$l(1+x) = A_1\left(\frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \&c.\right) [4]$$

Logarithms.

There still, however, remains one quantity A_1 indeterminate. This might have been expected, and indeed could not be otherwise, for the question to determine the logarithm of a given number is indeterminate, unless the base of the logarithm be given; and we shall find that the value of the quantity A_1 may be derived from the base of the system.

(286.) The series [4] is not always sufficiently convergent for the convenient determination of the logarithm. A series may, however, be derived from it which will be sufficiently so. Let x be changed into $-x$, and [4] becomes

$$l(1-x) = A\left(-\frac{x}{1} - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \&c.\right) [5]$$

By subtracting this from [4] we obtain

$$l\frac{1+x}{1-x} = 2A_1\left(\frac{x}{1} + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \&c.\right) [6]$$

Let $\dfrac{1-x}{1+x} = 1 + \dfrac{1}{z}$, $\therefore x = \dfrac{1}{2z+1}$, $\therefore$

$$l\left(1 + \frac{1}{z}\right)$$

$$= 2A_1\left(\frac{1}{2z+1} + \frac{1}{3(2z+1)^3} + \frac{1}{5(2z+1)^5} + \&c.\right)$$

or

$$l(1+z) - lz$$

$$= 2A_1\left(\frac{1}{2z+1} + \frac{1}{3(2z+1)^3} + \frac{1}{5(2z+1)^5} + \&c.\right) [7]$$

This series is sufficiently convergent, and gives the difference between the logarithms of two consecutive integers. Hence, by supposing z successively equal to 1, 2, 3, &c. we have

$$l\,2 = 2A_1\left(\frac{1}{3} + \frac{1}{3.3^3} + \frac{1}{5.3^5} + \frac{1}{7.3^7} + \&c.\right)$$

$$l\,3 - l\,2 = 2A_1\left(\frac{1}{5} + \frac{1}{3.5^3} + \frac{1}{5.5^5} + \frac{1}{7.5^7} + \&c.\right)$$

$$l\,4 - l\,3 = 2A_1\left(\frac{1}{7} + \frac{1}{3.7^3} + \frac{1}{5.7^5} + \frac{1}{7.7^7} + \&c.\right)$$

&c. &c.

(287.) Let it now be proposed to obtain the developement of a number in terms of its logarithm, or to develope a^x in a series of powers of x. Let

$$a^x = A_0 + A_1 x + A_2 x^2 + A_3 x^3 + \&c.$$

If $x = 0$ we have $1 = A_0$. Hence we have

$$a^x = 1 + A_1 x + A_2 x^2 + A_3 x^3 + \&c. \quad [1]$$

$$a^y = 1 + A_1 y + A_2 y^2 + A_3 y^3 + \&c. \quad [2]$$

By subtraction we obtain

$$a^x - a^y = A_1(x-y) + A_2(x^2-y^2)A_3 + (x^3-y^3) + \&c. [3]$$

In [1] changing x into $x - y$, we have

$$a^{x-y} = 1 + A_1(x-y) + A_2(x-y)^2 + A_3(x-y)^3 +, [4]$$

and since [3] may be written thus

$$a^y(a^{x-y} - 1) = A_1(x-y) + A_2(x^2-y^2) + A_3(x^3-y^3) + \&c.$$

we have

$$a^y\{A_1(x-y) + A_2(x-y)^2 + A_3(x-y)^3 + \&c.\}$$

$$= A_1(x-y) + A_2(x^2-y^2) + A_3(x^3-y^3) + \&c.$$

Dividing both members of this by $x - y$ we have

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$$a^x\{A_1 + A_2(x-y) + A_3(x-y)^2 + \dots\}$$
$$= A_1 + A_2(x+y) + A_3(x^2+xy+y^2) + \&c.$$

Let $y = x$, and this becomes

$$a^x . A_1 = A_1 + 2A_2 x + 3A_3 x^2 + 4A_4 x^3 + \&c.$$

and substituting for a^x its developement [1] we obtain

$$A_1(1 + A_1 x + A_2 x^2 + A_3 x^3 + \dots)$$
$$= A_1 + 2A_2 x + 3A_3 x^2 + \&c.$$

Hence we obtain

$$A_1^2 = 2A_2,\ \ A_1 A_2 = 3A_3,\ \ A_1 A_3 = 4A_4,\ \&c.$$

$$\because A_2 = \frac{A_1^2}{(2)},\ \ A_3 = \frac{A_1^3}{(3)},\ \ A_4 = \frac{A_1^4}{(4)},\ \&c.$$

where

$(2) = 1\ 2$, $(3) = 1.2.3$, $(4) = 1.2.3.4$, &c.

Hence

$$a^x = 1 + \frac{A_1 x}{1} + \frac{(A_1 x)^2}{(2)} + \frac{(A_1 x)^3}{(3)} + \frac{(A_1 x)^4}{(4)} + \&c.$$

In this case A_1 still remains undetermined. To determine it, let a be the base of the system, and let $a = 1 + b$, and let $(1+b)^x$ be developed by the binomial theorem. Hence we obtain

$$(1+b)^x = 1 + xb + \frac{x(x-1)}{(2)}b^2$$
$$+ \frac{x(x-1)(x-2)}{(3)}b^3 + \&c.$$

If the multipliers of the simple dimension of x in this series be collected, and their aggregate equated with that of x in [1,] we shall have

$$A_1 = \frac{b}{1} - \frac{b^2}{2} + \frac{b^3}{3} - \frac{b^4}{4} + \&c.$$

or

$$A_1 = \frac{(a-1)}{1} - \frac{(a-1)^2}{2}$$
$$+ \frac{(a-1)^3}{3} - \frac{(a-1)^4}{4} + \&c. \qquad [5]$$

Let the value of this series be called k. Hence

$$a^x = 1 + \frac{kx}{1} + \frac{k^2x^2}{(2)}$$
$$+ \frac{k^3x^3}{(3)} + \frac{k^4x^4}{(4)} + \&c. \qquad [6]$$

In this series x is independent of k, but k is dependent on a by [5.] Let $kx = 1$, or $x = \frac{1}{k}$, and we have

$$a^{\frac{1}{k}} = 1 + \frac{1}{1} + \frac{1}{(2)} + \frac{1}{(3)} + \frac{1}{(4)} + \&c.$$

This is a converging series, and its value obtained to seven decimal places is 2,7182818. Let this number be called e, and we have

$$a^{\frac{1}{k}} = e, \qquad \because a = e^k,$$
$$\because la = k\,le, \qquad \because k = \frac{la}{le}.$$

Thus the sum of the series [5] is obtained, and the dependence of k upon a exhibited more evidently.

If in [6] $a = e$, and $\because k = 1$, we have

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{(2)} + \frac{x^3}{(3)} + \frac{x^4}{(4)} + \&c.$$

By substituting for A_1 or k its value $\frac{la}{le}$ in [5] we obtain

$$la = le\left(\frac{a-1}{1} - \frac{(a-1)^2}{3} + \frac{(a-1)^3}{5} - \&c.\right)$$

But in the series [4] (285) if x be changed into $a - 1$ we have

$$la = A_1\left(\frac{a-1}{1} - \frac{(a-1)^2}{2} + \frac{(a-1)^3}{3} - \&c.\right)$$

therefore the indeterminate A_1 in that case becomes le, so that the series [7] (285) becomes

$$l(1+z) - lz$$
$$= 2\,le\left\{\frac{1}{2z+1} + \frac{1}{3(2z+1)^3} + \frac{1}{5(2z+1)^5} + \right\}$$

The logarithms may here be related to any base.

(288.) The logarithm of the number e in any system is called the *modulus* of the system.

(289.) A system of logarithms constructed with the base e is called the *Neperian logarithms*, (from *Neper*, the inventor of logarithms,) and sometimes *hyperbolic logarithms*.

Hyperbolic are sometimes distinguished from other logarithms by an accent placed over the letter thus, l'. Thus $l'a$ is the hyperbolic logarithm of a.

(290.) Let a be the base of a system of logarithms, and x being any number, we have

$$x = a^{lx}, \quad x = e^{l'x},$$
$$\because e^{l'x} = a^{lx}.$$

Taking the logarithms of both members related to the base a, we obtain

$$l'x\,le = lx, \quad \because l'x = \frac{lx}{le}.$$

Hence, if the logarithm of a number in any system be given, the Neperian, or hyperbolic logarithm of the same number may be found by dividing the given logarithm by the modulus.

(291.) If the hyperbolic logarithms of both members of $e^{l'x} = a^{lx}$ be assumed, we have

$$l'x = lx\,l'a,$$
$$\because \frac{lx}{l'x} = \frac{1}{l'a}, \quad \because le = \frac{1}{l'a}.$$

Hence the modulus of any system is equal to the reciprocal of the hyperbolic logarithm of its base.

If, therefore, the hyperbolic logarithms be given, the modulus of any system having a given base may be determined.

Hence, from the hyperbolic logarithms a system relative to any base may be immediately obtained by multiplying all the numbers by the hyperbolic logarithm of the given base.

It is evident that the modulus of hyperbolic logarithms is unity.

(292.) By the equation $l'x\ le = lx$, it follows that the logarithms of the same number in different systems are as their moduli. For let L denote another system, so that $l'x\ \mathrm{L}e = \mathrm{L}x$, $\because$

Algebra.

$$\frac{lx}{le} = \frac{\mathrm{L}\,x}{\mathrm{L}\,e}, \quad \therefore \frac{lx}{\mathrm{L}\,x} = \frac{le}{\mathrm{L}\,e}.$$

Hence it follows, that the logarithms of any one system being known, those of another system having any given base or modulus may be computed.

(293.) Let it be proposed to determine the error produced, by assuming that the difference of the numbers is proportional to the difference of their logarithms when the number of places in the numbers is 5, and their difference not greater than 1.

By the series

$$l\,(1+x) - lx = le\left\{\frac{1}{x} - \frac{1}{2\,x^2} + \frac{1}{3\,x^3} - \frac{1}{4\,x^4} + \right\}$$

it appears generally, that as the number x increases the difference of the logarithms diminishes. Also, since $\frac{1}{x}$ is greater than the remainder of the series, we have

$$l\,(1+x) - lx < \frac{le}{x}.$$

If the base be 10, $le = 0{,}4342 \ldots . < \frac{1}{2}$. Hence, in this case,

$$l\,(1+x) - lx < \frac{1}{2\,x}.$$

If x consist of five places, its least value is 10000. Therefore the greatest value of $l\,(1+x) - l\,x$ is less than $\frac{1}{20000} = 0{,}00005$.

Hence we may infer, that the logarithms of every two consecutive integers, consisting of five places, must agree in the first four decimal places at least.

Let

$$\Delta = l\,(1+x) - l\,x = l\,\frac{1+x}{x}$$

$$\Delta' = l\,(2+x) - l\,(1+x) = l\,\frac{2+x}{1+x}$$

$$\Delta - \Delta' = l\,\frac{1+x}{x} - l\,\frac{2+x}{1+x}$$

$$= l\,\frac{(1+x)^2}{x\,(2+x)} = l\left(1 + \frac{1}{x\,(2+x)}\right).$$

But by what has been already proved

$$l\left(1 + \frac{1}{y\,(2+y)}\right)$$

$$= le\left\{\frac{1}{y\,(2+y)} - \frac{1}{2\,y^2\,(2+y)^2} + \frac{1}{3\,y^3\,(2+y)^3} - \right\}$$

$$\therefore \Delta - \Delta' < \frac{1}{2\,y\,(2+y)}.$$

If y consist of five places, its least value is 10000, and therefore the greatest value of $\Delta - \Delta'$ is less than $\frac{1}{20000 \times 10002} = \frac{1}{200040000}$, which when reduced to a decimal has no significant digit within the first eight places. Hence, in tables which extend only to seven places, we may assume that $\Delta - \Delta' = 0$, or $\Delta = \Delta'$.

Thus we infer, that under the circumstances which have been supposed, the logarithms of numbers in arithmetical progression will themselves be in arithmetical progression.

Integral Functions.

Let n and $n+1$ be two consecutive integers, and $n + \frac{p}{q}$ an intermediate fraction. These may be looked upon as three terms of an arithmetical progression whose first term is n, and whose common difference is $\frac{1}{q}$; the number $n + \frac{p}{q}$ being the $(p+1)^{th}$ term, and $n+1$ the $(q+1)^{th}$ term. By what has been already established, the logarithms of the several terms of this series will also be in arithmetical progression. Let δ be their common difference. The $(p+1)^{th}$ term of this series will be

$$ln + p\,\delta,$$

which will be the logarithm of the $(p+1)^{th}$ term of the former series, $\therefore$

$$ln + p\,\delta = l\left(n + \frac{p}{q}\right).$$

Also the last term of the latter series, which will be

$$ln + q\,\delta,$$

will be the logarithm of the last term of the former series, $\therefore$

$$l\,(n+1) = ln + q\,\delta.$$

Hence we find

$$l\,(n+1) - ln = q\,\delta$$

$$l\left(n + \frac{p}{q}\right) - ln = p\,\delta$$

$$\therefore \frac{l\left(n + \frac{p}{q}\right) - ln}{l\,(n+1) - ln} = \frac{p}{q}.$$

But also

$$\frac{\left(n + \frac{p}{q}\right) - n}{(n+1) - n} = \frac{p}{q}.$$

Hence the differences of the logarithms are as the differences of the numbers.

SECTION XXVI.

Of Integral Functions.

(294.) When any quantity, as x, is connected with other quantities supposed known or constant by symbols indicating determinate operations to be effected on these quantities, the formula which represents the result of these operations is called a *function* of the quantity x. The quantity x is in this case usually called the *unknown quantity*, or the *variable*.

(295.) Functions are divided into classes, according to the nature of the operations by which the unknown quantity is connected with the known quantities.

If it be connected by any purely algebraical process, that is, by addition, subtraction, multiplication, division, involution, or evolution, the function is called an *algebraical function*. Thus, $a\,x^2 + b\,x + c$, $a\,x^m - b$, $\frac{a}{x}$, $(a+x)^m$, &c. are all algebraical functions of x.

If the unknown quantity enter any exponent, it is

Algebra. called an *exponential function*. Thus a^x, x^x, $(a+x^2)^x$ &c. are exponential fonctions of x.

If the logarithm of the onknown quantity, or any functioo of it occur, it is called a *logorithmic function*. Thus lx, $l(o+x)$, &c. are logarithmic functions of x.

(296.) Algebraical functions are divided ioto *rational* and *irrational*. A rational function is one io which the onknown quantity, whether alone or in connection with known quantities, is not affected by a radical or fractionni exponent, and an irrational function is one where it is so affected. Thus $o x^2 + x$, $a x + \frac{b}{x}$, $a x^m + b x^{m-1} + c x^{-n}$, ($m$ and n being integers, positive or negative) are rational functions; and $a \sqrt{x} + b$, $a + x^2 - \sqrt{a+x^2}$, $a + 10 x - \frac{b}{\sqrt{o\ x}}$ are irrational functions.

It should be observed, that a radical or fractional exponent does not render a functioo irrational unless it affects the onknown quantity. Thus $\sqrt{a} . x + \sqrt{b} . x^2$ is a rational functioo of x, although the coefficients of x and x^2 be irrational quantities.

(297.) Rational fuoctions are divided ioto integral and fractional. An integral function is a rational function io which the onknown quantity does not enter any denominator, or where, being in the numerator, its exponent is a positive integer. A fractional functioo is a rational fuoction in which the unknown quantity occurs in some denominator, or has a negative expoaeot in the numerator. Thus $a x^2 + b x + c$, $o x^2$, $a x^3 - b x^2$, &c. are integral functions, and $\frac{a + b x}{a' + b' x'}$, $a x^2 - \frac{b}{x}$, $a x^{-2}$, &c. are fractional functions.

It should also be observed here, that functions are not fractional, unless tha denominator of the fractioo inclnde the unknown quantity. Thus $\frac{o + b x}{c}$ is an integral function of x.

(298.) Integral fuoctions are said to be of the first, second, or n^{th} degree, according to the highest exponent of the unknowo quaotity. Every integral function of the first degree must come ooder the general form

$$A x + B.$$

Those of the second and third degrees under the form

$$A x^2 + B x + C$$
$$A x^3 + B x^2 + C x + D,$$

and io general one of the n^{th} degree under the form

$$A x^n + B x^{n-1} + C x^{n-2} + D x^{n-3} \ldots . S x^2 + T x + V.$$

In these general formulæ the literal coefficients A, B, C.... T, V are general representatives of any number, integral or fractiooal, rational or irrational. Any one or more of the coefficients may be $= 0$ in particular cases.

Thus $x^2 - 1$ is an integral functioo of the second degree, and the formola

$$A x^2 + B x + C$$

becomes identical with it by supposing $A = 1$, $B = 0$, $C = -1$. It should, however, be observed, that if the first coefficient be supposed $= 0$, the degree of the function is necessarily lowered. This is not the case with any other coefficient. Integral Functions.

(299.) One iotegral fuoction is said to divide or measure another, when the complete quote is an integral function of the sams quantity, or, which amounts to the same, an integral function A is said to divide or measure another C, when there is a third integral function B of the same quantity, such that $A \times B$ shall be identical with C.

(300.) If an iotegral function of x be multiplied or divided by any quantity K indepeodent of x, the product or quote will be an iotegral fuoction of x of the same degree. For let the function

$$A x^m + B x^{m-1} + C x^{m-2} \ldots . T x + V$$

be multiplied and divided by K, and the results are

$$K A x^m + K B x^{m-1} + K C x^{m-2} \ldots . K T x + K V,$$
$$\frac{A}{K} x^m + \frac{B}{K} x^{m-1} + \frac{C}{K} x^{m-2} \ldots . \frac{T}{K} x + \frac{V}{K}$$

each of which are integral functions of x of the m^{th} degree.

(301.) If one iotegral functioo of x (A) divide another (C) it will also divide it if it be multiplied or divided by any quaotity K independent of x. For let B be the integral fuoctioo of x, which multiplied by A produces C. Hence $A \times B = C$. Let this equality be expressed in either of the following ways:

$$\frac{A}{K} \times K B = C$$

$$K A \times \frac{B}{K} = C.$$

Since $\frac{A}{K}$ and K B are integral functions of x, (300,) it follows that $\frac{A}{K}$ measures C, and since K A and $\frac{B}{K}$ are integral functions of x, it follows that K A measures C.

(302.) Two integral functioos of x are said to be *prime* to one another with respect to x, when uo integral function of x measures both.

(303.) If ao integral functioo D be prime to another A, and measure the product of A and a third iotegral functioo B, it wili measure B.

If A be an absolute quantity independent of x, we bave, by bypothesis, $\frac{A \times B}{D}$, an integral function of x. If this then be divided by the quantity A, which is iodependent of x, the quote $\frac{B}{D}$ will be an integral function of x (300,) therefore D measures B.

Let us now suppose A to be a function of x of an higher degree than D. Let A be divided by D, and since they are prime there will be a remainder. Let this remainder be R, and the integral part of the quote be Q. We have then

$$A = D Q + R,$$
$$\therefore \frac{A B}{D} = B Q + \frac{B R}{D}$$
$$\therefore \frac{A B}{D} - B Q = \frac{B R}{D}$$

Algebra.

$\frac{AB}{D}$ is by hypothesis an integral function, and since the same is true of B Q, the quantity $\frac{BR}{D}$ is an integral function; therefore D measures B R. Now if R be independent of x, it follows that D measures B (301,) which was to be proved.

But if R be not independent of x, it must be an integral function of a lower degree than D. Let D be in this case divided by R, and let the quota and remainder be Q' and R', and we have

$$D = RQ' + R'.$$

There must in this case also be a remainder, otherwise R would be a common measure of D and A, contrary to hypothesis.

Multiplying the last equation by $\frac{B}{D}$, we have

$$B = \frac{BRQ'}{D} + \frac{BR'}{D}$$

$$\therefore B - \frac{BRQ'}{D} = \frac{BR'}{D}$$

But $\frac{BR}{D}$ has been already proved to be an integral function of x, and therefore $\frac{BRQ'}{D}$ must be an integral function. Hence $\frac{BR'}{D}$ must be an integral function. If in this case R' be independent of x, D must measure B (301,) which was to be proved; and if not, the same process must be continued. It will be observed, that in this process the successive remainders R, R', &c. are all integral functions of x, and each successive remainder is of a degree lower than that which preceded it. Also, since D and A are supposed prime it follows that no remainder can exactly measure that which preceded it. Hence it follows, that we must at last obtain a remainder independent of x, and since D will necessarily measure the product of that remainder and B, it must measure B.

In commencing this process, we supposed D a function inferior in degree to A. If A be inferior in degree to D, we should commence by dividing D by A, but in every other respect the process will be the same.

(304.) If an integral function of x divide a product, and be prime to all its factors but one, it must measure that one.

Let D measure A B C .... L M, and be prime to all but M, it must measure M. For since D measures A × B C .... L M, and is prime to A, it measures B C .... L M. Again, since it is prime to B, and measures B × C .... L M, it measures C .... M, and ultimately since it measures L M, and is prime to L, it measures M.

Hence, if an integral function measure another integral function, it cannot be prime to all the integral factors of that function.

(305.) If an integral function (D) of the first degree measure the product A × B of two integral functions, it must measure one of these functions. For it must either measure it or be prime to it, and it cannot be prime to both and measure their product, (304.)

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(306.) Hence every integral function (D) of the first degree which divides any power of an integral function A, must divide that function itself; and, also, if two integral functions be prime one to another, all their powers will be also prime one to another.

(307.) Every integral function A, which is divided by several integral functions D, D', D'', &c. which are prime to each other is also measured by the continued product D, D', D'', &c. of these functions.

By hypothesis $\frac{A}{D}$ is an integral function, let it be Q, so that A = D Q. Again, D' measures A or D Q, and is prime to D, ∵ it measures Q, suppose the quota Q', so that A = DD'Q'. Again, D'' measures A or DD'Q, and is prime to D, D', therefore it measures Q', and so on until we obtain A = the continued product of all the divisions D, D', D'', &c. into an integral function.

(308.) Hence, if any integral functions D, D', D'', &c. prime to each other, and another integral function A has certain powers of these D^n, $D'^{n'}$, $D''^{n''}$, &c. as divisors, it is evident that any powers of these divisors, with lower exponents than n, n', n'', &c. or products of which any combinations of these powers are factors, will be all divisors of A.

(309.) If any integral A function be resolved into the integral factors A', A'', A''', &c. every integral divisor of any of these factors, and every combination of such divisors by continued multiplication, will be divisors of the original integral function A. Also, each of these divisors multiplied or divided by any quantity independent of x will be a divisor of A, and it follows, that the original integral function A can have no other divisors except these.

These consequences are apparent from the preceding observations.

SECTION XXVII.

The General Theory of Equations.

(310.) A COMPLETE equation of the m^{th} degree, when cleared of fractions and radicals, and all the terms are brought into the first member, and divided by the coefficient of the highest dimension of x, is of the form,

$$x^m + Ax^{m-1} + Bx^{m-2} + Cx^{m-3} \ldots\ldots Tx + V = 0 \quad [1]$$

the coefficients A, B, C .... V being respectively any quantities whatever, positive or negative, integral or fractional, rational or irrational, or = 0.

(311.) Any quantity, whether numerical or algebraical, simple or complex, real or imaginary, which being substituted for x will change the equation into an identity, or make all its terms be such as *necessarily* to destroy each other, so that the aggregate shall = 0, is called a *root* of the equation.

(312.) If a be any root of the equation [1,] the first member of the equation is measured by $(x - a.)$

For let the first member by divided by $x - a$, by the ordinary process of division. The result is

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$$x - a)\ x^m + A x^{m-1} + B x^{m-2} + C x^{m-3} + \ldots\ldots T x + V\ \left(x^{m-1} \left.\begin{array}{l}+a\\+A\end{array}\right| x^{m-2} \left.\begin{array}{l}+a^2\\+Aa\\+B\end{array}\right| x^{m-3} \left.\begin{array}{l}+a^3\\+Aa^2\\+Ba\\+C\end{array}\right| x^{m-4} + \ldots\right.$$

$$x^m - a x^{m-1}$$

$$\left.\begin{array}{l}+a\\+A\end{array}\right| x^{m-1} + B x^{m-2}$$

$$\left.\begin{array}{l}+a\\+A\end{array}\right| x^{m-1} \left.\begin{array}{l}-a^2\\-Aa\end{array}\right| x^{m-2}$$

$$\left.\begin{array}{l}+a^2\\+Aa\\+B\end{array}\right| x^{m-2} + C x^{m-3}$$

$$\left.\begin{array}{l}+a^2\\+Aa\\+B\end{array}\right| x^{m-2} \left.\begin{array}{l}-a^3\\-Aa^2\\-Ba\end{array}\right| x^{m-3}$$

$$\left.\begin{array}{l}+a^3\\+Aa^2\\+Ba\\+C\end{array}\right| x^{m-3} + D x^{m-4}$$

&c. &c.

The coefficients of the several terms of the quote may be observed to be integral functions of a; that of the second term being of the first degree, that of the third of the second degree, and, in general, that of the n^{th} of the $(n-1)^{th}$ degree. In like manner, the successive remainders have the same coefficients to the highest power of x in them respectively, that of the first remainder being an integral function of a of the first degree, that of the second of the second degree, and, in general, that of the n^{th} remainder is an integral function of a of the n^{th} degree.

The number of terms in the original equation is evidently $m + 1$, and after proceeding with the division until the term V is brought down, the remainder with this annexed to it will be

$$\left.\begin{array}{l}+\ a^{m-1}\\+A a^{m-2}\\+B a^{m-3}\\+C a^{m-4}\\ \quad \&c.\\+T\end{array}\right| x + V$$

and, therefore, the corresponding term of the quote will be

$$a^{m-1} + A a^{m-2} + B a^{m-3} + \ldots.\ T,$$

which is independent of x. This, being multiplied by $x - a$, and subtracted from the former, gives for a remainder

$$a^m + A a^{m-1} + B a^{m-2} + C a^{m-3} + \ldots.\ T a + V. \ [2.]$$

But since, by hypothesis, a is a root of the equation; this, which is nothing more than the first member of the given equation, changing x into a, must $= 0$, and, therefore, the division is complete, and $x - a$ is proved to measure the first member.

(313.) The same process proves, that if $x - a$ measure the first member, a must be a root of the equation, for in that case the last remainder [2] must be $= 0$.

(314.) This principle gives a criterion for determining whether an integral function of x of the first degree $(x - a)$ is a divisor of any other given integral function of a, as A′. In A′ let x be changed into a, and if the result be identically 0, $x - a$ is a divisor, and otherwise not.

(315.) The law by which the successive coefficients of the quotient in (312) are obtained, should be observed. The coefficients of the several terms of the quote may be all obtained from the formula [2;] the coefficient of the second term of the quote is the first two terms of [2,] $(m - 1)$ being subtracted from each of the exponents; the coefficient of the third term of the quote is the first three terms of [2,] $(m - 2)$ being subtracted from each of the exponents; and in general the coefficient of the n^{th} term of the quote is the first n terms of [2,] $(m - (n - 1))$ being subtracted from the exponent.

Or, perhaps, a rule more easily impressed on the memory would be, that the coefficient of the n^{th} term of the quote is an integral function of a of the $(n - 1)^{th}$ degree, having the same coefficients as the original equation, and in the same order as far as the terms extend.

(316.) Every equation has as many roots as there are units in the number which marks its degree, and cannot have more.

We shall here take for granted, that the equation has at least one root, whether real or imaginary. Let the root be a. Hence, by what has been already proved, we have

$$x^m + A x^{m-1} + B x^{m-2} + C x^{m-3} + \ldots.\ T x + V = (x - a)\,(x^{m-1} - A' x^{m-2} + B' x^{m-3} + \&c.)$$

where A′, B′, &c. express the coefficients of the successive terms of the quote.

It is evident, that any number which is a root of the equation

$$x^{m-1} + A' x^{m-2} + B' x^{m-3} + \ldots. = 0,$$

must also be a root of the original equation; and as this equation must at least have one root, let it be a', so that we have

$$x^{m-1} + A' x^{m-2} + B' x^{m-3} + \ldots. = (x - a')\,(x^{m-2} + A'' x^{m-3} + \ldots.)$$

$$\therefore x^m + A x^{m-1} + B x^{m-2} + \ldots. = (x - a)\,(x - a')\,(x^{m-2} + A'' x^{m-3} + \ldots.)$$

For each simple factor thus found, the remaining factor of the integral function in the first member is lowered one degree, and by continuing the process through $(m - 1)$ steps, we should obtain an integral function of x of the first degree, and, therefore, of the form $x - a^{(m-1)}$. We should thus have the function in the first member resolved into m simple factors, viz. $x - a$, $x - a'$, $x - a''$, ... $(x - a^{(m-1)})$, whose continued product is

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equal to the integral function in the first member. By (311) it follows, that each of the quantities a, a', a'', &c. is a root of the equation; and since the function in the first member cannot have any other simple factor, the equation cannot have any other root. Thus, if there be *one* root there are m roots, and cannot be more.

We are not aware of any demonstration of the principle, that every equation must admit of one root of a nature such as could properly be introduced here.

(317.) If any number of the quantities a, a', a'', ... be equal, the corresponding factors will be equal. In this case the equation might be said to have a less number of roots than is due to its degree; but in order to generalize the principles, it is considered still to have the full number, but the two or more of them become equal. Thus the equation

$$x^2 - 2x + 1 = 0, \text{ or } (x-1)^2 = 0,$$

is said to have two roots each equal to 1.

(318.) Since the first member of every equation of the m^{th} degree admits of m divisors of the second degree, it must admit as many divisors of the second degree as there are combinations of two divisors of the first degree, since the product of every two factors of the first degree is a divisor of the second degree.

Hence there are $\frac{m(m-1)}{1 \ . \ 2}$ divisors of the second degree, and in like manner there are $\frac{m(m-1)(m-2)}{1 \ . \ 2 \ . \ 3}$ divisors of the third degree, and in general there are

$$\frac{m(m-1)(m-2)(m-3)\dots(m-(n-1))}{1 \ . \ 2 \ . \ 3 \ \dots\dots\dots\dots \ n}$$

divisors of the n^{th} degree, n being less than m. (243.)

These divisors of the higher degrees may become equal, like the factors of the first degree.

(319.) If the second member of the identity

$$x^m + A x^{m-1} + B x^{m-2} + \dots T x + V$$
$$= (x-a)(x-a')(x-a'')\dots(x-a^{(m-1)})$$

be developed and arranged by the dimensions of x, it will become (246)

$$x^m + A x^{m-1} + B x^{m-2} + \dots T x + V$$
$$= x^m - S(a) x^{m-1} + S(a)_2 x^{m-2} - S(a)_3 x^{m-3} + \dots$$
$$\dots S(a)_{m-1} \cdot x \pm S(a)_m.$$

the signs being alternately + and −, because an even number of negative factors give a positive, and an odd number a negative, product.

The meaning of the notation $S(a)_2$ &c. has been explained in (246.)

By equating the coefficients of the corresponding terms in both members, we have

$$A = -S(a), \ B = S(a)_2, \ C = -S(a)_3 \dots$$
$$V = \pm S(a)_m$$

the sign + being used when m is even, and − when m is odd.

Hence we find:

1. That the coefficient of the second term, its sign being changed, is the algebraical sum of the roots of the equation, with their signs changed.

2. That the coefficient of the third term is the sum of the products of every two roots, with their signs changed.

Greatest Common Measure of Algebraical Quantities.

3. That the coefficient of the fourth term, its sign being changed, is the sum of the products of every three roots, with their signs changed.

The last and absolute term is the product of all the roots, with their signs changed.

(320.) If the whole equation be divided by the last term, and arranged by the ascending dimensions of x, and the successive coefficients be A, B, C, &c. it assumes the form

$$1 + A x + B x^2 + C x^3 + \dots M x^{m-1} + N x^m = 0,$$

under this form it is evident, from what has been already proved,

1. That (A) the coefficient of the second term is the sum of the reciprocals of the roots.

2. That the coefficient B of the third term is the sum of the reciprocal products of every two roots.

3. That the coefficients of the fourth, fifth, and in general of the n^{th} term, is the sum of the reciprocal products of every three, four, &c. and $(n-1)$ roots; and the coefficient N of the last term, is the product of the reciprocals of all the roots.

(321.) If the last term of an equation arranged by the descending powers of the unknown quantity be unity, it will participate in both of the systems of properties we have just explained; for in this case it may, without dividing by any number, be arranged in either ascending or descending powers. In this case, the product of all the roots is unity. And since any system of quantities may be imagined to be the roots of an equation, we may infer, that if the continued product of m quantities be unity,

1. That the sum of the reciprocals of these quantities will be equal to the sum of the product of every combination of $(m-1)$ of the quantities.

2. That the sum of the reciprocal products of every two of them is equal to the sum of the products of every $(m-2)$ of them.

3. And in general, that the sum of the reciprocal products of n of them is equal to the sum of the products of $(m-n)$ of them.

SECTION XXVIII.

On the Greatest Common Measure of Algebraical Quantities.

(322.) Algebraical quantities being expressed by letters, their actual values are not apparent. In applying to these the principles already established respecting the greatest common measure of numbers, or any quantities of the same species, it will be necessary to explain the peculiar senses in which the terms are applied.

When two polynomes are arranged by the dimensions of the *same letter*, and considered as integral functions of that letter, one may be considered to measure the other exactly, if there be a third integral function of the same letter which being multiplied by the divisor will give a product identical with the dividend. In this sense the exactness of the division is not considered to be impaired, even though the *coefficients* of the dimen-

sions of the principal letter in the quote should be algebraical fractions. Hence, when a polynome is considered to be a function of any letter as x, it will in this sense be divisible by any other quantity, whether monome or polynome, which is independent of x.

When the different integral functions which divide or measure such a polynome are compared together, one is said to be greater or less than another, according as the highest exponent of the letter by whose dimensions they are arranged is higher or lower in the one than in the other.

(323.) Consequently the greatest common measure of two integral functions of the same letter is the highest integral function of that letter which measures both, in the sense already explained.

Two integral functions of x are said to be prime *with respect to* x, when no integral function of x measures both. It is evident from what has been said, that these functions may and *must* have many *common measures*, since every quantity *independent of* x measures them. But provided that no integral function of x measures them they are prime as respects x.

(324.) The greatest common measure of two integral functions of the same letter is found by a process exactly the same as that already established for other quantities. It is easy to see, that the successive remainders will be integral functions of x continually decreasing in degree. If any remainder measure the preceding one, that will be the greatest common measure, and is proved so exactly in the same manner as in the case of numbers. If there be, finally, a remainder independent of x, the fractions are prime with *respect to* x, since all their common measures must measure this remainder, and, therefore, none of them can be functions of x.

(325.) From the results of the last section it follows, that every integral function can be resolved into as many integral factors of the first degree, as there are units in the highest power of the principal letter which enters it. This decomposition into simple factors will be at once effected, if the equation obtained by equating the given integral function with 0 be solved, considering the principal letter as the unknown quantity. Each of the roots of this equation will determine a simple factor (316) of the integral function. Thus, the decomposition of an integral function into its factors, is reduced to the determination of the roots of an equation.

On the other hand, if by any means the first member of an equation, considered as an integral function of x, can be resolved into factors of the first or second degree, the roots will be immediately obtained by putting the factors severally $= 0$, and solving the equations thus obtained. Their roots will be the several roots of the proposed equation.

(326.) We shall now consider algebraical quantities and their measures in another sense.

A polynome is said to be integral and rational, when all its numeral coefficients are integers, and all its letters have positive and integral exponents. In fact, it is considered integral and rational *absolutely*, when it is integral and rational with respect to all the letters and coefficients which enter it. Thus

$$10\,a^2 - 3\,a\,b + b^2$$

is integral and rational. But

$$\sqrt{10}\,a^2 - 3\,a\,b + b^2$$

$$10\,a^2 - 3\,\frac{a}{b} + b^2$$

$$10\,a^2 - 3\sqrt{a\,b} + b^2$$

are not integral and rational.

(327.) It is evident, that if the product of two quantities be integral and rational, and that one of the factors be integral and rational, the other factor must be also integral and rational.

(328.) A quantity A is said to measure an integral and rational quantity B, when there is another integral and rational quantity C such that $A\,C = B$.

Hence it appears, (299,) that no quantity can measure an integral and rational quantity, except another integral and rational quantity.

(329.) Two integral and rational quantities are said to be *prime to each other*, when they have no common measure in the sense just explained.

(The student should observe the difference of the phrases " prime to each other," and " prime to each other with respect to a particular letter." In the use of the former the quantities are looked on as integral and rational quantities; but in the other, they are only considered integral and rational *with respect to a particular letter*.)

(330.) An integral and rational quantity is said to be *absolutely prime*, when it is not measured by any other integral and rational quantity.

Thus $c^2 - b\,c + a\,b$ is an *absolutely prime* polynome, although it be of the second degree with respect to c, and can therefore be decomposed into two simple factors. These factors though rational with respect to c, are irrational with respect to the other letters.

(331.) The greatest common measure of two rational and integral polynomes, is that common measure which has the greatest coefficients, or exponents, or both, or that whose terms have the highest dimensions.

(332.) If two rational and integral polynomes A and B be divided by their greatest common measure C, the quotes A′, B′ will be prime to each other. For if they have a common measure let it be c, and we have

$$A = A' \times C \qquad B = B' \times C$$

$$A' = A'' \times c \qquad B' = B'' \times c$$

$$\therefore A = A'' \times c \times C \qquad B = B'' \times c \times C.$$

Hence $c \times C$ is a common measure of A, B greater than C, because it must have greater exponents or coefficients, or both.

The following principles already established with respect to other quantities may also be extended to rational and integral polynomes.

1. All common divisors of two quantities are divisors of their greatest common divisor.

2. The greatest common divisor of two quantities is also the greatest common divisor of the lesser of those quantities and the first remainder, and also of the first and second remainders, and so on.

(333.) An example will best illustrate the method of determining the greatest common divisor of two algebraical quantities.

Let the two quantities be

$$a^3 - a^2\,b + 3\,a\,b^2 - 3\,b^3$$

$$a^2 - 5\,a\,b + 4\,b^2$$

The former, according to the criterion already explained, is the greater. Dividing it by the latter we obtain the integral part of the quote, aod the remainder as follows:

$$a^2-5ab+4b^2)\ a^3-a^2b+3ab^2-3b^3\ (a+4b$$
$$a^3-a^2b-16ab^2+16b^3$$
$$\overline{}$$
$$19ab^2-19b^3=19b^2(a-b)$$

Since the greatest commoo measure of the two proposed quantitles is also the greatest common measure of the dlvlsor and this remainder, and since no divisor of the factor $19b^2$ measures the divisor or lesser of the proposed quantities, it follows, that the greatest common measure of the proposed quantities must be the greatest commoo measure of the lesser quantity and the factor o $-b$, and the calculation may be disembarrassed of the simple factor. Upon the same principle, every simple factor of each remainder which is not a factor of the divisor may be removed; and any simple factor of one of the proposed quantities which is not also a simple factor of the other may be removed.

Upon nearly the same principle, any simple factor may be introduced into ooe of the proposed quantities, provided it be not a simple factor of the other. This is sometimes necessary io order to facilitate the process, as will be seen hereafter.

The problem in the exampls under consideration, is theo resolved to tbe iuvestigation of the greatest common measure of the quantities

$$a^2 \quad 5ab+4b^2$$
$$o-b.$$

Dividing the former by the latter, we have

$$o \quad b)\ a^2 \cdot \quad 5ob+4b^2\ (a-4b$$
$$a^2-5ab+4b^2$$
$$\overline{}$$
$$\text{''} \quad \text{''} \quad \text{''}$$

There beiog no remainder, it follows, that $a-b$ is the greatest common measure; aud, since this ls not measured by any other algebraical quantity, there is oo other common measure of the two proposed quantities.

(334.) Again, let the two quantities be

$$15a^5+10a^4b+4a^3b^2+6a^2b^3-3ab^4$$
$$12o^3b^2+38a^2b^3+16ab^4-10b^5.$$

On examining these quantities it appears, that o is a simple factor of the former which does not enter the latter, and $2b^2$ is a simple factor of the latter which does not enter the former. Neither of these can be factors of the greatest common measure, and may, therefore, be omitted in the iovestigatioo. By removing them, the quantities under consideration are reduced to

$$15a^4+10a^3b+4a^2b^2+6ab^3-3b^4$$
$$6a^3+19a^2b+8ab^2-5b^3.$$

The first term of the latter will not divide that of the former without introdocing fractional coefficients. This may, however, be avoided, by multiplying the former by such a quantity as will render the coefficient of the first term of the former a multiple of the coefficient of the first term of the latter; and such a multiplier not being a factor of the second quaotity, cannot affect the common measure which will result from the investigation.

If the former quantity be multiplied by 2, and the first division be effected, we have the followiog remainder

$$411a^2b^2+274ob^3-137b^4.$$

In this remainder there is the simpls factor $137b^2$, and as this does not eoter the lesser of the giveo quantities it may be omitted, and the other factor is

$$3a^2+2ab-b^2.$$

If the lesser of the proposed quantities be divided by this there will be no remainder, and an exact quote will be obtained. Hence this remaioder is the greatest commoo divisor.

The suppression of the simple factors which occur in the successive remainders, without occurring in the respective divisors, is not merely ao operation effected to expedite the process, but a matter of oecessity. For otherwise, in order to reoder the divisor divisible by the remainder, it would be necessary to multiply it by the simple factor, (for otherwise the quote would be fractional,) in which case it would be a commoo factor, and would, therefore, be also a factor of the commoo measure which would result from the process, and which would not, therefore, be a common measure of the proposed quantities.

If, however, the proposed quantities, or any subsequeot divisor and dividend, have any evident common measure, whether simple or complex, it may be set apart, and the investigatioo conducted as if it were suppressed. It must, however, be fioally multiplied by the common measure which results from the investigation, in order to find the greatest common measure of the proposed quantities.

In general, then, it appears, that the process for the determination of the greatest eommoo measure of two algebraical quantities should be conducted thus:

1. Let the two quantities be arranged according to the dimensions of the same letter.

2. Let any simple factor which is common, or any complex common factor which is apparent, be set apart to be multiplied by the common measure which is the result of the process.

3. Let any simple factors which are oot commoo be suppressed.

4. The quantities being thus prepared, let that which has the higher dimensions of the letter by which they are arranged be divided by the other, and if there be no remainder, this other multiplied by any common factors which may have been set apart is the greatest common measure. But if there be a remainder, this remainder and the divisor are to be treated in the same manner as the original quantities, and the process is to be contioued until there be no remainder, or one which is free of the letter by which the given quantities have been arranged. In the former case, the last remainder is the greatest common measure, and io the latter case there is no common measure.

(335.) It appears from what has been already proved, that every commoo factor of two polynomes is a factor of their greatest common measure. To investigate more particularly the compositioo of the greatest eommoo measure, let A be any rational and integral polyoome not absolutely prime; let it be supposed to be arranged according to the dimensions of the letter a. *In general*, such a polynome may be

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considered, in the first instance, as the product of three factors:

1. A monome factor A_1 common to all its terms. This factor is the greatest common measure of all its terms considered as simple quantities, and is formed by finding the greatest common measure of all the numeral coefficients, and multiplying this by the highest dimensions of the letters which are common to all the terms.

2. A polynome factor A_2 independent of the letter a, by which the proposed polynoma has been previously arranged, and which is the greatest common measure of the several polynomes, which are the coefficients of the several dimensions of a; the factor A_1, however, having been previously taken out.

3. The polynome factor A_3, arranged by the dimensions of a, which remains when the given polynoma has been divided by the two former factors. The several coefficients of this polynome A_3 are evidently prime to each other.

Hence the given polynome will be represented by the product

$$A_1 \times A_2 \times A_3.$$

If the coefficients of the several dimensions of a in the given polynome happen to be prime, we shall have $A_2 = 1$, $A_3 = 1$; and if the several monomes which compose the given polynome be prime, we shall have $A_1 = 1$.

(336.) Let A and B be two polynomes, whose common measure is to be investigated. By what has been just stated they may be resolved into the forms

$$A = A_1 \times A_2 \times A_3$$
$$B = B_1 \times B_2 \times B_3.$$

Let m_1 be the greatest common measure of A_1 and B_1, m_2 of A_2 and B_2, and m_3 of A_3 and B_3. It is evident that $m_1 \times m_2 \times m_3$ is a common measure of A and B. But it is also their greatest common measure; for every common measure of A and B, if it be a monome, must measure m_1; and if it be a polynome independent of a, must measure m_2; and if it be a polynome dependent on a, the coefficients of the powers of a being prime to each other, it must measure m_3. Hence, $m_1 \times m_2 \times m_3$ is the greatest common measure, and we have

$$A = m_1 \times m_2 \times m_3 \times A'$$
$$B = m_1 \times m_2 \times m_3 \times B',$$

A' and B' being prime to each other.

It appears, therefore, that the greatest common measure is the continued product of the greatest common monome factor, the greatest common polynome factor independent of the letter by which the given polynomes have been arranged, and the greatest common factor which is dependent on this letter, and, further, that every common measure whatever of A and B must measure this.

(337.) We shall now give a general demonstration of the second principle announced in (332,) that the greatest common measure of A and B is also the greatest common measure of the lesser B, and the remainder found on dividing the greater by the less.

Let us suppose that the polynomes being arranged by the dimensions of the same letters, the coefficients have all been divided by their greatest common factor, and are, therefore, prime. If then A and B be the two polynomes, let Q be the quote, and R the remainder. Let M be the greatest common measure of A and B, and M' of B and R. We have

$$A = B\,Q + R$$
$$\frac{A}{M} = \frac{B}{M}\,Q + \frac{R}{M}$$
$$\frac{A}{M'} = \frac{B}{M'}\,Q + \frac{R}{M'}$$

Greatest Common Measure of Algebraical Quantities.

By the second, since M measures A and B, M must also measure R; and by the third, since M' measures B and R, it must measure A. Hence, M' being a common measure of A and B, measures their greatest common measure M; and M being a common measure of B and R, measures their greatest common measure M'. Since M and M' measure each other, they must be equal; that is, the greatest common measure of two integral polynomes is also the greatest common measure of the lesser and remainder.

If the coefficients of the dimensions of a in the polynomes be not prime, let their greatest common measure be m. So that m A and m B will then be the original polynomes. The remainder will then be m R, the greatest common measure m M, and the greatest common measure of m B and m R will be m M'. Now M' has already been proved equal to M, $\therefore$ m M is equal to m M'.

Hence it follows, in general, that the greatest common measure of two integral polynomes is also the greatest common measure of the lesser and remainder.

(338.) If the greatest common measure of two integral polynomes can be determined, the greatest common measure of three or more can be found by a process precisely similar to that explained in (99,) and founded on the same reasoning.

(339.) Let us now investigate more particularly the process for determining the greatest common measure of two integral and rational polynomes, A and B.

First, let the common monome factor m_1 (if there be any such) be found. This factor is composed of the literal factors common to all the terms, and which appear on inspection, affected by the greatest common measure of all the numeral coefficients as a coefficient. This last is found by the rules established in Section VIII. This is one factor of the greatest common measure sought, and is set apart until the others are obtained. The monome factors common to the terms of the one, but not of the other, may be set aside, since they cannot enter in the common measure.

We shall now consider successively the cases in which the remaining factors of A and B include *one letter* only, *two letters*, and where they include *three* or more.

First Case. *To determine the greatest common measure of two integral polynomes arranged by the dimensions of one letter* (a,) *and whose coefficients are integers which have no common measure.*

Let that of the higher degree A' be divided, if possible, by the lower, B'. This will be possible if the coefficient of the highest dimension of a in A' be a multiple of the coefficient of the highest dimension of a in the lower B'. If this be not the case, the whole polynome A' must be multiplied by such an integer as will render the coefficient of the first term of A' a multiple of that of B'. Let m be this multiplier, so that the modified polynomes are m A' and B'. It is easy to see that this modification cannot affect the common measure. In other words, that if M be the greatest common measure

of m A′ and B′, it will also be the greatest common measure of A′ and B′. For since it is prime to m, and measures m A′, it measures A′; therefore it is a common measure of A′ and B′, and being so, it is evidently their greatest common measure.

Let the division be continued in this way, rendering the first term of each remainder when necessary a multiple of the first term of the divisor, until a remainder be obtained, in which the exponent of the highest power of a is less than the highest exponent in the divisor.

Let it then be determined whether the coefficients of this remainder have any common factor, and if so, lat it be suppressed, since it cannot be a factor of the common measure. This done, let the divisor be assumed as dividend, and the remainder as divisor, and proceed as before. Continue this process, making each remainder alternately divisor and dividend, until a remainder is found which exactly measures the preceding remainder. This remainder is then the greatest common divisor m_2 of the polyoomes A′, B′. If their coefficients previously had a common measure m_1, the greatest common measure would be $m_1 \times m_2$.

(340.) SECOND CASE. *To determine the greatest common measure of two integral polynomes which include but two letters,* a *and* b.

Let the common monome factor m_1, if there be such, be set apart, and also let any monome factors not common be removed, since they cannot enter the greatest common measure sought. Then let the two polynomes be arranged according to the dimensions of either of the letters, as a.

The coefficients of the several powers of a will in this case be integral polynomes, including no letter but b. Let the greatest common measure of all these coefficients in each polynome be found by the preceding case, and the principles which regulate the determination of the greatest common measure of several polynomes. Let these be M_2, N_2. Let the greatest common measure nf these be found, and it will evidently be the factor m_2 of the greatest common measure sought. The remaining factors of M_2 and N_2 not being common, cannot enter the greatest common measure, and may, therefore, be soppressed.

The two polynomes when thus divided by M_2 and N_2, will have their coefficients prime to each other. The principles established in the preceding case may then be applied to determine the common measure m_3, and thus the greatest common measure of the proposed polynomes $m_1 \times m_2 \times m_3$ will be determined.

(341.) THIRD CASE. *To determine the greatest common measure of two integral polynomes which include three,* a, b, c, *or more letters.*

Let them be arranged by the dimensions of one of the letters a. The coefficients of the powers of this letter will then be integral polynomes, including b and c. Let the greatest common monome factor m_1 be first found and set apart, and let any other monome factor of either polynome be suppressed. Let the greatest common measure M_2 of the several polynome coefficients of A be then found, and the same N_2 of the coefficients of B. This may be effected by the rules established in the second case, and in (99,) for the greatest common measure of several quantities. Let the greatest common measure m_2 of M_2 and N_2 be then found, and let all other factors of these be suppressed in A and B.

We shall thus have obtained two polynomes, of which the several coefficients are prime to each other; and the greatest common measure m_3 may be found by the principles already established. Thus the greatest common measure $m_1 \times m_2 \times m_3$ will be fonnd.

By pursuiog a similar method, the greatest commoo measure of a polynome, including any number of letters, may be found.

As an example of these prinoiples, let it be required to find tha greatest common measure of the polynomes,

$$a^2 d^2 - c^2 d^2 - a^2 c^2 + c^4$$

$$4 a^2 d - 2 a c^2 + 2 c^3 - 4 a c d.$$

There is here no common monome factor, $\because m_1 = 1$. The monome 2 is common to all the terms of the latter polynome, and shall therefore be suppressed. This being done, and the polynomes being arranged by the dimensions of d, they become

$$(a^2 - c^2)\, d^2 - a^2 c^2 + c^4$$

$$2 a (a - e) \;\; d - (a - c)\, c^2.$$

Since $- a^2 c^2 + c^4 = - c^2 (a^2 - c^2)$, it is evident that $a^2 - c^2$ is a factor of the coefficients of the former, and $a - e$ of the latter, so that $M_2 = a^2 - c^2$, $N_2 = a - c$. The common factor nf these is $a - c$, $\because m_2 = a - e$. The factors M_2 and N_2 being suppressed, the polynomes become

$$d^2 - c^2$$

$$2 a d - c^2$$

which are evidently prime, $\because m_3 = 1$. Hence the greatest common measure is $m_2 = a - c$.

The same result will be obtained if the quantities be arranged by the dimensions of a or e.

SECTION XXIX.

The Transformation of Equations.

(342.) THE resolution of equations of the higher degrees presents considerable difficulties to the analyst, and in cases where it can be effected at all requires the aid of peculiar analytical artifices. It frequently happens, that although the value of the unknown quantity in a proposed equation cannot be immediately determined, yet the value of some other unknown quantity, having a given relation to the required one, may be ascertained, and thus the required quantity finally may be found. The process by which this end is attained, is called the transformation of equations; and although, properly speaking, it is a particular case of the more general process of elimination to be treated of hereafter, yet, in order to introduce the abstract principles more gradually to the mind of the student, we shall so far invert the order of principles, as to investigate the principles of *transformation* before we enter upon the more general field of *elimination.*

Suppose that an equation of any degree be given, in which x is the unknown quantity, but which cannot immediately be solved. Soppose, also, that another equation be given, in which y is the unknown quantity, and which *can* immediately be solved. If it happen to be known that the unknown quantity y is a number which is greater or less than x by any given quantity,

as 5, it is evident that the first equation will thus be solved by means of the second. But any other known relation between x and y would equally attain the desired end; as, for example, if it were known that y, multiplied or divided by any given number, were equal to x, or that the sum of the squares of x and y were equal to a given number, &c.

There are here, then, in general, three things to be considered, the equation for x, the equation for y, and the relation between x and y. If any two of these be given, or assumed, the third may be found. Thus, if the equation for x (generally the proposed one) be given, and the relation between x and y be assumed, the equation for y may be found thus: by the assumed relation between x and y, we know what quantity composed of y and known quantities, or what *function* of y is equivalent to x. Let this be substituted for x in the proposed equation, and the result will be the equation for y. In this case the proposed equation is said to be *transformed*, and the unknown quantity x is said to be *eliminated*; and the process, which in general is called *elimination*, is in this particular application of it called *transformation*.

(343.) In general, the object of transforming an equation is to obtain another equation which may be resolved with greater facility. In this process let the proposed equation be called (A), the transformed equation (C), and the equation which expresses the relation between the two unknown quantities x and y (B). By the explanation of the process already given, it appears that the resolution of (A) depends on the resolution of both (B) and (C). The resolution of (C) determines the value of y. This value being substituted for y in (B), this equation (B) will determine the value of x. Generally, therefore, the equations (B) and (C) should be more simple and easy of solution than the equation (A), or the process is useless.

Transformation of Equations.

(344.) One of the most obvious simplifications which may be effected on an equation, is the diminution of the number of its terms. To investigate the means of effecting this, let the proposed equation be

$$x^m + A_1 x^{m-1} + A_2 x^{m-2} + \ldots A_{m-1} \cdot x + A_m = 0. \quad \text{(A)}$$

Let the assumed relation between x and y be such that x shall be equal to the algebraical sum of y, and another quantity k, to which we shall assign such a value as may be necessary to attain the end we propose. Thus we have

$$x = y + k. \qquad \text{(B)}$$

To determine the equation (C), let this value of x be substituted in (A), and the result, after each of the terms have been developed and arranged by the dimensions of y, will be

$$\begin{array}{l|l|l|l|l}
y^m + mk & y^{m-1} + \dfrac{m(m-1)}{1\,.\,2} k^2 & {}^{m-2} + \dfrac{m(m-1)(m-2)}{1\,.\,2\,.\,3} k^3 & y^{m-3} + \ldots\, k^m & = 0 \qquad \text{(C)} \\
\quad + A_1 & \qquad + (m-1) A_1 k & \quad \dfrac{(m-1)(m-2)}{1\,.\,2} A_1 k^2 & + A_1 k^{m-1} & \\
 & \qquad\quad + A^2 & \qquad (m-2) A_2 k & + A_2 k^{m-2} & \\
 & & \qquad\quad + A_3 & + \ldots & \\
 & & & \ldots\ldots & \\
 & & & \ldots\ldots & \\
 & & & + A_m &
\end{array}$$

The *law* which prevails among the coefficients of y is here easily perceived. The exponent of y in the n^{th} term is $m - (n-1)$, and the coefficient of this term is

$$\frac{m(m-1)(m-2) \ldots [m-(n-2)]}{1\,.\,2\,.\,3 \ldots\ldots\ldots (n-1)} \cdot k^{n-1}$$
$$+ \frac{(m-1)(m-2) \ldots [m-(n-2)]}{1\,.\,2\,.\,3 \ldots\ldots\ldots (n-2)} A_1 k^{n-2}$$
$$+ \frac{(m-2)(m-3) \ldots [m-(n-2)]}{1\,.\,2\,.\,3 \ldots\ldots\ldots (n-3)} \cdot A_2 k^{n-3}$$
$$+ \frac{(m-3)(m-4) \ldots [m-(n-2)]}{1\,.\,2\,.\,3 \ldots\ldots\ldots (n-4)} A_3 k^{n-4}$$
$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$+ [m-(n-2)] \cdot A_{n-2} \cdot k + A_{n-1} \cdot$$

Since the value of k is arbitrary, let such a value be assigned to it as will render the coefficient of the second term of (C) $= 0$. The equation (C) will then be more simple than A, as it will want the second term, or that which corresponds to $A_1 x^{m-1}$. To fulfil this condition we must have

$$mk + A_1 = 0, \quad \therefore k = -\frac{A_1}{m}.$$

Hence the equation (B) becomes

$$x = y - \frac{A_1}{m}.$$

(345.) Hence we derive the following rule for transforming an equation, so as to remove the second term: "Substitute for the unknown quantity x, the sum of another unknown quantity y, and the quote of the coefficient of the second term of the given equation by the exponent of x in its first term, with the sign of the coefficient being changed."

The process already explained for the solution of a complete quadratic equation (173) is an example of this principle. In this case the equation (A) is

$$x^2 + A_1 x + A_2 = 0, \qquad \text{(A)}$$

$$\therefore x = y - \frac{A_1}{2}, \qquad \text{(B)}$$

$$\therefore y^2 - \left(\frac{A_1^2}{4} - A_2\right) = 0, \qquad \text{(C)}$$

$$\therefore y = \pm\sqrt{\frac{A_1^2}{4} - A_2},$$

$$\therefore x + \frac{A_1}{2} = \pm\sqrt{\frac{A_1^2}{4} - A_2}.$$

$$\because x = -\frac{A_1}{2} \pm \sqrt{\frac{A_1^2}{4} - A_2},$$

which is the formula established in (173.)

(346.) Since the relation between k and the coefficients of the equation which is necessary to remove the second term is a simple equation, the value of k can always be determined, and is always real; and therefore this transformation can in every case be effected. This, however, will not be the case if we attempt to remove any of the subsequent terms of the equation. To remove the third term, we should have the condition

$$\frac{m(m-1)}{1 \,.\, 2} k^2 + (m-1) A_1 k + A_2 = 0.$$

If the roots k', k'' of this equation be real, the third term may be removed from the equation by substituting $y + k'$, or $y + k''$, for x in (C.)

To remove the fourth term would require the solution of a cubic equation for k, and in general to remove the n^{th} term would require the solution of an equation of the $(n-1)^{th}$ degree. The removal of the last term would require the solution of the proposed eqoation itself.

In all these cases the roots may be imaginary, and then the transformation will be impossible. It will, however, appear hereafter, that every equation whose degree is marked by an odd number, must have at least one real root; but those of even degrees may have all their roots imaginary, from whence it appears that it is always possible to remove the second, fourth, or any even term of an equation, but not always possible to remove the odd terms.

It is easy to see that the substitution of $y - \frac{A_1}{m}$ for x must remove the second term. For let A_1' be the coefficient of the second term of (C), and let S (a) be the sum of the roots of (A), and S (a') the eum of the roots of (C.) It is plain, that since each root of (A) is equal to the corresponding root of (C) $- \frac{A_1}{m}$, we have

$$S(a) = S(a') - m \,.\, \frac{A_1}{m} = S(a') - A_1,$$

but $\quad A_1 = -S(a),\ A_1' = -S(a'),\ \because$

$$-A_1 = -A_1' - A_1,\ \because A_1' = 0.$$

(347.) It may happen that the same condition which removes one term will also remova some other term. Let it be proposed to determine the relation which must subsist between the coefficients of the equation (A) in order that the same condition which removes the second term shall also remove the third term. In this case it is necessary that the same value of k shall satisfy the conditions

$$m k + A_1 = 0$$

$$\frac{m(m-1)}{1 \,.\, 2} k^2 + (m-1) k A_1 + A_2 = 0.$$

Let the value of k derived from the first be substituted in the second, and we obtain, after reduction,

$$(m-1) A_1^2 - 2 m A_2 = 0.$$

If then the exponent and coefficients of (A) are so related as to satisfy this condition, the same transformation will remova the first and second terms, but otherwise not.

In general, to determine whether the same transformation will remove any two terms, let the corresponding coefficients in (C) be put $= 0$, and let k be eliminated. If the resulting equation be an identity, tha effect will be produced, but otherwise not.

(348.) It ie sometimes necessary to consider the equation (C) arranged in ascending powers of y. In this case it assumes the form

$$\begin{array}{l|l|l|l|l}
k^m & + m k^{m-1} & y + \frac{m(m-1)}{1 \,.\, 2} k^{m-2} & y^2 + \ldots . m k & y^{m-1} + y^m = 0. \quad (D) \\
+ A_1 k^{m-1} & + (m-1) A_1 k^{m-2} & + \frac{(m-1)(m-2)}{1 \,.\, 2} A_1 k^{m-3} & + A_1 & \\
+ A_2 k^{m-2} & + (m-2) A_2 k^{m-3} & + \frac{(m-2)(m-3)}{1 \,.\, 2} A_2 k^{m-4} & & \\
+ \ldots . & + \ldots\ldots\ldots. & \ldots\ldots\ldots & & \\
\ldots\ldots & \ldots\ldots\ldots\ldots & \ldots\ldots\ldots & & \\
\ldots\ldots & \ldots\ldots\ldots\ldots & + A_{m-2} & & \\
+ A_{m-1} k & + A_{m-1} & & & \\
+ A_m & & & &
\end{array}$$

The coefficient of the n^{th} term in this case is

$$\frac{m(m-1)(m-2) \ldots . [m-(n-2)]}{1 \,.\, 2 \,.\, 3 \ldots\ldots\ldots . n-1} k^{m-(n-1)}$$
$$+ \frac{(m-1)(m-2) \ldots . [m-(n-1)]}{1 \,.\, 2 \,.\, 3 \ldots\ldots\ldots . n-1} A_1 k^{m-n}$$
$$+ \frac{(m-2)(m-3) \ldots\ldots\ldots [m-n]}{1 \,.\, 2 \,.\, 3 \ldots\ldots\ldots\ldots\ldots . n-1} A_2 k^{m-(n+1)}$$
$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
&c. &c.

It may be ohserved, that the coefficients of the successive powers of y may be deduced one from another, thus, "To find any coefficient multiply the successive terms of the preceding coefficient by the exponents of k, and then diminish the exponents of k by unity, and divide hy the nnmber of preceding terms." Thus, if any one term be known all tha succeeding terms can be foond. But the first term is the first member of the proposed equation changing x into k. Hence all the terms may be found.

(349.) It sometimes happens that the coefficients of an equation are some or all of tham fractional. If the equation be cleared of fractions by multiplying it hy

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the least common multiple of the denominators, the highest dimension of x may have a coefficient different from the unit, which it is desirable to avoid.

To determine a transformatioo which will remove the fractional coefficients, let the equation (B) be $x = \frac{y}{k}$. Hence the equatioo, (C,) after multiplying by k^m, will be

$$y^m + A_1 k . y^{m-1} + A_2 . k^2 . y^{m-2} + \ldots\ldots$$
$$A_{m-1} . k y^{m-1} + k^m A_m = 0. \qquad \text{(C)}$$

If the coefficients A_1, A_2 .... or any of them be supposed to be fractions in their least terms, it is necessary that their denominators respectively should measure k, k^2 .... in order that $A_1 k$, $A_2 k^2$ .... should be integral. This will be the case if k be on integer produced by the continued multiplication of all the prime factors of the denominators, each factor being repeated so as to render the powers k, k^2 .... multiples of the several denominators.

Let the given equation be

$$x^3 - \frac{7}{3} x^2 + \frac{11}{36} x - \frac{25}{72} = 0.$$

The prime factors are here 2 and 3, and $k = 6$, therefore the transformed equation is

$$y^3 - 14 y^2 + 11 x - 75 = 0.$$

SECTION XXX.

Transformation continued.—First Principles of Elimination.—Equation of Differences.

(350.) Before proceeding further in the theory of equations, it will be oecessary to explain the first principles of elimination. The more complete developement of this process, however, we shall reserve for a subsequent section.

Elimination is that process by which when two equations, (A,) (B,) each including two unknown quantities, x, y, are given, a third equation (C) is deduced from them, including but one unknown quantity, x.

Io general there are certaio *systems* of values of x and y which will satisfy the two eqoations (A,) (B.) There are, generally, an infinite number of systems of values (225) which will satisfy one of the equations, but only a limited nnmber which will satisfy both. If it be required to determine whether any particular number x' is a value of x, which, in conjunction with some corresponding value of y, will satisfy the equations, it is only necessary to substitute x' for x in the proposed equations; and then, considering y as the unknown quantity, if they have any common root, such a root will be the corresponding value of y, which, in conjunction with the proposed value of x, will satisfy the equations. It may even happen, that the equations will have several common roots, in which case there are several systems of values of x and y, in which the value of x is the same, and which will satisfy the proposed equations.

To determine the values of x which will satisfy the equations (A) and (B,) it is only necessary to find the roots of the equation (C.) This is called the *final equation*, and it appears from what has been just stated, that if any root of this equation be substituted for x in the proposed equations (A) and (B,) they must have at least one common root, and therefore, considered as polynomes, their first members most have some iotegral function (Y) of y as a commoo factor. If this function were known, the values of y corresponding to the proposed value of x would be the roots of the equatioo $Y = 0$.

Hence we may infer, that if such a value of x can be found as will make the polynomes, which form the first members of the equations (A,) (B) be divisible by the same integral functioo Y of y, these values of x being each united with the several values of y fouod from the equation $Y = 0$, will be systems of values of x and y, which will satisfy the proposed equations.

(351.) Before we proceed to explain the method of obtaining the equation (C) it is necessary to observe, that the polynomes which form the first members of the eqoations (A) and (B) cannot have a commoo divisor independent of particular values of the quaotities x and y, unless they be supposed indeterminate, and in fact equivalent to *one* equation.

For suppose that they admitted a common divisor D, and were of the forms

$$A' \times D = 0, \qquad B' \times D = 0.$$

1. If D be a function of x and y. In that case the two equations are equivalent to the equation $D = 0$, in which there being two unknown quantities is indeterminate, and therefore there are an unlimited number of systems of values which satisfy the proposed equation, (225.)

2. Let D be a fooction of one of the unknown quantities, as x. The equation $D = 0$ determines particular values of x, which will satisfy the proposed eqoations, independently of any particular value of y. Although, therefore, the values of x are determinate, those of y are absolutely unlimited, and therefore the equations are indeterminate.

3. Let D be independent of x and y. In that case D must be a common factor of the coefficients of the equations, and ought to be suppressed previously to their solution.

Hence we shall consider the equations as having no common factor, independently of particular values of x and y.

(352.) The equations then being sopposed determinate, let them be both arranged according to the descending powers of y; and let the process for finding the greatest common measure be pursued. Since they have no common measure by supposition, there will be at last a remainder obtained which is indepeodent of y. This remainder will then be a function of x; and if such a value be given to x as renders it $= 0$, the equations will necessarily admit a function of y as a common divisor. Hence the function of x, which is found in the last remainder, must be the first member of the final equation (C.) It is plain that the values of x thus determined, being substituted in the preceding remainder, render it a common measure of the first members of (A) and (B.)

Having by the resolution of the final equation (C) determined all the values of x, the corresponding values of y may be found by substituting these values for x in the preceding remaiader. By this substitution, this remainder will become the greatest common measure

of the first members of (A) and (B), the same values being substituted for x in them. These values of y will, therefore, be those which correspond to the values of x determined before.

The final equation to which we arrive in this way, is in general of an higher degree than the second, and, therefore, we must postpone for the present the actual investigation of the systems of values of the unknown quantities. It appears, however, from what has been stated, that we can eliminate one unknown quantity from two equations each containing two unknown quantities, and thereby obtain a final equation including but one unknown quantity *without the resolution* of any equation whatever.

(353.) If there be three equations including three unknown quantities, any one of them z is to be eliminated by the first and second equation, and also by the first and third. We shall then have two equations between x and y which are to be treated as above, and similar reasoning applies to four equations with four unknown quantities, &c.

(354.) As an application of these principles, let it be required "to find an equation whose roots have any given relation to any two roots of a given equation."

Let the unknown quantity in the new equation be u, and the combination or function of the two roots $x' x''$, to which u is supposed to be equal be expressed by $F(x', x'')$ that is

$$u = F(x', x''). \qquad [1]$$

Since also the values x', x'' both satisfy the given equation, we have

$$x'^m + A_1 x'^{m-1} + A_2 x'^{m-2} + \ldots . A_{m-1} x' + A_m = 0. \quad [2]$$

$$x''^m + A_1 x''^{m-1} + A_2 x''^{m-2} + \ldots . A_{m-1} x'' + A_m = 0. \quad [3]$$

The question then is, to eliminate x', x'' by these three equations, and the final equation will be the equation sought. This elimination will of course depend on the nature of the function in the second member of [1], or the relation which the roots of the sought equation are to have to those of the given equation. It should also be observed, that the final equation being independent of x' and x'' will be the same, whatever pair of roots may be assumed, and therefore its roots will be similarly related to every pair of roots of the proposed equation, and it will in general have as many roots as there are permuted combinations of two roots of the given equation. Hence the degree of the final equation must be at least $m(m-1)$.

Let the equation [1] be

$$u = x'' - x'.$$

Hence $x'' = u + x'$, which being substituted in [2], and the several terms developed and arranged by the ascending dimensions of u, give

$$X'_0 + X'_1 . \frac{u}{(1)} + X'_2 . \frac{u^2}{(2)} + X'_3 . \frac{u^3}{(3)} + \ldots + u^m = 0$$

where the notation (2), (3), &c. is used to express 1 . 2, 1 . 2 . 3, &c. and*

$$X'_0 = x'^m + A_1 x'^{m-1} + A_2 x'^{m-2} + A_3 x'^{m-3} + \ldots A_{m-1} x' + A^m$$
$$X'_1 = m x'^{m-1} + (m-1) A_1 x'^{m-2} + (m-2) A_2 x'^{m-3} +$$
$$X'_2 = m(m-1) x'^{m-2} + (m-1)(m-2) A_1 x'^{m-3} +$$
&c. &c.

But since x' is a root of the proposed equation $X'_0 = 0$. Observing this condition, and dividing the equation by u, we have

$$\frac{X'_1}{(1)} + \frac{X'_2}{(2)} u + \frac{X'_3}{(3)} . u^2 + \frac{X'_4}{(4)} . u^3 + \ldots . + u^{m-1} = 0.$$

The equation sought is, therefore, obtained by eliminating x' by this equation, and $X'_0 = 0$.

Hence to obtain the *equation of differences* it is only necessary to omit the last term of the proposed equation, to diminish each exponent of x by unity, and to change x into u, and the coefficients A_1, A_2, &c. into $\frac{X'_1}{(1)}$, $\frac{X'_2}{(2)}$, $\frac{X'_3}{(3)}$, &c., and to eliminate x by this and the given equation. It is unnecessary in this process to place the accent on x.

(355.) For example, let the proposed equation be

$$x^3 - 6x - 7 = 0.$$

Hence we have $X_0 = x^3 - 6x - 7$, $X_1 = 3x^2 \quad 6$, $X_2 = 6x$, $X_3 = 6$, $X_4 = 0$. Hence the equations for the elimination of x are

$$x^3 - 6x - 7 = 0$$
$$3x^2 - 6 + 3xu + u^2 = 0$$

which by elimination give

$$u^6 - 36u^4 + 324u^2 + 459 = 0,$$

which is the *equation of differences* sought.

(356.) Since $m(m-1)$ must always be an even number, the equation of differences must always be of an even order. But it is easy to prove that its alternate terms beginning with the second are wanted; in other words, that it includes only even powers of the unknown quantity. For since every *permuted combination* of the roots are to be combined by difference, it follows, that $x'' - x'$, and $x' - x''$, are both roots of this equation; and in general, if a, b, c, &c. are the numerical differences of the roots $+a$ and $-a$, $+b$ and $-b$, &c. are roots of the equations of differences. Hence the first member of this equation is

$(x-a)(x+a)(x-b)(x+b)(x-c)(x+c) \ldots$

or $\quad (x^2 - a^2)(x^2 - b^2)(x^2 - c^2) \ldots .$

If the square of x be taken as the unknown quantity, the equation will become

$$(z - a^2)(z - b^2)(z - c^2) \ldots \ldots = 0$$

the degree of which will be $\frac{m(m-1)}{1 \, . \, 2}$, z being put for x^2. This is called *the equation of the squares of the differences*. It has the advantage of being of a lower degree than the *equation of differences*.

SECTION XXXI.

Transformation continued.—Depression of Equations — Equal Roots.

(357.) WHEN particular relations are known to subsist between the roots of an equation, its resolution

* Those who are familiar with the differential calculus will perceive, that X_1, X_2, &c. are derived from X_0 by differentiation,

$$X_1 = \frac{d X_0}{d x'} \; X_2 = \frac{d^2 X_0}{d x'^2} \ldots . X_n = \frac{d^n X_0}{d x'^n}.$$

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may be reduced to that of another equation of an inferior degree; the process by which this reduction is effected, is called the depression of the equation.

If any root (a) of an equation be known, the degree may be depressed by dividing its first member by $x - a$. In like manner, if two roots (a, b) be known, the degree may be depressed by two units by dividing the first member by $(x - a)(x - b)$, and so on.

But even when no root is absolutely known, yet if a certain relation be known, or can be discovered to subsist between the roots, the equation may be shown to depend on the solution of an equation of a lower degree.

(358.) One of the most simple relations which can be imagioed to subsist between two or more roots of an equation is equality. This in the case when some of the binome factors of the first member are of the form $(x - a)^m$, $(x - b)^n$, &c.

Let X_0 be the first member of an equation of the m^{th} degree, and let a, b, c, &c. be its roots, ∴

$$X_0 = (x - a)(x - b)(x - c)\ldots\ldots$$

Let x be changed into $x + k$ and the result is

$$(x + k)^m + A_1(x + k)^{m-1} + A_2(x + k)^{m-2} + \ldots\ldots$$
$$= (x + k - a)(x + k - b)\ldots\ldots$$

or what is the same

$$(x + k)^m + A_1(x + k)^{m-1} + A_2(x + k)^{m-2} + \ldots\ldots$$
$$= (k + \overline{x - a})(k + \overline{x - b})\ldots\ldots$$

By developing both members, we obtain (354) for the first member

$$X_0 + X_1 . \frac{k}{(1)} + X_2 . \frac{k^2}{(2)} + X_3 . \frac{k^3}{(3)} + \ldots\ldots$$

and if the developement of the second member be arranged by the ascending powers of k, the first term or absolute quantity will be the continued product of $(x - a)$, $(x - b)$, &c., which by the above identity gives

$$X_0 = (x - a)(x - b)(x - c)\ldots\ldots$$

a result established already. The coefficients of k give

$$X_1 = \frac{X_0}{x - a} + \frac{X_0}{x - b} + \frac{X_0}{x - c} + \ \ldots\ldots$$

The equality of the coefficients of k^2 gives

$$\frac{X_2}{(2)} = \frac{X_1}{(x - a)(x - b)} + \frac{X_1}{(x - a)(x - c)} + \ldots\ldots$$

and so on.

If the original equation $X_0 = 0$ have equal roots, its first member X_0 will have equal factors of the form $(x - a)^m$, $(x - b)^n$, &c. Hence the several quotes to which X_1 is equal, will each have the quantities $(x - a)^{m-1}$, $(x - b)^{n-1}$ as factors. In fact, if the combinations of equal factors which enter X_0 are

$$(x - a)^m (x - b)^n \ldots\ldots$$

the combinations of the same factors which enter X_1 are

$$(x - a)^{m-1}(x - b)^{n-1}\ldots\ldots$$

Hence we infer, that "if the equation $X_0 = 0$ have equal roots, the polynomes X_0 and X_1 admit a common divisor."

It appears, also, that if the exponents m, n of any of the factors $x - a$, $x - b$ be greater than 2, they will also be divisors of X_2, and if they be greater than 3, they will be divisors of X_3, and so on.

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(359.) These principles being established, we are prepared to determine whether an equation $X_0 = 0$ have equal roots, and to determine these roots when it is possible so to do. Let the exponents of the factors $x - a$, $x - b$, $x - c$, &c. which occur more than once in X_0 be n, n', n'', &c., and let the factors which occur but once be $x - p$, $x - q$, &c. Hence

$$X_0 = (x - a)^n (x - b)^{n'} (x - c)^{n''} \ldots (x - p)(x - q)\ldots$$

The degree of the equation $X_0 = 0$ being m, it is plain from the value already found (358) for X_1, that it is equal to the som of the quotes of X_0 divided by each of its simple factors. Now as the factor $(x - a)$ occurs n times, the sum of the quotes for this alone must be $\frac{n X_0}{x - a}$. In like manner the sum of the quotee for $x \quad b$ is $\frac{n' X_0}{x - b}$, and so on. So that we have

$$X_1 = \frac{n X_0}{x - a} + \frac{n' X_0}{x - b} + \frac{n'' X_0}{x - c} + \ldots\ldots$$
$$\frac{X_0}{a - p} + \frac{X_0}{n - q}\ldots\ldots$$

Now it is plain, that the product $(x - a)^{n-1}(x - b)^{n'-1}(x - c)^{n''-1}\ldots\ldots$ is a common divisor of X_0 and X_1. But, further, it is the greatest common divisor, because it contains all the prime factors $(x - a)$, $(x - b)$, $(x - c)$, …. which are common to these polynomes.

Hence it follows, that if X_0 and X_1 have no common divisor, the equation X_0 has no equal roots. But if X_0 and X_1 have a common divisor, which can always be determined by the principles established in Section XXVIII., that common divisor is the product of the equal factors of X_0, the exponent of each being diminished by unity.

Let D be this common divisor. If it be of the first degree it may be reduced to the form $x - h$, and therefore $(x - h)^2$ is a factor of X_0, and h a root which occurs twice; and it follows, that in this case there are no other equal roots. By the division by $(x - h)^2$ the degree of the equation is depressed by two units.

If $D = 0$ be of the second degree there are two cases, either the roots of $D = 0$ are equal or unequal. If they be equal, D is of the form $(x - h)^2$; in which case h occurs three times as a root of $X_0 = 0$, and $(x - h)^3$ is a divisor of X_0, which will reduce the degree of the equation $X_0 = 0$ by three units. But if the roots of $D = 0$ be unequal, D is of the form $(x - h)(x - h')$, in which h and h' each occur twice as roots, and $(x - h)^2 (x - h')^2$ is a factor of X_0 which will depress the degree of $X_0 = 0$ by four units.

In general, it is necessary to resolve the equation $D = 0$, in order to determine the equal roots of $X_0 = 0$, and the number of times that each equal root occurs. Every root which occurs once in $D = 0$ will occur twice in $X_0 = 0$, every root which occurs twice in $D = 0$ will occur three times in $X_0 = 0$, and so on.

(360.) When we have obtained the equation $D = 0$, and that it is found to be of a degree above the second, it may be submitted to the process already described, to determine whether it have equal roots; and if it be found to have them, its degree may be depressed in the same manner as that of $X_0 = 0$, and so the process may be continued until an equation be found which has no

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equal roots. If the degree of this equation do not exceed the second it may be solved, and when solved its roots will furnish divisors which will depress the degrees of all the equations from which it was deduced.

But if the equation $D = 0$ have not equal roots, and that it exceed the second degree, each root will occur twice in $X_0 = 0$; and the methods of determining the roots will be explained hereafter.

(361.) We shall now show that the resolution of every equation $X_0 = 0$ which has equal roots can be made to depend on the resolution of a system of equations, of which the first includes the roots of the given equation which occur but once, the second those which occur twice, the third those which occur three times, and so on.

Let X' be the product of those simple factors of X_0 which occur in it but once, X'' the product of those which occur twice, and so on, so that we have

$$X_0 = X' . X''^2 . X'''^3 . X''''^4 \ldots .$$

and by what has been already proved

$$D = X'' . X'''^2 . X''''^3 \ldots .$$

Dividing the latter by the former, we have

$$\frac{X}{D} = Q = X' . X'' . X''' . X'''' \ldots .$$

which is the product of the simple factors, equal as well as unequal, of X_0.

Let the greatest common measure D' of D and Q be now found. It is evidently

$$D' = X'' . X''' . X'''' \ldots .$$

that is, the product of all the equal factors; each, however, being introduced but once.

If Q be divided by D', the quote is X', which is the product of all the factors of X_0 which occur but once. The equation $X_0 = 0$ may thus be cleared of all the equal roots, and considerably depressed in degree. The equation $X' = 0$ is the first of the system to which we proposed to reduce $X_0 = 0$.

By observing the form of the quantity D, it will be observed, that the equation $D = 0$, like the original equation, includes roots which occur once, twice, thrice, and so on. The product X'' of the roots which occur once, may be found by the same process applied to $D = 0$, as we have already applied to $X_0 = 0$. Hence we shall obtain the equation $X'' = 0$, which is the second of the proposed system; and by continuing the application of the same process, we shall obtain $X''' = 0$, $X'''' = 0$, &c. It may be observed, also, that the degree of the equation $X' = 0$ expresses the number of roots which occur but once in $X_0 = 0$, and its resolution gives the values of these roots. The degree of $X'' = 0$ represents the number of roots which occur twice, and its resolution gives the values of these roots, and so on.

(362.) By the principles which have been here established, we may obtain a criterion for determining whether a given polynome be a square, cube, or any perfect power. For this it is only necessary to derive from it another, in the same manner as X_1 was derived from X_0, and if this last be an exact measure of the first, the first is a perfect power, and otherwise not.

(363.) The results of this Section might be more simply and expeditiously established by the differential calculus. But as it is desirable that Algebra should be founded on principles independent of the calculus we shall here merely observe, that since $X_1 = \frac{d\,X_0}{d\,x}$ (354,) and

Depression of Equations.—Reciprocal Equations.

$$X_0 = (x-a)^n\,(x-b)^{n'}\,(x-c)^{n''} \ldots . (x-p)\,(x-q)$$

we have

$$X' = (x-a)^{n-1} \times \frac{X_0}{(x-a)^n} + (x-b)^{n'-1} \times \frac{X_0}{(x-b)^{n'}} + \text{\&c.}$$

from whence, and similar processes, the results may easily be obtained.

SECTION XXXII.

Depression of Equations continued.—Reciprocal Equations.

(364.) An equation in which the last term is unity, and of which the coefficients equidistant from the extreme terms are equal, is called a *reciprocal equation*, from a remarkable relation which subsists between its roots. The most general form under which such an equation can be expressed, is

$$x^m + A_1 x^{m-1} + A_2 x^{m-2} + \ldots . \; A_2 . x^2 + A_1 . x + 1 = 0.$$

Let $x\,y = 1$, and let each term of the equation be multiplied by that power of $x\,y$ whose exponent is the number of preceding terms. Hence we obtain

$$x^m + A_1 x^m y + A_2 x^m y^2 \ldots . \; + A_2 x^m y^{m-2} + A_1 x^m y^{m-1} + x^m y^m = 0,$$

which being divided by x^m, becomes

$$1 + A_1 y + A_2 y^2 + \ldots . \; A_2 y^{m-2} + A_1 y^{m-1} + y^m = 0,$$

which is the original equation, x being changed into y. Hence it appears, that y must be a root of the equation, and since $y = \frac{1}{x}$, it follows that if any number be a root of this equation, the reciprocal of that number must be also a root of the equation.

Hence we may also infer, that if the degree of the equation be expressed by an odd number, one of its roots must be unity. For by what has been just proved, if any number not unity be a root, its reciprocal must also be a root; and, consequently, the number of roots different from unity must be even; but since the total number is odd, there must be one root at least equal to unity. Such an equation can, therefore, always be reduced in degree, by dividing its first member by $x - 1$.

We shall, therefore, confine ourselves to the consideration of reciprocal equations of an even degree. Let $2\,m$ be the highest exponent, so that the equation is

$$x^{2m} + A_1 x^{2m-1} + A_2 x^{2m-2} + \ldots . \; A_2 x^2 + A_1 x + 1 = 0$$

Dividing the whole equation by x^m, and combining the extreme terms and those which are equally distant from them, we shall have

$$\left(x^m + \frac{1}{x^m}\right) + A_1\left(x^{m-1} + \frac{1}{x^{m-1}}\right) + A_2\left(x^{m-2} + \frac{1}{x^{m-2}}\right) + \ldots . \; + A_1\left(x + \frac{1}{x}\right) = 0.$$

Let $z = x + \frac{1}{x}$, ∴ $x^2 - zx + 1 = 0$, ∴ $x = \frac{z}{2} \pm \sqrt{\frac{z^2}{4} - 1}$. Hence we find

$$x + \frac{1}{x} = z \quad x^2 + \frac{1}{x^2} = z^2 - 2$$

$$x^3 + \frac{1}{x^3} = z^3 - 3z, \text{ \&c.}$$

which substitutions being made in the former, we obtain an equation of the m^{th} degree to determine z. For each value of z determined by this equation, we find two values of x by the formula $x = \frac{z}{2} \pm \sqrt{\frac{z^2}{4} - 1}$.

(365.) If the extreme terms of the equation, and those which are equally distant from them, have contrary signs, the equation will also have reciprocal roots when its degree is marked by an odd number. In this case the form of the equation is

$$x^m + A_1 x^{m-1} + A_2 x^{m-2} + \ldots - A_2 x^2 - A_1 x - 1 = 0.$$

Introducing $xy = 1$, and its powers as before, it becomes

$$1 + A_1 y + A_2 y^2 + \ldots - A_2 y^{m-2} - A_1 y^{m-1} - y^m = 0.$$

If the negative terms of the former were the same with the same coefficients as the positive terms, the latter equation becomes identical with the former, by changing y into x, and changing all the signs. This will be necessarily the case if the number of terms which is $m + 1$ be even, that is, if m be odd. And therefore in this case the former reasoning becomes applicable. But if m be even, there will be a middle term, and that term will have the same sign in both equations, while all the other terms differ in sign.

As an example of the application of these principles, let the proposed equation be $x^m - 1 = 0$. If this be divided by $x - 1$, we have

$$x^{m-1} + x^{m-2} + x^{m-3} \ldots x^3 + x^2 + x + 1 = 0,$$

which is a reciprocal equation of an even degree when m is odd, and of an odd degree when m is even.

Let $m = 5$, ∴

$$x^4 + x^3 + x^2 + x + 1,$$

$$\therefore \left(x^2 + \frac{1}{x^2}\right) + \left(x + \frac{1}{x}\right) + 1 = 0.$$

Let $x + \frac{1}{x} = z$, ∴ $x^2 + \frac{1}{x^2} = z^2 - 2$,

$$\therefore z^2 - 2 + z + 1 = 0,$$

$$\therefore z^2 + z - 1 = 0, \quad \therefore z = -\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{5},$$

$$x = \frac{z}{2} \pm \tfrac{1}{2}\sqrt{z^2 - 4}$$

$$\therefore x = -\tfrac{1}{4} \pm \tfrac{1}{4}\sqrt{5} \pm \tfrac{1}{4}\sqrt{10 \pm 2\sqrt{5}}.\ \sqrt{-1}.$$

SECTION XXXIII.

Of Symmetrical Functions of the Roots of an Equation.

(366.) When a quantity is a *function* of two or more quantities, it is called a *symmetrical* function when it is similarly related to each of these quantities on which its value depends. The test by which a symmetrical function may be known, is that its value will not be changed by changing any two of the quantities on which it depends, each into the other. Some examples will render this definition more clear. Let u be the function, and x and y the quantities on which it depends, and let the function be expressed by the letter F prefixed to x, y, so that $u = F(x, y)$. Now if the value of u remain the same when x is changed into y, and y into x, or $u = F(y, x)$, then is u a symmetrical function of x and y. Let $u = x + y$; this is evidently a symmetrical function, since $x + y = y + x$. But if $u = x - y$, u is not a symmetrical function, since $x - y$ is not equal to $y - x$. Again, let $u = xy$, or $u = x^2 + y^2$; these are symmetrical functions, because $xy = yx$, and $x^2 + y^2 = y^2 + x^2$. But, on the other hand, $x^m y^n$ is not a symmetrical function, because it is not equal to $y^m x^n$, unless $m = n$, in which case only it is a symmetrical function.

(367.) The most simple symmetrical function of any number of quantities is their *sum*, and the most simple class of such functions is that to which this belongs, viz. the sum of the n^{th} powers of those quantities. Let a_1, a_2, a_3, &c. be the quantities, the class of functions to which we allude, is

$$a_1 + a_2 + a_3 + a_4 + \ldots$$
$$a_1^2 + a_2^2 + a_3^2 + a_4^2 + \ldots$$
$$a_1^3 + a_2^3 + a_3^3 + a_4^3 + \ldots$$
$$\text{\&c.} \qquad \text{\&c.}$$
$$a_1^n + a_2^n + a_3^n + a_4^n \ \ldots$$

We shall express these functions severally by the notation $S(a)$, $S(a^2)$, $S(a^3)$, &c. We shall call these *symmetrical functions* of the *first kind*.

The class of symmetrical functions which are integral and rational, next in simplicity to the preceding, are symmetrical functions, each term of which is a product, into which two different literal factors enter. A function of this class is the sum of the products of all the letters of the function taken in permuted combinations of two with given numbers as exponents, the same number being the exponent of the first letter in each permuted combination. Thus,

$$a_1^2 a_2^3 + a_1^2 a_3^3 + a_2^2 a_3^3 + a_2^2 a_1^3 + a_3^2 a_1^3 + a_3^2 a_2^3,$$

is a symmetrical function of a_1, a_2, a_3, of the *second kind*.

The general form for such a function is

$$a_1^n a_2^{n'} + a_1^n a_3^{n'} + a_1^n a_4^{n'} + \ldots a_2^n a_1^{n'} + a_2^n a_3^{n'} + \ldots$$
$$a_3^n a_2^{n'} + a_3^n a_4^{n'} + \ldots$$

we shall represent such a function in general by $S(a^n a^{n'})$.

A symmetrical function of the third kind has a similar meaning, and is expressed by a similar notation $S(a^n a^{n'} a^{n''})$, and so on.

(368.) If n be the number of different letters which

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enter a symmetrical function, the number of terms in a symmetrical function of the first kind is evidently n. The number of terms in a function of the second kind is the number of permuted combinations of two letters which are obtained from n letters, *scil.* $n\ (n-1)$. In like manner, the number of terms in a symmetrical function of the third kind is $n\ (n-1)\ (n-2)$, and so on.

(*N. B.* There are an infinite variety of symmetrical functions of a given number of letters, but we confine ourselves in this place to the consideration of such as are algebraical, rational, and integral. Those which we have described are called *elementary symmetrical functions.*)

From the nature of symmetrical functions, it is evident that if any term be affected by a multiplier or divisor, all the terms must be affected by the same multiplier or divisor; and if A be such a coefficient, the function may be expressed $A \,.\, S\,(a^n)$, $A\, S\,(a^n a^n)$, &c.

(369.) Having thus explained the nature of the symmetrical functions we are about to consider, we shall proceed to investigate the method of determining such functions of the roots of an equation.

Symmetrical Functions of the Roots of an Equation.

Since every root of an equation must be similarly related to its coefficients, it follows that each of these coefficients must be a symmetrical function of the roots of the equation. Indeed, this follows immediately from the properties of the roots established in Sect. XXXII.

The coefficient of the second term is the sum of the roots with their signs changed, and is, therefore, the simplest species of symmetrical function of the first kind. The coefficient of the second term is the sum of the products of every two roots, and, therefore, is the simplest species of symmetrical function of the second kind, and so on.

Let it, however, be proposed to determine the other symmetrical functions of the first kind of the roots. Let the roots be a_1, a_2, a_3, &c. we have (312)

$$\frac{X}{x-a_1} = \begin{array}{l|l|l|l|l} x^{m-1} + a_1 & x^{m-2} + a_1^2 & x^{m-3} + a_1^3 & x^{m-4} + a_1^4 & x^{m-5} + \ldots\ldots + a_1^{m-1} \\ \quad + A_1 & \quad + A_1 . a_1 & \quad + A_1 . a_1^2 & \quad + A_1 . a_1^3 & \quad + A_1 . a_1^{m-2} \\ & \quad + A_2 & \quad + A_2 . a_1 & \quad + A_2 . a_1^2 & \quad + A_2 . a_1^{m-3} \\ & & \quad + A_3 & \quad + A_3 . a_1 & \quad \ldots\ldots \\ & & & \quad + A_4 & \quad \ldots\ldots \\ & & & & \quad + A_{m-1} \end{array}$$

$$\frac{X}{x-a_2} = \begin{array}{l|l|l|l|l} x^{m-1} + a_2 & x^{m-2} + a_2^2 & x^{m-3} + a_2^3 & x^{m-4} + a_2^4 & x^{m-5} + \ldots\ldots + a_2^{m-1} \\ \quad + A_1 & \quad + A_1 . a_2 & \quad + A_1 . a_2^2 & \quad + A_1 . a_2^3 & \quad + A_1\, a_2^{m-2} \\ & \quad + A_2 & \quad + A_2 . a_2 & \quad + A_2 . a_2^2 & \quad + A_2\, a_2^{m-3} \\ & & \quad + A_3 & \quad + A_3 . a_2 & \quad + \ldots\ldots \\ & & & \quad + A_4 & \quad \ldots\ldots \\ & & & & \quad + A_{m-1} \end{array}$$

and similar developements may be obtained for $\frac{X}{x-a_3}$, $\frac{X}{x-a_4}$, &c.

By adding all these developements we obtain $\frac{X}{x-a_1} + \frac{X}{x-a_2} + \frac{X}{x-a_3} + \ldots\ldots \frac{X}{x-a_m} =$

$$\begin{array}{l|l|l} m\, x^{m-1} + S\,(a) & x^{m-2} + S\,(a^2) & x^{m-3} + \ldots\ldots\ldots\ S\,(a^{m-1}) \\ \quad + m\, A_1 & \quad + A_1 . S\,(a) & \quad + A_1 . S\,(a^{m-2}) \\ & \quad + m\, A_2 & \quad + A_2 . S\,(a^{m-3}) \\ & & \quad + \ldots\ldots\ldots \\ & & \quad \ldots\ldots\ldots \\ & & \quad + m\, A_{m-1} \end{array}$$

But by (358) it appears that the first member of this equality is equal to X_1. By equating the several coefficients of the powers of x in the second member of the preceding equality with those of the same powers of x in the value of X_1 found in (354,) we find after reduction,

$$S\,(a) + A_1 = 0$$
$$S\,(a^2) + A_1\, S\,(a) + 2\, A_2 = 0$$
$$S\,(a^3) + A_1\, S\,(a^2) + A_2\, S\,(a) + 3\, A_3 = 0$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$S\,(a^{m-1}) + A_1 . S\,(a^{m-2}) + A_2\, S\,(a^{m-3}) + \ldots\ldots + (m-1)\, A_{m-1} . = 0.$$

The first of these equations gives the value of $S\,(a)$; this being found, and substituted in the second, we may find $S\,(a^2)$; this being known, the third gives $S\,(a^3)$, and so on. Thus the symmetrical functions of the first kind are determined as far as the $(m-1)^{th}$ degree.

To determine those of superior degrees, let a_1, a_2, a_3, &c. be successively substituted for x in the given equation, and let the results be multiplied by a_1^n, a_2^n, a_3^n, &c. respectively, and we obtain

$$S\,(a^{m+n}) + A_1\, S\,(a^{m+n-1}) + A_2\, S\,(a^{m+n-2}) + \ldots\ldots A_{m-1} . S\,(a^{n+1}) + A_m\, S\,(a^n) = 0.$$

In this equation, let 0, 1, 2, 3, &c. be successively substituted for n, and we obtain

$$S\,(a^m) + A_1 . S\,(a^{m-1}) + A_2 . S\,(a^{m-2}) + \ldots\ldots A_{m-1} . S\,(a) + m\, A_m = 0$$
$$S\,(a^{m+1}) + A_1 . S\,(a^m) + A_2 . S\,(a^{m-1}) + \ldots\ldots A_{m-1} . S\,(a^2) + A_m . S\,(a) = 0$$
$$S\,(a^{m+2}) + A_1 . S\,(a^{m+1}) + A_2 . S\,(a^m) + \ldots\ldots A_{m-1} . S\,(a^3) + A_m . S\,(a^2) = 0.$$
&c. &c.

The first of these determines the value of $S\,(a^m)$ where the functions $S\,(a^{m-1})$, $S\,(a^{m-2})$, &c. of inferior degree

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are known, and which can be found by the former process. The second determines $S(a^{m+1})$ when the functions of inferior degrees are known, and so on.

(370.) The symmetrical functions of the reciprocals of the roots may be determined by making n negative in the preceding formula. By this change it becomes

$$S(a^{m-n}) + A_1 S(a^{m-n-1}) + A_2 S(a^{m-n-2}) + \ldots .$$
$$A_{m-1} . S(a^{-n+1}) + A_m . S(a^{-n}) = 0.$$

Substituting for n in this, 1, 2, 3, &c. we obtain

$$S(a^{m-1}) + A_1 S(a^{m-2}) + \ldots . + A_{m-1} . S(a^0)$$
$$+ A_m . S(a^{-1}) = 0$$
$$S(a^{m-2}) + A_1 S(a^{m-3}) + \ldots . + A_{m-1} . S(a^{-1})$$
$$+ A_m . S(a^{-2}) = 0$$
$$S(a^{m-3}) + A_1 S(a^{m-4}) + \ldots . + A_{m-1} . S(a^{-2})$$
$$+ A_m . S(a^{-3}) = 0$$
&c. &c.

It is plain that $S(a^0) = m$, since $a^0 = 1$, and the number of terms in $S(a^0)$ is m. Hence all the terms of the first equation, except the last, have been previously determined, and therefore $S(a^{-1})$ can be found. By the second equation $S(a^{-2})$ may be determined, $S(a^{-1})$ being previously found, and so on.

Hence, in general, "When any equation of any degree is given, we may obtain the sum of the squares, cubes, &c. or any similar integral powers of its roots or the sum of the square, cubes, &c. or any similar integral powers of the reciprocals of its roots."

(371.) We shall now explain the method of determining the symmetrical functions of the second kind. These we shall express by the notation $S(a^n a^{n'})$. If $S(a^n)$ and $S(a^{n'})$ be multiplied, the product will evidently contain the $(n+n')^{th}$ powers of all the roots, and also the product of every combination of two roots in the n^{th} and n'^{th} powers. Hence we have

$$S(a^{n+n'}) + S(a^n a^{n'}) = S(a^n) \times S(a^{n'})$$
$$\therefore S(a^n a^{n'}) = S(a^n) . S(a^{n'}) - S(a^{n+n'})$$

The second member of this equation being composed of symmetrical functions of the first kind, which have already been determined, the first member is known.

If $n = n'$, the second member becomes $S(a^n)^2 - S(a^{2n})$. Of the $m(m-1)$ terms of the first member, the permuted combinations of the same letters become equal, and therefore the number of terms, when the equal terms are combined, becomes $\frac{m(m-1)}{2}$, and all of them have 2 as a common multiplier. Hence it is evident that the result is

$$S\left((a\,a)^n\right) = \frac{S(a^n)^2 - S(a^{2n})}{2}$$

The first member of this may be considered as a symmetrical function of the *first* kind, of the roots combined in products of two factors.

(372.) To determine the symmetrical functions of the third kind, let the values of $S(a^n a^{n'})$ and $S(a^{n''})$ be multiplied together. The terms of the product will be of three forms, $a^{n+n''} . a^{n'}$, $a^n . a^{n'+n''}$, and $a^n a^{n'} a^{n''}$, and we have evidently

$$S(a^{n+n''} . a^{n'}) + S(a^n . a^{n'+n''}) + S(a^n a^{n'} a^{n''})$$
$$= S(a^n a^{n'}) . S(a^{n''})$$
$$\therefore S(a^n a^{n'} a^{n''}) = S(a^n a^{n'}) . S(a^{n''}) - S(a^n . a^{n'+n''})$$
$$- S(a^{n+n''} . a^{n'}).$$

The second member of this being composed of functions of the second kind, has been already determined.

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If $n' = n''$, the terms of the first member become equal in pairs, which being united, the whole will be affected by the common factor 2. Hence we have

$$S(a^n a^{n'} a^{n'}) = \tfrac{1}{2}[S(a^n a^{n'}) . S(a^{n'}) - S(a^n . a^{2n'})$$
$$- S(a^{n+n'} . a^{n'})]$$

The first member of this may be considered as a function of the second kind of the roots themselves, and their combinations in pairs. The number of terms in it is evidently $\frac{m(m-1)(m-2)}{1\,.\,2\,.\,3}$.

If $n = n' = n''$, the terms of the first member will be the n^{th} powers of every permuted combination of three roots. The terms containing the permuted combinations of the same letters are equal, and as the number of such terms is 1. 2. 3. this will be a common factor. But also in the second member since $n = n'$ $\therefore S(a^n a^{n'}) = 2S((aa)^n)$, and the terms $S(a^{n+n''} . a^{n'})$ and $S(a^n . a^{n'+n''})$ become identical. Hence we have

$$S((a\,a\,a)^n) = \tfrac{1}{3}[S((a\,a)^n) . S(a^n) - S(a^{2n} a^n)]$$

By pursuing a similar method, symmetrical functions of all higher kinds may be determined.

(373.) All symmetrical functions whatever, which are integral and rational, must be combinations of those already determined, and hence we may in general infer, "That any integral and rational symmetrical function whatever of the roots of an equation may be determined when the coefficients of the equation are known."

A symmetrical fractional function, if all its terms be reduced to the same denominator, and added, will become a fraction, whose numerator and denominator are integral symmetrical functions. Hence the preceding inference may be extended to all rational symmetrical functions whatever.

If the symmetrical functions of the forms $S(a)$, $S(a\,a)$, $S(a\,a\,a)$, &c. be called *primary* symmetrical functions, we may infer in general, without immediate reference to equations, that *if the primary symmetrical functions of any number of quantities be given, all rational symmetrical functions of the same quantities may be found.* For the primary symmetrical functions are the coefficients of an equation, of which the quantities themselves are the roots.

(374.) Let us now apply the preceding principles to the solution of the following problem, "to find an equation whose roots are the sums of every pair of roots of a given equation."

Let the given equation be $X = 0$, and a_1, a_2, a_3, &c. its roots.

Let the sought equation be $Y = 0$, and $a_1 + a_2$, $a_1 + a_3$, &c. its roots. The number of roots of Y being the number of combinations of two letters which can be made from m letters, the degree of Y will be $\frac{m(m-1)}{1\,.\,2}$.

The coefficient of its second term will be the sum of the quantities $a_1 + a_2$, $a_1 + a_3$, ... that of the third term will be the sum of their products in combinations of two, that of the fourth the sum of their products in combinations of three, and so on. These, being all symmetrical functions of the roots, may be determined by the preceding principles.

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And in general an equation may be found whose roots are any symmetrical functions of the roots of the given equation taken in combinations of two, or of three, &c. For the degree of the sought equation will be determined by the number of combinations, and its coefficients being symmetrical functions of its roots, which are themselves symmetrical functions of the roots of the given equations, and these again being functions of the coefficients of the proposed equation, it follows, that in every case the coefficients of the sought equation may be derived from those of the given equation.

The equation of the squares of the differences determined in (356) is an example of this, since the squares of the differences are symmetrical functions of the roots. The coefficients of this equation may easily be determined on the principles established in the present section.

(375.) We shall now apply the properties of symmetrical functions to the solution of the following important analytical problem: "To determine the degree of the final equation resulting from the elimination of one of the unknown quantities by two equations of the m^{th} and n^{th} degrees, including two unknown quantities."

A general equation of the m^{th} degree between two unknown quantities in which the sum of the exponents of the unknown quantity in that term in which it is highest is equal to m. Such an equation should include terms containing every combination of powers of the unknown quantities, the sum of whose exponents does not exceed m. Hence if it be arranged according to the dimensions of x, and the coefficients be A_0, A_1, A_2, &c.

$$A_0 x^m + A_1 x^{m-1} + A_2 x^{m-2} + \ldots . A_{m-1} . x + A_m = 0$$

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the several coefficients, the first excepted, will be integral and rational functions of y; but they must be such that their dimensions when combined with the power of x will not exceed m. The first coefficient must, therefore, be independent of x and y, and therefore a known quantity; and the *forms* of the successive coefficients must be respectively

$$\begin{array}{ll} A_1 \ldots\ldots\ldots & a y + b \\ A_2 \ldots\ldots\ldots & a y^2 + b y + c \\ A_3 \ldots\ldots\ldots & a y^3 + b y^2 + c y + e \\ \&c. & \&c. \\ A_{m-1} \ldots\ldots & a y^{m-1} + b y^{m-2} \ldots . l \\ A_m \ldots\ldots\ldots & a y^m + b y^{m-1} + \ldots k y + l \end{array}$$

Now let the two given equations be arranged according to the powers of x, and let the coefficients be understood to be functions of y, such as those just described, and let the equations be

$$A_0 x^m + A_1 x^{m-1} + A_2 x^{m-2} + \ldots . A_{m-1} . x + A_m = 0$$
$$B_0 x^m + B_1 x^{m-1} + B_2 x^{m-2} + \ldots . B_{m-1} . x + B_m = 0$$

Let their first members be called A and B.

Let the former equation be imagined to be solved, as if y was a known quantity, and let the roots be a_1, a_2, a_3, &c. These will be respectively functions of y. If any one of these functions be substituted for x in $A = 0$, it will convert the equation into an *identity*, and it will be true for every value whatever of y. This will not, however, be the case if any of the same values be substituted in $B = 0$. By successively substituting the functions $a_1, a_2, a_3, \ldots$ for x in $B = 0$, the first member becomes

$$\left.\begin{array}{l} B_0 a_1^m + B_1 a_1^{m-1} + B_2 a_1^{m-2} + \ldots\ldots\ldots + B_{m-1} . a_1 + B_m \\ B_0 . a_2^m + B_1 . a_2^{m-1} + B_2 a_2^{m-2} + \ldots\ldots\ldots + B_{m-1} . a_2 + B_m \\ B_0 . a_3^m + B_1 . a_3^{m-1} + B_2 . a_3^{m-2} + \ldots\ldots\ldots + B_{m-1} . a_3 + B_m . \\ \qquad\qquad \&c. \qquad\qquad\qquad\qquad \&c. \end{array}\right\} \quad [1]$$

Since $B_0, B_1, B_2, \ldots$ are rational functions of y, and $a_1, a_2, a_3, \ldots$ are *in general* irrational functions of y, it follows, that those polynomes are in general irrational functions of y. We shall now prove that any value of y which renders any one of these polynomes $= 0$, will, in combination with the corresponding value of x, satisfy the proposed equations $A = 0$, $B = 0$. Since any of the functions $a_1, a_2, a_3, \ldots$ will satisfy $A = 0$ independently of y, they will also satisfy it when y has such a value as renders one of the above polynomes $= 0$. Let this value be y'. Now let y' be a value which renders the first of the polynomes $= 0$. Let y' be substituted for y in the function a_1, and let the corresponding value of a_1 be x'. It follows then, that y' x' are a system of values of y and x which satisfy the equation $A = 0$. But they also satisfy the equation $B = 0$. For since y' renders the first of the above polynomes $= 0$, and this polynome is, in fact, the first member of $B = 0$, a_1 being substituted for x, it follows, that if y' and x' be substituted for y and x in the first member of $B = 0$, it will become an identity. Hence, in general, y' x' is a system of values of y and x, which satisfies both of the given equations.

It is easy to perceive, also, that every value of y, which, in combination with a value of x, will satisfy both of the given equations, must render one of the preceding polynomes $= 0$. For the value of x which in conjunction with that of y satisfies the equations, must be one of the functions $a_1, a_2, a_3, \ldots$ the value of y being substituted for it; and hence it is evident, that the corresponding polynome becomes an identity.

Hence we may infer, that the equation whose first member is the product of all the polynomes [1], must contain among its roots all the values of y, which can satisfy both the equations $A = 0$, $B = 0$. If these several polynomes be expressed by $A^{(1)}$, $A^{(2)}$, $A^{(3)}$, &c. the equation which thus gives the values of y independently of x is

$$A^{(1)} . A^{(2)} . A^{(3)} \ldots\ldots A^{(m)} = 0. \qquad [2]$$

The first member of this equation is evidently a symmetrical function of the roots $a_1, a_2, a_3, \ldots\ldots$ for if any one of the roots be changed into any other in it, no other change will be produced than a change in the *order* of its factors.

Now as *every* symmetrical function of the roots can be determined by the principles established in this section, the present one may also be obtained; and hence an equation will be established in which the unknown quantity y alone will appear, x being eliminated; which is, in effect, a new process of *elimination*.

We shall not here go through the process for determining the form of the function in the first member of [2]; our present object is merely to determine the degree of the final equation [2].

The object then is to determine the highest dimen-

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sion of y which is found among its terms. Let $L\,a_1^{p}$, $L'\,a_2^{p'}$, $L''a_3^{p''}$, &c. be any terms of each of the polynomes of [1]. The continued product of these will be a term of the first member of [2] when developed, and this term is therefore

$$L \,.\, L' \,.\, L'' \ldots\ldots \times a_1^{p}\, a_2^{p'}\, a_3^{p''} \ldots.$$

Now as products of the same combination of letters will be permuted in every possible way in the first member of [2], it follows, that

$$L \,.\, L' \,.\, L'' \ldots\ldots \times S\,(a_1^{p} \,.\, a_2^{p'} \,.\, a_3^{p''} \ldots.)$$

will necessarily be a part of this first member. The question then is, to find what is the highest power of y which can enter such a function as this.

The quantities $L, L', L'', \ldots.$ being coefficients of the equation $B = 0$, the highest dimension of y which enters any one of them is such that, when added to the exponent of the power of x which it multiplies, it will give a sum equal to n. If then p be the exponent of x, $n - p$ will be the highest corresponding exponent of y, in each of the quantities $L, L', L'', \ldots.$ and as the number of these quantities is that of the roots of $A = 0$ or m, the highest exponent of y in the product $L\,L'\,L'' \ldots.$ is $m\,(n - p)$.

To determine the highest exponent of y in the function $S\,(a_1^{p} \,.\, a_2^{p'} \ldots\ldots)$ we must refer to the values of $S\,(a)$, $S\,(a^2)$, $S\,(a^3) \ldots.$ established in (369). From these, and from the forms of the coefficients of $A = 0$, and $B = 0$, it appears that the dimensions of y in the functions $S\,(a)$, $S\,(a^2)$, $S\,(a^3) \ldots.$ are the 1st, 2d, 3d, &c. Hence, it follows, that the dimensions of y in $S\,(a_1^{p} \,.\, a_2^{p'} \,.\, a_3^{p''} \ldots.)$ are $p + p' + p'' \ldots.$ But p, $p', p'', \ldots.$ being the exponents of x in $B = 0$, they must be such, that when added to the highest exponent of y in $L\,L'\,L'' \ldots.$ the sum will not exceed n. If this exponent be l, the highest value of p will be $n - l$; and since the number of these which are contained as factors in the product $S\,(a_1^{p}\, a_2^{p'}\, a_3^{p''} \ldots.)$ is m, the highest dimensions of y is $m\,(n - l)$. This added to the dimensions $m\,(n - p)$ in $L\,L'\,L'' \ldots$ will give the highest dimensions of y in [2] $m\,(2\,n - p - l)$, but $p + l = n$, $\because$ the highest degree of y in [2] is $m\,n$.

SECTION XXXIV.

Numerical Equations—limits of the Roots.

(376.) The various properties of the roots of equations established in the preceding sections are applicable to all equations whatever, whether their coefficients be literal or numeral, that is, whether the equations be algebraic or numerical. The solution of the problem to determine the roots of a general algebraic equation of a degree higher than the fourth has never yet been effected. And even in the cases in which some analysts have succeeded in discovering the formulæ for the roots, the results are always complicated, and frequently inapplicable in practice. The species of equations which, however, most frequently occur in philosophical investigations are *numerical*, and although we may be unable to assign the general forms of the roots, yet we can always determine their values where the numerical values of the coefficients are known. In the present and succeeding sections, we propose to develope the methods of finding the roots of numerical equations.

Limits of the Roots of Numerical Equations.

Let the first member of a numerical equation of the m^{th} degree be expressed as before, thus

$$x^m + A_1\, x^{m-1} + A_2\, x^{m-2} \ldots.\, A_{m-1} \,.\, x + A_m$$

the letters used here to represent the several coefficients are to be understood as expressing *particular numbers.*

Any number whatever being substituted for x, let the value of this polynome corresponding to that number be y; hence we have

$$y = x^m + A_1 \,.\, x^{m-1} + A_2 \,.\, x^{m-2} + \ldots\, A_{m-1} \,.\, x + A_m, \quad [1]$$

the roots of the proposed equation are those numbers which being substituted for x will render $y = 0$. In general, a particular value being substituted for x must render y either > 0, $= 0$, or < 0. Let two particular values x', x'' be substituted for x, and let the corresponding values of y be y', y''. These values y', y'' must either have different signs or the same sign.

(377.) 1. *If y' and y'' have different signs, there is at least one real root included between the numbers x' and x'', and, in general, there may be an odd number of real roots between them.*

(By the numbers included between two given numbers, is meant numbers greater than the lesser, and less than the greater. It is necessary, however, to attend to the effect of their signs (188)).

Let y' be negative, and y'' positive. Let X be the sum of the positive terms in the value of y, and X′ the sum of the negative terms, so that we have

$$y = X - X'.$$

Since X and X′ each consist of integral powers of x with positive numerical coefficients, it is evident that if we suppose the value of x continually to increase from x' to x'', each of the quantities X and X′ must also continually increase. But when $x = x'$, $y = y' < 0$, $\therefore X < X'$, and when $x = x''$, $y = y'' > 0$. Hence, as x continually increases from x' to x'', X and X′ both increase; but X increases more rapidly than X′, since it is in the first instance less than X′, and afterwards surpasses it. As the increase is *continual*, it follows, that before X surpasses X′ it must become equal to it, and when it does, $X - X' = 0$, $\therefore y = 0$, and the value of x which corresponds to this state is a real root. Hence there is one real root at least between x' and x''.

But it may happen, that between the values of X and X′ which correspond to x' x'', the value of X first increases so as to exceed X′, then the rate of increase of X becoming slower than that of X′, the latter may again surpass X, so that X - X′ again becomes negative, and, finally, X may again increase more rapidly than X′, and become greater than X′ before x becomes equal to x''. In this case, while x is increasing gradually in value from x' to x'', X first increases from being $<$ X to be $>$ X′, then X′ increases from being $<$ X to be $>$ X, and, finally, X again increases so as to be $>$ X′. X must be equal to X′ in *three cases*; and, therefore, there will be three real roots between x' and x''.

By generalizing this reasoning, it appears, that the rates of increase of X and X′ may alternately exceed each other, while x is increasing from x' to x''. But that by these changes X and X′ *must* be equal at least *once*, and *may* be equal an odd number of times. From whence it follows, that between x' x'' there *must* be one real root, and *may* be any odd number of real roots.

This reasoning is applicable when either or both of the values x' x'' is negative, and when either of them $= 0$.

(378.) 2. *If y' and y'' have the same sign, there are either no real roots or an even number of them between x' and x''.*

As before, X and X' increase continually by the continual increase of x. If the common sign of y' and y'' be +, X is greater than X' at the two limiting values corresponding to x' and x'', and may, therefore, be greater than it for all intermediate values. If the common sign of y' and y'' be −, X is less than X' for both the limiting values, and may, therefore, be less than it for all intermediate values. In both cases, therefore, there may be no real root between the limits x' and x'', since it is not necessary that X should = X'.

But it may so happen that the rates of increase of X and X' may so change between the limits, that each will alternately surpass the other, and in every such change they must become equal. Now, since at the limiting values X and X' are similarly related to one another, it is plain that if there be any changes of relation as to magnitude, there must be an even number; for otherwise the result of the whole would change their relative magnitudes contrary to hypothesis. Hence, it follows, that between the limits x' and x'' there must be an even number of real roots, or none.

(379.) *A value may always be assigned to* x *in the second member of* [1], *such that* $y > 0$, *and so that all values greater than the assigned value will also render* $y > 0$.

Let the equality, [1] be expressed in the form

$$y = x^m\left(1 + \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{A_3}{x^3} \ldots\ldots \frac{A_{m-1}}{x^{m-1}} + \frac{A_m}{x^m}\right)$$

A value may be assigned to x such as will render each of the terms within the parenthesis, except the first, less than any assigned value; and therefore such a value may be given to x as will render the sum of these terms < 1. It is evident that one such value being found, every *greater* value of x will render the sum of these terms still less than 1. Hence, this value of x, and all greater values, will render the parenthesis positive; and since x^m is positive, y will be necessarily positive.

Hence, it is evident, that no root of the equation can be greater than such a value of x.

(380.) *To determine a number, which, being substituted for* x, *will render the first member of an equation positive, and such that all greater numbers will also render it positive.*

The number sought must be such as will render the first term x^m greater than the algebraical sum of all the succeeding terms. Let S be this algebraical sum, and let S' be the arithmetical sum, and let K be the greatest numerical coefficient. It is plain that S' is generally greater, and cannot be less than S $\because$ if $x^m > S'$ we must also have $x^m > S$. Also it is plain that S' cannot be greater than

$$K\,(x^{m-1} + x^{m-2} + x^{m-3} \ldots\ldots x + 1).$$

Since K is, by hypothesis, the greatest coefficient in S', and the several terms are affected by the same powers of x. The problem will then be solved by any value of x which satisfies the condition

$$x^m > K\,(x^{m-1} + x^{m-2} + x^{m-3} \ldots\ldots x + 1).$$

But the quantity within the parenthesis is a geometrical series whose first term is 1, the common multiplier x, and the number of terms m. Hence the sum of the series is $\frac{x^m - 1}{x - 1}$, by which the above inequality becomes

$$x^m > K\,.\,\frac{x^m - 1}{x - 1},$$

$$\because x^{m+1} - x^m > K\,x^m - K,$$

$$\because x^{m+1} > (K + 1)\,x^m - K,$$

$$\because x > (K + 1) - \frac{K}{x^m},$$

a condition which will evidently be fulfilled if $x = K + 1$. Hence we may infer that the greatest coefficient in the equation, taken with a positive sign, and increased by unity, is greater than the greatest root of the equation.

In obtaining this superior limit we have taken an extreme case, *scil.* that in which all the terms of the equation, except the first, are negative. This seldom happens, and therefore the limit thus obtained is, in general, too wide. To obtain a nearer limit, let the exponent of the highest power of x, which has a negative coefficient, be $m - n$. Let S be the algebraical sum of this and all the succeeding terms. It is evident that any value of x which renders $x^m > S$ will be a superior limit. Let S' be the arithmetical sum of the terms of S. As before, S' is generally greater and cannot be less than S. Let K be the greatest numerical coefficient of S; as before, S cannot be greater than

$$K\,(x^{m-n} + x^{m-n-1} + \ldots\ldots x + 1).$$

Hence the value of x will be a superior limit, if it fulfil the condition

$$x^m > K\,(x^{m-n} + x^{m-n-1} + \ldots\ldots x^2 + x + 1).$$

The sum within the parenthesis is $\frac{x^{m-n+1} - 1}{x - 1}$. Hence the condition becomes

$$x^m > K\,.\,\frac{x^{m-n+1} - 1}{x - 1}.$$

Hence it follows that the condition will be fulfilled by the value of x determined by

$$x^m > \frac{K\,x^{m-n+1}}{x - 1},$$

$$\because x^{n-1} > \frac{K}{x - 1},$$

$$\because x^{n-1}\,(x - 1) > K.$$

Let $x - 1 = p$, $\because x = p + 1$, $\because$

$$(p + 1)^{n-1}\,.\,p > K.$$

This inequality is evidently satisfied by $p^n = K$, for

$$(p + 1)^{n-1}\,.\,p > p^{n-1}\,.\,p = p^n.$$

Hence $\quad x - 1 = K^{\frac{1}{n}}, \quad \because x = K^{\frac{1}{n}} + 1.$

Hence we infer, that "that root of the greatest numerical coefficient whose exponent is the number of terms preceding the first negative term increased by unity, is a major limit of the roots of the equation."

If all the terms of the equation, except the first, be negative, this limit is equivalent to the former one.

(381.) The limits just determined are major limits of the positive roots. It remains to determine their minor limits, and also the major and minor limits of the negative roots.

If the equation be transformed by the substitution $x = \frac{1}{y}$, and the new equation cleared of fractions, it will be one whose greatest positive root is the least positive root of the former; since the roots of the two equations are reciprocals. The coefficients in the two equations will also be the same, but occurring in an opposite order. Hence the major limit of the roots in the transformed equation is the minor limit of the positive roots in the original equation.

(382.) To determine the major limit of the negative roots, let x in the proposed equation be changed into $-y$, and the positive roots of the transformed equation are equal to the negative roots of the original equation. Hence the major limit of the positive roots of the transformed equation is the major limit of the negative roots of the original equation.

(383.) To determine the minor limit of the negative roots, let $-\frac{1}{y}$ be substituted for x, and the major limit of the positive roots of the transformed equation will be the minor limit of the negative roots of the given equation.

Hence it appears that the method of determining the major limit of the positive roots being known, the other limits may be found.

Examples. Determine the major limits of the positive roots of the following equations:

$$x^4 - 5x^3 + 37x^2 - 3x + 39 = 0,$$

$$\sqrt[n]{K} + 1 = 5 + 1 = 6;$$

$$x^5 + 7x^4 - 12x^3 - 49x^2 + 52x - 13 = 0,$$

$$= \sqrt{49} + 1 = 8;$$

$$x^4 + 11x^2 - 25x - 67 = 0,$$

$$= \sqrt[3]{67} + 1 = 6;$$

$$3x^3 - 2x^2 - 11x + 4 = 0,$$

$$= \frac{11}{3} + 1 = 5.$$

(384.) In particular cases it happens that transformations present themselves which expedite the process and give nearer limits than the general method.

If the first member of the equation be such as can be resolved into a series of products, one factor of each being a monome, and the other a binome, whose second term is a particular number and negative. Such is the second of the preceding examples, which may be written thus,

$$x^2(x^3 - 49) + 7\left(x - \frac{12}{7}\right) + 52\left(x - \frac{1}{4}\right) = 0.$$

A limit will here be obtained by finding a value of x, which will render all the binome factors positive. Such is $x = \sqrt[3]{49}$. Hence 4 is a limit nearer than 8.

(385.) There is another method of finding limits to the roots of equations, the discovery of which is due to NEWTON. Let $x' + u$ be substituted for x, and the transformed equation will become (354)

$$X'_0 + X'_1 \cdot \frac{u}{(1)} + X'_2 \cdot \frac{u^2}{(2)} + X'_3 \cdot \frac{u^3}{(3)} + \dots u^n = 0.$$

Let such a value be assigned to x' as will render the several polynomes X'_0, X'_1, X'_2 .... positive, and this value will be a major limit of the positive roots. For in that case the transformed equation cannot have any positive root, since a polynome, all whose terms are positive, cannot $= 0$. Hence the real values of u must be essentially negative. But $x = x' + u$, $\therefore x' = x - u$. Since u is essentially negative, $-u$ is essentially positive, $\therefore x - u > x$, $\therefore x' > x$. Hence x' is a major limit.

(386.) If all the terms of an equation be positive, it cannot have a real positive root, for the sum of any number of positive monomes cannot $= 0$. For a similar reason, if the terms be alternately positive and negative, it cannot have a real negative root; for in this case if the degree of the equation were even, all the terms of the first member would be positive monomes; and if the degree were odd, all the terms would be negative monomes. In the one case, the first member would be the sum of several positive monomes, and in the other, it would be the sum of several negative monomes. In neither case could it be $= 0$.

SECTION XXXV.

On the Real Roots of Numerical Equations.

(387.) *Every equation whose degree is characterised by an odd number, and whose coefficients are real, has at least one real root, whose sign is different from that of its last term.*

If in the equation

$$y = x^m + A_1 x^{m-1} + A_2 x^{m-2} + \dots A_{m-1} \cdot x + A_m$$

we suppose $x = 0$, we have $y = A_m$; and, on the other hand, a value may be assigned to x such that x^m will be *numerically* greater than all the succeeding terms together; if such a value be assigned to x, with a sign different from that of A_m, the sign of y will be different from that of A_m. Hence, then, for $x = 0$, the sign of y is the same as that of A_m, and for the other value of x it is different. Hence one real root must be between those values.

(388.) *Every equation of an even degree in which the least term is negative, and whose coefficients are real, must have at least two real roots with different signs.*

For if $x = 0$, $y = -A_m$; and, on the other hand, such a value may be assigned to x as will render x^m numerically greater than the sum of all the succeeding terms. Whether this value of x be positive or negative, x^m will be positive, since m is even, and therefore the value of y will be positive. Hence it follows, that between this value of x, taken with a positive and negative sign, and $x = 0$, there is in each case a real root, the one positive and the other negative.

It is evident, that in the former case the real root lies between $K + 1$ and 0, and that in the latter case the positive root is comprised between $K + 1$ and 0, and the negative root between $-(K + 1)$ and 0.

Hence, the principle assumed in (316,) that "every equation has at least one root, is established for all equations, except those of an even order, in which the last term is positive."

(389.) *Imaginary roots enter equations by pairs*; that is to say, there must be an even number of them, or none.

Let the first member be divided by all the simple factors which correspond to the real roots; the quotient must be rational, and its coefficients must be real. Its roots will, by hypothesis, be all the imaginary roots of the proposed equation, and no others. Its degree must therefore be even, since it can have no real root (388;) and since its degree is even, the number of its roots is even therefore, &c.

Hence, an equation whose roots are all imaginary must be of an even degree.

(390.) *The first member of an equation whose roots are imaginary will be positive whenever a real value is ascribed to* x. For if for any such value it were negative, there will be another value $(K + 1)$ for which it will be positive, and these values will include between them at least one real root contrary to the hypothesis.

It is evident also, that in such an equation the last term must be positive, (388.)

(391.) *When the last term of an equation is positive, the number of real and positive roots is even; and when the last term is negative the number is odd.*

1. Let the last term be positive.

For when $x = 0$, y is positive; and such a value $K+1$ may be assigned to x as will render y positive. Hence there must be an even number of real and positive roots comprised between $x = 0$ and $x = K + 1$, or none.

2. If the last term be negative. When $x = 0$, y is negative; and when $x = K + 1$, y is positive. Hence, between these limits there must be an odd number of positive roots.

In the same manner it is evident, that if the number of real and positive roots be even, the last term is positive; and if it be odd, the last term is negative.

(392.) *No equation can have a greater number of positive roots than there are changes of sign among its successive terms, nor a greater number of negative roots than there are successive repetitions of the same sign.*

This rule, which was first established by Descartes, is known by the name of *Descartes rule of signs.*

By "changes of sign," and "successive repetitions of the same sign," is meant each successive pair of terms which have the same sign, and each successive pair of terms which have different signs. The number of changes, together with the number of successive repetitions, must be one less than the number of terms, and therefore must be equal to the exponent of the degree of the equation. Thus, if the equation be

$$x^7 + A_1 x^6 - A_2 x^5 - A_3 x^4 + A_4 x^3 + A_5 x^2 + A_6 x - A_7 = 0,$$

there are four successive repetitions, and three changes.

We shall establish the rule of Descartes by showing, that for every positive root which is introduced into an equation one additional change of sign at least is also introduced, and for every negative root which is introduced one additional successive repetition at least is also introduced.

Let the equation be

$$x^m + A_1 x^{m-1} + A_2 x^{m-2} + \ldots . A_{m-1} . x + A_m = 0. \quad [1]$$

To introduce into this an additional positive root $(+ a)$ it is only necessary to multiply it by $x - a$, and the result is

$$\begin{array}{r|r|r|r|r|l} x^{m+1} + A_1 & x^m + A_2 & x^{m-1} + A_3 & x^{m-2} + \ldots . + A_{m-1} & x^2 + A_m x & \\ - a & - A_1 a & - A_2 a & - A_{m-2} . a & - A_{m-1} . a & - A_m a = 0. \end{array} \quad [2]$$

This equation is one degree higher than the former, and contains one term more. Each coefficient is composed of two parts, the first part being the coefficient of the term which holds the same order in the former equation, and the second part the coefficient of the term which precedes that in the former equation multiplied by $- a$. Thus, if A_{n-1} be the coefficient of the n^{th} term of the former equation $(A_{n-1} - A_{n-2} . a)$ will be the coefficient of the n^{th} term in the latter equation.

The signs of the successive coefficients of the equation [2] depend in some cases on the signs alone of the successive terms of [1], and in some cases on the values of the coefficients, and the root a.

If the $(n - 1)^{th}$ and n^{th} coefficients of [1] have the same sign, and therefore form a successive repetition, the two parts A_{n-1} and $A_{n-2} . a$ of the coefficient of the n^{th} term of [2] will necessarily have different signs, and in this case the sign of the whole coefficient will depend on the particular values of the numbers A_{n-1}, A_{n-2}, and a.

But if the $(n - 1)^{th}$ and n^{th} coefficients of [1] have different signs, then the common sign of the parts of the n^{th} coefficient of [2] will be that of the n^{th} coefficient of [1]; this common sign will then be the same as the sign of the n^{th} term of [2].

Thus it appears, that each successive repetition in [1] gives a doubtful sign in [2]; doubtful as far as it can be determined by the signs alone of [1], and each change of sign in [1] gives to the corresponding term of [2] the sign of [1]. There will then be in [2] as many doubtful signs as there are successive repetitions in [1], and all the other signs will be the same with those of the corresponding terms in [2]. The sign of the last term of [2] will be evidently different from that of the last term of [1].

Our object is now to prove that the number of changes of sign in [2] must be at least one more than in [1]. To establish this, let the doubtful signs be replaced in the manner least favourable to the production of changes, which is to make every doubtful sign, or succession of doubtful signs, the same as the sign which immediately precedes or follows it. It should here be observed, that when a doubtful sign is immediately preceded and followed by determinate signs, these determinate signs must be different; this necessarily follows from the consideration that determinate signs in [2] are produced by *changes* in [1], and doubtful signs by repetitions. Hence it follows, that whether a doubtful sign be replaced by the preceding or following sign, it must be the means of introducing at least one change of sign into [2]. In the same manner it follows, that if several doubtful signs succeed each other in [2], the signs which immediately precede and follow the series must be different; and therefore whether the doubtful signs be replaced by the preceding or following sign, one change at least must be introduced.

Hence it follows, that if all the doubtful signs be replaced by determinate signs in the manner just described, there will be the same number of changes and of repetitions in the first m terms of [2] as there are in [1]. But the m^{th} and $(m+1)^{th}$ term of [2] will necessarily give an additional change. For if it be immediately preceded by a determinate sign the change is manifest, since A_m and $-A_m a$ must have different signs, and the sign of A_m must be that of the penultimate term of [2]. But if the sign of the penultimate term of [2] be doubtful, and that it be replaced by the sign which precedes it, the change in the last term is also apparent, since the term which precedes it must have a sign different from that of the last term. If, on the other hand, it be replaced by the sign of the last term, the additional change will fall upon the penultimate term. The same reasoning evidently applies, *mutatis mutandis*, to the case in which the penultimate term is the last of a succession of doubtful signs.

As an example of this, let the succession of signs in [1] be

$$+++--+-+\cdot--+.$$

Let the doubtful sign be expressed by , and the succession of signs in [2] will be

$$+,,-,+-+-,,+-$$

Now if each doubtful or succession of doubtful signs be replaced by the sign which precedes it, we shall have

$$+++--+-+---+-.$$

In this case the signs are the same as in the first, as far as the penultimate. Between that and the last is a change.

If the doubtful places were filled by the signs which follow them, we should have

$$+---++-+-+++-.$$

Here are seven changes, while there are but six in the first.

Since each positive root which is introduced necessarily adds one to the number of changes, it follows that there cannot be more positive roots than there are changes of sign in the equation.

By reasoning exactly similar, it is proved, that the multiplication of [1] by the factor $x+a$ necessarily introduces at least one repetition more; and that, therefore, the number of negative roots cannot exceed the number of repetitions.

(393.) Hence it follows, that if the roots of the equation be all real, the number of positive roots is equal to the number of changes of sign; and the number of negative roots is equal to the number of repetitions of sign.

(394.) If any power of x which is admissible in an equation of the m^{th} degree be wanted, it may be conceived to be supplied with a coefficient which $=0$. The term in this case may be conceived to be affected indifferently with the sign $+$ or $-$. The number of real positive roots will be determined by the number of repetitions of the same sign in each case when the roots are all real. Now if this number be different when the deficient term is supposed to have the sign $+$ from what it is when it has the sign $-$, a contradiction arises from the supposition that all the roots are real. In such a case, therefore, we may infer the existence of imaginary roots. But if either sign, which may be attributed to the deficient term, satisfies the condition established in the preceding paragraph, we cannot infer the existence of imaginary roots.

Thus, in the equation

$$x^3+px+q=0,$$

if the deficient term be supplied thus

$$x^3\pm 0\,.\,x^2+px+q=0,$$

if the upper sign be taken, we infer, that if all the roots be real they must be all positive; and if the lower sign be taken, we infer, that two must be negative and one positive, which is a contradiction. Hence we infer, that in this case all the roots of the equation cannot be real; and since only an even number of imaginary roots can occur, it follows that but one can be real.

But if the equation be

$$x^3-px+q=0,$$

the deficient term being supplied, we have

$$x^3\pm 0\,.\,x^2-px+q=0.$$

In this case, whichever sign be attributed to the deficient term, the number of repetitions and changes are the same. Hence we cannot infer the existence of imaginary roots.

Hence, a test for proving the existence of imaginary roots is this, that the change in the sign of the deficient term should alter the number of repetitions and changes.

(395.) *An equation whose roots are all real has as many positive roots, whose values are between* 0 *and* $+$ a, *as there are repetitions of sign in the equation obtained by substituting* x $-$ a *for* x.

All the roots of the proposed equation which are between 0 and $+a$ will necessarily be negative when $x-a$ is substituted for x; therefore as many changes of sign in the original equation as are equal to the number of roots between 0 and a will necessarily be changed into repetitions of sign in the transformed equation. The reverse of this may also be easily established, *scil.* An equation whose roots are all real cannot have any positive roots between 0 and $+a$, unless the equation found by substituting $x-a$ for x in the proposed equation has a greater number of repetitions than the proposed equation.

SECTION XXXVI.

Method of Determining the Rational Roots of Numerical Equations.

(396.) The determination of all rational roots may be reduced to that of integral roots. For we have already (349) shown, that if an equation have any fractional coefficients a transformation may be effected which will remove them, and give an equation with integral coefficients, that of the first term being unity. Every rational root of such an equation must be an integer; for let a fraction $\frac{a}{b}$ be substituted for x in its first member, and it becomes

Algebra.

$$\frac{a^n}{b^n} + A_1 \cdot \frac{a^{n-1}}{b^{n-1}} + A_2 \cdot \frac{a^{n-2}}{b^{n-2}} + \dots$$
$$+ A_{n-1} \cdot \frac{a}{b} + A_n = 0.$$

Multiply the whole by b^{n-1} and we obtain

$$\frac{a^n}{b} + A_1 \cdot a^{n-1} + A_2 \cdot a^{n-2} \cdot b + \dots$$
$$+ A_{n-1} \cdot a\, b^{n-2} + A_n \cdot b^{n-1} = 0.$$

The first term of this is a fraction which is irreducible. For since b is prime to a it is prime to a^n, (95;) and all the other terms are integers, whence we should find a fraction equal to an integer.

Hence the equation cannot have a fractional root, and therefore every rational root must be integral.

It may be observed here, that if all the terms of an equation but one be integral, that one must also be integral.

(397.) We shall therefore consider the equation as having only integral coefficients, and as having no rational roots but integers. Let a be a root, and be substituted in its first member, and the result divided by a gives

$$a^{n-1} + A_1 a^{n-2} + A_2 a^{n-3} + \dots\dots$$
$$+ A_{n-1} \cdot + \frac{A_n}{a} = 0.$$

Since A_n is the continued product of all the roots with their signs changed, a is a factor of it, and since a is an integral root by hypothesis $\dfrac{A_n}{a}$ is an integer. Let it be Q_1, so that

$$a^{n-1} + A_1 a^{n-2} + A_2 a^{n-3} + \dots\dots$$
$$A_{n-2} \cdot a + A_{n-1} + Q_1 = 0.$$

Let this be divided by a, and we obtain

$$a^{n-2} + A_1 a^{n-3} + A_2 a^{n-4} + \dots\dots$$
$$A_{n-3} \cdot a + A_{n-2} + \frac{A_{n-1} + Q_1}{a} = 0.$$

Since all the terms of this but the last are integers, the last must also be an integer, and therefore a measures $A_{n-1} + Q_1$. By the continuance of this process we obtain the following results,

$$\frac{A_n}{a} = Q_1, \qquad \frac{A_{n-1} + Q_1}{a} = Q_2,$$
$$\frac{A_{n-2} + Q_2}{a} = Q_3, \qquad \frac{A_{n-3} + Q_3}{a} = Q_4,$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\frac{A_2 + Q_{n-2}}{a} = Q_{n-1}, \qquad \frac{A_1 + Q_{n-1}}{a} = -1.$$

(398.) Hence, to determine the integral roots it is necessary to determine, in the first instance, the integral factors of the last term. Among these, all the integral roots must be found. To determine whether any one of these be a root, it is only necessary to substitute it in the proposed equation, and if it converts this into an identity it is a root, and otherwise not. But this process is generally tedious, and when the last term contains several factors must be repeated for each factor. In the cases where the factors are *not* roots, they may be determined not to be so more expeditiously by the several criterions which we have just established.

Rational Roots of Numerical Equations.

1. *Let the last term be divided by the proposed factor* a, *and let the quote be added to the preceding coefficient,* this sum must be divisible by a.

2. *Let this new quote be added to the coefficient of* x^2, *and the sum must be divisible by* a, and so on.

Now if any of these sums be not divisible by a, it is sufficient to prove a not a root, without continuing the process further.

But if upon continuing the process the factor a be found to measure each sum, and if upon finally adding to A_1 the quote Q_{n-1} of the preceding sum, we obtain a result which is equal to a with a different sign, then a is a root of the proposed equation, and not otherwise.

(399.) The practical process for obtaining the rational roots of an equation is then as follows:

1. If the equation have any fractional coefficients, let the transformation in (349) be effected, and one obtained which will have integral coefficients.

2. Let the integral factors of the last term be found.

3. Let such of these factors as are included within the limits of the positive and negative roots be written down in succession.

4. Let the last term be divided by each of these, and let the quotes be written under them respectively.

5. Under these quotes let the coefficient of x be written.

6. Let this coefficient be added respectively to the member immediately over it, and let the sum be placed immediately under it.

7. Let each of these sums be divided by the first term in each column, and if the quote be an integer let it be written under the last term of the column. If not, the process may be stopped in that column in which the quote is fractional; and in this way the process may be continued, until either every column is stopped by fractional quotes, or until some of them arrive at the coefficient A_1.

(400.) We shall now apply this process to an example. Let the equation be

$$x^4 - x^3 - 13\,x^2 + 16\,x - 48 = 0.$$

The major limit of the positive roots is $13 + 1 = 14$. For the last two terms may be reduced to the form $16\,(x - 3)$. The major limit of the negative roots is $-(1 + \sqrt{48})$ or -8. The divisors of 48 are 1, 2, 3, 4, 6, 8, 12. Neither 1 nor -1 will satisfy the equation, since the last term alone is greater than the numerical sum of the other coefficients. Hence we have the following calculation:

Value of A_m	−48	−48	−48	−48	−48	−48	−48	−48	−48	−48
Factors of A_m	12	8	6	4	3	2	− 2	− 3	− 4	− 6
Values of Q_1	− 4	− 6	− 8	−12	−16	−24	+24	+16	+12	+ 8
Values of A_{m-1}	16	16	16	16	16	16	16	16	16	16
$A_{m-1} + Q_1$	12	10	8	4	0	− 8	40	32	28	24
Q_2	1	,	,	1	0	− 4	−20	,	− 7	− 4
A_{m-2}	−13	,	,	−13	−13	−13	−13	,	−13	−13
$A_{m-2} + Q_2$	−12	,	,	−12	−13	−17	−33	,	−20	−17
Q_3	− 1	,	,	− 3	,	,	,	,	5	,
A_1	− 1	,	,	− 1	,	,	,	,	− 1	,
$A_1 + Q_3$	− 2	,	,	− 4	,	,	,	,	4	,
Q_m	,	,	,	− 1	,	,	,	,	− 1	,

Hence the integral roots in this case are $+ 4$ and $- 4$. The equation is therefore divisible by $(x + 4)$ $(x - 4) = x^2 - 16$, which reduces it to

$$x^2 - x + 3 = 0,$$

the roots of which are imaginary.

(401.) It may however happen, that the equation which is obtained by dividing the given equation by the simple factors corresponding to the roots found by the preceding process, may have one or more integral roots. It is true, that the investigation already given determines all the different integral roots which the proposed equation can have; but it does not indicate whether any of these roots are more than once repeated in the equation. If they be so, it is evident that they will occur again as roots of the equation obtained by dividing the given one by the simple factors. It is proper, therefore, to submit this equation to the same process as the first, in order to detect the existence of these repeated roots. If the number of different integral roots of the first equation be not great, the repetition of them may be detected at once by dividing the resulting equation again by the same simple factor.

Also it follows, that the roots cannot be repeated if they be not factors of the last term of the new equation.

(402.) When the number of integral factors of the last term which are included between the limits of the positive and negative roots is considerable, the process by which those which are not roots may be determined may be shortened.

If a be a root, the first member is divisible by $x - a$, and the quote gives

$$X = (x - a)\,(x^{m-1} + A'_1 x^{m-2} + A'_2\, x^{m-3} + \ldots).$$

The forms of the coefficients A'_1, A'_2, &c. have been determined in (315). This equation must be fulfilled, whatever be the value of x. Let $x = 1$, and the polynome X becomes equal to the algebraical sum of its coefficients. The same is true of the polynome in the second member. Hence we have

$$\frac{1 + A_1 + A_2 + \ldots A_m}{1 - a} = 1 + A'_1 + A'_2 + \ldots$$

By the forms of the coefficients A'_1, A'_2 .... established in (315) it appears that they must all be integers. Hence it follows, that the algebraical sum of the coefficients of the proposed equation must be divisible by $1 - a$, if a be a root.

In like manner, if $- 1$ be substituted for x, we may prove that what the first member becomes by this substitution is divisible by $- 1 - a$. Hence the rule,

Substitute successively $+ 1$ and $- 1$ for x in the proposed equation, and let the numerical values of the results be M and M′.

1. Every positive factor of the last term which, being diminished by 1, does not divide M, and every negative factor which, being increased by 1, does not divide M′, must be rejected, not being roots.

2. Every negative factor whose numerical value, increased by 1, does not divide M, and every positive factor which, diminished by 1, does not divide M′, must be rejected, not being roots of the equation.

(403.) The investigation of the real and rational roots of equation is equivalent to the investigation of the real and rational factors of the first degree of their first members. After all the rational factors of the first degree have been found, although the remaining factors of the *first degree* be not rational, yet when combined in pairs they may form rational factors of the *second degree*. Before we conclude this section we shall therefore offer some remarks on factors of this kind.

Let any rational factor of the second degree be represented by $x^2 + p\,x + q$, and let p and q be considered as indeterminate quantities, whose values are to be ascertained in rational numbers.

For this purpose let the first member X of the equation be divided by $x^2 + p\,x + q$, and let the division be continued until a remainder be found which is of a lower degree than the divisor, and therefore of the form $M\,x + N$. In order that X should be exactly divisible by $x^2 + p\,x + q$, it is necessary that this remainder should $= 0$, independently of x; and, therefore, that $M = 0$ and $N = 0$. But M and N are quantities whose values depend on the numerical coefficients of X, and the indeterminates p and q. These latter, therefore, must have such values as will fulfil the two conditions $M = 0$ and $N = 0$. In these equations, therefore, let p and q be considered as unknown quantities; and either of them being eliminated gives a final equation including only the other. Such roots of this equation as are rational, being substituted in $M = 0$ or $N = 0$, give corresponding values of the other; and such systems of values as are rational being substituted for p and q in $x^2 + p\,x + q$, will give so many rational quadratic factors of the first member X of the proposed equation.

Since the general process here described must give every quadratic factor, it is evident that the final equation which determines the indeterminate p or q, must be of the $\frac{m\,(m-1)^{th}}{1\,.\,2}$ degree, since this is the number of different combinations of two factors. It must be apparent, therefore, that this process would be attended with great difficulties in practice, and is therefore rarely resorted to.

SECTION XXXVII.

On the Determination of the Real and Irrational Roots of Numerical Equations.

(404.) By the methods established in the preceding section, the rational roots of an equation being determined, its first member may be divided by the several corresponding simple factors, the result will be an equation whose roots are severally either irrational or imaginary. We propose to devote the present section to explaining the methods of determining the irrational roots, and we shall accordingly consider the equation as having been previously cleared of its rational roots.

The general form for these roots is not known, and can only be determined when some general method for the solution of equations of the higher degrees shall have been found. The want of these methods, however, in no wise impedes the progress of practical science, for we can always obtain the irrational roots with any required degree of approximation, and if we had their general forms we could do no more.

The numerical value of an irrational root, when reduced to decimal expression, will in general consist of two parts, the integral part a which precedes the decimal point, and the decimal part u which follows it. To express the decimal part u exactly, would require an infinite series of decimal places; for if the series were finite, or even periodic, the decimal would be equivalent to a rational number.* All, therefore, which can be done in this case is to determine as many places of u as may be necessary to give the requisite approximation, and this can always be done.

We shall, however, first consider the method of determining the integral part a of the root.

(405.) Let the major limit of the positive roots be $+$ L, and that of the negative roots $-$ L′; the more narrow these limits are determined, the more expeditious will be the process. Substitute for x in the equation the successive integers from 0 to $+$ L with positive signs, and from 0 to $-$ L′ with negative signs. When two successive substitutions give different signs to the first member of the equation, one at least, and in general an odd number of real roots must be comprised between the two successive integers, and the lower of the two integers is evidently the integral part a of the corresponding roots. If two successive substitutions give the same sign to the first member of the proposed equation, there will either be no real root comprised between the two integers, or there will be an even number of them. In the latter case, the lower of the two integers will be the integral part of all the intermediate roots.

Before, therefore, we can determine what integers between the limits $+$ L and $-$ L′ belong to irrational roots, it will be necessary to determine what number of roots are intercepted between each pair of successive integers.

We have already determined an equation which may always be deduced from the proposed equation, and of which the squares of the differences of the roots of the proposed equation are the roots. Since the square of a real quantity must always be positive, it follows, that the negative roots of this equation, if it have any, must be the squares of the differences of the imaginary roots. Let the minor limit of the positive roots of this equation be found, and let its square root be extracted. Let D be this root, or any number less than it.

If D $>$ 1, which will be the case if the minor limit of the positive roots of the equation of differences be greater than unity, it follows that no two real roots of the proposed equation can be contained between two successive integers, and, therefore, that if two successive integers substituted for x give the first member of the equation different signs, one, and but one, real root will be included between them, and the integral part of this root will be equal to the lesser of the two integers so substituted.

If two successive integers substituted for x give the first member the same sign, no real root can be included between them. Thus, in this case, we determine the number of incommensurable real roots, and the integral part of each.

If this number be equal to the exponent of the degree of the equation, there will be no imaginary roots. But if it be less than that exponent, there will be a number of imaginary roots equal to their difference.

If D $<$ 1, several real roots of the proposed equation may be intercepted between two successive integers. To determine if this be the case, let

$$0,\ 0 + \mathrm{D},\ 1 + \mathrm{D},\ 2 + \mathrm{D}, \ldots\ (\mathrm{L} - 1) + \mathrm{D},$$
$$0\ - \mathrm{D}, - 1 - \mathrm{D}, - 2 - \mathrm{D}, \ldots\ - (\mathrm{L}' - 1) - \mathrm{D},$$

be successively substituted for x in the first member of the supposed equation. Any two successive substitutions which give the first member different signs, must contain between them one, and but one real root; and any two successive substitutions which give the first member the same sign, can contain between them no real root. Hence the number of real roots is exactly obtained, and the integer next below each real root is known. This is the integral part of the root.

If the number of real roots in this case be equal to the exponent of the degree of the equation, there will be no imaginary roots; but if the number be less than that exponent, there will be a number of imaginary roots equal to their difference.

In this reasoning we have proceeded on the hypothesis, that the equation has been cleared of its *equal roots*. For if there were equal roots in the proposed equation, one of the roots of the equation of the squares of the differences would be $= 0$. Thus the minor limit D would $= 0$, and the process of substitution already explained would not be applicable. Indeed it is evident, that if there were equal roots we could not in any case infer that the change of sign on the substitution of two consecutive integers inferred but one intermediate root, nor that the identity of sign inferred none.

The equation may be cleared of its equal roots by the process explained in Sect. XXXI.

(406.) The methods which we shall explain for obtaining the decimal part u of the root, require that there should not be more than one real root between two successive integers. It will be therefore necessary in the case in which D $<$ 1 to effect a transformation on the equation, such as will render D $>$ 1. Let the denominator of D be k, and let $x = \frac{y}{k}$. By this substi-

* See Arithmetic, p. 499.

tution an equation will be obtained, whose roots are k times greater than the roots of the proposed equation, and, therefore, whose differences are k times greater. For if x, x'' be two roots of the proposed, we have

$$x' = \frac{y'}{k} \quad x'' = \frac{y''}{k} \quad \therefore (x' - x'')\, k = y' - y''.$$

Hence the least difference of the roots of the transformed equation will be k D, and as k is the denominator of D, k D cannot be less than unity. Hence, in the transformed equation more than one real root cannot be intercepted between two consecutive integers.

(407.) Having thus explained the methods of ascertaining the total number of irrational roots, the integral part of each of them, and of so transforming the equation that no two roots shall have the same integral part, we shall now proceed to explain the methods of determining the decimal part u, and in so doing we shall suppose that this transformation has been previously effected.

(408.) The first method of approximation which we shall explain is that of Lagrange.

Let X be the first member of the equation, and a the integral part of the root. Let $a + u$ be substituted for x in $X = 0$, and the result arranged by the dimensions of u is of the form established in (354.) If in this $u = \frac{1}{y}$, and the result be cleared of fractions, it becomes $Y = 0$, where Y expresses a polynome of the form $A\, y^m + B\, y^{m-1} + \ldots$ whose coefficients, however, are those found in (354,) x' being changed into a.

Since $x = a + \frac{1}{y}$ should determine all the values of x when those of y are known, and no others, it follows that $\frac{1}{y}$ must have one, and but one, real value < 1, and $\therefore y$ must have one, and but one, real value > 1; for were it supposed that y had more than one real and positive value > 1, then x would have more than one real value between a and $a + 1$, which is contrary to hypothesis.

If then the successive integers 1, 2, 3, ... be severally substituted for y in $Y = 0$, it must happen that some two successive substitutions will produce a change of sign, and between the two integers which produce this change of sign the value of y must be placed. Let these two integers be b and $b + 1$, and let $b + \frac{1}{y'}$ be substituted for y in Y, and let the transformed equation be Y′. This equation, as before, must have one, and but one, real and positive root > 1. And the integers c and $c + 1$, between which it lies, will be determined as before.

Again, substituting in $Y' = 0$, $c + \frac{1}{y''}$ for y', we obtain another transformed equation $Y'' = 0$, which, as before, must have one, and but one, real and positive root > 1. And so the process may be indefinitely continued.

Hence we have

$$x = a + \frac{1}{y} \quad y = b + \frac{1}{y'} \quad y' = c + \frac{1}{y''}$$

$$y'' = d + \frac{1}{y'''} \quad \ldots\ldots\ldots$$

$$\therefore x = a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \cfrac{1}{e + \&c}}}}$$

By continuing this fraction we may approximate indefinitely to the value of x, (Sect. XX.) It is evident, that in the process this fraction can never terminate, for if it did, the value of x would be rational, which is contrary to hypothesis. None of the transformed equations $Y' = 0$, $Y'' = 0$. .... can therefore have a positive and integral root.

If, however, the root were not irrational, it might be determined exactly by this method; for in that case some of the transformed equations would have a positive and integral root, in which case the continued fraction would terminate.

(409.) There is another method of approximation proposed by Newton, which is more expeditious than that of Lagrange, which we have just explained.

In the method of Newton a first approximation to within 0,1 of the value of the root is obtained by a tentative process. The root being between the integers a and $a + 1$, let $a + 0{,}5$ be substituted for x, and if this and a give the first member different signs, the root is between a and $a + 0{,}5$, but if they give it the same sign, the root is between $a + 0{,}5$ and $a + 1$.

If the root be between $a + 0{,}5$ and $a + 1$, by substituting $a + 0{,}6$, $a + 0{,}7$, $a + 0{,}8$, &c. two results will be found with different signs, and the root will, therefore, be between these, and either of them will differ from the root by a quantity less than 0,1. But it is rarely necessary to go through all these substitutions, as it most generally happens that the first two will determine the root within 0,1 of its exact value.

The root being thus far determined, let the value found be x', so that $x = x' + u$, u being a quantity $< 0{,}1$. By substituting this in the proposed equation, we obtain (354)

$$X'_1 + X'_2 \cdot \frac{u}{1} + X'_3 \cdot \frac{u^2}{(2)} + \ldots = 0$$

$$\therefore u = -\frac{X'_1}{X'_2} - \frac{X'_3}{X'_2} \cdot \frac{u^2}{(2)} - \frac{X'_4}{X'_2} \cdot \frac{u^3}{(3)} - \ldots$$

Since $u < 0{,}1$, $\therefore u^2 < 0{,}01$. The terms of this series which succeed the first are, then, in general much less than 0,01. If then we assume $u = -\frac{X'_1}{X'_2}$, the assumed value differs from the true by less than 0,01, and therefore $x' - \frac{X'_1}{X'_2}$ differs from the true value of x by less than 0,01. Let this value be x'' and let $x' = x'' + u$. In this case, $u < 0{,}01$. Substituting, as before, $x'' + u$ for x in the proposed equation, we obtain a result of a form exactly similar to the last; and assuming $u = -\frac{X'_1}{X'_2}$, the assumed value differs from the true by less than 0,0001; and in the same manner, another approximation will differ from the exact value of x by $< 0{,}00000001$, and so on.

To approximate to the negative roots, it is only necessary to change x into $-x$ in the proposed equa-

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tion, and treat them as positive roots. This method sometimes fails: see Lagrange, on *Numerical Equations*.

(410.) After each approximation it should, therefore, be determined whether the desired end has been attained. This may be easily done. Let the approximate value obtained at any stage of the process be, for example, 3,1858. Substitute this in the equation for x, and if it give the first member a different sign from that which it receives from the substitution of 3 for x, the root must be $< 3,1858$. Substitute then for x, 3,1857; and if the result have the same sign with that which proceeds from the substitution of 3, the root is between the numbers 3,1857 and 3,1858, and, therefore, the requisite approximation has been obtained; but if the results of the substitutions of 3,1857 and 3,1858 had the same sign, the requisite approximation would not have been obtained, and it would be necessary to diminish the last digit of the decimal.

The same observations, *mutatis mutandis*, apply to the case where the substitution of 3,1858 and 4 give the first member different signs.

(411.) Of these two methods of approximation, that of Lagrange has the advantage of giving a nearer degree of approximation at each step, which Newton's may not; and also Lagrange's method extends to the exact determination of rational roots. Newton's method, however, is in general more expeditious.

The method of determining rational roots, explained in Section XXXVI., is only applicable when the coefficients of the equation are rational numbers. Lagrange's method may be applied when the coefficients are irrational. It is often very advantageous to apply both methods in the same investigation. Thus we may employ Lagrange's method to obtain the roots to within 0,1 or 0,01 of their exact value, and continue the approximation by Newton's method.

There are also other methods, but the developement of them would lead us into details unsuitable to the present Treatise. See Lagrange, *Traité de la Résolution des Equations Numériques*; *Nouvelle Méthode pour résoudre les Equations Numériques*, by Budan; *Théorie des Nombres*, by Legendre.

SECTION XXXVIII.

Elimination applied to two Numerical Equations between two Unknown Quantities.

(412.) When the conditions of any problem reduced to an algebraical statement give two numerical equations of any degrees between two unknown quantities, every pair of particular numbers which, being substituted for the unknown quantities in the equations, convert these equations into identities, are to be considered as a solution of the proposed problem.

In general, let the first members of the two equations be A and B, and the equations being

$$A = 0 \qquad B = 0.$$

Let x' and y' be any particular numbers which, being substituted for x and y in A and B, render the several terms of these polynomes such as will destroy each other. Such a system of values we shall call *conjugate values* of x and y.

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In order that any particular number should be a conjugate value of y, it is necessary that when it is substituted for y in the proposed equations, that they, having then no unknown quantity but x, should have a common root; for if not, they would be inconsistent. This common root will be the value of x, conjugate to the assumed value of y.

It may so happen, that when a particular number is substituted for y, there will be more common roots, or several values of x, which will convert both equations into identities. In this case the same value of y will have several different conjugate values of x.

When a conjugate value of y is substituted for it in the given equations, their first members must admit a common divisor which is a function of x. If this function of x be of the first degree, there is but one value of x conjugate to the assumed value of y; if it be of the second degree there are two, and, in general, if it be of the n^{th} degree there are n values of x conjugate to the same value of y.

If, however, on the substitution of a particular number for y, the first members of the proposed equations admit of no common measure, there will be no corresponding value of x, and in this case the assumed value of y is not a conjugate value.

(413.) *Elimination*, properly so called, is that process by which, from the two given equations an equation is deduced, which includes but one of the two unknown quantities, and whose roots are the several conjugate values of that unknown quantity, and which has no root which is not a conjugate value. Such an equation is properly called the *final equation*.

The number of roots in this equation should be equal to the number of systems of conjugate values which the proposed equations admit. If x be the unknown quantity which has been eliminated, the roots of the final equation should be the several values of y. The number of *unequal* roots should, therefore, be the same as the number of different conjugate values of y. But we have observed, that it may so happen that the same conjugate value of y may have several different conjugate values of x. In this case we must consider the several repetitions of the value of y with the different conjugate values of x to be so many different conjugate values of y, which have become equal, and, therefore, in this case the value in question should be one of several *equal roots* of the final equation. Hence, in general, the *degree* of the final equation must be equal to the number of different systems of conjugate values of the unknown quantities in the proposed equations.

(414.) We shall now consider how far the final equation, obtained by the method founded on the process for obtaining the greatest common measure, fulfils these conditions. It is necessary to show: 1. that every conjugate value of y is found among its roots; 2. that it has no root which is not a conjugate value of y, or if it have it is necessary, 3. to show how such roots may be distinguished, and how the equation may be disembarrassed from them.

The first members of the proposed equations being arranged by the dimensions of x, let the process for determining the greatest common measure be instituted. Let the multipliers which are successively introduced, in order to render the first terms of the several dividends exact multiples of those of the divisors, be a', a'', a''', ... These will be, in general, functions of y. Let the successive quotes be Q', Q'', Q''', ..., and the

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remainders R', R'', R''',...... By the nature of the process we have the following identities:

$$a' \mathrm{A} = \mathrm{B\,Q'} + \mathrm{R'} \qquad [1]$$
$$a'' \mathrm{B} = \mathrm{R'\,Q''} + \mathrm{R''} \qquad [2]$$
$$a''' . \mathrm{R'} = \mathrm{R''} . \mathrm{Q'''} + \mathrm{R'''} \qquad [3]$$
$$\ldots\ldots = \ldots\ldots\ldots$$
$$\ldots\ldots = \ldots\ldots\ldots$$
$$a^{(n)} . \mathrm{R}^{(n-2)} = \mathrm{R}^{(n-1)} . \mathrm{Q}^{(n)} + \mathrm{R}^{(n)} \qquad [4]$$

By [1] it appears, that every system of conjugate values of $x\,y$ in $\mathrm{A}=0$, $\mathrm{B}=0$, are also conjugate in $\mathrm{B}=0$, $\mathrm{R}'=0$.

By [2] it appears, that every system which is conjugate in $\mathrm{B}=0$ and $\mathrm{R}'=0$ is also conjugate to $\mathrm{R}'=0$ and $\mathrm{R}''=0$.

By [3] it follows, that every system which is conjugate in $\mathrm{R}'=0$ and $\mathrm{R}''=0$, is also conjugate in $\mathrm{R}''=0$ and $\mathrm{R}'''=0$, and so on.

By this reasoning we infer, that every system of values of x, y which is conjugate in $\mathrm{A}=0$, $\mathrm{B}=0$ is also conjugate in $\mathrm{R}^{(n-1)}=0$ and $\mathrm{R}^{(n)}=0$.

Since $\mathrm{R}^{(n)}$ is independent of x, *all the conjugate values of y in $\mathrm{A}=0$, $\mathrm{B}=0$ must be roots of $\mathrm{R}^{(n)}=0$.*

But since a' is in general a function of y, it also follows from [1] that the conjugate values of x, y in $a=0$, $\mathrm{B}=0$ are also conjugate values in $\mathrm{B}=0$, $\mathrm{R}'=0$, and, therefore, by what has been already established are conjugate values in $\mathrm{R}^{(n-1)}=0$, $\mathrm{R}^{(n)}=0$. But as $\mathrm{R}^{(n)}$ is a function of y alone, it follows, that every value of y which is conjugate in $\mathrm{R}^{(n-1)}=0$, $\mathrm{R}^{(n)}=0$ must be a root of $\mathrm{R}^{(n)}=0$.

In the same manner it follows, that every value of y which is conjugate in $a''=0$, $\mathrm{R}'=0$, is a root of $\mathrm{R}^{(n)}=0$, and in the same way the conjugate values of y in $a'''=0$, $\mathrm{R}''=0$, &c. are roots of $\mathrm{R}^{(n)}=0$.

Thus, in general, we may conclude, that the conjugate values of y in the following pairs of equations are roots of $\mathrm{R}^{(n)}=0$:

$$\left.\begin{matrix}\mathrm{A}=0\\ \mathrm{B}=0\end{matrix}\right\} \left.\begin{matrix}\mathrm{B}=0\\ a'=0\end{matrix}\right\} \left.\begin{matrix}\mathrm{R}'=0\\ a''=0\end{matrix}\right\} \left.\begin{matrix}\mathrm{R}''=0\\ a'''=0\end{matrix}\right\} \begin{matrix}..\\ ..\end{matrix} \left.\begin{matrix}\mathrm{R}^{(n-1)}=0\\ a^{(n)}=0\end{matrix}\right\} \begin{matrix}..\\ ..\end{matrix}$$

It is, however, only those values of y which are conjugate in the first pair which are the proposed equations which should be roots of the true final equation. As in the succeeding pairs there *may* be conjugate values of y which are not conjugate to the first pair, it follows that all such values will be roots of $\mathrm{R}^{(n)}=0$, and that, therefore, before $\mathrm{R}^{(n)}=0$ can represent the true final equation, these roots must be determined, and the equation $\mathrm{R}^{(n)}$ cleared of them.

(415.) It will be remembered, that the functions a', a'', a''', ... are the multipliers which are successively introduced, in order to render the first terms of the successive dividends A, B, R', ... exactly divisible by the first terms of the successive divisors B, R', R'', ..., and, therefore, from the nature of the process it follows, that a', a'', a''', ... must be integral functions of y and independent of x. On the other hand, the quantities B, R', R'', ... are functions of y and x, and are supposed to be arranged according to the dimensions of x. To determine, therefore, the conjugate values of x, y, in any pair of the equations already mentioned, except the first, it will be necessary first to determine the roots of $a^{(n)}=0$, and these must be substituted for y in $\mathrm{R}^{(n-1)}=0$. The coefficients of the powers of x in $\mathrm{R}^{(n-1)}$, being previous to the substitution functions of y, will now become numerical, and if the component parts of these coefficients, or any of them, be not $=0$, the equation $\mathrm{R}^{(n)}=0$ will assume the form

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$$\mathrm{A}' + \mathrm{B}'x + \mathrm{C}'x^2 + \ldots\ldots = 0,$$

where A', B', C' .... are particular numbers. This equation will give particular numerical values for x, and, therefore, the value of y thus substituted is conjugate to $a^{(n)}=0$, $\mathrm{R}^{(n-1)}=0$, and is, therefore, a root of $\mathrm{R}^{(n)}=0$. If this value of y be not conjugate in $\mathrm{A}=0$, $\mathrm{B}=0$, it will be necessary to clear the equation $\mathrm{R}^{(n)}=0$ of it before it can be considered the true final equation. The same may be said of every value of y determined in this way, in each pair of the equations

$$\left.\begin{matrix}\mathrm{B}=0\\ a'=0\end{matrix}\right\} \quad \left.\begin{matrix}\mathrm{R}'=0\\ a''=0\end{matrix}\right\} \quad \begin{matrix}\ldots\ldots\\ \ldots\ldots\end{matrix}$$

But it may so happen, that a value of y deduced from $a^{(n)}=0$, when substituted in $\mathrm{R}^{(n-1)}=0$, or

$$\mathrm{A}' + \mathrm{B}'x + \mathrm{C}x^2 + \ldots\ldots = 0,$$

may render the coefficients B', C', D', ... each $=0$, in which case the equation $\mathrm{R}^{(n-1)}=0$ will not be fulfilled, whatever be the value of x. In this case, and in this case only, the value of y deduced from $a^{(n)}=0$ is not a conjugate value in $a^{(n)}=0$, $\mathrm{R}^{(n-1)}=0$, and, therefore, not a root of $\mathrm{R}^{(n)}=0$.

It may also happen, that a value of y deduced from $a^{(n)}=0$, shall render all the quantities A', B', C' .... $=0$. In this case $\mathrm{R}^{(n-1)}$ will $=0$, whatever be the value of x. In this case, the quantity $\mathrm{R}^{(n-1)}$ must have an integral function of y, independent of x as a factor, and by the principles which have been already established respecting the process for finding the greatest common measure it follows, that this function of y must be a common factor of the proposed equations $\mathrm{A}=0$, $\mathrm{B}=0$, so that they become

$$\mathrm{A}' \times \mathrm{Y} = 0 \qquad \mathrm{B}' \times \mathrm{Y} = 0,$$

if Y be the common factor. Now, both of these equations are satisfied by $\mathrm{Y}=0$, independently of x. The equations therefore are indeterminate, since, although, the values of y are limited in number by the equation $\mathrm{Y}=0$, the value of x is absolutely indeterminate. To render the equations determinate, it would be necessary to disembarrass them of the common factor $\mathrm{Y}=0$.

To distinguish, therefore, the roots from which the equation $\mathrm{R}^{(n)}$ is to be cleared, in order to obtain the true final equation, it is necessary to determine successively the roots of the several equations $a'=0$, $a'=0$, $a'''=0$, ...; and to select such of these roots as do not render $=0$ the several coefficients of the equations $\mathrm{B}=0$, $\mathrm{R}'=0$, $\mathrm{R}''=0$, ...; and such of these values as are not conjugate values in $\mathrm{A}=0$, $\mathrm{B}=0$, should be cleared from $\mathrm{R}^{(n)}=0$, and the result will be the true final equation.

In cases, however, where the equations $a'=0$, $a''=0$, $a'''=0$, ... are of the higher degrees, the determination of their roots may be attended with some difficulty. In this case we can have recourse to a process which will render the determination of their roots unnecessary.

It should be observed, that the roots of $a'=0$ are always conjugate values of y in $a'=0$, $\mathrm{B}=0$, except in the particular case in which the value of y deduced from $a'=0$, renders $=0$ the coefficients of all the

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powers of x in $B=0$. To determine whether this be the case, it will be only necessary to find whether these coefficients severally, and a', admit any function of y as a common measure. If they do, then the values of y found by putting this function $=0$, are not conjugate values, and are not roots of $R^{(n)}=0$. The equation $a'=0$ is then to be cleared of this function of y by division; and if the quote be a function of y, its roots will necessarily be roots of $R^{(n)}=0$, and such of them as are not conjugate in $A=0$, $B=0$ must be removed by division from $R^{(n)}=0$.

If it should happen that the common measure of all the coefficients of the powers of x in $B=0$ should be also a measure of its absolute quantity, then the original equations will be indeterminate, for this same function of y will be a common factor of them.

(416.) The observations just made, concerning the equations $a'=0$, $B=0$, will equally apply to $a''=0$, $R'=0$, to $a'''=0$, $R''=0$, ... By these means, the equation $R^{(n)}=0$ may be successively cleared of all the factors, or roots, which do not correspond to conjugate values of y in the equations $A=0$, $B=0$.

If the last remainder $R^{(n)}$ be an absolute quantity independent of y, there is no value of x which would render the two polynomes divisible by the same function of y, and, therefore, there are no conjugate values of x, y, and the given equations are inconsistent, or contradictory.

If $R^{(n)}=0$, independently of y, it follows, that the value of x which satisfies the two equations is independent of any value of y, that is, the two functions A, B are divisible by a common function of x. Let this be X. Both equations are satisfied by the roots of $X=0$, whatever be the value of y. Hence, in this case, they are indeterminate.

Before we proceed further with this abstract reasoning, we shall illustrate it by its application to the following examples.

Let

$$A=y^3x^2-3y^3x-y^2+2$$

$$B=(y^2-3y+2)x^2+(y-1)x-3y+1$$

The first multiplier a' is

$$a'=y^2-3y+2$$

$$\therefore R'=(-3y^5+8y^4-5y^3)x+2y^4+2y^3-6y+4$$

$$-3y^5+8y^4-5y^3=-y^3(y-1)(3y-5).$$

In this case it is necessary to take

$$a''=y^6(y-1)\ (3y-5)^2$$

$$\therefore R''=R^{(n)}=27y^{10}-136y^9+214y^8-112y^7+65y^6$$

$$-100y^5+30y^4-24y^3+120y^2-112y+32.$$

This polynome includes all the conjugate values of y. But before these can be determined, it is necessary to determine what factors have been introduced by the multipliers a', a''. We have

$$a'=y^2-3y+2=(y-1)(y-2)$$

$$B=(y^2-3y+2)+(y-1)x-3y+1=0.$$

If $a'=0$, $\therefore y=1$, or $y=2$. If $y=1$, $B=-2$. Hence this root does not enter $R^{(n)}=0$. If $y=2$, $B=x-5=0$, $\therefore x=5$. The value $y=2$ is not a conjugate value in $A=0$, $B=0$; for if 2 and 5 be substituted for y and x in A, we have $A=78$. Hence it is necessary to divide $R''=0$ by $y-2$.

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If $a''=0$, $\therefore y=0$, or $y=1$, or $y=\frac{5}{3}$. But $y=0$ does not satisfy $R'=0$, $\therefore y$ is not a factor of $R^{(n)}$. In like manner $y=1$ does not satisfy $R'=0$, $\therefore y-1$, as before, is not a factor of $R^{(n)}$. The same observation applies to $y=\frac{5}{3}$. Thus it appears, that the only factor of which $R^{(n)}$ is to be cleared is $y-2$. Being divided by this the quote becomes.

$$27y^9-82y^8+50y^7-12y^6+41y^5-18y^4-6y^3$$

$$-36y^2+48y-16=0.$$

The roots of this equation are the conjugate values of y, and the only ones in $A=0$, $B=0$. These roots being determined, and successively substituted in $R'=0$, will determine the conjugate values of x.

(417.) It may be observed, that in general the last remainder $R^{(n)}$ being a function of y independent of x, the preceding remainder is of the form

$$Mx+N$$

where x occurs only in the first degree. The values of y being determined by the equation $R^{(n)}=0$, and successively substituted for y in the functions M and N, the equation

$$Mx+N=0$$

will determine all the conjugate values of x without having recourse to the original equations at all. In fact, any value of y which renders $R^{(n)}=0$ must necessarily render $R^{(n-1)}$, or $Mx+N$ a common measure of the first members A, B of the proposed equations, which are therefore satisfied by

$$Mx+N=0.$$

If any root of $R^{(n)}=0$ renders $N=0$ the conjugate value of $x=0$. If it render $M=0$, $x=\infty$, and if it render both $M=0$ and $N=0$, it follows, that since $R^{(n-1)}=0$ independently of x, the preceding remainder $R^{(n-2)}$ must be a common measure of A and B. Therefore, if in this remainder we substituted the same value of y, the roots of the equation $R^{(n-2)}=0$ will be the conjugate values of x. In this case $R^{(n-2)}=0$ will be an equation of the second degree, and there will be two values of x conjugate to the same value of y.

(418.) It may happen, that the value of y in question also renders $R^{(n-2)}=0$ independently of x. In this case the preceding remainder $R^{(n-3)}$ will be a common measure of the quantities A, B, and the conjugate values of x will be the roots of $R^{(n-3)}=0$. This will be an equation of the third degree, and, therefore, there will be three values of x conjugate to the same value of y. In the same manner, R^{n-3} may $=0$ independently of x, in which case $R^{(n-4)}=0$ will give four conjugate values of x, and so on.

It is evident, that whenever for the same value of y there are several conjugate values of x, several successive remainders must be $=0$ independently of y; for otherwise, for each value of y there would be but one value of x determined by $R^{(n-1)}=0$, which is always of the first degree in x.

(419.) It may be observed, that the method of elimination by the greatest common divisor always gives the true final equation, when the given equations do not exceed the second degree. For, in this case,

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$$A = a x^2 + b x + c = 0,$$
$$B = a' x^2 + b' x + c' = 0.$$

Here a, a' must be numerical coefficients, for if they included any dimension of y the equations would exceed the second degree. These, being the factors first introduced to render either divisible by the other, cannot introduce any root into the final equation.

The last multiplier $a^{(n)}$ which is introduced cannot in this case, nor in any other, be the means of introducing a root into the final equation; any value of y deduced from $a^{(n)} = 0$ would render $= 0$ the coefficient of x in $R^{(n-1)}$, and would reduce this to a numerical quantity which would not in general $= 0$.

The degree of the equation $R^{(n)} = 0$ may frequently indicate the existence in it of roots which are not conjugate values of y. If it exceed the product of the numbers which mark the degrees of the two equations $A = 0$, $B = 0$, there must be at least as many roots which are not conjugate values as the units by which the degree of $R^{(n)} = 0$ exceeds that product.

(420.) We cannot, however, on the contrary infer, that if its degree be equal to the product of the degrees of $A = 0$ and $B = 0$, that there are, therefore, no roots but conjugate values of y. Because, although the highest degree the final equation can have, is the product of the degrees of the original equations, yet, in particular cases, it may have a lower degree.

SECTION XXXIX.

On the Imaginary Roots of Equations.

(421.) By the principles which have been already established, we are enabled to clear an equation of its real and rational roots. But, although we may approximate at pleasure to the irrational roots, yet unless we could obtain them exactly, it would be impossible to clear the equation of them by division. We shall, therefore, in the present section consider the equation as having irrational and imaginary roots, but no rational roots. Our object will be to determine the imaginary roots.

We have already proved, that in an equation with real coefficients there must always be either an even number of imaginary roots or none. We propose now to establish a more general theorem which includes this, *scil.*, *Every imaginary root of an equation must be of the form $a \pm b\sqrt{-1}$, and if $a + b\sqrt{-1}$ be an imaginary root of any equation, $a - b\sqrt{-1}$ must be also an imaginary root of the same equation, a and b being real quantities.*

Let

$$X = x^m + A_1 x^{m-1} + A_2 x^{m-2} + \dots A_{m-1} \cdot x + A_m = 0.$$

Let $a + b\sqrt{-1}$ be substituted for x in $X = 0$. By (259) it appears, that if $(a + b\sqrt{-1})^m$ be expanded by the binomial theorem, the alternate terms beginning with the first will be real, and alternately $+$ and $-$, and the alternate terms beginning from the second will be affected with the imaginary factor $\sqrt{-1}$, and alternately $+$ and $-$. Observing this, it is evident, that the substitution in X will produce a series of real, and a series of imaginary, terms. Let the sum of the real terms be M, and that of the coefficients of $\sqrt{-1}$ in the imaginary terms N, the result will be

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$$M + N\sqrt{-1} = 0$$
$$\therefore M = 0 \qquad N = 0.$$

These two equations will determine the values of a and b.

If $a - b\sqrt{-1}$ had been substituted for x, the result would have been

$$M - N\sqrt{-1} = 0$$
$$\therefore M = 0 \qquad N = 0,$$

which would give the same values for a and b as before. Hence, if $a + b\sqrt{-1}$ be a root of $X = 0$, $a - b\sqrt{-1}$ will also be a root of it.

(422.) Before we proceed to show that every imaginary root must have the form $a \pm b\sqrt{-1}$, it will be first necessary to establish the principle, that every algebraic function of $a \pm b\sqrt{-1}$ may be reduced to the form $M \pm N\sqrt{-1}$.

Let
$$u = a \pm b\sqrt{-1}$$
$$u' = a' \pm b'\sqrt{-1}$$
$$u'' = a'' \pm b''\sqrt{-1}$$
&c. &c.

Let $\Sigma(u)$, $\Sigma(a)$, $\Sigma(b)$, signify the algebraical sums of $u, u', u'' \dots\ a, a', a'' \dots\ b, b', b'' \dots$ By addition we have

1. $$\Sigma(u) = \Sigma(a) \pm \Sigma(b) \cdot \sqrt{-1} = M \pm N\sqrt{-1}.$$

2. $$u u' = (a a' - b b') \pm (a' b + a b')\sqrt{-1}$$
$$u u' = M \pm N\sqrt{-1}.$$

3. $$\frac{u}{u'} = \frac{u(a' \mp b'\sqrt{-1})}{u'(a' \mp b'\sqrt{-1})} = \frac{(a a' + b b') \pm (a' b - a b')\sqrt{-1}}{a'^2 + b'^2}$$
$$\frac{u}{u'} = M \pm N\sqrt{-1}$$

4. $$u^m = (a \pm b\sqrt{-1})^m.$$

In this case whether m be positive or negative, integral or fractional, its developement may be reduced to the form $M \pm N\sqrt{-1}$, by what has been already proved.

(423.) We shall now show that every imaginary root of $X = 0$ can be reduced to the form $a \pm b\sqrt{-1}$.

Let $a_1, a_2, a_3 \dots a_m$ be the roots of $X = 0$. By the principles established in Section XXX. an equation may be found, whose roots will be functions of each pair of roots of $X = 0$, of the form

$$a_1 + a_2 + k\, a_1 a_2$$

k being any integer whatever. Let this equation be $Z = 0$. It will, in general, have as many roots as there are different combinations of two roots among the m roots of the proposed equation. This is $\frac{m(m-1)}{1 \cdot 2}$, which is, therefore, the degree of $Z = 0$.

The number m being by hypothesis even is divisible

by 2, and, in general, has the form $2^n . m'$, m' being an odd integer.

1. Let $n = 1$, $\because \frac{m}{2} = m'$, and since this is odd, and also $m - 1$ is odd, it follows, that $\frac{m(m-1)}{2}$ is odd, and therefore $Z = 0$ must at least have one real root. Let this be z', and let

$$z' = a_1 + a_2 + k\,a_1\,b_2.$$

For each integral value which is ascribed to k there will be a different equation $Z = 0$, and each of these equations will have one real root, at least, which must be a function of some pair of roots of the proposed equation of the form already mentioned. Since the number of combinations of pairs is limited, it follows, that after a certain number of substitutions for k, the real root of the equation must be a function of some pair, of which the real root of the equation resulting from some former substitution was also a similar function. Let the two real roots which are functions of $a_1\,a_2$, corresponding to the values $k\,k'$ be z, z', and we have

$$z = a_1 + a_2 + k\,a_1\,a_2$$
$$z' = a_1 + a_2 + k'\,a_1\,a_2$$
$$\because z - z' = (k - k')\,a_1\,a_2$$
$$\because a_1\,a_2 = \frac{z - z'}{k - k'}$$
$$a_1 + a_2 = \frac{z\,k' - z'\,k}{k' - k}.$$

Hence it follows, that in this case $a_1 + a_2$ and $a_1\,a_2$ are real quantities, and, therefore,

$$(x - a_1)(x - a_2) = x^2 - (a_1 + a_2)\,x + a_1\,a_2,$$

which is a quadratic factor of $X = 0$, is real.

2. Let $n = 2$, $\because \frac{m}{2} = 2\,m'$, $\because \frac{m(m-1)}{1 \,.\, 2} = 2\,m'$ $(m - 1)$. Hence, in this case, the equation $Z = 0$ is of an even degree, but its exponent $2\,m'\,(m - 1)$ if divided by 2 gives an odd number for a quote. Hence, by the last case, it follows, that $Z = 0$ must have a real factor of the second degree. Let this be

$$z^2 + A\,z + B,$$

and let its simple factors be z', z''. These quantities z', z'' must, in general, be of the form $a \pm b\,\sqrt{-1}$.

Let $z' = a_1 + a_2 + k\,a_1\,a_2$. By the same reasoning as in the former case, we can prove, that there is another value of k by which another root which is a simple factor of a real quadratic factor of Z may be found. Let this be z'', so that we have

$$z' = a_1 + a_2 + k\,a_1\,a_2$$
$$z'' = a_1 + a_2 + k'\,a_1\,a_2.$$

The values of $z' + z''$ and $z'\,z''$, deduced from these, being algebraical functions of z', z'' must also be of the form $a' \pm b'\,\sqrt{-1}$. So that we shall have a quadratic factor of the form

$$x^2 - (p \pm q\,\sqrt{-1})\,x + p' \pm q'\,\sqrt{-1}.$$

The values of x which render this $= 0$ being algebraic functions of the coefficients must be reducible to the form $p \pm q\,\sqrt{-1}$. We shall then have a simple factor of X of the form $x - (p \pm q\,\sqrt{-1})$, and, therefore, another of the form $x - (p \mp q\,\sqrt{-1})$, and hence we obtain a quadratic factor of the form

$$x^2 + p\,x + q,$$

which will be a real quadratic factor of X.

Similar reasoning will apply when $n = 3$, $n = 4$, &c. Hence we infer, in general, *That the first member of every equation of an even order admits, at least, one real quadratic factor.*

This being proved, it easily follows, that *the first member admits of being resolved into as many real quadratic factors as there are units in* $\frac{m}{2}$, *or half the exponent of its degree.* For, since it admits of one real factor, this may be removed by division, and an equation of an even degree lower by 2 will be the result. This, again, must admit of a real quadratic factor.

Hence the first member of *an equation whose degree is even, may be considered as the continued product of as many real quadratic factors as there are units in half the exponent.*

And since an equation of an odd degree must always have one real root, *its first member may be considered as the continued product of one real simple factor, and as many real quadratic factors as there are units in half of the exponent of the degree diminished by unity*, *or* $\frac{m-1}{2}$.

The form of the imaginary roots being thus determined, their actual values may be found by the equation

$$M = 0 \qquad N = 0$$

in (421) which will give the values of the indeterminates a and b.

Two imaginary roots, such as $a + b\,\sqrt{-1}$, $a - b\,\sqrt{-1}$, which differ only in the sign of the imaginary part, are called *conjugate imaginary roots.*

(424.) The equation of the squares of the differences of the roots of an equation, has a connection with the imaginary roots which it may be useful to trace.

The difference between any two real roots must be real, and either positive or negative; in either case its square will be positive, and must, therefore, be a real and positive root of the equation of differences. Hence the equation of squares of differences must have, at least, as many real and positive roots as there are combinations of two real roots in the proposed equation.

The difference of two conjugate imaginary roots being of the form $\pm 2\,\sqrt{-1}\,.\,b$, its square must in every case be negative. Hence the equation of squares of differences must have, at least, as many negative roots as there are real quadratic factors, whose simple factors are imaginary in the proposed equation.

The difference of two imaginary roots which are not conjugate, is in general

$$(a - a') \pm (b - b')\,\sqrt{-1}.$$

The square of this is *in general* imaginary, and of the same form as each of the roots; and, therefore, there will be an imaginary root in the equation of the squares of differences for each pair of imaginary roots *whose rational and irrational parts are respectively unequal.* But if the rational parts a, a' be equal, the difference will be $(b - b')\,\sqrt{-1}$, the square of which will be negative, but always real; and if the imaginary

parts be equal, the difference will be $a - a'$, the square of which must be positive and real.

Hence the real and positive roots of the equation of the squares of differences must contain among them the squares of the differences of those pairs of imaginary roots (if there be any such) in which the imaginary parts are equal, and the real and negative roots must contain among them the squares of the differences of those imaginary roots in which the real parts are equal.

The difference between a real and an imaginary root being of the form

$$(a - a') \pm b\sqrt{-1}$$

its square must in general be imaginary, and when so, the corresponding root of the equation of squares of differences will be imaginary. But if the real root be equal to the real part of the imaginary root, then the difference will be of the form $\pm b\sqrt{-1}$, the square of which is negative, and therefore in this case the corresponding root of the equation of the squares of differences will be negative.

If we suppose that the two equations have no two imaginary roots whose real parts are equal, nor any real root equal to the real part of an imaginary one, it follows that every negative root of the equation of the squares of differences will be equal to $-4b^2$, or four times the square of the coefficient of $\sqrt{-1}$ in the imaginary part of one of the roots taken with a negative sign. Let $-a$ be a negative root of this equation, ∵

$$a = 4b^2 \because b = \frac{\sqrt{a}}{2}$$

the value of b being thus determined, let it be substituted in $M = 0$ or $N = 0$, and the corresponding values of a will be the real part of the root.

Whether $-a$ be a root proceeding from either of the circumstances just mentioned, *scil.* the equality of the real parts of two different roots, whether both imaginary or one real and one imaginary, may be known by finding whether the value of b thus determined will give the equations $M = 0$, $N = 0$ a common root. If there be a common value of a, which satisfies both, then the value of b will belong to conjugate roots, and otherwise not.

It follows from what has been inferred here, and what has been established in (392,) that there are at least as many changes of sign in the equation of the squares of differences, as there are combinations of two real roots in the proposed equation. Also it must have at least as many successive repetitions of sign as there are pairs of conjugate imaginary roots in the proposed equation, or, in other words, it cannot have a less number of successive repetitions of sign than half the number of imaginary roots in the proposed equation.

Hence we may infer, that if the equation of differences have its terms alternately positive and negative, and therefore have no successive repetition of sign, there can be no imaginary root in the proposed equation.

SECTION XL.

On the Resolution of Algebraic Equations of the Third and Fourth Degrees.

(425.) The general problem to determine the roots of an algebraic equation of the m^{th} degree as functions of its literal coefficients, has long engaged the attention of analysts. The converse of this problem, *scil.* the determination of the coefficients as functions of the roots, was solved in an early stage of the algebraic analysis; but the general problem of the resolution of literal equations has baffled the powers of the most refined modern analysis. When it is considered, however, that all the equations which present themselves in actual philosophical investigations, are numerical equations, the particular data of the problem furnishing the values of the numeral coefficients, the general problem must be considered of an interest rather speculative than practical.

We shall, however, in the present section explain the methods of resolving general equations of the third and fourth degrees, which is the utmost extent, except in very particular instances, to which the solution of algebraical equations has been yet as carried.

By the transformation explained in (345,) it is possible in every equation to remove the second term. We shall, therefore, consider equations of the third degree in general represented by

$$x^3 + a x + b = 0.$$

Let $x = y + z$ ∵

$$x^3 = y^3 + z^3 + 3yz(y + z)$$

$$\because x^3 = y^3 + z^3 + 3yzx$$

$$\because x^3 - 3yzx - y^3 - z^3 = 0.$$

Comparing this with the proposed equation in order to identify them, it will be necessary that

$$-3yz = a \qquad y^3 + z^3 = -b$$

$$\because y^3 z^3 = -\frac{a^3}{27}$$

$$y^3 + z^3 = -b.$$

Since the sum of y^3 and z^3 is $-b$, and their product $-\frac{a^3}{27}$, they must be the roots of the equation (176)

$$x'^2 + b x' - \frac{a^3}{27} = 0$$

$$\because x' = -\frac{b}{2} \pm \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}$$

$$\because y^3 = -\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}$$

$$z^3 = -\frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}$$

$$\because x = \left(-\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right)^{\frac{1}{3}}$$

$$+ \left(-\frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right)^{\frac{1}{3}}$$

Let a', a'' signify the arithmetical values of the third roots of the values found above for y^3, z^3, when the particular numbers which b and a may signify are substituted for them, and let m', m'' signify the two imaginary third roots of unity. The three values of x in the proposed equation will then be

$$x = a' + a'' \quad x = m'(a' + a'') \quad x = m''(a' + a'').$$

It is evident that the two roots a', a'' ought to be so taken that their product should he $-\frac{a}{3}$.

If a', a'' be substituted for y, z in the equation

$$x^3 - 2yzx - y^3 - z^3 = 0,$$

and the result

$$x^3 - 2a'a''x - a'^3 - a''^3 = 0,$$

divided by $x - (a' + a'')$, the quote will be

$$x^2 + (a' + a'')x + a'^2 - a'a'' + a''^2 = 0,$$

which solved for x, gives

$$x = -\frac{a' + a''}{2} \pm \sqrt{\left(\frac{a' + a''}{2}\right)^2 - a''^2 + a'a'' - a'^2}$$

which values may be reduced to the forme

$$x = -\tfrac{1}{2}(a' + a'') + \tfrac{1}{2}(a' - a'')\sqrt{-3}$$
$$x = -\tfrac{1}{2}(a' + a'') - \tfrac{1}{2}(a' - a'')\sqrt{-3}$$

The identity of these two forms with $m'(a' + a'')$ and $m''(a' + a'')$ ie obvious, by attending to the values of m', m'',

$$m' = \frac{-1 + \sqrt{-3}}{2} \quad m'' = \frac{-1 - \sqrt{-3}}{2}$$

In considering the neture of the roots we shall successively examine the cases in which

$$\frac{b^2}{4} + \frac{a^3}{27} > 0, \ = 0, \ < 0.$$

1. If $\frac{b^2}{4} + \frac{a^3}{27} > 0$. In this case the values a', a'' must be necessarily real, and, therefore, $a' + a''$ must also be real. The other two roots of the proposed equation

$$x = -\tfrac{1}{2}(a' + a'') \pm \tfrac{1}{2}(a' - a'')\sqrt{-3}$$

are necessarily imaginary since the coefficient of $\sqrt{-3}$ is real, and not $= 0$. The eign of the real root ie in this case different from that of b.

2. Let $\frac{b^2}{4} + \frac{a^3}{27} = 0$. In this case $a' = a'' = -\sqrt{\frac{b}{2}}$ ∵

$$x = -2\sqrt[3]{\frac{b}{2}}$$

$$x = -\tfrac{1}{2}(a' + a'') = \sqrt[3]{\frac{b}{2}}$$

which last is the common value of the two roots, which io the last case were imaginary, and have under the present condition become equal. The common value of the equal roots is, therefore, half the first root with a contrary eign.

If $b > 0$ the first root is negative and the other two positive, and if $b < 0$ the first ie positive and the other two negative.

In this case it may be observed, that a must be negative in the original equation, for otherwise $\frac{b^2}{4} + \frac{a^3}{27}$ would be composed of two terme eseentially negative, and therefore could not $= 0$.

3. Let $\frac{b^2}{4} + \frac{a^3}{27} < 0$. In this case a must also be negative, and the quantities a', a'' will be imaginary.

The first root $x = a' + a''$ assumes the form

$$x = (p + q\sqrt{-1})^{\frac{1}{3}} + (p - q\sqrt{-1})^{\frac{1}{3}}$$

This, although it includes imaginary terms, is essentially real, eince if its parts be developed by the binomial theorem, the imaginary parts will mutnally destroy each other (259.)

It follows also from the same principles (259) that $\frac{(a' - a'')}{\sqrt{-1}}$ is real, and, therefore, that $(a' - a'')\sqrt{-3}$ is real Hence the roots

$$x = -\tfrac{1}{2}(a' + a'') \pm \tfrac{1}{2}(a' - a'')\sqrt{-3}$$

are real.

Thue in thie case the roots are all real.

This case of equations of the third degree is commonly called the *irreducible case*. Because, although the formula obtaioed for the roots is their true algebraical expression, yet it can only be cleared of imaginary quantitiee by converting it into a series, end as thie series ie seldom convergent, it ie useless for the actual determination of the roots; and therefore we must always have recourse to the methods of approximating to the roots of numerical equations.

For other methods of solving cubic equations see TRIGONOMETRY.

(426.) We shall now proceed to explain the methods of resolving equations of the fourth degree. The second term being capable of being removed by a transformation, we may consider all equatione of this degree included under the form

$$x^4 + px^2 + qx + r = 0.$$

Following a method of investigation aoalogous to that adopted in the case of equatious of the third degree, let

$$x = y + z + u$$
$$\because x^2 = y^2 + z^2 + u^2 + 2(yz + yu + zu)$$
$$\therefore x^4 - 2(y^2 + z^2 + u^2)x^2 + (y^2 + z^2 + u^2)^2$$
$$= 4(y^2z^2 + y^2u^2 + z^2u^2) + 8yzu(y + z + u)$$
$$\because x^4 - 2(y^2 + z^2 + u^2)x^2 - 8yzux + (y^2 + z^2 + u^2)^2$$
$$- 4(y^2z^2 + y^2u^2 + z^2u^2) = 0.$$

To ldentify thie equation with the proposed one, the following conditione will he necessary:

1. $p = -2(y^2 + z^2 + u^2) \quad \because y^2 + z^2 + u^2 = -\frac{1}{2}p$
2. $r = (y^2 + z^2 + u^2)^2 - 4(y^2z^2 + y^2u^2 + z^2u^2)$

$$\because y^2z^2 + y^2u^2 + z^2u^2 = \frac{p^2 - 4r}{16}$$

3. $q = -8yzu \quad \because yzu = -\frac{q}{8} \quad \because y^2z^2u^2 = \frac{q^2}{64}$

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By this it appears, that of the three quantities y^2, z^2, u^2, the *sum*, the *sum of the products in pairs*, and the continued *product of all three*, are severally given. They are therefore the roots of the equation

$$t^3 + \frac{p}{2}\,t^2 + \frac{p^2 - 4\,r}{16}\,t - \frac{q^2}{64} = 0.$$

This being transformed into another which will be free of fractions by substituting $\frac{s}{4}$ for t, the equation becomes

$$s^3 + 2\,p\,s^2 + (p^2 - 4\,r)\,s - q^2 = 0.$$

This being an equation of the third degree, its roots may be determined by the methods already explained. Let them be s', s'', s'''. Hence

$$y = \pm\tfrac{1}{2}\sqrt{s'} \qquad z = \pm\tfrac{1}{2}\sqrt{s''} \qquad u = \pm\tfrac{1}{2}\sqrt{s'''}.$$

Since $x = y + z + u$, the values of y, z, and u being combined in every possible manner by addition, would give eight values of x. But since $y\,z\,u = -\frac{q}{8}$ it is necessary that they be so combined that their product shall have a different sign from that of q.

Hence when q is negative, either two or none of the values of y, z, u must be negative; hence the values of x are in this case

$$x = +\tfrac{1}{2}\sqrt{s'} + \tfrac{1}{2}\sqrt{s''} + \tfrac{1}{2}\sqrt{s'''}$$
$$x = +\tfrac{1}{2}\sqrt{s'} - \tfrac{1}{2}\sqrt{s''} - \tfrac{1}{2}\sqrt{s'''}$$
$$x = -\tfrac{1}{2}\sqrt{s'} - \tfrac{1}{2}\sqrt{s''} + \tfrac{1}{2}\sqrt{s'''}$$
$$x = -\tfrac{1}{2}\sqrt{s'} + \tfrac{1}{2}\sqrt{s''} - \tfrac{1}{2}\sqrt{s'''}$$

When q is positive it is necessary that either one or three of the values of y, z, u be negative. Hence in this case the values of x are

$$x = -\tfrac{1}{2}\sqrt{s'} - \tfrac{1}{2}\sqrt{s''} - \tfrac{1}{2}\sqrt{s'''}$$
$$x = -\tfrac{1}{2}\sqrt{s'} + \tfrac{1}{2}\sqrt{s''} + \tfrac{1}{2}\sqrt{s'''}$$
$$x = +\tfrac{1}{2}\sqrt{s'} - \tfrac{1}{2}\sqrt{s''} + \tfrac{1}{2}\sqrt{s'''}$$
$$x = +\tfrac{1}{2}\sqrt{s'} + \tfrac{1}{2}\sqrt{s''} - \tfrac{1}{2}\sqrt{s'''}$$

(427.) The nature of the roots of the proposed equation evidently depends on that of the roots s', s'', s'''. These must either be all real, or one real and the other two imaginary.

If they be all real, they must either be all positive, or one positive and the other two negative, since the last term $-q^2$ of the equation of which they are the roots is essentially negative.

If s', s'', s''' be all positive, all the values of x are necessarily real.

If s' be positive, and s'', s''' negative, all the values of x are imaginary, except in the particular cases where two imaginary terms happen to be equal, and therefore destroy each other when united with opposite signs. In that case two roots will be real and two imaginary.

If one of the values s', s'', s''' be real, and the other two imaginary, the real value is necessarily positive, since the last term of the equation of which they are roots is $-q^2$ essentially negative. The other two being conjugate imaginary roots, must be of the forms (423,)

$$a + b\sqrt{-1}, \qquad a - b\sqrt{-1},$$

and these must enter the values of x in one or other of the forms

Series for Sines, &c. of Multiple Arcs.

$$(a + b\sqrt{-1})^{\frac{1}{2}} + (a - b\sqrt{-1})^{\frac{1}{2}},$$
$$(a + b\sqrt{-1})^{\frac{1}{2}} - (a - b\sqrt{-1})^{\frac{1}{2}}.$$

The former is a real, and the latter an imaginary quantity. (259.)

Hence it easily appears, that in this case two of the values of x must be real, and two imaginary.

SECTION XLI.*

Of the Developement of the Sines and Cosines of Multiple Arcs in Powers of the Sines and Cosines of the Simple Arcs.

(428.) NOTWITHSTANDING the elementary nature of the trigonometrical analysis, and the attention which has been devoted to its various details, from the time of Euler to the present day, by the greatest mathematicians, yet the analysis of angular sections remained until a late period in a most imperfect state. Formulæ expressing relations between the sine and cosine of an arc, and those of its multiples, were established by Euler, and subsequently confirmed by the searching analysis of Lagrange, which have since been proved inaccurate, or true only under particular conditions; and it was only within the last three years that the complete exposition of this theory has been published, and general formulæ assigned expressing those relations. In the year 1811, Poisson detected an error in a formula of Euler, expressing the relation between the power of the sine or cosine of an arc, and the sines and cosines of certain multiples of the same arc. But the most complete discussion of this subject which has hitherto appeared, is contained in a Memoir read before the Academy of Sciences at Paris by Poinsot,† an eminent French mathematician, in the year 1823, and further developed by him in another Memoir published in the year 1825.

The developements respecting multiple arcs may be divided into two distinct classes. The first includes all series in which the sine or cosine of a multiple arc is expressed in powers of those of the simple arc; and the second, those in which a power of the sine or cosine of a simple arc is expressed in a series of sines or cosines of its multiples: to the former we shall devote the present section, reserving the latter for the following one.

The series in powers of the sine, cosine, &c. may be either ascending or descending, and accordingly the several problems into which our analysis resolves itself may be enumerated as follow:

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* This and the following Section are extracted, by the permission of the Publisher and the Author, from Dr. Lardner's Treatise on the *Analysis of Angular Sections* in the third part of his work on *Plane and Spherical Trigonometry*.

† This mathematician has rendered himself distinguished by the invention of the "theory of couples," (*Théorie des couples*,) a most powerful instrument of investigation in analytical mechanics, and one which has not yet received the attention which it deserves from mathematical writers, either here or on the continent, and which we venture to predict it must ultimately command.

To develope

1. $\left.\begin{array}{l}\cos m\,x\\ \sin m\,x\end{array}\right\}$ in ascending powers of $\cos x$.
2. $\left.\begin{array}{l}\sin m\,x\\ \cos m\,x\end{array}\right\}$ in ascending powers of $\sin x$.
3. $\left.\begin{array}{l}\sin m\,x\\ \cos m\,x\end{array}\right\}$ in ascending powers of $\tan x$.
4. $\left.\begin{array}{l}\cos m\,x\\ \sin m\,x\end{array}\right\}$ in descending powers of $\cos x$.
5. $\left.\begin{array}{l}\sin m\,x\\ \cos m\,x\end{array}\right\}$ in descending powers of $\sin x$.

(429.) *To develope* $\cos m\,x$ *in a series of ascending powers of* $\cos x$.

Let $\cos x = y$, and let

$$z = y + \sqrt{y^2 - 1},$$

$$\therefore \frac{1}{z} = y - \sqrt{y^2 - 1}.$$

But also (see TRIGONOMETRY)

$$2 \cos m\,x = z^m + \frac{1}{z^m}.$$

If then z^m be obtained in ascending powers of y, and z^{-m} deduced from it by changing the sign of m, we shall thence obtain $2 \cos m\,x$ in a series of the required form.

Let

$$z^m = u = A_0 + A_1\,y + A_2\,y^2 + A_3\,y^3 + \ldots.$$

The solution of the question will be effected if the values of the coefficients of this series can be obtained without introducing any condition which restricts the generality of the problem.

Let the series assumed to express u be twice differentiated, and the results will be

$$\frac{d\,u}{d\,y} = A_1 + 2 A_2\,y + 3\,A_3\,y^2 + 4\,A_4\,y^3 + \ldots.$$

$$\frac{d^2 u}{d\,y^2} = 2\,A_2 + 2\,.\,3\,A_3\,y + 3\,.\,4\,A_4\,y^2 + \ldots.$$

Also, let

$$u = (y + \sqrt{y^2 - 1})^m$$

be twice successively differentiated, and the results are

$$\left(\frac{d\,u}{d\,y}\right)^2 (y^2 - 1) - m^2 u = 0,$$

$$\left(\frac{d\,u}{d\,y}\right)\left(\frac{d^2 u}{d\,y^2}\right)(y^2 - 1) + \left(\frac{d\,u}{d\,y}\right)^2 y - \left(\frac{d\,u}{d\,y}\right) m^2\,u = 0,$$

which, divided by $\frac{d\,u}{d\,y}$, gives

$$\frac{d^2 u}{d\,y^2}(y^2 - 1) + \left(\frac{d\,u}{d\,y}\right) y - m^2\,u = 0.$$

Let the values of u, $\frac{d\,u}{d\,y}$, $\frac{d^2 u}{d\,y^2}$, derived from differentiating the assumed series, be substituted in the last equation, and let the result be arranged according to the ascending powers of y. We shall thus obtain the following series:

$$\begin{array}{l}
A_0\,m^2 + 2\,A_2, \\
+ [A_1\,(m^2 - 1) + 2\,.\,3\,A_3]\,y, \\
+ [A_2\,(m^2 - 4) + 3\,.\,4\,A_4]\,y^2, \\
+ [A_3\,(m^2 - 9) + 4\,.\,5\,A_5]\,y^3, \\
+ [A_4\,(m^2 - 16) + 5\,.\,6\,A_6]\,y^4. \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
+ \{\,A_{n-2}\,[m^2 - (n-2)^2] + (n-1)\,(n)\,A_n\,\}\,y^{n-2} \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = 0.
\end{array}$$

Since this must be fulfilled independently of y, the coefficients must severally $= 0$. Hence we find

$$A_2 = -\frac{m^2}{2}\,A_0,$$

$$A_3 = -\frac{m^2 - 1}{2\,.\,3}\,A_1,$$

$$A_4 = -\frac{m^2 - 4}{3\,.\,4}\,A_2,$$

$$A_5 = -\frac{m^2 - 9}{4\,.\,5}\,A_3,$$

$$\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots$$

$$A_n = -\frac{m^2 - (n-2)^2}{(n-1)\,n}\,A_{n-2},$$

$$\ldots = \ldots\ldots\ldots\ldots\ldots$$

Hence we obtain the following conditions:

$$A_2 = -\frac{m^2}{2}\,A_0,$$

$$A_3 = -\frac{m^2 - 1}{2\,.\,3}\,A_1,$$

$$A_4 = +\frac{m^2\,(m^2 - 4)}{2\,.\,3\,.\,4}\,A_0,$$

$$A_5 = +\frac{(m^2 - 1)\,(m^2 - 9)}{2\,.\,3\,.\,4\,.\,5}\,A_1,$$

$$A_6 = -\frac{m^2\,(m^2 - 4)\,(m^2 - 16)}{2\,.\,3\,.\,4\,.\,5\,.\,6}\,A_0,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

The law of which is evident. These conditions, however, fail to determine the first two coefficients A_0, A_1. To find these, let $y = 0$ in the series for u and $\frac{d\,u}{d\,y}$, and also in the values

$$u = z^m = (y + \sqrt{y^2 - 1})^m,$$

$$\frac{d\,u}{d\,y} = \frac{m\,u}{\sqrt{y^2 - 1}};$$

and equating the results, we obtain

$$A_0 = (\sqrt{-1})^m = (-1)^{\frac{m}{2}},$$

$$A_1 = m\,(\sqrt{-1})^{m-1} = m\,(-1)^{\frac{m-1}{2}},$$

whence we find

$$A_2 = -\frac{m^2}{2}\,(-1)^{\frac{m}{2}},$$

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$$A_3 = -\frac{m^2 - 1^2}{2.3}.m(-1)^{\frac{m-1}{2}},$$

$$A_4 = +\frac{m^2(m^2 - 2^2)}{2.3.4}(-1)^{\frac{m}{2}},$$

$$A_5 = +\frac{(m^2 - 1^2)(m^2 - 3^2)}{2.3.4.5}.m(-1)^{\frac{m-1}{2}},$$

$$A_6 = -\frac{m^2(m^2 - 2^2)(m^2 - 4^2)}{2.3.4.5.6}(-1)^{\frac{m}{2}},$$

$$.. = - \dots\dots\dots\dots\dots\dots$$

$$.. = + \dots\dots\dots\dots\dots\dots$$

Hence we find

$$z^m = (-1)^{\frac{m}{2}}\Big\{1 - \frac{m^2}{1.2}y^2 + \frac{m^2(m^2 - 2^2)}{1.2.3.4}y^4 - \frac{m^2(m^2-2^2)(m^2-4^2)}{1.2.3.4.5.6}y^6 + \frac{m^2(m^2-2^2)(m^2-4^2)(m^2-6^2)}{1.2.3.4.5.6.7.8}y^8 - \dots\dots\Big\}$$

$$+ m(-1)^{\frac{m-1}{2}}\Big\{y - \frac{(m^2-1^2)}{1.2.3}y^3 + \frac{(m^2-1^2)(m^2-3^2)}{1.2.3.4.5}y^5 - \frac{(m^2-1^2)(m^2-3^2)(m^2-5^2)}{1.2.3.4.5.6.7}y^7 + \dots\Big\}$$

To find the series for z^{-m}, it is only oecessary to change the sign of m in the result whicb has just been obtained. Since neither of the series in this result contains any odd power of m, this change produces no other effect than to change the sign of the coefficient of the second parenthesis. Let the series in the first parenthesis be called for hrevity S, and that in the second S′, and we have

$$z^m = (-1)^{\frac{m}{2}}.S + m(-1)^{\frac{m-1}{2}}.S',$$

$$z^{-m} = (-1)^{-\frac{m}{2}}.S + m(-1)^{-\frac{m-1}{2}}.S';$$

since $-m(-1)^{-\frac{m-1}{2}} = m(-1)^{-\frac{m-1}{2}}$.

Hence, hy additioo we obtain,

$$z^m + z^{-m} = [(-1)^{\frac{m}{2}} + (-1)^{-\frac{m}{2}}]S + [(-1)^{\frac{m-1}{2}} + (-1)^{-\frac{m-1}{2}}]mS',$$

$$\therefore 2\cos mx = [(-1)^{\frac{m}{2}} + (-1)^{-\frac{m}{2}}]S + [(-1)^{\frac{m-1}{2}} + (-1)^{-\frac{m-1}{2}}]mS' \dots [1],$$

which is the developement sought.

(430.) The form of the coefficients of this formula may be changed. By Trigonometry we have $(\cos x + \sqrt{-1}\sin x)^m = \cos m(2n\pi \pm x) + \sqrt{-1}\sin m(2n\pi \pm x)$, n being any positive integer. Let $x = \frac{1}{2}\pi$, ∴

$$(-1)^{\frac{m}{2}} = \cos\tfrac{1}{2}m(4n\pm 1)\pi + \sqrt{-1}\sin\tfrac{1}{2}m(4n\pm 1)\pi,$$

$$\therefore (\sqrt{-1})^{-m} = \cos\tfrac{1}{2}m(4n\pm 1)\pi - \sqrt{-1}\sin\tfrac{1}{2}m(4n\pm 1)\pi$$

$$\therefore (-1)^{\frac{m}{2}} + (-1)^{-\frac{m}{2}} = 2\cos\tfrac{1}{2}m(4n\pm 1)\pi,$$

$$(-1)^{\frac{m-1}{2}} + (-1)^{-\frac{m-1}{2}} = 2\cos\tfrac{1}{2}(m-1)(4n\pm 1)\pi.$$

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Hence the series for $\cos mx$ becomes

$$\cos mx = \cos\tfrac{1}{2}m(4n\pm 1)\pi.S + \cos\tfrac{1}{2}(m-1)(4n\pm 1)\pi.mS' \dots\dots [2].$$

In this formula n is an iudeterminate integer for each value of which the second member has two values corresponding to the double sign $\pm$. The successive terms of the series

$$0,\ 1,\ 2,\ 3,\ \dots\dots$$

being substituted for n in $\cos\frac{1}{2}m(4n+1)\pi$, it will successively assume different values until the number substituted for n is equal to the denominator of m; for this value of n the value of $\cos\frac{1}{2}m(4n+1)\pi$ will be equal to that obtained by substituting 0 for n; and all integers greater than the denominator of m will in like manner give a constant repetition of values before obtained by substituting for n values less than the denominator of m. It follows, therefore, that $\cos\frac{1}{2}m(4n+1)\pi$ is in general susceptible of as many different values as there are units io the denominator of m, and no more. In like manner, $\cos\frac{1}{2}m(4n-1)\pi$ is susceptible of the same number of values; and therefore the coefficient of S is susceptible of twice as many values as there are units in the denominator of m, and a like observation applies to the coefficient of m S′.

Since S and S′ involve no functions of x, except $\cos x$, the change of x into $2n\pi \pm x$ makes no change in their value; and it follows, therefore, that for a given value of $\cos x$ the second member of [2] is susceptible of twice as many values as there are units io the denominator of m. It is therefore necessary to show how $\cos mx$ can have several values corresponding to a given value of $\cos x$. The angle x being changed into $2n'\pi \pm x$, n' being an integer, makes no change in $\cos x$, but changes $\cos mx$ ioto $\cos m(2n'\pi \pm x)$, which has twice as many values as there are units in the denominator of m. Hence the formula [2] will be more generally and correctly expressed thus,

$$\cos m(2n'\pi \pm x) = \cos\tfrac{1}{2}m(4n\pm 1)\pi.S + \cos\tfrac{1}{2}(m-1)(4n\pm 1)\pi.mS',$$

where both members have the same number of values, and where the values of the indeterminate integers n', n are supposed to be less than the denominator of m.

It still remains, however, to show the values of each member which correspond respectively to those of the other. Since the value of each member changes by ascribing different values to the integers n' and n, this question only amounts to the determination of the relation between any two corresponding values of these integers.

Let $x = \frac{1}{2}\pi$, and therefore $S = 1$, $S' = 0$. Hence

$$\cos m(2n'\pi \pm \tfrac{1}{2}\pi) = \cos\tfrac{1}{2}m(4n\pm 1)\pi,$$

$$\text{or } \cos\tfrac{1}{2}m(4n'\pm 1)\pi = \cos\tfrac{1}{2}m(4n\pm 1)\pi.$$

Since n and n' are oot supposed to receive any value greater than the denominator of m (for all the values of the cosine after that would only be repetitions of former values,) this last condition can only be satisfied by

$$n = n'.$$

Hence the formula becomes*

$$\cos m (2 n \pi \pm x) = \cos \tfrac{1}{2} m (4 n \pm 1) \pi \,.\, S$$
$$+ \cos \tfrac{1}{2} (m - 1) (4 n \pm 1) \pi \,.\, m S' \;\ldots\; [3].$$

(431.) It does not always happen that the formula expressing the value of $\cos m x$ includes both terms of the second member; for the angles whose cosines are the multipliers of S and S' in [3] may one or other of them be an odd multiple of a right angle, in which case the multiplier will be $= 0$, and the term will disappear.

To determine the conditions under which this can occur, it is necessary to consider when either of the numbers

$$\tfrac{1}{2} m (4 n \pm 1) \pi, \quad \tfrac{1}{2} (m - 1) (4 n \pm 1) \pi,$$

is an exact *odd* multiple of $\frac{1}{2} \pi$. This evidently takes place when either of the numbers

$$m (4 n \pm 1), \quad (m - 1) (4 n \pm 1),$$

is an odd integer.

Let $m = \dfrac{m'}{n'}$, and let I be any odd integer. That the first of the above numbers be an odd integer, it is necessary that

$$m' (4 n \pm 1) = n' \mathrm{I}.$$

Since m' and n' are prime, one or other must be an odd number; but since $4 n \pm 1$ and I are also odd, it is necessary that both m' and n' should be odd.

Also, since m' is prime to n', and measures n' I, it must measure I. Let $\dfrac{\mathrm{I}}{m'} = i$, which must be an odd integer, since both I and m' are odd. Hence

$$4 n \pm 1 = n' i,$$
$$\therefore \frac{4 n \pm 1}{n'} = i.$$

But since n is supposed to receive no value greater than n', i cannot be greater than 4; and since it is an odd integer, it must be either 1 or 3. The two corresponding values of n are

$$n = \frac{n' \mp 1}{4}, n = \frac{3 n' \mp 1}{4}.$$

The denominator n' being odd, must be either of the form $4 t + 1$, or $4 t - 1$.

If n' be of the form $4 t + 1$, the two values of n must be

$$n = \frac{n' - 1}{4}, n = \frac{3 n' + 1}{4};$$

since 4 evidently would not measure $n' + 1 = 4 t + 2$, nor $3 n' - 1 = 12 t + 3 - 1 = 12 t + 2$.

These values of n being substituted in [3], and m being changed into $\dfrac{m'}{n'}$, and the sign $+$ only being used for the first, and $-$ for the second, give

$$\left.\begin{aligned} \cos \frac{m'}{n'} \left(\frac{n' - 1}{2} \pi + x\right) &= \cos \tfrac{1}{2} m' \pi \,.\, S \\ &+ \cos \tfrac{1}{2} (m' - n') \pi \,.\, \frac{m'}{n'} S' \\ \cos \frac{m'}{n'} \left(\frac{3 n' + 1}{2} \pi - x\right) &= \cos \tfrac{3}{2} m' \pi \,.\, S \\ &+ \cos \tfrac{3}{2} (m' - n') \pi \,.\, \frac{m'}{n'} S' \end{aligned}\right\} \ldots [4].$$

Since m' and n' are odd,

$$\cos \tfrac{1}{2} m' \pi = 0, \quad \cos \tfrac{1}{2} (m' - n') \pi = \pm 1,$$
$$\cos \tfrac{3}{2} m' \pi = 0, \quad \cos \tfrac{3}{2} (m' - n') \pi = \pm 1,$$

$$\left.\begin{aligned} \therefore \cos \frac{m'}{n'} \left(\frac{n' - 1}{2} \pi + x\right) &= \pm \frac{m'}{n'} S' \\ \cos \frac{m'}{n'} \left(\frac{3 n' + 1}{2} \pi - x\right) &= \pm \frac{m'}{n'} S' \end{aligned}\right\} \ldots\ldots [5].$$

the sign $+$ being used when $\frac{1}{2} (m' - n')$ is even, and $-$ when odd.

If n' be of the form $4 t - 1$, the two values of n are

$$n = \frac{n' + 1}{4}, n = \frac{3 n' - 1}{4},$$

for it is evident that 4 would not in this case measure $n' - 1$, or $3 n' + 1$.

These values being substituted in [3], and m being changed as before into $\dfrac{m'}{n'}$, we obtain

$$\left.\begin{aligned} \cos \frac{m'}{n'} \left(\frac{n' + 1}{2} \pi - x\right) &= \cos \tfrac{1}{2} m' \pi \,.\, S \\ &+ \cos \tfrac{1}{2} (m' - n') \pi \,.\, \frac{m'}{n'} S' \\ \cos \frac{m'}{n'} \left(\frac{3 n' - 1}{2} \pi + x\right) &= \cos \tfrac{3}{2} m' \pi \,.\, S \\ &+ \cos \tfrac{3}{2} (m' - n') \pi \,.\, \frac{m'}{n'} S' \end{aligned}\right\} \ldots [6].$$

Hence, as before, we find

$$\left.\begin{aligned} \cos \frac{m'}{n'} \left(\frac{n' + 1}{2} \pi - x\right) &= \pm \frac{m'}{n'} S' \\ \cos \frac{m'}{n'} \left(\frac{3 n' - 1}{2} \pi + x\right) &= \pm \frac{m'}{n'} S' \end{aligned}\right\} \ldots\ldots [7].$$

The signs $+$ and $-$ being used as before.

(432.) The condition under which

$$(m - 1) (4 n \pm 1) = \frac{m' - n'}{n'} (4 n \pm 1)$$

should be an odd integer, may be immediately derived from those of the last case by changing m' into $m' - n'$. Hence the two values of n are the same as those already found, and n' and $m' - n'$, must be odd integers. Hence m' is even. Hence we have

$$\cos \tfrac{1}{2} m' \pi = \pm 1, \quad \cos \tfrac{1}{2} (m' - n') \pi = 0,$$
$$\cos \tfrac{3}{2} m' \pi = \pm 1, \quad \cos \tfrac{3}{2} (m' - n') \pi = 0.$$

* In clearing the formula [1] of imaginary quantities, *Lagrange* has fallen into an error which was lately detected by *Poinsot*, and the difficulty explained as above. *Lagrange's* mistake arose from assuming that

$$\sqrt{(-1)} = \cos \tfrac{1}{2} m \pi + \sqrt{-1} \sin \tfrac{1}{2} m \pi,$$

which is evidently erroneous, since the first member has as many different values as there are units in the denominator of m, and the second member has but one value, he forgot to take into account, that while the change of x into $2 n \pi + x$ produces no change on

$$(\cos x + \sqrt{-1} \sin x)^m,$$

it does produce a change on

$$\cos m x + \sqrt{\pm 1} \sin m x.$$

In fact, without this consideration, *Moivre's formula* itself is involved in the absurdity of one member having a greater number of different values than the other.

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Hence the formulæ [4] and [6] become

$$\left.\begin{aligned}
\cos\frac{m'}{n'}\left(\frac{n'-1}{2}\pi + x\right) &= \pm S \\
\cos\frac{m'}{n'}\left(\frac{3n'+1}{2}\pi - x\right) &= \pm S \\
\cos\frac{m'}{n'}\left(\frac{n'+1}{2} - x\right) &= \pm S \\
\cos\frac{m'}{n'}\left(\frac{3n'-1}{2} + x\right) &= \pm S
\end{aligned}\right\}\dots\dots[8]$$

the sign + being used when $\frac{1}{2}m'$ is even, and − when odd.

(433.) From the preceding observations it appears, that when the denominator of m is odd there are always two values of an angle x whose cosine is given, of which the cosine of the multiple $m x$ admits of being expressed by a single series of ascending powers of the given cosine;* but that for all other values of the arc whose cosine is given, the cosine of the same multiple requires the combination of both series S and S′.

If the denominator of m be even, there is no value whatever of the angle whose cosine is given, which allows of cos $m x$ being expressed by a single series.

(434.) The case in which m is an integer comes under the cases where the denominator of m is of the form $4t+1$, t being in this case $= 0$. If m be odd, we have by [5]

$$\cos m x = \pm m S',$$

the sign + being used when $\frac{1}{2}(m'-1)$ is even, and − when odd. If m be even, we have by the first of [8]

$$\cos m x = \pm S,$$

the sign + being used when $\frac{1}{2}m$ is even, and − when odd.

(435.) The laws of the two series S and S′ are easily defined. Let T be the r^{th} term of S and T′ of S′; by attending to the forms of the coefficients and exponents we find

$$T = \pm\frac{m^2(m^2-2^2)(m^2-4^2)(m^2-6^2)\dots(m^2-(2r-4)^2)}{1.2.3.\dots\dots\dots 2(r-1)}y^{2(r-1)}$$

$$T' = \pm\frac{(m^2-1^2)(m^2-3^2)(m^2-5^2)\dots(m^2-(2r-3)^2)}{1.2.3\dots\dots\dots 2r-1}y^{2r-1}$$

It is evident from the forms of these terms, that the series S can only terminate when m is an even integer, and S′ when m is an odd integer.

(436.) To determine the number of terms in each series when it is finite, let n be the sought number. The $(n+1)^{th}$ term must therefore $= 0$. Substituting $n+1$ for r in T and T′, and putting the results $= 0$, we obtain

$$m^2 - (2n+2-4)^2 = 0,$$

$$\therefore n = \frac{m}{2} + 1$$

the number of terms in S; and

$$m^2 - (2n+2-3)^2 = 0,$$

$$\therefore n = \frac{m+1}{2},$$

the number of terms in S′.

(437.) To obtain the last term (z) of S, it is only necessary to substitute the value of n in place of r in T, and the result is

$$z = \pm\frac{m^2(m^2-2^2)(m^2-4^2)\dots(m^2-(m-2)^2)}{1.2.3\dots\dots\dots m}y^m.$$

Each factor of the numerator may be resolved into two, thus

$$m^2 = m \times m,$$
$$(m^2-2^2) = (m+2)\times(m-2),$$
$$(m^2-4^2) = (m+4)\times(m-4),$$
$$\dots\dots\dots\dots$$
$$\dots\dots\dots\dots$$
$$(m^2-(m-2)^2) = (2m-2)\times 2.$$

The second factors of these, beginning from the lowest, are obviously the even integers from 2 to m inclusive, and the first factors, beginning from the highest, are the even integers from m to $2m-2$ inclusive. Thus the simple factors of the numerator are all the even integers from 2 to $2m-2$ inclusive, the factor m being twice repeated. The numerator of z may therefore be written thus,

$$2.4.6\dots(2m-2)\times m,$$

which is equivalent to

$$1.2.3\dots(m-1)\times m\times 2^{m-1}$$

The factors of the denominator destroying all these except 2^{m-1}, we have

$$z = \pm 2^{m-1}y^m.$$

+ being taken when $\frac{m}{2}$ is even, and − when odd.

(438.) To determine the last term z' of S′, let $\frac{m+1}{2}$ be substituted for r in the general term, and we obtain

$$z' = \pm\frac{(m^2-1^2)(m^2-3^2)\dots(m^2-(m-2)^2)}{1.2.3\dots\dots\dots m}y^m.$$

Each of the factors of the numerator may, as before, be resolved into two, thus

$$(m^2-1^2) = (m+1)\times(m-1),$$
$$(m^2-3^2) = (m+3)\times(m-3),$$
$$\dots = \dots\dots$$
$$\dots = \dots\dots$$
$$(m^2-(m-2)^2) = (2m-2)\times 2.$$

The last factors of each of these, beginning from the lowest, are the even integers from 2 to $m-1$ inclusive, and the first, beginning from the highest, are the even integers from $m+1$ to $2m-2$ inclusive. Hence the factors of the numerators may be expressed thus,

* Before the publication of Poinsot's Memoir, these cases were not noticed. *Lagrange* expressly states, that whenever m is a fraction, both terms of the second member of [3] are *necessary*.

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$$2.4.6 \ldots\ldots\ldots\ldots (2m-2)$$
$$= 1.2.3 \ldots\ldots\ldots\ldots (m-1).2^{m-1}$$

Hence

$$z' = \pm \frac{1}{m} 2^{m-1} y^m.$$

(439.) *To develope* sin m x *in ascending powers of* cos x.

By subtracting the value of z^{-m} obtained in (429) from that of z^m, and the result being disengaged from the imaginary symbols by the method used in (430) becomes

$$\sin m(2n\pi \pm x) = \sin \tfrac{1}{2} m (4n \pm 1)\pi S + \sin \tfrac{1}{2}(m-1)(4n \pm 1)\pi . m S'. \quad [9]$$

All the preceding observations are equally applicable here. When the denominator of m is an odd integer there are always two values of an angle x whose cosine is given, which are such that $\sin m x$ will be expressed by only one of the two series in [9].

(440.) To determine the conditions under which this will happen, it is necessary to determine when either of the numbers

$$m(4n \pm 1) \qquad (m-1)(4n \pm 1)$$

is an even integer.

To find the values of n which will render $m(4n \pm 1)$ an even integer, let

$$m(4n \pm 1) = I$$
$$\because\ m'(4n \pm 1) = I n'$$
$$\because\ 4n \pm 1 = \frac{I}{m'} n'.$$

Hence $\frac{I}{m'} n'$ is an odd integer, therefore $\frac{I}{m'}$ must be odd integer, therefore m' must be even. Let $\frac{1}{m'} = i$,

$$\because\ 4n \pm 1 = i n'.$$

It may be proved, as in the former case, that i must be either 1 or 3, and that when n' has the form $4t+1$, the values of n are

$$n = \frac{n'-1}{4},\ n = \frac{3n'+1}{4};$$

and when n' has the form $4t - 1$, the values are

$$n = \frac{n'+1}{4},\ n = \frac{3n'-1}{4}.$$

(441.) In like manner, in order that $(m-1)(4n \pm 1)$ be an even integer, the same values of n are obtained, and it is necessary that n' should be odd, and $m' - n'$ even, and therefore m' odd.

(442.) Hence if m' be even, and the values of n obtained above be substituted for it in [9], we obtain

$$\left.\begin{aligned}
\sin \frac{m'}{n'}\left(\frac{n'-1}{2}\pi + x\right) &= \sin \tfrac{1}{2} m'\pi . S \\
&\quad + \sin \tfrac{1}{2}(m'-n')\pi . \frac{m'}{n'} S' \\
\sin \frac{m'}{n'}\left(\frac{3n'+1}{2}\pi - x\right) &= \sin \tfrac{1}{2} m'\pi . S \\
&\quad + \sin \tfrac{1}{2}(m'-n')\pi . \frac{m'}{n'} S' \\
\sin \frac{m'}{n'}\left(\frac{n'+1}{2}\pi - x\right) &= \sin \tfrac{1}{2} m'\pi . S \\
&\quad + \sin \tfrac{1}{2}(m'-n')\pi . \frac{m'}{n'} S' \\
\sin \frac{m'}{n'}\left(\frac{3n'-1}{2}\pi + x\right) &= \sin \tfrac{1}{2} m'\pi . S \\
&\quad + \sin \tfrac{1}{2}(m'-n')\pi . \frac{m'}{n'} S'
\end{aligned}\right\} \ldots . [10].$$

But since m' is even, and n' odd,

$$\sin \tfrac{1}{2} m'\pi = 0, \quad \sin \tfrac{1}{2}(m'-n')\pi = \pm 1,$$
$$\sin \tfrac{1}{2} m'\pi = 0, \quad \sin \tfrac{1}{2}(m'-n')\pi = \pm 1,$$

Hence

$$\left.\begin{aligned}
\sin \frac{m'}{n'}\left(\frac{n'-1}{2}\pi + x\right) &= \pm \frac{m'}{n'} S' \\
\sin \frac{m'}{n'}\left(\frac{3n'+1}{2}\pi - x\right) &= \pm \frac{m'}{n'} S'
\end{aligned}\right\} \ldots\ldots [11].$$

$$\left.\begin{aligned}
\sin \frac{m'}{n'}\left(\frac{n'+1}{2}\pi - x\right) &= \pm \frac{m'}{n'} S' \\
\sin \frac{m'}{n'}\left(\frac{3n'-1}{2}\pi + x\right) &= \pm \frac{m'}{n'} S'
\end{aligned}\right\} \ldots\ldots [12].$$

The two first being true when n' is of the form $4t+1$, and the last when of the form $4t-1$. The sign $+$ is used when $\frac{1}{2}(m' - n' + 1)$ is odd, and $-$ when even.

(443.) If m' be odd,

$$\sin \tfrac{1}{2} m'\pi = \pm 1, \quad \sin \tfrac{1}{2}(m'-n')\pi = 0,$$
$$\sin \tfrac{1}{2} m'\pi = \pm 1, \quad \sin \tfrac{1}{2}(m'-n')\pi = 0.$$

Hence the formulæ [10] become

$$\left.\begin{aligned}
\sin \frac{m'}{n'}\left(\frac{n'-1}{2}\pi + x\right) &= \pm S \\
\sin \frac{m'}{n'}\left(\frac{3n'+1}{2}\pi - x\right) &= \pm S \\
\sin \frac{m'}{n'}\left(\frac{n'+1}{2}\pi - x\right) &= \pm S \\
\sin \frac{m'}{n'}\left(\frac{3n'-1}{2}\pi + x\right) &= \pm S
\end{aligned}\right\} \ldots\ldots\ldots [13],$$

the sign $+$ being used when $\frac{1}{2}(m'+1)$ is odd, and $-$ when even.

(444.) The series S and S' in [9] being the same as those in [3], their law and properties when m is an integer have been already determined.

It is obvious that when m is even, we have

$$\sin m x = \pm m S',$$

$+$ being used when $\frac{1}{2} m$ is even, and $-$ when odd.

And when m is odd

$$\sin m x = \pm S,$$

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$+$ being used when $\frac{1}{2}(m+1)$ is odd, and $-$ when even.

(445.) Another form for the developement of $\sin m x$ in ascending powers of the $\cos x$, may be established by differentiating the series found for $\cos m x$ in (430.) By this process we obtain

$$m \sin m (2 n \pi \pm x) = - \cos \tfrac{1}{2} m (4 n \pm 1) \pi \cdot \frac{d S}{d x}$$

$$- \cos \tfrac{1}{2} (m - 1)(4 n \pm 1) \pi \, . \, m \frac{d S'}{d x},$$

$$\frac{d S}{d y} = - \frac{m^2}{1} y + \frac{m^2 (m^2 - 2^2)}{1 \, . \, 2 \, . \, 3} y^3 -$$

$$\frac{m^2 (m^2 - 2^2)(m^2 - 4^2)}{1 \, . \, 2 \, . \, 3 \, . \, 4 \, . \, 5} y^5 + \ldots . = - m R,$$

$$\frac{d S'}{d y} = 1 - \frac{m^2 - 1^2}{1 \, . \, 2} y^2 + \frac{(m^2 - 1^2)(m^2 - 3^2)}{1 \, . \, 2 \, . \, 3 \, . \, 4} y^4$$

$$- \ldots \ldots = R',$$

$$\frac{d y}{d x} = - \sin x,$$

$$\therefore \frac{d S}{d x} = m R \sin x, \quad \frac{d S'}{d x} = - R' \sin x,$$

$$\because m \sin m (2 n \pi \pm x) = - \sin x \, [\cos \tfrac{1}{2} m (4 n \pm 1)$$

$$\pi \, . \, m R - \cos \tfrac{1}{2} (m - 1)(4 n \pm 1) \pi \, . \, R'] \ldots [13].$$

This being deduced directly from the formula [3] is liable to the various modifications which have been shown to be incident to [3], on assigning particular values to m and n. The several modifications of [13] which correspond to these, may be deduced by differentiating the several series [5], [7], [8], &c. &c.

(446.) The laws of the series R and R' are easily defined.

Let T and T' be their r^{th} terms respectively,

$$T = \pm \frac{m^2 (m^2 - 2^2)(m^2 - 4^2) \ldots . [m^2 - (2r - 2)^2]}{1 \, . \, 2 \, . \, 3 \ldots . \, 2 r - 1} y^{2r-1},$$

$$T' = \pm \frac{(m^2 - 1^2)(m^2 - 3^2) \ldots . [m^2 - (2r - 3)^2]}{1 \, . \, 2 \, . \, 3 \ldots . \, 2 (r - 1)} y^{2(r-1)}$$

The number of terms in R is only finite when m is an even integer, and in R' when it is an odd integer. The number in R is evidently one less than in S when it is finite, and is therefore equal to $\frac{m}{2}$. But the number in R' when it is finite is the same as in S', and is therefore $\frac{m+1}{2}$.

The last terms of R and R' in these cases may be obtained by differentiating those of S and S', and dividing the one by $m \sin x$, and the other by $- \sin x$.

(447.) *To develope the cosine or sine of a multiple arc in ascending powers of the sine of the simple arc.*

Let $y = \sin x$,

$$z = \sqrt{1 - y^2} + y \sqrt{-1},$$

$$z = (\sqrt{1 - y^2} + y \sqrt{-1})^m;$$

and since

$$2 \cos m x = z^m + z^{-m},$$

$$2 \sqrt{-1} \sin m x = z^m - z^{-m},$$

Series for Sines, &c. of Multiple Arcs.

the problem will be solved by obtaining the developement of z^m in ascending powers of y.

Let

$$z^m = A_0 + A_1 y + A_2 y^2 + \ldots .$$

By proceeding exactly as in (429), we shall obtain

$$z^m = A_0 \left\{ 1 - \frac{m^2}{1 \, . \, 2} y^2 + \frac{m^2 (m^2 - 2^2)}{1 \, . \, 2 \, . \, 3 \, . \, 4} y^4 \right.$$

$$\left. - \frac{m^2 (m^2 - 2^2)(m^2 - 4^2)}{1 \, . \, 2 \, . \, 3 \, . \, 4 \, . \, 5 \, . \, 6} y^6 + \ldots . \right\}$$

$$+ A_1 \left\{ y - \frac{m^2 - 1^2}{1 \, . \, 2 \, . \, 3} y^3 + \frac{(m^2 - 1^2)(m^2 - 3^2)}{1 \, . \, 2 \, . \, 3 \, . \, 4 \, . \, 5} y^5 \right.$$

$$\left. - \ldots \ldots \right\}$$

The values of A_0 and A_1 may be determined by making, as in (429), $y = 0$ in the two values of z^m and $\frac{d(z^m)}{d y}$, and equating the results, which gives*

$$A_0 = (1)^{\frac{m}{2}}, \quad A_1 = m \sqrt{-1} \, (1)^{\frac{m-1}{2}}.$$

The value of z^{-m} may be deduced from that of z^m by changing the sign of m. Hence, if the series which enter these values be Q, Q', we obtain

$$z^m = (1)^{\frac{m}{2}} Q + \sqrt{-1} \, (1)^{\frac{m-1}{2}} m Q',$$

$$z^{-m} = (1)^{-\frac{m}{2}} Q - \sqrt{-1} \, (1)^{-\frac{m-1}{2}} m Q',$$

$$\because 2 \cos m x = [(1)^{\frac{m}{2}} + (1)^{-\frac{m}{2}}] Q + \sqrt{-1}$$

$$[(1)^{\frac{m-1}{2}} + (1)^{-\frac{m-1}{2}}] m Q',$$

$$2 \sqrt{-1} \sin m x = [(1)^{\frac{m}{2}} - (1)^{-\frac{m}{2}}] Q + \sqrt{-1}$$

$$[(1)^{\frac{m-1}{2}} + (1)^{-\frac{m-1}{2}}] m Q'.$$

It will be observed, that by changing x into $2 n \pi + x$, no change is made on the series Q and Q'; but there is a change made upon the first member of each equation. The coefficients of Q and Q' have exactly as many different values as the first members of the equation. This is a circumstance which has been hitherto overlooked.†

The above formula can be cleared of imaginary quantities by the usual method,

$$(1)^{\frac{m}{2}} = \cos n m \pi + \sqrt{-1} \sin n m \pi,$$

$$(1)^{\frac{m-1}{2}} = \cos n (m - 1) \pi + \sqrt{-1} \sin n (m - 1) \pi,$$

the number n being an indeterminate integer. All the arcs which have the same sine may be included under the formula $n \pi \pm x$, x being taken with the sign $+$ when n is even, and $-$ when n is odd. Hence the formulæ become

* Lagrange, and all mathematicians after him, have fallen into an error in the determination of these coefficients. *Poinsot* has lately corrected it.

† *Poinsot*, 1825.

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$$\cos m\,(n\,\pi \pm x) = \cos n\,m\,\pi\,.\,Q - \sin n\,(m-1)\,\pi\,.\,m\,Q' \ldots\ [14],$$

$$\sin m\,(n\,\pi \pm x) = \sin n\,m\,\pi\,.\,Q + \cos n\,(m-1)\,\pi\,.\,m\,Q' \ldots\ [15].$$

(448.) There are certain values of n, for which each of the coefficients of these formulæ $= 0$. To determine these, let $m = \frac{m'}{n'}$, and let it be remembered that no value is supposed to be assigned to n greater than n'. We have theoce the following conditions:

$$\cos n\,m\,\pi = 0, \quad \because\ n = \frac{n'}{2}, \text{ or } n = \frac{3\,n'}{2}$$

$$\cos n\,(m-1)\,\pi = 0, \quad \because\ n = \frac{n'}{2}, \text{ or } n = \frac{3\,n'}{2},$$

$$\sin n\,m\,\pi = 0, \quad \because\ n = 0, \text{ or } n = n',$$

$$\sin n\,(m-1)\,\pi = 0, \quad \because\ n = 0, \text{ or } n = n'.$$

The first two conditions can only be satisfied when the denominator (n') of m is even. Hence it follows, that of all the arcs whose sines have any given value, there are always two (X) for which the formulæ [14], [15], are reduced tn a single series. These two arcs are of the forms $\frac{n'}{2}\,\pi + x$, $\frac{3\,n'}{2}\,\pi - x$, or $\frac{n'}{2}\,\pi - x$, $\frac{3\,n'}{2} + x$. For these two values of n we have

$$\cos m\,X = \pm\,m\,Q', \quad \sin m\,X = \pm\,Q.$$

The last two conditions can be fulfilled, whatever be the value of n', and the formulæ [14], [15], become

$$\cos m\,X = \pm\,Q, \quad \sin m\,X = \pm\,m\,Q';$$

where X is an arc of the form x or $n'\,\pi + x$ when n' is even, and x or $n'\,\pi - x$ when n' is odd.

It appears, therefore, that among the values of an arc whose sine is given there are always two, the cosioes and the sines of whose multiples admit of beiog expressed by a single series. In this respect, the developemeots by the powers of the sine differ from those by the powers of the cosine, in which, when the denominator of m is even, there is no value of the simple arc, the cosine or sine of whose multiple can be developed in a single series.

(449.) If m be ao integer, one of the coefficients of each of the formulæ [14], [15], must necessarily $= 0$.

This comes within the case in which m has an odd denominator, sioce the denominator is unity, and since no value is supposed to be given to n greater than n', it is in this case necessarily $= 1$. Hence in this case

$$\cos m\,x = \pm\,Q, \quad \sin m\,x = \pm\,m\,Q'.$$

The double sign applies to the two values of x, *scil.* x and $\pi - x$, which have the same sine. The value of cos $m\,x$ with the sign $+$ is used when m is eveo, and that with the sign $-$ when m is odd; and in the value of sio $m\,x$ the sign $+$ is used when m is odd, aod $-$ when m is even.

When m is even, the series Q is finite and Q′ infinite, and when m is odd, Q′ is finite and Q iufinite. The form of these series being the same as the series S, S′, in (429) the law, the number of terms when finite, and the last term is determined in the same manner.

Series for Sines, &c of Multiple Arcs.

(450.) *To develope the sine and cosine of a multiple arc in a series of ascending powers of the tangent of the simple arc.*

By developing the formula

$$\cos m\,x + \sqrt{-1}\,\sin m\,x = (\cos x + \sqrt{-1}\,\sin x)^m;$$

by the binomial theorem we shall obtain

$$\cos m\,x + \sqrt{-1}\,\sin m\,x = R + \sqrt{-1}\,R' \ldots\ [16],$$

where R represents the sum of the odd, and R′ of the eveo terms of the developement, and therefore

$$R = \cos^m x - A_2 \cos^{m-2} x \sin^2 x + A_4 \cos^{m-4} x \sin^4 x - \ldots$$

$$R' = A_1 \cos^{m-1} x \sin x - A_3 \cos^{m-3} x \sin^3 x + A_5 \cos^{m-5} x \sin^5 x - \ldots$$

where A_1, A_2, A_3, ... represent the coefficients of the second and succeeding terms of the expanded binomial, whose exponent is m.

As each side of the equation [16] consists partly of real, and partly of imaginary quantities, it is equivalent to two distinct equations, between each separately. If we consider R composed exclusively of real, and $\sqrt{-1}$ R′ of imaginary quantities, we should therefore have

$$\left.\begin{array}{l}\cos m\,x = R\\ \sin m\,x = R'\end{array}\right\}\ [17].$$

These formulæ, which were first published by John Bernouilli in the Leipsie Acts, 1701, have been, even to the present day, considered as exact and general. This, however, is not the case.

To explain this, let

$$T = 1 - \tan A_2 \tan^2 x + A_4 \tan^4 x - \ldots$$

$$T' = A_1 \tan x - A_3 \tan^3 x + A_5 \tan^5 x - \ldots$$

$$\because\ R = \cos^m x\,.\,T,$$

$$R' = \cos^m x\,.\,T'.$$

By changing x into $2\,n\,\pi + x$, the factors T, T′ of the second members of

$$\cos m\,x = \cos^m x\,.\,T,$$

$$\sin m\,x = \cos^m x\,.\,T',$$

undergo no change, since these arcs have the same tangent, and since T, T′ include no powers except iotegral powers of tao x, they can have each but one value for an arc, whose sine and cosine are given. The first factor $\cos^m x$ has, however, as maoy different values as there are units in the denominator of m, of which two, at most, can be real, and all the others must be imaginary. Oo the other hand, for an arc whose sine and cosine are given, and which is of the form $2\,n\,\pi + x$, n being any integer, the first members of these equations have as many different values as there are units in the denominator of m, and *all* these values are *real*. Thus the two members of the equations are inconsistent.

It is not difficult to perceive, that this absurdity has arisen from the false assumption, that the real and imaginary parts of the second member of [16] were R and $\sqrt{-1}$ R′. We shall find, upon consideration, that neither of these quantities are altogether real, or altogether imaginary, but that each of them is composed partly of real and partly of imaginary quantities, and is of the form $a + \sqrt{-1}\,.\,b$.

In the formula

$$\cos m x + \sqrt{-1} \sin m x = \cos^m x \,(\mathrm{T} + \sqrt{-1}\, \mathrm{T}'),$$

let the absolute, real, or arithmetical, value of $\cos^m x$, $\cos x$ being considered merely as a number, be P. It is plain that its several algebraical values will be expressed by the formula $\mathrm{P}\,(\pm 1)^m$. And since

$$(\pm 1)^m = \cos m n \varpi + \sqrt{-1} \sin m n \varpi,$$

$$\therefore \cos^m x = \mathrm{P}\,(\cos m n \varpi + \sqrt{-1} \sin m n \varpi),$$

the indeterminate integer n being even when $\cos x$ is positive, and odd when it is negative.

Making this substitution in the former equation, and in place of x, substituting the general formula $n'\varpi \pm x$ for all arcs having the same cosine, in which the sign $+$ is used when n' is even, and $-$ when it is odd, we obtain

$$\cos m\,(n'\varpi \pm x) + \sqrt{-1} \sin m\,(n'\varpi \pm x)$$
$$= \mathrm{P}\,(\mathrm{T} \cos m n \varpi - \mathrm{T}' \sin m n \varpi)$$
$$+ \sqrt{-1}\,.\,\mathrm{P}\,(\mathrm{T} \sin m n \varpi + \mathrm{T}' \cos m n \varpi).$$

Here the real and imaginary parts are separated on each side, and equating them, we have

$$\cos m\,(n'\varpi \pm x) = \mathrm{P}\,(\mathrm{T} \cos m n \varpi - \mathrm{T}' \sin m n \varpi),$$
$$\sin m\,(n'\varpi \pm x) = \mathrm{P}\,(\mathrm{T} \sin m n \varpi + \mathrm{T}' \cos m n \varpi).$$

Each member of these equations is susceptible of as many different values as there are units in the denominator of m. But it remains still to be determined, which of the values of the second members correspond or are equal to those of the first severally. In other words, it is necessary to determine what relation subsists between the indeterminate integers n' and n, neither of which are supposed to exceed twice the denominator of m. To determine this, let $x = 0$, $\therefore \mathrm{P} = 1$, $\mathrm{T} = 1$, $\mathrm{T}' = 0$. Hence

$$\cos m n' \varpi = \cos m n \varpi,$$
$$\sin m n' \varpi = \sin m n \varpi,$$
$$\therefore n' = n$$

These integers are, therefore, always equal, and the formulæ become

$$\cos m\,(n \varpi \pm x) = \mathrm{P}\,(\mathrm{T} \cos m n \varpi - \mathrm{T}' \sin m n \varpi) \quad [18],$$
$$\sin m\,(n \varpi \pm x) = \mathrm{P}\,(\mathrm{T} \sin m n \varpi + \mathrm{T}' \cos m n \varpi) \quad [19].$$

Whether the odd or even integers are to be substituted for n in these formulæ, and whether x is to be taken with $+$ or $-$, is to be determined by the signs of $\sin x$ and $\cos x$, which are supposed to be given. If $\cos x$ be positive, the values of n are to be selected from the series

$$0, 2, 4, 6, \ldots\ldots;$$

if it be negative, they are to be selected from

$$1, 3, 5. \ldots\ldots$$

If $\sin x$ be positive, x is to be taken with $+$, and if negative with $-$. In all cases, however, the coefficient P in the second members is to be considered as an abstract number independent of any sign.

If m be an integer, the formulæ are reduced to the forms

$$\cos m x = \cos^m x\,\mathrm{T}, \quad \sin m x = \cos^m x\,.\,\mathrm{T}',$$

which have hitherto been taken to be general for all values of m.

There are, however, particular values of n' even when m is a fraction, for which one or other of the series by which $\cos m x$ and $\sin m x$ are expressed will disappear. In order that $\cos m n \varpi$ should $= 0$, it is necessary that $m n$ should be a fraction whose denominator is 2, and, therefore, whose numerator is an odd number. This can only happen when m is a fraction with an even denominator, and therefore an odd numerator, and when n is equal to half the denominator. Also in this case, if half the denominator of m be an even number, it is necessary that $\cos x$ should be positive, (otherwise n should be odd,) and if half the denominator be an odd number, it is necessary that $\cos x$ should be negative, for otherwise n should be even. Hence we may conclude, that if m be a fraction with an even denominator, there is always one arc, whose cosine has any given positive value when half the denominator of m is even, and whose cosine has any given negative value when half the denominator is odd, which is such, that each of the formulæ [18], [19], are reduced to a single series, since under the conditions just stated,

$$\cos m n \varpi = 0, \quad \sin m n \varpi = \pm 1.$$

In order that $\sin m n \varpi = 0$, it is necessary that $m n$ should be an integer, and therefore that n should be equal, either to the denominator of m, or to twice the denominator. In each case $\sin m n \varpi = 0$, and $\cos m n \varpi = \pm 1$. If $\cos x$ be positive, n must be even, and in this case, if the denominator of m be even, there are two values of n, which will reduce the formulæ [18], [19], to a single series; but if it be odd, since n must be even, there is but one value will satisfy this condition. If $\cos x$ be negative, n must be odd, and, therefore, when the denominator of m is odd, there is but one value of n, which will reduce the formulæ to a single series, and when m is even, there is no value of n will effect this.

It appears, therefore, that when m is an integer, $\cos m x$, and $\sin m x$, can always be expressed in a single series of powers of the tangent; but that when m is a fraction, there are only certain values of an arc of a given sine and cosine, which admit of a developement without both the series of [9], [10], and that in some cases there is no arc which admits it.

If the two formulæ [18], [19], be divided one by the other, we shall obtain

$$\tan m\,(n \varpi \pm x) = \frac{\mathrm{T} \cos m n \varpi - \mathrm{T}' \sin m n \varpi}{\mathrm{T} \sin m n \varpi + \mathrm{T}' \cos m n \varpi},$$

$$= \frac{\mathrm{T} - \mathrm{T}' \tan m n \varpi}{\mathrm{T} \tan m n \varpi + \mathrm{T}'} \ldots\ldots\ldots [20];$$

which, when m is an integer, and in the particular cases already mentioned when m is a fraction, becomes

$$\tan m x = \frac{\mathrm{T}}{\mathrm{T}'},$$

$$\text{or } \tan m x = \frac{\mathrm{T}}{\mathrm{T}'}.$$

(451.) *To develope the cosine or sine of a multiple arc in descending powers of the cosine of the simple arc.*

This problem was investigated by Euler, and subsequently by Lagrange, and both obtained the same result, although they proceeded on different principles

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and by different methods. The series which were the results of their investigations, and which have, even to the present time, been received as general and exact, are the following,

$$2\cos m x = (2y)^m - m(2y)^{m-2} + \frac{m(m-3)}{1\,.\,2}(2y)^{m-4}$$

$$-\frac{m(m-4)(m-5)}{1\,.\,2\,.\,2}(2y)^{m-6} + \frac{m(m-5)(m-6)(m-7)}{1\,.\,2\,.\,3\,.\,4}$$

$$(2y)^{m-8}\ \ldots\ [21],$$

$$-\ \ldots\ldots\ldots\ldots\ +\ \ldots\ldots\ldots\ldots$$

$$+(2y)^{-m} + m(2y)^{-m-2} + \frac{m\,.\,m+3}{1\,.\,2}(2y)^{-m-4},$$

$$+\frac{m(m+4)(m+5)}{1\,.\,2\,.\,3}(2y)^{-m-6} + \frac{m(m+5)(m+6)(m+7)}{1\,.\,2\,.\,3\,.\,4}$$

$$(2y)^{-m-8}$$

$$+\ \ldots\ldots\ldots\ldots\ +\ \ldots\ldots\ldots\ldots$$

&c. &c.

where $y = \cos x$. The series for $\sin m x$ was deduced from this by differentiation.

In the memoir already cited, Poinsot has examined the analysis by which these results were obtained, and shown that it is fallacious, and that the results themselves are false. To render this refutation intelligible, it would be necessary to detail the process by which Euler and Lagrange established the formulæ, which would lead to investigations unsuited to the purposes of the present treatise. As, however, the results of Lagrange have been hitherto universally received as correct, it is proper to make the student aware of the fact of their having been proved erroneous; and if he be desirous to examine the details of the process, he is referred to the memoir itself.

We shall confine ourselves here to that part of the memoir in which the true developement of $\cos m x$ and $\sin m x$ is investigated.

Let $p = \cos x$ and $q = \sin x$. We have

$$\cos m x = p^m\left(1 - A_2\frac{q^2}{p^2} + A_4\frac{q^4}{p^4} - A_6\frac{q^6}{p^6} + \ldots\right)$$

where 1, A_2, A_4, ... are the coefficients of the binomial series, m being the exponent. We have

$$q^2 = 1 - p^2,\ q^4 = 1 - 2p^2 + p^4,\ \ldots$$

Let these values be substituted for q^2, q^4, &c., and let the results be arranged according to the descending powers of p, and we have

$$\cos m x = A p^m - B p^{m-2} + \frac{1}{2}C p^{m-4}$$

$$-\frac{1}{1.2.3}D p^{m-6} + \&c.$$

where

$$A = 1 + A_2 + A_4 + A_6 \ldots$$
$$B = A_2 + 2A_4 + 3A_6 + 4A_8 + 5A_{10} \ldots$$
$$\tfrac{1}{2}C = A_4 + 3A_6 + 6A_8 + 10A_{10} + \ldots$$
$$\frac{1}{1.2.3}D = A_6 + 4A_8 + 10A_{10} + 20A_{12}$$
$$\ldots = \ldots\ldots\ldots\ldots$$

&c. &c.

Series for Sines, &c. of Multiple Arcs.

The law by which these coefficients are formed is evident, but it is necessary to obtain finite expressions for them as functions of m. For this purpose, let us suppose that the successive terms of the first coefficient A were multiplied by the successive powers of an arbitrary quantity y, so that it becomes

$$1 + A_2 y + A_4 y^2 + A_6 y^3 + \ldots$$

$$\text{or } 1 + \frac{m(m-1)}{1\,.\,2}y + \frac{m(m-1)(m-2)(m-3)}{1\,.\,2\,.\,3\,.\,4}y^2 +$$

But this last is equivalent to

$$\frac{(1+\sqrt{y})^m + (1-\sqrt{y})^m}{2} = U;$$

so that U becomes equal to A when $y = 1$. It is not difficult to perceive, that the other coefficients are what the successive differential coefficients of U taken with respect to y as a variable become when $y = 1$. We have

$$U = 1 + A_2 y + A_4 y^2 + A_6 y^3 + \ldots$$

$$\frac{dU}{dy} = A_2 + 2A_4 y + 3A_6 y^2 + 4A_8 y^3 \ldots$$

$$\frac{1}{2}\frac{d^2U}{dy^2} = A_4 + 3A_6 y + 6A_8 y^2 + \ \ldots$$

$$\ldots = \ldots\ldots\ldots\ldots$$

&c. &c.

When $y = 1$, the second members of these equations become equal severally to A, B, $\frac{1}{2}$C, $\frac{1}{1.2.3}$. D.... Let the values of the function U and its successive differential coefficients when $y = 1$ be called Y, Y′, Y″, Y‴, &c.; we have hence

$$A = Y = \frac{1}{2}\left\{2^m + 0^m\right\},$$

$$B = Y' = \frac{1}{2^2}\left\{m(2^{m-1} - 0^{m-1})\right\},$$

$$C = Y'' = \frac{1}{2^3}\left\{m(m-1)(2^{m-2} + 0^{m-2})\right.$$
$$\left. - m(2^{m-1} - 0^{m-1})\right\},$$

$$D = Y''' = \frac{1}{2^4}\left\{m(m-1)(m-2)(2^{m-3} - 0^{m-3})\right.$$
$$\left. - 3m(m-1)(2^{m-2} + 0^{m-2}) + 3m(2^{m-1} - 0^{m-1})\right\},$$

$$E = Y'''' = \frac{1}{2^5}\left\{m(m-1)(m-2)(m-3)(2^{m-4} + 0^{m-4})\right.$$
$$- 6m(m-1)(m-2)(2^{m-3} - 0^{m-3}) + 15m(m-1)$$
$$\left.(2^{m-2} + 0^{m-2}) - 15m(2^{m-1} - 0^{m-1})\right\}, \text{ \&c. \&c.}$$

In these analytical expressions for the coefficients of the sought series, it is necessary to preserve the terms 0^m, 0^{m-1}, 0^{m-2}, &c. because each of these powers of 0 become either unity, 0, or infinite, according as the exponent of the power is = 0, positive or negative.

The true developement, therefore, of $\cos m x$ in descending powers of $\cos x$ or p, the angle x being sup-

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posed less than a right angle, and only considering a single value of $\cos m x$ relative tn the arc x, is

$$\cos m x = \mathrm{Y} p^{m} - \mathrm{Y}' p^{m-2} + \tfrac{1}{2} \mathrm{Y}'' p^{m-4} - \frac{1}{2.3} \mathrm{Y}''' p^{m-6} + .$$

If m be a positive integer, this series will be finite, since all the terms beynnd a certain term will $= 0$, and it will thus give the exact value of $\cos m x$. Thus when $m = 0$, or $m = 1$, we find that the first coefficient only has a finite value, and all the others $= 0$. For $m = 2$ and $m = 3$, the first two coefficients are finite, and all the rest $= 0$. Fnr $m = 4$, $m = 5$, there are three terms finite, and all the rest equal nothing; and in general, if m be an even integer, the number nf finite terms is $\frac{m}{2} + 1$, and if it be odd, $\frac{m+1}{2}$

But if m be a fractinn, the series never terminates, and the coefficients nnly continue finite as lnng as the exponent of 0 which nccurs in them is nnt negative. After this happens, all the succeeding coefficients are infinite. Thus, if m be a fraction between 0 and 1, the first coefficient alone is finite, and all the rest infinite. If m be between 1 and 2, the first twn coefficients are finite, and all the rest infinite, and so nn. If m be a fractinn between $n - 1$ and n, the first n terms are finite, and all the rest infinite. The series, therefnre, in these cases is useless and absurd, and the same happens when m is negative. From whence we may conclude, that the developement of the cosine of a multiple arc in descending powers of that of the simple arc is never possible, except when the coefficient nf the multiple is a positive integer; and in this case, since the number of terms is finite, the series is nothing more than the series already nbtained in ascending powers, the nrder nf the terms being reversed. So that in effect, the only case in which the develnpement by descending pnwers is possible, it is useless.

It is wnrthy nf remark, that in the analytical expression for the coefficients A, B, $\frac{1}{2}$ C, &c. if the pnwers 0^{m}, 0^{m-2}, 0^{m-4}, &c. be neglected, the coefficients will be exactly those of the series [21], which has been hitherto considered exact. Whence may be seen the reason why this series gives false values for $\cos m x$, and also why, in the particular case in which m is an integer, the value resulting from it will be exact, if we retnin in it nnly the pnsitive powers nf p, fnr that is, in effect, rejecting all that part nf the true developement which becomes $= 0$.

(452.) The series for $\cos m x$ in descending powers of $\cos x$ or p, m being supposed to be an integer, is therefore

$$2 \cos m x = (2p)^{m} - m (2p)^{m-2} + \frac{m(m-3)}{1.2} (2p)^{m-4}$$
$$- \frac{m(m-4)(m-5)}{1.2.3} (2p)^{m-6}$$
$$+ \frac{m(m-5)(m-6)(m-7)}{1.2.3.4} (2p)^{m-8} - \ldots . \text{ [22]}.$$

(453.) To define the law of this series, let the r^{th} term be T,

$$\mathrm{T} = \pm \frac{m(m-r)(m-r-1)\,.\,.\,(m-2r+4)(m-2r+3)}{1\,.\,2\,.\,3 \ldots\ldots\ldots\ldots\ldots (r-1)} (2p)^{m-2(r-1)}.$$

To determine the last term z, let the values of n already found be substituted for r in this formula.

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If m be even, let $\frac{m}{2} + 1$ be substituted for r, and the result is

$$z = \pm \frac{m\left(\frac{m}{2} - 1\right)\left(\frac{m}{2} - 2\right) \ldots . 3.2.1}{1.2.3 \ldots . \left(\frac{m}{2} - 1\right) \frac{m}{2}} (2y)^{m-m}$$

All the factors nf the numerator, except the first, destrny all the factors of the denominator, except the last, nud therefore

$$z = \pm 2,$$

$+$ being taken when $\frac{m}{2} + 1$ is odd, and $-$ when $\frac{m}{2} + 1$ is even

If m be odd, let $\frac{m+1}{2}$ be substituted fnr n, and the result is

$$z = \pm \frac{m\left(\frac{m-1}{2}\right)\left(\frac{m-3}{2}\right) \ldots\ldots\ldots 3.2.1}{1.2.3 \ldots\ldots\ldots \left(\frac{m-3}{2}\right)\left(\frac{m-1}{2}\right)} (2y).$$

The factors of the denominatnr destroying those of the numeratnr, except the first, we obtain

$$z = \pm 2 m y,$$

$+$ being taken when $\frac{m+1}{2}$ is odd, and $-$ when it is even.

(454.) *To develope* sin m x *in descending powers of* cos x.

To effect this, it is only necessary to differentiate the series [22]. This being done, and the result divided by $2m$, and nbserving that $dy = d \cos x = -\sin x\, dx$, we nbtain

$$\frac{\sin m x}{\sin x} = (2y)^{m-1} - (m-2)(2y)^{m-3}$$
$$+ \frac{(m-3)(m-4)}{1.2} (2y)^{m-5} - \ldots . \text{ [23]}.$$

This developement, like the last, is nnly possible when m is an integer.

When m is an even integer, the number of terms in the series for $2 \cos m x$ being $\frac{m}{2} + 1$, and the last term $z = \pm 2$, it follows, since $dz = 0$, that in the present case the number of terms must be $\frac{m}{2}$.

The r^{th} term in the present case is evidently

$$\pm \frac{(m-r)(m-r-1)\,.\,.\,(m-2r+3)(m-2r+2)}{1.2.3 \ldots\ldots\ldots\ldots\ldots r-1} (2y)^{m-(2r-1)}.$$

Hence the last term, m being an even integer, may be found by substituting $\frac{m}{2}$ for r in this formula,

which gives

$$z = \pm \frac{\frac{m}{2}\left(\frac{m}{2} - 1\right) \ldots\ldots\ldots 3 \,.\, 2}{2 \,.\, 3 \ldots\ldots\ldots \left(\frac{m}{2} - 1\right)} (2\, y).$$

$$\therefore z = \pm\, m\, y.$$

If m be odd, the last term in the series for $2 \cos m x$ being $\pm\, m\, (2\, y)$, that of the series for $\frac{\sin m x}{\sin x}$ is

$$z = \pm\, 1,$$

the number of terms being $\frac{m+1}{2}$, and $+$ being taken when this is odd, and $-$ when it is even.

(455.) *To develope the cosine and sine of a multiple arc in descending powers of the sine of the simple arc.*

In [22] and [23] let x be changed into $\frac{\pi}{2} - x$, and the two series being expressed by M and M′, and p being understood to express $\sin x$ instead of $\cos x$, we shall have

$$2 \cos m \left(\frac{\pi}{2} - x\right) = \text{M},$$

$$\sin m \left(\frac{\pi}{2} - x\right) = \cos x \,.\, \text{M}'.$$

In this case, as in the former, m must be an integer.

If m be even,

$$\cos m \left(\frac{\pi}{2} - x\right) = \pm \cos m x,$$

$$\sin m \left(\frac{\pi}{2} - x\right) = \mp \sin m x,$$

$+$ being taken when $\frac{1}{2}\, m$ is even, and $-$ when odd. Hence, in these cases,

$$2 \cos m x = \pm\, \text{M},$$

$$2 \sin\, m x = \mp \cos x \,.\, \text{M}'.$$

If m be odd

$$\cos m \left(\frac{\pi}{2} - x\right) = \pm \sin m x,$$

$$\sin m \left(\frac{\pi}{2} - x\right) = \pm \cos m x,$$

$+$ being used if $\frac{m-1}{2}$ be even, and $-$ if odd.

Hence

$$\sin m x = \pm\, \text{M},$$

$$\cos m x = \pm \cos x \,.\, \text{M}'.$$

SECTION XLII

Of the Developement of a Power of the Sine or Cosine of an Arc in a Series of Sines or Cosines of its Multiples.

(456.) *To develope* $\cos^m x$ *in a series of cosines or sines of multiples of* x.

We have (Trigonometry)

$$2 \cos x = e^{x\sqrt{-1}} + e^{-x\sqrt{-1}},$$

$$\therefore 2^m \cos^m x = (e^{x\sqrt{-1}} + e^{x \cdot -\sqrt{-1}})^m.$$

If this be developed by the binomial theorem, we obtain

$$2^m \cos^m x = e^{mx\sqrt{-1}} + \text{A}\, e^{(m-2)x\sqrt{-1}} + \text{B}\, e^{(m-4)x\sqrt{-1}} + \ldots.$$

where $\quad 1, \text{A}, \text{B}, \text{C}, \ldots.$

are the coefficients of the binomial series.

Eliminating e by the general formula,

$$\cos m x + \sqrt{-1} \sin m x = e^{mx\sqrt{-1}},$$

we obtain

$$2^m \cos^m x = \cos m x + \text{A} \cos (m-2)\, x$$
$$+ \text{B} \cos (m-4)\, x + \ldots.$$
$$+ \sqrt{-1}\, [\sin m x + \text{A} \sin (m-2)\, x$$
$$+ \text{B} \sin (m-4)\, x + \ldots.].$$

Let the first series be P_x, and the second Q_x, and we have

$$(2 \cos x)^m = \text{P}_x + \sqrt{-1}\, \text{Q}_x.$$

Let $\cos x$ be first supposed to be positive, and in that case $(2 \cos x)^m$ must have at least one real value. Let this be X, and all its other values will be found by multiplying X by the values of $(1)^m$. They are, therefore, all expressed by the formula

$$\text{X} (\cos 2\, m\, n\, \pi + \sqrt{-1} \sin 2\, m\, n\, \pi),$$

n being an integer not exceeding the denominator of m.

Also, in

$$(2 \cos x)^m = \text{P}_x + \sqrt{-1}\, \text{Q}_x,$$

no change is made in the first member by changing x into $2\, n\, \pi + x$, and therefore

$$(2 \cos x)^m = \text{P}_{2n\pi+x} + \sqrt{-1}\, \text{Q}_{2n\pi+x}.$$

Hence

$$\text{x} \cos 2\, m\, n\, \pi + \sqrt{-1} \,.\, \text{x} \sin 2\, m\, n\, \pi = \text{P}_{2n\pi+x}$$
$$+ \sqrt{-1}\, \text{Q}_{2n\pi+x} \ldots. [1].$$

Equating the real and imaginary parts of this equation, we find

$$\text{X} = \frac{1}{\cos 2\, m\, n\, \pi} \text{P}_{2n\pi+x}, \quad \text{X} = \frac{1}{\sin 2\, m\, n\, \pi} \text{Q}_{2n\pi+x} \ldots [2]$$

Hence it appears that the real and positive value X of $(2 \cos x)^m$ can be indifferently expressed, either in a series of powers of the cosines or sines of the multiples of x, and that the two series differ from one another only in the constant coefficients.

Between the two series thus found, there subsists a constant relation,

$$\frac{\cos 2\, m\, n\, \pi}{\sin 2\, m\, n\, \pi} = \frac{\text{P}_{2n\pi+x}}{\text{Q}_{2n\pi+x}};$$

by which it appears that these series have a constant ratio, whatever be the value ascribed to x, for 0 to $\frac{\pi}{2}$.

If $n = 0$, we obtain by [2]

$$\text{X} = \text{P}_x,$$

which is therefore perfectly general, provided x be supposed less than $\frac{\pi}{2}$, and X confined to the real and positive value of $(2 \cos x)^m$.

The second formula of [2] gives $\sin 2\,m\,n\,\pi = 0$, $\because$

$$X = \frac{0}{0}.$$

This fails in giving any value of X, but shows that $Q_x = 0$ for all values of x from 0 to $\pm\frac{\pi}{2}$.

(457.) If the $\cos x$ be negative, let $(2 \cos x)^m$ be expressed thus,

$$(-2 \cos x)^m = (2 \cos x)^m (-1)^m = X(-1)^m.$$

But since

$$(-1)^m = \cos m(2n+1)\pi + \sqrt{-1} \sin m(2n+1)\pi.$$

Hence

$$X \cos m(2n+1)\pi + \sqrt{-1}\, X \sin m(2n+1)\pi$$
$$= P_{2n\pi+x} + \sqrt{-1}\, Q_{2n\pi+x}.$$

By equating the real and imaginary parts, we find

$$X = \frac{1}{\cos m(2n+1)\pi} P_{2n\pi+x} \quad \ldots\ldots [3].$$

$$X = \frac{1}{\sin m(2n+1)\pi} Q_{2n\pi+x} \quad \ldots\ldots [4].$$

In which the integer n is susceptible of any value from 0 to the denominator of m.

If $n = 0$, we have

$$X = \frac{1}{\cos m\pi} P_x,\ X = \frac{1}{\sin m\pi} Q_x,$$

which give developements of the real value of $(2 \cos x)^m$ when $\cos x$ is negative.

From this it appears, that Q_x is not $= 0$, as in the former case, where $\cos x$ was supposed positive.

But although $Q_{2n\pi+x}$ may not $= 0$ when $n = 0$, yet there may be some other value of n, which will render this series $= 0$. To discover this, let it be determined what value of n will satisfy the condition,

$$\sin m(2n+1)\pi = 0,$$
$$\because \cos m(2n+1)\pi = \pm 1.$$

That these conditions be fulfilled, it is necessary that $m(2n+1)$ be an integer. Let $m = \frac{m'}{n'}$, and let I be any integer, $\because$

$$m'(2n+1) = \mathrm{I}\,n';$$

but m' being prime to n' measures I. Let $\frac{\mathrm{I}}{m'} = i$, $\because$

$$2n+1 = i\,n'.$$

Since $2n+1$ is odd, both i and n' must be odd. But since n is supposed not to exceed n', i must be $= 1$.

Hence

$$n = \frac{n'-1}{2},$$

which is therefore the only value of n which can satisfy the proposed condition.

Hence, if m be a fraction with an odd denominator (n'), we have

$$X = \pm P_{(n'-1)\pi+x},\ Q_{(n'-1)\pi+x} = 0,$$

$+$ being used when m' is even, and $-$ when odd.

But if m be a fraction with an even denominator, there is no arc $(2n+1)\pi$ which can render $\cos m(2n+1)\pi = \pm 1$; and, consequently, no arc $2\,m\,n\,\pi + x$ for which the series P_x can become equal to the real value of $(2 \cos x)^m$.

By the formulæ [3], [4], it follows that when $\cos x$ is negative, the real and positive value of $(2 \cos x)^m$ may be expressed either in a series of sines or cosines of the multiples of x, and that the two developements differ only in the coefficients; and, finally, that their ratio is the same for all values of x between $\frac{\pi}{2}$ and $\frac{3\pi}{2}$.

(458.) If m be a positive integer $Q_x = 0$, and we have

$$(2 \cos x)^m = P_x.$$

The number of terms in P_x is $m + 1$, being those of the binomial series. Hence the last term must be

$$\cos(m - 2m)x = \cos m x,$$

which is equal to the first. And, in like manner, the penultimate term is equal to the second, and every pair of terms equidistant from the extremes are equal.

It follows, therefore, that when m is odd, and $\because m+1$ even, the first half of the series S is equal to $\frac{1}{2}(2^m \cos^m x) = 2^{m-1} \cos^m x$; and when m is even, and therefore $m + 1$ odd, the first $\frac{m}{2}$ terms together with half the $\left(\frac{m}{2}+1\right)^{th}$ term is equal to $2^{m-1} \cos^m x$.

Hence we conclude,

1. When m is odd,

$$2^{m-1} \cos^m x = \cos m x + \mathrm{A} \cos(m-2)x$$
$$+ \mathrm{B} \cos(m-4)x + \ldots\ldots$$

continued to $\frac{m+1}{2}$ terms.

The last term of this series is

$\mathrm{M} \cos\left[m - 2\left(\frac{m+1}{2} - 1\right)\right]x = \mathrm{M} \cos x$, M being the coefficient of the $\left(\frac{m+1}{2}\right)^{th}$ term of an expanded binomial. From the law of the binomial series we have

$$\mathrm{M} = \frac{m \,.\, m-1 \,.\, m-2 \,..\left(m - \frac{m-3}{2}\right)}{1 \,.\, 2 \,.\, 3 \,\ldots\ldots\ldots\ldots\ldots \frac{m-1}{2}}.$$

This may, however, be reduced to a somewhat simpler form. Let both terms of the fraction be multiplied by $2^{\frac{m-1}{2}}$, the operation being effected on the denominator by doubling each of its factors; the result is

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$$M = \frac{m \,.\, m-1 \,.\, m-2 \,.\,.\, \left(m - \frac{m-3}{2}\right)}{2 \,.\, 4 \,.\, 6 \ldots\ldots\ldots\ldots (m-1)} \,.\, 2^{\frac{m-1}{2}}.$$

Again, multiplying both numerator and denominator by the odd integers from 1 to m inclusive, in order to complete the series of factors in the denominator,

$$M = \frac{m \,.\, m-1 \,.\, m-2 \ldots \left(m - \frac{m-3}{2}\right)}{1 \,.\, 2 \,.\, 3 \ldots\ldots\ldots\ldots (m-1)\, m} 2^{\frac{m-1}{2}} (1 \,.\, 3 \,.\, 5 \ldots m)$$

Expunging from both numerator and denominator the descending factors from m to $m - \frac{m-3}{2}$ inclusive, we obtain

$$M = \frac{1 \,.\, 3 \,.\, 5 \ldots\ldots m}{1 \,.\, 2 \,.\, 3 \ldots\ldots \frac{m+1}{2}} 2^{\frac{m-1}{2}}.$$

Hence the last term z is

$$z = \frac{1 \,.\, 3 \,.\, 5 \ldots\ldots m}{1 \,.\, 2 \,.\, 3 \ldots\ldots \frac{m+1}{2}} 2^{\frac{m-1}{2}} \cos x.$$

2. If m be even,

$$2^{m-1} \cos^m x = \cos m x + A \cos (m-2) x + \ldots\ldots$$

continued to $\frac{m}{2} + 1$ terms, the coefficient of the last term being half that of the $\left(\frac{m}{2}+1\right)^{th}$ term of the expanded binomial. Let z be the last term.

$$z = \tfrac{1}{2} M \cos (m-m) x = \tfrac{1}{2} M,$$

$$M = \frac{m \,.\, m-1 \,.\, m-2 \ldots (m - \frac{m}{2} + 1)}{1 \,.\, 2 \,.\, 3 \ldots\ldots\ldots\ldots \frac{m}{2}}.$$

Multiplying both numerator and denominator by $2^{\frac{m}{2}}$ in the same manner as in the last case, and introducing the deficient factors $1 \,.\, 3 \,.\, 5 \ldots m-1$, we obtain

$$M = \frac{m \,.\, m-1 \,.\, m-2 \,.\,.\, (m - \frac{m}{2} + 1)}{1 \,.\, 2 \,.\, 3 \ldots\ldots\ldots\ldots m} 2^{\frac{m}{2}} (1 \,.\, 3 \,.\, 5 \ldots m-1)$$

Expunging from the numerator and denominator the descending factors from m to $(m - \frac{m}{2} + 1)$ inclusive, we obtain

$$M = \frac{1 \,.\, 3 \,.\, 5 \ldots (m-1)}{1 \,.\, 2 \,.\, 3 \ldots\ldots \frac{m}{2}} 2^{\frac{m}{2}},$$

$$\therefore z = \frac{1 \,.\, 3 \,.\, 5 \ldots (m-1)}{1 \,.\, 2 \,.\, 3 \ldots\ldots \frac{m}{2}} 2^{\frac{m}{2}-1},$$

which is the value of the last term.

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(459.) The developement which has been thus obtained, gives the value of the m^{th} power of the cosine of an arc in a series of cosines or sines of its multiples. Similar series for the m^{th} power of the sine may be obtained in a similar way.

By expanding

$$(2 \sin x)^m (\sqrt{-1})^m = (e^{x\sqrt{-1}} - e^{-x\sqrt{-1}})^m,$$

and eliminating e by the formula,

$$\cos m x \pm \sqrt{-1} \sin m x = e^{\pm m x \sqrt{-1}}$$

we obtain

$$(2 \sin x)^m (\sqrt{-1})^m = \cos m x - A \cos (m-2) x + B \cos (m-4) x - \ldots + \sqrt{-1}$$
$$[\sin m x - A \sin (m-2) x + B \sin (m-4) x - \ldots].$$

If the series be called P_s and Q_s, we have

$$(2 \sin x)^m (-1)^{\frac{m}{2}} = P_s + \sqrt{-1}\, Q_s.$$

This formula being treated in a manner similar to that for $(2 \cos x)^m$, will give similar results.

(460.) If m be a positive integer, the number of terms in each of the series P_s and Q_s will be $m+1$, and one or other of them will $= 0$. We shall consider successively the cases in which m is even and odd.

1. Let m be even.

The number of terms in Q_s being $(m+1)$ and $\because$ odd, the sign of the last term is by the law of the series $+$, and it is therefore

$$+ \sin (m - 2m) x = - \sin m x.$$

The penultimate term is

$$- A \sin [m - 2(m-1)] x = - A \sin (-m+2) x$$
$$= + A \sin (m-2) x,$$

and by continuing the process, it appears that the extreme terms, and those equally distant from them, destroy each other. Hence $Q_s = 0$, and therefore

$$2^m (\sqrt{-1})^m \sin^m x = P_s.$$

But since m is even,

$$(-1)^{\frac{m}{2}} = \pm 1,$$

$+$ being taken when $\frac{m}{2}$ is even, and $-$ when odd. Therefore

$$\pm 2^m \sin^m x = P_s.$$

In the same manner as in the former case, it follows that in the series P_s the extreme terms, and those which are equidistant from them, are equal, and have the same sign, and hence, as before, we find

$$\pm 2^{m-1} \sin^m x = P_s,$$

the number of terms being $\frac{m}{2} + 1$, and the last term being the same as for $2^{m-1} \cos^m x$ when m is even.

2. Let m be an odd integer.

In this case the number of terms being $m+1$, the sign of the last term of P_s is by the law of the series $-$, and it is therefore

$$- \cos (m - 2m) x = - \cos m x,$$

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and the penultimate term is

$$+ \mathrm{A} \cos [m-2(m-1)] x = + \mathrm{A} \cos (m-2) x,$$

and by continuing the process, it appears that the extreme terms, and those which are equidistant from them, are equal with different signs, and therefore destroy each other. Hence $\mathrm{P}_x = 0$, and

$$2^m (\sqrt{-1})^m \sin^m x = \sqrt{-1}\, \mathrm{Q},$$

$$\therefore 2^m (\sqrt{-1})^{m-1} \sin^m x = \mathrm{Q}_x.$$

But since $m - 1$ is even,

$$(\sqrt{-1})^{m-1} = \pm 1,$$

$$\therefore \pm \mathrm{Q}^m \sin^m x = \mathrm{Q}_x,$$

$+$ being taken when $\frac{m-1}{2}$ is even, and $-$ when it is odd.

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In the same manner as before, it may be shown that the extreme terms of Q_x are equal and have the same sign. Hence we find

$$2^{m-1} \sin^m x = \mathrm{Q}_x$$

continued to $\frac{m+1}{2}$ terms, the last term being

$$z = \frac{1.3.5 \ldots . m}{1.2.3 \ldots . \frac{m+1}{2}} 2^{\frac{m-1}{2}} \sin x.$$

GEOMETRICAL ANALYSIS.

Geometrical Analysis.

Introduction.

SECTION I.

(1.) ANALYSIS, or *resolution*, is a process by which, commencing with what is *sought* as if it were *given*, a chain of relations is pursued which terminates in what is *given*, (or may be obtained,) as if it were *sought*. SYNTHESIS, or *composition*, is a process the very reverse of this; being one in which the series of relations exhibited commences with what is given, and ends with what is sought. Consequently *analysis* is the instrument of invention, and *synthesis* that of instruction.

The analysis of the ancients is distinguished from that of the moderns by being conducted without the aid of any calculus, or the use of any principles except those of Geometry, the latter being conducted entirely by the language and principles of Algebra. The ancient is, therefore, called the *Geometrical Analysis*. For its origin and history, the reader is referred to our HISTORY OF ANALYSIS.

Ancient analysis compared with the modern.

The interest which the Geometrical Analysis derives from its antiquity, and from having been the instrument by which the splendid results of the ancient Geometry were obtained, would alone be sufficient to render it an object of attention even after the discovery of the more powerful agency of Algebra. But this is not its only nor its principal claim upon our notice. Its inferiority, compared with the modern analysis, in power and facility, is balanced by its *extreme* purity and rigour; and though its value as an instrument of discovery be lost, yet it must ever be considered as a most useful exercise for the mind of a student; and it may be fairly questioned, whether it may not be more conducive to the improvement of the mental faculties than the modern analysis, unless the latter be pursued much farther than it usually is in the common course of academical education, in which the student acquires little more than a knowledge of its notation. Newton was fully aware of the advantages attending the cultivation of this branch of mathematical science, and in many parts of his works laments that the study of it has been so much abandoned. He considered, that, however inferior in power and despatch the ancient method might be, it had greatly the advantage in rigour and purity; and he feared, that by the premature and too frequent use of the modern analysis the mind would become debilitated and the taste vitiated. We must however confess, that the pretensions of the ancient method to superior *rigour* do not seem to us to be as well founded as they are sometimes considered. It would be no very difficult matter to expunge the algebraical symbols from a modern investigation, and substitute for them their meaning expressed in the language used in geometrical investigations; but would such a change confer upon them greater rigour, or would it give to the conclusions greater validity? And yet this is precisely what Newton himself has done in many parts of his great work, the *Principia*. His theorems are, evidently, investigated algebraically; but in demonstrating them, the process is disguised by the substitution of lines and geometrical figures for the algebraical species and formulæ. It cannot but excite astonishment, that a man of his extraordinary sagacity could so far deceive himself, as to suppose that by such a proceeding his reasoning acquired greater *rigour*.

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But, without reference to the modern analysis, we conceive that the ancient method has sufficient claims to our attention on the score of its own intrinsic beauty. It has this further advantage, that we can enter at once upon its most interesting discussions without the repelling task of learning any new language or system of notation.

No general rules in Geometrical Analysis.

In the application of the Geometrical Analysis to the solution of problems, or the demonstration of theorems, no general rules nor invariable directions can be given which will apply in all cases. The previous construction to be used, and the preparatory steps to be taken, depend on the particular circumstances of the question, and must be determined by the sagacity of the analyst; and his skill and taste will be evinced in the selection of the properties or affections of the given or sought quantities on which he founds his analysis; for the same question may frequently be investigated in many different ways.

Analysis of a problem.

In submitting a *problem* to analysis, its solution, in the first instance, is assumed; and from this assumption a series of consequences are drawn, until at length something is found which may be done by established principles, and which *if done* will necessarily lead to the execution of what is required in the problem. Such is the *analysis*. In the *synthesis*, then, or the *solution*, we retrace our steps; beginning by the execution of the construction indicated by the final result of the analysis, and ending with the performance of what is required in the problem, and which constituted the first step of the analysis.

Of a theorem.

When a *theorem* is submitted to analysis, the thing to be determined is, whether the statement expressed by it be true or not. In the analysis this statement is, in the first instance, assumed to be true; and a series of consequences are deduced from it until some result is obtained, which either is an established or admitted truth, or contradicts an established or admitted truth. If the final result be an established truth, the theorem proposed may be proved by retracing the steps of the investigation, commencing with that final result, and concluding with the proposed theorem. But if the final result contradict an established truth, the proposed theorem must be false, since it leads to a false conclusion.

These general observations on the nature of the Geometrical Analysis, and the methods of proceeding in it, will be more clearly apprehended after the inves-

tigations contained in the subjoined treatise have been examined.

SECTION II.

Miscellaneous Problems.

(2.) *Definition.* A *point* is said to be *given* when its situation is either given or may be determined.

(3.) *Definition.* A *right line* is said to be *given in position* when it is either actually exhibited and drawn, or may be exhibited and drawn by previously established principles.

PROPOSITION.

(4.) *To draw from a given point a right line intersecting two right lines given in position, so that the segments between the point and the right lines shall have a given ratio.*

Fig. 1. Let the given point be P, A B and C D the right lines given in position, and $m : n$ the given ratio.

Let P M : P N :: $m : n$. If any other line as P L be drawn intersecting A B and C D, and a parallel to C D be drawn from N, that parallel will divide P L similarly to P M, and therefore in the required ratio. This parallel may, or may not, coincide with the line N K. First, let us suppose that it does. In that case the two lines given in position will be parallel, and the line P L, or any other line, drawn intersecting them, will be cut similarly to P M, and therefore all such lines will be cut in the required ratio. Hence it appears, that in this case the problem is indeterminate, since every line which can be drawn intersecting the given lines will equally solve it.

Secondly, if the given lines A B, C D be not parallel, let the parallel to C D from N meet P L in O, so that P L : P O :: $m : n$. But P L may be drawn, and the point O therefore may be determined; and since the direction of C D is given, the direction of O N is determined, and therefore the point N may be found. Hence, the solution is as follows: let any line P L be drawn. If P L : P K :: $m : n$, the problem is solved. If not, let P L be cut at O, so that P L : P O :: $m : n$, and from O draw O N parallel to C D, meeting A B in N, and through N draw P N M. Then P M : P N :: P L : P O :: $m : n$.

(5.) *Cor.* 1. The same solution will apply if the line A B be a curve of any kind.

(6.) *Cor.* 2. If the parallel to C D through O do not meet the line A B, the solution is impossible. If A B be a right line, this happens when it is parallel to C D. And therefore we conclude in general, that when the two right lines A B and C D are parallel, the problem is either indeterminate or impossible.

PROPOSITION.

(7.) *From two given points to draw to the same point in a right line given in position, two lines equally inclined to it.*

Fig. 2 Let the given points be A and B, and let C D be the line given in position. Let P be the sought point, so that the angle A P C shall be equal to the angle B P D.

Produce the line B P beyond P, until P E is equal to P A, and join A E. The angles B P D and E P C are equal; but also (hyp.) B P D and A P C are also equal, therefore the angle A P C is equal to the angle E P C. But also the sides P A and P E are equal, and the side P F is common to the triangles A P F and E P F. Therefore the angles A F P and E F P are equal, and therefore are right angles, and also the A F is equal to E F.

But since A and C D are given, the perpendicular A F is given, and hence the solution of the problem may be derived.

From either of the given points A draw a perpendicular A F to the given right line C D, and produce it through F, until F E is equal to A F, and draw the right line E B meeting the line C D in P. Draw A P, and the lines A P and B P are those which are required. For since A F and F E are equal, and P F common to the triangles A F P and E F P, and the angles A F P and E F P are equal, the angles A P F and E P F are equal. But B P D and E P F are also equal, therefore the angles A P F and B P D are equal.

Scholium. If the given points lie at different sides of the given right line, the problem is solved by merely joining the points.

PROPOSITION.

(8.) *To inscribe a square in a triangle.*

Let A B C be the triangle, and D F E the required square. Draw the perpendicular B G, and draw A E to meet a parallel B H to A C at H. It is easy to see that D F : F E :: G B : B H; for the triangles A F D and A B G, A F E and A B H are respectively similar each to each. Hence, since D F is equal to F E, G B is also equal to B H. But G B is given in magnitude and position, and therefore B H is given in magnitude and position. To solve the problem therefore it is only necessary to draw B H and join A H, and the point E where A H meets B C will be the vertex of the angle of the square. Fig. 3.

(9.) *Cor.* 1. It is evident that the same analysis will solve the more general problem, "To inscribe in a triangle a rectangle given in species." For in this case the ratio B H : B G is given, and therefore B H is as before given in position and magnitude.

(10.) *Schol.* If B H be drawn equal to B G and on the same side of the vertex with A, then it will be necessary to produce A H and C B, in order to obtain their point of intersection E. In this case, however, D F E will still be a square, for the corresponding triangles will be similar, B G A to F D A, and H B A to E F A. Hence G B : B H :: D F : F E. Fig. 4.

(11.) *Cor.* 2. In the same manner the more general problem, "To inscribe a rectangle given in species," may be extended.

PROPOSITION.

(12.) *To draw a line from the vertex of a given triangle to the base, so that it will be a mean proportional between the segments.*

Let A B C be the triangle, and let B D be a mean proportional between A D and D C. Produce B D to E, so that D E shall be equal to B D, and join C E. Since Fig. 5.

A D : B D :: E D : D C,

and the angles B D A and E D C are equal, the triangles B D A and C D E are similar. Therefore the angles E and A are equal, and are in the same segment of a circle described on C B. If from the centre of this circle F D be drawn, the angle F D B will be a right angle, and the point F will therefore be in a circle described on F B as diameter. But the point F is given, since it is the centre of a circle circumscribed about the given triangle, and the line F B is therefore given, and the circle on it as diameter is given, and therefore the point D is given. The solution of the problem is therefore effected by circumscribing a circle about the given triangle, and drawing from its centre to the angle B a radius. On that radius, as diameter, describe a circle; and to a point D, where this circle meets the base, draw the line B D, and it will be a mean proportional between the segments. For the angle B D F in a semicircle is right, therefore B D = D E; and therefore the square of B D is equal to the rectangle under A D and D C.

If the circle on B F intersect A C, there will be two points in the base to which a line may be drawn, which will be a mean proportional between the segments. If this circle touch the base there will be but one such line, and it may happen that the circle may not meet the base at all, in which case the solution is impossible.

Fig. 6. If the centre F be upon the base A C, the angle A B C will be right, and the point F itself is one of the points which solve the problem; for in that case A F, B F, and C F are equal. The other point D is the foot of a perpendicular B D from the vertex on the base.

(13.) *Cor.* Hence, in a right angled triangle, the perpendicular on the hypothenuse is a mean proportional between the segments; and it is the only line which can be drawn from the right angle to the hypothenuse which is a mean, except the bisector of the hypothenuse.

Schol. It has been observed, that the solution of the problem to draw a line to the base which shall be a mean proportional between the segments is impossible when the vertical angle is acute. That this is erroneous, must be evident from the preceding analysis. For let one circle be described upon the radius of another as diameter. Let any line, as A C, be drawn not passing through F, but intersecting the inner circle; and so that the point of contact B and the centre F shall lie at the same side of it. Draw A B and C B, and also B D. It is evident that B D is a mean proportional between A D and C D, and yet the angle A B C is acute, being in a segment greater than a semicircle.

The possibility of the solution of this problem does not at all depend on the magnitude of the vertical angle. It may be obtuse, right, or acute, and may be equal in fact to any given angle, and yet the solution be possible.

Fig. 7. Let it be required to determine the conditions on which the solution is possible. If the circle on B F meet the base, the perpendicular distance of its centre from the base must be less than its radius; that is, less than half the radius of the circle which circumscribes the given triangle. From F and B draw perpendiculars F I and B H on A C, and from the centre of the lesser circle G draw the perpendicular G K. Since G F is equal to G B, G K is equal to half the sum of F I and B H. Hence it follows, that the solution will only be possible when half the sum of F I and B H is not greater than B G, or when the sum of F I and B H is not greater than B F; that is, when the sum of the perpendiculars on the base from the vertex and the centre of the circumscribed circle is not greater than the radius of that circle.

PROPOSITION.

(14.) *Right lines being drawn bisecting the internal and external angles of a triangle, and being produced to meet the base, and the production of the base to determine the conditions on which the rectangle under the sides of the triangle will be a geometric, arithmetic, or harmonic mean between the rectangle under the segments of the base by the internal bisector, and the rectangle under the segments of the base by the external bisector.*

Let A B C be the triangle, B D the bisector of the internal angle, and B E the bisector of the external angle. By the principles of Geometry we have Fig. 8.

$$AE : CE :: AB : BC,$$

also

$$AD : DC :: AB : BC.$$

Hence it follows, that the three rectangles A E × C E, A B × B C, A D × D C are similar.

1. Let the rectangle under A B and B C be a geometric mean between the other two. If three similar figures be in geometrical progression, their homologous sides must also be in geometrical progression; hence C E : C B : C D. But since the angle D B E is equal to A B D and E B F together, it is a right angle, and therefore since B C is a mean proportional between D C and C E, B C A must be a right angle, (12.) Hence the rectangle under the sides is a geometric mean, when either of the base angles is right.

2. Let the rectangle A B × B C be an arithmetic mean between the other two. In that case the rectangle A E × E C should exceed A B × B C by as much as this last exceeds A D × D C. But by Geometry the excess of A E × E C above A B × B C is the square of B E, and the excess of A B × B C above A D × B C is the square of B D. Hence in the present instance the squares of B E and B D are equal, and therefore the lines themselves are equal. Hence the angles B D C and B E C are equal, and since D B E is a right angle, B D C must be half a right angle, and therefore the difference between B D C and B D A is a right angle. But since by adding to each of the base angles B A D and B C D the equal halves of the vertical angles, we obtain sums equal to the angles B D C and B D A, it follows that the difference between the base angles B C D and B A D is a right angle. Hence when the difference of the base angles is right, the rectangle A B × B C is an arithmetic mean between the other two rectangles.

3. Let the rectangle A B × B C be an harmonic mean. In that case, by the nature of harmonic proportion, we have

$$AE \times EC : AD \times DC :: AE \times EC - AB \times BC : AB \times BC - AD \times DC;$$

that is, the first rectangle is to the third as the difference between the first and second is to the difference between the second and third. But these differences are the squares of the lines B E and B D, and therefore

we have the rectangle $AE \times EC$ to the rectangle $AD \times DC$ as the square of BE is to the square of BD. Since then the similar rectangles of which CE and CD are homologous sides, are proportional to the squares of BE and BD, these lines themselves are proportional. Therefore

$$BE : BD :: CE : CD.$$

Hence the line BC bisects the angle DBE; but since DBE is right, CBD is half a right angle, and therefore ABC is a right angle. Hence if the bisected angle be right, the rectangle $AB \times BC$ is an harmonic mean between the other two rectangles.

PROPOSITION.

(15.) *To draw a right line from the vertex of a triangle to the base, or to the base produced, so that its square shall be equal to the difference between the rectangle under the sides, and the rectangle under the segments into which it divides the base.*

Fig. 9. Let the triangle be ABC, and let the required line be BD, and let a circle be circumscribed about the triangle.

1. Let the line be drawn to the base itself, and let it be produced to meet the opposite circumference at E, and draw CE. By hypothesis, the square of BD, together with the rectangle $AD \times DC$, is equal to the rectangle $AB \times BC$. But the rectangle $AD \times DC$ is equal to the rectangle $BD \times DE$. Add to both the square of BD; and the rectangle $AD \times DC$, together with the square of BD, is equal to the rectangle $BD \times DE$, together with the square of BD. But the former rectangle and square are together equal to the rectangle $AB \times BC$, and the latter rectangle and square are together equal to the rectangle $BE \times BD$. Hence the rectangle $AB \times BC$ is equal to the rectangle $BE \times BD$. Hence we have

$$AB : BD :: EB : BC;$$

and the angles A and E are equal. Therefore in the triangles ABD and EBC the sides AB, BD are proportional to EB, BC, and the angles opposite to one pair of homologous sides BD and BC are equal, and therefore the angles opposite the other pair of homologous sides must be either equal or supplemental. If they be equal, the triangles ABD and EBC are similar, and therefore the line BD bisects the angle ABC. If the angles BDA and BCE be supplemental, the sum of the arcs which they subtend must be equal to the whole circumference. Hence the arcs BA, AE, BA, and CE are together equal to the circumference. But BA, AE, BC, and CE are also together equal to the circumference. Take away from both the arcs BA, AE, and CE, and the remaining arcs BC and BA are equal; and therefore their chords are equal, and therefore the triangle is isosceles.

Hence we infer, that "if a line be drawn from the vertex of a triangle to the base, so that its square, together with the rectangle under the segments, shall be equal to the rectangle under the sides, that line will bisect the vertical angle, except when the triangle is isosceles, in which case any line drawn from the vertex to the base will have the required property."

Fig. 10. 2. Let the line BD meet the base produced. By hypothesis, the rectangle $AD \times DC$ is equal to the rectangle $AB \times BC$, together with the square of BD. But the rectangle $AD \times DC$ is equal to the rectangle $ED \times BD$, which is equal to the rectangle $EB \times BD$, together with the square of BD. From these equals take away the square of BD and the remainders, the rectangles $EB \times BD$ and $AB \times BC$ are equal. Hence

$$EB : BC :: AB : BD.$$

Draw CE, and the angles E and A are equal. Hence in the triangles EBC and ABD there are two sides EB and BC proportional to two AB, BD, and the angles opposite one pair of homologous sides equal, and therefore the angles opposite to the other homologous sides must be either equal or supplemental. If they be equal, take ABC from both, and the remainders EBA and CBD are equal; but EBA and FBD are also equal, and therefore BD bisects the external angle CBF of the given triangle.

If the angles ABD and EBC be supplemental. Since the angles ABD and FBD are also supplemental, we should have the angles FBD and EBC equal; and therefore EBA and EBC equal; and therefore the point B cannot in this case lie between E and D. It must therefore be placed as in fig. 11. Here the square of BD is manifestly greater than the rectangle $CD \times DA$, and therefore the proposed condition must be that the rectangle $CD \times DA$, together with the rectangle $AB \times BC$, is equal to the square of BD. But the rectangle $CD \times DA$ is equal to the rectangle $BD \times DE$; and taking these equals from the former, the remainders, *viz.* the rectangles $AB \times BC$ and $BD \times BE$ are equal. Hence Fig. 11.

$$EB : BC :: AB : BD.$$

Draw CE, and in the triangles EBC and ABD the two sides EB, BC are proportional to two AB, BD, and the angles BEC and BAD opposite to one pair of homologous sides are supplemental, (for BAC and BEC are equal,) and therefore the angles BCE and D opposite the other pair of homologous sides are equal. Hence the difference of the arcs subtended by D is equal to the arc subtended by BCE, that is, the difference between the arcs BC and AE is equal to the arc BE; or the arcs BE and AE together, that is, the arc AEB is equal to the arc AB, and therefore their chords are equal, but their chords are the sides AB, BC of the triangle, which is therefore isosceles.

Hence it follows, that "if a line be drawn from the vertex of a triangle to the produced base, so that its square, together with the rectangle under the sides, shall equal the rectangle under the segments of the base, that line will bisect the vertical angle, except when the given triangle is isosceles, in which case there is no line which has the required property. In this case, however, the square of every line drawn from the vertex to the produced base is equal to the sum of the rectangles under the sides and segments."

SECTION III.

Of the Contact of Right Lines and Circles.

(16.) PROBLEMS of contact of right lines and circles furnished the ancients with an extensive subject for the exercise of the Geometrical Analysis. In general

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three conditions are necessary to determine a circle. In the class of problems to which we allude, one at least of these conditions is, that it should touch a given right line or a given circle. The other data may be, that it should pass through one or two given points, or that it should have a given radius or centre, or that the locus of its centre should be a given right line or circle. It would not be easy to enumerate all the problems of this class; but by combining the following data for the determination of a circle, a considerable number of them may be found.

To describe a circle

1. Passing through a given point
2. Passing through two given points.
3. Passing through three given points.
4. Touching a given right line.
5. Touching two given right lines.
6. Touching three given right lines.
7. Touching a given circle.
8. Touching two given circles.
9. Touching three given circles.
10. Having a radius given in magnitude.
11. Having its centre on a given right line.
12. Having its centre on a given circle.
13. Having a given centre.

Every combination of three which can be formed from these data, may be taken as the limiting circumstances in problems for the determination of a circle. In the invention of such problems it should however be observed, that 2, 5, 8, and 13 are each to be counted as two data, and 3, 6, 9 are each to be counted as three data. Each of the latter is, therefore, itself sufficient to determine the circle, but each of the former ought to be combined with some one of the data 1, 4, 7, 10, 11, 12.

We cannot here enter at large on this class of problems, we shall therefore confine ourselves to a few examples.

PROPOSITION.

(17.) *To describe a circle passing through two given points, and touching a right line given in position.*

If the given points be at different sides of the given line, the solution is manifestly impossible.

Fig. 12. Let them then be A, B at the same side of the given right line C D. Let the required circle be A B C, and let A B be produced to meet the right line at D.

The square of C D is equal to the rectangle A D × D B. But this rectangle is given, therefore the square of C D is given, and therefore C D itself is given in magnitude and position, and hence the point C is given. But also the points A, B being given, the circle through these points A, B, C is given.

The solution, therefore, is effected by producing A B to D, and taking D C equal to a mean proportional between A D and D B, and then describing a circle through A, B, C.

But it may happen, that the line A B is parallel to C D, and will not meet it when produced.

Fig. 13. In this case draw A C and B C. The angle B C D is equal to the angle A in the alternate segment, and also equal to the alternate angle B. Hence the angles A and B are equal, and therefore the sides A C and B C are equal. Draw C E perpendicular to A B, and A E and B E are equal. The point E is, therefore, given, and the perpendicular E C is given in position, and therefore the point C is given.

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To solve the problem in this case therefore, bisect A B at E, and draw the perpendicular through E, intersecting C D in C. A circle passing through A, B, C will be that which is required.

PROPOSITION.

(18.) *To describe a circle passing through a given point, and touching two right lines given in position.*

1. Let the given right lines be parallel. In this case it is necessary that the point should be between them, for otherwise the solution would be impossible.

Let the lines be A B, C D, and the point be P. Let A P C be the required circle, and draw A P and the diameter A C. Through P draw P P′ parallel to the given right lines, and describe any circle B P′ D, touching the right lines at B, D, and intersecting the parallel at P′, and draw P′ B. Since the circle B P′ D may be drawn, the point P′ is given, and therefore the line P′ B is given in magnitude and position. But the triangles B P′ D and A P C are similar, and since B D and A C are parallel, B P′ and A P are parallel. Therefore the line P A is given in direction, and since the point P is given, it is also given in position. Hence the given points A and C are given, and therefore the circle A P C is given. Fig. 14

To solve the problem therefore, describe any circle touching the two lines, and draw the parallel through P to meet it at P′. From P′ draw P′ B, and draw P A parallel to it. Draw A C perpendicular to A B, and it will be the diameter of the required circle.

2. Let the given lines A B, C D intersect at E.

As before, describe any circle B P′ D touching the right lines, and from E draw E P intersecting this circle at P′. Draw the radii G A, G P, F B, and F P′. Since G A is parallel to F B, we have Fig. 15.

$$G A : F B :: G E : F E.$$

Therefore $G P : F P' :: G E : F E.$

Therefore $G P : G E :: F P' : F E.$

Hence the lines G P and F P′ are parallel. But F P′ is given in position, and therefore G P is given in direction, but P is given, and therefore G P is given in position. But the line E G bisects the angle A E C under the given lines, and is therefore given in position, and therefore the point G where it intersects P G is given. Hence the centre G and the radius G P of the required circle are given, and therefore the circle itself is given.

To solve the problem, draw E P, and also E G, bisecting the angle E. Describe any circle B P′ D touching the given right lines, and draw P′ F. Through P draw P G parallel to P′ F, meeting the bisector E G in G. With G as centre and G P as radius, let a circle be described. This circle will touch the right lines. The demonstration is obvious.

It is evident, that in each of the preceding cases there may be two circles drawn, which will solve the problem. This circumstance arises from the line P P′ meeting the circle B P′ D in two points. The principle used in the solution of both cases is the same. The parallel in the first case corresponds to the bisector of the angle in the second.

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PROPOSITION.

(19.) *To describe a circle passing through two given points, and touching a given circle.*

Fig. 16 and 17. Let A and B be the given points, and let C be the centre, and C D the radius of the given circle. Let D be the point of contact sought. Draw A D E, B D F, and F E. Also, let a tangent F G at F be drawn, and from B draw B C I.

By the properties of the circle it appears that A B and F E are parallel, and therefore the angles A and E are equal. But also the angle E is equal to the angle G F B, and therefore G F B is equal to the angle A, and therefore the triangles ABD and GFB are similar. Hence we have

A B : B D : : F B : B G.

Therefore the rectangle A B × B G is equal to the rectangle B D × B F. But also the rectangle B D × B F is equal to the rectangle B I × B H. Hence the rectangle A B × B G is equal to the rectangle B I × B H. But since the point B and the circle C are given, the rectangle B I × B H is given, and therefore the rectangle A B × B G is given in magnitude.

But one side A B is given, and therefore also the other side B G is given, hence the point G is given. Hence the line G B is given in magnitude and position, and the point of contact D where it intersects the given circle is given. The circle through this point D, and the given points A, B is therefore given.

The problem is therefore solved by taking B G, a fourth proportion to A B, B I, and B H; and from G drawing the tangent G F, and from F the point of contact drawing the line FB. The point D where this line intersects the given circle is the point where the sought circle through A, B touches it.

SECTION IV.

Trisection of the Angle.—Investigation of Two Mean Proportionals.—Delian Problem.

PROPOSITION.

(20.) *To trisect a given angle.*

Trisection of an angle. Fig. 18. Let A B C be the given angle, and from any point A in one leg draw a perpendicular A C to the other, and from the same point A draw a parallel A D to the other leg B C. Let B D be the line which cuts off the angle C B D one third of the given angle A B C. Hence the angle A D B, which is equal to D B C, is one third of the given angle A B C, and the angle A B D is two thirds of A B C, and therefore is double the angle A D B

Draw A E, making E A D equal to E D A, and therefore A E is equal to D E, and the angle A E B is equal to twice the angle A D B. Hence the angle A E B is equal to the angle A B E, and A B is equal to A E. But also since A F E together with A D E is equal to a right angle, and also F A D is a right angle; if from these equals the equal angles F D A and D A E be taken, the remaining angles F A E and A F E will be equal, and therefore A E is equal to E F, and therefore F D is equal to twice A E, or to twice A B. But A B is given, and therefore D F is given in magnitude.

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The problem to trisect an angle is therefore reduced to the inflection of a line of given magnitude between the legs of a right angle, and passing through a given point. This is a problem not capable of solution by the right line and circle.

The condition may also be reduced to the inflection of a right line from a given point in the circumference of a circle, so that the part intercepted between the circle and a diameter produced passing through another point shall have a given magnitude. For if with the centre A and the radius A B or A E a circle be described, it will be sufficient if from B a line B D be inflected on A D, so that the external part D E shall be equal to the radius. This condition is, in effect, the same as the former.

PROPOSITION.

(21.) *To trisect a given ratio, or to find two continued mean proportionals between two lines.*

Trisection of a ratio. This, like the last, is a problem the solution of which surpasses the powers of Plane Geometry. We can, however, investigate the conditions on which its solution depends.

Fig. 19. Let the terms of the ratio, expressed by lines, be placed at right angles, and the rectangle A C B D completed, let C E and B F on the produced sides of this rectangle be the two means, so that

A C : C E : B F : A B.

By the similar triangles formed by the sides of the rectangle we have

F D : D E : : A C : C E,

therefore F D : D E : : C E : B F.

Hence the rectangle F D × B F is equal to D E × C E. Let a circle be circumscribed round the rectangle intersecting F E in G. The rectangle D F × F B is equal to the rectangle A F × F G, and the rectangle D E × C E is equal to the rectangle E A × G E. But the rectangles D F × F B and D E × C E have been proved equal, and therefore the rectangles A F × F G and G E × A E are also equal. But G A the difference of the sides of these rectangles is common, and therefore the sides are respectively equal, *viz.* G E is equal to A F, and F G is equal to A E.

Hence it follows that two mean proportionals will be found, if through the point A a line can be drawn, so that the parts F G and A E intercepted between the circle and the produced sides of the rectangle be equal.

The same problem leads also to a different condition.

Fig. 20 Let the former construction remain, and on B D construct an isosceles triangle whose side K B or K D is equal to half of D C. Bisect D C at N, and draw K F. The square of K F is equal to the square of K B, together with the rectangle D F × F B. But also the square of N E is equal to the rectangle D E × E C, together with the square of N C. Since N C is equal to K B, (*Constr.*) and the rectangle D E × C E has been already proved to be equal to the rectangle D F × F B, it follows that the square of N E is equal

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to the square of K F, and therefore these lines themselves are equal. Since

$$DE : CE :: DF : DB,$$

therefore

$$DE + CE : DC :: DF + DB : BF.$$

But D E + C E is equal to twice N E, or to twice K F; and if D L be produced equal to B D, D F + D B is equal to L F, and D C is equal to twice K B. Hence the preceding proportion becomes

$$2KF : 2KB :: LF : BF.$$

Draw L K and B M parallel to it through B. Hence we have

$$KF : MF :: LF : BF,$$

or

$$2KF : 2MF :: LF : BF.$$

Therefore twice M F is equal to twice K B, and therefore M F is equal to K B, and therefore to half of D C.

Hence it follows that the insertion of two means between A B and A C depends on the inflection of a line across the sides of the angle F B M, so that it shall pass through the given point K; and the part M F intercepted by the sides of the angle shall be of a given magnitude, *viz.* equal to half of A B, one of the given lines.

This condition is similar to that required for the trisection of an angle, so that if one of these problems could be solved the other would also be solved.

Duplication of the cube. The insertion of two mean proportionals is necessary to solve the celebrated problem of "the duplication of the cube," or to find a cube which doubles a given cube. The general proposition, of which this is a particular case, is to construct a solid of a given species, and bearing a given ratio to a given solid of that species. This problem is thus solved. Find a line to which any edge of the given solid has the given ratio. Between this line and that edge find two mean proportionals, and with the first of these means as an edge construct a solid similar to the given one. This will be that which is required. For similar solids are in the triplicate ratio of their homologous edges; and therefore the given solid is to the constructed one as its edge is to the fourth continued proportional, that is, in the given ratio.

Thus on this principle depends the change of the scale of solids in any required proportion.

The problem of the "duplication of the cube" is called the Delian problem. See History of Geometry; also History of Analysis.

SECTION V.

Geometric Loci.

Geometric loci. (22.) When a point is required to be determined in a problem with data which are insufficient for its solution, the problem is said to be indeterminate, because the position of the point cannot be found from it. But although the position cannot be absolutely determined, yet it may be so restricted by the conditions which are prescribed in the problem, that it may be known to be on some line, the nature of which may frequently be determined. This line is called the *locus* of the point.* This will easily be understood by the following examples: suppose that the base of a triangle were given in magnitude and position, and that its area were given in magnitude, to determine its vertex. In this case, it is evident, that the problem is indeterminate, since innumerable triangles may be constructed on each side of the given base having equal areas. But since the area is equal to the rectangle under the perpendicular and half the base, it follows that the perpendiculars from the vertices of all these triangles on the base must be equal, and therefore these vertices must all lie on parallels to the base at such a perpendicular distance that the rectangle under it, and half the base shall be equal to the given magnitude.

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The *locus* of the vertex is therefore in this case two right lines parallel to the base, and at equal perpendicular distances at opposite sides of it.

If the base of a triangle be given in magnitude and position, and the vertical angle be given in magnitude, to determine the vertex, the problem is evidently indeterminate; for an unlimited number of different triangles may be constructed on the same base whose vertical angles are equal. But the vertices of all the triangles on the same side of the base will in this case be placed on the arc of a circle containing an angle equal to the given angle. Hence the *locus* will be two segments of circles containing an angle equal to the given angle, and constructed on opposite sides of the given base.

The investigation of *loci* is of very extensive use in the solution of determinate problems. In cases where the determination of a point is required from certain data, by omitting any one of the data the point will have a *locus* which may be found by the remaining data. This being successively applied to two of the data, two loci will be found, the intersection of which will determine the point.

This may be illustrated by the examples already given. Let the base of a triangle be given in magnitude and position, and the area and vertical angle in magnitude, to determine the vertex. If we omit the vertical angle, the locus is the parallels already described. If we omit the area, the locus is the segments of the circle. The vertex being then at the same time on both loci must be at the intersection of the two loci, and will therefore be at the points where the parallels meet the circle. In general there will be in the present case four such points, and consequently four triangles, but these triangles will differ only in position, being equal as to their sides and angles.

The following propositions will illustrate the theory of Geometric loci.

Proposition.

(23.) *Given in magnitude and position the base of a triangle, and the difference of the squares of its sides, to find the locus of the vertex.*

Let A B be the given base, and C be a point of the sought locus. Draw A C, B C, and from C draw the perpendicular C D. The difference of the squares of the sides A C, B C is equal to the difference of the squares of the segments A D, D B, which is therefore given. The points at which the perpendicular meets the base are therefore given, and therefore the perpen- Fig. 21

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dicular itself is given in position; and since the vertex must be on the perpendicular, the locus is determined. To construct the locus, it is therefore only necessary to cut the base at D, so that the difference of the squares of the segments shall be equal to the given difference of the squares of the sides, and the perpendicular C D through the point of section will be the locus sought. It is evident that there are in general four points D at which the line may be cut as required, two on the line itself and two in its production, and that these points are respectively equally distant from the middle point.

Proposition.

(24.) *Given the base of a triangle in magnitude and position, and the sum of the squares of the sides, to find the locus of the vertex.*

Fig. 22 Let the base be A B, and let C be any point of the locus. Draw C D to the middle point of the base, and draw C A and C B. The sum of the squares of C A and C B is equal to twice the sum of the squares of C B and D B. But the sum of the squares of C A and C B is given, and therefore also twice the sum of the squares of C D and D B is given, and therefore the sum of the squares of C D and D B is given. But the square of D B (half the given base A B) is given; therefore the square of C D and C D itself are given. The point C, whose locus is sought, is therefore at a given distance from the middle point D of the base, and its locus is therefore a circle whose centre is the middle point of the base, and whose radius is the given distance. This radius is evidently a line whose square is half the difference between the given sum of the squares of the sides, and double the square of half the base.

This proposition is only a particular case of the following more general one: "Any number of points being given, to find the locus of a point such that the sum of the squares of its distances from the several given points shall be given." * If the given and sought points be in the same plane, the locus will be a circle; but if they be not limited to the same plane, the locus will be the surface of a sphere. In this case the centre of the sphere will be the centre of gravity of equal masses placed at the several points, or that point which is mathematically denominated the centre of mean distances.

Proposition.

(25.) *Given in magnitude and position the base of a triangle, and the ratio of the sides, to determine the locus of the vertex.*

Let A B be the base of the given triangle, and let C be a point of the sought locus, and let the given ratio be $m : n$. Draw A C, B C. Also draw C D, C D′, bisecting the internal and external angles at C. Hence

$$AD : DB :: AC : BC :: m : n$$

$$AD' : D'B :: AC : BC : \ m : n.$$

The ratio of the segments into which the line A B is cut at D and D′ is therefore given, and therefore the points D and D′ are given. The bisectors C D and C D′ form a right angle at C, and therefore the point C must be placed upon a circle whose diameter is D D′, and therefore this circle is the locus of the vertex of the triangle sought. Section V.

As there are two points D at which the line may be divided in the given ratio, and as it may be produced through either end, the locus, strictly speaking, is two circles.

Proposition.

(26.) *Given the base of a triangle, the sum of the squares of the sides and the vertical angle, to construct the triangle.*

If the base and the sum of the squares of the sides be given, the locus of the vertex is found by (24,) and if the base and vertical angle be given, the locus of the vertex is found by (22.) The intersection of these loci will determine the vertex.

It may happen, that the loci do not intersect. In this case the solution is impossible, and the data are inconsistent.

It may also happen, that the two loci are identical, in which case the problem is indeterminate, and the data are not distinct. This happens in the present instance, when the sum of the squares of the sides is equal to the square of the base, and the vertical angle is right. Either of these data follows necessarily from the other, and the two loci are the same circle.

(27.) These observations, however, apply to all determinate problems solved by two loci, *viz.* when the loci do not meet, the problem is impossible, and the data contradictory; and when they become identical, the problem is indeterminate, and the data not independent.

Proposition.

(28.) *Given the base of a triangle, the ratio of the sides, and the difference of their squares, to determine the triangle.*

This problem is solved by the intersection of the loci determined in (25) and (23), and is subject to the observations in (26.)

Proposition.

(29.) *A circle is given in magnitude and position, and a chord passes through a given point, to find the locus of the intersection of tangents through the extremities of the chord.*

Let C B A be the circle, P the given point, A B any chord through it, and D the corresponding point of the locus. Draw C D, which will evidently bisect B A at right angles, and we have by the known properties of the circle C E : C F : C D. Hence the rectangle D C × C E is equal to the square of the radius C F. Draw D G perpendicular to C P produced, and the angles G and E being right, the quadrilateral D E P G may be circumscribed by a circle; therefore the rectangle D C × C E is equal to the rectangle G C × C P, and therefore the rectangle G C × C P is equal to the square of the radius. Hence the point G is independent of the point D, and a perpendicular from any point of the locus will meet C P produced at the same point D. Hence to construct the locus, find a third proportional Fig. 24 and 25.

4 O 2

* See Lardner's *Algebraic Geometry*, p. 113.

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to C P and the radius, and take C G equal to this third proportional, and through G draw a perpendicular to C G. This perpendicular will be the locus sought.

The nearer the given point P is to the centre, the more remote will be the locus G D, and when P coincides with the centre, C G will become infinite, so that in this case the locus may be considered a right line at an infinite distance.

There will be no difficulty in establishing the converse of this principle, *viz.* "if tangents be drawn from each point in a given right line to a given circle, the chords joining the points of contact will all pass through a certain given point."*

SECTION VI.

Porisms.

(30.) The term *porism* † has been variously defined by Geometers. *Pappus* states, that Euclid wrote three books on porisms, (which have been lost,) but is so obscure and indistinct on the subject, that it is impossible merely from what he has stated to determine to what species of Geometrical proposition the Ancients applied this term.‡ It is certain, that it was sometimes used synonimously with *corollary*; thus Euclid, in his *Elements*, calls the corollaries of his propositions *πορισματα*. In an elaborate dissertation on the subject of *porisms*, in the *Transactions of the Royal Society of Edinburgh*, Playfair has, however, succeeded in giving the word a meaning more worthy of the importance which is evidently attached to this class of propositions. The porisms of Euclid are said to be "*collectio artificiosissima multorum rerum quæ spectant ad analysin difficiliorum et generalium problematum.*"

According to Playfair, a porism is "a problem in which the data are so related to each other that it becomes indeterminate, and admits of numberless solutions."

It is easily conceived that a problem which in general is determinate will, when its data are submitted to certain conditions, become indeterminate. In such cases it becomes a *porism*; and it may be proposed in a porism to determine what condition or restriction will render a determinate problem indeterminate.

Thus, if it be required to draw a right line through a given point, subject to some given condition, the problem may be in general determinate; and it may be possible to draw but one such right line. But, on the other hand, such a position may be selected for the given point, as that every line passing through it will fulfil the given condition. When this position is assigned to the point, the problem becomes a porism. The following examples will render these observations more intelligible.

* A numerous collection of Local problems will be seen in Lardner's *Algebraic Geometry*. The solutions there given are, however, by the Algebraical Analysis.

† From *πορίζω*, *I establish*; or, according to some, from *πόρος*, *a transition*.

‡ Pappus defines a porism to be something between a theorem and problem, or that in which something is proposed to be investigated. Simson follows Pappus, and says, that a porism is a theorem or problem in which it is proposed to investigate or demonstrate something.

Section VI

PROPOSITION.

(31.) *To draw a line passing through a given point, and crossing a given triangle, in such a manner that the sum of the perpendiculars on it from the two vertices on one side of it shall be equal to the perpendicular on it from the other vertex placed on the other side of it.*

Let D be the given point, and A B C the given triangle, and let D E be the required line, so that A E and B G taken together are equal to C F. Draw C H from C to the middle point H of A B, and draw H K perpendicular to D E. Fig 26.

In the trepezium A E G B, the parallels A E, H K, and B G are in arithmetical progression; therefore the sum of A E and B G is equal to twice H K; but this sum is also equal to C F. Therefore C F is equal to twice H K. Since C F and H K are parallel, the triangles H L K and C F L are similar, and therefore

$$CF : HK :: CL : LH.$$

But C F is equal to twice H K, and therefore C L is equal to twice L H, or L H is one third of C H. Since C H is given in magnitude and position, the point L is given. Hence the problem is solved by drawing a line from any angle C of the triangle, bisecting the opposite side A B, and taking on this one third of it H L. The line drawn from the given point D through the point L will be that which is required.

If the given point happen to be the point L itself, any line whatever passing through it will have the proposed property, and hence we have the following porism: "A triangle being given in position, a point may be determined, such that any line being drawn through it, the sum of the perpendiculars from two angles of the triangle placed on one side of it, shall be equal to the perpendicular from the remaining angle and the other side."

The point L is evidently the centre of gravity of equal masses placed at the three vertices, or, considered mathematically, it is the centre of the mean distances of the three points A B C.

This porism is only a particular case of a much more general one; "any number of points being given in the same plane, a point may be found through which any line whatever being drawn, it will pass amongst the points in such a manner, that if perpendiculars be drawn from them upon the line the sum of the perpendiculars at the one side will be equal to the sum of the perpendiculars on the other side." In this case, as in the former, the sought point is the centre of mean distances.

The same porism may receive another modification which generalizes it further. "Any number of points being given in the same plane, to determine the condition under which a right line may be drawn amongst them, so that the sum of the perpendiculars from the points on one side shall exceed the sum of the perpendiculars from the points on the other side by a given line."*

In this case, it may be proved that the line must be a tangent to a circle, whose centre is the centre of mean distances, and whose radius is equal to the given line divided by the number of given points.

If the given points be not in the same plane, the

* See Lardner's *Algebraic Geometry*, p. 34.

porism may be made still more general: "Given any number of points in space, to determine a plane passing among them, so that the sum of the perpendiculars from the points on one side shall exceed the sum of the perpendiculars from the points on the other side by a given line."

In this case the plane must touch a sphere whose centre is the centre of mean distances, and whose radius is the given line divided by the number of points.

If the sum of the perpendiculars on one side be equal to those on the other, the given line and the radius of the sphere vanish, and the sphere is reduced to its centre, *i. e.* the centre of mean distances. Hence, "if a plane be drawn through the centre of mean distances, the sum of the perpendicular from the points on the one side is equal to the sum of the perpendiculars from the points on the other side."

Proposition.

(32.) *A circle and a straight line being given in position, a point may be found such that any right line drawn from it to the given line shall be a mean proportional between the parts of the same line, intercepted between the given right line and the circumference of the given circle.*

Let A B be the given right line, H K F the given circle, and D the sought point. Draw G D I perpendicular to A B through D, and also any other line C D F Also join C I and draw H K.

Fig. 27 The square of C D is equal to the rectangle C E × C F; but it is also equal to the squares of C G and G D, and the rectangle C E × C F is equal to the rectangle C K × C I. Hence the rectangle C K × C I is equal to the sum of the squares of C G and G D. The square of G D is equal to the rectangle G H × G I; therefore the rectangle G H × G I, together with the square of C G, is equal to the rectangle C K × C I. Also the square of C I is equal to the sum of the squares of C G and G I. But the square of C I is equal to the rectangle C K × C I, together with C I × K I, and the sum of the squares of C G and G I is equal to the square of C G, together with the rectangles G H × G I and G I × H I. Taking away from these equals the rectangle C K × C I, and its equivalent the rectangle G H × G I, together with the square of G C the remainders, the rectangles C I × I K and G I × I H are equal. Hence, we have

G I : I C :: I K : I H.

Therefore, in the triangles C I G and H I K the angle I is common, and the sides which include it are proportional, and therefore the triangles are similar; but G is a right angle, and therefore H K I is a right angle, and therefore H I is a diameter. Since, then, H I passes through the centre of the given circle, and is perpendicular to A B, the given right line, it is given in position. Also G H and G I are given in magnitude, and therefore G D, which is a mean proportional between them, is given in magnitude, and therefore the point D is given in position.

(33.) There is between local theorems and porisms a close analogy. In fact, every local theorem may be converted into a porism; but, on the contrary, every porism cannot be converted into a local theorem. In local propositions the indeterminate is always a point, the position of which is restricted, but not absolutely fixed by the given conditions. Such may always be expressed as a porism. But this class of propositions is more general than geometric loci; the indeterminate may be a *line*, the direction of which is not restricted by the conditions, but which is otherwise limited, as, for example, to pass through a given point, or to touch a given circle. It may also be a plane similarly restricted to pass through a given point, or to touch a given sphere. Instances of these have been given in (31.)

Porisms, in common with geometric loci, take their rise from the conditions of a problem becoming indeterminate. This may happen in two ways. The number of conditions may not be sufficient, or among the given conditions there may exist some particular relation, by which some one or more of them may be deduced from the others. Thus, for the determination of a triangle three conditions are necessary, and such a problem becomes manifestly indeterminate if only two conditions be given. But even though three be given, the problem will still be indeterminate, if any one of the three can be inferred from the other two. For example, suppose the base of a triangle, the point where the perpendicular intersects it, and the difference of the squares of the sides be given, the problem to determine the triangle is indeterminate, because the difference of the squares of the sides is equal to the difference of the squares of the segments of the base, and may, therefore, be inferred from the base and the point of section.

The geometrical circumstances by which determinate problems in Geometry are converted into porismatic and local problems, are precisely similar to those under which the solution of an algebraical question becomes indeterminate. In such a question there should be as many equations as unknown quantities, and the problem is indeterminate evidently if there be less. But it may also be indeterminate, even if the number of equations be equal to that of the unknown quantities, and will be so when any one of the equations can be deduced from the others. It may in general be observed, both in geometrical and algebraical problems, that the number of *independent conditions* should be equal to the number of quantities sought, and should neither be more nor less. If they be more, the results may be inconsistent, and if they be less, the solution will be indeterminate.

THEORY OF NUMBERS.

The Theory of Numbers is a branch of Analysis by which we investigate the properties, dependencies, and relations of integral numbers, as by Geometry we inquire into the dimensions, position, and relations of lines; and as in the latter science a combination of lines, or a certain disposition of them, receives particular denominations, so in this branch of Analysis, numbers are distinguished into classes, according to the nature and dependence of the integral parts of which they are composed. It will be convenient, therefore, to proceed in this case, as in the other, by definitions and propositions.

I. *Introduction, showing the forms, properties, and relations of simple Integral Numbers.*

Definitions.

1. An *integer*, or *integral number*, is an unit, or any number of units.

2. The *factors* of a number, are those numbers by the multiplication of which the former number is produced; and the number thus formed, is called the *product* of those factors.

3. The *multiple* of a number is the product of that number by some integral factor.

4. *Even numbers* are those which can be divided into two equal parts; and *uneven*, or *odd*, numbers are those which cannot be so divided.

5. A *composite number* is any number produced by the multiplication of integral factors.

6. A *prime number* is that which cannot be produced by the multiplication of any integral factors, or that cannot be divided into any equal integral parts greater than unity.

7. *Commensurable numbers* are any two or more numbers having a common integral divisor; and *incommensurable numbers* are those which have not a common divisor. The latter numbers are also said to be *prime to each other.*

8. A *square*, or second power, is the product of two equal factors. A *cube*, or third power, the product of three equal factors; and, generally, the n^{th} power of a number is the continued product of n equal integral factors; and the number from the multiplication of which any power is produced, is called the *root of that power.*

9. The *forms of numbers*, or *formulæ*, are certain algebraical expressions under which those numbers are contained.

Thus, every even number is of the form $2n$, and every odd number of the form $2n \pm 1$; because an even number may be divided by 2, and will produce an integral quotient which may be represented by n, and, consequently, the number itself by $2n$; and an even number increased or diminished by unity is an odd number; therefore all odd numbers may be expressed by, or are of the form, $2n \pm 1$.

In a similar manner, numbers may be classed according to any other measure or modulus, as $4n \pm 1$, $6n \pm 1$, &c.

10. *Numbers of the same form* with respect to any modulus, are all those which can be represented by the same formula. Thus, 13, 17, 21, &c. are all of the form $4n+1$; and 19, 25, 31, &c. of the form $6n+1$; 4 and 6 being the moduli.

The forms and relations of Integral Numbers, and of their sums, differences, and products.

1. The sum or difference of any two even numbers is an even number.

For let $A = 2n$ and $B = 2n$ be any two even numbers: then

$$A \pm B = 2n \pm 2n' = 2(n \pm n') = 2n'',$$

which being of the form $2n$ is an even number.

2. The sum or difference of two odd numbers is even, but the sum of three odd numbers is odd.

Let $A = 2n+1$, $B = 2n'+1$, and $C = 2n''+1$ be three odd numbers: then

$$A + B = 2n + 2n' + 2 = 2n'',$$

and $A + B + C = 2n + 2n' + 2n'' + 3 = 2n''' + 1$;

the former being the form of an even, and the latter of an odd number.

In a similar way it may be shown:

(1.) That the sum of any number of even numbers is even.

(2.) That any even number of odd numbers is even, but that any odd number of odd numbers is an odd number.

(3.) That the sum of an even and odd number is an odd number.

(4.) That the product of any number of factors, one of which is even, will be an even number, but the product of any number of odd numbers is odd; and hence, again,

(5.) Every power of an even number is even, and every power of an odd number is an odd number.

(6.) Hence the sum and difference of any power and its root is an even number.

For the power and root will be either both even or both odd, and the sum or difference in either case is an even number.

3. If an odd number divide an even number, it will also divide the half of it.

Let $A = 2n$, $B = 2n'+1$ be any even and odd number, such that B is a divisor of A; let the division be made, and call the quotient p, then we have

$$2n = p(2n'+1),$$

consequently p is even, or of the form $2n''$, hence

$$2n = 2n''(2n'+1),$$

and $$\frac{n}{2n'+1} = n'',$$

that is, $n = \frac{1}{2}A$ is divisible by B, if A itself be so.

Theory of Numbers.

4. If a number p divide each of two numbers a and b, it will divide their sum and difference, or the sum and difference of any multiples of them.

Let $$\frac{a}{p} = q \text{ and } \frac{b}{p} = q', \text{ then}$$
$$\frac{a \pm b}{p} = q + q' = q'',$$

which is an integer, because q and q' are both integers.

In like manner, na, mb being multiples of a and b, we have

$$\frac{na + mb}{p} = nq + mq' \text{ an integer.}$$

Deductions.

It follows from the preceding propositions:

(1.) That if a number divide the whole of another number, and a part of it, it will also divide the other part.

(2.) It follows, also, that if a number consist of many parts, and each of these parts be divisible by another number, that the whole number, or the parts taken collectively, will be divisible by the same number.

5. If a and b be any two numbers prime to each other, their sum $a + b$ is prime to each of them.

For if $(a + b)$ and a had a common divisor, their difference $(a + b) - a = b$ would have the same divisor; that is, a and b would have a common measure, which is contrary to the supposition; and, in the same way, it may be shown that $a + b$, and b, cannot have a common measure.

Deductions.

(1.) In like manner, it may be demonstrated, that if a and b be prime to each other, their difference $a - b$ will also be prime to each of them, if $a - b > 1$.

(2.) Conversely, if a number consist of two parts, and be prime to one of those parts, it will be prime to the other.

(3.) And if a number consist of many parts, and each of those parts but one be divisible by another number p, then the whole number taken collectively is not divisible by p.

6. If a and b be two numbers prime to each other, their sum and difference will be prime to each other, or they can have only the common measure 2.

For if $a + b$ and $a - b$ have a common measure, their sum and difference $2a$ and $2b$ will have the same; but a and b are prime to each other, therefore $2a$ and $2b$ can only have the common measure 2; therefore $a + b$ and $a - b$ can only have the common measure 2; and if one of these numbers a or b be even and the other odd, then $a + b$ and $a - b$ are both odd; in this case, therefore, they are prime to each other; but if a and b are both odd, then their sum and difference will have the common measure 2, but no other

7. If a and p be any numbers prime to each other, a being the greater, then may a be always represented by the formula $a = np + r$, in which r shall be less than p and prime to it.

Let a be divided by p, and give a quotient n, and remainder r, which makes

$$a = np + r,$$

Introduction.

where r is obviously less than p, n being supposed the greatest quotient.

And r is prime to p; because if p and r had a common measure np and r, as also $np + r$, and r would have the same common measure, but $a = np + r$; therefore a and p would have the same, which is contrary to the supposition, these being prime to each other.

The same expression may be employed if a be less than p, but in this case $n = 0$ and $a = r$.

8. The same conditions being made with respect to a and p, it is always possible to express a by the formula

$$a = np \pm r,$$

in which r shall be less than $\frac{1}{2}p$. For if in the formula

$$a = np + r,$$

r, which is less than p, be less than $\frac{1}{2}p$, the formula agrees with the enunciation of this proposition; and if r should be greater than $\frac{1}{2}p$, then we may make

$$a = (n + 1)p - (p - r)$$

or making $n + 1 = n'$ and $p - r = r'$,

$$a = n'p - r',$$

and here since $r > \frac{1}{2}p$, $r' = p - r' < \frac{1}{2}p$. The same formula applies to all numbers whatever, except that r and p in this case are not necessarily prime to each other.

9. If a and p be any two numbers prime to each other, there cannot be another number b prime to a which will render the product ab divisible by p. Or if a number p be prime to two other numbers a and b, it will be prime to their product ab.

First, if there be such a number b as will render ab divisible by p, let us suppose it to be the least of all those that will make ab divisible by p; and since p is prime to b, let

$$p = nb + b',$$

so that b' shall be less than b, and also prime both to p and b. Then, multiplying both sides by a, we have

$$ap = anb + ab', \text{ or}$$
$$ap - anb = ab'.$$

If therefore ab' be divisible by p, anb, and consequently $ap - anb$, as also its equal ab' will be so likewise.

But b is by the supposition the least number that renders ab divisible by p, whereas we have now found a less b', which is absurd. There cannot, therefore, be a number which is the least that renders ab divisible by p, but if there were any such numbers one of them must be the least; therefore there is no such number; that is, if p be prime both to a and b it is prime to their product ab.

Deductions.

(1.) From this it follows, that if a number p be prime to any number of factors a, b, c, d, &c., it is also prime to their product $a \cdot b \cdot c \cdot d$; and if p be prime to any number whatever, it is prime to all its factors.

(2.) If those factors are all equal, then the product becomes a power; if therefore p be prime to a, it is prime to any power of a, as a^n.

(3.) Hence again, conversely, a power can only have the same prime divisors as its root.

(4.) Consequently if p divide the product ab, but is prime to one of its factors, it must be a divisor of the other; and if p be a divisor of a continued product

$a \cdot b \cdot e \cdot d$, &c., and is prime to one of the factors a, it must be a divisor of the other factors b, e, d, &c.

(5.) If a be prime to p, and b less than p, then, whether b be prime or not, the product ab is not divisible by p.

(6.) If there be any number of factors a, b, e, &c. respectively prime to any other factors p, q, r, then will the products $a \cdot b \cdot e \cdot p \cdot q \cdot r$, be prime to each other.

(7.) If a product ab be divisible by p, and one of those factors as a be prime to p, then will the quotient be divisible by a.

10. Neither the sum nor the difference of two fractions which are in their lowest terms, and of which the denominator of the one contains a factor not common with the other, can be equal to an integer.

Let $\frac{a}{A}$ and $\frac{b}{B\,t}$ be any two fractions in their lowest terms, and of which the denominator of the one, as $\frac{b}{B\,t}$, contains a factor t not contained in A, then the equation

$$\frac{a}{A} \pm \frac{b}{B\,t} = e \text{ an integer}$$

is impossible.

For $$\frac{a}{A} \pm \frac{b}{B\,t} = \frac{a\,B\,t \pm A\,b}{A\,B\,t},$$

which cannot be an integer unless A b be divisible by t; but A and b are each prime to t; their product A b is therefore also prime to t. Consequently, $\frac{a\,B\,t \pm A\,b}{A\,B\,t}$ cannot be an integer: that is,

$$\frac{a}{A} \pm \frac{b}{B\,t} = e \text{ an integer}$$

is impossible under the conditions of the proposition.

Deductions

(1.) The same is also true if the first fraction be not in its lowest terms, if t be prime to A and $\frac{b}{B\,t}$ a fraction in its lowest terms.

(2.) The sum or difference of two fractions each in its lowest terms is also in its lowest terms, provided the denominators be prime to each other: that is, if $\frac{a}{A}$ and $\frac{b}{B}$ be in their lowest terms, and A prime to B, then will $\frac{a\,B \pm b\,A}{A\,B}$ be also in its lowest terms.

(3.) If two fractions are each in its lowest terms, their product is in its lowest terms.

11. Every integral number may be represented by the formula $a^n\, b^m\, c^q$ &c.

First, if p be a prime, then $b = 1$, $c = 1$, &c. and n, m, q, &c. may also be supposed $= 1$, and we shall have $p = a$.

Secondly, if p be not a prime, divide it first by the highest power a^n of one of its prime factors contained in it; and the quotient again by the highest power of b, as b^m, and the new quotient by the highest power of one of its factors, as c^q, and so on. Then ultimately we shall obtain

$$p = a^n\, b^m\, c^q, \text{ \&c.}$$

where a, b, c, &c. are all prime numbers.

Deductions.

(1.) Since every number is of the above form, the root of any square number is of that form, and therefore every square number is of the form

$$p^2 = a^{2n}.\, b^{2m}.\, c^{2q}, \text{ \&c.}$$

(2.) If $p = a^n\, b^m\, c^q$, and any one of the exponent n, m, q, be an odd number, p is not a square number. And if n, m, q, &c. be not each divisible by 3, p is not a cube, and so on in the higher powers.

(3.) Hence a square multiplied by a square will produce a product which is a square; but a square multiplied by a factor which is not a square, will give a product which is not a square, and so on with the higher powers.

12. If any square p^2 can be divided once by some other number p', and after that, neither by p' nor by any factor of p', then is p' also a square.

For let p be resolved into the form $p = a^n.\, b^m.\, c^q$.

then $p^2 = a^{2n}\, b^{2m}\, c^{2q}$,

and since p^2 is divisible by p', this last must contain some of the prime factors of p, that is, p' must have the form

$$p' = a^r.\, b^s, \text{ \&c.}$$

and $$\frac{p^2}{p'} = \frac{a^{2n}\, b^{2m}\, c^{2q}}{a^r\, b^s} = a^{2n-r}.\, b^{2m-s}, \text{ \&c.}$$

which latter quotient will still be divisible by a, b, &c., unless $r = 2n$, $s = 2m$, &c.; and since, by the supposition, this quotient is not again divisible either by p' or by any factor of p', it follows, that $p' = a^{2n}.\, b^{2m}$, &c.

Deductions.

(1.) In the same manner, if any power p^n be divisible once by some other number p', and after that neither by p' nor by any factor of it, then will p' itself be a complete n^{th} power.

(2.) It follows from this, that no product arising from any number of different prime numbers can be a square; for let p' be one of those prime numbers; then the product may be divided once by p', and only once, therefore that product is not a square.

(3.) The same is true of any two or more numbers prime to each other, unless they be all squares.

(4.) Therefore, conversely, the product of the square roots of non-quadrate numbers prime to each other cannot produce an integer.

For if p and q be two such numbers, and

$$\sqrt{p} \times \sqrt{q} = r, \text{ then } p\,q = r^2,$$

which we have seen is impossible.

13. The square root of an integer that is not a complete square cannot be expressed by a fraction.

If it be possible, let $\sqrt{a} = \frac{m}{n}$; $\frac{m}{n}$ being supposed in its lowest terms, so that m is prime to n, then $a = \frac{m^2}{n^2}$; and consequently m^2 must be divisible by n^2, which is impossible, because m and n are prime to each other.

DEDUCTIONS.

From the two preceding propositions it follows:

(1.) That any root of a number which cannot be expressed by an integer, cannot be expressed by a rational fraction.

(2.) The product of the square roots of any two or more non-quadrate numbers, cannot be expressed by any rational fraction.

(3.) And, generally, if $\sqrt[n]{a}$ and $\sqrt[n]{b}$ be neither of them expressible in integers, and if a be prime to b, then can $\sqrt[n]{a} \times \sqrt[n]{b}$ be neither expressed in integers nor in rational fractions.

14. Neither the sum nor the difference of the square roots of two numbers which are not both squares, can be expressed by any rational quantity.

Let p and q be two such numbers, and if possible, let

$$\sqrt{p} \pm \sqrt{q} = c,$$

then $$p + q \pm 2\sqrt{pq} = c^2,$$

and $$\sqrt{pq} = \frac{c^2 - p - q}{2} \text{ a rational fraction,}$$

which is impossible.

DEDUCTIONS.

(1.) In the same way it may be shown, that $\sqrt{p} \pm \sqrt{q} = \sqrt{c}$ is impossible.

For then $$\sqrt{pq} = \frac{c - p - q}{2},$$

which is impossible.

(2.) If p and q be prime to each other, then $\sqrt{p} \pm \sqrt{q} = \sqrt{r} \pm \sqrt{s}$ is impossible.

For squaring both sides and reducing we obtain

$$\pm \sqrt{pq} \pm \sqrt{rs} = \frac{r + s - p - q}{2},$$

which is impossible, whether $\sqrt{rs}$ be rational or irrational.

I. *On the divisors of Composite Numbers.*

15. To find the number of divisors of any given number.

Let N be the given number, let N be resolved into the form $N = a^m b^n c^p d^q$, &c. then will the number of its divisors be expressed by the formula

$$(m + 1)(n + 1)(p + 1)(q + 1) \text{ \&c.}$$

For it is evident that N will be divisible by a, and by every power of a to a^m, that is, by every term in the series

$$a, a^2, a^3, a^4, \text{\&c. } a^m,$$

and also by b, and by every power of b to b^n, that is, by every term in the series

$$b, b^2, b^3, b^4, \text{\&c. } b^n,$$

and in the same manner by c, and every power of c to c^p; by d and every power of d to d^q, &c.

And also by every possible combination of the terms of the above series; that is, by every term in the continued product

$$(1 + a + a^2 \ldots\ldots a^m) \times (1 + b + b^2 + \text{\&c. } b^n)$$
$$(1 + c + c^2 + \text{\&c. } c^p) \times (1 + d + d^2 + \text{\&c. } d^q)$$

but the numbers of terms in this series, since no two of them can be the same, is truly expressed by the formula

$$(m + 1)(n + 1)(p + 1)(q + 1) \text{ \&c.}$$

which is, therefore, the number of the divisors sought, unity and N being both included as divisors.

Thus $360 = 2^3 . 3^2 . 5^1$.

Has $(3 + 1)(2 + 1)(1 + 1) = 24$ divisors.

And $1000 = 2^3 . 5^3$.

Has $(3 + 1)(3 + 1) = 16$ divisors.

DEDUCTIONS.

(1.) As the number $N = a^m b^n c^p d^q$ has

$$(m + 1)(n + 1)(p + 1)(q + 1) \text{ divisors,}$$

it is obvious that the number of ways in which it can be divided into two factors will be expressed by

$$\tfrac{1}{2}(m + 1)(n + 1)(p + 1)(q + 1) \text{ \&c.}$$

being equal to half the number of its divisors.

(2.) If it be required, in how many ways a number, $N = a^m b^n c^p$, &c., may be resolved into two factors prime to each other, it is evident, that this number no longer depends upon the value of the exponents m, n, p, &c., but will be the same as if N was simply resolved into the factors a, b, c, &c.; and is, therefore, equal to

$$\frac{(1 + 1) . (1 + 1) . (1 + 1), \text{ \&c.}}{2},$$

hence, if k represents the number of prime factors, a, b, c, d, &c., then will 2^{k-1} be the number of ways in which N may be resolved into two factors prime to each other. Thus, for example, 360 has twenty-four divisors (example 1,) and, consequently, may be resolved into factors twelve different ways; but it has only three prime factors, 2, 3, and 5, and can, therefore, be resolved into factors prime to each other only, $2^2 = 4$, different ways.

16. To find a number that shall have any given number of divisors.

Let w represent the given number of divisors, and resolve w into factors, as $w = x \times y \times z$. Take $m = x - 1$, $n = y - 1$, $p = z - 1$, &c.; so shall

$$a^m b^n c^p, \text{ \&c.}$$

be the number required, as is evident from the foregoing proposition, where a, b, c, &c. may be taken any prime numbers whatever.

Thus, to find a number that shall have 30 divisors, we have $30 = 2 \times 3 \times 5$. Wherefore $x = 2$, $y = 3$, $z = 5$, and $m = x - 1 = 1$, $n = y - 1 = 2$, $p = z - 1 = 4$, and $a^m b^n c^p$ is the number sought, a, b, c being any prime numbers whatever.

If $a = 5$, $b = 3$, $c = 2$ we have

$$5 . 3^2 . 2^4 = 720, \text{ the number sought,}$$

and this is the least of all numbers having 30 divisors, because a, b, c are the least three prime numbers, and that which is involved to the highest power is the least.

17. To find the sum of all the divisors of any given number.

Let N be the number, and make $N = a^m b^n c^p$, &c., then the sum of all the divisors of N is expressed by the formula

$$\left(\frac{a^{m+1}-1}{a-1}\right)\left(\frac{b^{n+1}-1}{b-1)}\right)\left(\frac{c^{p+1}-1}{c-1}\right) \text{ \&c.}$$

For we have seen that the formulas

$$(1+a+a^2 \text{ \&c. } a^m) \quad (1+b+b^2+\text{\&c. } b^n)$$
$$(1+c+c^2 \text{ \&c. } c^p) \quad (1+d+d^2+\text{\&c. } d^q)$$

include all the divisors of N, and by the laws of arithmetical series,

$$1+a+a^2+\text{\&c. } a^m = \frac{a^{m+1}-1}{a-1}$$
$$1+b+b^2+\text{\&c. } b^n = \frac{b^{n+1}-1}{a-1}$$
$$\text{\&c.} \qquad \text{\&c.}$$

Consequeotly, this product is equal to

$$\left(\frac{a^{m+1}-1}{a-1}\right)\times\left(\frac{b^{n+1}-1}{b-1}\right)\times\left(\frac{c^{p+1}-1}{c-1}\right) \text{ \&c.,}$$

which, therefore, expresses the sum of all the divisors of N.

In this expression, N is considered as a divisor of itself; because, from the developement of the above product, the last term will evidently be $a^m b^n c^p$, &c.; that is, the last term of the product will be the number N itself.

Required the sum of all the divisors of 360.

First, $360 = 2^3 . 3^2 . 5$; therefore,

$$\left(\frac{2^4-1}{2-1}\right)\times\left(\frac{3^3-1}{3-1}\right)\times\left(\frac{5^2-1}{5-1}\right)$$
$$= 15 . 13 . 6 = 1170;$$

which is the sum of all the divisors of 360, itself being considered as one of them.

18. If $N = a^m b^n c^p$ &c. represent any number, a, b, c, &c. being its prime factors, then will

$$N \times \frac{a-1}{a} \times \frac{b-1}{b} \times \frac{c-1}{c}, \text{ \&c.,}$$

express the number of integers that are less than N, and also prime to it.

First, if N be a prime number, or $N = a$, then we know, that all numbers less than a are also prime to it; and, consequently, $N \times \frac{a-1}{a} = a - 1$ is the real expression for the number of them in this case.

And if N be any power of a prime number, or $N = a^m$, then, in the series of numbers

$$1, 2, 3, 4, 5, \text{ \&c., } a^m,$$

every ath term is a multiple of a, these forming of themselves the series

$$a, 2a, 3a, 4a, 5a, \text{ \&c., } a^{m-1},$$

and therefore, from the a^m terms in the first series, we must deduct the a^{m-1} terms in the last, and the remainder will be the number of those terms in the first, that are prime to N, or to a^m; that is, $a^m - a^{m-1}$ are the number of integers prime to N; but since $N = a^m$ we have

$$a^m - a^{m-1} = a^m \times \frac{a-1}{a} = N \times \frac{a-1}{a},$$

for the number of those integers; which is likewise the form in question.

Again, if $N = a^m b^n$, it is evident, from the same consideration as before, that we shall have

$$a^{m-1} b^n, \quad \text{terms divisible by } a;$$
$$a^m b^{n-1}, \quad \text{terms divisible by } b;$$
$$a^{m-1} b^{n-1}, \text{ terms divisible by } a\,b.$$

But as the first expression includes all numbers divisible by a, and the second all those divisible by b, it follows, that the latter expression is included in each of the former; and therefore we have

$$a^{m-1} b^n - a^{m-1} b^{n-1}, \text{ terms divisible by } a \text{ only};$$
$$a^m b^{n-1} - a^{m-1} b^{n-1}, \text{ terms divisible by } b \text{ only};$$
$$a^{m-1} b^{n-1}, \text{ terms divisible by } a\,b;$$

and these, together, include all those terms of the series

$$1, 2, 3, 4, 5, \text{ \&c., } a^m b^n,$$

that have any common divisor with $a^m b^n$, or with N; and, consequently, their sum, taken from N, will be the number of those that are prime to it: hence, then, we have

$$a^m b^n - a^{m-1} b^n - a^m b^{n-1} + a^{m-1} b^{n-1} =$$
$$(a^m - a^{m-1})\, b^n - (a^m - a^{m-1})\, b^{n-1} =$$
$$(a^m - a^{m-1}) \times (b^n - b^{n-1}) =$$
$$\left(a^m \times \frac{a-1}{a}\right)\times\left(b^n \times \frac{b-1}{b}\right)$$
$$= N \times \frac{a-1}{a} \times \frac{b-1}{b},$$

which is again the formula in question.

Let, now, $N = a^m b^n c^p$, then, on the same principles as above, we shall have

$$P = a^{m-1} b^n c^p, \qquad \text{terms divisible by } a;$$
$$Q = a^m b^{n-1} c^p, \qquad \text{terms divisible by } b;$$
$$R = a^m b^n c^{p-1}, \qquad \text{terms divisible by } c;$$
$$S = a^{m-1} b^{n-1} c^p, \quad \text{terms divisible by } a\,b;$$
$$T = a^{m-1} b^n c^{p-1}, \quad \text{terms divisible by } a\,c;$$
$$V = a^m b^{n-1} c^{p-1}, \quad \text{terms divisible by } b\,c;$$
$$W = a^{m-1} b^{n-1} c^{p-1}, \text{ terms divisible by } a\,b\,c.$$

But since all the terms W are necessarily included in those of S, T, and V, and these last again in P, Q, and R, we shall have, by subtraction,

$$S - W, \text{ divisible by } a\,b \text{ only};$$
$$T - W, \text{ divisible by } a\,c \text{ only};$$
$$V - W, \text{ divisible by } b\,c \text{ only}:$$

and then, again,

$$P - S - T + 2\,W - W; \text{ or,}$$
$$P - S - T + W, \text{ divisible by } a \text{ only};$$
$$Q - S - V + W, \text{ divisible by } b \text{ only};$$
$$R - T - V + W, \text{ divisible by } c \text{ only};$$
$$W, \text{ divisible by } a\,b\,c \text{ only.}$$

And, consequently, the sum of all these expressions will be the number of terms that have a common divisor with $a^m b^n c^p$, or with N; and, therefore, N *minus* this sum will be the number of integers prime to N, and less than itself; which, by addition and subtraction, will be expressed as follows:

$$N - P - Q - R + S + T + V - W.$$

And by reestablishing again the values of P, Q, R, &c., it becomes

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$$(a^m b^n c^p - a^{m-1} b^n c^p) - (a^m b^{n-1} c^p - a^{m-1} b^{n-1} c^p) -$$
$$(a^m b^n c^{p-1} - a^{m-1} b^n c^{p-1}) +$$
$$(a^m b^{n-1} c^{p-1} - a^{m-1} b^{n-1} c^{p-1}) =$$
$$(a^m - a^{m-1})(b^n c^p - b^{n-1} c^p - b^n c^{p-1} + b^{n-1} c^{p-1}) =$$
$$(a^m - a^{m-1}) \times (b^n - b^{n-1}) . (c^p - c^{p-1}) =$$
$$\therefore \quad N \times \frac{a-1}{a} \times \frac{b-1}{b} \times \frac{c-1}{c}$$

the same form as before.

And, exactly in the same manner, if N were the product of a greater number of factors, we should still find, that the number of integers less than, and prime to N, would be represented by

$$N \times \frac{a-1}{a} \times \frac{b-1}{b} \times \frac{c-1}{c} \times \frac{d-1}{d}, \text{ \&c.}$$

Where it is only necessary to observe, that unity is included as one of those integers.

Find how many numbers there are under 100 that are prime to it.

First, $100 = 2^2 5^2$; therefore,

$$100 \times \frac{2-1}{2} \times \frac{5-1}{5} = 40,$$

the number sought: these being as follows; *viz.*

1	13	27	39	51	63	77	89
3	17	29	41	53	67	79	91
7	19	31	43	57	69	81	93
9	21	33	47	59	71	83	97
11	23	37	49	61	73	87	99

Again, how many numbers are there less than 360 which are also prime to it.

$360 = 2^3 . 3^2 . 5^1$, therefore

$$360 \times \frac{2-1}{2} \times \frac{3-1}{3} \times \frac{5-1}{5} = 96,$$

the number sought.

II. *Of Figurate Numbers, &c.*

19. The theory of figurate, amicable, and polygonal numbers must be admitted to be rather a subject of curiosity than of utility, we shall confine ourselves, therefore, almost entirely to a definition of them, and to a statement of some of their properties, and for the investigations we shall be content to refer to Barlow's *Theory of Numbers.*

Definitions.

20. A *Perfect Number* is that which is equal to the sum of all its aliquot parts, or of all its divisors.

Thus $6 = \frac{6}{2} + \frac{6}{3} + \frac{6}{6}$, and is, therefore, a perfect number.

21. *Amicable Numbers* are those pairs of numbers each of which is equal to all the aliquot parts of the other: thus 284, and 220, are a pair of amicable numbers; for it will be found, that all the aliquot parts of 284 are equal to 220; and all the aliquot parts of 220 are equal to 284.

22. *Figurate Numbers* are all those which fall under the general expression

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$$\frac{n\,(n+1)\,(n+2)\,(n+3)\text{ \&c. } \overline{n+m}}{1\,.\,2\,.\,3\,.\,4\text{ \&c. } \overline{m+1}};$$

and they are said to be of the 1st, 2d, 3d, &c. order, according as $m = 1, 2, 3,$ &c.

23. *Polygonal Numbers* are the sums of different and independent arithmetical series, and are termed *lineal* or *natural, triangular, quadrangular* or *square, pentagonal,* &c., according to the series from which they are generated.

24. *Natural Numbers* are formed from a series of units; thus,

Units,	1	1	1	1	1,	&c.
Nat. Numbers,	1	2	3	4	5,	&c.

25. *Triangular Numbers* are the successive sums of an arithmetical series, beginning with unity, the common difference of which is 1; thus,

Arith. Series,	1	2	3	4	5,	&c.
Trian. Num.	1	3	6	10	15,	&c.

26. *Quadrangular,* or *Square, Numbers* are the sums of an arithmetical series, beginning with unity, the common difference of which is 2; thus,

Arith. Series,	1	3	5	7	9	11,	&c.
Quadrang. or Square Num.	1	4	9	16	25	36,	&c.

27. *Pentagonal Numbers* are the sums of an arithmetical series, beginning with unity, the common difference of which is 3; thus,

Arith. Series,	1	4	7	10	13	16, &c.
Pentagonal Numbers,	1	5	12	22	35	51, &c.

And universally, the *m-gonal Series of Numbers* is formed from the successive sums of an arithmetical progression, beginning with unity, the common difference of which is $m - 2$.

28. *Perfect Numbers* are expressed, or determined, as follows:

Find $2^n - 1$ a prime number, then will $N = 2^{n-1} (2^n - 1)$ be a perfect number. For from what has been demonstrated in the preceding section, the sum of all the divisors of this formula will be represented by

$$\frac{2^n - 1}{2 - 1} \times \frac{(2^n - 1)^2 - 1}{(2^n - 1) - 1};$$

because $2^n - 1$ is a prime by hypothesis. But in this expression 1 is included as a divisor, which must be excluded in the case of perfect numbers; exclusive of this, therefore, the formula will be

$$\frac{2^n - 1}{2 - 1} \times \frac{(2^n - 1)^2 - 1}{(2^n - 1) - 1} - 2^{n-1} - 1\,(2^n - 1) =$$
$$(2^n - 1) \times (2^n - 1 + 1) - 2^{n-1}(2^n - 1) =$$
$$2\,(2^n - 1) . 2^{n-1} - 2^{n-1}(2^n - 1) = 2^{n-1} . (2^n - 1) = N,$$

that is, the sum of all the aliquot parts of N exclusive of itself, or of 1 as a divisor, is equal to N, and is, therefore, by the definition a perfect number.

The only perfect numbers known are the following eight:

6	33550336
28	8589869056
496	137438691328
8128	2305843008139952128.

29. To find a pair of amicable numbers N and M, or such a pair that each shall be respectively equal to all the divisors of the other.

Make $N = a^m\, b^n\, c^p$, &c., and $M = a^\mu\, \beta^\nu\, \gamma^\pi$, then, according to the definition, and from what has been demonstrated in the last section, we must have

$$\frac{a^{m+1}-1}{a-1} \times \frac{b^{n+1}-1}{b-1} \times \frac{c^{p+1}-1}{c-1} = N + M,$$

$$\frac{a^{\mu+1}-1}{a-1} \times \frac{\beta^{\nu+1}-1}{\beta-1} \times \frac{\gamma^{\pi+1}-1}{\gamma-1} = M + M.$$

Find, therefore, such a power of 2 as 2^r, that

$$3\,.\,2^r - 1,\ 6\,.\,2^r - 1, \text{ and } 18\,.\,2^{2r} - 1$$

may be all prime numbers, then will

$$N = 2^{r+1}\,.\,d, \text{ and } M = 2^{r+1}\, b\, c$$

be the pair of amicable numbers sought.

The least three pair of amicable numbers are,

284	220
17296	18416
9363583	9437056.

III. *Of the forms and properties of Prime Numbers.*

30. If a number cannot be divided by some other number, which is equal to, or less than, the square root of itself, that number is a prime.

For every number p, that is not a prime, may be represented by $p = a\,b$. Now if $a = b$, then a and b are each equal to $\sqrt{p}$; and, consequently, p, which is not a prime, is divisible by $\sqrt{p}$. Again, if $a > \sqrt{p}$, then will $b < \sqrt{p}$; for otherwise, we should have $a \times b = a\,b > p$, which is contrary to the supposition; therefore, if $a > \sqrt{p}$, then will $b < \sqrt{p}$; and if $b > \sqrt{p}$, then will $a < \sqrt{p}$; and, consequently, since p is divisible both by a and b, it is divisible by a number less than the square root of itself: and this is evidently true of all numbers that can be resolved into the form $p = a\,b$; that is, of all numbers that are not primes: therefore, if a number cannot be so divided, that number is a prime.

Hence, in order to ascertain whether a given number be a prime number or not, we must attempt the division of it by all the prime numbers less than the square root of itself; and if it be not divisible by any of them, it is a prime. It is obvious, that we need only essay the division by prime numbers; for if it be divisible by a composite number, it is evidently also divisible by the prime factors of that divisor. This method, however, although it admits of some contractions, is, notwithstanding, extremely laborious for large numbers; nor has any easy, practical rule been yet discovered, for ascertaining whether a given number be a prime or not.

31. Of the different linear forms of prime numbers.

Every prime number greater than 2, is of one of the forms $4\,n + 1$, or $4\,n - 1$. For every number divided by 4 will leave a remainder 1, 2 or 3; that is, every number whatever is included in one of the four forms

$$4\,n,\quad 4\,n + 1,\quad 4\,n + 2,\quad 4\,n + 3;$$

but the first and third of these are not primes, being both even or divisible by 2, therefore all prime numbers must fall under one of the other two, viz. $4\,n + 1$, or $4\,n + 3$; but $4\,n + 3 = 4\,(n + 1) - 1 = 4\,n' - 1$, therefore all prime numbers are included in the general formula $4\,n \pm 1$.

Deductions.

(1.) In a similar way it may be shown, that all prime numbers are included in the forms

$8\,n \pm 1$	$8\,n \pm 3$
$6\,n \pm 1$	———
$12\,n \pm 1$	$12\,n \pm 5$,
&c.	&c.

(2.) It may be proper just to observe, that although all prime numbers are included in these sets of formula, the prime number 2 only excepted, yet the converse is not true, viz. that all numbers contained in these forms are prime numbers; indeed no algebraical formula whatever can be found that includes prime numbers only. This is demonstrated in the following proposition.

32. No algebraical formula can contain prime numbers only.

Let $\quad p + q\,x + r\,x^2 + s\,x^3 + \text{\&c.}$

represent any general algebraical formula. It is to be demonstrated that such values may be given to x, that the formula in question shall not with that value produce a prime number, whatever values are given to p, q, r, &c.

For suppose, in the first place, that by making $x = m$, the formula

$$P = p + q\,m + r\,m^2 + s\,m^3 +, \text{\&c.},$$

is a prime oomber.

And, if now we assume $x = m + \phi\,P$, we have

$$\begin{array}{lll} p & = & \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\ p \\ q\,x & = & \dots\dots\dots\dots\dots\dots\dots\dots\dots\ q\,m + q\,\phi\,P \\ r\,x^2 & = & \dots\dots\dots\dots\ r\,m^2 + 2\,r\,m\,\phi\,P + r\,\phi^2\,P^2 \\ s\,x^3 & = & \dots\ s\,m^3 + 3\,s\,m^2\,\phi\,P + 3\,s\,m\,\phi^2\,P^2 + s\,\phi^3\,P^3 \\ \text{\&c.} & & \qquad\qquad \text{\&c.} \end{array}$$

Or,

$p + qx + rx^2 + sx^3 = (p + qm + rm^2 + sm^3 +, \text{\&c.}) + P\,(q\,\phi + 2\,r\,m\,\phi + 3\,s\,m^2\,\phi) + P^2\,(r\,\phi^2 + 3\,s\,m\,\phi^2) + s\,\phi^3\,P^3 = P + P\,(q\,\phi + 2\,r\,m\,\phi + 3\,s\,m^2\,\phi) + P^2\,(r\,\phi^2 + 3\,s\,m\,\phi^2) + s\,\phi^3\,P^3.$

But this last quantity is divisible by P; and, consequently, the equal quantity

$$p + q\,x + r\,x^2 + s\,x^3 +, \text{\&c.}$$

is also divisible by P, and cannot, therefore, be a prime number. Hence, then, it appears, that in any algebraical formula, such a value may be given to the indeterminate quantity, as will render it divisible by some other number; and, therefore, no algebraical formula can be found that contains prime numbers only.

But although no algebraical formula can be found that contains prime numbers only, there are several remarkable ones that contain a great many; thus $x^2 + x + 41$, by making successively $x = 0, 1, 2, 3, 4$, &c., will give a series 41, 43, 47, 53, 61, 71, &c., the first forty terms of which are prime numbers. The above formula is mentioned by Euler in the *Memoirs of Berlin*, (1772, p. 36.)

To the above we may add the following $x^2 + x + 17$, and $2x^2 + 29$, the former has seventeen of its first terms primes, and the latter twenty-nine.

Fermat asserted that the formula $2^m + 1$ was always a prime, while m was taken any term in the series 1, 2, 4, 8, 16, &c.; but Euler found that $2^{32} + 1 = 641 \times 6700417$ was not a prime.

33. The number of prime numbers is infinite.

For if not, let the number of them be represented by n, and the greatest of all those primes by p; then it is evident that the continued product of all the prime numbers not exceeding p, as

$$2 \,.\, 3 \,.\, 4 \,.\, 5, \text{\&c. } p$$

will be divisible by each of those numbers, and, therefore, if 1 be added to the product, the sum will be divisible by no one of them; consequently, if the formula

$$(2 \,.\, 3 \,.\, 4 \,.\, 5, \text{\&c. } p) + 1$$

be divisible by any prime number, it must be by some one greater than p, and if not it will be itself a prime, and, consequently, greater than p. Hence there must be a prime number greater than p, and, consequently, a greater number of prime numbers than n; and the same may be shown, however great n and p may be, therefore the number of prime numbers is infinite.

34. If a and b be any two numbers prime to each other, and each of the terms of the series

$$b,\ 2b,\ 3b,\ 4b, \text{\&c. } (a-1)\,b$$

be divided by a, they will each leave a different position remainder.

For if any two of these terms when divided by a leave the same remainder, let them be represented by xb, yb; then it is obvious, that $xb - yb$ would be divisible by a, or $(x-y)\,b$ would be divisible by a. But this is impossible, because a is prime to b, and $x - y$ is less than a, (art. 9,—5:) therefore $b\,(x-y)$ is not divisible by a; but it would be so divisible if the terms xb, yb left the same remainder; these do not, therefore, leave the same remainder, consequently every term of the series

$$b,\ 2b,\ 3b, \text{\&c. } (a-1)\,b$$

divided by a, will leave a different remainder.

Deductions.

(1.) Since the remainders arising from the division of each term in the series

$$b,\ 2b,\ 3b, \text{\&c. } (a-1)\,b$$

by a, are different from each other, and $a - 1$ in number, and each of them necessarily less than a, it follows that these remainders include all numbers from 1 to $a - 1$.

(2.) Hence, again, it appears, that some one of the above terms will leave a remainder 1; and that, therefore, if b and a be any two numbers prime to each other, a number $x < a$ may be found that will render $bx - 1$ divisible by a; or, the equation $bx - ay = 1$ is always possible if a and b are numbers prime to each other.

And it is always impossible if a and b have any common measure, as is evident; because one side of the equation $bx - ay = 1$ would be divisible by this common measure; but the other side, 1, would not be so: therefore, in this case, the equation is impossible.

(3.) We have seen, in the foregoing deduction, that the equation $bx - ay = 1$ is always possible, when a and b are prime to each other; and the same is evidently true of the equation $bx - ay = -1$, for $a - 1$ is one of the remainders in the above series, so that a value $x < a$ may be found, that renders $bx - (a-1)$ divisible by a; or the equation $bx - ay = a - 1$ is always possible; but this is the same as $bx - a\,(y-1) = -1$; or, making $y - 1 = y'$, $bx - ay' = -1$ is always possible; and, consequently, the equation $ax - by = \pm 1$ is always possible, when a and b are prime to each other.

35. If a be any prime number, then will the formula

$$1 \,.\, 2 \,.\, 3 \,.\, 4 \,.\, 5, \text{\&c. } (a-1) + 1$$

be divisible by a.

For it is demonstrated in our preceding second deduction, that, if a and b be any two numbers prime to each other, another number x may be found $< a$, that renders the product $bx - 1 \rightharpoondown$ * a; or, which is the same thing, $bx = ya + 1$; and that there is only one such value of $x < a$ may be shown as follows:

The foregoing equation gives, by transposition,

$$bx - ay = 1;$$

and, if it be possible, let also

$$bx' - ay' = 1;$$

and make $x' = x \pm m$, and $y' = y \pm n$, where m is necessarily less than a, because both x and x' are so by the supposition. Now, by this substitution, we have

$$(bx \pm bm) - (ay \pm an) = 1; \text{ but}$$
$$bx \qquad - ay \qquad = 1;$$

therefore $\pm bm = \mp an$, or $bm \rightharpoondown a$; but this is impossible, since b is prime to a, and $m < a$. (art. 9,—5.)

There cannot, therefore, be two values of x less than a, that renders the equation $bx - ay = 1$ possible.

But in the series of integers

$$1, 2, 3, 4, 5, \ldots\ldots\ a - 1,$$

every term is prime to a, except the first, a being itself a prime; if, therefore, we write successively, $b = 2$, $b' = 3$, $b'' = 4$, &c., a corresponding term x, in the same series, may be found for each distinct value of b, that renders the product $xb \rightleftharpoons ay + 1$, $x'b' \rightleftharpoons ay' + 1$, $x''b'' \rightleftharpoons ay + 1$, &c.; and it is evident, that no one of these values of x can be equal either to 1, or $a - 1$; for, in the first case, we should have $1 \times b = ay + 1$, which is impossible, because $b < a$; and the second would give $(a-1)\,b = ay + 1$, or $a\,(b-y) = b + 1$; that is, $b + 1 \rightharpoondown a$; which can only be when $b = a - 1$, or when $b = x$, which case is excepted, because we suppose two different terms of the series. In fact, since $(a-1)^2 \rightleftharpoons ay + 1$, there can be no other term, in the same series, that is of this form; for if $x^2 \rightleftharpoons ay' + 1$, then $(a-1)^2 - x^2$ would be divisible by a, or $(a - 1 + x) \times (a - 1 - x) \rightharpoondown a$, which is impossible, since each of these factors is prime to a, as is evident, because $x < a$, and a is a prime number.

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* To save the repetition of the words *divisible by*, which frequently occur, the sign $\rightharpoondown$ is used to express them; and for the same reason the symbol $\rightleftharpoons$ is introduced, to express the words *of the form of*, which are also of frequent occurrence.

Hence, then, our product

$$1 . 2 . 3 . 4 . 5 \ldots\ldots (a-1), \text{ becomes}$$

$$1 . b\,x . b'x' . b''x'' \ldots . \; a - 1;$$

but each of these products, $b\,x$, $b'x'$, $b''x''$, &c., is, as we have seen, of the form $a\,y+1$; therefore, their continued product will have the same form, and the whole product, including 1 and $a-1$, will be

$$\equiv (a\,y+1) \times (a-1) \equiv a^2 y + a\,y + a - 1,$$

to which, if unity be added, the result will be evidently divisible by a, that is, the formula

$$1 . 2 . 3 . 4 . 5 \ldots\ldots (a-1) + 1$$

is always divisible by a, when a is a prime number.

Deductions.

(1.) The product,

$$1 . 2 . 3 . 4 . 5 \ldots\ldots (a-1),$$

is the same as

$$1\,(a-1)\,2\,(a-2)\,3\,(a-3), \text{ \&c., } \left(\frac{a-1}{2}\right)^2;$$

and this product, with regard to its remainder, when divided by a, is the same as

$$\pm 1^2 . 2^2 . 3^2 . 4^2 \ldots\ldots \left(\frac{a-1}{2}\right)^2;$$

the ambiguous sign being *plus* (+) when $a-1$ is even, and *minus* (−) when $a-1$ is odd; that is, + when a is a prime number of the form $4\,n+1$, and − when a is a prime number of the form $4\,n-1$; also this product,

$$\pm 1^2 . 2^2 . 3^2 . 4^2 \ldots\ldots \left(\frac{a-1}{2}\right)^2,$$

is the same as

$$\pm \left(1 . 2 . 3 . 4 \ldots\ldots \frac{a-1}{2}\right)^2;$$

and, consequently, from what is said above relating to the ambiguous sign, we shall have

$$\left\{\left(1 . 2 . 3 . 4 \ldots\ldots \frac{a-1}{2}\right)^2 + 1\right\} \div a,$$

when $a \equiv 4\,n+1$; and

$$\left\{\left(1 . 2 . 3 . 4 \ldots\ldots \frac{a-1}{2}\right)^2 - 1\right\} \div a,$$

when $a \equiv 4\,n-1$.

Whence it follows, that every prime number of the form $4\,n+1$ is a divisor of the sum of two squares.

Again, the latter form may be resolved into the two factors

$$\left\{\left(1 . 2 . 3 . 4 \ldots\ldots \frac{a-1}{2}\right) + 1\right\} \times$$

$$\left\{\left(1 . 2 . 3 . 4 \ldots\ldots \frac{a-1}{2}\right) - 1\right\},$$

which product, being divisible by a, it follows, that a is a divisor of one or other of these factors, when it is a prime number of the form $4\,n-1$.

(2.) From the first product, which we have demonstrated to be divisible by a, viz.

$$\frac{1 . 2 . 3 . 4, \text{ \&c., } (a-1) + 1}{a} = e, \text{ an integer},$$

we may derive a great many others; as

$$\frac{1^2 . 2^2 . 3 . 4 . 5, \text{ \&c., } (a-3)\,(a-1) + 1}{a} = e, \text{ an integer};$$

$$\frac{1^2 . 2^2 . 3^2 . 4 . 5, \text{ \&c., } (a-4)\,(a-1) + 1}{a} = e, \text{ an integer};$$

and so on, till we arrive at the same form as that in the first deduction.

The theorem above demonstrated was first proposed by Sir John Wilson, as we are informed by Waring, in his *Meditationes Algebraicæ*, p. 380; but, notwithstanding the simple principles on which its demonstration is founded, it escaped the observation of these two celebrated mathematicians; the latter of whom speaks of it, at the place above quoted, as an extremely difficult proposition to demonstrate, on account of our having no formula for expressing prime numbers. Lagrange was the first who demonstrated this theorem, in the *New Memoirs of the Academy of Berlin*, 1771, (which demonstration is, as might be expected from the celebrity of its author, very ingenious;) and, afterwards, Euler gave a different demonstration of the same proposition, in his *Opusc. Analyt.* tom. i. p. 329, which is upon a similar principle to the foregoing; and, finally, Gauss, in his *Disquisitiones Arithmeticæ*, extended the theorem by demonstrating, that "*The product of all those numbers less than, and prime to a given number* $a \pm 1$ *is divisible by* a;" the ambiguous sign being −, when a is of the form p^m, or $2\,p^m$, p being any prime number greater than 2; and, also, when $a = 4$; but positive in all other cases, (*Recherches Arithmétiques*, p. 57.)

The theorem of Sir John Wilson furnishes us with an infallible rule, *in abstracto*, for ascertaining whether a given number be a prime or not; for it evidently belongs exclusively to those numbers, as it fails in all other cases, but is of no use in a practical point of view, on account of the great magnitude of the product even for a few terms.

IV. *On the forms of Square Numbers.*

36. Every square number is of one of the forms $4\,n$, or $4\,n+1$.

Every number is either even or odd, that is, every number is of one of the forms $2\,n$, or $2\,n+1$; and, consequently, every square is of one of the forms

$$4\,n^2 \equiv 4\,n$$

$$4\,n^2 + 4\,n + 1 \equiv 4\,n + 1.$$

Deductions.

(1.) Every even square number is divisible by 4.

(2.) Since every odd square by the above is of the form $4\,(n^2+n)+1$; and since n^2+n is necessarily even, it follows, that every odd square is of the form $8\,n+1$. And, consequently, no number of the forms $8\,n+3$, $8\,n+5$, $8\,n+7$, can be a square number.

(3.) The sum of two odd squares cannot be a square; for

$$(8\,n+1) + (8\,n+1) \equiv 4\,n + 2,$$

which is an impossible form.

37. Every square number is of one of the forms $5n$, or $5n \pm 1$.

For all numbers compared by the modulus 5, are of one of the forms

$$5n, \quad 5n \pm 1, \quad 5n \pm 2,$$

and all squares, therefore, are of one of the forms

$$25n^2 \quad \backsimeq 5n$$
$$25n^2 \pm 10n + 1 \backsimeq 5n + 1$$
$$25n^2 + 20n + 4 \backsimeq 5n + 4, \text{ or } 5n - 1.$$

Therefore all squares are of one of the forms $5n$, or $5n \pm 1$.

Deductions.

(1.) If a square number be divisible by 5, it is also divisible by 25; and, if a number be divisible by 5, and not by 25, it is not a square.

(2.) No number of the form $5n + 2$, or $5n + 3$, is a square number.

(3.) If the sum of two squares be a square, one of the three is divisible by 5, and, consequently, also by 25. For all the possible combinations of the three forms $5n$, $5n + 1$, and $5n - 1$, are as follows:

$$(5n + 1) + (5n' + 1) \backsimeq 5n + 2,$$
$$(5n - 1) + (5n' - 1) \backsimeq 5n - 2 \backsimeq 5n + 3,$$
$$5n \quad + 5n' \quad \backsimeq 5n,$$
$$5n \quad + (5n' + 1) \backsimeq 5n + 1,$$
$$5n \quad + (5n' - 1) \backsimeq 5n - 1,$$
$$(5n + 1) + (5n' - 1) \backsimeq 5n.$$

Now of these six forms, the latter four have one of the squares divisible by 5, and, therefore, also by 25. And the two first are each impossible forms for square numbers; that is, neither of these two combinations can produce squares: therefore, if the sum of two squares be a square, one of the three squares is divisible by 25.

(4.) In a similar way it may be shown, that all square numbers compared by modulus 10, are of one of the forms

$$10n, \; 10n + 5, \; 10n + 1, \; 10n + 6,$$
$$10n + 4, \text{ or } 10n + 9.$$

Therefore all square numbers terminate with one of the digits 0, 1, 4, 5, 6, or 9; and hence, again, no number terminating with 2, 3, 7, or 8, can be a square number.

(5.) By examining, in like manner, the forms of squares to modulus 100, we may deduce the following properties.

(6.) A square number cannot terminate with an odd number of cyphers.

(7.) If a square number terminate with a 4, the last figure but one must be even.

(8.) If a square number terminate with a 5, it must terminate with 25.

(9.) If the last digit of a square be odd, the last digit but one must be even; and if it terminate with any even digit except 4, the last but one must be odd.

(10.) A square number cannot terminate with more than three equal digits, unless they are 0's; nor can it terminate with three, unless they are 4's.

38. All square numbers are of the same form with regard to any modulus a, as the squares

$$0^2, 1^2, 2^2, 3^2, \&c. \; (\tfrac{1}{2}a)^2, \; a \text{ being even},$$

and as

$$0^2, 1^2, 2^2, 3^2, \&c. \left(\frac{a-1}{2}\right), \; a \text{ being odd}.$$

For every number may be represented by the formula $an \pm r$, in which r shall never exceed $\frac{1}{2}a$, (Art. 8.)

Now $\quad (an \pm r)^2 = a^2n^2 \pm 2arn + r^2,$

where it is obvious that r^2 and $(an \pm r)^2$ will leave the same remainder, when divided by a; therefore $(an \pm r)^2$ and r^2 will be of the same form compared by modulus a; but r never exceeds $\frac{1}{2}a$, therefore all numbers compared by modulus a are of the same forms as

$$0^2, 1^2, 2^2, 3^2, \&c. \; r^2,$$

or, as the squares

$$0^2, 1^2, 2^2, 3^2, \&c. \; (\tfrac{1}{2}a)^2, \text{ when } a \text{ is even},$$

and as

$$0^2, 1^2, 2^2, 3^2, \&c. \left(\frac{a-1}{2}\right)^2, \text{ when } a \text{ is odd}.$$

Deductions.

(1.) When a is even, the general formula

$$a^2n^2 \pm 2anr + r^2 \text{ becomes}$$
$$4a'^2n^2 \pm 4a'nr + r^2$$
$$\backsimeq 4a'(a'n^2 \pm nr) + r^2.$$

Therefore all square numbers are of the same form to modulus $4a$, as the squares

$$0^2, 1^2, 2^2, 3^2, \&c. \; a^2;$$

and hence we see immediately, that all square numbers to modulus 8, must be of the same forms as the squares

$$0^2, \; 1^2, \; 2^2;$$

that is, they are all of the form

$$8n, \; 8n + 1, \; 8n + 4,$$

as we have already demonstrated.

(2.) The following tables exhibit the possible and impossible forms of square numbers for all moduli from 2 to 10.

Possible formulæ.

$2n$,	$2n + 1$,			
$3n$,	$3n + 1$,			
$4n$,	$4n + 1$,			
$5n$,	$5n \pm 1$,			
$6n$,	$6n + 1$,	$6n + 3$,	$6n + 4$,	
$7n$,	$7n + 1$,	$7n + 2$,	$7n + 4$,	
$8n$,	$8n + 1$,	$8n + 4$,	———	
$9n$,	$9n + 1$,	$9n + 4$,	$9n + 7$,	
$10n$,	$10n \pm 1$,	$10n \pm 4$,	$10n \pm 5$.	

Impossible formulæ.

$3n$,			
$4n$,	$4n + 3$,		
$5n$,	$5n + 3$,		
$6n$,	$6n + 5$,		
$7n$,	$7n + 5$,	$7n + 6$,	
$8n$,	$8n \pm 3$,	$8n + 7$,	
$9n$,	$9n \pm 3$,	$9n + 5$,	$9n + 8$,
$10n$,	$10n \pm 3$.		

V. *Of the possible and impossible forms of Indeterminate Equations of the first and second degree.*

39. If a and b be any two numbers prime to each other, the equation

$$a x - b y = \pm c$$

is always possible; and an infinite number of different values may be given to x and y, that answer the condition of the equation in Integers.

By (Art. 34,—2) it appears the equation

$$a x - b y = 1$$

is always possible while a and b are prime to each other, and, consequently,

$$a c x - b c y = \pm c, \text{ or } a x' - b y' = \pm c,$$

by making $c x = x'$, and $c y = y'$;

and we have, evidently, the same result if we write

$$a (x' \pm m b) \text{ for } a x'$$
$$b (y' \pm m a) \text{ for } b y',$$

for these still give

$$a (x' \pm m b) - b (y' \pm m a) = \pm c.$$

Or, again, making

$$x' \pm m b = x$$
$$y' \pm m a = y,$$

our equation becomes

$$a x - b y = \pm c,$$

which is therefore always possible when a and b are prime to each other.

And it is evident, that by means of the indeterminate sign $\pm$, and indeterminate quantity m, the formulas

$$x' \pm m b = x$$
$$y' \pm m a = y,$$

will furnish an indefinite number of values of x and y, which will answer the conditions of the problem.

It is also obvious, that m may be so assumed that x shall be less than b, and y less than a.

Deductions.

(1.) In any of our future investigations we may, therefore, when the state of the question requires such an artifice, substitute $t x - u y = c$; t and u being numbers prime to each other, and c any number whatever prime to each of them, without inquiring about the particular values of x and y; it being sufficient for our purpose, in many cases, to know that the equation is possible.

(2.) But if t and u have any common measure, then such a substitution cannot be made, unless c has the same common measure.

40. The equation $a x + b y = c$ is always possible, if a and b be prime to each other, and

$$c > (a b - a - b).$$

For let $c = (a b - a - b) + r$, then the equation becomes

$$a x + b y = (a b - a - b) + r;$$

the possibility of which depends upon

$$x = \frac{a b - a - b - b y + r}{a}$$

being an integer. Now this equation is the same as

$$x = b - 1 - \frac{(y+1) b - r}{a};$$

and, therefore, it depends upon the possibility of

$$\frac{(y+1) b - r}{a} = x' \text{ being an integer;}$$

or, which is still the same, by calling $y + 1 = y'$, upon the possibility of the equation $y' b - a x' = r$; which we have seen may always be established, so that $y' < a$, or $y + 1 < a$; by the foregoing proposition.

Since, then, in the equation

$$\frac{(y+1) b - r}{a} = x',$$

$y + 1$ is less than a, x' must necessarily be less than b, and, consequently,

$$x = b - 1 - \frac{(y+1) b - r}{a} = b - 1 - x';$$

and since $x' < b$, therefore $x = b - 1 - x' = 0$, or some integer number: whence the equation

$$a n + b y = c$$

is always possible when a and b are prime to each other.

41. Investigation relative to indeterminate integral equations of the form

$$a t^2 + b u^2 = w^2.$$

First, in an equation of this form, we may always consider a and b as quantities that have no square factor, or divisor; for, if $a = a' \phi^2$, and $b = b' \theta^2$, our equation becomes $a' \phi^2 t^2 \pm b \theta^2 u^2 = w^2$; or, making $\phi t = t'$, and $\theta u = u'$, we have $a' t'^2 \pm b' u'^2 = w^2$; and, consequently, if the above equation obtain when the quantities a and b, or either of them, have a square divisor, it may always be put in another form, $a' t'^2 \pm b' u'^2 = w^2$, in which the similar quantities a' and b' have not a square divisor; and, therefore, in what follows, with regard to the possibility or impossibility of equations of the form $a t^2 \pm b u^2$, we may always consider a and b as not having a square divisor.

Again, if the equation $a t^2 \pm b u^2 = w^2$ be possible, when t^2, u^2, and w^2, have a common square divisor ϕ^2, it is also possible when divided by it; thus, if

$$a \phi^2 t'^2 \pm b \phi^2 u'^2 = \phi^2 w'^2 \text{ be possible, so also is}$$
$$a t'^2 \pm b u'^2 = w'^2,$$

which is a similar equation to the first, and in which t'^2, u'^2, and w'^2, have now no common square divisor. And it is evident, that no two of these squares can have a common divisor, unless the third square has the same. For, if it be possible, let $t^2 = t'^2 \phi^2$, and $u^2 = u'^2 \phi^2$; then, $a t'^2 \phi^2 \pm b u'^2 \phi^2 = w^2$, where the first side of the equation is divisible by ϕ^2, but the second is not, by the supposition, and yet it is equal to the first, which is absurd: and the same may be demonstrated if any other two of those squares are supposed to contain a square divisor, not common with the third; a and b having no square divisor, as is shown above.

Hence, then, we may draw this conclusion, in any case where we are investigating the possibility of an equation of the form $a t^2 \pm b u^2 = w^2$, the quantities

a and b may be considered as not containing a square divisor; and also the three quantities t, u, and w, as being prime to each other: for if the equation be possible under these conditions, it is possible when those quantities have a common measure; and if it be impossible under the former case, it is also impossible under the latter.

And it may be farther observed, that if any equation of the form $a\,t^2 \pm b\,u^2 = w$ be impossible in integers, it is so likewise in fractions; for make

$$t = \frac{r}{s},\ u = \frac{y}{v},\ \text{and } w = \frac{x}{z},\ \text{then it becomes}$$

$$a\,\frac{r^2}{s^2} \pm u\,\frac{y^2}{v^2} = \frac{x^2}{z^2};\ \text{which reduces it to this}$$

$$a\,r^2 v^2 \pm b\,s^2 y^2 = \frac{s^2 v^2 x^2}{z^2};\ \text{or, making}$$

$$r^2 v^2 = t^2,\ s^2 y^2 = u^2,\ \text{and } \frac{s^2 v^2 x^2}{z^2} = w^2,$$

which last must evidently be an integral square, we have again $a\,t^2 \pm b\,u^2 = w^2$; so that the possibility of any fractional equation of this kind depends upon a similar integral equation, and if, therefore, an equation be impossible, in integers, with any specified value of a and b, it is also impossible in fractions.

Deduction.

All that has been proved of the equation $a\,t^2 \pm b\,u^2 = w^2$ is also true of the equation $a\,t^3 \pm b\,u^3 = w^3$, and generally of the equations $a\,t^n \pm b\,u^n = w^n$, it being always understood, that neither a nor b contain any factor that is a complete n^{th} power.

42. The equation $(3p + 2)\,t^2 \pm 3\,q\,u^2 = w^2$ is always impossible either in integers or fractions. We have seen in the foregoing article, that it will be sufficient to consider t and u as integers, and that we always suppose t^2, u^2, w^2 to be prime to each other. Now since $3\,q\,u^2$ is always of the form $3\,n$, whatever may be the form of u^2, and since t^2 must be one of the forms $3\,n$ or $3\,n + 1$, (Art. 38,) we shall either have

First $\quad (3p + 2)\,3\,n \pm 3\,q\,u^2 = w^2$, or

Second $\quad (3p + 2)(3\,n + 1) \pm 3\,q\,u^2 = w^2$.

But in the first equation, where we suppose $t^2 = 3\,n$, we have the first side of the equation divisible by 3, and, consequently, the other side w^2 is also divisible by 3; that is, both t^2 and w^2 are divisible by 3, which cannot be, because they are prime to each; therefore the equation, when t^2 is of the form $3\,n$, is impossible.

Again, in the second equation, in which we suppose $t^2 = 3\,n + 1$, we have

$$(3p + 2) \times (3\,n + 1) \pm 3\,q\,u^2 = w^2,\ \text{or}$$
$$9\,p\,n + 6\,n + 3\,p + 2 \pm 3\,q\,u^2 = w^2,\ \text{or}$$
$$3\,(3\,p\,n + 2\,n + p + q\,u^2) + 2 = w^2,\ \text{or}$$
$$w^2 = 3\,n + 2,$$

which is impossible, (Art. 38;) therefore the equation $(3p + 2)\,t^2 + 3\,q\,u^2 = w^2$ is impossible, under the limitations of the problem.

Deductions.

(1.) By means of this general form we may derive many particular cases of impossible equations, by giving different values to p and q; thus is $q = 1$, and

$$p = 0,\ \text{then } 2\,t^2 \pm 3\,u^2 = w^2,$$
$$p = 1\ \ldots\ldots\ 5\,t^2 \pm 3\,u^2 = w^2,$$
$$p = 2\ \ldots\ldots\ 8\,t^2 \pm 3\,u^2 = w^2,$$

are all impossible equations.

And if $q = 2$, then

$$p = 0\ \text{gives } 2\,p^2 \pm 6\,u^2 = w^2,$$
$$p = 1\ \ldots\ldots\ 5\,p^2 \pm 6\,u^2 = w^2,$$
$$p = 2\ \ldots\ldots\ 8\,p^2 \pm 6\,u^2 = w^2,$$

which are all impossible equations.

(2.) In a similar manner it may be demonstrated, that the general equations

$$(5p \pm 2)\,t^2 \pm 5\,q\,u^2 = w^2,$$
$$(7p + 3)\,t^2 \pm 7\,q\,u^2 = w^2,$$
$$(7p + 5)\,t^2 \pm 7\,q\,u^2 = w^2,$$
$$(7p + 6)\,t^2 \pm 7\,q\,u^2 = w^2,$$

are all impossible equations, either in integers or fractions, under the same limitations as before.

And from these general forms we readily deduce the following particular cases,

$2\,t^2 \pm 5\,u^2 = w^2$,	$2\,t^2 \pm 10\,u^2 = w^2$,
$3\,t^2 \pm 5\,u^2 = w^2$,	$3\,t^2 \pm 10\,u^2 = w^2$,
$7\,t^2 \pm 5\,u^2 = w^2$,	$7\,t^2 \pm 10\,u^2 = w^2$,
$8\,t^2 \pm 5\,u^2 = w^2$,	$8\,t^2 \pm 10\,u^2 = w^2$,
&c.	&c.
$3\,p^2 \pm 7\,u^2 = w^2$,	$3\,p^2 \pm 14\,u^2 = w^2$,
$5\,p^2 \pm 7\,u^2 = w^2$,	$5\,p^2 \pm 14\,u^2 = w^2$,
$6\,p^2 \pm 7\,u^2 = w^2$,	$6\,p^2 \pm 14\,u^2 = w^2$,
$10\,p^2 \pm 7\,u^2 = w^2$,	$10\,p^2 \pm 14\,u^2 = w^2$,
$12\,p^2 \pm 7\,u^2 = w^2$,	$12\,p^2 \pm 14\,u^2 = w^2$,
$13\,p^2 \pm 7\,u^2 = w^2$,	$13\,p^2 \pm 14\,u^2 = w^2$,
&c.	&c.

which are all impossible equations.

(3.) By examining the above impossible forms it will be seen, that the multipliers of t^2 are all impossible forms with regard to that particular prime modulus to which they are referred, thus

$$3p + 2 \qquad \text{to modulus 3,}$$
$$5p \pm 2 \qquad \text{to modulus 5,}$$
$$\left.\begin{matrix} 7p + 3 \\ 7p + 5 \\ 7p + 6 \end{matrix}\right\} \qquad \text{to modulus 7;}$$

and we are hence led to an inference, that the same is true for any other prime modulus: that is, the

Equations
$$(11p + 2)\,t^2 \pm 11\,q\,u^2 = w^2,$$
$$(11p + 6)\,t^2 \pm 11\,q\,u^2 = w^2,$$
$$(11p + 7)\,t^2 \pm 11\,q\,u^2 = w^2,$$
$$(11p + 8)\,t^2 \pm 11\,q\,u^2 = w^2,$$
$$(11p + 10)\,t^2 \pm 11\,q\,u^2 = w^2,$$

are all impossible, while q is taken prime to 11.

Also,
$$(13p \pm 2)\,t^2 \pm 13\,q\,u^2 = w^2,$$
$$(13p \pm 5)\,t^2 \pm 13\,q\,u^2 = w^2,$$
$$(13p \pm 6)\,t^2 \pm 13\,q\,u^2 = w^2,$$

when q is taken prime to the modulus 13.

And
$$(17p \pm 3)\,t^2 \pm 17\,q\,u^2 = w^2,$$
$$(17p \pm 5)\,t^2 \pm 17\,q\,u^2 = w^2,$$
$$(17p \pm 6)\,t^2 \pm 17\,q\,u^2 = w^2,$$
$$(17p \pm 7)\,t^2 \pm 17\,q\,u^2 = w^2,$$

when q is taken prime to the modulus 17.

Likewise
$$(19p + 2)\,t^2 \pm 19\,q\,u^2 = w^2,$$
$$(19p + 3)\,t^2 \pm 19\,q\,u^2 = w^2,$$
$$(19p + 8)\,t^2 \pm 19\,q\,u^2 = w^2,$$
$$(19p + 10)\,t^2 \pm 19\,q\,u^2 = w^2,$$
$$(19p + 12)\,t^2 \pm 19\,q\,u^2 = w^2,$$
$$(19p + 13)\,t^2 \pm 19\,q\,u^2 = w^2,$$
$$(19p + 14)\,t^2 \pm 19\,q\,u^2 = w^2,$$
$$(19p + 15)\,t^2 \pm 19\,q\,u^2 = w^2,$$
$$(19p + 18)\,t^2 \pm 19\,q\,u^2 = w^2,$$

when q is prime to 19; are all impossible forms of equations in rational numbers.

These latter forms are only deduced from observation, upon the supposition that the product of a possible and impossible form is also of an impossible form; which property may be satisfactorily demonstrated; we shall not, however, enter upon the inquiry in this place, but refer the reader who is desirous of following out this proposition, to Barlow's *Theory of Numbers*, (Art. 51 and 52.) We shall here content ourselves with the induction, and proceed to a practical application of the theorem in question.

43. To ascertain the possibility or impossibility of any equation of the form

$$a\,x^2 + b\,y^2 = c\,z^2.$$

First, since a possible and impossible form multiplied together always produce an impossible form, it follows, that $a\,x^2$ is always of the same form as a, with regard to possible or impossible; and, in the same manner, $b\,y^2$ is of the same form as b, and $c\,z^2$ of the same form as c. Now $a\,x^2 = n\,a$, therefore $c\,z^2 - b\,y^2$ must be also of the form $n\,a$; and, consequently, $c\,z^2$ must leave the same remainder, when divided by a, as $b\,y^2$ does when divided by the same: it is evident, therefore, that these remainders must be both of the class of possible remainders, or both impossible, for otherwise they could not be equal; but these remainders will be of the same classes as c and b are; and hence it follows, that, if c and b are both found among the remainders to modulus a, or neither of them are found there, the equation may be possible; but if one of them is found there, and the other not, the equation is certainly impossible. And, in the same manner, if a and c be both found among the remainders to modulus b, or if neither of them be found there, the equation may be possible; but if one is found there, and the other not, the equation is certainly impossible. And, for the same reason, a and $-b$, or, which is equivalent, a and $c - b$, must be either both found among the remainders of modulus c, or neither of them, if the equation be possible. Having thus shown the principle of the rule, it may be delivered more briefly thus:

Find the forms of all squares to modulus a, or, which is the same, the remainders arising from dividing the squares,

$$1^2,\ 2^2,\ 3^2,\ 4^2,\ \&c.\ (\tfrac{1}{2}\,a)^2, \text{ by } a;$$

and if b and c are both found in this series of remainders, or if neither of them be found there, the equation may obtain; but if one of them be found there, and the other not, the equation is certainly impossible, and it will be needless to proceed any farther in the investigation. But if one of the two first conditions have place, then find the remainders of

$$1^2,\ 2^2,\ 3^2,\ 4^2,\ \&c.\ (\tfrac{1}{2}\,b)^2, \text{ divided by } b;$$

and these remainders must be submitted to the same test, with regard to a and c; and if one of them be found there, and the other not, the equation is impossible, and we need proceed no farther in the investigation. But if this be not the case, find the remainders of

$$1^2,\ 2^2,\ 3^2,\ 4^2,\ \&c.\ (\tfrac{1}{2}\,c)^2, \text{ divided by } c;$$

and if a and $(c - b)$ be both found in this series, or if neither of them be found there, the equation is possible, supposing the same to have had place in the other two series; but otherwise the equation is certainly impossible.

It is to be observed, that when any one of those three quantities is greater than the modulus, with the remainders of which it is compared, it must be divided by the modulus and the remainder used, instead of the quantity itself. It may be also farther observed, that if any one of the three quantities, a, b, or c, be unity, only two trials will be necessary, and if two of them be unity, but one.

These operations will be considerably abridged by means of the following table, which exhibits the remainders to every modulus, from 2 to 51, excepting only those numbers that contain square factors, because a, b, and c, contain no square factors (by Art. 41;) and hence the possibility or impossibility of any equation, in which the coefficients do not exceed 50, may be ascertained by inspection.

Table of the Remainders of Squares to every Modulus, from 2 to 51.

Moduli.	*Remainders.*
2	1
3	1
5	1 4
6	1 3 4
7	1 2 4
10	1 4 5 6 9
11	1 3 4 5 9
13	1 3 4 9 10 12
14	1 2 4 7 8 9 11
15	1 4 6 9 10
17	1 2 4 8 9 13 15 16
19	1 4 5 6 7 9 11 16 17
21	1 4 7 9 15 16 18
22	1 3 4 5 9 11 12 14 15 16 20
23	1 2 3 4 6 8 9 12 13 16 18
26	1 3 4 9 10 12 14 16 17 22 23 25
29	1 4 5 6 7 9 13 16 20 22 23 24 25 28
30	1 4 6 9 10 15 16 19 21 24 25
31	1 2 4 5 7 8 9 10 14 16 18 19 20 25 28
33	1 3 4 9 12 15 16 22 25 27 31
34	1 2 4 8 9 13 15 16 17 18 19 21 25 26 30 32 33
35	1 4 9 11 14 15 16 21 25 29 30
37	1 3 4 7 9 10 11 12 16 21 25 26 27 28 30 33 34 36
38	1 4 5 6 7 9 11 16 17 19 20 23 24 25 26 28 30 35 36
39	1 3 4 9 10 12 13 16 22 25 27 30 36
41	1 2 4 5 8 9 10 16 18 20 21 23 25 31 32 33 36 37 39 40
42	1 4 7 9 15 16 18 21 22 25 28 30 36 37 39
43	1 4 6 9 10 11 13 14 15 16 17 21 23 24 25 31 35 36 38 40 41
46	1 2 3 4 6 8 9 12 13 16 18 23 24 25 26 27 29 31 32 35 36 39 41
47	1 2 3 4 6 7 8 9 12 14 16 17 18 21 24 25 27 28 32 34 36 37 42
51	1 4 9 13 15 16 18 19 21 25 30 33 34 36 42 43 49

Example 1. It is required to ascertain, whether the equatioo $7x^2 \pm 11y^2 = 13z^2$ be possible or impossible.

$$11 \equiv 7n + 4, \text{ and } 13 \equiv 7n + 6.$$

Now 4 is found in the table to belong to modulus 7, but 6 is not found there, whence the equation is impossible.

Example 2. Find whether the equation

$$7x^2 + 11y^2 = 23z^2 \text{ be possible or impossible.}$$

$$11 \equiv 7n + 4, \text{ and } 23 \equiv 7n + 2.$$

And 4 and 2 being both found to belong to modulus 7, ths equation may be possible.

Again,

$$7 \equiv 11n + 7, \text{ and } 23 \equiv 11n + 1.$$

Now one of these remainders, 1, belongs to modulus 11, but 7 does not, therefore the equation is impossible.

Example 3. Find whether the equation

$$14x^2 + 6y^2 = 17z^2 \text{ be possible or impossible.}$$

$$6 \equiv 14n + 6, \text{ and } 17 \equiv 14n + 3.$$

And neither 6 nor 3 belongs to modulus 14, therefore the equation may be possible.

Again,

$$14 \equiv 6n + 2, \text{ and } 17 \equiv 6n + 5.$$

And neither 2 nor 5 belongs to modulus 6, the equatioo therefore may still be possible.

Also,

$$14 \equiv 7n + 14, \text{ and } 17 - 6 \equiv 17n + 11.$$

And neither 11 nor 14 belongs to modulus 17, therefore tha equation is possible. In fact,

$$14 \,.\, 11^2 + 6 \,.\, 1^2 = 17 \,.\, 10^2.$$

These examples will be quite sufficient for explaining our operation; it may oot, however, be superfluous to add, that, when an equation appears under the form $a x^2 - b y^2 = c z^2$, it is immediately transformed to the sort of equation we bave been investigating, by writing it $c z^2 + b y^2 = a x^2$. Tha cases io which one or two of the coefficients become unity, are evideotly involved in the general form above given, and, therefore, need no examples.

44. The eqoatioo $x^2 - y^2 = a z^2$ is always possible io iotegers.

For, if we resolve $x^2 - y^2$ into its factors $x + y$, and $x - y$, (which are the only two literal factors that the formula admits of,) and also $a z^2$ into any two factors $a m t^2$, and $m u^2$, we have, by comparison,

$$\left.\begin{aligned} x + y &= a m t^2, \\ x - y &= m u^2, \end{aligned}\right\} \quad \text{or} \quad \left\{\begin{aligned} x + y &= m u^2, \\ x - y &= a m t^2, \end{aligned}\right.$$

which, by moltiplication, becomes $x^2 - y^2 = a m^2 t^2 u^2$, or $x^2 - y^2 = a z^2$, by making $z = m t u$.

Now these equations give,

$$\text{1st, } x = \frac{a m t^2 + m u^2}{2}, \text{ aod } y = \frac{a m t^2 - m u^2}{2};$$

$$\text{2d, } x = \frac{m u^2 + a m t^2}{2}, \text{ and } y = \frac{m u^2 - a m t^2}{2}.$$

On making $m = 2$, in order to clear the expressions of fractions, they become,

$$\text{1st, } x = a t^2 + u^2, \text{ and } y = a t^2 - u^2;$$
$$\text{2d, } x = u^2 + a t^2, \text{ and } y = u^2 - a t^2;$$

therefore the equation is always possible in integers.

We may also take $m = 1$, or any odd number, only observing, that if a be odd, we mast have t and u both odd; for otherwise x and y would not be integers. And if a be even, then u must be even likewise.

Deductions.

(1.) If a be a prime number, the solution above given is the only ooe the equatiou admits of in integers, for $x + y$ and $x - y$ are the only literal factors of $x^2 - y^2$; and $a m t^2$ and $m u^2$ are the only factors of $a z^2$, with regard to form; and, consequently, one of the two equalities must obtain: bot the quantities t and u being indeterminate, they will furnish an infinite number of numerical solutions. But if a be a composite number, then the equatioo may have, besida the two solotioos given above, as many different literal solutions as there are different ways of producing a by two factors; thus, if $a = b c$, we may have

$$\text{1st, } \left\{\begin{aligned} x + y &= a m t^2, \\ x - y &= m u^2, \end{aligned}\right\} \text{ or } \left\{\begin{aligned} x + y &= m u^2, \\ x - y &= a m t^2; \end{aligned}\right. \text{ and,}$$

$$\text{2d, } \left\{\begin{aligned} x + y &= b m t^2, \\ x - y &= c m u^2, \end{aligned}\right\} \text{ or } \left\{\begin{aligned} x + y &= c m u^2, \\ x - y &= b m t^2. \end{aligned}\right.$$

(2.) The equation $x^2 - y^2 = a z^2$ includes the two forms $x^2 - a z^2 = y^2$, and $x^2 + a z^2 = y^2$; for, by transposition, the first of these becomes $x^2 - y^2 = a z^2$, and tha latter $y^2 - x^2 = a z^2$, which are evidently both of the same form.

Therefore, if it be required to make $x^2 + a z^2 = y^2$ a square, we may bave $x = a t^2 - u^2$, or $= u^2 - a t^2$, and $z = 2 t u$; whence $x^2 + a z^2 = (a t^2 + u^2)^2$; or we may have $x = \dfrac{a t^2 - u^2}{2}$, and $z = t u$, which give

$$x^2 + a z^2 = \left(\frac{a t^2 + u^2}{2}\right)^2.$$

And to make $x^2 - a z^2 = y^2$ a square, we may assume $x = a t^2 + u^2$, and $z = 2 t u$, which giva

$$x^2 - a z^2 = (a t^2 - u^2)^2, \text{ or } = (u^2 - a t^2)^2;$$

or we may take

$$x = \frac{a t^2 + u^2}{2}, \text{ and } z = t u.$$

(3.) But if $a = 1$, and the equation become $x^2 + z^2 = y^2$, then we may have indifferently $x = t^2 - u^2$, and $z = 2 t u$, or $x = 2 t u$, aod $z = t^2 - u^2$, unless there be any thing in the natnre of the equation wh'ch limits these forms: as, for example, if it be necessary that one of the quantities, x or z, be even; then it is obvious, that the even quantity must have the form $2 t u$.

With regard to the equation $x^2 - z^2 = y^2$, it gives either $x = t^2 + u^2$, and $z = 2 t u$, or $z = t^2 - u^2$, both of which values of z answer the required conditions of the equation.

Example. Fiod the values of x, y, and z, in the equation $x^2 - y^2 = 30 z^2$.

Here the following substitutions may be made,

$$1. \left\{\begin{aligned} x + y &= m t^2, \\ x - y &= 30 m u^2, \end{aligned}\right. \text{ or } \left\{\begin{aligned} x + y &= 30 m t^2, \\ x - y &= m u^2. \end{aligned}\right.$$

$$2. \left\{\begin{aligned} x + y &= 3 m t^2, \\ x - y &= 10 m u^2, \end{aligned}\right. \text{ or } \left\{\begin{aligned} x + y &= 10 m t^2, \\ x - y &= 3 m u^2. \end{aligned}\right.$$

3. $\begin{cases} x+y = 2mt^2, \\ x-y = 15mu^2, \end{cases}$ or $\begin{cases} x+y = 15mt^2, \\ x-y = 2mu^2 \end{cases}$

4. $\begin{cases} x+y = 5mt^2, \\ x-y = 6mu^2, \end{cases}$ or $\begin{cases} x+y = 6mt^2, \\ x-y = 5mu^2. \end{cases}$

And making, io each of these, $m = 2$, in order to avoid fractions, we have the following general integral values of x end y:

1. $\begin{cases} x = t^2 + 30u^2, \\ y = t^2 - 30u^2, \end{cases}$ or $\begin{cases} x = 30t^2 + u^2, \\ y = 30t^2 - u^2. \end{cases}$

2. $\begin{cases} x = 3t^2 + 10u^2, \\ y = 3t^2 - 10u^2, \end{cases}$ or $\begin{cases} x = 10t^2 + 3u^2, \\ y = 10t^2 - 3u^2. \end{cases}$

3. $\begin{cases} x = 2t^2 + 15u^2, \\ y = 2t^2 - 15u^2, \end{cases}$ or $\begin{cases} x = 15t^2 + 2u^2, \\ y = 15t^2 - 2u^2. \end{cases}$

4. $\begin{cases} x = 5t^2 + 6u^2, \\ y = 5t^2 - 6u^2, \end{cases}$ or $\begin{cases} x = 6t^2 + 5u^2, \\ y = 6t^2 - 5u^2. \end{cases}$

In which formulæ, t and u may be any integer numbers whatever.

45. The two indeterminate equations,

$$x^2 + y^2 = z^2, \text{ and } x^2 - y^2 = w^2,$$

cannot both obtain, with the same values of x and y.

For, in the first place, x and y may be considered prime to each other, (art. 41,) and therefore x aod y odd, or one even and ooe odd; and we eee, immediately, that it is y that must be even: for if $x^2 = 4n + 1$, and $y = 4n + 1$, then $x^2 + y^2 = 4n + 2$, which cannot be a square; and if $x^2 = 4n$, and $y^2 = 4n + 1$, then $x^2 - y^2 = 4n + 3$, which is also an impossible form; therefore x is odd, and y even.

Hence, then (art. 44, —3) we must have,

let, $\begin{cases} x = r^2 - s^2, \\ y = 2rs. \end{cases}$ 2d, $\begin{cases} x = t^2 + u^2, \\ y = 2tu. \end{cases}$

Which furnish the following equations:

$$\begin{cases} r^2 - s^2 = t^2 + u^2, \\ rs = tu. \end{cases}$$

Now, in these eqnatione, r ie prime to s, and t prime to u; for otherwise x and y would have a common measure, which ie contrary to the eupposition; and, farther, as $x = r^2 - s^2$ is odd, one of these quantities, r or s, is even, and the other odd; and the same is also true of t and u, because $t^2 + u^2 = x$ ie an odd number.

Again, since $\frac{rs}{t} = u$ ie an integer, either r or s, or both, must contain the factors of t; for otherwise the quotient would not be an ioteger: we may, therefore, make $t = ab$, supposing a, b, to be its two factors, which may always be done, because, io the case of t being a prime, we have only to make one of these two factors equal to unity: and, eince these factors are also contained in rs, we may write $r = ar'$, and $s = bs'$, whence $u = r's'$; and oow, eubstituting these values for r, s, t, and u, the above equation becomes

$$a^2 r'^2 - b^2 s'^2 = a^2 b^2 + r'^2 s'^2.$$

And here, since r is prime to s, and t to u; r', s', a, and b, are all prime among themselves, ae ie evident; for if we suppose any two of the quantitiee to have a common measure, as, for example, a and b, then, since a and b enter, either seperately or connectedly, into three of the above quantities, the fourth, $r's'$, must have the same common measure, that is, $t = ab$, and $u = r's'$, would have a common measure, whereas we have eeen that they are prime to each other; and, consequently, r', s', a, and b, are all prime to one another.

Now, by transpoeitiou, this equatioo becomes

$$a^2 r'^2 - s'^2 r'^2 = a^2 b^2 + s'^2 b^2, \text{ or}$$

$$(a^2 - s'^2) r'^2 = (a^2 + s'^2) b^2, \text{ or}$$

$$\frac{a^2 + s'^2}{a^2 - s'^2} = \frac{r'^2}{b^2}.$$

And here, eince a^2 is prime to s'^2, $a^2 + s'^2$ is prime to $a^2 - s'^2$, or they have only the common measure 2; and we have, therefore, these two cases to consider separately. First, suppose $a^2 + s'^2$ and $a^2 - s'^2$ to be prime to each other, then the fraction $\frac{a^2 + s'^2}{a^2 - s'^2}$ is in its lowest terme, as is also $\frac{r'^2}{b^2}$, because r' is prime to b; and hence, the two fractions being equal to each other, and in their lowest terms, we must have, as resulting from the first supposition,

$$\begin{cases} a^2 + s'^2 = r'^2, \\ a^2 - s'^2 = b^2. \end{cases}$$

Again, let $a^2 + s'^2$ and $a^2 - s'^2$ have a common measure 2, then

$$\frac{\frac{1}{2}(a^2 + s'^2)}{\frac{1}{2}(a^2 - s'^2)} = \frac{a^2 + s'^2}{a^2 - s'^2} = \frac{r'^2}{b^2};$$

the first and last of which fractions are in their lowest terms, and, consequently,

$$\left.\begin{array}{l} \frac{1}{2}(a^2 + s'^2) = r'^2, \\ \frac{1}{2}(a^2 - s'^2) = b^2, \end{array}\right\} \text{ or } \begin{cases} a^2 + s'^2 = 2r'^2, \\ a^2 - s'^2 = 2b^2; \end{cases}$$

the last of which gives

$$\begin{cases} a^2 = r'^2 + b^2, \\ s'^2 = r'^2 - b^2. \end{cases}$$

Now these two resulte in both cases are exactly eimilar to the originel equations, only here the quantitiee are much emaller than in that, at least r', s' and b, a, are less than y, because $y = r's'ab$.

Hence, then, it follows, that if the equations

$$\begin{cases} x^2 + y^2 = z^2, \\ x^2 - y^2 = w^2, \end{cases}$$

were both possible, with the eame values of x and y, it would also be possible to find eimilar equations,

$$\begin{cases} x'^2 + y'^2 = z'^2, \\ x'^2 - y'^2 = w'^2; \end{cases}$$

which would also be possible, and in which $y' < y$. And, in the same manner, if these last were possible, we might still find others,

$$\begin{cases} x''^2 + y''^2 = z''^2, \\ x''^2 - y''^2 = w''^2, \end{cases}$$

where $y'' < y$, and so on of others, *ad infinitum.*

But it is impossible for a series of positive integers,

$$y', y'', y''', y'''', \text{\&c.},$$

to go on decreasing to infinity, without becoming zero; in which case our equations are

$$\begin{cases} x^2 = z^2, \\ x^2 = w^2. \end{cases}$$

And, consequently, the two proposed equations can never obtain, with the same values of x and y, except when $y = o$; that is, the double equality

$$\begin{cases} x^2 + y^2 = z^2, \\ x^2 - y^2 = w^2, \end{cases}$$

is impossible.

DEDUCTIONS.

(1.) Hence, also, it appears, that the two equations,

$$\begin{cases} x^2 + y^2 = 2\,z^2, \\ x^2 - y^2 = 2\,w^2, \end{cases}$$

are impossible, with the same values of x and y, for these may be reduced to

$$\begin{cases} x'^2 = z^2 + z'^2, \\ y'^2 = z^2 - w^2; \end{cases}$$

and the two last being impossible, the former are impossible also.

(2.) The two equations

$$\begin{cases} 2\,x^2 + y^2 = z^2, \\ 2\,x^2 - y^2 = w^2, \end{cases}$$

are both impossible, with the same values of x and y.

For we may consider x and y as prime to each other; and therefore both odd, or one even and one odd; but they cannot be both odd, for then

$$2\,x^2 + y^2 = 2\,(4\,n + 1) + (4\,n' + 1) = 4\,n + 3,$$

which cannot be a square. Neither can x be even and y odd, for then

$$2\,x^2 - y^2 = 2\,(4\,n) - (4\,n' + 1) = 4\,n + 3,$$

which is an impossible form. And if y were even and x odd, then

$$2\,x^2 + y^2 = 2\,(4\,n + 1) + 4\,n' = 4\,n + 2,$$

which is also impossible; and therefore the two given equations cannot both obtain.

(3.) And this, again, shows the impossibility of the two equations

$$\begin{cases} x^2 + 2\,y^2 = 2\,z^2, \\ x^2 - 2\,y^2 = 2\,w^2; \end{cases}$$

for, by doubling these, we have

$$\begin{cases} 2\,x^2 + (2\,y)^2 = (2\,z)^2, \\ 2\,x^2 - (2\,y)^2 = (2\,w)^2, \end{cases}$$

which we have seen are impossible.

(4.) By a very similar mode of reasoning it may be proved, that the two equations

$$x^2 + 2\,y^2 = w^2,$$
$$x^2 - 2\,y^2 = z^2,$$

are both impossible with the same values of x and y, as are also the two equations

$$2\,x^2 + y^2 = w^2,$$
$$2\,x^2 - y^2 = z^2.$$

(5.) In this way the following table of impossible forms *in pairs* have been deduced, *viz.*

1. $\begin{cases} x^2 + y^2 = z^2, \\ x^2 - y^2 = w^2. \end{cases}$ 2. $\begin{cases} x^2 + y^2 = 2\,z^2, \\ x^2 - y^2 = 2\,w^2. \end{cases}$

3. $\begin{cases} 2\,x^2 + y^2 = z^2, \\ 2\,x^2 - y^2 = w^2. \end{cases}$ 4. $\begin{cases} x^2 + 2\,y^2 = 2\,z^2, \\ x^2 - 2\,y^2 = 2\,w^2. \end{cases}$

5. $\begin{cases} x^2 + 2\,y^2 = z^2, \\ x^2 - 2\,y^2 = w^2. \end{cases}$ 6. $\begin{cases} 2\,x^2 + y^2 = 2\,z^2, \\ 2\,x^2 - y^2 = 2\,w^2. \end{cases}$

7. $\begin{cases} x^2 + y^2 = z^2, \\ x^2 + 2\,y^2 = w^2. \end{cases}$ 8. $\begin{cases} x^2 - y^2 = z^2, \\ x^2 - 2\,y^2 = w^2. \end{cases}$

9. $\begin{cases} x^2 + y^2 = z^2, \\ x^2 + 3\,y^2 = w^2. \end{cases}$ 10. $\begin{cases} x^2 - y^2 = z^2, \\ x^2 - 3\,y^2 = w^2. \end{cases}$

11. $\begin{cases} x^2 + 2\,y^2 = z^2, \\ x^2 + 3\,y^2 = w^2. \end{cases}$ 12. $\begin{cases} x^2 - 2\,y^2 = z^2, \\ x^2 + 3\,y^2 = w^2. \end{cases}$

13. $\begin{cases} x^2 - y^2 = z^2, \\ x^2 + 2\,y^2 = u^2. \end{cases}$ 14. $\begin{cases} x^2 + y^2 = z^2, \\ x^2 - 2\,y^2 = w^2. \end{cases}$

&c. &c.

And, generally, the pair of equations

$$x^2 \pm c\,y^2 = z^2,$$
$$x^2 \pm \ y^2 = w^2$$

are impossible, if the two equations

$$m^2 \pm c\,n^2 = (c - 1)\,p^2,$$
$$m^2 \pm \ n^2 = (c - 1)\,q^2$$

be impossible; and, conversely, if these two be possible so also are the former.

46. The difference of two biquadrates cannot be equal to a square, or the equation $x^4 - y^4 = z^2$ is impossible. For

$$x^4 - y^4 = (x^2 + y^2)\,(x^2 - y^2),$$

and since x and y are prime, or may be supposed prime to each other, these factors are either prime to each other, or have only the common measure 2; and, therefore, if their product be a square we must have either

$$\left.\begin{array}{l} x^2 + y^2 = r^2, \\ x^2 - y^2 = s^2, \end{array}\right\} \text{ or } \begin{cases} x^2 + y^2 = 2\,r^2, \\ x^2 - y^2 = 2\,s^2, \end{cases}$$

for otherwise their product would not be a square, or they would have a greater common measure than 2.

But these are both impossible forms, by the last article, therefore the equation

$$x^4 - y^4 = z^2$$

is also impossible.

DEDUCTIONS.

(1.) In a similar way it may be shown, that

$$x^4 + 4\,y^4 = z^2$$

is impossible.

(2.) And that $x^4 + y^4 = 2\,z^2$ is impossible.

47. The sum of two biquadrates cannot be equal to a square; or the equation

$$x^4 + y^4 = z^2$$

is impossible.

For (art. 54,—2) if $x^4 + y^4$ be a square, we must have either

$$\left.\begin{array}{l} x^2 = t^2 - u^2, \\ y^2 = 2\,t\,u, \end{array}\right\} \text{ or } \begin{cases} y^2 = t^2 - u^2, \\ x^2 = 2\,t\,u, \end{cases}$$

which are similar expressions; it will therefore be sufficient for our purpose to prove that either pair of them are impossible, and, as we may suppose x and y prime to each other, (art. 41,) it follows, that t and u are also prime to each other; and, consequently, since $2\,t\,u = y^2$, one of these quantities must be a square, and the other double a square; let then $t = 2\,x'^2$, and $u = y'^2$, whence $t^2 - u^2 = 4\,x'^4 - y'^4$; that is, $4\,x'^4 - y'^4 = x^2$. Or, making $t = x'^2$ and $u = 2\,y'^2$, the equation becomes $x'^4 - 4\,y'^4 = x^2$. We have, therefore, to examine the two cases

$$\begin{cases} x'^4 - 4y'^4 = z^2, \\ 4x^4 - y^4 = z^2, \end{cases}$$

one of which conditions must obtain, if the original equation be possible.

Now these are resolvable into

1. $x'^4 - 4y'^4 = (x''^2 + 2y''^2)(x''^2 - 2y''^2)$.
2. $4x'^4 - y'^4 = (2x''^2 + y''^2)(2x''^2 - y''^2)$.

And since x is prime to y, and t to u, it follows, x is prime to y', and therefore these factors are prime to each other, or can have only the common measure 2. And, moreover, as their product is a square we must have either

$$\left.\begin{aligned} x''^2 + 2y''^2 = r''^2, \\ x''^2 - 2y''^2 = s''^2, \end{aligned}\right\} \text{ or } \begin{cases} x''^2 + y''^2 = 2r''^2, \\ x''^2 - y''^2 = 2s''^2, \end{cases}$$

in the first case, and

$$\left.\begin{aligned} 2x''^2 + y''^2 = r''^2, \\ 2x''^2 - y''^2 = s''^2, \end{aligned}\right\} \text{ or } \begin{cases} 2x''^2 + y''^2 = 2r''^2, \\ 2x''^2 - y''^2 = 2s''^2, \end{cases}$$

in the second.

But each of these forms, taken in pairs, has been demonstrated to be impossible, consequently the original equation, whence they have been derived, is impossible also.

Deductions.

(1.) Hence also it follows, that the two equations

$$\begin{cases} x^4 - 4y^4 = z^2, \\ 4x^4 - y^4 = z^2, \end{cases}$$

as is evident from the preceding investigation.

(2.) Since $x^4 + y^4 = z^2$ is impossible, *a fortiori*, $x^4 + y^4 = z^4$ is impossible.

48. The area of a rational right angled triangle cannot be equal to a square.

For this would require the two equations

$$\left.\begin{aligned} x^2 + y^2 = z^2 \\ \tfrac{1}{2}xy = w^2 \end{aligned}\right\}$$

to be both possible together.

Multiply the latter by 4, and add and subtract it from the first, and we shall have

$$z^2 + 4w^2 = (x+y)^2,$$
$$z^2 - 4w^2 = (x-y)^2;$$

but these are impossible; therefore the area of a rational right angled triangle cannot be a square number.

Deduction.

In a rational right angled triangle

$$x^2 + y^2 = z^2,$$

we must therefore have

$$\left.\begin{aligned} x = r^2 - s^2, \\ y = 2rs. \end{aligned}\right\}$$

And, consequently, if in the fraction $\frac{r^2 - s^2}{2rs}$ or $\frac{2rs}{r^2 - s^2}$, the numerator and denominator be taken for the sides of a right angled triangle, it will be a rational one; and in these expressions we may give any values at pleasure to r and s. If, in the second fraction $\frac{2rs}{r^2 - s^2}$, we make $r = s + 1$, it becomes

$$\frac{2s^2 + 2s}{2s+1} = s + \frac{s}{2s+1};$$

and in this expression, by making successively

$$s = 1, 2, 3, 4, \&c.,$$

we have the following remarkable series,

$$s + \frac{s}{2s+1} = 1\frac{1}{3},\ 2\frac{2}{5},\ 3\frac{3}{7},\ 4\frac{4}{9},\ 5\frac{5}{11},\ 6\frac{6}{13}, \&c.;$$

each of which expressions, reduced to an improper fraction, gives the sides of a rational right angled triangle. And if in the fraction $\frac{r^2 - s^2}{2rs}$ we make $s = 1$, and $r = 2n + 2$, our expression becomes

$$\frac{4n^2 + 8n + 3}{4n + 4} = n + \frac{4n+3}{4n+4};$$

and here, making $n = 1, 2, 3, 4$, &c., we have this other series,

$$n + \frac{4n+3}{4n+4} = 1\frac{7}{8},\ 2\frac{11}{12},\ 3\frac{15}{16},\ 4\frac{19}{20},\ 5\frac{23}{24}, \&c.,$$

which has the same property as the former.

VI. *Of the possible and impossible forms of Cubes and Higher Powers.*

49. All cube numbers are of one of the forms $4n$ or $4n \pm 2$.

Every number is of one of the forms

$$4n,\ 4n \pm 1, \text{ or } 4n + 2,$$

therefore all cubes fall in one of the forms

$$(4n)^3 \equiv 4n$$
$$(4n \pm 1)^3 \equiv 4n \pm 1$$
$$(4n + 2)^3 \equiv 4n.$$

Therefore all cubes are of one of the forms $4n$ or $4n \pm 1$.

Deductions.

(1.) By subdividing these, we deduce the forms to modulus 8, as follow. All cubes fall in one of the forms

$$8n,\ 8n \pm 1,\ 8n \pm 3.$$

(2.) Therefore, conversely, no numbers of the form

$$4n + 2,\ 8n \pm 2,\ 8n + 4,$$

can be cubes.

(3.) In a similar way we may deduce the possible forms of cubes to the moduli 7 and 9, *viz.*

$$7n,\ 7n \pm 1,\ 9n,\ 9n \pm 1.$$

50. All cube numbers are of the same form to any modulus a as the cubes

$$0^3, 1^3, 2^3, 3^3, \&c.\ (a-1)^3.$$

For every number may be reduced to the form $an + r$, such that r shall be less than a. Consequently, $(an \pm r)^3$ divided by a will leave the same remainder as a, but r is either zero, or some number less than a, whence the truth of the proposition is manifest.

DEDUCTIONS.

(1.) By means of this general proposition, the possible forms of cube numbers to any modulus are easily deduced. If we essay modulus 10 we find all the following possible forms, *viz.*

$$10\,n,\ 10\,n+1,\ 10\,n+2,\ 10\,n+3,\ \&c.\ 10\,n+9,$$

no number is therefore excluded by this modulus, consequently, a cube number may terminate with any digit.

(2.) To modulus 6 all cubes are of the same forms as their roots, consequently, the difference between any cube number and its root is divisible by 6.

51. The equation $(4\,p+2)\ t^3 \pm 4\,q\,u^3 = w^3$ is always impossible in integers, while q is prime to 4.

By art 41, the three cubes t^3, u^3, w^3 may be considered prime to each other; and since all cubes are of one of the forms $4\,n$, or $4\,n \pm 1$, and $4\,q\,u^3$ is always of the form $4\,n$,

$$(4\,p+2)\ t^3 \pm 4\,q\,u^3$$

must be (when t^3 is of the form $4\,n$) of the form

$$(4\,p+2)\ 4\,n \pm 4\,q\,u^3 \equiv 4\,n,$$

that is, t^3 and w^3 are both of the form $4\,n$, which is absurd, because they are prime to each other.

And if t^3 be supposed of the form $4\,n \pm 1$, then the equation is of the form

$$(4\,p+2)\,(4\,n \pm 1) \pm 4\,q\,u^3 \equiv 4\,n+2,$$

which is an impossible form.

Therefore $(4\,p+2)\ t^3 \pm 4\,q\,u^3 = w^3$ is impossible, q being prime to 4.

DEDUCTIONS.

(1.) By giving different values to p and q, we obtain the following impossible forms

$$\begin{array}{ll} 2\,p^3 \pm 4\,u^3, & 2\,p^3 \pm 12\,u^3, \\ 6\,p^3 \pm 4\,u^3, & 6\,p^3 \pm 12\,u^3, \\ 10\,p^3 \pm 4\,u^3, & 10\,p^3 + 12\,u^3. \\ \&c. & \&c. \end{array}$$

(2.) In a similar way we may show, that

$$\begin{array}{l} (7\,p \pm 2)\ t^3 \pm 7\,q\,u^3 = w^3, \\ (7\,p \pm 3)\ t^3 \pm 7\,q\,u^3 = w^3, \\ (9\,p \pm 2)\ t^3 \pm 9\,q\,u^3 = w^3, \\ (9\,p \pm 3)\ t^3 \pm 9\,q\,u^3 = w^3, \\ (9\,p \pm 4)\ t^3 \pm 9\,q\,u^3 = w^3, \\ \&c. \qquad \&c. \end{array}$$

q being in the first two prime to 7, and in the latter three prime to 9; and from these an indefinite number of impossible forms may be deduced.

52. All 4th powers are of the same form with regard to any number a as a modulus, as the 4th powers

$$0^4,\ 1^4,\ 2^4,\ 3^4,\ \&c.,\ (\tfrac{1}{2}\,a)^4,$$

when a is even; and as

$$0^4,\ 1^4,\ 2^4,\ 3^4,\ \&c.,\left(\frac{a-1}{2}\right)^4,$$

when a is odd.

For every number whatever may be represented by the formula $a\,n \pm r$, where r never exceeds $\frac{1}{2}\,a$, (art. 10.) But

$$(an \pm r)^4 = a^4n^4 \pm 4\,a^3n^3r + 6\,a^2n^2r^2 \pm 4\,anr^3 + r^4,$$

and all the terms, but the last, of this expression, being divisible by a, the whole quantity is evidently of the same form, with regard to a as a modulus, as the last term r^4; but r never exceeds $\frac{1}{2}\,a$, therefore every 4th power to modulus a is of the same form as the 4th powers.

$$0^4,\ 1^4,\ 2^4,\ 3^4,\ \&c.\ (\tfrac{1}{2}\,a)^4,\ a \text{ being even,}$$

$$0^4,\ 1^4,\ 2^4,\ 3^4,\ \&c.\left(\frac{a-1}{2}\right)^4,\ a \text{ being odd.}$$

By means of which result, tables of possible and impossible forms of both powers may be obtained to any indefinite extent, and amongst other curious results it will be found by examining these series, that all 4th powers are of one of the forms $16\,n$, or $16\,n+1$.

53. The two indeterminate equations

$$\begin{cases} x^4 \pm \ \ y^4 = z^4, \\ x^4 \pm a^2\,y^4 = z^4, \end{cases}$$

are both impossible.

For we have seen, (arts. 46, 47,) that the equation $x^4 \pm y^4 = z^2$ is impossible in integers; and therefore, *a fortiori*, the equation $x^4 \pm y^4 = z^4$ is also impossible.

Again, we have, by transposition, in the second equation,

$$x^4 - z^4 = (a\,y^2)^2,$$

which is also impossible, (art. 46;) and, consequently, the two given equations are impossible in integers.

DEDUCTIONS.

(1.) Hence it follows, that the equation

$$x^4 \pm 4\,y^4 = z^4$$

is impossible; and, in like manner,

$$\begin{cases} x^4 + y^4 = 2\,z^4, \\ 2\,x^4 - y^4 = z^4, \\ 4\,x^4 - y^4 = z^4, \end{cases}$$

are all impossible equations.

(2.) In a manner very similar to that employed in the case of squares and cubes, it may be demonstrated that

$$\begin{array}{l} (5\,p+2)\ t^4 \pm 5\,q\,u^4 = w^4, \\ (5\,p+3)\ t^4 \pm 5\,q\,u^4 = w^4, \\ (5\,p+4)\ t^4 \pm 5\,q\,u^4 = w^4, \\ \&c. \qquad \&c. \end{array}$$

are all impossible equations, as is also the general form

$$(16\,p+r)\ t^4 + v\,u^4 = w^4,$$

r and v being so taken that $r+v < 16$.

54. Every 5th power is terminated with the same digit as its root. Or all 5th powers are of the same form, with regard to modulus 10, as the roots of those powers.

For all numbers to modulus 10 are of one of the following forms:

$$\begin{array}{lll} (10\,n \quad\ )^5 \equiv 10^5\,n^5 & & \equiv 10\,n'', \\ (10\,n+1)^5 \equiv 10\,n'+1 & & \equiv 10\,n''+1, \\ (10\,n+2)^5 \equiv 10\,n'+2^5 & & \equiv 10\,n''+2, \\ (10\,n+3)^5 \equiv 10\,n'+3^5 & & \equiv 10\,n''+3, \\ (10\,n+4)^5 \equiv 10\,n'+4^5 & & \equiv 10\,n''+4, \end{array}$$

$$(10\,n+5)^5 \doteq 10\,n' + 5^5 \doteq 10\,n'' + 5,$$
$$(10\,n+6)^5 \doteq 10\,n' + 6^5 \doteq 10\,n'' + 6,$$
$$(10\,n+7)^5 \doteq 10\,n' + 7^5 \doteq 10\,n'' + 7,$$
$$(10\,n+8)^5 \doteq 10\,n' + 8^5 \doteq 10\,n'' + 8,$$
$$(10\,n+9)^5 \doteq 10\,n' + 9^5 \doteq 10\,n'' + 9.$$

Where the latter formulæ are evidently the same as the first; and, consequently, the powers have the same forms to modulus 10 as the roots of those powers, or they are terminated with the same digits.

Deduction.

It has been demonstrated, (art. 50,—2,) that all cubes have the same forms as their roots to modulus 6; and, in the above proposition, that all 5th powers have the same forms as their roots to modulus 10; and the same is universally true for prime powers, namely, that they are of the same form as their roots to modulus double the exponent of the power, *viz.* all 7th powers are of the same form as their roots to modulus 14, and 11th powers of the same form as their roots to modulus 22: and so on for any other prime powers.

VII. *Of the divisors and forms of the Integral Powers of Numbers.*

55. The difference of two equal integral powers is divisible by the difference of their roots.

Let x and y be two numbers, then will

$$\frac{x^n - y^n}{x-y} = \text{M, an integer,}$$

or $$x^n - y^n \doteq \text{M}\,(x-y).$$

Let $x = y + d$, or $x - y = d$, then we have to prove that

$$\frac{(y+d)^n - y^n}{d} = \text{M, an integer.}$$

Make $$1,\ n,\ m,\ p, \text{ \&c. } n,\ 1,$$

to represent the integral coefficients of $y + d$, raised to the n^{th} power, then the above numerator is expressed by

$$d^n + n\,d^{n-1}\,y + m\,d^{n-2}\,y^2 + p\,d^{n-3}\,y^3, \text{ \&c.,}$$

every term of which is obviously divisible by d, and, consequently, the whole number is so, that is,

$$x^n - y^n \text{ is always divisible by } x - y,$$

or, $$x^n - y^n \doteq \text{M}\,(x-y).$$

56. The difference of two equal integral powers is always divisible by the sum of the roots, if the index of the power be an even number; that is

$$x^n - y^n \doteq \text{M}\,(x+y)$$

when n is an even number.

Make $$x + y = s, \text{ or } x = s - y,$$

then, as in the preceding proposition, writing

$$1,\ n,\ m,\ p, \text{ \&c. } n,\ 1,$$

for the coefficient of $\overline{s-y}\,^n$, we have

$$(s-y)^n - y^n =$$
$$s^n - n\,s^{n-1}\,y + m\,s^{n-2}\,y^2 - p\,s^{n-3}\,y^3 + \text{\&c.}$$
$$+ m\,s^2\,y^{n-2} - n\,s\,y^{n-1} + y^n - y^n.$$

In which, as the last two terms destroy each other, and the others are each divisible by s, the whole quantity is divisible by s, that is

$$x^n - y^n \doteq \text{M}\,(x+y)$$

when n is an even number.

57. The sum of two equal odd powers is always divisible by the sum of their roots, or $x^n + y^n \doteq \text{M}\,(x+y)$ when n is an odd number.

Make $x + y = s$, or $x = s - y$, then $x^n + y^n$ becomes

$$(s-y)^n + y^n =$$
$$s^n - n\,s^{n-1}\,y + m\,s^{n-2}\,y^2 - \text{\&c.} - m\,s^2\,y^{n-2} + n\,s\,y^{n-1}$$
$$- y^n + y^n.$$

In which, as before, the last two terms destroy each other, and each of the remaining terms is divisible by s, and therefore the whole remainder is divisible by it; that is,

$$x^n + y^n \doteq \text{M}\,(x+y),$$

when n is an odd number.

Deduction.

By means of the above propositions, we are also enabled to ascertain the divisors of the sum or difference of unequal powers of the same root, *viz.*

$$(x^m - x^n) = \text{M}\,(x-1), \text{ and M}\,(x+1),$$

when $m - n$ is even, or of the form $2\,n'$, for

$$x^m - x^n = x^n \times (x^{m-n} - 1),$$

and since $m - n \doteq 2\,n'$, therefore,

$$(x^{m-n} - 1) = (x^{2n'} - 1^{2n'}) = \text{M}\,(x-1), \text{ and M}\,(x+1);$$

and, consequently,

$$x^n \times (x^{m-n} - 1) = (x^m - x^n) = \text{M}\,(x-1), \text{ and M}\,(x+1).$$

Again, if $n - m$ be odd, or of the form $2\,n' + 1$, then

$$(x^m - x^n) = \text{M}\,(x-1), \text{ and}$$
$$(x^m + x^n) = \text{M}\,(x+1).$$

For

$$(x^m - x^n) = x^n \times (x^{m-n} - 1), \text{ and}$$
$$(x^m + x^n) = x^n \times (x^{m-n} + 1);$$

also, since $m - n \doteq 2\,n' + 1$, therefore,

$$(x^{m-n} - 1) = (x^{2n'+1} - 1^{2n'+1}) = \text{M}\,(x-1), \text{ and}$$
$$(x^{m-n} + 1) = (x^{2n'+1} + 1^{2n'+1}) = \text{M}\,(x+1);$$

and, consequently,

$$x^n \times (x^{m-n} - 1) = (x^m - x^n) = \text{M}\,(x-1),$$
$$x^n \times (x^{m-n} + 1) = (x^m + x^n) = \text{M}\,(x+1).$$

58. If m be a prime number, and x any number not divisible by m, then will the remainder arising from the division of x by m be the same as that from the division of x^m by m.

It is necessary first to show, that if m be any prime number, each of the coefficients of the expanded binomial $(a+1)^m$ is divisible by m, except the first and last. For each of these coefficients is of the form

$$\frac{m\,.\,(m-1)\,(m-2)\,(m-3)\,\text{\&c.}}{1\,.\,2\,.\,3\,.\,4\,\text{\&c.}} = p$$

an integer, or

$$m \times \frac{(m-1)\,(m-2)\,(m-3)\,\text{\&c.}}{2\,.\,3\,.\,4\,\text{\&c.}} = p.$$

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Where the quantity in the parenthesis is obviously integral, because the whole quantity is so, and m is not divisible by any of the factors of the denominator. We have, therefore, $m\,N = p$, consequently, p is always divisible by m when m is a prime number.

This being premised, make $x = x' + 1$, then we have

$$x^m = (x'+1)^m = x'^m + m\,x'^{m-1} + m\,a\,x'^{m-2} + \&c.\ m\,x' + 1.$$

And since each of the terms of this expanded binomial, except the first and last, is divisible by m, it follows, that the remainder from the division of $(x'+1)^m$ by m, is the same as the remainder from the division of $x'^m + 1$ by m, which, by rejecting the multiples of m, may be expressed thus,

$$x^m = (x'+1)^m = x'^m + 1.$$

Making now $x' = x'' + 1$, we shall have, on the same principles,

$$x^m = (x'+1)^m = x'^m + 1 = (x''+1)^m + 1 = x''^m + 2.$$

Again, let $x'' = x''' + 1$, and we obtain

$$x^m = x'^m + 1 = x''^m + 2 = x'''^m + 3.$$

And thus, by continual substitutions, we have

$$x^m = x'^m + 1 = x''^m + 2 = x'''^m + 3 = \&c.;\ \text{or,}$$

$$\begin{cases} x^m = (x-1)^m + 1 = (x-2)^m + 2 = (x-3)^m + 3 \\ \quad \&c.\ (x-x)^m + x, \end{cases}$$

the last of which terms is equal to x; whence it follows, that the remainder arising from the division of x by m is the same as that from the division of x^m by m.

59. If m be a prime number, and x any number not divisible by m, then will the formula $x^{m-1} - 1$ be divisible by m, or, which is the same,

$$(x^{m-1} - 1) = M\,(m).$$

For, by the foregoing proposition, the remainder of $\frac{x}{m}$ is the same as the remainder of $\frac{x^m}{m}$; and, consequently, the difference $x^m - x$ is divisible by m. But $x^m - x = x\,(x^{m-1} - 1)$, and since this product is divisible by m, and the factor x is prime to m, it must be the other factor, *viz.* $(x^{m-1} - 1)$, that is divisible by m.

Deductions.

(1.) Since $x^{m-1} - 1$ is always divisible by m, if x be prime to m, and m itself a prime; there are necessarily $m - 1$ values of x less than m that will satisfy the equation

$$\frac{x^{m-1} - 1}{m} = e, \text{ an integer,}$$

that is, x may be any number in the series

$$1 \,.\, 2 \,.\, 3 \,.\, 4, \&c.\ m - 1,$$

because all these numbers are necessarily prime to m; and since $m - 1$ is an even number, we shall have also $m - 1$ values of x' comprised between the limits $-\frac{1}{2}m$ and $+\frac{1}{2}m$, that is, x may be any number in the series,

$$\pm 1, \pm 2, \pm 3, \pm \&c. \pm \frac{m-1}{2},$$

so that in both cases we have $m - 1$ values of $x < m$, which render the equation

$$\frac{x^{m-1} - 1}{m} = e, \text{ an integer.}$$

Sect. VII. Integral Powers.

(2.) Since $x^{m-1} - 1$ is always divisible by m under the limitations of the proposition, therefore $x^{m-1} \doteqdot a\,m + 1$, and, consequently, every power whose exponent plus 1 is a prime as m, will be of the form $a\,m$ or $a\,m + 1$, and we may thus ascertain the forms of many of the higher powers; thus

$$\begin{aligned} x^4 &\doteqdot 5\,n, \text{ or } 5\,n + 1, \\ x^6 &\doteqdot 7\,n, \text{ or } 7\,n + 1, \\ x^{10} &\doteqdot 11\,n, \text{ or } 11\,n + 1, \\ x^{12} &\doteqdot 13\,n, \text{ or } 13\,n + 1, \\ &\&c., \qquad \&c. \end{aligned}$$

Again, since m is a prime number, if it be greater than 2, it is an odd number; and, consequently, $m - 1$ an even number; and, therefore,

$$x^{m-1} - 1 = \left(x^{\frac{m-1}{2}} + 1\right) \times \left(x^{\frac{m-1}{2}} - 1\right);$$

and, since this product,

$$\left(x^{\frac{m-1}{2}} + 1\right) \times \left(x^{\frac{m-1}{2}} - 1\right),$$

is divisible by m, and m is a prime number, one of these factors must be divisible by m; that is,

$$x^{\frac{m-1}{2}} \doteqdot m\,a \pm 1;$$

and, consequently, every power, the double of whose exponent plus 1 is a prime number, as (m), is of one of the forms

$$a\,m, \text{ or } a\,m \pm 1;$$

and hence, again, we derive the forms of many other higher powers; thus,

$$\begin{aligned} x^3 &\doteqdot 7\,n, \text{ or } 7\,n \pm 1, \\ x^5 &\doteqdot 11\,n, \text{ or } 11\,n \pm 1, \\ x^6 &\doteqdot 13\,n, \text{ or } 13\,n \pm 1, \\ x^8 &\doteqdot 17\,n, \text{ or } 17\,n \pm 1, \\ x^9 &\doteqdot 19\,n, \text{ or } 19\,n \pm 1, \\ x^{11} &\doteqdot 23\,n, \text{ or } 23\,n \pm 1, \\ &\&c. \qquad \&c. \end{aligned}$$

(3.) And hence we have the following forms of all powers from 2 to 12, the 7th power only excepted, which cannot be introduced, because neither $7 + 1$, nor $2 \,.\, 7 + 1$, is a prime number.

Table of the possible forms of Powers from 2 to 12.

$$\begin{aligned} x^2 &\doteqdot 3\,n, \text{ or } 3\,n + 1 \doteqdot 5\,n, \text{ or } 5\,n \pm 1, \\ x^3 &\doteqdot \ldots\ldots\ldots\ldots \doteqdot 7\,n, \text{ or } 7\,n \pm 1, \\ x^4 &\doteqdot 5\,n, \text{ or } 5\,n + 1 \doteqdot \ldots\ldots\ldots\ldots \\ x^5 &\doteqdot \ldots\ldots\ldots\ldots \doteqdot 11\,n, \text{ or } 11\,n \pm 1, \\ x^6 &\doteqdot 7\,n, \text{ or } 7\,n + 1 \doteqdot 13\,n, \text{ or } 13\,n \pm 1, \\ x^8 &\doteqdot \ldots\ldots\ldots\ldots \doteqdot 17\,n, \text{ or } 17\,n \pm 1, \\ x^9 &\doteqdot \ldots\ldots\ldots\ldots \doteqdot 19\,n, \text{ or } 19\,n \pm 1, \\ x^{10} &\doteqdot 11\,n, \text{ or } 11\,n + 1 \doteqdot \ldots\ldots\ldots\ldots \\ x^{11} &\doteqdot \ldots\ldots\ldots\ldots \doteqdot 23\,n, \text{ or } 23\,n \pm 1, \\ x^{12} &\doteqdot 13\,n, \text{ or } 13\,n + 1 \doteqdot \ldots\ldots\ldots\ldots \end{aligned}$$

By means of the above table, we may frequently prove the impossibility of equations of the form

$$a\,x^n \pm b\,y^n = d\,z^n;$$

but it does not follow of course, if they fall within the possible forms, that they are actually resolvable in integers; thus

$$x^3 \pm y^3 = z^3,$$
$$x^4 \pm y^4 = z^4,$$

are impossible, and generally

$$x^n \pm y^n = z^n$$

is impossible if n be greater than 2, although these may not fall under the impossible forms of the table.

60. If m be a prime number, and P be made to represent any polynomial of the n^{th} degree, as

$$P = x^n + a\,x^{n-1} + b\,x^{n-2} + c^{n-3} + \ldots\ldots q,$$

then there cannot be more than n values of x, between the limits $+\frac{1}{2}m$, and $-\frac{1}{2}m$, which render this polynomial divisible by m.

For let k be the first value of x, which renders P divisible by m, so that

$$A\,m = k^n + a\,k^{n-1} + b\,k^{n-2} + e\,k^{n-3} + \ldots\, q;$$

then, by subtraction, we have

$$\begin{cases} P - A\,m = (x - k^n) + a\,(x^{n-1} - k^{n-1}) + \\ \quad b\,(x^{n-2} - k^{n-2}) + \&c. \end{cases}$$

But the latter side of this equation, being divided by $x - k$, (art. 55,) we shall have for a quotient a polynomial of the degree $n - 1$; which, being represented by P′, gives

$$P - A\,m = (x - k)\,P', \text{ or } P = (x - k)\,P' + A\,m.$$

Let now k' be a second value of x, which renders P divisible by m, then it follows, that $(x - k)\,P' + A\,m$ is also divisible by m; and, consequently, $(x - k)\,P'$ divisible by m, but the factor $x - k$, which now becomes $(k' - k)$, cannot be divisible by m, because both k' and k are less than $\frac{1}{2}m$; therefore P cannot be divisible a second time by m, unless P′ be divisible by m.

The polynomial P is therefore only once more divisible by m than the polynomial P′; and, in the same manner, it may be shown, that P′, of the degree $n - 1$, is only once more divisible by m, than P″ of the $n - 2$ degree, &c.; and hence it follows, that P being a polynome of the n degree, there can be only n different values of x, comprised between the limits $+\frac{1}{2}m$, and $-\frac{1}{2}m$, which renders it divisible by m.

Deduction.

We have seen, that if m be a prime number, the formula $x^{m-1} - 1$ has $m - 1$ values of x, between the limits $+\frac{1}{2}m$ and $-\frac{1}{2}m$, which renders it divisible by m. Now this being put under the form

$$\left(x^{\frac{m-1}{2}} + 1\right) \times \left(x^{\frac{m-1}{2}} - 1\right)$$

it follows, that each of the factors has $m - 1$ values of x, between the limits $+\frac{1}{2}m$ and $-\frac{1}{2}m$, which renders them divisible by m. For neither of them can have more than $\frac{m-1}{2}$ such values, by the foregoing proposition, and since their product has $m - 1$, it is obvious they have each the same number of values of x between the above limits, and that this number is therefore $\frac{m-1}{2}$.

VIII. *Of the products and transformations of Algebraical Formulæ referable to the forms of Numbers.*

61. The product of the sum and difference of two quantities, is equal to the difference of their squares.

For $\quad (x + y) \times (x - y) = x^2 - y^2,$

as is evident.

62. The product of the sum of two squares by double a square, is also the sum of two squares, or

$$(x^2 + y^2) \times 2\,z^2 = x'^2 + y'^2.$$

For $(x^2 + y^2) \times 2\,z^2 = (x + y)^2\,z^2 + (x - y)^2\,z^2,$

which is evidently $= x'^2 + y'^2$.

Deduction.

Hence, if a number be the sum of two squares, its double is the sum of two squares; and if N be the sum of two squares, 2^n N will be so likewise.

Thus $\quad 5 = 2^2 + 1^2,\ 5 \times 2 = 10 = 3^2 + 1^2,$

$$10 \times 2 = 20 = 4^2 + 2^2, \text{ and } 40 = 6^2 + 2^2.$$

63. The product arising from the sum of two squares by the sum of two squares, is also the sum of two squares.

Or $\quad (x^2 + y^2)\,(x'^2 + y'^2) = x''^2 + y''^2.$

For

$$(x^2 + y^2)\,(x'^2 + y'^2) = \begin{cases} (x\,x' + y\,y')^2 + (x\,y' - x'\,y)^2, \\ \text{or } (x\,x' - y\,y')^2 + (x\,y' + x'\,y)^2, \end{cases}$$

as will appear from the developement of these expressions, and, consequently,

$$(x^2 + y^2)\,x'^2 + y'^2) = x''^2 + y''^2.$$

Deduction.

Hence the product may be divided into two squares two different ways. And if this product be again multiplied by another, that is the sum of two squares, the resulting product may be divided into two squares four different ways; and, generally, if a number N be the product of n factors, each of which is the sum of two squares, then will N be the sum of two squares, and may be resolved into two squares 2^n different ways.

For example, $\quad 5 = 2^2 + 1^2$

$$13 = 3^2 + 2^2$$

then the product $65 = 8^2 + 1^2$, or $7^2 + 4^2$.

Again, $\quad 17 = 4^2 + 1^2$

the product $\begin{cases} 1105 = 32^2 + 9^2 = 33^2 + 4^2 = 31^2 + 12^2 \\ \quad = 24^2 + 23^2. \end{cases}$

And this resolution of the given product into square parts, is readily effected by the foregoing theorem; for

$$\begin{cases} (8^2 + 1)\,(4^2 + 1^2) = (4\,.\,8 + 1)^2 + (8\,.\,1 - 4\,.\,1)^2 = \\ \quad (4\,.\,8 - 1)^2 + (8\,.\,1 + 4\,.\,1)^2, \text{ and} \end{cases}$$

$$\begin{cases} (7^2 + 4^2)\,(4^2 + 1) = (4\,.\,7 + 1\,.\,4)^2 + (4\,.\,4 - 7\,.\,1)^2 = \\ \quad (4\,.\,7 - 1\,.\,4)^2 + (7\,.\,1 + 4\,.\,4)^2. \end{cases}$$

And in the same manner may any other product, arising from factors of this form, be resolved into its square parts.

64. The product of the sum of three squares by the sum of two squares, is the sum of four squares; or

$$(x^2 + y^2 + z^2)(x'^2 + y'^2) \rightleftharpoons w''^2 + x''^2 + y''^2 + z''^2.$$

For

$$(x^2 + y^2 + z^2)(x'^2 + y'^2) =$$
$$(xx' + yy')^2 + (xy' - yx')^2 + x'^2 z^2 + y'^2 z^2,$$

as will appear from the developement of these formula, and, consequently,

$$(x^2 + y^2 + z^2)(x'^2 + y'^2) \rightleftharpoons w''^2 + x''^2 + y''^2 + z''^2.$$

Thus $\quad 14 = 3^2 + 2^2 + 1^2$

$\qquad\quad 5 = 2^2 + 1^2$

the product $\begin{cases} 70 = (3\,.\,2 + 2\,.\,1)^2 + (2\,.\,2 - 3\,.\,1)^2 \\ \quad + 2^2 + 1^2 = 8^2 + 1^2 + 2^2 + 1^2, \end{cases}$

and a like decomposition may be effected on any other similar product.

65. The product of the snm of four squares by the sum of two squares, is the sum of four squares; that is,

$$(w^2 + x^2 + y^2 + z^2) \times (x'^2 + y'^2) \rightleftharpoons w''^2 + x''^2 + y''^2 + z''^2.$$

For

$$(w^2 + x^2)(x'^2 + y'^2) \rightleftharpoons w''^2 + x''^2,$$
$$(y^2 + z^2)(x'^2 + y'^2) \rightleftharpoons y''^2 + z''^2;$$

consequently,

$$(w^2 + x^2 + y^2 + z^2) \times (x'^2 + y'^2) \rightleftharpoons w''^2 + x''^2 + y''^2 + z''^2.$$

66. The product of the sum of four squares by the sum of four squares, is also of the same form; or

$$\begin{cases} (w^2 + x^2 + y^2 + z^2)(w'^2 + x'^2 + y'^2 + z'^2) \rightleftharpoons \\ \quad w''^2 + x''^2 + y''^2 + z''^2). \end{cases}$$

For

$$(w^2 + x^2 + y^2 + z^2)(w'^2 + x'^2 + y'^2 + z'^2) =$$
$$(ww' + xx' + yy' + zz')^2 + (wx' - xw' + yz' - zy')^2 +$$
$$(wy' - xz' - yw' + zx')^2 + (wz' + xy' - yx' - zw')^2,$$

as will appear Immediately from the davelopement of the above formulæ; and, consequently, the product in question $\rightleftharpoons (w''^2 + x''^2 + y''^2 + z''^2)$.

Deductions.

(1.) As in this product there are only complete squares enter, we may change at pleasure the signs of the simple quantities; and, consequently, there will result several different formulæ equal to the same product, and each equal to the sum of four squares; and in su many different ways may any number that arises frn the product of the factors of the above form, be resolved intn the sum of four squares.

(2.) This proposition may be reudered more general by the following enunciation:

The product of the two formulæ,

$$(w^2 - bx^2 - cy^2 + bcz^2)(w'^2 - bx'^2 - cy'^2 + bcz'^2) \rightleftharpoons$$
$$(w''^2 - bx''^2 - cy''^2 + bcz''^2).$$

For

$$(w^2 - bx^2 - cy^2 + bcz^2)(w'^2 - bx'^2 - cy'^2 + bcz'^2) =$$
$$\begin{cases} (ww' + bxx' \pm cyy' \pm bczz')^2 - \\ b(wx' + w'x \pm cyz' \pm cy'z)^2 - \\ c(wy' - bxz' \pm yw' \mp bzx')^2 + \\ bc(xy' - wz' \pm zw' \mp yx')^2, \end{cases}$$

as will appear from the developement, and, consequently, the product in question is of the same form as each of its factors.

67. The product of the two formulæ $(x^2 - ay^2)$ and $(x'^2 - ay'^2)$ is of the same form as each of them.

For

$$(x^2 - ay^2)(x'^2 - ay'^2) = \begin{cases} (xx' + ayy')^2 - a(xy' + yx')^2 \\ \text{or } (xx' - ayy')^2 - a(xy' - yx')^2, \end{cases}$$

consequently,

$$(x^2 - ay^2)(x'^2 - ay'^2) \rightleftharpoons x''^2 - ay''^2.$$

Hence the product of any number of factors of this form is of the same form as each of its factors.

68. The two formulæ

$$x^2 + y^2 + z^2, \text{ and } x^2 + y^2 + 2z^2,$$

are so related to each other, that the double of the one produces the other; that is

$$2(x^2 + y^2 + z^2) \rightleftharpoons x'^2 + y'^2 + 2z'^2,$$
$$2(x^2 + y^2 + 2z^2) \rightleftharpoons x'^2 + y'^2 + z'^2.$$

For

$$2(x^2 + y^2 + z^2) = (x + y)^2 + (x - y)^2 + 2z^2, \text{ and}$$
$$2(x^2 + y^2 + 2z^2) = (x + y)^2 + (x - y)^2 + (2z)^2,$$

as is obvious.

For example, $\quad 4 = 3^2 + 2^2 + 1^2$

multiplied by $\quad 2$

the product $\begin{cases} = 28 = (3 + 2)^2 + (3 - 2)^2 + 2\,.\,1^2 \\ = 5^2 + 1^2 + 2. \end{cases}$

And $\quad 15 = 3^2 + 2^2 + 2\,.\,1^2$

multiplied by $\quad 2$

the product $\begin{cases} = 30 = (3 + 2)^2 + (3 - 2)^2 + 2^2 \\ = 5^2 + 1^2 + 2^2. \end{cases}$

And the same of all other numbers of these forms.

69. The formula $x^2 - 2y^2$ may be always transformed to another of the form $2x'^2 - y'^2$, and this last may be converted into the former; that is,

$$\begin{cases} x^2 - 2y^2 \rightleftharpoons 2x'^2 - y'^2, \\ 2x^2 - y^2 \rightleftharpoons x'^2 - 2y'^2. \end{cases}$$

For

$$x^2 - 2y^2 = 2(x \pm y)^2 - (x \pm 2y)^2 \rightleftharpoons 2x'^2 - y'^2, \text{ and}$$
$$2x^2 - y^2 = (x \pm 2y)^2 - 2(x \pm y)^2 \rightleftharpoons x'^2 - 2y'^2;$$

as is evident from the developement of these formulæ; and, consequently, a number that is of one of these forms is also of the other.

For example, $\quad 14 = 2\,.\,3^2 - 2^2 = 4^2 - 2\,.\,1^2$;

also, $\quad 28 = 6^2 - 2\,.\,2^2 = 2\,.\,4^2 - 2^2$.

And the same of any other numbers of either of these forms.

70. The formula $x^2 - 5y^2$ may be always transformed to another of the form $5x'^2 - y'^2$, and this last may be converted into the former; that is,

$$\begin{cases} x^2 - 5y^2 \rightleftharpoons 5x'^2 - y'^2, \\ 5x^2 - y^2 \rightleftharpoons x'^2 - 5y'^2. \end{cases}$$

For

$$x^2 - 5y^2 = 5(x \pm 2y)^2 - (2x \pm 5y)^2 \rightleftharpoons 5x'^2 - y'^2, \text{ and}$$
$$5x^2 - y^2 = (5x \pm 2y)^2 - 5(2x \pm y)^2 \rightleftharpoons x'^2 - 5y'^2;$$

and, consequently, any number that is of one of these forms is also of the other.

For example $29 = 7^2 - 5.2^2 = 5.11^2 - 24^2 = 5.3^2 - 4^2$;
and $41 = 5.3^2 - 2^2 = 19^2 - 5.8^2 = 11^2 - 5.4^2$.

And a similar transformation may be made on any other number falling under either of the above forms.

71. If a be any number of the form $b^2 + 1$, then will the formula $x^2 - a y^2$ be resolvable into another of the form $a x'^2 - y'^2$; and, conversely, this last may be transformed into the former; that is,

$$\begin{cases} x^2 - (b^2 + 1) y^2 = (b^2 + 1) x'^2 - y'^2, \text{ and} \\ (b^2 + 1) x^2 - y^2 = x'^2 - (b^2 + 1) y'^2. \end{cases}$$

For

$$x^2 - (b^2+1)y^2 = (b^2+1)(x \pm by)^2 - \{ bx \pm (b^2+1)y \}^2,$$

and

$$(b^2+1)x^2 - y^2 = \{ (b^2+1)x \pm by \}^2 - (b^2+1)(bx+y)^2,$$

the first of which transformed formulæ ie evidently

$$= (b^2 + 1) x'^2 - y'^2; \text{ also the latter}$$
$$= x'^2 - (b^2 + 1) y'^2; \text{ and, consequently,}$$
$$x^2 - a y^2 = a x'^2 - y'^2, \text{ and}$$
$$a x^2 - y^2 = x'^2 - a y'^2, \text{ when}$$
$$a = b^2 + 1.$$

Deduction.

These general formulæ furnish us with many particular cases, which have the singular property of being convertible from one to the other; such are

$$\begin{cases} x^2 - 2y^2 = 2x'^2 - y'^2. \\ 2x^2 - y^2 = x'^2 - 2y'^2. \end{cases}$$
$$\begin{cases} x^2 - 5y^2 = 5x'^2 - y'^2. \\ 5x^2 - y^2 = x'^2 - 5y'^2. \end{cases}$$
$$\begin{cases} x^2 - 10y^2 = 10x'^2 - y'^2. \\ 10x^2 - y^2 = x'^2 - 10y'^2. \end{cases}$$
$$\begin{cases} x^2 - 17y^2 = 17x'^2 - y'^2. \\ 17x^2 - y^2 = x'^2 - 17y'^2. \end{cases}$$

&c. &c.

72. If m and n be the two roots of the quadratic equation $\phi^2 - a\phi + b = 0$, then will the product of the two formulæ $(x + m y)$, and $(x + n y)$, be equal to $x^2 + a x y + b y^2$.

This is evident from the actual multiplication of the factors $(x + m y)$ and $(x + n y)$.

For

$$(x + my)(x + n y) = x^2 + (m + n) x y + m n y^2;$$

and, since m and n are the roots of the equation $\phi^2 - a\phi + b = 0$, we have, from the nature of equations, $m + n = a$, and $m n = b$; and, consequently, the above product becomes

$$x^2 + a x y + b y^2.$$

Deduction.

Hence, conversely, every quantity of the form $x^2 + a x y + b y^2$ may be considered as the product arising from the multiplication of two factors, $(x + m y)$ and $(x + n y)$, m and n being the roots of the quadratic equation

$$\phi^2 - a\phi + b = 0;$$

or, which is the same, m and n being such as to answer the conditions, $m + n = a$, and $m n = b$.

73. The product arising from the multiplication of the two formulæ

$$x^2 + a x y + b y^2, \text{ and } x'^2 + a x' y' + b y'^2,$$

is of the same form as each of them; that is,

$$(x^2 + a x y + b y^2)(x'^2 + a x' y' + b y'^2) = (x''^2 + a x'' y'' + b y''^2).$$

For

$$x^2 + a x y + b y^2 = (x + m y)(x + n y), \text{ and}$$
$$x'^2 + n x' y' + b y'^2 = (x' + m y')(x' + n y');$$

and, therefore, the product in question is the same as the continued product of the four latter factors.

Now,

$$(x + m y)(x' + m y') = x x' + m(x y' + x' y) + m^2 y y',$$

but since m is one of the roots of the equation

$$\phi^2 - a\phi + b = 0,$$

we have $m^2 - a m + b = 0$, whence $m^2 = a m - b$; and substituting this value of m^2, in the above formula, it becomes

$$x x' - b y y' + m(x y' + x' y + a y y').$$

And if, in order to simplify, we make

$$X = x x' - b y y',$$
$$Y = x y' + y x' + a y y',$$

the product of the two factors,

$$(x + m y)(x' + m y') = X + m Y;$$

and, in the same manner, we find

$$(x + n y)(x' + n y') = X + n Y;$$

and, consequently, the whole product will be

$$(X + m Y)(X + n Y) = X^2 + a X Y + b Y^2;$$

that is, the product

$$\begin{cases} (x^2 + a x y + b y^2)(x'^2 + a x' y' + b y'^2) = \\ (x''^2 + a x'' y'' + b y''^2). \end{cases}$$

Deduction.

Hence it follows, that the product of any number of factors of this form; as

$$x^2 + a x y + b y^2,$$
$$x'^2 + a x' y' + b y'^2,$$
$$x''^2 + a x'' y'' + b y''^2,$$

&c. &c.

will always be of the same form as those factors.

Therefore if we make $x = x'$, and $y = y'$, we shall have $X = x^2 - b y^2$, and $Y = 2 x y + n y^2$; and, consequently,

$$(x^2 + a x y + b y^2)^2 = X^2 + a X Y + b Y^2.$$

And, therefore, if it were required to make a square of the expression

$$X^2 + a X Y + b Y^2,$$

we shall only have to give to X and Y the preceding values, whence we readily obtain for the root of the square required the formula

$$x^2 + a x y - b y^2,$$

where x and y may be any numbers at pleasure.

Example 1. Find the values of x and y in the equation

$$x^2 + 3 x y + 5 y^2 = z^2.$$

Here $a = 3$ and $b = 5$, therefore the general values of x and y are

$$\begin{cases} x = t^2 - 5u^2, \\ y = 2t^2 u + 3u^2, \end{cases}$$

where, for distinction sake, we write t and u, in the above formulæ, instead of x and y. Whence, by assuming successively,

$$t = 3, 4, 5, 6, \&c.,$$

$$u = 1, 1, 1, 1, \&c.,$$

we shall have the following corresponding values of x and y:

$$x = 4, 11, 20, 31, \&c.,$$

$$y = 9, 11, 13, 15, \&c.$$

Example 2. Find the values of x and y in the equation

$$x^2 - 7xy + 3y^2 = z^2.$$

Here, since $a = -7$ and $b = 3$, the general values of x and y are

$$\begin{cases} x = t^2 - 3u^2, \\ y = 2tu - 7u^2. \end{cases}$$

And making now

$$t = 4, 5, 6, 7, 8, \&c.,$$

$$u = 1, 1, 1, 1, 1, \&c.,$$

we obtain

$$x = 13, 22, 33, 46, 61, \&c.,$$

$$y = 1, 3, 5, 7, 9, \&c.$$

Each of which corresponding values of x and y answer the required conditions of the equation; and it is manifest, that an infinite number of other values might be obtained, by changing those of t and u.

IX. *On the Quadratic Divisors of Algebraical Formulæ.*

74. If in the indeterminate formula

$$py^2 + 2pqyz + rz^2 = \phi,$$

the coefficients p, q, and r have not all three the same common divisor, and y and z be any numbers whatever prime to each other; and if $2q > p$. or $> r$, this formula may always be transformed to a similar one,

$$p'y'^2 + 2p'q'z'y' + r'z'^2 = \phi,$$

which shall be equal to the same quantity ϕ, and in which $2q'$ shall not exceed either p' or r'.

Let us suppose, first, $2q > p$; and in the case in which also $2q > r$, let p be the least of the two numbers p and r, abstracting from their signs.

Make $y = y' - mz$, m being an indeterminate coefficient; and, substituting for this value of y in the given equation, we have

$$p(y' - mz)^2 + 2qz(y' - mz) + rz^2 = \phi, \text{ or}$$

$$py'^2 - 2(pm - q)y'z + (pm^2 - 2qm + r)z^2 = \phi,$$

where we may always take the indeterminate m, so that $\pm(pm - q) < p$. Calling therefore $\pm(pm - q) = q'$, and $(pm^2 - 2qm + r) = r'$, the transformed formula will be

$$py'^2 + 2q'y'z + r'z^2 = \phi,$$

in which $2q' < p$ (this sign not excluding equality) and in which $pr' - q'^2 = pr - q^2$, for

$$q'^2 = p^2m^2 - 2pqm + q^2,$$

$$pr' = p^2m^2 - 2pqm + rp,$$

therefore, by subtraction,

$$pr' - q'^2 = pr - q^2,$$

where these quantities will always have the same sign.

Since, then, we have $2q > p$, $2q' < p$, it follows that $q' < q$. Hence we have now an equation

$$py'^2 + 2q'y'2 + r'z^2 = \phi,$$

in which the mean coefficient $2q'$ does not exceed the extreme coefficient p; and if at the same time it does not exceed the other extreme coefficient r', the formula is transformed as required. But if $2q$, although $<$ than p, be $> r'$, we may proceed, in a similar manner, to obtain a new transformation, in which the mean coefficient (which we may denote by $2q''$) shall be less than q', and so on again for others, in which the mean coefficient $2q'''$ shall be less than $2q''$. But the series of integers

$$q, q', q'', q''', \&c.$$

cannot go on continually decreasing, without becoming finally less than the extreme coefficients; and, therefore, by continuing these transformations, we must necessarily arrive at that which admits not of any further reduction; and which will be consequently such, that the mean coefficient is less than either of the extremes, or at least not greater than the least of them; for with any formula in which this is not the case further reduction may be made. Therefore every formula

$$py^2 + 3qyz + rz^2$$

in which the mean coefficient $2q$ exceeds either, or both, of the extreme coefficients, may be transformed to another in which the mean coefficient $2q'$ shall be less than either of the extreme coefficients, or at least not greater than the least of them.

Deductions.

(1.) In the successive transformations of the formula

$$py^2 + 2qyz + rz^2, \text{ to}$$

$$py'^2 + 2q'y'z + r'z^2, \text{ to}$$

$$p'y'^2 + 2q''y'z' + r'z'^2, \&c.;$$

we have always

$$pr - q^2 = pr' - q'^2 = p'r' - q''^2, \&c.,$$

each of these quantities having the same sign, as is obvious from the form of the preceding transformations.

(2.) As an example of the reduction stated in the foregoing proposition, let it be proposed to transform

$$35y^2 + 172yz + 210z^2 = \phi,$$

in which the mean coefficient 172 is greater than the extreme coefficient 35 to another equal and similar one, in which the mean coefficient shall be less than either of the extremes.

First, put $y = y' - mz$, which value of y, being substituted in the given formula, gives

$$35y'^2 - (70m - 172)y'z + (35m^2 - 172m + 210)z^2.$$

And now, in order that $70m - 172 < 35$, take $m = 2$, which reduces the above to

$$35y'^2 + 32y'z + 6z^2 = \phi,$$

in which the mean coefficient 32, though < 35, is still > 6; and, therefore, we must proceed to another similar reduction.

Let, then, $z = z' - m y'$, and the second transformed formula will become

$$6 z'^2 - (12 m - 32) y' z' + (6 m^2 - 32 m + 35) y'^2.$$

And here, taking $m^2 = 3$ in order that $12 m - 32 < 6$, we obtain

$$6 z'^2 - 4 z' y' - 7 y'^2 = \phi,$$

and this last formula has the required conditions; because $4 < 6$ and < 7.

And moreover, in these transformations, we have

$$p r - q^2 = p r' - q'^2 = p' r' - q''^2, \text{ or}$$

$$35 . 210 - (66)^2 = - 46,$$

$$35 . \quad 6 - (16)^2 = - 46,$$

$$-6 . \quad 7 - (\ 2)^2 = - 46,$$

all equal, and with the same sign, as observed in the foregoing deduction.

75. Every divisor of the formula $t^2 + a u^2$, in which t and u are prime to each other, and a any integer number whatever, positive or negative, is also a divisor of the formula $q^2 + a$.

For let p represent any divisor of the formula $t^2 + a u^2$, so that

$$t^2 + a u^2 = p p',$$

then it is evident, that p is prime to u, for otherwise t and u must have the same common measure, which is contrary to the hypothesis, because t is prime to u; we may, therefore, find two other numbers, q and y, such that $t = p y + q u$, q being $+$ or $-$ as the case may require: and if now we substitute this value of t, in the above expression, we obtain

$$p^2 y^2 + 2 p q y u + (q^2 + a) u^2 = p p';$$

or, dividing by p, we have

$$p y^2 + 2 q y u + \left(\frac{q^2 + a}{p}\right) u^2 = p';$$

and, consequently, since p' is an integer, $(q^2 + a) u^2$ is divisible by p, but we have seen that u is prime to p, and, therefore, it must be the other factor, $(q^2 + a)$, that is divisible by p, therefore, if p be a divisor of the formula $t^2 + a u^2$, t and u being prime to each other, it is also a divisor of the more simple formula $q^2 + a$.

Hence, conversely, if p be not a divisor of the formula $q^2 + a$, in which there is only one indeterminate quantity q, it cannot be a divisor of the more general formula $t^2 + a u^2$, in which there are two indeterminates prime to each other.

76. Every divisor of the formula $t^2 + a u^2$, in which t and u are prime to each other, is of the form $p y^2 + 2 q y u + r u^2$; and in this formula $p r - q^2 = u$ $2 q < p$ and $< r$, or not greater than p or r

By the foregoing proposition we have

$$p y^2 + 2 q y z + \left(\frac{q^2 + a}{p}\right) u^2 = p';$$

and since $\frac{q^2 + a}{p}$ is an integer, make $\frac{q^2 + a}{p} = r$, then the above becomes

$$p y^2 + 2 q y u + r u^2 = p';$$

that is, the factor

$$p' = p y^2 + 2 q y u + r u^2;$$

but p' may equally represent any one of the factors or divisors of $t^2 + a u^2$, and, consequently, any factor or divisor of the formula $t^2 + a u^2$ is of the form

$$p y^2 + 2 q y u + r u^2.$$

And, again, since $\frac{q^2 + u}{p} = r$, $p r - q^2 = a$, and we have seen how every indeterminate formula

$$p y^2 + 2 q y u + r u^2$$

may be transformed to a similar and equal formula, so that $2 q < p$ or $< r$, and in which $p r - q^2$ is always equal to the same constant quantity. Consequently every divisor of the formula $t^2 + a u^2$ has the property stated in the head of the proposition.

Deductions.

(1.) Because $2 q < p$, and $2 q < r$, independently of the signs of these quantities, we have $4 q^2 < p r$; and since $p r - q^2 = a$, it follows, that when a is negative, p or r, that is $p r$ is also negative, for otherwise $p r - q^2$ would not have the same sign as a; which we have seen is always the case in every transformation. Hence

(2.) Every divisor of the formula $t^2 + a u^2$, when a is positive, may be represented by the formula

$$p y^2 + 2 q y z + r z^2,$$

in which $p r - q^2 = a$, $2 q < p$, $2 q < r$, and, consequently, $4 q^2 < p r$, and therefore $p r - q^2 = a > 3 q^2$, or $q < \sqrt{\frac{a}{3}}$, as is evident.

(3.) And every divisor of the formula $t^2 - a u^2$ may be represented by the formula $p y^2 + 2 q y z - r z^2$, in which $p r - q^2 = - a$, or $p r + q^2 = a$; and here, since $p r < 4 q^2$, we must have $q < \sqrt{\frac{a}{5}}$.

(4.) We may have cases in which $p = r = 2 q$, as, for example, when $p = 2$, $q = 1$, and $r = 2$; for then $2 q$ does not exceed either p or r, neither are p, q, and r, divisible by the same number, which condition is, therefore, strictly within the limits of the proposition; and hence it follows, that we must not consider the sign $<$ in the two expressions $q < \sqrt{\frac{a}{3}}$ and $q < \sqrt{\frac{a}{5}}$, to exclude equality.

77. Every divisor of the formula $t^2 + u^2$, t and u being prime to each other, is always of the same form $y^2 + z^2$. Or the sum of two squares, which are prime to each other, can only be divided by numbers that are also the sum of two squares.

For by deduction 2 of the foregoing proposition, every divisor of the formula $t^2 + a u^2$ is included in the formula

$$p y^2 + 2 q y z + r z^2,$$

and in which $q < \sqrt{\frac{a}{3}}$, and $p r - q^2 = a$.

Now in the present case $a = 1$, therefore,

$q < \sqrt{\frac{1}{3}}$, or $q = 0$, there being no integer $< \sqrt{\frac{1}{3}}$; and, since $p r - q^2 = 1$, we have $p r = 1$

and therefore $p = 1$, and $r = 1$; and, consequently, the above formula, which includes all the divisors of $t^2 + u^2$, becomes

$$y^2 + z^2;$$

that is, every divisor of the formula $t^2 + u^2$ is of the form $y^2 + z^2$, or every divisor of the sum of two squares, prime to each other, is also the sum of two squares.

Deductions.

(1.) As an example, $65 = 64 + 1$, or $8^2 + 1^2$ is only divisible by $13 = 3^2 + 2^2$, and by $5 = 2^2 + 1$.

(2.) And $50 = 7^2 + 1^2$ is only divisible by $5 = 2^2 + 1^2$, by $10 = 3^2 + 1^2$, by $2 = 1^2 + 1^2$, and by $25 = 4^2 + 3^2$; and the same obtains with the divisors of every number that is the sum of two squares prime to each other.

78. Every divisor of the formula $t^2 + 2u^2$, t and u being prime to each other, is of the same form $y^2 + 2z^2$; or the divisors of the sum of a square, and double a square, are also each equal to the sum of a square and double a square.

For every divisor of this formula $t^2 + a u^2$ is contained in the formula

$$p y^2 + 2 q y z + r z^2,$$

in which $q < \sqrt{\frac{a}{3}}$, and $p r - q^2 = a$, (art. 76,—2.)

But in this case $a = 2$, therefore $q < \sqrt{\frac{2}{3}}$, or $q = 0$; also, since $p r - q^2 = 2$, we have $p r = 2$, whence $p = 2$, and $r = 1$, or $p = 1$, and $r = 2$; therefore, the above formula becomes

$$\begin{cases} 2y^2 + z^2, & \text{in the first case, and} \\ y^2 + 2z^2, & \text{in the second,} \end{cases}$$

which are two identical forms, by changing y into z, and z into y; consequently, every divisor of the formula $t^2 + 2u^2$ is also of the same form as itself.

With regard to the divisor 2, it can only be of the form $y^2 + 2z^2$, when $y = 0$ and $z = 1$; so that, in this case, we have $0^2 + 2 . 1^2$.

As an example to this proposition, we may take $99 = 1 + 2 . 7^2$, which can only be divided by

$$3 = 1^2 + 2 . 1^2,$$
$$9 = 1^2 + 2 . 2^2,$$
$$11 = 3^2 + 2 . 1^2,$$
$$33 = 5^2 + 2 . 2^2;$$

and it is the same with every number that is contained under the above form.

79. Every divisor of the formula $t^2 - 2u^2$, t and u being prime to each other, is of the same form $y^2 - 2z^2$.

For since every divisor of the formula $t^2 - a u^2$ is contained in the formula

$$p y^2 + 2 q y z - r z^2,$$

in which $p r + q^2 = a$, and also $q < \sqrt{\frac{a}{5}}$, or $< \sqrt{\frac{2}{5}}$, (art. 76,—3,) it follows, that in this case $q = 0$, whence also $p r = 2$, and therefore $p = 2$, $r = 1$, or $p = 1$, and $r = 2$; consequently, the above formula becomes either

$$2y^2 - z^2, \text{ or } y^2 - 2z^2,$$

which two forms are the same; because

$$2y^2 - z^2 = (2y \pm z)^2 - 2(y \pm z)^2,$$

which is the same form Therefore every divisor of the form $t^2 - 2u^2$ is of the same form, or the divisors of the difference between a square and double a square is also the difference between a square and double a square.

Thus, $98 = 10^2 - 2 . 1^2$, has for divisors

$$2 = 2^2 - 2 . 1^2,$$
$$7 = 3^2 - 2 . 1^2,$$
$$14 = 4^2 - 2 . 1^2,$$
$$49 = 9^2 - 2 . 4^2;$$

and the same obtains with all numbers falling under the form $t^2 - 2u^2$.

80. Every odd divisor of the formula $p^2 + 3u^2$ is of the same form, *viz.* $y^2 + 3z^2$.

For since all its divisors are contained in the formula

$$p y^2 + 2 q y z + r z^2,$$

in which $p r - q^2 = a$, or $p r - q^2 = 3$, and also $q =$ or $< \sqrt{\frac{3}{3}}$, we must have $q = 1$, or $q = 0$; therefore, in the first case, since $2q$ is not greater than p, or r, and $p r - q^2 = 3$, we must have $p = 2$, and $r = 2$, whence the formula becomes

$$2y^2 + 2 q y z + 2z^2;$$

but as this is evidently an even divisor, it does not belong to the class at present under consideration, which only relates to the odd divisors of the given formula.

In our case, therefore, $q = 0$, and, consequently, $p r - q^2 = 3$, or $p r = 3$; therefore $p = 3$ and $r = 1$, or $p = 1$ and $r = 3$, whence the above formula is reduced to

$$3y^2 + z^2, \text{ or } y^2 + 3z^2;$$

which are identical as to their form, and therefore every odd divisor of the formula $t^2 + 3u^2$ is of the same form $y^2 + 3z^2$.

Deduction.

When the divisor $= 3$, then $y = 0$, but in all other cases y and z are real quantities.

For example, $133 = 5^2 + 3 . 6^2$.

its divisors $\begin{cases} 19 = 4^2 + 3 . 1^2, \\ 7 = 2^2 + 3 . 1^2. \end{cases}$

Again, $1209 = 3^2 + 3 . 20^2$.

its divisors, $\begin{cases} 13 = 1^2 + 3 . 2^2, \\ 31 = 2^2 + 3 . 3^2, \\ 39 = 6^2 + 3 . 1^2, \\ 93 = 9^2 + 3 . 2^2. \end{cases}$
&c. &c.

81. Every odd divisor of the formula $t^2 - 5u^2$ is also of the same form $y^2 - 5z^2$.

For all its divisors are contained in the formula

$$p y^2 + 2 q y z - r z^2,$$

in which $-pr - q^2 = -a$, or $pr + q^2 = 5$, and $q =$ or $< \sqrt{\frac{5}{5}}$; and, consequently, $q = 1$ or 0; but the first case gives only even divisors, the same as in the foregoing proposition; and the latter case of $q = 0$ reduces the above formula to

$$5y^2 - z^2, \text{ or } y^2 - 5z^2,$$

which are identical forms; because

$$5y^2 - z^2 = (5y \pm 2z)^2 - 5(2y \pm z)^2;$$

and, consequently, every odd divisor of the formula $t^2 - 5u^2$ is itself of the same form.

As examples, we have

$$95 = 10^2 - 5 \,.\, 1^2.$$

$$\text{its divisors} \begin{cases} 5 = 5^2 - 5 \,.\, 2^2, \\ 19 = 7^2 - 5 \,.\, 2^2. \end{cases}$$

$$\text{Again,} \quad 395 = 20^2 - 5 \,.\, 2^2.$$

$$\text{its divisors} \begin{cases} 5 = 5^2 - 5 \,.\, 2^2, \\ 79 = 18^2 - 5 \,.\, 7^2. \end{cases}$$

&c. &c.

Deductions.

(1.) From the foregoing proposition it appears, that all numbers which are comprised in the following formulæ,

$$\left.\begin{matrix} t^2 + u^2 \\ t^2 + 2u^2 \end{matrix}\right\} \left.\begin{matrix} t^2 - 2u^2 \\ t^2 + 3u^2 \end{matrix}\right\} \text{and } t^2 - 5u^2,$$

t and u being prime to each other, are of the same form as the numbers they divide, excepting only the two latter, $t^2 + 3u^2$ and $t^2 - 5u^2$, when these are the doubles of an odd number.

(2.) It frequently happens, that a number falls under two or more of the above forms, in which case its divisors are also of the same double or treble forms; and in some cases we have numbers that belong to each of the forms above given. Thus,

$$\left.\begin{matrix} 241 = 15^2 + 4^2 \\ 241 = 13^2 + 2.6^2 \end{matrix}\right\} \left.\begin{matrix} = 21^2 - 10^2 \\ = 7^2 + 3.8^2 \end{matrix}\right\} = 31^2 - 5.12^2$$

X. *Of the classification of Prime Numbers, according to their quadratic forms.*

82. We have already treated of the linear forms of prime numbers, but there are several curious properties of these numbers, depending on their quadratic forms, which ought to find a place in an article of this kind; several of these are the immediate consequence of some of our preceding propositions, and others deducible from them. Of the former, the following theorems may be enumerated, which, however, applies to all odd numbers whatever.

(1.) Every odd number which is the sum of two squares, is of the form $4n + 1$; that is, every odd number represented by the formula

$$y^2 + z^2 = 4n + 1.$$

(2.) Every odd number represented by the formula

$$y^2 + 2z^2 = 8n + 1, \text{ or } 8n + 3.$$

(3.) Every odd number represented by the formula

$$y^2 - 2z^2 = 8n + 1, \text{ or } 8n + 7;$$

and from these arise, by way of exclusion, the three following:

(4.) No number of the form $4n - 1$ can be represented by the formula $y^2 + z^2$.

(5.) No number of the form $8n + 5$, or $8n + 7$, can be represented by the formula $y^2 + 2z^2$.

(6.) No number of the form $8n + 3$, or $8n + 5$, can be represented by the formula $y^2 - 2z^2$.

83. Every prime number of the form $4n + 1$ is the sum of two squares, or is contained in the formula $y^2 + z^2$.

For let m represent a prime number of this form, or $m = 4n + 1$; then (art. 59)

$$(x^{m-1} - 1) = \text{M}(m), \text{ or } (x^{4n} - 1) = \text{M}(m).$$

But $x^{4n} - 1 = (x^{2n} + 1)(x^{2n} - 1)$, and each of these factors has $2n$ values of x contained between the limits $+\frac{1}{2}m$ and $-\frac{1}{2}m$, that render them divisible by m, (art. 60,) whence the factor $x^{2n} + 1$ is divisible by m; but $x^{2n} + 1$ is the sum of two squares, and therefore its divisor m is also the sum of two squares; because every divisor of the formula $t^2 + u^2$ is itself of the same form.

Deductions.

(1.) As the form $4n + 1$ includes the two, $8n + 1$ and $8n + 5$; therefore every prime number contained in these two latter forms is also the sum of two squares.

Thus, 5, 13, 17, 29, 37, and 41, are prime numbers of the form $4n + 1$, and each of these is the sum of two squares; for $5 = 2^2 + 1^2$, $13 = 3^2 + 2^2$, $17 = 4^2 + 1$, $29 = 5^2 + 2^2$, $37 = 6^2 + 1^2$, and $41 = 5^2 + 4^2$; and so on for all other prime numbers of this form.

(2.) We have seen (art. 63) that every number, which is produced from the multiplication of factors that are the sums of two squares, is itself of the same form, and may be resolved into two squares different ways, according to the number of its factors; and hence we may find a number, that is resolvable into two squares as many ways as we please, by multiplying together different prime numbers of the form $4n + 1$.

84. Every prime number $8n + 1$ is of the three forms $\quad y^2 + z^2,\ y^2 + 2z^2,\ y^2 - 2z^2.$

Let m be any prime number of this form, or

$$m = 8n + 1;$$

and as the first case has been demonstrated in the preceding proposition, we need here only attend to the two latter.

Since $(x^{m-1} - 1) = \text{M}(m)$, or $x^{8n} - 1 = \text{M}(m)$, (art. 59,) we may put this under the form

$$(x^{4n} + 1)(x^{4n} - 1),$$

and each of these factors will have $4n$ values of $x < \frac{1}{2}m$ that render them divisible by m, (art. 60;) there are, therefore, so many different values of x that render the binomial $x^{4n} + 1$ divisible by m; but this may be put under the form

$$(x^{2n} - 1)^2 + 2x^{2n};$$

and m being a divisor of this formula, it is itself of the same form $y^2 + 2z^2$, (art. 78.) We may also put the same quantity $x^{4n} + 1$ under the form $(x^{2n} + 1)^2 - 2x^{2n}$, and m being also a divisor of this formula is itself of the same form $y^2 - 2z^2$, (art. 79.) Hence every prime number of the form $8n + 1$ is of the three forms $\quad y^2 + z^2,\ y^2 \pm 2z^2.$

Thus $\begin{cases} 41 = 5^2 + 4^2 = 3^2 + 2 \,.\, 4^2 = 7^2 - 2 \,.\, 2^2 \\ 73 = 8^2 + 3^2 = 1^2 + 2 \,.\, 6^2 = 9^2 - 2 \,.\, 2^2 \end{cases}$

85. Every prime number $8n + 3$ is of the form $y^2 + 2z^2$.

Let m be a prime number of this form, or $m = 8n + 3$, then we have (art. 59)

$$(x^{m-1} - 1) = \mathrm{M}\,(m), \text{ or } x^{8n+2} - 1 = \mathrm{M}\,(m).$$

And there are $8n + 2$ values of x less than $8n + 2$, which reoder this formula divisible by m.

Now $2^{8n+2} - 1 = (2^{4n+1} + 1)\,(2^{4n+1} - 1) = \mathrm{M}\,(m)$, therefnre one of these factors is divisible by m, and it cannot be the latter, because

$$2^{4n+1} - 1 = 2 \,.\, 2^{4n} - 1 \doteq 2t^2 - u^2, \text{ or } t^2 - 2u^2,$$

and if m were a divisor of this it would be of the same form, or $m \doteq y^2 - 2z^2$, but this formula cannot represent any number of the form $8n + 3$, (art. 82.) Consequently, m must be a divisor of the other factor $2^{4n+1} + 1$. But

$$2^{4n+1} + 1 = 2 \,.\, 2^{4n} + 1 \doteq 2t^2 + u^2.$$

Consequently its divisor m is of the same form; that is,

$$m \doteq 2y^2 + 1z^2 \doteq y^2 + 2z^2.$$

As examples, we have

$$11 = 3^2 + 2 \,.\, 1^2,\ 19 = 1^2 + 2 \,.\, 3^2,\ 43 = 5^2 + 2 \,.\, 3^2, \text{ \&c.}$$

86. Every prime number $8n + 7$ is of the form $y^2 - 2z^2$.

Let $m = 8n + 7$, then we have

$$x^{m-1} - 1 = x^{8n+6} - 1 = \mathrm{M}\,(m).$$

Heoce, therefore, as above

$$2^{8n+6} - 1 = (2^{4n+3} + 1)\,(2^{4n+3} - 1) = \mathrm{M}\,(m),$$

one of these factors is divisible by m; and, consequently, m will also be a divisor of one of them when doubled; that is, it is a divisor of one of the two quantities

$$2\,(2^{4n+3} + 1), \text{ or } 2\,(2^{4n+3} - 1),$$

which two expressions thus become

$$2^{4n'} + 2 \,.\, 1^2, \text{ and } 2^{4n'} - 2 \,.\, 1^2,$$

and m is necessarily a divisor of one of them. But it cannot be a divisor of the first, because this being of the form $t^2 + 2u^2$, if m were a divisor of it, we should have $m \doteq y^2 + 2z^2$, (art 78;) but $m \doteq 8n + 7$, and no odd number of the form $y^2 + 2z^2$ is of the form $8n + 7$, (art. 82:) since, therefore, m is not a divisor of this factor, it must necessarily be a divisor of the other factor $2^{4n'} - 2 \,.\, 1^2$, which is of the form $t^2 - 2u^2$; and, consequently, its divisor m is also of the same form, (art. 79;) that is, $m \doteq y^2 - 2z^2$.

For example, $31 = 7^2 - 2 \,.\, 3^2$, and $47 = 7^2 - 2 \,.\, 1^2$; and the same of all other prime numbers in this form.

Deductions.

From the last four propositions we may draw the following *theorems:*

(1.) All prime numbers of the form $8n + 1$, and $8n + 5$, are, exclusively of all others, contained in the formula $y^2 + z^2$.

(2.) All prime numbers of the form $8n + 1$, and $8n + 3$, are, exclusively of all others, contained in the formula $y^2 + 2z^2$.

(3.) All prime numbers of the form $8n + 1$, and $8n + 7$, are, exclusively of all others, contained in the formula $y^2 - 2z^2$.

(4.) All prime numbers of the form $8n + 1$, are at the same time of the three forms

$$y^2 + z^2,\ y^2 + 2z^2,\ y^2 - 2z^2.$$

87. If a be eny prime number, and the series of squares

$$1^2, 2^2, 3^2, 4^2, \text{\&c.}, \left(\frac{a-1}{2}\right)^2$$

be divided by a, they will each leave a different positive remainder.

This is, in fact, only a particular case of the general proposition demonstrated (art. 38;) for, by making $\phi = 1$, the series of squares,

$$\phi^2, 2^2\phi^2, 3^2\phi^2, 4^2\phi^2, \text{\&c.}, \left(\frac{a-1}{2}\right)\phi^2,$$

becomes

$$1^2, 2^2, 3^2, 4^2, \text{\&c.}, \left(\frac{a-1}{2}\right)^2,$$

each of which, when divided by a, will leave e different remainder, as is demonstrated in that article.

Deductions.

(1.) The same is evidently true of the negative remainders, which arise from taking the quotieots in excess.

(2.) Hence, also, we may see in what cases the positive and negative remainders are equal to each other, for then it is evideot, that a will be a divisor of the sum of two squares, and we shall have $\dfrac{r^2 + s^2}{a} = e$, an integer.

Therefore when a is nnt a divisor of the sum of two squares, the positive and negative remainders are all different from each other, and include every number from 1 to $a - 1$.

(3.) When a is not the divisor of the sum of two ..ares, that is, when ell the positive and negative remainders are different from each other, then some of each of these remainders are *greater* and some *less* than $\frac{1}{2}a$. For all the consecutive squares under a will be found amongst the positive remaioders, and some of these squares must necessarily be greater and some less than $\frac{1}{2}a$; and since the positive and negative remainders together include all numbers from 1 to $a - 1$, the same is manifestly true of the negative remaioders.

88. If a be a prime number, it is always possible to find four squares, w^2, x^2, y^2, z^2, the roots of each of which shall be less than $\frac{1}{2}a$, such that their sum may be divisible by a, or the equation

$$w^2 + x^2 + y^2 + z^2 = a\,a'$$

is always possible, a being any prime number what ever.

First, when the prime number a is a divisor of the sum of two squares, the proposition is evident; and it will, therefore, only be necessary to consider the case in which a is not e divisor of the sum of two squares, end, consequently, when all the remainders of the consecutive squares are differeot from each other (art. 87,—2.)

Now, in this case, we shall find some of the positive

remainders *greater*, and some *less*, than $\frac{1}{2}a$; and the same of the negative remainders, (art. 87,—3.) It is, therefore, always possible to find two squares, such that each being divided by a, the positive remainder of the one shall exceed the negative remainder of the other, by unity: and also two other squares in the same series, such that each being divided as before, the negative remainder of the one shall exceed the positive remainder of the other, by unity; that is, the equations $w^2 + x^2 - 1 = m\,a$, and $y^2 + z^2 + 1 = n\,a$, are always possible, which may be demonstrated as follows:

Let p be the least negative remainder, then $p + 1$ must be found amongst either the positive or negative remainders; if it be found amongst the positive remainders, we have at once a positive remainder, that exceeds a negative remainder, by unity; and if it be not found amongst the positive, then $p + 1$ is still negative: and $p + 2$ must be either a positive or negative remainder; if it be positive, we have a positive remainder exceeding a negative one, by unity, but if not, $p + 2$ is still negative, and $p + 3$ must be either positive or negative; and proceeding thus, we must necessarily (as some of each of these remainders are greater and some less than $\frac{1}{2}a$) arrive at that negative remainder p', such that $p' + 1$ shall be a positive one; and, consequently, the equation $w^2 + x^2 - 1 = m\,a$ is always possible; and, in the same manner, the possibility of the equation $y^2 + z^2 + 1 = n\,a$ may be demonstrated. Having thus proved the possibility of the equation $w^2 + x^2 - 1 = m\,a$, and $y^2 + z^2 + 1 = n\,a$, we have

$$\frac{w^2 + x^2 + y^2 + z^2}{a} = m + n, \text{ an integer,}$$

or the equation

$$w^2 + x^2 + y^2 + z^2 = a\,a'$$

is always possible.

Deduction.

(1.) It is obvious from the foregoing demonstration, that the roots w, x, y, z are each less than $\frac{1}{2}a$, because we have only considered the squares contained in the series

$$1^2,\ 2^2,\ 3^2,\ 4^2,\ \&c., \left(\frac{a-1}{2}\right)^2.$$

But independently of this limitation it may be readily shown, that if a be the divisor of the sum of any four squares $w^2 + x^2 + y^2 + z^2$, each of which is prime to a, that it is also a divisor of the sum of the four squares

$$(w - \alpha\,a)^2 + (x - \beta\,a)^2 + (y - \gamma\,a)^2 + (z - \delta\,a)^2,$$

in which it is obvious, that $\alpha, \beta, \gamma, \delta$, may always be so taken as to make the roots less than $\frac{1}{2}a$

89. Every prime number a is the sum of two, three, or four squares.

For, by the foregoing proposition, the equation

$$w^2 + x^2 + y^2 + z^2 = a\,a'$$

is always possible, each of the roots of these squares being less than $\frac{1}{2}a$; and, consequently, each of the squares less than $\frac{1}{4}a^2$, whence we have $a\,a' < a^2$, or $a' < a$. Now, if $a' = 1$, it is evident that

$$w^2 + x^2 + y^2 + z^2 = a,$$

and the proposition will be demonstrated.

But if $a' > 1$, then, because a' is a divisor of the formula

$$w^2 + x^2 + y^2 + z^2,$$

it is also a divisor of the formula

$$(w - \alpha\,a')^2 + (x - \beta\,a')^2 + (y - \gamma\,a')^2 + (z - \delta\,a')^2,$$

where each of the roots is less than $\frac{1}{2}a'$, (art. 88,—1;) assuming, therefore,

$$(w-\alpha\,a')^2 + (x-\beta\,a')^2 + (y-\gamma\,a')^2 + (z-\delta\,a')^2 = a''\,a',$$

we shall have, for the same reason as above,

$$a''\,a' < a'^2, \text{ or } a'' < a'.$$

Now, by means of the formula (art. 66,) if we multiply together the values of $a\,a'$, and $a''\,a'$, we shall find a product that is the sum of four squares, and of which each is divisible by a'^2; and having performed this division, we obtain

$$a''a = (a - \alpha w - \beta x - \gamma y - \delta z)^2 + (\alpha x - \beta w + \gamma z - \delta y)^2$$
$$+ (\alpha y - \gamma w + \delta x - \beta z)^2 + (\alpha z - \delta w + \beta y - \gamma x)^2;$$

or, for the sake of abridging this expression,

$$w'^2 + x'^2 + y'^2 + z'^2 = a''\,a;$$

and here we have $a'' < a'$. If now $a'' = 1$, the above becomes

$$w'^2 + x'^2 + y'^2 + z'^2 = a,$$

and the proposition will be demonstrated; but if a'', though $< a'$, be > 1, we may proceed, in the same manner, to find a new product,

$$w''^2 + x''^2 + y''^2 + z''^2 = a'''\,a,$$

and in which $a''' < a''$; and by continuing thus the decreasing series of integers $a, a', a'', a''', a'''', \&c.$, we must necessarily, finally, arrive at a term $a^{(m)}$ equal to unity, and then we shall have a equal to the sum of four squares.

90. Every integral number whatever is either a square, or the sum of two, three, or four squares.

This follows immediately from the foregoing proposition, and the formula, (art. 65;) for every number is either a prime, or produced by the multiplication of prime factors; and since every prime number is of the form

$$(w^2 + x^2 + y^2 + z^2),$$

and the product of two or more such formulæ being still of the same form, (art. 65,) it necessarily follows, that every integral number whatever is of the form

$$(w^2 + x^2 + y^2 + z^2).$$

But it is to be observed, that no limitation in the course of the demonstration of the foregoing proposition was made, that could prevent any one or more of these squares from becoming zero; therefore, every integral number whatever is either a square, or the sum of two, three, or four squares.

Deductions.

(1.) All that has been proved in the foregoing proposition for integral numbers, is equally true of fractions; for every fraction may be expressed by an equivalent one having a square denominator; therefore, every fraction is of the form

$$\frac{w^2 + x^2 + y^2 + z^2}{m^2} = \frac{w^2}{m^2} + \frac{x^2}{m^2} + \frac{y^2}{m^2} + \frac{z^2}{m^2};$$

this curious property, therefore, extends to every rational number whatever.

(2.) The theorem that we have demonstrated, in the two foregoing propositions, forms a part of a general property of polygonal numbers, discovered by Fermat; which is this, "Every number is either a triangular number, or the sum of two or three triangular numbers. A square, or the sum of two, three, or four squares. A pentagonal, or the sum of two, three, four, or five pentagonals. And so on for hexagonals," &c. Or the same may be more generally expressed thus: If m represent the denomination of any order of polygonals, then is every number N the sum of m polygonals, of that order; it being understood that any of these polygonals may become zero.

Let, therefore, N be any given number, and x, y, z indeterminate quantities; then the different parts of the general theorem may be detailed in the following order:

$$\text{1st, } N = \frac{x^2+x}{2} + \frac{y^2+y}{2} + \frac{z^2+z^2}{2};$$

$$\text{2d, } N = w^2 + x^2 + y^2 + z^2;$$

$$\text{3d, } N = \frac{3u^2-u}{2} + \frac{3w^2-w}{2} + \frac{3x^2-x}{2} + \frac{3y^2-y}{2} + \frac{3z^2-z}{2};$$

4th, &c. &c.

The second form which relates to the squares has been demonstrated in the foregoing proposition, and Legendre has also demonstrated the first case, for triangular numbers; but all the other cases, past the second, still remain without demonstration, notwithstanding the researches and investigations of many of the ablest mathematicians of the present time, and of others now no more: amongst the former we may mention Lagrange, Legendre, and Gauss; and of the latter, Euler, Waring, and Fermat himself; the latter of whom, however, as appears from one of his notes on Diophantus, was in possession of the demonstration, although it was never published, which circumstance renders the theorem still more interesting to mathematicians, and the demonstration of it the more desirable.

We have demonstrated the second case, but this carries us no farther, whereas, if we had demonstrated the first, the second would flow from it as a corollary; and it may not be uninteresting to show in what manner these different parts of the same theorem are connected with each other.

First, let us suppose the possibility of the equation

$$N = \frac{x^2+x}{2} + \frac{y^2+y^2}{2} + \frac{z^2+z}{2}$$

to have been demonstrated, from which may be drawn this,

$$8N + 3 = (2x+1)^2 + (2y+1)^2 + (2z+1)^2, \text{ or}$$

$$8N + 3 = x'^2 + y'^2 + z'^2, \text{ or}$$

$$8N + 4 = x'^2 + y'^2 + z'^2 + 1;$$

and since these four squares are all odd, the numbers $x' + y'$, $x' - y'$, $x' + 1$, and $z' - 1$, are all even; and hence we have, in integers,

$$4N + 2 = \left(\frac{x'+y'}{2}\right)^2 + \left(\frac{x'-y'}{2}\right)^2 + \left(\frac{z'+1}{2}\right)^2 + \left(\frac{z'-1}{2}\right)^2,$$

or, for the sake of abridging,

$$4N + 2 = w''^2 + x''^2 + y''^2 + z''^2;$$

of which squares two are even and two odd, for otherwise their sum could not have the form $4N + 2$; we may therefore write

$$4N + 2 = 4r^2 + 4s^2 + (2t+1)^2 + 2v + 1)^2;$$

from which we deduce

$$2N + 1 = (r+s)^2 + (r-s)^2 + (t+v+1)^2 + (t-v)^2;$$

that is, every odd number is the sum of four squares, and the double of a number, that is, the sum of four squares, is itself the sum of four squares, for

$$\begin{cases} 2(m^2 + n^2 + p^2 + q^2) = \\ (m+n)^2 + (m-n)^2 + (p+q)^2 + (p-q)^2; \end{cases}$$

and, therefore, every number is the sum of four squares.

If, therefore, the case which relates to triangular numbers was demonstrated, that which relates to squares would be readily deduced from it; but the converse has not place; that is, we cannot deduce the first case from the second.

The third case gives

$$N = \frac{3u^2-u}{2} + \frac{3w^2-w}{2} + \frac{3x^2-x}{2} + \frac{3y^2-y}{2} + \frac{3z^2-z}{2}, \text{or}$$

$$24N + 5 =$$
$$(6u-1)^2 + (6w-1)^2 + (6x-1)^2 + (6y-1)^2 + (6z-1)^2.$$

So that the enunciation of this particular part returns to this,

Every number of the form $24N + 5$ *is the sum of five squares, of which each of the roots is of the form* $6n - 1$.

The fourth case returns to this,

Every number of the form $8N + 6$ *may be decomposed into six squares, of which the roots are of the form* $4n - 1$.

And, in general, the proposition is always reducible to the decomposition of a number into squares, and all the partial propositions that we have considered are included in the general form,

$$8aN + (a+2)(a-2)^2 =$$
$$(2ax - a + 2)^2 + (2ay - a + 2)^2 + (2az - a + 2)^2 + \&c.$$

the number of squares on the latter side of the equation being $(a + 2)$.

TRIGONOMETRY.

Trigonometry. Sect. I. Definitions.

TRIGONOMETRY, (Τριγωνομετρία, from τριγωνόν, a triangle, and μετρέω, I measure,) the Science of Triangles, the branch of Mathematics which treats of the application of Arithmetic to Geometry. The term was originally restricted to signify the science which gives the relation of the parts of triangles described on a plane or spherical surface; but it is now understood to comprehend all theorems respecting the properties of angles and circular arcs, and the lines belonging to them. This latter department is frequently called the Arithmetic of Sines.

In the application of Mathematics to Physics, no branch is more important than Trigonometry. It is the connecting link by which we are enabled to combine, in their fullest extent, the practical exactness of Arithmetical calculations with the hypothetical accuracy of Geometrical constructions. Without it, the former could never have been applied to Physics, and the limit of the errors of the latter would have depended on the skill of the practical Geometer. By the substitution of numerical calculations for graphical constructions, we are enabled to obtain results to any desired degree of accuracy. With Trigonometry, in fact, Astronomy first received such a degree of exactness as justly to merit the name of Science; and every improvement that has been made in Trigonometry to the present time, has been attended with corresponding improvements in all parts of Physical Science.

The following will be the arrangement of the present Treatise: The first section will contain the definitions of the terms most frequently in use; in the second will be given the principal theorems relating to Trigonometrical lines; the third will explain the use of subsidiary angles; the fourth will contain all the most important propositions of Plane Trigonometry; the fifth, those of Spherical Geometry; and the sixth, those of Spherical Trigonometry. In the seventh will be given formulæ for small corresponding variations of the parts of triangles; and the eighth will contain some theorems which require for their investigation a more refined analysis. The ninth will treat of some expressions peculiar to Geodetic operations; and the tenth will explain the construction of Trigonometrical tables.

SECTION I.

Definitions.

Fig. 1. (1.) LET A B (fig. 1) be a circular arc, of which C is the centre, and let C A, C B be joined. The arc A B is proportional to the angle A C B, and either of these can therefore be used as the measure of the other, provided the arc A B is less than half the circumference, or the angle A C B less than two right angles. Since this holds with regard to all the angles of triangles, we shall, in treating of them, use indifferently the terms *arc* and *angle* to express the inclination of two lines.

(2.) But in the higher parts of the science it is by no means a matter of indifference which term we employ. It is evident, that an arc can be conceived to exceed, not only half a circumference, but even a whole circumference, or any number of circumferences; while an angle cannot be greater than two right angles. Much obscurity has frequently arisen from neglecting to observe, that when we speak of an angle greater than two right angles, we mean merely an arc greater than half a circumference; and that, when we consider trigonometrical lines as functions of such an angle, we intend nothing more than that they are functions of the corresponding arc of a circle. The reader, therefore, will be careful to recollect, that all trigonometrical lines are considered to be functions of the circular arc to which they correspond, the radius being given; and that there is no limit whatever to the extension of this arc.

(3.) The circumference of the circle has usually been divided into 360 equal parts, called *degrees*; each of these subdivided into 60, called *minutes*; each of these into 60, called *seconds*; the seconds are sometimes divided each into 60 *thirds*, the thirds into 60 *fourths*, &c., but they are more usually divided decimally. But in most of the French treatises lately published the circumference is divided into 400 equal parts, or *grades*, each grade into 100 minutes, and each minute into 100 seconds. Degrees, minutes, and seconds are commonly marked $^{\circ}$, $'$, $''$; grades and their subdivisions sometimes thus, ^{g}, $^{`}$, $^{``}$. Thus, $38^{\circ}\ 17'\ 22''$ is read thirty-eight degrees, seventeen minutes, twenty-two seconds; $44^{g}\ 76^{`}\ 27^{``}$, or $44^{g},7627$, is forty-four grades, seventy-six minutes, twenty-seven seconds.

(4.) In most of the following investigations we shall consider the radius of the circle as the unit of linear measure. The semi-circumference is then $= 3,141592653590$; its logarithm $= 0,4971498726$; one degree $= 0,017453292520$; one minute $= 0,000290888209$; one second $= 0,000004848137$; their logarithms increased by 10 are 8,2418773675; 6,4637261171, and 4,6855748667. One grade $=$ 0,0157079632679; its logarithm increased by $10 = 8,1961198769$; from which the values for a minute and second are immediately found. The number of degrees contained in the radius is 57,29577; the number of grades is 63,66197. The value of the semi-circumference to radius 1 is generally denoted by π; $\frac{\pi}{2}$ is therefore the value of the quadrant, and 2π that of the circumference.

(5.) The defect of an arc from 180° is called its *supplement*; its defect from 90° is called its *complement*.

(6.) Join A B, (fig. 1;) draw B D and C F perpendicular to A C; at A and F draw lines touching the circle, which will therefore be parallel to C F, C A; produce C B to cut these lines in E and G. Then A B

is the *chord* of the arc A B, B D is the *sine*, C D is the *cosine*, A E is the *tangent*, C E is the *secant*, F G is the *cotangent*, C G the *cosecant*, A D the *versed sine*. D H has been called by some the *suversed sine*.

(7.) These definitions suppose the arc to be less than a quadrant. If it be greater than a quadrant and less than a semicircle, as A B′, the same construction gives for the sine, versed sine, cosecant, cosine, tangent, secant, and cotangent, the lines B′D′, A D′, C G′, C D′, A E′, C E′, F G′. The four last of these, it will be observed, are measured in directions opposite to those in which the corresponding lines for arcs less than a quadrant were measured, and are therefore considered negative.* We shall show that, by this convention, formulæ which have been found to be true for arcs less than a quadrant may be made to apply to arcs greater than a quadrant.

(8.) If the arc be greater than two quadrants, and less than three, as A F H B″, (fig. 2,) making the same construction, we find that the sine, cosine, secant, and cosecant, are negative. And if the arc be greater than three quadrants, and less than four, as A F H B‴, it appears that the sine, tangent, cotangent, and cosecant are negative. The remark at the end of (7) applies to these. The versed sine and suversed sine are positive for all values of the arc. Fig. 2.

(9.) Thus it appears, that, while the arc increases from 0 to a quadrant, the sine increases from 0 to radius, (its greatest value,) and the cosine diminishes from radius (its greatest value) to 0. While the arc increases to a semicircle, the sine diminishes to 0; and the cosine, whose sign is now negative, increases in magnitude till it $= -$ radius. As the arc increases to three quadrants, the sine is negative, and its magnitude increases from 0 till it $= -$ radius, while the negative value of the cosine diminishes till it $= 0$. From three quadrants to four the sine, still negative, diminishes its negative value till it $= 0$, while the cosine, become positive, increases till it is $=$ radius, as at first.

(10.) The tangent, while the arc increases from 0 till it is $\frac{\pi}{2}$, increases so as to become greater than any assigned quantity; when the arc $= \frac{\pi}{2}$, or $\frac{3\pi}{2}$, there is really no tangent, as the lines, by whose intersection the tangent is defined, do not meet; then, until the arc $= \pi$ the tangent is negative, and diminishes from a value indefinitely great to 0; then, for the third and fourth quadrants the values are the same as for the first and second. And the secant, while the arc increases from 0 to $\frac{\pi}{2}$, increases from radius to a value greater than any assignable; it then becomes negative, and diminishes from a value indefinitely great to radius, which it reaches when the arc $= \pi$; for the third and fourth quadrants its values are the same as for the first and second, with the sign changed.

(11.) If the arc, instead of being $=$ A B, were $=$ A B increased by any number of whole circumferences, the values of the several trigonometrical lines would be the same as those for the arc A B.

(12.) The definitions of the complement and supplement, without some extension, will not apply to arcs greater than 90° or 180° respectively. It is only necessary to consider the defect of the arc from 90° or 180° as being negative when the arc is greater than either of those values; and all the theorems relating to these defects will be comprehended under the same formula.

(13.) Since we have considered positive arcs as measured from A towards F, we may consider negative arcs as measured in the opposite direction. Let A B, A B′ (fig. 3) be equal arcs positive and negative; their sines B D, B′D will evidently be in the same straight line; A E′= A E, F G′= F G, C E′= C E, C G′= C G. Hence for a negative arc, the cosine, versed sine, and secant, are the same as those for an equal positive arc; the sine, tangent, cotangent, and cosecant, are equal in respect of magnitude, but are affected with different signs. Our figure supposes A B less than a quadrant, but it will be seen that the same is true if A B be greater than a quadrant. Fig. 3.

(14.) The whole of what we have assumed with regard to the signs to be affixed to the expressions for lines according to their directions, is purely arbitrary. Its utility is this: a single formula, as we shall show by induction, will now comprehend several cases for which as many separate formulæ would otherwise have been necessary. This, we conceive, is in all cases the true foundation for the use of the negative sign.

SECTION II.

Relations of Trigonometrical Lines.

(15.) In the succeeding articles we shall use the abbreviations Sin, Cos, Tan, Sec, Cot, Cosec, Vers, to denote the sine, cosine, &c. to the radius r; and sin, cos, &c. to denote them supposing the radius $= 1$.

(16.) If C K L be drawn perpendicular to A B, (fig. 1,) A K = K B, the angle A C K = B C K, and the arc A L = B L, therefore A B = 2 . A K. But A K is evidently the sine of A L, or $\frac{1}{2}$ A B. And the straight line A B is the chord of the arc A B. Hence Chord A B $= 2 \,.\, \text{Sin}\,\frac{\text{A B}}{2}$, and chord A B $= 2 \sin \frac{\text{A B}}{2}$.

(17.) A D = A C − C D, or Vers A B $= r -$ Cos A B, and therefore vers A B $= 1 -$ cos A B. By the convention established with regard to signs it will be found, that this equation applies to arcs terminated in all quadrants of the circle.

* The secant is negative, because it is not measured from the centre in the direction of the radius through the extremity of the arc, but in the opposite direction.

(18.) By similar triangles, (Geometry, book iv. prop. 20,) $\mathrm{A\,E} = \frac{\mathrm{D\,B} \times \mathrm{C\,A}}{\mathrm{C\,D}}$, or $\mathrm{Tan\,A\,B} = \frac{r.\,\mathrm{Sin\,A\,B}}{\mathrm{Cos\,A\,B}}$, and $\tan \mathrm{A\,B} = \frac{\sin \mathrm{A\,B}}{\cos \mathrm{A\,B}}$.

(19.) By similar triangles, $\mathrm{F\,G} = \frac{\mathrm{C\,D} \times \mathrm{C\,F}}{\mathrm{D\,B}}$, or $\mathrm{Cot\,A\,B} = \frac{r.\,\mathrm{Cos\,A\,B}}{\mathrm{Sin\,A\,B}}$, and $\cot \mathrm{A\,B} = \frac{\cos \mathrm{A\,B}}{\sin \mathrm{A\,B}}$.

(20.) Multiplying together these expressions, $\mathrm{Tan\,A\,B} \times \mathrm{Cot\,A\,B} = r^2$, and $\tan \mathrm{A\,B} \cot \mathrm{A\,B} = 1$.

(21.) By similar triangles, $\mathrm{C\,E} = \frac{\mathrm{C\,A} \times \mathrm{C\,B}}{\mathrm{C\,D}}$, or $\mathrm{Sec\,A\,B} = \frac{r^2}{\mathrm{Cos\,A\,B}}$, and $\sec \mathrm{A\,B} = \frac{1}{\cos \mathrm{A\,B}}$.

(22.) By similar triangles, $\mathrm{C\,G} = \frac{\mathrm{C\,F} \times \mathrm{C\,B}}{\mathrm{D\,B}}$, or $\mathrm{Cosec\,A\,B} = \frac{r^2}{\mathrm{Sin\,A\,B}}$, and $\mathrm{cosec\,A\,B} = \frac{1}{\sin \mathrm{A\,B}}$.

(23.) Suppose $\mathrm{H\,B'} = \mathrm{A\,B}$; then $\mathrm{A\,B'}$, or $180° - \mathrm{H\,B'}$, is the supplement of A B. And $\mathrm{B'D'} = \mathrm{B\,D}$, $\mathrm{C\,D'} = \mathrm{C\,D}$, $\mathrm{A\,E'} = \mathrm{A\,E}$, $\mathrm{C\,E'} = \mathrm{C\,E}$, $\mathrm{F\,G'} = \mathrm{F\,G}$, $\mathrm{C\,G'} = \mathrm{C\,G}$, $\mathrm{A\,D} = \mathrm{H\,D'}$. Hence the sine and cosecant of any arc are the same as those of its supplement; the cosine, tangent, cotangent, and secant, are equal in magnitude, with different signs; and the versed sine of one is the suversed sine of the other.

(24.) If $\mathrm{A}b = \mathrm{F\,B}$, and bd, $\mathrm{C}eg$, be drawn as before, it is plain that $bd = \mathrm{C\,D}$, $\mathrm{C}d = \mathrm{B\,D}$, $\mathrm{A}e = \mathrm{F\,G}$, $\mathrm{F}g = \mathrm{A\,E}$, $\mathrm{C}e = \mathrm{C\,G}$, $\mathrm{C}g = \mathrm{C\,E}$. But bd, $\mathrm{C}d$, $\mathrm{A}e$, $\mathrm{F}g$, $\mathrm{C}e$, $\mathrm{C}g$, are the sine, cosine, tangent, cotangent, secant, and cosecant of $\mathrm{A}b$ or B F; and B F is the complement of A B. Hence the sine, cosine, tangent, cotangent, secant, and cosecant, of the complement of an arc, are respectively equal to the cosine, sine, cotangent, tangent, cosecant, and secant of the arc.

(25.) All these theorems have been proved for arcs less than a quadrant. If, however, we make use of the convention established with regard to signs, it will be found that they apply to every case. For example, when the arc, as A F H B''', fig. 2, is greater than three quadrants and less than four, the sine is negative, the cosine is positive; therefore the tangent $= \frac{\text{sine}}{\text{cosine}}$ (18) ought by the formula to be negative; which from the figure it appears to be. The magnitude is determined by the same proportion as before, and cannot be erroneous. The secant $= \frac{1}{\text{cosine}}$ (21) ought to be positive; and the cosecant $= \frac{1}{\text{sine}}$ (19) ought to be negative; as they are found to be. The same, it will be found, is true for every other case.

(26.) By similar triangles, the following proportions will easily be verified. Radius : Sin A B :: Sec A B : Tan A B; therefore $\mathrm{Sin\,A\,B} = \frac{r.\,\mathrm{Tan\,A\,B}}{\mathrm{Sec\,A\,B}}$, and $\sin \mathrm{A\,B} = \frac{\tan \mathrm{A\,B}}{\sec \mathrm{A\,B}}$. Radius : Cos A B :: Cosec A B :: Cot A B; therefore $\mathrm{Cos\,A\,B} = \frac{r.\,\mathrm{Cot\,A\,B}}{\mathrm{Cosec\,A\,B}}$, and $\cos \mathrm{A\,B} = \frac{\cot \mathrm{A\,B}}{\mathrm{cosec\,A\,B}}$.

(27.) Since $(\mathrm{Sec\,A\,B})^2 = r^2 + (\mathrm{Tan\,A\,B})^2$, (Geometry, book iv. prop. 13,) or $\sec^2 \mathrm{A\,B} = 1 + \tan^2 \mathrm{A\,B}$, and $\mathrm{cosec}^2 \mathrm{A\,B} = 1 + \cot^2 \mathrm{A\,B}$, we may thus express these values; $\sin \mathrm{A\,B} = \frac{\tan \mathrm{A\,B}}{\sqrt{1 + \tan^2 \mathrm{A\,B}}} = \frac{\sqrt{\sec^2 \mathrm{A\,B} - 1}}{\sec \mathrm{A\,B}}$; $\cos \mathrm{A\,B} = \frac{\cot \mathrm{A\,B}}{\sqrt{1 + \cot^2 \mathrm{A\,B}}} = \frac{\sqrt{\mathrm{cosec}^2 \mathrm{A\,B} - 1}}{\mathrm{cosec\,A\,B}}$. And the equations of (21) and (22) may be thus expressed; $\cos \mathrm{A\,B} = \frac{1}{\sqrt{1 + \tan^2 \mathrm{A\,B}}}$; $\sin \mathrm{A\,B} = \frac{1}{\sqrt{1 + \cot^2 \mathrm{A\,B}}}$.

(28.) In the same way, observing that $\sin^2 \mathrm{A\,B} + \cos^2 \mathrm{A\,B} = 1$, we find from (18) and (19), $\tan \mathrm{A\,B} = \frac{\sin \mathrm{A\,B}}{\sqrt{1 - \sin^2 \mathrm{A\,B}}} = \frac{\sqrt{1 - \cos^2 \mathrm{A\,B}}}{\cos \mathrm{A\,B}}$; $\cot \mathrm{A\,B} = \frac{\cos \mathrm{A\,B}}{\sqrt{1 - \cos^2 \mathrm{A\,B}}} = \frac{\sqrt{1 - \sin^2 \mathrm{A\,B}}}{\sin \mathrm{A\,B}}$. These are the principal formulæ of the relations of trigonometrical lines belonging to one arc.

(29.) We proceed to one of the most important propositions of Trigonometry. To find the sine and cosine of the sum and difference of two arcs in terms of the sine and cosine of the simple arcs. Let A B, fig. 4, be the longer arc $=$ A; $\mathrm{B\,E} = \mathrm{B\,F} = \mathrm{B}$; then $\mathrm{A\,E} = \mathrm{A} + \mathrm{B}$, $\mathrm{A\,F} = \mathrm{A} - \mathrm{B}$. Draw E G, F G, perpendicular to C B, which will meet at G and be in the same straight line, and will be equal; also draw B D, E H, F K, G L, perpendicular to A C, and G M, F N, perpendicular to E H, G L. Then E H or $\mathrm{G\,L} + \mathrm{E\,M} = \sin \overline{\mathrm{A} + \mathrm{B}}$; F K or $\mathrm{G\,L} - \mathrm{G\,N} = \sin \overline{\mathrm{A} - \mathrm{B}}$; C H or $\mathrm{C\,L} - \mathrm{G\,M} = \cos \overline{\mathrm{A} + \mathrm{B}}$; C K or $\mathrm{C\,L} + \mathrm{F\,N} = \cos \overline{\mathrm{A} - \mathrm{B}}$. Now the angle $\mathrm{E\,G\,M} = 90° - \mathrm{M\,G\,C} = \mathrm{C\,G\,L} = \mathrm{C\,B\,D}$; also E M G and C D B are right angles, therefore the triangles E G M, B C D, are similar, and C B : C D :: E G : E M, or Radius : Cos A :: Sin B : E M $= \frac{\mathrm{Cos\,A\,.\,Sin\,B}}{r} = \mathrm{G\,N}$. And C B : B D :: E G : G M, or Radius : Sin A :: Sin B : G M $= \frac{\mathrm{Sin\,A\,.\,Sin\,B}}{r} = \mathrm{F\,N}$. Also

$CB : BD :: CG : GL = \frac{BD.CG}{CB} = \frac{\operatorname{Sin} A . \operatorname{Cos} B}{r}$; and $CB : CD :: CG : CL = \frac{CD.CG}{CB} = \frac{\operatorname{Cos} A . \operatorname{Cos} B}{r}$. Substituting these values $\operatorname{Sin} \overline{A+B} = \frac{\operatorname{Sin} A . \operatorname{Cos} B + \operatorname{Cos} A . \operatorname{Sin} B}{r}$; $\operatorname{Sin} \overline{A-B} = \frac{\operatorname{Sin} A . \operatorname{Cos} B - \operatorname{Cos} A . \operatorname{Sin} B}{r}$; $\operatorname{Cos} \overline{A+B} = \frac{\operatorname{Cos} A . \operatorname{Cos} B - \operatorname{Sin} A . \operatorname{Sin} B}{r}$; $\operatorname{Cos} \overline{A-B} = \frac{\operatorname{Cos} A . \operatorname{Cos} B + \operatorname{Sin} A . \operatorname{Sin} B}{r}$. Or, if the radius be the unit of measure,

$\sin \overline{A+B} = \sin A . \cos B + \cos A . \sin B$; $\quad \sin \overline{A-B} = \sin A . \cos B - \cos A . \sin B$;

$\cos \overline{A+B} = \cos A . \cos B - \sin A . \sin B$; $\quad \cos \overline{A-B} = \cos A . \cos B + \sin A . \sin B$.

(30.) It is here supposed that A is greater than B, and that A is less than 90°. If these conditions should not hold, it would still be found that, by virtue of our conventions with regard to the signs of arcs and straight lines, the same formulæ would apply. We shall leave it to the reader to examine in this manner every distinct case, and shall, merely as an example, suppose A greater than 180°, B greater than 90°. Let $AF'B' = A$; $B'E' = B'F' = B$. Make the same construction in every respect as before; then $E'H' = E'M' - G'L' = \frac{CD'.E'G'}{CB'} - \frac{B'D'.CG'}{CB'}$. But, by (7) and (8), since $AF'B'E' = A + B$, $E H'$ is $= - \operatorname{Sin} \overline{A+B}$; $CD' = - \operatorname{Cos} A$; $E'G' = \operatorname{Sin} B$; $B'D' = \operatorname{Sin} A$; $CG' = - \operatorname{Cos} B$; thus the equation becomes $- \operatorname{Sin} \overline{A+B} = \frac{- \operatorname{Cos} A . \operatorname{Sin} B - \operatorname{Sin} A . \operatorname{Cos} B}{r}$, or $\operatorname{Sin} \overline{A+B} = \frac{\operatorname{Sin} A . \operatorname{Cos} B + \operatorname{Cos} A . \operatorname{Sin} B}{r}$; the same as for arcs less than 90°. And the same will be found to be true for every different case.

(31.) From these expressions,

$$\sin \overline{A+B} + \sin \overline{A-B} = 2 \sin A . \cos B,$$
$$\sin \overline{A+B} - \sin \overline{A-B} = 2 \cos A . \sin B,$$
$$\cos \overline{A+B} + \cos \overline{A-B} = 2 \cos A . \cos B,$$
$$\cos \overline{A-B} - \cos \overline{A+B} = 2 \sin A . \sin B.$$

(32.) Let $A + B = C$; $A - B = D$; then

$$\sin C + \sin D = 2 \sin \frac{C+D}{2} . \cos \frac{C-D}{2},$$
$$\sin C - \sin D = 2 \cos \frac{C+D}{2} . \sin \frac{C-D}{2},$$
$$\cos C + \cos D = 2 \cos \frac{C+D}{2} . \cos \frac{C-D}{2},$$
$$\cos D - \cos C = 2 \sin \frac{C+D}{2} . \sin \frac{C-D}{2}.$$

(33.) Let $B = A$; then $\sin 2A = 2 \sin A \cos A$; and $\cos 2A = \cos^2 A - \sin^2 A = 1 - 2 \sin^2 A = 2 \cos^2 A - 1$. From these, $\sin A = \sqrt{\frac{1 - \cos 2A}{2}} = \sqrt{\frac{\operatorname{versin} 2A}{2}}$; $\cos A = \sqrt{\frac{1 + \cos 2A}{2}}$. If in these values we put $\sqrt{1 - \sin^2 2A}$ for $\cos 2A$, $\sin A = \sqrt{\frac{1 - \sqrt{1 - \sin^2 2A}}{2}} = \frac{1}{2}\left(\sqrt{1 + \sin 2A} - \sqrt{1 - \sin 2A}\right)$; $\cos A = \frac{1}{2}\left(\sqrt{1 + \sin 2A} + \sqrt{1 - \sin 2A}\right)$.

(34.) Again, $\cot A + \tan A = \frac{\cos A}{\sin A} + \frac{\sin A}{\cos A} = \frac{\cos^2 A + \sin^2 A}{\sin A \cos A} = \frac{1}{\sin A \cos A} = \frac{2}{2 \sin A \cos A} = \frac{2}{\sin 2A} = 2 \operatorname{cosec} 2A$. Similarly, $\cot A - \tan A = \frac{\cos^2 A - \sin^2 A}{\sin A \cos A} = \frac{2 \cos 2A}{\sin 2A} = 2 \cot 2A$.

(35.) Since $\sin A = \sqrt{\frac{1 - \cos 2A}{2}}$, and $\cos A = \sqrt{\frac{1 + \cos 2A}{2}}$, we have $\tan A = \frac{\sin A}{\cos A} = \sqrt{\frac{1 - \cos 2A}{1 + \cos 2A}}$. Hence $\cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A}$.

(36.) Since $\sin A = \frac{\tan A}{\sqrt{1 + \tan^2 A}}$, and $\cos A = \frac{1}{\sqrt{1 + \tan^2 A}}$, (27,) $\sin 2 A = 2 \sin A \cos A = \frac{2 \tan A}{1 + \tan^2 A}$.

(37.) $\frac{1 - \cos 2 A}{\sin 2 A} = \frac{2 \sin^2 A}{2 \sin A \cos A} = \frac{\sin A}{\cos A} = \tan A$. Similarly, $\frac{\sin 2 A}{1 + \cos 2 A} = \tan A$.

(38.) From (31,) $\sin \overline{A + B} = 2 \sin A . \cos B - \sin \overline{A - B}$. Let $A = n B$; then $\sin \overline{n + 1}\, B = 2 \sin n B . \cos B - \sin \overline{n - 1}\, B$. Making n successively $= 2, 3$, &c., we form the following table:

$$\begin{aligned}
\sin B &= \sin B, \\
\sin 2 B &= 2 \sin B \cos B, \\
\sin 3 B &= 3 \sin B - 4 \sin^3 B, \\
\sin 4 B &= (4 \sin B - 8 \sin^3 B) \cos B, \\
\sin 5 B &= 5 \sin B - 20 \sin^3 B + 16 \sin^5 B, \\
\&c. \quad & \quad \&c.
\end{aligned}$$

Again, from (31,) $\cos \overline{A + B} = 2 \cos A . \cos B - \cos \overline{A - B}$. Let $A = n B$, therefore $\cos \overline{n + 1}\, B = 2 \cos n B . \cos B - \cos \overline{n - 1}\, B$. Making n successively $= 2, 3$, &c.

$$\begin{aligned}
\cos B &= \cos B, \\
\cos 2 B &= 2 \cos^2 B - 1, \\
\cos 3 B &= 4 \cos^3 B - 3 \cos B, \\
\cos 4 B &= 8 \cos^4 B - 8 \cos^2 B + 1, \\
\cos 5 B &= 16 \cos^5 B - 20 \cos^3 B + 5 \cos B, \\
\&c. \quad & \quad \&c.
\end{aligned}$$

(39.) $\sin \overline{A + B} . \sin \overline{A - B}$, by (31,) (putting $A + B$ for A, and $A - B$ for B) $= \frac{1}{2} (\cos 2 B - \cos 2 A) = \frac{1}{2} (1 - 2 \sin^2 B - 1 + 2 \sin^2 A)$; by (33,) $= \sin^2 A - \sin^2 B$, or $= \cos^2 B - \cos^2 A$. And $\cos \overline{A + B} . \cos \overline{A - B} = \frac{1}{2} (\cos 2 B + \cos 2 A) = \frac{1}{2} (1 - 2 \sin^2 B + 2 \cos^2 A - 1) = \cos^2 A - \sin^2 B$, or $= \cos^2 B - \sin^2 A$.

(40.) $\frac{\sin \overline{A + B}}{\sin \overline{A - B}} = \frac{\sin A \cos B + \cos A . \sin B}{\sin A \cos B - \cos A . \sin B} = \frac{\tan A + \tan B}{\tan A - \tan B}$, or $= \frac{\cot B + \cot A}{\cot B - \cot A}$; and similarly, $\frac{\cos \overline{A + B}}{\cos \overline{A - B}} = \frac{\cot B - \tan A}{\cot B + \tan A}$, or $= \frac{\cot A - \tan B}{\cot A + \tan B}$.

(41.) $\frac{\sin A + \sin B}{\sin A - \sin B} = \frac{2 \sin \frac{A + B}{2} \cos \frac{A - B}{2}}{2 \cos \frac{A + B}{2} \sin \frac{A - B}{2}} = \frac{\tan \frac{A + B}{2}}{\tan \frac{A - B}{2}}$; $\frac{\cos B + \cos A}{\cos B - \cos A} = \frac{2 \cos \frac{A + B}{2} \cos \frac{A - B}{2}}{2 \sin \frac{A + B}{2} \sin \frac{A - B}{2}} = \cot \frac{A + B}{2} . \cot \frac{A - B}{2}$.

(42.) $\tan A + \tan B = \frac{\sin A}{\cos A} + \frac{\sin B}{\cos B} = \frac{\sin A \cos B + \cos A \sin B}{\cos A . \cos B} = \frac{\sin \overline{A + B}}{\cos A . \cos B}$. Similarly, $\tan A - \tan B = \frac{\sin \overline{A - B}}{\cos A . \cos B}$; $\cot A + \cot B = \frac{\sin \overline{A + B}}{\sin A . \sin B}$; $\cot B - \cot A = \frac{\sin \overline{A - B}}{\sin A . \sin B}$.

(43.) To find an expression for the tangent of the sum or difference of two arcs: $\tan \overline{A + B} = \frac{\sin \overline{A + B}}{\cos \overline{A + B}} = \frac{\sin A . \cos B + \cos A . \sin B}{\cos A . \cos B - \sin A . \sin B}$; which, dividing the numerator and denominator by $\cos A . \cos B$, and observing that $\frac{\sin A}{\cos A} = \tan A$, gives $\tan \overline{A + B} = \frac{\tan A + \tan B}{1 - \tan A . \tan B}$. Similarly, $\tan \overline{A - B} = \frac{\tan A - \tan B}{1 + \tan A . \tan B}$. If $B = A$, $\tan 2 A = \frac{2 \tan A}{1 - \tan^2 A}$.

(44.) Hence, $\tan \overline{A + B + C} = \frac{\tan \overline{A + B} + \tan C}{1 - \tan \overline{A + B} . \tan C} = \frac{\tan A + \tan B + \tan C - \tan A . \tan B . \tan C}{1 - \tan A . \tan B - \tan A . \tan C - \tan B . \tan C}$. If $C = B = A$, $\tan 3 A = \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}$. If $A + B + C = \pi$, $\tan \overline{A + B + C} = 0$, (10:) hence in that case we have this remarkable equation, $\tan A + \tan B + \tan C = \tan A . \tan B . \tan C$.

(45.) These are the most important relations that subsist generally between different arcs. As there are some which depend upon the numerical expression for the lines belonging to particular arcs, we shall proceed to investigate their values.

(46.) Let B C D, fig. 5, be half a right angle, or A B = $45^\circ = \frac{\pi}{4}$; therefore the angle C B D = half a right angle = B C D, therefore B D = C D, therefore $1 = \sin^2\frac{\pi}{4} + \cos^2\frac{\pi}{4} = 2\sin^2\frac{\pi}{4}$, therefore $\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ Fig. 5.
$= \cos\frac{\pi}{4}$; $\tan\frac{\pi}{4} = 1 = \cot\frac{\pi}{4}$; $\sec\frac{\pi}{4} = \sqrt{2} = \operatorname{cosec}\frac{\pi}{4}$.

(47.) Let A E = $60^\circ = \frac{\pi}{3}$; then, since the sum of the three angles of the triangle A C E = two right angles $= 180^\circ$, the sum of those at A and E $= 120^\circ$; and as they are equal each $= 60^\circ = \frac{\pi}{3}$; therefore the triangle is equilateral, and C F = A F. Hence $\cos\frac{\pi}{3} = \frac{1}{2}$; $\sin\frac{\pi}{3} = \sqrt{1-\frac{1}{4}} = \frac{\sqrt{3}}{2}$; $\tan\frac{\pi}{3} = \sqrt{3}$; $\cot\frac{\pi}{3} = \frac{1}{\sqrt{3}}$; $\sec\frac{\pi}{3} = 2$; $\operatorname{cosec}\frac{\pi}{3} = \frac{2}{\sqrt{3}}$.

(48.) Let A G = $36^\circ = 2 \times 18^\circ$; then the complement of A G = $54^\circ = 3 \times 18^\circ$; therefore, (24,) $\sin\overline{2 \times 18} = \cos\overline{3 \times 18^\circ}$, or $2 . \sin 18^\circ \cos 18^\circ = 4\cos^3 18 - 3\cos 18^\circ$, by (38;) or, dividing by $\cos 18^\circ$, $2\sin 18^\circ = 4\cos^2 18^\circ - 3$. Let $\sin 18^\circ = x$; therefore $2x = 1 - 4x^2$, from the solution of which equation x or $\sin 18^\circ = \frac{-1+\sqrt{5}}{4} = \cos 72^\circ$; $\cos 36^\circ = 1 - 2\sin^2 18^\circ\ (33) = \frac{1+\sqrt{5}}{4} = \sin 54^\circ$.

(49.) From these values, $\sin\overline{45^\circ + A} = \sin 45^\circ . \cos A + \cos 45^\circ . \sin A = \frac{\cos A + \sin A}{\sqrt{2}}$; $\cos\overline{45^\circ + A} = \cos 45^\circ . \cos A - \sin 45^\circ . \sin A = \frac{\cos A - \sin A}{\sqrt{2}}$; $\tan\overline{45^\circ + A} = \frac{\tan 45^\circ + \tan A}{1 - \tan 45^\circ . \tan A} = \frac{1+\tan A}{1-\tan A}$; $\tan\overline{45^\circ - A}$, similarly $= \frac{1-\tan A}{1+\tan A}$; from which $\tan\overline{45^\circ + A} - \tan\overline{45^\circ - A} = \frac{4\tan A}{1-\tan^2 A} = 2\tan 2A$, (43.) Also $\sin\overline{60^\circ + A} - \sin\overline{60^\circ - A} = 2\cos 60^\circ . \sin A = \sin A$. And

$$\sin\overline{72^\circ + A} - \sin\overline{72^\circ - A} = 2\cos 72^\circ . \sin A = \frac{\sqrt{5}-1}{2}\sin A,$$

$$\sin\overline{36^\circ + A} - \sin\overline{36^\circ - A} = 2\cos 36^\circ . \sin A = \frac{\sqrt{5}+1}{2}\sin A.$$

Subtracting the upper from the lower, and transposing

$$\sin\overline{36^\circ + A} + \sin\overline{72^\circ - A} = \sin A + \sin\overline{36^\circ - A} + \sin\overline{72^\circ + A}.$$

If we had taken $\cos\overline{72^\circ + A} + \cos\overline{72^\circ - A}$, &c., we should have found

$$\cos\overline{36^\circ + A} + \cos\overline{36^\circ - A} = \cos A + \cos\overline{72^\circ + A} + \cos\overline{72^\circ - A}.$$

(50). These are the principal formulæ of the Arithmetic of Sines. Many of them may be proved geometrically, but we have preferred the algebraical investigations, as less cumbrous, and not less satisfactory.

(51.) The values of the trigonometrical lines which have occurred in these theorems, (the numerical calculation of which we shall treat of hereafter,) for different arcs, have, with their logarithms, been collected in tables. The sines, tangents, &c. themselves are very seldom used, almost all calculations being now conducted by means of their logarithms. With regard to these it is necessary to observe, that the sines and cosines of all arcs, and the tangents of arcs less than 45°, being less than 1, their logarithms are negative; the use of which would be extremely inconvenient. To avoid this, the logarithms of the tables are made greater by 10 than the real logarithms of the numbers; which it is always necessary to keep in mind in using the tables. For instance, (using l for the true logarithm, and L for the logarithm of the tables,) since $\tan A = \frac{\sin A}{\cos A}$, therefore $l . \tan A = l . \sin A - l . \cos A$, therefore $L . \tan A - 10 = L \sin A - 10 - L\cos A + 10$, or $L \tan A = L \sin A - L \cos A + 10$. The natural sines, &c. are usually given to radius 10000, but upon removing the decimal point four places to the left they are adapted to radius 1.

(52.) In all expressions involving the length of an arc, deduced from operations by the differential calculus, or from series in terms of the sines, &c., radius is supposed to be the unit of measure. To obtain the number of seconds, we must divide the length by 0,000004848137; or add to its logarithm 5,3144251 to find the logarithm of the number of seconds.

SECTION III.

On the use of Subsidiary Angles.

(53.) The possession of trigonometrical tables, ready calculated, frequently enables us to shorten very much numerical calculations which have no relation whatever to Trigonometry. The angles which are used in this process, being employed simply to expedite a calculation, are called *Subsidiary Angles*. Their use will be best elucidated by examples.

(54.) Suppose it is wished to calculate $x = \sqrt{a^2 - b^2}$, and suppose that the logarithms of a and b have already occurred in our operations. Here $x = a \sqrt{1 - \frac{b^2}{a^2}}$. If $\frac{b}{a}$ were the sine of an angle θ, x would be $a \times \cos\theta$. Determine θ therefore by the condition $\frac{b}{a} = \sin\theta$, or L $\sin\theta = \log b + 10 - \log a$ (51,) and having found θ in the tables, x will be found from the expression $\log x = \log a + \text{L} \cos\theta - 10$.

(55.) It is required to calculate the expression $x = a\cos\phi + b\sin\phi$. If we make $\frac{a}{b} = \tan\theta$, this can be put under the form $b\,(\tan\theta\,.\,\cos\phi + \sin\phi) = \frac{b}{\cos\theta}\,(\sin\theta\cos\phi + \cos\theta\sin\phi) = \frac{b\,.\,\sin\overline{\theta+\phi}}{\cos\theta}$.

Determine θ by the equation L $\tan\theta = \log a + 10 - \log b$, and then $\log x = \log b + \text{L}\sin\overline{\theta+\phi} - \text{L}\cos\theta$, or $= \log b + \text{L}\sin\overline{\theta+\phi} + \text{L}\sec\theta - 20$.

(56.) It is required to find the logarithm of $a + b$, the logarithms of a and b being known. If a and b are of such a nature that both are in all cases positive, $a + b = a\left(1 + \frac{b}{a}\right)$; make $\frac{b}{a} = \tan^2\theta$, then $a\left(1 + \frac{b}{a}\right) = a\sec^2\theta$. In logarithms, $2\,\text{L}\tan\theta = \log b + 20 - \log a$; log required $= \log a + 2\,\text{L}\sec\theta - 20$

If, however, a and b may be sometimes positive and sometimes negative, the following method must be used: $a + b = \sqrt{2}\,.\,\frac{a+b}{\sqrt{2}} = \sqrt{2}\,.\,(a\cos 45^\circ + b\sin 45^\circ)$. Let $\frac{a}{b} = \tan\theta$, or L $\tan\theta = \log a + 10 - \log b$; then $\overline{a+b} = \sqrt{2}\,.\,b\,.\,(\tan\theta\,.\,\cos 45^\circ + \sin 45^\circ) = \sqrt{2}\,.\,\frac{b}{\cos\theta}\,(\sin\theta\,.\,\cos 45^\circ + \cos\theta\sin 45^\circ) = \frac{b\sqrt{2}}{\cos\theta}$ $\sin\overline{\theta + 45^\circ}$, and $\log\overline{a+b} = ,1505150 + \log b + \text{L}\sin\overline{\theta+45^\circ} - \text{L}\cos\theta$.

(57.) In Physical Astronomy the following expression occurs: $\text{P} = (1 + e').(1 + e'').(1 + e''')$.&c., where

$$e' = \frac{1 - \sqrt{1 - e^2}}{1 - \sqrt{1 + e^2}}, \quad e'' = \frac{1 - \sqrt{1 - e'^2}}{1 + \sqrt{1 - e'^2}}, \text{ \&c.}$$

Let
$$e = \sin\theta,\ \sqrt{1 - e^2} = \cos\theta,\ \frac{1 - \sqrt{1 - e^2}}{1 + \sqrt{1 - e^2}} = \frac{1 - \cos\theta}{1 + \cos\theta} = \tan^2\frac{\theta}{2};$$

$1 + e' = \sec^2\frac{\theta}{2}$. Similarly, making e' or $\tan^2\frac{\theta}{2} = \sin\theta'$, $1 + e'' = \sec^2\frac{\theta'}{2}$, &c. Hence, $\log \text{P} = 2\left(\text{L}\sec\frac{\theta}{2} + \text{L}\sec\frac{\theta'}{2} + \text{\&c.} - 10 - 10 - \text{\&c.}\right)$ This computation would be almost impracticable in any other way.

(58.) The roots of the quadratic $x^2 - px - q = 0$, being

$$\frac{p}{2} \pm \sqrt{\frac{p^2}{4} + q}, \text{ or } \sqrt{q}\,.\left\{\frac{p}{2\sqrt{q}} \pm \sqrt{\frac{p^2}{4q} + 1}\right\}, \text{ let } \frac{p}{2\sqrt{q}} = \cot\theta;$$

the roots are $\sqrt{q}\,.\,(\cot\theta \pm \operatorname{cosec}\theta) = \sqrt{q}\,.\,\frac{\cos\theta \pm 1}{\sin\theta} = $ by (37) $-\sqrt{q}\,.\tan\frac{\theta}{2}$ and $\sqrt{q}\,.\cot\frac{\theta}{2}$.

The roots of $x^2 - px + q = 0$, being $\frac{p}{2}\left(1 \pm \sqrt{1 - \frac{4q}{p^2}}\right)$, let $\frac{4q}{p^2} = \sin^2\theta$, and the roots are $\frac{p}{2}(1 \pm \cos\theta) = p\cos^2\frac{\theta}{2}$ and $p\sin^2\frac{\theta}{2}$.

(59.) The possible root of the cubic $x^3 - q\,x - r = 0$ is

$$\sqrt[3]{\left\{\frac{r}{2} + \sqrt{\frac{r^2}{4} - \frac{q^3}{27}}\right\}} + \sqrt[3]{\left\{\frac{r}{2} - \sqrt{\frac{r^2}{4} - \frac{q^3}{27}}\right\}}$$

$$= \sqrt{\frac{q}{3}} \cdot \left(\sqrt[3]{\sqrt{\frac{27\,r^2}{4\,q^3}} + \sqrt{\frac{27\,r^2}{4\,q^3} - 1}} + \sqrt[3]{\sqrt{\frac{27\,r^2}{4\,q^3}} - \sqrt{\frac{27\,r^2}{4\,q^3} - 1}}\right).$$

Let $\frac{27\,r^2}{4\,q^3} = \text{cosec}^2\,\theta$, the root

$$= \sqrt{\frac{q}{3}}\left\{\sqrt[3]{\frac{1+\cos\theta}{\sin\theta}} + \sqrt[3]{\frac{1-\cos\theta}{\sin\theta}}\right\} = \sqrt{\frac{q}{3}}\left\{\sqrt[3]{\cot\frac{\theta}{2}} + \sqrt[3]{\tan\frac{\theta}{2}}\right\}.$$

Let $\sqrt[3]{\tan\frac{\theta}{2}} = \tan\phi$, the root $= \sqrt{\frac{q}{3}}\,(\cot\phi + \tan\phi) = \sqrt{\frac{4\,q}{3}}\,.\,\text{cosec}\ 2\,\phi$. If $\frac{q^3}{27}$ be greater than $\frac{r^2}{4}$, let x be assumed $= a\cos\theta$, or $\frac{x}{a} = \cos\theta$; then $\cos 3\,\theta = \frac{4\,x^3}{a^3} - \frac{3\,x}{a}$, by (38,) or $x^3 - \frac{3\,a^2}{4}\,x - \frac{a^3}{4}\cos 3\,\theta = 0$; making this coincide with the given equation, $\frac{3\,a^2}{4} = q$, $\frac{a^3}{4}\cos 3\,\theta = r$, which determine a and θ; and $a\cos\theta$, or x, is then immediately found. The equation will also be satisfied by making $x = a\cos\overline{\theta + \frac{2\,\pi}{3}}$, or $x = a\,.\cos\overline{\theta + \frac{4\,\pi}{3}}$, for these give $x^3 - \frac{3\,a^2}{4}\,x$ equal to $\frac{a^3}{4}\cos\overline{3\,\theta + 2\,\pi}$ and $\frac{a^3}{4}\cos\overline{3\,\theta + 4\,\pi}$, which by (11) are each equal to $\frac{a^3}{4}\cos 3\,\theta$.

SECTION IV.

Plane Trigonometry.

(60.) A TRIANGLE consists of six parts, *viz.* three sides and three angles; and if any three of these be given, the triangle is completely defined. The case must be excepted in which the three angles are given; as then the proportion only of the sides can be found, the absolute magnitudes remaining unknown. To determine in number the values of three parts from those of three given parts, is the special object of Plane Trigonometry.

(61.) Suppose the triangle right-angled, let a and b be the sides containing the right angle, c the third side, A, B, C the angles opposite, (fig. 6.) If the hypothenuse and the angle B be given, describe a circle D E Fig. 6. to radius 1; draw D F and E G perpendicular to B C; then D F is sin B, B F is cos B, E G is tan B. And A B : B C :: D B : B F, or c : a :: 1 : cos B, therefore $a =$ c cos B. Also A B : A C :: D B : D F, or c : b :: 1 : sin B, therefore $b =$ c sin B. And the angle $A = \frac{\pi}{2} - B$.

(62.) If a and B be given, B C : C A :: B E : E G, or a : b :: 1 : tan B, therefore $b = a$ tan B. And B C : B A :: B E : B G, or a : c :: 1 : sec B, therefore $c = a$ sec B. If b and B be given, $a = \frac{b}{\tan B} = b$ cot B; $c = \frac{b}{\sin B} = b$ cosec B.

(63.) If a and c be given, $b = \sqrt{c^2 - a^2}$, $\cos B = \frac{a}{c} = \sin A$. If a and b be given, $c = \sqrt{a^2 + b^2}$, $\tan B = \frac{b}{a} = \cot A$.

(64.) Now, suppose the triangle to be any whatever, we shall first prove this general proposition: The sides of a triangle are in the same proportion as the sines of the angles opposite. In fig. 7 and 8 draw B D a Fig. 7, 8. perpendicular from B on A C, or A C produced; then B D = A B sin A, (61,) and B D also = C B . sin B C A, whether B C A be greater or less than 90°, (23;) therefore A B . sin A = C B . sin B C A, or A B : C B :: sin B C A : sin B A C.

(65.) Suppose the three sides of a triangle given, to find the angles. In fig. 7, B A² = B C² + C A² − 2 A C . C D, (GEOMETRY, book iv. prop. 16;) in fig. 8, B A² = B C² + C A² + 2 A C . C D. (GEOMETRY, book iv. prop. 15.) Now in the former case, by (61,) C D = B C . cos C; in the latter, C D = B C . cos $\overline{\pi - C}$ = − B C cos C, (23;) therefore, generally, A B² = B C² + C A² − 2 A C . B C . cos C, or $c^2 = a^2 +$

$b^2 - 2\,a\,b\,.\cos C$. Hence $\cos C = \frac{a^2 + b^2 - c^2}{2\,a\,b}$. This formula is very inconvenient for logarithmic computation.

(66.) $1 + \cos C$, or $2\cos^2\frac{C}{2}$ (33)

$$= \frac{\overline{a+b}^2 - c^2}{2\,a\,b} = \frac{\overline{a+b+c}\,.\,\overline{a+b-c}}{2\,a\,b} = 2\,\frac{\frac{a+b+c}{2}\,.\,\overline{\frac{a+b+c}{2}-c}}{a\,b}$$

Let $\frac{a+b+c}{2} = S$, therefore $\cos^2\frac{C}{2} = \frac{S\,.\,\overline{S-c}}{a\,b}$. Also $1 - \cos C$, or $2\sin^2\frac{C}{2}$ (33) $= \frac{c^2 - \overline{a-b}^2}{2\,a\,b}$ $= \frac{\overline{c+a-b}\,.\,\overline{c-a+b}}{2\,a\,b} = 2\,.\,\frac{\overline{S-b}\,.\,\overline{S-a}}{a\,b}$; therefore $\sin^2\frac{C}{2} = \frac{\overline{S-b}\,.\,\overline{S-a}}{a\,b}$. Dividing this by the last, $\tan^2\frac{C}{2} = \frac{\overline{S-b}\,.\,\overline{S-a}}{S\,.\,\overline{S-c}}$. Multiplying the product of $\sin^2\frac{C}{2}$ and $\cos^2\frac{C}{2}$ by 4, since $\sin C = 2\sin\frac{C}{2}\cos\frac{C}{2}$, (33,) $\sin^2 C = \frac{4\,.\,S\,.\,\overline{S-a}\,.\,\overline{S-b}\,.\,\overline{S-c}}{a^2\,b^2}$. All these expressions, bnt more particularly the second, are very convenient for the application of logarithms. If two or three angles were required, the formula for $\tan^2\frac{C}{2}$ would probably be most convenient, as the same numbers would be used for the three calculations; or when one angle is found, the theorem of (64) may be applied.

(67.) From the last expression we derive the formula for the area of a triangle in terms of the sides. For the area $= \frac{CA\,.\,BD}{2}$ (GEOMETRY, book iv. prop. 8) $= \frac{a\,b\sin C}{2} = \sqrt{S\,.\,\overline{S-a}\,.\,\overline{S-b}\,.\,\overline{S-c}}$.

(68.) Suppose now two sides and the angle they contain (a, b, C) to be given, to find the other angles. By (64,) $\frac{\sin A}{\sin B} = \frac{a}{b}$, therefore $\frac{\sin A + \sin B}{\sin A - \sin B} = \frac{a+b}{a-b}$, or (41,) $\frac{\tan\frac{A+B}{2}}{\tan\frac{A-B}{2}} = \frac{a+b}{a-b}$.

Now $A + B + C = \pi$, therefore $\frac{A+B}{2} = \frac{\pi}{2} - \frac{C}{2}$, therefore, by (24,) $\tan\frac{A+B}{2} = \cot\frac{C}{2}$, and therefore $\tan\frac{A-B}{2} = \frac{a-b}{a+b}\cot\frac{C}{2}$. When the logarithms of a and b are known, the operation is facilitated thus. Let $\frac{a}{b} = \tan\theta$, therefore $\frac{a-b}{a+b} = \frac{\tan\theta - 1}{\tan\theta + 1} = \frac{\tan\theta - \tan 45^\circ}{1 + \tan\theta\,.\,\tan 45^\circ} = \tan\overline{\theta - 45^\circ}$, and $\tan\frac{A-B}{2} = \tan\overline{\theta - 45^\circ}\,.\,\cot\frac{C}{2}$. Or thus, if b be less than a, let $\frac{b}{a} = \cos\phi$, then $\frac{a-b}{a+b} = \frac{1-\cos\phi}{1+\cos\phi} = \tan^2\frac{\phi}{2}$ (35,) and $\tan\frac{A-B}{2} = \tan^2\frac{\phi}{2}\,.\,\cot\frac{C}{2}$. $\frac{A+B}{2}$ and $\frac{A-B}{2}$ being known, their sum gives the value of A, and their difference that uf B. The third side may be found by the proportion of (64.)

(69.) Sometimes, however, it is desirable to find the third side without finding the two remaining angles. In this case, by (65,) $c^2 = a^2 + b^2 - 2\,a\,b\cos C = a^2 + 2\,a\,b + b^2 - 2\,a\,b\,(1 + \cos C) = \overline{a+b}^2 \left\{1 - \frac{4\,a\,b}{(a+b)^2}\cos^2\frac{C}{2}\right\}$. Let $\frac{4\,a\,b}{(a+b)^2}\cos^2\frac{C}{2} = \sin^2\theta$; then $c = \overline{a+b}\,.\,\cos\theta$. Or $c^2 = a^2 - 2\,a\,b + b^2 + 2\,a\,b\,(1 - \cos C) = \overline{a-b}^2\,.\left\{1 + \frac{4\,a\,b}{(a-b)^2}\sin^2\frac{C}{2}\right\}$; let $\frac{4\,a\,b}{(a-b)^2}\sin^2\frac{C}{2} = \tan^2\theta$, then $c = \overline{a-b}\,.\sec\theta$. Or, since $\cos C = \cos^2\frac{C}{2} - \sin^2\frac{C}{2}$, by (33,) and $1 = \cos^2\frac{C}{2} + \sin^2\frac{C}{2}$, $c^2 = \overline{a-b}^2\,.\cos^2\frac{C}{2} + \overline{a+b}^2\,.\sin^2\frac{C}{2} = \overline{a-b}^2\,.\cos^2\frac{C}{2}\left(1 + \left(\frac{a+b}{a-b}\right)^2.\tan^2\frac{C}{2}\right)$; let $\frac{a+b}{a-b}\tan\frac{C}{2} = \tan\theta$, then $c = \overline{a-b}\,.\cos\frac{C}{2}\,.\sec\theta$. All these are easily calculated by logarithms.

(70.) If two sides and an angle opposite one of them (a, b, A) be given, the angla B is found by the proportion $a : b :: \sin A : \sin B$, (64;) then $C = \pi - A - B$, and $a : c :: \sin A : \sin C$.

(71.) If c, A, B be given, $C = \pi - A - B$; and the three angles and one side being known, the other rides are easily found by (64.)

(72.) If A, B, a be given, $C = \pi - A - B$, and $b = \frac{a \cdot \sin B}{\sin A}$, $c = \frac{a \cdot \sin C}{\sin A}$. These forms comprehend all the cases of Plane Trigonometry.

(73.) In using these formulæ we must, however, observe, that we shall in certain cases arrive at results, the meaning of which is apparently doubtful. These are called the ambiguous cases. We proceed to distinguish those in which the ambiguity is apparent, from those in which it is real.

(74.) First, then, we may observe, that the lengths of lines determined by the formulæ above, since they are the results of simple multiplication and division, and are not given by the solution of quadratic equations, are perfectly free from ambiguity.

(75.) In the next place, an angle when determined by the value of its cosine, versed sine, tangent, cotangent, or secant, is not ambiguous. For the values of the tangent and cotangent, which correspond to the arc A, correspond also to the arc $\pi + A$ (10) and to no smaller arc; the values of the cosine, versed sine, and secant, belong to the arc $2\pi - A$, and to no smaller arc, by (9) and (10;) and these being greater than π, or 180°, cannot be used in calculations of triangles.

(76.) But if an angle be determined by the value of its sine or cosecant, since these by (23) belong equally to the arc A and $\pi - A$, both of which, when the sine is positive, are less than π, the value of the arc is apparently doubtful. We will examine every case in which these expressions are found.

(77.) In right-angled triangles the angles must be less than $\frac{\pi}{2}$, and there is therefore no ambiguity. When the angle C in (66) is found by the expression for $\sin \frac{C}{2}$, since C must be less than π, $\frac{C}{2}$ must be less than $\frac{\pi}{2}$, and there is no ambiguity. If found by the expression for sin C, it must be observed that C is greater or less than $\frac{\pi}{2}$, according as c^2 is greater or less than $a^2 + b^2$. In the case of two sides and an angle opposite one being given (70,) if a be greater than b, there is no ambiguity; for in the triangle A C B (fig. 9) the angle B must be less than A, and must therefore be less than $\frac{\pi}{2}$, (as if A be greater than $\frac{\pi}{2}$, sin B being less than sin A, of the arcs corresponding to it one is less than $\frac{\pi}{2}$, the other greater than A.) But if a be less than b, the angle A being less than $\frac{\pi}{2}$, (fig. 10,) there is nothing to determine whether B is greater or less than $\frac{\pi}{2}$; that is, whether the triangle A C B or A C B′ is to be taken. In this case, then, and in this alone, there is a real ambiguity. Fig. 9. Fig. 10

SECTION V.

Spherical Geometry.

In our Paper on GEOMETRY, book ix. a comprehensive Treatise of Spherical Geometry has been given. As it is necessary, however, for our present purpose, to state some of the propositions with slight alterations and additions, and as a small number only are wanted here, we have thought it best, at the risk of some repetition, to premise all the Geometrical propositions that may be necessary.

(78.) A *sphere* is a solid bounded by a surface of which every point is equally distant from a point within it, called the *centre*. A straight line drawn from the centre to the surface, is called a *radius*; if produced both ways to meet the surface, it is a *diameter*.

(79.) Every section of a sphere by a plane is a circle. Let A B (fig. 11) be any section of a sphere made by a plane; from the centre O draw O C perpendicular to this plane; take D, K, any points in the section, and join C D, O D, C K, O K. Since O C is perpendicular to the plane, it is perpendicular to every line which meets it in the plane; therefore O C D, O C K are right angles, and $CD = \sqrt{OD^2 - OC^2}$, $CK = \sqrt{OK^2 - OC^2}$. But $OK = OD$, therefore $CK = CD$, or the section is a circle of which C is the centre. Fig. 11

(80.) A *great circle* is one whose plane passes through the centre of the sphere; a *small circle* is one whose plane does not pass through the centre. Hence a radius of a great circle is a radius of the sphere. Two circles are said to be parallel when their planes are parallel.

(81.) A great circle may be drawn through any two points on the surface of a sphere, but not generally through more than two. For the plane of a great circle must also pass through the centre of the sphere; and a plane may be made to pass through any three given points, but not generally more than three, (GEOMETRY, book vi. prop. 2, cor. 1.) A small circle may be drawn through any three given points.

(82.) Two great circles bisect each other. For the intersection of their planes, being a straight line passing through the centre, is a diameter of the sphere, and is therefore a diameter of both circles; and the circles are therefore bisected.

(83.) The *inclination* of two great circles, is the angle made by their tangents at the point of intersection. Since each of these tangents is perpendicular to the radius in which the planes of the circles intersect, the same angle measures the inclination of the planes of the circles, (GEOMETRY, book vi. def. 4.)

(84.) If through the centre of a circle, whether great or small, a straight line be drawn perpendicular to its plane, the point in which, if produced, it meets the surface of the sphere, is called the *pole* of that circle. Thus,
Fig. 11. in fig. 11, F C E being perpendicular to the plane of A B D, and passing through its centre C, E and F are the poles of A D B. From the demonstration of (79) it is evident, that this line will always pass through the centre of the sphere. In a small circle the term *pole* is more usually applied to that point only, as E, which is nearest to the circle.

(85.) If a great circle be made to pass through D and E, and another through K and E, and if the chords D E, K E be drawn; then, since C D is equal to C K, and C E is common, and the angle E C D = E C K, both being right angles, the chord E D is equal to the chord E K, and the arc E D = arc E K. Hence the pole of a circle is equally distant from every point of that circle; the distances being measured by arcs of great circles.

(86.) If E be the pole of the great circle G H, since the centre of this circle is the same with the centre of the sphere, E O G is a right angle, and E G is a quadrant. The distance, therefore, of every point of a great circle from its pole is a quadrant of a great circle. Since E O is perpendicular to G O H, the plane E O G is perpendicular to the plane G O H, and the angle E G H is therefore, by (83,) a right angle. And the tangent of G M at G is perpendicular to the tangent of G E; and it is also perpendicular to G O, therefore it is perpendicular to the plane E O G, ((GEOMETRY, book vi. prop. 4;) so also is the tangent of D B at D, which is parallel to it, (GEOMETRY, book vi. prop. 7;) therefore the tangent of D B at D is perpendicular to the tangent of D E.

(87.) The inclination of E G, E H, which is measured by the inclination of the tangents at G and H, since these tangents are parallel to O G and O H respectively, is also measured by the angle G O H, or the arc G H.

(88.) Since a line which is perpendicular to two lines meeting it in a plane is perpendicular to that plane, if a point E can be found such that its distance, measured by a great circle, from each of two points G and H not in the same diameter, is a quadrant, that point is the pole of the great circle passing through G and H.

(89.) If in a plane perpendicular to another plane a line be drawn at right angles to their common intersection, it will be perpendicular to the second plane, (GEOMETRY, book vi. prop. 17.) Hence, if G E be drawn, so that E G H is a right angle, and G E be made = a quadrant, E will be the pole of the circle L G M.

(90.) If D K be a small circle parallel to G H, the line O C is perpendicular to both their planes, and therefore, by (84,) E is the pole of both. And the angle D C K is equal to the angle G O H. Hence D K, the part of the small circle A B intercepted between the two great circles E D G, E K H, passing through their common pole : H G, the part of the great circle L M intercepted in the same manner : : O G : C D : : radius : sin E D. If the radius of the sphere = 1, this ratio becomes 1 : sin E D.

(91.) A *spherical triangle* is a portion of the surface of a sphere contained by three arcs of great circles.

(92.) Any two sides of a spherical triangle taken together are greater than the third. For the arcs A B, B C,
Fig. 12. C A, fig. 12, being arcs of circles whose radii are equal, are measures of the angles A O B, B O C, C O A, at the centre; and when a solid angle is formed by three plane angles, any two of these taken together are greater than the third, (GEOMETRY, book vi. prop. 19;) hence, any two of the sides A B, B C, C A, taken together, are greater than the third.

(93.) The sum of the three angles A O B, B O C, C O A, is less than four right angles, (GEOMETRY, book vi. prop. 20:) and, consequently, the sum of the sides A B, B C, C A, is less than a whole circumference, or 2π.

(94.) The surface of the sphere included between E G F, E H F, fig. 11, is proportional to the angle H E G. For if the angle H E G be repeated any number of times, it is quite evident that the area will be repeated as often, and therefore the whole area will be proportional to the number of the repetitions, or to the whole angle. Hence the area E H F G E is to the whole surface as H E G is to four right angles, or 2π. Now the surface of a sphere whose radius is r is $4\pi . r^2$; hence the surface E H F G E $= 2r^2 \times$ H E G.

(95.) Produce all the sides of the spherical triangle A B C, fig. 12, so as to form complete circles; let D, E, F, be the points of their intersections. Now, (82) the arc A D = semicircle = C F, therefore A C = D F. Similarly, A B = D E, B C = E F. And the angle at A = the angle at D, since (83) each of these is the same as the inclination of the planes A B D, A C D; similarly, the angles at B and C are equal to those at E and F respectively. Hence the triangle A B C is in every respect similar and equal to D E F, and therefore encloses an equal surface. Similarly, A F E = B D C, B F D = A E C. Let the area of A B C or D E F $= x$; that of B D C or A F E $+$ P; that of A E C or B F D = Q; that of A F B = R. Then, by (94,) since x and P together make up the space included by A B D, A C D, we have

$$x + P = 2r^2 \times A.$$

Similarly,

$$x + Q = 2r^2 \times B,$$

$$x + R = 2r^2 \times C.$$

(A being taken to represent the arc corresponding to the angle at A, to radius 1.)

Adding them, $2x + x + P + Q + R = 2r^2 \times (A + B + C)$. But $x + P + Q + R =$ E D F + A F E + B D F + B F A = area defined by B D E A = surface of hemisphere $= 2\pi . r^2$, therefore $2x + 2\pi r^2 = 2r^2 (A + B + C)$, therefore $x = r^2 (A + B + C - \pi)$. If $r = 1$, $x = A + B + C - \pi$. The area of a spherical

triangle, therefore, is proportional to the excess of the sum of its angles above two right angles. This is usually called the *spherical excess*.

(96.) Suppose great circles E F, F D, D E, fig. 13, to be described, of which A, B, C are respectively the poles; they will intersect in points D, E, F, and form a spherical triangle, called the *polar* or *supplemental triangle.* Now, since A is the pole of E F, the arc joining A and F is a quadrant, by (86;) since B is the pole of D F, the arc joining B and F is also a quadrant; hence F is the pole of A B, (88.) Similarly, D and E are the poles of B C, A C, and therefore the triangle A B C is the polar triangle to D E F. Fig. 13.

(97.) Produce the sides of A B C, if necessary, to meet the sides of the polar triangle. Now, D being the pole of K B C, D K = quadrant; similarly, E H = quadrant, therefore D E = D K + E H − H K = semicircle − H K. But as C H and C K are each = a quadrant, H K is the measure of the angle at C, by (87;) hence the sides of the polar triangle are supplements of the angles of the original triangle. Similarly, since the relation between the triangles is reciprocal, the angles of the polar triangle are supplements of the sides of the original triangle.

(98.) The sum of the sides of the polar triangle and the angles of the original triangle $= 3\pi$. Now, the sides of the polar triangle must have some magnitude, and their sum (93) is less than 2π; hence the sum of the angles of the original triangle must be less than 3π, and greater than π.

(99.) A *right-angled spherical triangle* is a spherical triangle having at least one of its angles a right angle.

(100.) If we describe the polar triangle corresponding to a right-angled triangle, one at least of its sides will $= \frac{\pi}{2}$, (97.) This is called a *quadrantal triangle.*

(101.) Let A B C be a triangle right-angled at C, fig. 14; produce the sides A B, C B, to D and E, making $AD = CE = \frac{\pi}{2}$; join F D, and produce it to meet A C produced in F; E B D is called the *complemental triangle.* Since $EC = \frac{\pi}{2}$, and A C E is a right angle, E is the pole of A C, and $FA = EF = \frac{\pi}{2}$, by (89) and (86.) And because $AE = AD = \frac{\pi}{2}$, A is the pole of E D, and $AF = \frac{\pi}{2}$. Since A F and A D each $= \frac{\pi}{2}$, D F measures the angle A, (87.) But E D is the complement of D F, therefore E D is the complement of A. Similarly, the angle E is the complement of A C. And the side B D is evidently the complement of the hypothenuse A B. The angle A D E being a right angle, the complemental triangle is also a right-angled triangle. Fig. 14.

SECTION VI.

Spherical Trigonometry.

(102.) The sines of the sides of a spherical triangle are proportional to the sines of the opposite angles. Let A B C, fig. 15, be any spherical triangle: from C draw C D perpendicular on the plane A O B, meeting it in D; and from D draw in that plane D E, D F perpendicular to A O, B O, and join C E, C F, D O. Now, $CE^2 = CD^2 + DE^2 = CO^2 - OD^2 + DE^2$ (since C D being perpendicular to the plane A O B is perpendicular to D E, D O) $= CO^2 - OE^2$; therefore the angle C E O is a right angle, and the angle C E D (83) = A, and C E is the sine of A C. Hence $CD = CE \,.\, \sin CED = \sin AC \,.\, \sin A$. Similarly, $CD = \sin CB \,.\, \sin B$. Hence $\sin CA \,.\, \sin A = \sin CB \,.\, \sin B$, or $\sin CA : \sin CB :: \sin B : \sin A$. Fig. 15

(103.) To find the cosine of one angle of a spherical triangle when the three sides are given. Let A B C, fig. 16, be the triangle; draw C D, C E, tangents to C A, C B, and O D, O E secants; join D E. Then (83) the angle made by D C, E C, is the angle C; also, the angle D O E is measured by A B. Now, $DE^2 = DC^2 + CE^2 - 2\,DC \,.\, CE \,.\, \cos DCE$, (65,) and $DE^2 = DO^2 + OE^2 - 2\,DO \,.\, OE \,.\, \cos DOE$. Comparing these values, and substituting for D C, &c., $\tan^2 AC + \tan^2 BC - 2 \tan AC \,.\, \tan BC \,.\, \cos C = \sec^2 AC + \sec^2 BC - 2 \sec AC \,.\, \sec BC \,.\, \cos AB$. But $\sec^2 AC = 1 + \tan^2 AC$, $\sec^2 BC = 1 + \tan^2 BC$; subtracting from both sides $\tan^2 AC + \tan^2 BC$, $-\,2 \tan AC \,.\, \tan BC \,.\, \cos C = 2 - 2 \sec AC \,.\, \sec BC \,.\, \cos AC$; or $-\frac{2 \sin AC \,.\, \sin BC \,.\, \cos C}{\cos AC \,.\, \cos BC} = 2 - \frac{2 \cos AB}{\cos AC \,.\, \cos BC}$; from which $\cos C = \frac{\cos AB - \cos AC \,.\, \cos BC}{\sin AC \,.\, \sin BC}$. It is convenient to denote the sides opposite to the angles A, B, C, by the letters *a*, *b*, *c*; then $\cos C = \frac{\cos c - \cos a \,.\, \cos b}{\sin a \,.\, \sin b}$. Fig. 16.

(104.) This is the fundamental formula of Spherical Trigonometry: the theorem of (102) may be deduced from it, but as the process is rather long, and as the geometrical proof is very simple, we have preferred establishing it on an independent demonstration. We shall now proceed to investigate the formulæ best adapted for the logarithmic computation of spherical triangles; the general problem being, as in Plane Trigonometry, from any three given parts (sides or angles) to find the other three. And we shall begin with right-angled triangles.

(105.) Let A B C, fig. 14, be the triangle, having the angle at C a right angle. By the formulæ of (103,) $\cos C = \frac{\cos c - \cos a \,.\, \cos b}{\sin a \,.\, \sin b}$; but $C = 90°$, $\cos C = 0$, therefore $\cos c = \cos a \,.\, \cos b$.

(106.) Hence in the complemental triangle E B D, which is right-angled, $\cos d = \cos b' \,.\, \cos e$; by the relation given in (101) this is immediately transformed into $\sin a = \sin c \,.\, \sin A$; similarly, $\sin b = \sin c \,.\, \sin B$. This might have been proved by (102.)

(107.) Since $\sin b' = \sin d \,.\, \sin B$, we have $\cos A = \cos a \,.\, \sin B$. And $\cos B = \cos b \,.\, \sin A$. Multiplying these equations, $\cos B \,.\, \cos A = \cos b \,.\, \cos a \,.\, \sin B \,.\, \sin A$, or $\cot A \,.\, \cot B = \cos c$.

(108.) Hence $\cot E \,.\, \cot B = \cos d$, or $\tan b \,.\, \cot B = \sin a$. Similarly, $\tan a \,.\, \cot A = \sin b$.

(109.) From this, $\tan e \,.\, \cot E = \sin b'$, or $\cot c \,.\, \tan b = \cos A$; and $\cot c \,.\, \tan a = \cos B$.

(110.) These equations comprehend every case of right-angled spherical triangles; that is, if any two parts besides the right angle be given, any one of the remaining parts can be found by a short logarithmic calculation In the opinion of Delambre (and no one was better qualified by experience to give an opinion) these theorems are best recollected by the practical calculator in their unconnected form. For common purposes, however, a technical memory has been invented, under the title of Naper's rules for Circular Parts, which we shall now describe.

(111.) The five circular parts are the two sides, the complement of the hypothenuse, and the complements of the angles. Any one of these is called a *middle part*; the two next it are then called the *adjacent parts*, and the two remaining ones the *opposite parts*. The two rules are then as follows: the sine of the middle part = product of tangents of adjacent parts; and the sine of the middle part = product of cosines of opposite parts.

(112.) These rules are proved to be true only by showing that they comprehend all the equations which we have just found. We shall leave to the reader the labour of examining every case.

(113.) It was observed in (100) that the polar triangle, corresponding to a right-angled triangle, is a quadrantal triangle. Naper's rules then may be applied to quadrantal triangles, if we take for the circular parts the complements of the sides, the complement of the angle opposite the quadrant, and the two angles. But as there is some difficulty in the determination of the signs, it will, perhaps, be found more convenient to make use of the general formulæ of (102) and (103,) which for this case are always much simplified.

(114.) We shall now examine whether any of these solutions are ambiguous. And for this purpose, as before, we shall attend only to those whose values are given by the values of their sines. Now it is easily seen, that if A and a be given, B, b, and C are all given by their sines; and this case therefore is ambiguous, there being nothing which will enable us to determine whether the smallest corresponding arcs, or their supplements,

Fig. 17.

are to be taken. In fact, the triangles A B C and A′ B C, fig. 17, will equally satisfy the given conditions, since the angle at A′ = that at A.

(115.) If A and c be given, a is given by its sine. Since, however, $\tan a = \sin b \,.\, \tan A$, and the tangent becomes negative when the arc is greater than 90°, and since $\sin b$ is always positive, (as b must be less than 180°,) a must be greater or less than 90°, as A is greater or less than 90°, which removes the apparent ambiguity. If a and c be given to find A, the same remark applies.

(116.) We proceed to find formulæ of solution for all spherical triangles. Given the three sides to find the angles. We have seen (103) that $\cos C = \frac{\cos c - \cos a \,.\, \cos b}{\sin a \,.\, \sin b}$. This formula is not adapted to logarithmic calculation. But $1 + \cos C$, or $2\cos^2 \frac{C}{2} = \frac{\cos c - (\cos a \,.\, \cos b - \sin a \,.\, \sin b)}{\sin a \,.\, \sin b} = \frac{\cos c - \cos \overline{a+b}}{\sin a \,.\, \sin b} =$

$\frac{2 \sin \frac{a+b+c}{2} \,.\, \sin \frac{a+b-c}{2}}{\sin a \,.\, \sin b}$, or putting $S = \frac{a+b+c}{2}$, $\cos^2 \frac{C}{2} = \frac{\sin S \,.\, \sin \overline{S-c}}{\sin a \,.\, \sin b}$. Again, $1 - \cos C$, or $2 \sin^2 \frac{C}{2} = \frac{(\cos a \,.\, \cos b + \sin a \,.\, \sin b) - \cos c}{\sin a \,.\, \sin b} = \frac{\cos \overline{a-b} - \cos c}{\sin a \,.\, \sin b} = \frac{2 \sin \frac{b+c-a}{2} \,.\, \sin \frac{a+c-b}{2}}{\sin a \,.\, \sin b}$

$\sin^2 \frac{C}{2} = \frac{\sin \overline{S-a} \,.\, \sin \overline{S-b}}{\sin a \,.\, \sin b}$. Dividing this by the former, $\tan^2 \frac{C}{2} = \frac{\sin \overline{S-a} \,.\, \sin \overline{S-b}}{\sin S \,.\, \sin S - c}$. Multiplying them together, since $\sin C = 2 \sin \frac{C}{2} \cos \frac{C}{2}$, $\sin^2 C = \frac{4 \,.\, \sin S \,.\, \sin \overline{S-a} \,.\, \sin \overline{S-b} \,.\, \sin S - c}{\sin^2 a \,.\, \sin^2 b}$. With all these forms logarithms can conveniently be used.

(117.) Given two sides (a, b) and the included angle (C) to find the other parts. From the expressions just found, $\tan \frac{A}{2} = \sqrt{\frac{\sin \overline{S-b} \,.\, \sin \overline{S-c}}{\sin S \,.\, \sin \overline{S-a}}}$; $\tan \frac{B}{2} = \sqrt{\frac{\sin \overline{S-a} \,.\, \sin \overline{S-c}}{\sin S \,.\, \sin \overline{S-b}}}$, therefore $\tan \frac{A+B}{2} =$

$$\frac{\tan\frac{A}{2}+\tan\frac{B}{2}}{1-\tan\frac{A}{2}\,.\,\tan\frac{B}{2}}=\sqrt{\frac{\sin\overline{S-c}}{\sin S}}\,.\,\frac{\sqrt{\frac{\sin\overline{S-b}}{\sin\overline{S-a}}}+\sqrt{\frac{\sin\overline{S-a}}{\sin\overline{S-b}}}}{1-\frac{\sin\overline{S-c}}{\sin S}}=\sqrt{\frac{\sin S\,.\,\sin\overline{S-c}}{\sin\overline{S-a}\,.\,\sin\overline{S-b}}}\times$$

$$\frac{\sin\overline{S-b}+\sin\overline{S-a}}{\sin S-\sin\overline{S-c}}=\cot\frac{C}{2}\,.\,\frac{2\sin\frac{C}{2}\,.\,\cos\frac{a-b}{2}}{2\cos\frac{a+b}{2}\,.\,\sin\frac{C}{2}}=\frac{\cos\frac{a-b}{2}}{\cos\frac{a+b}{2}}\cot\frac{C}{2}.$$ Similarly, $\tan\frac{A-B}{2}=$

$$\frac{\tan\frac{A}{2}-\tan\frac{B}{2}}{1+\tan\frac{A}{2}\tan\frac{B}{2}}=\cot\frac{C}{2}\times\frac{\sin\overline{S-b}-\sin\overline{S-a}}{\sin S+\sin\overline{S-c}}=\frac{\sin\frac{a-b}{2}}{\sin\frac{a+b}{2}}\cot\frac{C}{2}.$$ The sum and difference of A and B being thus found, A and B will be determined. The third side will be found by the proportion of (102.)

(118.) It is, however, very frequently desirable to find the third side without finding the angles. Now, $\cos c$ (103) $=\cos a\,.\,\cos b+\sin a\,.\,\sin b\,.\,\cos C$, or $\operatorname{versin} c=1-\cos c=1-\cos a\,.\,\cos b-\sin a\,.\,\sin b\,.\,\cos C$ $=1-(\cos a\,.\,\cos b+\sin a\,.\,\sin b)+\sin a\,.\,\sin b\,.\operatorname{versin} C=\operatorname{versin}\overline{a-b}+\sin a\,.\,\sin b\,.\operatorname{versin} C$ $=\operatorname{versin}\overline{a-b}\,.\left(1+\frac{\sin a\,.\,\sin b\,.\operatorname{versin} C}{\operatorname{versin}\overline{a-b}}\right)$. Make $\frac{\sin a\,.\,\sin b\,.\operatorname{versin} C}{\operatorname{versin}\overline{a-b}}=\tan^2\theta$; then $\operatorname{versin} c=\operatorname{versin}\overline{a-b}\,.\sec^2\theta$. Or thus, $\cos C=2\cos^2\frac{C}{2}-1$; therefore $\cos c=\cos a\,.\,\cos b-\sin a\,.\,\sin b+2\sin a\,.\,\sin b\,.\,\cos^2\frac{C}{2}=\cos\overline{a+b}+2\sin a\,.\,\sin b\,.\,\cos^2\frac{c}{2}$; therefore $1-\cos c$, or $2\sin^2\frac{C}{2}=1-\cos\overline{a+b}-2\sin a\,.\,\sin b\,.\,\cos^2\frac{C}{2}$, and $\sin^2\frac{c}{2}=\sin^2\frac{a+b}{2}-\sin a\,.\,\sin b\,.\,\cos^2\frac{C}{2}$. Let $\sin a\,.\,\sin b\,.\,\cos^2\frac{C}{2}=\sin^2\theta$; then $\sin^2\frac{c}{2}=\sin^2\frac{a+b}{2}-\sin^2\theta=$, by (39,) $\sin\overline{\frac{a+b}{a}+\theta}\,.\,\sin\overline{\frac{a+b}{2}-\theta}$.

(119.) The following theorem is frequently useful. We have found $\cos A=\frac{\cos a-\cos b\,.\,\cos c}{\sin b\,.\,\sin c}$; also $\cos c=\cos a\,.\,\cos b+\sin a\,.\,\sin b\,.\,\cos C$; substituting this in the numerator, $\cos A=$

$$\frac{\cos a-\cos^2 b\,.\cos a-\sin b\,.\,\cos b\,.\sin a\,.\,\cos C}{\sin b\,.\,\sin c}=\frac{\cos a\,.\,\sin b-\cos b\,.\,\sin a\,.\,\cos C}{\sin c};$$

and $\sin c=\frac{\sin c\,.\,\sin a}{\sin A}$, therefore $\cos A=\sin\frac{\cos a\,.\,\sin b-\cos b\,.\,\sin a\,.\,\cos C}{\sin C\,.\,\sin A}$, or $\cot A.\sin C=\cot a\,.\sin b-\cos C\,.\cos b$. This formula is chiefly useful for finding the corresponding small variations of the parts of a spherical triangle. It may also be used to determine A: thus, $\cot A=\frac{\cot a}{\sin C}\times\left(\sin b-\frac{\cos C}{\cot a}\cos b\right)$; let $\frac{\cos C}{\cot a}=\tan\theta$, then $\cot A=\frac{\cot a}{\sin C\,.\,\cos\theta}(\sin b\cos\theta-\cos b\sin\theta)=\frac{\cot a\,.\,\sin\overline{b-\theta}}{\sin C\,.\,\cos\theta}$.

(120.) Suppose two angles and the included side (A, B, c) given, To find the remaining parts. Take the polar triangle; let a', b', c', be the sides of which the points A, B, C, are the poles; A', B', C', the opposite angles.

Then, (97,) $c'=\pi-C$, $C'=\pi-c$. Then $\tan\frac{A'+B'}{2}=\cot\frac{C'}{2}\,.\,\frac{\cos\frac{a'-b'}{2}}{\cos\frac{a'+b'}{2}}$, (117,) or $-\tan\frac{a+b}{2}$ $=\tan\frac{c}{2}\,.\,\frac{\cos\frac{B-A}{2}}{-\cos\frac{A+B}{2}}$, or $\tan\frac{a+b}{2}=\tan\frac{c}{2}\,.\,\frac{\cos\frac{A-B}{2}}{\cos\frac{A+B}{2}}$. Similarly, $\tan\frac{A'-B'}{2}=\cot\frac{C'}{2}\,.\,\frac{\sin\frac{a'-b'}{2}}{\sin\frac{a'+b'}{2}}$, or $\tan\frac{a-b}{2}=\tan\frac{c}{2}\,.\,\frac{\sin\frac{A-B}{2}}{\sin\frac{A+B}{2}}$. The sides being thus found, the third angle may be found by the pro-

portion of (102.) If it be wished to have the third angle independently, the formulæ of (118) may be adapted in the same way.

(121.) If we divide one of the equations in (117,) or (120,) by the other, we find $\dfrac{\tan\dfrac{a+b}{2}}{\tan\dfrac{a-b}{2}} = \dfrac{\tan\dfrac{A+B}{2}}{\tan\dfrac{A-B}{2}}$.

(122.) If two sidee be given and an angle adjacent to one, then another angle is found by (102,) and the third side by (120,) or the third angle by (117.) In this case the solution is ambiguous under the same circumstances as in the correeponding case of plane triangles. If two angles and an adjacent side, B, C, b, fig. 18, be given, the process is the same. In this case, when C is greater than B, either of the triangles C A B, C A B′ (in which B′ A produced makes A D = A B) satisfies the given conditions. These are the only ambiguous cases of oblique-angled spherical triangles. Fig. 18.

(123.) If the three angles be given, the formulæ of (116) may be applied to the polar triangle, and the eides of the given triangle may be found. This, however, is a case which never occurs in any applicatione of Trigonometry.

SECTION VII.

On small corresponding Variations of the Parts of Triangles.

(124.) It is frequently desirable to ascertain the effect which will be produced on one part of a triangle by the variation of another, all the rest remaining unvaried. To estimate the probable effect of error in observation; to reduce observatinns made in one situation to what they would be in a situation little distant; to take account of refraction, parallax, &c., this theory is absolutely necessary. We shall, therefore, give the general method of finding these corresponding variations.

(125.) In almost all cases expressions may be conveniently found by writing dnwn two equations, one of which results from giving to the quantities contained in the other the variations which they are supposed to undergo, and then taking their difference. And this method has the advantage of showing precisely the magnitude of the error made by any farther simplification. It will be best illustrated by examples.

(126.) The height of a building is found by measuring a horizontal line from its base, and at the extremity observing the apparent altitude; and the angle is liable to a small error of observatinn. In this case, if a be the measured distance, θ the angle, x the height, we have $x = a \,.\, \tan\theta$. And if giving to θ the variation $\delta\theta$ would produce in x the variation δx, we have $x + \delta x = a \,.\, \tan\overline{\theta + \delta\theta}$. Subtracting the former equation, $\delta x = a\,(\tan\overline{\theta+\delta\theta} - \tan\theta) =$, by (42,) $a\,\dfrac{\sin\delta\theta}{\cos\theta \,.\, \cos\overline{\theta+\delta\theta}}$. Nnw, if we suppose $\delta\theta$ to be very small, we may put $\delta\theta$ instead of $\sin\delta\theta$, and $\cos\theta$ instead nf $\cos\overline{\theta+\delta\theta}$, without sensible error; then $\delta x = \dfrac{a\,\delta\theta}{\cos^2 .\, \theta}$. Here $\delta\theta$ is supposed to be expressed by the length of the corresponding arc to radius 1. If it $= n$ seconds, then fnr $\delta\theta$ we must put $n \times 0{,}000004848$, (4,) and $\delta x = \dfrac{a \,.\, n \,.\, 0{,}000004848}{\cos^2\theta}$ very nearly.

(127.) If it were wished to determine a, so that the error should be a *minimum*, it must be observed that a though determinate is not constant, but $= x \,.\, \cot\theta$, whence $\delta x = \dfrac{x\,\delta\theta}{\sin\theta \,.\, \cos\theta} = \dfrac{2\,x\,\delta\theta}{\sin 2\theta}$, which is least when $\sin 2\theta$ is greatest, or $2\theta = \dfrac{\pi}{2}$, nr $\theta = \dfrac{\pi}{4}$.

(128.) Suppose in a right-angled spherical triangle, C being the right angle, A is given, To find the variatinn of a when c receives a small variation. Here (106) $\sin a = \sin A \,.\, \sin c$; hence $\sin\overline{a+\delta a} = \sin A \,.\, \sin\overline{c+\delta c}$; taking the difference, $\sin\overline{a+\delta a} - \sin a = \sin A\,(\sin\overline{c+\delta c} - \sin c)$, or $2\cos\overline{a+\dfrac{\delta a}{2}} \,.\, \sin\dfrac{\delta a}{2} = 2\sin A \,.\, \cos\overline{c+\dfrac{\delta c}{2}} \,.\, \sin\dfrac{\delta c}{2}$, and $\sin\dfrac{\delta a}{2} = \dfrac{\sin A \,.\, \cos\overline{c+\dfrac{\delta c}{2}}}{\cos\overline{a+\dfrac{\delta a}{2}}}\sin\dfrac{\delta c}{2}$, or if δa and δc be very small,

$\delta a = \frac{\sin A . \cos c}{\cos a} \delta c = \sin A . \cos b . \varrho c$, or $= \frac{\tan a}{\tan c} \delta c$. If m be the number of seconds in δa, n that in δc, $m = \frac{\tan a}{\tan c} n$.

(129.) The consideratinn of particular cases of this last problem shnws that we must be cantious in applying to any extent the simplifications which were there introduced from considering δc as small. Suppose $c = 90°$; it would seem that $\delta a = 0$. Taking, however, the original expression $\sin \frac{\delta a}{2} = \frac{\sin A . \cos \overline{c + \frac{\delta c}{2}}}{\cos \overline{a + \frac{\delta a}{2}}} . \sin \frac{\delta c}{2}$, we may observe that, when $c = \frac{\pi}{2}$, $\cos \overline{c + \frac{\delta c}{2}} = -\sin \frac{\delta c}{2}$, by (23) and (34;) therefore $\sin \frac{\delta a}{2} = \frac{-\sin A}{\cos \overline{a + \frac{\delta a}{2}}} \sin^2 \frac{\delta c}{2}$. Making $\frac{\delta c}{2}$ very small, $\frac{\delta a}{2} = -\frac{\sin A}{\cos a} . \frac{\overline{\delta c}|^2}{4}$, or $\delta a = \frac{\sin A}{2 \cos a} \overline{\delta c}|^2$. Here then $m = -\frac{\sin A \times 0{,}000004848}{2 \cos a} n^2 = -\tan A \times 0{,}000002424 \times n^2$, since a now $= A$.

(130.) Given two sides (a, b) of a spherical triangle, and the included angle (C) to find the variation produced in c by the variation of C. Here $\cos c = \cos a . \cos b + \sin a . \sin b . \cos C$, (103,) and $\cos \overline{c + \delta c} = \cos a . \cos b + \sin a . \sin b . \cos \overline{C + \delta C}$. Subtracting the latter, $2 \sin \overline{c + \frac{\delta c}{2}} . \sin \frac{\delta c}{2} = 2 \sin a . \sin b . \sin \overline{C + \frac{\delta C}{2}} . \sin \frac{\delta C}{2}$. If δC be small, and if C or c be not small, then $\sin c . \delta c = \sin a . \sin b . \sin C . \delta C$ nearly, or $\delta c = \frac{\sin a . \sin b . \sin C}{\sin c} \times \delta C = \sin B . \sin a . \delta C$. If m and n be the number of seconds in δc and δC, $m = \sin B . \sin a . n$. If $C = 0$, then $2 \sin \overline{c + \frac{\delta c}{2}} . \sin \frac{\delta c}{2} = 2 \sin a . \sin b . \sin^2 \frac{\delta C}{2}$; and supposing δC very small, $\sin c . \frac{\delta c}{2} = \sin a . \sin b . \frac{\overline{\delta C}|^2}{4}$, or $\delta c = \frac{\sin a . \sin b}{2 \sin c} \overline{\delta C}|^2$, or $m = \frac{\sin a . \sin b \times 0{,}000004848}{2 \sin c} \times n^2$.

(131.) With the same data, to find the variation in A. Here (119) $\cot A . \sin C = \cot a . \sin b - \cos C . \cos b$, and $\cot \overline{A + \delta A} . \sin \overline{C + \delta C} = \cot a . \sin b - \cos . \overline{C + \delta C} . \cos b$; subtracting the former, $\cot \overline{A + \delta A} . \sin \overline{C + \delta C} - \cot A . \sin C = \cos b . (\cos C - \cos \overline{C + \delta C})$. Now $\cot \overline{A + \delta A} . (\sin \overline{C + \delta c} - \sin C) = \cot \overline{A + \delta A} . 2 \cos \overline{C + \frac{\delta C}{2}} . \sin \frac{\delta C}{2}$; also $\sin C (\cot \overline{A + \delta A} - \cot A) = -\sin C \frac{\sin . \delta A}{\sin A . \sin \overline{A + \delta A}}$; adding these together, $\cot \overline{A + \delta A} . \sin \overline{C + \delta C} - \cot A . \sin C = 2 \cot \overline{A + \delta A} . \cos \overline{C + \frac{\delta c}{2}} . \sin \frac{\delta c}{2} - \sin C . \frac{\sin \delta A}{\sin A . \sin \overline{A + \delta A}}$. And $\cos C - \cos \overline{C + \delta C} = 2 \sin \overline{C + \frac{\delta C}{2}} . \sin \frac{\delta C}{2}$; substituting in the equation, and supposing δC and δA very small, $\cot A . \cos C . \delta C - \frac{\sin C}{\sin^2 A} \delta A = \cos b . \sin C . \delta C$, and $\delta A = \frac{\sin^2 A}{\sin C} (\cot A \cos C - \cos b . \sin C) . \delta C$; or if p be the number of seconds in δA, $p = \frac{\sin^2 A}{\sin C} (\cot A \cos C - \cos b . \sin C) \times n$. Putting for cot A its value, this is easily changed into $p = -\frac{\sin^2 A}{\sin C} \cot B . \frac{\sin b}{\sin a} n = -\frac{\sin A . \cos B}{\sin C} n$.

(132.) The principle and the mode of its application is now sufficiently evident. We must, hnwever, remark that in many cases the corresponding variations may be easily fonnd by geometrical considerations. Thus, for the problem of (130,) let A B C, fig. 19, be the triangle, and by the variation of C let it be changed to A B′ C; Fig. 19

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if B x be supposed to be drawn perpendicular to A B′, then A x will ultimately = A B, and therefore x B′ = δc. Now δc = B B′ sin B′ B x = B B′ sio C B A (since C B B′ is a right angle, as C is the pole of B B′, (86,) and therefore C B B′ = x B A); but B B′ = sin a . B C B′ by (90) = sin a . δ C, therefore δc = sio a . sio C B A . δ C = sin B . sin a . δ C, as in (130.) Aod if the variation of A were required, we should have $\delta A = \frac{B\,x}{\sin c} = \frac{B\,B' . \cos B' B\,x}{\sin c} = \frac{\sin a . \cos B . \delta C}{\sin c}$, or $= \frac{\sin A . \cos B}{\sin C} \delta C$ for the quantity hy wbieh A is diminished, as in (131.)

(133.) The geometrical method then can be applied with great ease to those examples in which the variation of one element is expressed in terms of the first power of the variation of another elemeot, but it ean very seldom be applied to those cases in which as in (129) the variation of one depends on the square of the variation of the other. Another method will hereafter be described, not however preferable in general to the first given here.

SECTION VIII.

Investigations requiring a higher Analysis than the preceding.

(134.) The preeeding sections have referred to nothing more difficult than the most commoo propositions of Plane Geometry and Algebra, and one or two theorems of Solid Geometry. In this section it is proposed to compreheod some of those expressions which require for their demonstration some of the higher parts of analysis, particularly the Differential Calculus, and the Calculus of Finite Differences.

(135.) To express generally cos $n\,x$ in a series proceeding hy powers of cos x. If we observe the manner in which the expressions of (38) are successively formed, we shall easily see that cos $n\,x$, n being a positive integer, will always be expressed hy this form, $2^{n-1} \cos^n x + a \cos^{n-2} x + b \cos^{n-4} x +$ &c.; a, b, &c. being functions of n. Also there will be no second term till $n = 2$; no third term till $n = 4$, &c. Let $2 \cos n\,x = u_n$; $2 \cos x = p$; then $u_{n+1} = p . u_n - u_{n-1}$. Assume then $u_n = p^n + A_n . p^{n-2} + B_n . p^{n-4} + C_n . p^{n-6} +$ &c., A_n, B_n, &c. being functions of n to be determined; then $u_{n+1} = p^{n+1} + A_{n+1} . p^{n-1} + B_{n+1} . p^{n-3} + C_{n+1} . p^{n-5} +$ &c.; $u_{n-1} = p^{n-1} + A_{n-1} . p^{n-3} + B_{n-1} . p^{n-5} + C_{n-1} . p^{n-7} +$ &c. Substituting these expressioos io the equation above, and equating the coefficients of similar powers, we have these equations; $A_{n+1} = A_n - 1$; $B_{n+1} = B_n - A_{n-1}$; $C_{n+1} = C_n - B_{n-1}$, &c.; or since $A_{n+1} - A_n = \Delta . A_n$, &c.; $\Delta . A_n = -1$; $\Delta . B_n = - A_{n-1}$; $\Delta C_n = - B_{n-1}$, &c. Integrating the first, we have $A_n = -n + C$; and sinee $2 \cos 2\,x = 2\,\overline{\cos x}|^2 - 2$, we must have $A_2 = -2$, therefore $C = 0$, and $A_n = -n$. (It will be remarked, that we have not found the correction hy making $n = 1$, because the equation $A_2 = A_1 - 1$ is not true, the value of u_0 or $2 \cos 0$ being not 1 but 2. After this, however, the equation $A_{n+1} = A_n - 1$ always holds; A_2 therefore is the first quantity to which the general value of A_n can be applied. The other equations $B_{n+1} = B_n - A_{n-1}$, &c. are true without any exception.) Hence $A_{n-1} = -\overline{n-1}$, therefore $\Delta B_n = \overline{n-1} = \overline{n-2} + 1$; integrating, $B_n = \frac{\overline{n-2} . \overline{n-3}}{2} + n + C'$; making this $= 0$ when $n = 3$, $B_n = \frac{\overline{n-2} . \overline{n-3}}{2} + \overline{n-3} = \frac{n . \overline{n-3}}{2}$. Henee $- B_{n-1} = \frac{-\overline{n-1} . \overline{n-4}}{2} = \frac{-\overline{n-3} . \overline{n-4}}{2} - 2\,\frac{n-4}{2} = \Delta . C_n$; therefore $C_n = - \frac{\overline{n-3} . \overline{n-4} . \overline{n-5}}{2 . 3} - \frac{\overline{n-4} . \overline{n-5}}{2} = - \frac{n . \overline{n-4} . \overline{n-5}}{2 . 3}$, which needs no correction, as it vanishes when $n = 5$. Continuing the proeess, we find $D_n = \frac{n . \overline{n-5} . \overline{n-6} . \overline{n-7}}{2 . 3 . 4}$, &c.; hence $u_n = p^n - n . p^{n-2} + \frac{n . \overline{n-3}}{2} p^{n-4} -$ &c.; or $2 \cos n\,x = (2 \cos x)^n - n\,(2 \cos x)^{n-2} + \frac{n . \overline{n-3}}{2} (2 \cos x)^{n-4} - \frac{n . \overline{n-4} . \overline{n-5}}{2 . 3} (2 \cos x)^{n-6} + \frac{n . \overline{n-5} . \overline{n-6} . \overline{n-7}}{2 . 3 . 4} (2 \cos x)^{n-8} -$ &c. Of this important theorem we believe this is the simplest demonstration that has yet beeo given.

(136.) If n be even and $= 2\,m$, the last term will be $\pm \frac{2\,m . \overline{m-1} . \overline{m-2} \ldots . 1}{2 . 3 \ldots m} = 2$; the last but one $= \mp \frac{2\,m . m . \overline{m-1} \ldots . 4 . 3}{2 . 3 \ldots . \overline{m-1}} (2 \cos x)^2 = \mp m^2 . (2 \cos x)^2$; the last but two $= \pm \frac{2\,m . \overline{m+1} . m . \overline{m-1} . . 6 . 5}{2 . 3 \ldots . \overline{m-2}} (2 \cos x)^4 = \pm \frac{2\,m^2 . \overline{m^2 - 1}}{2 . 3 . 4} (2 \cos x)^4$, &c. Hence $\cos n\,x = \pm \left\{ 1 - \frac{n^2}{1 . 2} \cos^2 x + \frac{n^2 . \overline{n^2 - 4}}{1 . 2 . 3 . 4} \cos^4 x - \frac{n^2 . \overline{n^2 - 4} . \overline{n^2 - 16}}{1 . 2 . 3 . 4 . 5 . 6} \cos^6 x + \text{\&c.} \right\}$, the upper sign to be taken when m is even or n divisible by 4

If n be odd or of the form $\overline{2m+1}$, the last term will be $\pm \frac{\overline{2m+1}\,.\,m\,.\,\overline{m-1}\,\ldots\,2}{2\,.\,3\,\ldots\,m}\,2\cos x = \pm\,\overline{2m+1}$. $2\cos x = \pm n\,.\,2\cos x$; the last but one $= \mp \frac{\overline{2m+1}\,.\,\overline{m+1}\,.\,m\,.\,\overline{m-1}\,\ldots\,4}{2\,.\,3\,\ldots\,\overline{m-1}}\,(2\cos x)^3$ $= \mp \frac{\overline{2m+1}\,.\,\overline{m+1}\,.\,m}{2\,.\,3}\,(2\cos x)^3 = \mp \frac{n\,.\,\overline{n^2-1}}{2\,.\,3}\,2\cos^3 x$, &c.; hence when n is odd $\cos n x$ $= \mp \left\{ n\cos x - \frac{n\,.\,\overline{n^2-1}}{1\,.\,2\,.\,3}\cos^3 x + \frac{n\,.\,\overline{n^2-1}\,.\,\overline{n^2-9}}{1\,.\,2\,.\,3\,.\,4\,.\,5}\cos^5 x - \text{\&c.} \right\}$, the upper sign being taken when n is of the form $4s+1$, and the lower when of the form $4s+3$.

(137.) Let $x = \frac{\pi}{2} - y$; then $\cos x = \sin y$, and $\cos n x = \cos\left(\frac{n\pi}{2} - n y\right) = \cos\frac{n\pi}{2}\ \cos n y + \sin\frac{n\pi}{2}$ $\sin n y$. If n be divisible by 4, this $= \cos n y$; if only divisible by 2, it $= -\cos n y$. Hence in all cases, n being even, $\cos n y = 1 - \frac{n^2}{1\,.\,2}\,.\,\sin^2 y + \frac{n^2\,(n^2-4)}{1\,.\,2\,.\,3\,.\,4}\,.\sin^4 y - \text{\&c.}$ If n be of the form $4s+1$, $\cos n x$ $= \sin n y$; if of the form $4s+3$, $\cos n x = -\sin n y$. Hence in all cases, n being odd, $\sin n y = n \sin y$ $- \frac{n\,(n^2-1)}{1\,.\,2\,.\,3}\sin^3 y + \text{\&c.}$

(138.) Differentiating the first equation of the last article we find, n being even, $\sin n y = \cos y \Big\{ n \sin y$ $- \frac{n\,(n^2-4)}{1\,.\,2\,.\,3}\sin^3 y + \frac{n\,(n^2-4)\,(n^2-16)}{1\,.\,2\,.\,3\,.\,4\,.\,5}\sin^5 y - \text{\&c.}\Big\}$. By similar operations we may from these deduce other formulæ.

(139.) Let $n\,y = z$; then (n even) $\cos z = 1 - \frac{z^2}{1\,.\,2}\,.\,\frac{\sin^2 y}{y^2} + \frac{z^4}{1\,.\,2\,.\,3\,.\,4}\,.\,\frac{\left(1-\frac{4}{n^2}\right)\sin^4 y}{y^4}$ $- \frac{z^6}{1\,.\,2\,.\,3\,.\,4\,.\,5\,.\,6}\,.\,\frac{\left(1-\frac{4}{n^2}\right)\left(1-\frac{16}{n^2}\right)\sin^6 y}{y^6} + \text{\&c.}$ Suppose now n to be increased without limit; the expressions $1 - \frac{4}{n^2}$, $1 - \frac{16}{n^2}$, &c. approach to 1 as their limit; the fraction $\frac{\sin y}{y}$ also has 1 for its limit. Hence $\cos z = 1 - \frac{z^2}{1\,.\,2} + \frac{z^4}{1\,.\,2\,.\,3\,.\,4} - \frac{z^6}{1\,.\,2\,.\,3\,.\,4\,.\,5\,.\,6} + \text{\&c.}$

(140.) Again, (n odd) $\sin z = z\,.\,\frac{\sin y}{y} - \frac{z^3}{1\,.\,2\,.\,3}\,.\,\frac{\left(1-\frac{1}{n^2}\right)\sin^3 y}{y^3} + \frac{z^5}{1\,.\,2\,.\,3\,.\,4\,.\,5}$ $.\,\frac{\left(1-\frac{1}{n^2}\right)\left(1-\frac{9}{n^2}\right)\sin^5 y}{y^5} - \text{\&c.}$ Increasing n without limit $\sin z = z - \frac{z^3}{1\,.\,2\,.\,3} + \frac{z^5}{1\,.\,2\,.\,3\,.\,4\,.\,5} - \text{\&c.}$

(141.) Now we may remark, that if we expand $e^{x\sqrt{-1}}$ and $e^{-x\sqrt{-1}}$ (e being the base of Naperian logarithms $= 2{,}7182818$) in the same way in which we expand e^x, we have

$$e^{x\sqrt{-1}} = 1 + \frac{x\sqrt{-1}}{1} - \frac{x^2}{1\,.\,2} - \frac{x^3\sqrt{-1}}{1\,.\,2\,.\,3} + \frac{x^4}{1\,.\,2\,.\,3\,.\,4} + \text{\&c.}$$

$$e^{-x\sqrt{-1}} = 1 - \frac{x\sqrt{-1}}{1} - \frac{x^2}{1\,.\,2} + \frac{x^3\sqrt{-1}}{1\,.\,2\,.\,3} + \frac{x^4}{1\,.\,2\,.\,3\,.\,4} - \text{\&c.}$$

Adding them, $e^{x\sqrt{-1}} + e^{-x\sqrt{-1}} = 2\left(1 - \frac{x^2}{1\,.\,2} + \frac{x^4}{1\,.\,2\,.\,3\,.\,4} - \text{\&c.}\right) = 2\cos x$, or $\cos x = \frac{e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}}{2}$. Subtracting, $e^{x\sqrt{-1}} - e^{-x\sqrt{-1}} = 2\sqrt{-1}\left\{x - \frac{x^3}{1\,.\,2\,.\,3} + \frac{x^5}{1\,.\,2\,.\,3\,.\,4\,.\,5} - \text{\&c.}\right\} = 2\sqrt{-1}\,.\,\sin x$, or $\sin x$ $= \frac{e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}}{2\sqrt{-1}}$. These expressions are to be regarded as having no other meaning than this; if expanded according to the rules by which we expand possible algebraic quantities, they would produce the series for $\cos x$ and $\sin x$.

(142.) From these equations we have $e^{x\sqrt{-1}} = \cos x + \sqrt{-1}\,.\sin x$; $e^{-x\sqrt{-1}} = \cos x - \sqrt{-1}\,.\sin x$. Similarly, $e^{y\sqrt{-1}} = \cos y + \sqrt{-1}\,.\sin y$; multiplying this by $e^{x\sqrt{-1}}$, $e^{(x+y)\sqrt{-1}} = (\cos x + \sqrt{-1}\sin x)$ $(\cos y + \sqrt{-1}\,.\sin y)$. But $e^{(x+y)\sqrt{-1}} = \cos \overline{x+y} + \sqrt{-1}\,.\sin \overline{x+y}$; hence we have this very remarkable formula, $(\cos x + \sqrt{-1}\,.\sin x)\,.\,(\cos y + \sqrt{-1}\,.\sin y) = \cos \overline{x+y} + \sqrt{-1}\,.\sin \overline{x+y}$. The same result will be obtained by actually multiplying together the two factors. If we suppose y successively $= x, 2x$, &c., we shall have $(\cos x + \sqrt{-1}\,.\sin x)^n = \cos n x + \sqrt{-1}\,.\sin n x$, which implies that n is an integer. Or thus, $e^{nx\sqrt{-1}} = (e^{x\sqrt{-1}})^n$, that is, $\cos n x + \sqrt{-1}\,.\sin n x = (\cos x + \sqrt{-1}\,.\sin x)^n$, whether n be whole or fractional. Similarly, $\cos n x - \sqrt{-1}\,.\sin n x = (\cos x - \sqrt{-1}\,.\sin x)^n$. This theorem is due to Demoivre.

(143.) Expanding the two last expressions, adding them together, and dividing by 2, we have

$$\cos n x = \cos^n x - \frac{n\,.\,\overline{n-1}}{2}\cos^{n-2} x\,.\,\sin^2 x + \frac{n\,.\,\overline{n-1}\,.\,\overline{n-2}\,.\,\overline{n-3}}{2\,.\,3\,.\,4}\,.\cos^{n-4} x\,.\,\sin^4 x - \text{\&c.}$$

$$= \cos^n x\left\{1 - \frac{n\,.\,\overline{n-1}}{2}\tan^2 x + \frac{n\,.\,\overline{n-1}\,.\,\overline{n-2}\,.\,\overline{n-3}}{2\,.\,3\,.\,4}\tan^4 x - \text{\&c.}\right\}.$$

Subtracting and dividing by $2\sqrt{-1}$,

$$\sin n x = n\,.\cos^{n-1} x\,.\,\sin x - \frac{n\,.\,\overline{n-1}\,.\,\overline{n-2}}{2\,.\,3}\cos^{n-3} x\,.\,\sin^3 x + \text{\&c.}$$

$$= \cos^n x\left\{n \tan x - \frac{n\,.\,\overline{n-1}\,.\,\overline{n-2}}{2\,.\,3}\tan^3 x + \text{\&c.}\right\}.$$

Dividing the latter by the former,

$$\tan n x = \frac{n \tan x - \dfrac{n\,.\,\overline{n-1}\,.\,\overline{n-2}}{2\,.\,3}\tan^3 x + \text{\&c.}}{1 - \dfrac{n\,.\,\overline{n-1}}{2}\tan^2 x + \dfrac{n\,.\,\overline{n-1}\,.\,\overline{n-2}\,.\,\overline{n-3}}{2\,.\,3\,.\,4}\tan^4 x - \text{\&c.}}.$$

(144.) In (142) suppose such a value to be given to $n x$ that $\sin n x = 0$, $\cos n x = 1$; then $n x = 0$, or 2π, or 4π, &c., and $x = 0$, or $\frac{2\pi}{n}$, or $\frac{4\pi}{n}$, &c. Hence the following equations are true, $\overline{1}|^n = 1$; $\left(\cos\frac{2\pi}{n} \pm \sqrt{-1}\sin\frac{2\pi}{n}\right)^n = 1$; $\left(\cos\frac{4\pi}{n} \pm \sqrt{-1}\,.\sin\frac{4\pi}{n}\right)^n = 1$, &c. The quantities within the brackets are therefore roots of the equation $z^n = 1$, or $z^n - 1 = 0$. Hence we have for simple divisors of that equation, $z - 1$, $z - \cos\frac{2\pi}{n} - \sqrt{-1}\,.\sin\frac{2\pi}{n}$, $z - \cos\frac{2\pi}{n} + \sqrt{-1}\,.\sin\frac{2\pi}{n}$, &c.; or grouping in pairs the corresponding factors, the factors of $z^n - 1$ are $z - 1$, $z^2 - 2z\,.\cos\frac{2\pi}{n} + 1$, $z^2 - 2z\,.\cos\frac{4\pi}{n} + 1$, &c., to be continued till the number of dimensions $= n$. If n be even, the last factor will be $z + 1$. In a similar way, we have for the factors of $z^n + 1$, $z^2 - 2z\cos\frac{\pi}{n} + 1$, $z^2 - 2z\cos\frac{3\pi}{n} + 1$, &c. to n dimensions. If n be odd, the last factor will be $z + 1$.

(145.) If we put $\frac{w}{a}$ for z, we have $w^n - a^n = (w - a)\,.\left(w^2 - 2wa\cos\frac{2\pi}{n} + a^2\right).\left(w^2 - 2wa\cos\frac{4\pi}{n} + a^2\right)$ &c. to n dimensions, the last factor being $w + a$ if n be even. And $w^n + a^n = \left(w^2 - 2wa\cos\frac{\pi}{n} + a^2\right).\left(w^2 - 2wa\cos\frac{3\pi}{n} + a^2\right)$ &c. to n dimensions, the last factor being $w + a$ if n be odd. This is called, from the inventor, Cotes's theorem.

(146.) It is required to express $(\cos x)^n$ by the cosines of multiples of x. Here $(\cos x)^n = \frac{1}{2^n}(e^{x\sqrt{-1}} + e^{-x\sqrt{-1}})^n$ $= \frac{1}{2^n}\left\{e^{nx\sqrt{-1}} + n\,.\,e^{\overline{n-2}.x\sqrt{-1}} + \frac{n\,.\,\overline{n-1}}{2}e^{\overline{n-4}.x\sqrt{-1}} + \text{\&c.} + \frac{n\,.\,\overline{n-1}}{2}e^{-\overline{n-4}x\sqrt{-1}} + n\,.\,e^{-\overline{n-2}x\sqrt{-1}} + e^{-nx\sqrt{-1}}\right\}$, n being an integer, $= \frac{1}{2^n}.\left\{e^{nx\sqrt{-1}} + e^{-nx\sqrt{-1}} + n(e^{\overline{n-2}x\sqrt{-1}} + e^{-\overline{n-2}x\sqrt{-1}}) + \frac{n\,.\,\overline{n-1}}{2}(e^{\overline{n-4}x\sqrt{-1}} + e^{-\overline{n-4}x\sqrt{-1}}) + \text{\&c.}\right\}$

$= \frac{1}{2^{n-1}}\left\{\cos n x + n \cos \overline{n-2} \,.\, x + \frac{n \,.\, \overline{n-1}}{2} \cos \overline{n-4} \,.\, x + \&c.\right\}$. The coefficients are the same as those in the first half of the expansion of $(a+b)^n$; but if n be even, the last term, which does not multiply a cosine, is half of the middle term in the expansion of the binomial.

(147.) As this formula is demonstrated entirely by means of imaginary symbols, we shall endeavour to explain how it happens that operations conducted by imaginary expressions can give correctly a real result. We know that $\left(\frac{e^y + e^{-y}}{2}\right)^n = \frac{1}{2^n} \,.\, \{ e^{ny} + e^{-ny} + n\,(e^{\overline{n-2}y} + e^{-\overline{n-2}y}) + \&c. \}$, or $\left(1 + \frac{y^2}{1\,.\,2} + \frac{y^4}{1\,.\,2\,.\,3\,.\,4} + \&c.\right)^n = \frac{1}{2^{n-1}} \,.\, \left\{1 + \frac{n^2 y^2}{1\,.\,2} + \frac{n^4 y^4}{1\,.\,2\,.\,3\,.\,4} + \&c. + n\left(1 + \frac{\overline{n-2}|^2 \,.\, y^2}{1\,.\,2} + \frac{\overline{n-2}|^4 \,.\, y^4}{1\,.\,2\,.\,3\,.\,4} + \&c.\right) + \&c.\right\}$. Now this is true for all values of y; and, consequently, if both sides were expanded, the expanded expressions would be *identically* equal; and therefore there would still be an equality if instead of y^2 we put $-x^2$ and operated upon it by the rules of common algebra. This would give us $\left(1 - \frac{x^2}{1\,.\,2} + \frac{x^4}{1\,.\,2\,.\,3\,.\,4} - \&c.\right)^n = \frac{1}{2^{n-1}} \left\{1 - \frac{n^2 x^2}{1\,.\,2} + \frac{n^4 x^4}{1\,.\,2\,.\,3\,.\,4} - \&c. + n\left(1 - \frac{\overline{n-2}|^2 x^2}{1\,.\,2} + \frac{\overline{n-2}|^4 x^4}{1\,.\,2\,.\,3\,.\,4} - \&c.\right) - \&c.\right\}$, or $(\cos x)^n = \frac{1}{2^{n-1}} \left\{\cos n x + n \cos \overline{n-2}\, x + \frac{n \,.\, \overline{n-1}}{2} \cos \overline{n-4}\, x + \&c.\right\}$.

(148.) In this formula for x put $\frac{\pi}{2} - y$; then $(\sin y)^n =$

$$\frac{1}{2^{n-1}}\left\{\cos \overline{\frac{n\pi}{2} - n y} + n \,.\, \cos \overline{\frac{\overline{n-2}\,\pi}{2} - \overline{n-2}\, y} + \frac{n \,.\, \overline{n-1}}{2} \,.\, \cos \overline{\frac{\overline{n-4}\,\pi}{2} - \overline{n-4}\, y} + \&c.\right\}.$$

Let $n = 4p$; then $\cos \overline{\frac{n\pi}{2} - n y} = \cos \overline{2 p \pi - n y} = \cos 2 p \pi \,.\, \cos n y + \sin 2 p \pi \,.\, \sin n y = \cos n y$;

$\cos \overline{\frac{\overline{n-2}\,\pi}{2} - \overline{n-2}\, y} = \cos \overline{2p-1} \,.\, \pi \,.\, \cos \overline{n-2}\, y + \sin \overline{2p-1}\, \pi \,.\, \sin \overline{n-2}\, y = -\cos \overline{n-2}\, y$, &c.;

therefore in this case $(\sin y)^n = \frac{1}{2^{n-1}} \left\{\cos n y - n \,.\, \cos \overline{n-2}\, y + \frac{n \,.\, \overline{n-1}}{2} \cos \overline{n-4}\, y - \&c.\right\}$.

Let $n = \overline{4p+2}$; then in the same manner it is found, that

$$(\sin y)^n = \frac{1}{2^{n-1}}\left\{-\cos n y + n \,.\, \cos \overline{n-2}\, y - \frac{n \,.\, \overline{n-1}}{2} \cos n - 4 y + \&c.\right\}.$$

Let $n = \overline{4p+1}$; $\cos \overline{\frac{n\pi}{2} - n y} = \cos \overline{2p + \tfrac{1}{2}}\, \pi \,.\, \cos n y + \sin \overline{2p + \tfrac{1}{2}}\, \pi \,.\, \sin n y = \sin n y$;

$\cos \overline{\frac{\overline{n-2}\,\pi}{2} - \overline{n-2}\, y} = \cos \overline{2p - \tfrac{1}{2}}\, \pi \,.\, \cos \overline{n-2} \,.\, y + \sin \overline{2p - \tfrac{1}{2}}\, \pi \,.\, \sin \overline{n-2}\, y = -\sin \overline{n-2}\, y$, &c.;

and therefore in this case

$$(\sin y)^n = \frac{1}{2^{n-1}}\left\{\sin n y - n \,.\, \sin \overline{n-2}\, y + \frac{n \,.\, \overline{n-1}}{2} \sin \overline{n-4}\, y - \&c.\right\}.$$

Let $n = 4p + 3$; then in the same way

$$(\sin y)^n = \frac{1}{2^{n-1}}\left\{-\sin n y + n \,.\, \sin \overline{n-2} \,.\, y - \frac{n \,.\, \overline{n-1}}{2} \,.\, \sin \overline{n-4}\, y + \&c.\right\}.$$

(149.) When n is even, the last term in the expression for $(\cos x)^n$ and for $(\sin y)^n$ is $\frac{n \,.\, \overline{n-1} \ldots\ldots \overline{\frac{n}{2}+1}}{1 \,.\, 2 \ldots\ldots\ldots\ldots \frac{n}{2}} \, \frac{1}{2^n}$

$$= \frac{1}{2^n} \,.\, \frac{1\,.\,2\,.\,3 \ldots\ldots \overline{n-1} \,.\, n}{\left(1\,.\,2 \ldots\ldots \frac{n}{2}\right)^2} = \frac{1\,.\,2\,.\,3 \ldots\ldots \overline{n-1} \,.\, n}{(2\,.\,4 \ldots\ldots\ldots\ldots n)^2} = \frac{1\,.\,3\,.\,5 \ldots\ldots \overline{n-1}}{2\,.\,4\,.\,6 \ldots\ldots n}.$$

(150.) One of the principal uses of these expressions is the simplification of integrals taken between two values of x or y that differ by a circumference. Since $\int_x \cos p x$ or $\int \cos p x \,.\, d x$, as well as $\int_x \sin p x$,

(p being an integer,) always vanishes between two such values, it appears that through a whole circumference $\int_\pi (\cos x)^n$ or $\int_\pi (\sin x)^n$ is $= 0$ when n is odd, and $= 2\pi \cdot \frac{1 \cdot 3 \cdot 5 \ldots\ldots \overline{n-1}}{2 \cdot 4 \cdot 6 \ldots\ldots n}$ when n is even.

(151.) Since ϕ ($\cos x$) can generally be expanded in integral powers of $\cos x$, it can generally be expanded in cosines of multiples of x. This in most cases can be effected with the greatest ease by particular artifices, and especially by the use of the imagioary expression for $\cos x$, &c., as we proceed to show by examples.

(152.) Suppose $\tan \phi = n \tan \theta$; it is required to find a series for ϕ in terms of θ. If in $\tan \phi$ and $\tan \theta$ we put the values for $\sin \phi$, $\cos \phi$, &c., found in (141) we have $\frac{e^{\phi\sqrt{-1}} - e^{-\phi\sqrt{-1}}}{e^{\phi\sqrt{-1}} + e^{-\phi\sqrt{-1}}} = n \cdot \frac{e^{\theta\sqrt{-1}} - e^{-\theta\sqrt{-1}}}{e^{\theta\sqrt{-1}} + e^{-\theta\sqrt{-1}}}$, or $\frac{e^{2\phi\sqrt{-1}} - 1}{e^{2\phi\sqrt{-1}} - 1}$ $= n \cdot \frac{e^{2\theta\sqrt{-1}} - 1}{e^{2\theta\sqrt{-1}} + 1}$, whence $e^{2\phi\sqrt{-1}} = e^{2\theta\sqrt{-1}} \cdot \frac{1 - \frac{n-1}{n+1} e^{-2\theta\sqrt{-1}}}{1 - \frac{n-1}{n+1} e^{2\theta\sqrt{-1}}}$. Let $\frac{n-1}{n+1} = k$, and take the logarithms of both sides; then $2\phi\sqrt{-1} =$

$$2\theta\sqrt{-1} + \log(1 - k \cdot e^{-2\theta\sqrt{-1}}) - \log(1 - k \cdot e^{2\theta\sqrt{-1}}) =$$

$$2\theta\sqrt{-1} + k\,(e^{2\theta\sqrt{-1}} - e^{-2\theta\sqrt{-1}}) + \frac{k^2}{2}(e^{4\theta\sqrt{-1}} - e^{-4\theta\sqrt{-1}}) + \frac{k^3}{3}(e^{6\theta\sqrt{-1}} - e^{-6\theta\sqrt{-1}}) + \&c.$$

Dividing by $2\sqrt{-1}$, $\phi = \theta + k \sin 2\theta + \frac{k^2}{2} \sin 4\theta + \frac{k^3}{3} \sin 6\theta + \&c.$; a theorem of great utility. The truth of the process is to be proved as in (147.) In the same manner we might find a series if $\tan \phi = \frac{n \,\text{sio}\, \theta}{1 - n \cos \theta}$, n being less than 1.

(153.) To expand $(a^2 - 2ab\cos\theta + b^2)^n$ in a series proceeding by cosines of multiples of θ, b being less than a. Since $2\cos\theta = e^{\theta\sqrt{-1}} + e^{-\theta\sqrt{-1}}$, this expression $= \{(a - b \cdot e^{\theta\sqrt{-1}}) \cdot (a - b \cdot e^{-\theta\sqrt{-1}})\}^n$ $= a^{2n} \cdot \left(1 - \frac{b}{a} e^{\theta\sqrt{-1}}\right)^n \cdot \left(1 - \frac{b}{a} e^{-\theta\sqrt{-1}}\right)^n$

Now $\left(1 - \frac{b}{a} e^{\theta\sqrt{-1}}\right)^n = 1 - n\frac{b}{a} e^{\theta\sqrt{-1}} + \frac{n \cdot \overline{n-1}}{2} \cdot \frac{b^2}{a^2} \cdot e^{2\theta\sqrt{-1}} - \frac{n \cdot \overline{n-1} \cdot \overline{n-2}}{2 \cdot 3} \cdot \frac{b^3}{a^3} \cdot e^{3\theta\sqrt{-1}} + \&c.$

$\left(1 - \frac{b}{a} e^{-\theta\sqrt{-1}}\right)^n = 1 - n\frac{b}{a} e^{-\theta\sqrt{-1}} + \frac{n \cdot \overline{n-1}}{2} \cdot \frac{b^2}{a^2} e^{-2\theta\sqrt{-1}} - \frac{n \cdot \overline{n-1} \cdot \overline{n-2}}{2 \cdot 3} \cdot \frac{b^3}{a^3} \cdot e^{-3\theta\sqrt{-1}} + \&c.$

The product of these (observing that $e^{2\theta\sqrt{-1}} + e^{-2\theta\sqrt{-1}} = 2\cos 2\theta$, &c.) $=$

$$1 + n^2 \frac{b^2}{a^2} + \left(\frac{n \cdot \overline{n-1}}{2}\right)^2 \cdot \frac{b^4}{a^4} + \left(\frac{n \cdot \overline{n-1} \cdot \overline{n-2}}{2 \cdot 3}\right)^2 \cdot \frac{b^6}{a^6} + \&c.$$

$$-\left(n\frac{b}{a} + n \cdot \frac{n \cdot \overline{n-1}}{2} \cdot \frac{b^3}{a^3} + \frac{n \cdot \overline{n-1}}{2} \cdot \frac{n \cdot \overline{n-1} \cdot \overline{n-2}}{2 \cdot 3} \cdot \frac{b^5}{a^5} + \&c.\right) 2\cos\theta$$

$$+\left(\frac{n \cdot \overline{n-1}}{2} \cdot \frac{b^2}{a^2} + n \cdot \frac{n \cdot \overline{n-1} \cdot \overline{n-2}}{2 \cdot 3} \cdot \frac{b^4}{a^4} + \&c.\right) 2\cos 2\theta$$

$$-\left(\frac{n \cdot \overline{n-1} \cdot \overline{n-2}}{2 \cdot 3} \cdot \frac{b^3}{a^3} + \&c.\right) 2\cos 3\theta$$

$$+ \&c.$$

Multiplying this by a^{2n} we have the series required.

(154.) To find $\log(1 - n\cos\theta)$, n being less than 1, in a similar series. Let $1 - n\cos\theta =$ $(a - b \cdot e^{\theta\sqrt{-1}})\,(a - b \cdot e^{-\theta\sqrt{-1}}) = a^2 + b^2 - 2ab\cos\theta$, therefore $a + b = \sqrt{1+n}$, $a - b = \sqrt{1-n}$. The log

$$= 2\log a + \log\left(1 - \frac{b}{a} e^{\theta\sqrt{-1}}\right) + \log\left(1 - \frac{b}{a} e^{-\theta\sqrt{-1}}\right) = \left.\begin{array}{r} 2\log a - \frac{b}{a} \cdot e^{\theta\sqrt{-1}} - \frac{b^2}{2a^2} \cdot e^{2\theta\sqrt{-1}} - \&c. \\ - \frac{b}{a} \cdot e^{-\theta\sqrt{-1}} - \frac{b^2}{2a^2} \cdot e^{-2\theta\sqrt{-1}} - \&c. \end{array}\right\} =$$

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$2 \log a - \frac{2b}{a} \cos\theta - \frac{2b^2}{2a^2} \cos 2\theta -$ &c. And $a = \frac{1}{2}(\sqrt{1+n} + \sqrt{1-n})$, $a^2 = \frac{1}{2}(1 + \sqrt{1-n^2})$; $\frac{b}{a} = \frac{\sqrt{1+n} - \sqrt{1-n}}{\sqrt{1+n} + \sqrt{1-n}} = \frac{n}{1+\sqrt{1-n^2}}$; whence $\log(1 - n\cos\theta) = \log . \frac{1+\sqrt{1-n^2}}{2} - \frac{1}{1} . \frac{n}{1+\sqrt{1-n^2}} . \cos\theta - \frac{1}{2}\left(\frac{n}{1+\sqrt{1-n^2}}\right)^2 \cos 2\theta - \frac{1}{3}\left(\frac{n}{1+\sqrt{1-n^2}}\right)^3 \cos 3\theta -$ &c.

(155.) To express sin x by a continued product. We have seen in (145) that $x^{2n} - a^{2n} = \overline{x-a} . \overline{x^2 - 2ax\cos\frac{\pi}{n} + a^2} . \overline{x^2 - 2ax . \cos\frac{2\pi}{n} + a^2} \ldots (\overline{n-1} \text{ terms}) \ldots \overline{x+a}$; dividing by $\overline{x-a}$, $x^{2n-1} + x^{2n-2} . a +$ &c. $+ a^{2n-1} = \overline{x^2 - 2ax\cos\frac{\pi}{n} + a^2} . \overline{x^2 - 2ax\cos\frac{2\pi}{n} + a^2} \ldots (\overline{n-1} \text{ terms}) \ldots \overline{x+a}$; this is true if x differ from a, however small the difference may be. By making that difference very small, and making $a = 1$, we have this equation for the limit of that above; $2n = 2 . \overline{1 - \cos\frac{\pi}{n}} . 2 . \overline{1 - \cos\frac{2\pi}{n}} \ldots (\overline{n-1} \text{ terms}) \ldots 2 = 2^{2n-1} . \sin^2\frac{\pi}{2n} . \sin^2\frac{2\pi}{2n} . \sin^2\frac{3\pi}{2n} \ldots (\overline{n-1} \text{ terms})$. Again, let $x = 1 + \frac{z}{2n}$, $a = 1 - \frac{z}{2n}$; then $x^2 + a^2 = 2 + 2\left(\frac{z}{2n}\right)^2$; $2ax = 2 - 2\left(\frac{z}{2n}\right)^2$, and the first equation becomes $\left(1 + \frac{z}{2n}\right)^{2n} - \left(1 - \frac{z}{2n}\right)^{2n} = \frac{2z}{2n} . 2\left(\overline{1 - \cos\frac{\pi}{n}} + \left(\frac{z}{2n}\right)^2 . \overline{1 + \cos\frac{\pi}{n}}\right) . 2\left(\overline{1 - \cos\frac{2\pi}{n}} + \left(\frac{z}{2n}\right)^2 \overline{1 + \cos\frac{2\pi}{n}}\right) \ldots (\overline{n-1} \text{ terms}) \ldots \times 2$. Or, since $1 - \cos\frac{\pi}{n} = 2\sin^2\frac{\pi}{2n}$, and $\frac{1+\cos\frac{\pi}{2n}}{1-\cos\frac{\pi}{2n}} = \cot^2\frac{\pi}{2n}$, $\left(1+\frac{z}{2n}\right)^{2n} - \left(1-\frac{z}{2n}\right)^{2n} = 2^{2n} . \sin^2\frac{\pi}{2n} . \sin^2\frac{2\pi}{2n} . \sin^2\frac{3\pi}{2n} \ldots (\overline{n-1} \text{ terms}) \ldots \frac{z}{2n} . \left(1+\left(\frac{z}{2n}\right)^2\cot^2\frac{\pi}{2n}\right) . \left(1 + \left(\frac{z}{2n}\right)^2 . \cot^2\frac{\pi}{2n}\right) \ldots (\overline{n-1} \text{ terms})$, which the former equation reduces to $\left(1+\frac{z}{2n}\right)^{2n} - \left(1-\frac{z}{2n}\right)^{2n} = 2z\left(1+\left(\frac{z}{2n}\right)^2 . \cot^2\frac{\pi}{2n}\right) . \left(1+\left(\frac{z}{2n}\right)^2\cot^2\frac{2\pi}{2n}\right) . . (\overline{n-1} \text{ terms})$. Now suppose n indefinitely great; since $\left(1 + \frac{z}{2n}\right)^{2n} = 1 + 2n . \frac{z}{2n} + \frac{2n . 2\overline{n-1}}{1 . 2} . \frac{z^2}{4n^2} +$ &c., or $= 1 + z + \frac{1 - \frac{1}{2n}}{1 . 2} z^2 +$ &c., the limit of the first side is $2\left(z + \frac{z^3}{1.2.3} + \frac{z^5}{1.2.3.4.5} + \text{\&c.}\right)$; since $\frac{z}{2n} . \cot\frac{\pi}{2n} = \frac{z}{\pi} . \frac{\frac{\pi}{2n}}{\tan\frac{\pi}{2n}}$ = ultimately $\frac{z}{\pi}$, the limit of the second side is $2z\left(1 + \frac{z^2}{\pi^2}\right) . \left(1 + \frac{z^2}{4\pi^2}\right)$. &c. indefinitely continued. Dividing both by $2z$, $1 + \frac{z^2}{1.2.3} + \frac{z^4}{1.2.3.4.5} +$ &c. $= \left(1 + \frac{z^2}{\pi^2}\right)\left(1 + \frac{z^2}{4\pi^2}\right)$. &c.; therefore, as in (147,) $1 - \frac{x^2}{1.2.3} + \frac{x^4}{1.2.3.4.5} -$ &c. $= \left(1 - \frac{x^2}{\pi^2}\right) . \left(1 - \frac{x^2}{4\pi^2}\right)$. &c.; and multiplying both sides by x, $\sin x = x\left(1 - \frac{x^2}{\pi^2}\right) . \left(1 - \frac{x^2}{4\pi^2}\right) . \left(1 - \frac{x^2}{9\pi^2}\right)$. &c. *ad infinitum*

(156.) To express cos x by a continued product. By (145,) $x^{2n} + a^{2n} = \left(x^2 - 2ax\cos\frac{\pi}{2n} + a^2\right) . \left(x^2 - 2ax\cos\frac{3\pi}{2n} + a^2\right)$. &c. to n terms. Let $x = 1$, $a = 1$; then $2 = 2\left(1 - \cos\frac{\pi}{2n}\right) . 2$

$\left(1 - \cos\frac{3\pi}{2n}\right) \ldots$ (n terms) $= 2^{2n} \,.\, \sin^2\frac{\pi}{4n} \,.\, \sin^2\frac{3\pi}{4n} \ldots$ (n terms). Again, let $x = 1 + \frac{z}{2n}$, $a = 1 - \frac{z}{2n}$; then as before $\left(1 + \frac{z}{2n}\right)^{2n} + \left(1 - \frac{z}{2n}\right)^{2n} = 2^{2n} \,.\, \sin^2\frac{\pi}{4n} \,.\, \sin^2\frac{3\pi}{4n} \ldots$ (n terms) $\ldots \times \left(1 + \left(\frac{z}{2n}\right)^2 . \cot^2\frac{\pi}{4n}\right)$ $.\left(1 + \left(\frac{z}{2n}\right)^2 . \cot^2\frac{3\pi}{4n}\right) \ldots$ (n terms), and the equation just found reduces this to $\left(1 + \frac{z}{2n}\right)^{2n} + \left(1 - \frac{z}{2n}\right)^{2n}$ $= 2\left(1 + \left(\frac{z}{2n}\right)^2 \cot^2\frac{\pi}{4n}\right)\left(1 + \left(\frac{z}{2n}\right)^2 \cot^2\frac{3\pi}{4n}\right) \ldots$ (n terms); and taking the limit of each side when n is indefinitely increased, $1 + \frac{z^2}{1\,.\,2} + \frac{z^4}{1\,.\,2\,.\,3\,.\,4} + \&c. = \left(1 + \frac{4z^2}{\pi^2}\right)\left(1 + \frac{4z^2}{9\pi^2}\right) . \&c.$, therefore $1 - \frac{x^2}{1\,.\,2} + \frac{x^4}{1\,.\,2\,.\,3\,.\,4} - \&c. = \left(1 - \frac{4x^2}{\pi^2}\right)\left(1 - \frac{4x^2}{9\pi^2}\right) . \&c. = \cos x$.

(157.) Taking the differential coefficient with respect to x of the logarithm of the expression for $\cos x$ we find $\tan x = \frac{8x}{\pi^2 - 4x^2} + \frac{8x}{9\pi^2 - 4x^2} + \&c.$ Similarly, from the expression for $\sin x$, $\cot x = \frac{1}{x} - \frac{2x}{\pi^2 - x^2}$ $- \frac{2x}{4\pi^2 - x^2} + \&c.$

(158.) The following theorems we shall find useful hereafter, $x^{2n} - 2x^n \cos a + 1 = (x^n - \overline{\cos a + \sqrt{-1}\,.\sin a})$ $.(x^n - \overline{\cos a - \sqrt{-1}\,.\,\sin a})$. If we solve the equation $x^n - \overline{\cos a + \sqrt{-1}\,.\,\sin a} = 0$, we have $x = (\cos a + \sqrt{-1}\,.\,\sin a)^{\frac{1}{n}}$; the different values of which, as will be seen upon applying the theorems of (142,) and (11,) are $\cos\frac{a}{n} + \sqrt{-1}\,.\,\sin\frac{a}{n}$; $\cos\frac{2\pi + a}{n} + \sqrt{-1}\,.\,\sin\frac{2\pi + a}{n}$, &c.; and the factors of $x^n - \overline{\cos a + \sqrt{-1}\,.\,\sin a}$ are therefore $x - \overline{\cos\frac{a}{n} + \sqrt{-1}\,.\,\sin\frac{a}{n}}$, $x - \overline{\cos\frac{2\pi + a}{n} + \sqrt{-1}\,.\,\sin\frac{2\pi + a}{n}}$, &c. Similarly, the factors of $x^n - \overline{\cos a - \sqrt{-1}\,.\,\sin a}$ are $x - \overline{\cos\frac{a}{n} - \sqrt{-1}\,.\,\sin\frac{a}{n}}$, $x - \overline{\cos\frac{2\pi + a}{n} - \sqrt{-1}\,.\,\sin\frac{2\pi + a}{n}}$, &c. Combining the similar factors, $x^{2n} - 2x^n\,.\,\cos a + 1 = \left(x^2 - 2x \cos\frac{a}{n} + 1\right) . \left(x^2 - 2x\cos\frac{2\pi + a}{n} + 1\right) . \left(x^2 - 2x\,.\,\cos\frac{4\pi + a}{n} + 1\right)$ &c. to n terms.

(159.) Now let $x = 1$; $x^{2n} - 2x^n\,.\,\cos a + 1$ becomes $2 - 2\cos a = 4\sin^2\frac{a}{2}$; $x^2 - 2x\cos\frac{a}{n} + 1$ becomes $2 - 2\cos\frac{a}{n} = 4\sin^2\frac{a}{2n}$, &c., and the equation is changed to this; $4\sin^2\frac{a}{2} = 4^n\,.\,\sin^2\frac{a}{2n} \cdot \sin^2\frac{2\pi + a}{2n}\,.$ $\sin^2\frac{4\pi + a}{2n} \ldots$ (n terms), or $\sin\frac{a}{2} = 2^{n-1}\,.\,\sin\frac{a}{2n}\,.\,\sin\frac{2\pi + a}{2n}\,.\,\sin\frac{4\pi + a}{2n} \ldots$ (n terms). Let $\frac{a}{2n} = \beta$; then $\sin n\beta = 2^{n-1}\,.\,\sin\beta\,.\,\sin\overline{\beta + \frac{\pi}{n}}\,.\,\sin\overline{\beta + \frac{2\pi}{n}} \ldots$ (n terms).

(160.) In the equation $x^{2n} - 2x^n\,.\,\cos a + 1 = \left(x^2 - 2x\,.\,\cos\frac{a}{n} + 1\right)\left(x^2 - 2x\,.\,\cos\frac{2\pi + a}{n} + 1\right)$&c., the coefficient of x^{2n-1} must $= -2\left(\cos\frac{a}{n} + \cos\frac{2\pi + a}{n} + \cos\frac{4\pi + a}{n} + \&c.\ (n \text{ terms})\right)$. But this coefficient $= 0$; therefore, putting γ for $\frac{a}{n}$, $\cos\gamma + \cos\left(\gamma + \frac{2\pi}{n}\right) + \cos\left(\gamma + \frac{4\pi}{n}\right) + \&c.\ (n \text{ terms}) = 0$. If n be even, this is an identical equation. If n be odd, the terms are all different, and observing that the cosine of an arc is the same as that of its defect from 2π, the equation, supposing γ less than $\frac{\pi}{n}$, may be put under this form

$$\left.\begin{array}{l}\cos\gamma+\cos\overline{\frac{2\pi}{n}+\gamma}+\cos\overline{\frac{4\pi}{n}+\gamma}+\&c.\\ +\cos\overline{\frac{2\pi}{n}-\gamma}+\cos\overline{\frac{4\pi}{n}-\gamma}+\&c.\end{array}\right\}=0,$$

where each line is to be continued to that value of the arc which is next less than π. By transferring to the second side those terms that are negative, this is easily changed into the following,

$$\left.\begin{array}{l}\cos\gamma+\cos\overline{\frac{2\pi}{n}+\gamma}+\cos\overline{\frac{4\pi}{n}+\gamma}+\&c.\\ +\cos\overline{\frac{2\pi}{n}-\gamma}+\cos\overline{\frac{4\pi}{n}-\gamma}+\&c.\end{array}\right\}=\left\{\begin{array}{l}\cos\overline{\frac{\pi}{n}+\gamma}+\cos\overline{\frac{3\pi}{n}+\gamma}+\&c.\\ +\cos\overline{\frac{\pi}{n}-\gamma}+\cos\overline{\frac{3\pi}{n}-\gamma}+\&c.\end{array}\right.$$

in which, γ being supposed less than $\frac{\pi}{2n}$, each series is to be continued till the angle reaches its greatest value next below 90°. If n be made $=5$, it will easily be seen that the last theorem of (49) is but a particular case of this.

(161.) In (124) and the following articles we explained a method of finding the corresponding small variations of parts of triangles. This may sometimes be abridged by the Differential Calculus. For if a a function of c receive the variation $\delta\,a$ in consequence of c receiving the variation $\delta\,c$, then $\delta\,a=\frac{d\,a}{d\,c}\delta\,c+\frac{d^2\,a}{d\,c^2}\cdot\frac{(\delta\,c)^2}{1\,.\,2}$ + &c. If $\delta\,c$ be very small, then $\delta\,a=\frac{d\,a}{d\,c}\,\delta\,c$ nearly. If, however, $\frac{d\,a}{d\,c}=0$, then $\delta\,a=\frac{d^2\,a}{d\,c^2}\cdot\frac{(\delta\,c)^2}{2}$ nearly. Thus, in the case of (128,) $\sin a=\sin A\,.\,\sin c$; $\cos a\,.\,\frac{d\,a}{d\,c}=\sin A\,.\,\cos c$, or $\frac{d\,a}{d\,c}=\frac{\sin A\,.\,\cos c}{\cos a}$ therefore $\delta\,a=\frac{\sin A\,.\,\cos c}{\cos a}\,\delta\,c$ nearly. This is 0 when $c=\frac{\pi}{2}$; taking the second differential coefficient, $\cos a\,.\,\frac{d^2\,a}{d\,c^2}-\sin a\,.\left(\frac{d\,a}{d\,c}\right)^2=-\sin A\,.\,\sin c$, or $\cos a\,.\frac{d^2\,a}{d\,c^2}=\frac{\sin a\,.\,\sin^2 A\,.\,\cos^2 c}{\cos^3 a}-\sin A\,.\,\sin c$. Make $c=\frac{\pi}{2}$, $a=A$; $\cos A\,.\frac{d^2\,a}{d\,c^2}=-\sin A$; $\frac{d^2\,a}{d\,c^2}=-\tan A$; and $\delta\,a=-\tan A\,.\,\frac{(\delta\,c)^2}{2}$ nearly, as in (129.)

(162.) This example sufficiently illustrates the use of this principle. For the cases in which the first differential coefficient does not vanish, and in which the neglect of the other terms will certainly introduce no error, it is convenient; but when a particular value makes the first differential coefficient vanish, or when it is necessary to examine the terms after the first, the method of (125) is generally preferable.

(163.) In our solutions of triangles it will be remarked, that we have frequently given several formulæ for the same case. The reason is, that in particular cases the value of an angle cannot at all by the tables be found exactly from its logarithmic sine or cosine; and in other cases it cannot be found exactly without much trouble. To provide, then, for all cases several formulæ are sometimes necessary. We shall now show in what cases these difficulties occur.

(164.) The ratio of the small variation of any function of an arc to the variation of the arc being ultimately the differential coefficient, we shall have $\delta\,.\,\log\sin\theta=\frac{d\,.\,\log\sin\theta}{d\,\theta}\,\delta\,\theta$ nearly $=M\cot\theta\,.\,\delta\,\theta$, M being the modulus of common logarithms $=0.43429448$. Now when θ is near 90°, $\cot\theta$ is very small, and a large variation of the arc is attended by a small variation of its log sine. A small error then in the log sine will produce a great error in the arc; or if the tables be not carried to many decimals, the same log sine will correspond to several successive values of the arc. Consequently an arc cannot be found accurately from its log sine when it is near 90°.

(165.) If now the arc be very small, $M\cot\theta$ becomes large; the second differential coefficient also $(=-M\operatorname{cosec}^2\theta)$ is very great. It may happen then that the second differences of the log sines (of which the expression is $\frac{d^2\,.\,\log\sin\theta}{d\,\theta^2}(\delta\,\theta)^2$ + &c.) become large; and we must have the labour of interpolating by second differences. This, however, is commonly avoided by constructing tables for a few of the first degrees of the quadrant to every second, or to smaller intervals than the rest of the tables; $\delta\,\theta$ is thus made so small that the second differences are seldom sensible. But it is still better avoided by the use of a small table giving the logarithm of $\frac{\sin\theta}{\theta}$ for a few degrees. For $\frac{\sin\theta}{\theta}$, by (140,) $=1-\frac{\theta^2}{1\,.\,2\,.\,3}+\frac{\theta^4}{1\,.\,2\,.\,3\,.\,4\,.\,5}-$ &c.; its logarithm $=-M\left(\frac{\theta^2}{6}+\frac{\theta^4}{180}+\&c.\right)$; the differential coefficient of which, or $-M\left(\frac{\theta}{3}+\frac{\theta^3}{45}+\&c.\right)$ is very

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small when θ is small. And if $\theta = n''$, $\log \theta = \log n + \log 1'' = \log n + 4.6855749$, of which the first part can be found to any accuracy by common tables, and the second is constant; thus, when θ is small, $\log \sin \theta$ can be found accurately. The most convenient tables contain a table of $\log \frac{\sin \theta \times 1''}{\theta}$; let the number in this table corresponding to n'' be a, then $\log \sin n'' = a + \log n$.

(166.) Conversely from a given value of $\log \sin \theta$, θ when small is found with great ease. For subtracting from $\log \sin \theta$ the logarithm of $1''$, or 4.6855749, we have, nearly, the log of the number of seconds, by which we find in the table the $\log \frac{\sin \theta}{\theta}$ or the $\log \frac{\sin \theta \times 1''}{\theta}$; and though the number of seconds is not theoretically exact, yet from the very slow variation of $\log . \frac{\sin \theta}{\theta}$, the error in the result will not be sensible. Then log true number of seconds $= \log \sin \theta - \log \frac{\sin \theta \times 1''}{\theta}$.

(167.) In the want of such tables, this method is convenient, $\frac{\sin \theta}{\theta} = 1 - \frac{\theta^2}{6}$ nearly $= \left(1 - \frac{\theta^2}{12}\right)^{\frac{1}{3}} = (\cos \theta)^{\frac{1}{3}}$ nearly; therefore $\log \frac{\sin \theta}{\theta} = \frac{1}{3} \log \cos \theta$ nearly. Hence, $\log \sin \theta = \log \theta + \frac{1}{3} \log \cos \theta$ nearly $= \log \theta - \frac{1}{3}$ arithmetical complement of $\log \cos \theta$.

(168.) The same remarks in all respects apply to the tangent of a small arc. The series for the $\tan \theta = \frac{\sin \theta}{\cos \theta}$

$$= \frac{\theta\left(1 - \frac{\theta^2}{6} + \&c.\right)}{1 - \frac{\theta^2}{2} + \&c.} = \theta\left(1 + \frac{\theta^2}{3} + \&c.\right), \text{ therefore } \frac{\tan \theta}{\theta} = 1 + \frac{\theta^2}{3} = \left(1 - \frac{\theta^2}{1.2}\right)^{-\frac{2}{3}}, \text{ and } \log \tan \theta =$$

$\log \theta + \frac{2}{3}$ ar. comp. $\log \cos \theta$ nearly. These expressions can be used without sensible error till $\theta = 8°$. Since the differential coefficient of $\log \tan \theta \left(= \frac{M}{\sin \theta . \cos \theta}\right)$ is never small, we can never meet with difficulties in the use of it like that mentioned in (164.(

(169.) In this way, then, we find that an arc cannot be determined accurately from its sine or cosecant when it is near 90°, from its cosine or secant when very small, or from its versed sine when near 180°; but from its tangent it can always be found with accuracy. Of the expressions, therefore, in (66) and (116) the first must not be used when $\frac{C}{2}$ is small or C is small; the second must not be used when C is near 180°, nor the fourth when C is near 90°. The third may always be used. In (63) cos $B = \frac{a}{c}$, which is inaccurate if B is small; but this expression may then safely be used; $\frac{1 - \cos B}{1 + \cos B}$, or $\tan^2 \frac{B}{2} = \frac{c - a}{c + a}$. In (70) if B is near 90°, let $B = 90° \pm x$; then $\cos x = \frac{b}{a} \sin A$, and $\tan^2 \frac{x}{2} = \frac{1 - \frac{b}{a} \sin A}{1 + \frac{b}{a} \sin A}$. Now $\frac{b}{a}$ in all cases of difficulty will be greater than 1, and less than $\frac{1}{\sin A}$; let $\frac{a}{b} = \sin \theta$; then $\tan^2 \frac{x}{2} = \frac{\sin \theta - \sin A}{\sin \theta + \sin A} = \tan \frac{\theta - A}{2} . \cot \frac{\theta + A}{2}$, which can be calculated with accuracy. In (105) if a and b be very small, (a case which often occurs,) c cannot be accurately found from that formula; we must therefore take $\tan A = \frac{\tan a}{\sin b}$, and $\tan c = \frac{\tan b}{\cos A}$, by which c is found to the greatest accuracy. In (109,) $\cos A = \frac{\tan b}{\tan c}$; if A be small, $\frac{1 - \cos A}{1 + \cos A} = \frac{\tan c - \tan b}{\tan c + \tan b}$, or $\tan^2 \frac{A}{2} = \frac{\sin \overline{c - b}}{\sin \overline{c + b}}$, which is not liable to inaccuracy. In (118,) if c should be near 180°, use this expression, $1 + \cos c = 1 + \cos a . \cos b + \sin a . \sin b - \sin a . \sin b \; (1 - \cos C,)$ or $\cos^2 \frac{c}{2} = \cos^2 \frac{a - b}{2} - \sin a . \sin b . \sin^2 \frac{C}{2}$; make $\sin a . \sin b . \sin^2 \frac{C}{2} = \sin^2 \theta$, then $\cos^2 \frac{c}{2} = \cos^2 \frac{a - b}{2} - \sin^2 \theta = \cos . \overline{\frac{a - b}{2} + \theta}$

$\cos \overline{\frac{a-b}{2} - \theta}$. We have given, we believe, the most important cases; but in any others the same principle may easily be applied.

(170.) We shall conclude our remarks on this subject with the solution of the following problem: To find how far the tables are sufficiently exact. This will be done by giving to the arc the variation 1″, or any other, according to the degree of accuracy required, and finding at what limit the corresponding variation of the tabular numbers is equal to one unit in the last place of decimals. Thus, for log sines: by (164,) the variation of log sin θ for $1'' = 0.4343 \times \cot \theta \times 0.000004848$. If the tables be carried to 7 decimals, cot θ at the limit $= \frac{0.0000001}{0.4343 \times 0.000004848}$; if to 10, $\cot \theta = \frac{0.0000000001}{0.4343 \times 0.000004848}$. The former gives $\theta = 87^\circ\ 17'$; the latter gives $\theta = 89^\circ\ 59'\ 50''$; and beyond these the tables of log sines cannot be trusted to seconds. The same principle may be applied to any other tables.

SECTION IX.

Formulæ peculiar to Geodetic Operations.

(171.) The Trigonometrical surveys, which have been carried on for the two objects of mapping an extensive country, and determining the figure and dimensions of the earth, afford the best exemplifications of most of the theorems both in plane and in Spherical Trigonometry. For some of the reductions, however, they require peculiar formulæ; these we shall give, after describing generally the course of operations.

(172.) The first part is the measurement of a base, for which a plain of four or five miles in extent is generally chosen; the line is measured with the most scrupulous exactness. In England, rods of deal, tubes of glass, and steel chains, have been used; the temperature being always noticed, and the proper correction applied for expansion. In the late surveys in France, the measuring rods consisted each of a rod of platina and a rod of brass, lying one upon the other, and connected at one extremity; the expansion of these metals being different, the difference of the expansions was observed, and the whole expansion of one bar found by a simple proportion.

Other bases are measured in different situations, called bases of verification, and their measure, compared with their length, as found by calculation, serves for a criterion of the correctness of the observations. Thus, for the French surveys of 1740, 17 bases were measured; but in the late surveys there, two only were used; and in the operations in Hindoostan, carried over a greater extent of country, five only were employed.

(173.) Proper situations for signals being selected, the country is divided into triangles by lines joining the stations; and the angles of the triangles, that is, the angles which two signals subtend, as seen from a third, are measured, (the first observation being made from the extremities of the base;) and here the nature of the instruments used, modifies the calculation in a considerable degree. For the late French survey, repeating circles were employed, by means of which the angle between the two signals was observed; but since the signals are seldom seen exactly in the horizon, a calculation is necessary to find from this the horizontal angle. In England and India the horizontal angle was observed immediately by a theodolite.

(174.) In all the principal triangles each of the three angles is observed; and the error, if it is found from their sum that any exists, is divided among them in the most probable proportion. The sum of the errors in the nicer observations has seldom amounted to 2″. For the smaller triangles it is sufficient to measure two angles.

(175.) Beginning now with the measured base, we have the length of the base and the observed angles at its extremities, to determine the distance of a signal from its extremities, and the angle at the signal; that is, we have one side and the two adjacent angles to determine the other parts. Thus we determine A C, B C, fig. 20. Fig. 20
Similarly, A D, B D, are found; then C D is determined. Then in the triangle C E D we have similar data. And this process we extend to any number of triangles, till we arrive at a base of verification.

(176.) It is generally thought proper to choose the stations such that the sides of the triangles are greater than 10 and less than 20 miles. In the English survey, however, the distance from Beachy-Head to Dunnose, which formed one side of a triangle, is more than $64\frac{1}{2}$ miles. And in the extension of the French survey to Spain, to connect Iviza with the continent, a triangle was formed, of which one side was nearly 100 miles. The calculations are verified either by comparing the calculated length of a base of verification with the measured length, or by comparing the distance between two signals as calculated from two chains of triangles, beginning either from the same base or from different bases. Thus, in England by a series of triangles, extending more than 200 miles, from Dunnose in the Isle of Wight to Clifton in Yorkshire, it was found that the error in a line of 22 miles does not exceed six feet. And in some of the English bases of verification of four or five miles in length, the difference between the computed and measured lengths has not exceeded one or two inches.

(177.) The latitudes and longitudes of the principal stations (those of one being known) are then determined accurately, and those of the minor objects which have been observed by a more expeditious method. This is for the purpose of mapping; if it is intended to ascertain the length of a degree of latitude, the distance of two places in the direction of the meridian must be ascertained, and the latitude of each must be observed. This was the object of the late French survey; their purpose being to determine the length of the terrestrial quadrant, of

which the 10,000,000th part, or *metre*, was made the standard of linear measure. For the determination of a degree of longitude (a calculation which implies the spheroidal form of the earth) methods are used of which it would be foreign to our purpose to treat.

(178.) This is a general explanation of the usual process: we shall now give the mathematical theorems connected with it. In the French bases, the line measured was not straight, but consisted of two parts, as a and

Fig. 21. b, fig. 21, forming a small angle θ, (when largest it was 49′.) To find the correction, $c^2 = a^2 + b^2 - 2\,a\,b\,.\cos\overline{\pi - \theta} = a^2 + b^2 + 2\,a\,b\cos\theta$; but since θ is very small, $\cos\theta = 1 - \frac{\theta^2}{2}$ nearly; therefore $c^2 = \overline{a + b}^2 - a\,b\,.\,\theta^2$, and $c = a + b - \frac{a\,b}{2\,(a+b)}\,\theta^2$ nearly, or the correction is $\frac{a\,b}{2\,(a+b)}\,\theta^2$. If $\theta = n$ seconds $= n \times 0{,}000004848$, the correction $= \frac{a\,b\,.\,n^2}{a+b} \times 0{,}000{,}000{,}000{,}01175$.

(179.) Supposing the three angles of a triangle observed, and one side, as a, known, To find its figure, that the lengths of the other sides may be least affected by the errors of observation. Let A be the observed angle opposite to a, B and C the angles adjacent, and b and c the sides opposite to them. Suppose the errors of A, B, and C, to be δ A, δ B, and δ C; then, as the sum of the angles (if erroneous) is supposed to be corrected by altering each of the angles by the same quantity, $\delta\,\mathrm{A} = -\,(\delta\,\mathrm{B} + \delta\,\mathrm{C})$. Then the true value of c is $a\,.\,\frac{\sin\overline{\mathrm{C} + \delta\,\mathrm{C}}}{\sin\overline{\mathrm{A} - \delta\,\mathrm{B} - \delta\,\mathrm{C}}}$; but $\sin\overline{\mathrm{C} + \delta\,\mathrm{C}} = \sin\mathrm{C}\,.\cos\delta\,\mathrm{C} + \cos\mathrm{C}\,.\sin\delta\,\mathrm{C} = \sin\mathrm{C} + \delta\,\mathrm{C}\,.\cos\mathrm{C}$ nearly; putting a similar expression for the denominator, and observing that $\sin\mathrm{B} = \sin\overline{\pi - \mathrm{B}} = \sin\overline{\mathrm{A} + \mathrm{C}} = \sin\mathrm{A}\cos\mathrm{C} + \cos\mathrm{A}\,.\sin\mathrm{C}$, we find $c = a\left(\frac{\sin\mathrm{C}}{\sin\mathrm{A}} + \frac{\sin\mathrm{B}}{\sin^2\mathrm{A}}\,\delta\,\mathrm{C} + \frac{\sin\mathrm{C}\,.\cos\mathrm{A}}{\sin^2\mathrm{A}}\,\delta\,\mathrm{B}\right)$, or the error of c is $a\left(\frac{\sin\mathrm{B}}{\sin^2\mathrm{A}}\,\delta\,\mathrm{C} + \frac{\sin\mathrm{C}\,.\cos\mathrm{A}}{\sin^2\mathrm{A}}\,\delta\,\mathrm{B}\right)$. Similarly, the error of b is $a\left(\frac{\sin\mathrm{C}}{\sin^2\mathrm{A}}\,\delta\,\mathrm{B} + \frac{\sin\mathrm{B}\,.\cos\mathrm{A}}{\sin^2\mathrm{A}}\,\delta\,\mathrm{C}\right)$. Now it is impossible to assign exactly the chances of the errors δ B, δ C, and δ A, or $-\,(\delta\,\mathrm{B} + \delta\,\mathrm{C})$, and our reasoning must therefore be vague. It is evident, however, that sin A must not be small; it is largest when $\mathrm{A} = 90°$. But it is equally evident, that there is a greater chance that the signs of δ B and δ C are different, than that they are the same; since in the three pairs that we can form of δ A, δ B, δ C, two will have errors of different signs, and one will have errors of the same sign. And if δ B and δ C have different signs, the errors of b and c will be diminished by giving cos A a positive value. A therefore ought to be less than 90°; and if δ B and δ C are probably not very different, B and C should be nearly equal. These conditions will be satisfied by a triangle differing not much from an equilateral triangle.

(180.) If two angles only, A and B, be observed, the expression for the errors will be as above; but we have now no reason to think them of different signs rather than of the same sign. In this case, then, we shall probably have our errors smallest, if $\cos\mathrm{A} = 0$, or $\mathrm{A} = 90°$; the remaining angle of the triangle ought therefore to be as nearly as possible a right angle.

(181.) The elevations or depressions of signals being small, the correction to be applied to their measured

Fig. 22. angular distance in order to obtain the horizontal angle is thus found. Suppose O A, O B, (fig. 22,) to be the directions in which two signals are seen from O; the angle A O B is measured. If a sphere be supposed described about O as centre, and if through Z the point vertical to O great circles O A C, O B D be drawn, and C O D be the horizontal plane, then C O D or Z, since Z is the pole of C D, is the horizontal angle required. Now $\cos\mathrm{Z} = \frac{\cos\mathrm{A\,B} - \cos\mathrm{Z\,A}\,.\cos\mathrm{Z\,B}}{\sin\mathrm{Z\,A}\,.\sin\mathrm{Z\,B}}$. Let $\mathrm{A\,B} = \mathrm{D}$, $\mathrm{Z} = \mathrm{D} + x$; $\cos\mathrm{Z} = \cos\mathrm{D}\,.\cos x - \sin\mathrm{D}\,.\sin x = \cos\mathrm{D} - \sin\mathrm{D}\,.\,x$ nearly, let $\mathrm{A\,C} = h$, $\mathrm{B\,D} = h'$, then $\cos h = 1 - \frac{h^2}{2}$, $\cos h' = 1 - \frac{h'^2}{2}$, $\sin h = h$, $\sin h' = h'$, nearly; and the equation becomes $\cos\mathrm{D} - x\sin\mathrm{D} = \frac{\cos\mathrm{D} - h\,h'}{1 - \frac{h^2}{2} - \frac{h'^2}{2}}$ nearly $= \cos\mathrm{D} - h\,h' + \frac{\cos\mathrm{D}}{2}\,(h^2 + h'^2)$; therefore $x = \frac{h\,h'}{\sin\mathrm{D}} - \frac{\cos\mathrm{D}}{2\sin\mathrm{D}}\,(h^2 + h'^2)$. Let $h + h' = p$; $h - h' = q$; therefore

$$h\,h' = \frac{p^2 - q^2}{4};\quad h^2 + h'^2 = \frac{p^2 + q^2}{2},\ \text{and}\ x = \frac{1}{4}\left(\frac{p^2 - q^2}{\sin\mathrm{D}} - \frac{(p^2 + q^2)\cos\mathrm{D}}{\sin\mathrm{D}}\right)$$

$$= \frac{1}{4}\left(p^2\,\frac{1 - \cos\mathrm{D}}{\sin\mathrm{D}} - q^2\,\frac{1 + \cos\mathrm{D}}{\sin\mathrm{D}}\right) = \frac{1}{4}\left(p^2\tan\frac{\mathrm{D}}{2} - q^2\cot\frac{\mathrm{D}}{2}\right).$$

For observations with the theodolite, this is not necessary.

(182.) The horizontal angles being thus found, all our triangles are converted into spherical triangles, the sides of which are small compared with the radius of the sphere. For the solution of these triangles, three different methods are used. The first is to solve them as spherical triangles, taking for the sines of the sides

the expressions in (165) and (167.) Knowing nearly the radius of the earth, the angle subtended at the centre by an arc of given length is known, and hence log $\frac{\sin a}{a}$ can be taken from a table where a is expressed in feet or toises; adding log a, log sin a is found. This method is, by Delambre, preferred to the others. The second is to find from the angles of the spherical triangles the angles formed by their chords, and to solve this as a plane triangle. Let C be nne spherical angle, $C - x$ the angle contained by the chnrds, then $\cos \overline{C - x}$

$$\frac{(\text{chord of } a)^2 + (\text{chnrd of } b)^2 - (\text{chnrd nf } e)^2}{2 \text{ chnrd of } a \,.\, \text{chord of } b} = \frac{\sin^2 \frac{a}{2} + \sin^2 \frac{b}{2} - \sin^2 \frac{e}{2}}{2 \sin \frac{a}{2} \,.\, \sin \frac{b}{2}}.$$ But $\cos C$

$$= \frac{\cos e - \cos a \,.\, \cos b}{\sin a \,.\, \sin b} = \frac{1 - 2\sin^2 \frac{c}{2} - \left(1 - 2\sin^2 \frac{a}{2}\right)\left(1 - 2 \sin^2 \frac{b}{2}\right)}{4 \sin \frac{a}{2} \cos \frac{a}{2} \sin \frac{b}{2} \cos \frac{b}{2}}$$

$$= \frac{\sin^2 \frac{a}{2} + \sin^2 \frac{b}{2} - \sin^2 \frac{c}{2}}{2 \sin \frac{a}{2} \sin \frac{b}{2} \cos \frac{a}{2} \cos \frac{b}{2}} - \frac{\sin \frac{a}{2} \,.\, \sin \frac{b}{2}}{\cos \frac{a}{2} \,.\, \cos \frac{b}{2}};$$ therefore $\cos \overline{C - x} = \cos \frac{a}{2} \,.\, \cos \frac{b}{2} \cos C + \sin \frac{a}{2} \,.\, \sin \frac{b}{2} = \cos C + x \sin C$; therefore $x \sin C = \sin \frac{a}{2} \,.\, \sin \frac{b}{2} - \left(1 - \cos \frac{a}{2} \,.\, \cos \frac{b}{2}\right) \cos C = \frac{a\,b}{4} - \frac{a^2 + b^2}{8} \cos C$. Let $a + b = e$, $a - b = f$; therefore $a^2 + b^2 = \frac{e^2 + f^2}{2}$, $a\,b = \frac{e^2 - f^2}{4}$; and $x \sin C = \frac{e^2 - f^2}{16} - \frac{e^2 + f^2}{16} \cos C$, or $x = \frac{1}{16}\left(e^2\, \frac{1 - \cos C}{\sin C} - f^2 . \frac{1 + \cos C}{\sin C}\right) = \frac{1}{16}\left(e^2 \tan \frac{C}{2} - f^2 \cot \frac{C}{2}\right)$. All these expressions suppose the angles to be expressed in numbers considering the radius as 1; if $e = n$ feet, then for e we must put $\frac{n}{\text{number of feet in radius}}$; if $x = m$ seconds, for x we must put $m \times 0{,}000004848$. This method was used in the English surveys.

(183.) This principle of the third method is, by applying a correctinn to the angles nf the spherical triangle to treat it as a plane triangle. Let a, b, e be the sides tn radius r; then

$$\cos C = \frac{\cos \frac{c}{r} - \cos \frac{a}{r} \cos \frac{b}{r}}{\sin \frac{a}{r} \,.\, \sin \frac{b}{r}} = \frac{1 - \frac{c^2}{2\,r^2} + \frac{c^4}{24\,r^4} - \left(1 - \frac{a^2}{2\,r^2} + \frac{a^4}{24\,r^4}\right)\left(1 - \frac{b^2}{2\,r^2} + \frac{b^4}{24\,r^4}\right)}{\frac{a\,b}{r^2}\left(1 - \frac{a^2}{6\,r^2}\right)\left(1 - \frac{b^2}{6\,r^2}\right)}$$

$$\text{nearly} = \frac{a^2 + b^2 + c^2 - \frac{1}{12\,r^2}\left\{2\,a^2\,b^2 + 2\,a^2\,e^2 + 2\,b^2\,c^2 - a^4 - b^4 - c^4\right\}}{2\,a\,b}.$$ But if $C - x$ be the angle in the triangle considered as plane, then $\cos \overline{C - x}$, or $\cos c + x \sin C = \frac{a^2 + b^2 - c^2}{2\,a\,b}$; therefore $x \sin C = \frac{1}{24\,r^2\,a\,b}\left\{2\,a^2\,b^2 + 2\,a^2\,c^2 + 2\,b^2\,c^2 - a^4 - b^4 - c^4\right\}$. The part within the brackets $= 4\,a^2\,b^2 - (a^2 + b^2 - c^2)^2$ $= \left\{2\,a\,b + \overline{a^2 + b^2 - c^2}\right\} . \left\{2\,a\,b - \overline{a^2 + b^2 - c^2}\right\} = \left\{(a + b)^2 - c^2\right\} . \left\{c^2 - (a - b)^2\right\} = (a + b + c)$ $(a + b - c)\,(a + c - b)\,(b + c - a) = 16$ (area nf triangle)^2. But $a\,b \sin C = 2$. area; therefore $x = \frac{\text{area of triangle}}{3\,r^2}$; or if $x = n$ seconds, $n = \frac{\text{area of triangle}}{3\,r^2 \times 0{,}000004848}$. This is Legendre's theorem. If the area of the triangle be found in feet, the logarithm of the divisor is 9,8038940, a degree on the earth's surface being considered = 365155 feet. This is due to General Roy. The dimensions of the triangle are always known accurately enough to find the area with sufficient exactness. The correction is the same for each nf the angles; it is therefore nne-third nf the excess nf the sum of the three angles above 180°, commonly called the spherical excess. Tha sphericsl excess seldom amounts to 5″; in the largest triangls joining Iviza with the coast of Spain it amounted however to 39″.

Fig. 23. (184.) The sides and angles of the triangles being found by some of these methods, and the relative situation of the signals being found, it is necessary to determine the angle which some one of the lines makes with the meridian. In the English surveys this was done by observing with the theodolite the horizontal angle between a signal and the pole-star, at the time when it was known to be at its greatest azimuth. Let Z, fig. 23, be the zenith, P the pole, S the pole-star, Z S a great circle. Then cot Z . sin Z P S = cot S P . sin Z P − cos Z P S . cos Z P. Suppose a small error in the time, this would create a small error in the angle Z P S. Now, as in (131,) we find that the simplified expression for the error of Z vanishes when cos S is 0, or S is a right angle. Returning then to the original expression, and observing that cos Z P = cos Z . cos P; and putting for cot $\overline{Z + \delta Z}$, &c. their approximate values, we find at length $\delta Z = -\frac{\sin Z . \cos Z}{\sin^2 P} . \frac{(\delta P)^2}{2}$. Now with the pole-star sin Z is small, and δ P very small; hence a small error in time will not produce a sensible error in the azimuth.

(185.) In the French surveys the azimuth was found by observing the angle between the signal and the sun when near the horizon; also by taking the angular distance of the signal from the pole-star when nearest to the signal, or farthest from it. To allow the observations to be repeated, a correction was investigated not very dissimilar to that of the last article, to be applied to the observations made when the pole-star was a little removed from the point nearest to, or farthest from, the signal. From this distance the azimuth is found by a right-angled spherical triangle. But in Spain, a transit instrument being adjusted to a mark nearly in the meridian, the intervals of the transits of different stars were observed: comparing these intervals with those that ought to have been observed in the meridian, the azimuth of the mark was determined by a formula common in practical Astronomy. From this the azimuth of any signal was easily found.

Fig. 24. (186.) The direction of one side being known, we have now to solve this problem. Given P A, fig. 24, the colatitude of A, and the angle P A B, and the length of A B; to find P B the colatitude of B, and the angle B, and the difference of longitude P; A B being small (seldom = 1°.) Here cos $\overline{B P}$ = cos A P . cos A B + sin A P . sin A B . cos A; let B P = A P − x; cos $\overline{A P - x}$ − cos A P, or 2 sin $\overline{A P - \frac{x}{2}}$. sin $\frac{x}{2}$ = sin A P . sin A B . cos A − cos A P (1 − cos A B) = A B . sin A P . cos A − cos A P . $\frac{A B^2}{2}$ nearly; therefore sin $\frac{x}{2}$

$$= \frac{A B . \sin A P . \cos A - \frac{A B^2}{2} \cos A P}{2 \sin \overline{A P - \frac{x}{2}}}.$$

An approximate value of $\frac{x}{2}$ is $\frac{A B . \cos A}{2}$; substituting this in the denominator, $x = 2 \sin \frac{x}{2}$ nearly $= A B \cos A - \frac{\cot A P . \sin^2 A}{2} A B^2$. If greater accuracy is desired, this may be again substituted in the denominator; then $A B^3$ must be taken in the numerator; and observing that $\frac{x}{2} = \sin \frac{x}{2} + \frac{1}{6} \left(\sin \frac{x}{2}\right)^3$ nearly, $x = A B \cos A - \frac{\cot A P . \sin^2 A}{2} A B^2$ $\frac{(1 + 3 \cot^2 A P) \sin^4 A . \cos A}{6} A B^3$. Then sin P $= \frac{\sin A B . \sin A}{\sin P B}$, and sin B $= \frac{\sin A P . \sin A}{\sin P B}$. Or, if a series be preferred, $P = \frac{A B . \sin A}{\sin P A} - \frac{A B^2 . \sin A . \cos A . \cot P A}{\sin P A}$ − &c.; $B = 180^\circ - A +$

$$\frac{A \cdot B . \sin A . \sin \frac{P A + P B}{2}}{\sin P B}.$$

(187.) For the points of less consequence, which have been observed from two stations, the distances being found considering the triangles as plane, the value $x = A B \cos \theta$ is sufficiently accurate; and then $P = \frac{A B . \sin A}{\sin P A}$ nearly.

These are the principal formulæ of Trigonometry that are used for surveys on a large scale. We have treated of them at some length, as we know not any book in the English language in which any account of them is to be found. We have confined ourselves to what appeared to be strictly connected with the subject of this Treatise; for the explanation of the methods used in different hypotheses of the figure of the earth, and for the results deduced from them, we refer to our article on the Figure of the Earth.

SECTION X.

On the Construction of Trigonometrical Tables.

(188.) The construction of tables naturally divides itself into two parts: the first is, the determination of values of the function to be tabulated for certain values of the arc, at large intervals; the second is, the filling up of the tables by inserting the values included between these. In this order we propose to consider the formation of tables of the values of Trigonometrical lines and their logarithms.

(189.) The method which first suggests itself for the determination of netural sines, is to take some erc whose sine and cosine are known, (as 30°, 45°, 18°, 54°, &c.) end determine the cosine of half the arc by the formula $\cos a = \sqrt{\frac{1 + \cos 2a}{2}}$, and after repeated applications of it to determine the sine hy tha form $\sin a = \sqrt{\frac{1 - \cos 2a}{2}}$. Or tha sine and cosine may he determined by the formula $\sin a = \frac{1}{2}\{\sqrt{1 + \sin 2a} - \sqrt{1 - \sin 2a}\}$, $\cos a = \frac{1}{2}\{\sqrt{1 + \sin 2a} + \sqrt{1 - \sin 2a}\}$. This method, when $2a$ is small, is more accurate than the former. For when $\frac{1 - \cos 2a}{2}$ is very small $= v$, suppose x to be the error to which it is liable, or the value of the figures rejected; then its square root will be liable to the error $\frac{x}{2\sqrt{v}}$ nearly, which, when v is small, is very considerable. On the contrary, in the other method, $1 + \sin 2a$, and $1 - \sin 2a$, being nearly $= 1$, upon extracting their roots we are not liable to tha same error. In this manner find the sine of $\frac{30^\circ}{2^n} = 52'' \, 44''' \, 3'''' \, 45'''''$. Now by observation of the sines of this arc, and of the double of this arc, it will be seen that the sines of small arcs are nearly as the arcs; and therefore $52'' \, 44''' \, 3'''' \, 45''''' : 1' : :$ sine found : sine of 1'. From this the cosine of 1' is found; and the sines and cosines of 2', 3', 4', &c. are found hy the formulæ of (38.)

(190.) But tha same thing may be done in this manner, with fewer (though more laborious) operations, and without the proportion used in tha last article. It was found that $\sin 5a = 5 \sin a - 20 \sin^3 a + 16 \sin^5 a$; conversely, tha solution of the equation $5x - 20x^3 + 16x^5 = \sin 5a$ will give the value of $\sin a$. Thus, from sin 15° (found by bisection) we may by approximation find sin 3°. Again, $\sin 3b = 3 \sin b - 4 \sin^3 b$; solving this equation we have the value of $\sin b$ from $\sin 3b$, and therefore from sin 3° we find sin 1°. By a repetition of the same operations we descend to sin 30', sin 15', sin 3', sin 1'; and then ascend as before. In the same way we might have begun from 18°, or any arc whose sine is known.

(191.) But in a process of this kind, where an error in the calculation of one number would affect all the following ones, it is clearly desirable to compute independently some numbers in the series et convenient intervals to serve as verifications for the rest. Thus, from sin 30° we may by trisection find sin 10°; from this we get cos 10° or sin 80°; then $\sin 20^\circ = 2 \sin 10^\circ . \cos 10^\circ$ is found; then since $\sin \overline{60^\circ + A} - \sin \overline{60^\circ - A} = \sin A$, we have $\sin 80^\circ - \sin 40^\circ = \sin 20^\circ$, whence sin 40° is found; thence sin 50° or cos 40° is found; and $\sin 70^\circ = \sin 50^\circ + \sin 10^\circ$. The sines for every 10° of the quadrant being found, those of every degree should then be calculated as verifications for those of every minute, &c. Tha following is the best method of performing these calculations: $\sin \overline{n + 1}\, b = 2 \cos b . \sin nb - \sin \overline{n - 1}\, b$, therefore $\sin \overline{n + 1}\, b - \sin nb = \sin nb - \sin \overline{n - 1}\, b - (2 - 2 \cos b) \sin nb$. But $\sin \overline{n + 1}\, b - \sin nb =$ difference of $\sin nb$; $\sin nb - \sin \overline{n - 1}\, b =$ the preceding difference; hence the difference is less than the preceding difference by $(2 - 2 \cos b) \sin nb$, or $4 \sin^2 \frac{b}{2} \sin nb$; that is, the second difference is $- 4 \sin^2 \frac{b}{2} \sin nb$. Now, since $\sin nb$ is already found, this can be calculated; and the operation will not be long, for the multiplier $4 \sin^2 \frac{b}{2}$ being the same every time, a table of its products by the 9 digits may be prepared. Thus then we have $\sin 12^\circ - \sin 11^\circ = \sin 11^\circ - \sin 10^\circ - 4 \sin^2 30' . \sin 11^\circ$, &c. In this way the sines for every degree may be found; if the values for sin 10°, sin 20°, &c. are not the same as those found before, it shows that there is some error in the computation.

(192.) But the natural sines for these arcs, et least for 10°, 20°, &c. or more conveniently for 9°, 18°, &c. may be calculated independently thus. We found for $\sin x$ the series $x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - $ &c.,

let $x = \frac{m}{n} \cdot \frac{\pi}{2}$; then $\frac{\pi}{2}$ being fonnd by the differential calculus to $= 1,570796326794897$, we have $\sin \frac{m\pi}{2n} =$

$$\frac{m}{n} \times 1,570796326794827 \qquad - \frac{m^3}{n^3} \times 0,645964097506246$$
$$+ \frac{m^5}{n^5} \times 0,079692626246167 \qquad - \frac{m^7}{n^7} \times 0,004681754135319$$
$$+ \frac{m^9}{n^9} \times 0,000160441184787 \qquad \frac{m^{11}}{n^{11}} \times 0,000003598843235$$
$$+ \frac{m^{13}}{n^{13}} \times 0,000000056921729 \qquad - \frac{m^{15}}{n^{15}} \times 0,000000000668804$$
$$+ \frac{m^{17}}{n^{17}} \times 0,000000000006067 \qquad - \frac{m^{19}}{n^{19}} \times 0,000000000000044$$

Similarly, as $\cos x = 1 - \frac{x^2}{1 \cdot 2} + \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} -$ &c. we have $\cos . \frac{m}{n} \cdot \frac{\pi}{2} =$

$$1,000000000000000 \qquad - \frac{m^2}{n^2} \times 1,233700550136170$$
$$+ \frac{m^4}{n^4} \times 0,253669507901048 \qquad - \frac{m^6}{n^6} \times 0,020863480763353$$
$$+ \frac{m^8}{n^8} \times 0,000919260274839 \qquad - \frac{m^{10}}{n^{10}} \times 0,000025202042373$$
$$+ \frac{m^{12}}{n^{12}} \times 0,000000471087478 \qquad - \frac{m^{14}}{n^{14}} \times 0,000000006386603$$
$$+ \frac{m^{16}}{n^{16}} \times 0,000000000065660 \qquad - \frac{m^{18}}{n^{18}} \times 0,000000000000529$$
$$+ \frac{m^{20}}{n^{20}} \times 0,000000000000003$$

The cosine of an arc being the sine of its complement, $\frac{m}{n}$ will never exceed $\frac{1}{2}$; and a few terms of these series will give the natural sioes with great ease to 15 decimals.

(193.) Wheo the sines for every degree are calculated, they should be verified; and for this purpose the last equation of (160) will be found extremely useful. By giviog to γ and n different values, we may with great ease examine the accuracy of as many calculated sines as we wish.

(194.) The sines for degrees being found, those for smaller divisions, as mioutes, are generally found by differences. And a remarkable relation between the differences of successive orders enables us to determine the differences with which we must begin our table, from knowing the two first of them. Let it be supposed that the arc x is formed by successive additions of h; then $\Delta \sin x = \sin \overline{x+h} - \sin x = 2 \sin \frac{h}{2} . \cos \overline{x + \frac{h}{2}}$; $\Delta^2 \sin x = 2 \sin \frac{h}{2} \left(\cos \overline{x + \frac{3h}{2}} - \cos \overline{x + \frac{h}{2}}\right) = -4 \sin^2 \frac{h}{2} . \sin \overline{x+h}$; and, consequently, $\Delta^2 \sin \overline{x-h} = -4 \sin^2 \frac{h}{2} . \sin x$. Hence $\Delta^4 . \sin \overline{x-h} = -4 \sin^2 \frac{h}{2}\, \Delta^2 \sin x = 16 \sin^4 \frac{h}{2} . \sin \overline{x+h}$, and, therefore, $\Delta^4 \sin . \overline{x-2h} = 16 \sin^4 \frac{h}{2} . \sin x$. Similarly, $\Delta^6 . x \sin \overline{x-3h} = -64 \sin^6 \frac{h}{2} . \sin x$, &c. Also $\Delta^3 \sin x = -8 \sin^3 \frac{h}{2} . \cos \overline{x + \frac{3h}{2}}$, therefore $\Delta^3 . \sin \overline{x-h} = -8 . \sin^3 \frac{h}{2} . \cos \overline{x + \frac{h}{2}}$; similarly, $\Delta^5 \sin \overline{x-2h} = 32 \sin^5 \frac{h}{2} . \cos \overline{x + \frac{h}{2}}$, &c. Now if we arrange these in tables in the usual order, as below,

$\sin \overline{x-3h}$	$\Delta \sin \overline{x-3h}$	$\Delta^2 \sin \overline{x-3h}$	$\Delta^3 \sin \overline{x-3h}$	$\Delta^4 \sin \overline{x-3h}$	$\Delta^5 \sin \overline{x-3h}$
$\sin \overline{x-2h}$	$\Delta \sin \overline{x-2h}$	$\Delta^2 \sin \overline{x-2h}$	$\Delta^3 \sin \overline{x-2h}$	$\Delta^4 \sin \overline{x-2h}$	$\Delta^5 \sin \overline{x-2h}$
$\sin \overline{x-h}$	$\Delta \sin \overline{x-h}$	$\Delta^2 \sin \overline{x-h}$	$\Delta^3 \sin \overline{x-h}$		
$\sin x$	$\Delta \sin x$				
$\sin \overline{x+h}$					

we shall remark that $\sin x$, $\Delta^2 . \sin \overline{x-h}$, $\Delta^4 \sin \overline{x-2h}$, &c. are in one horizontal line, and that $\Delta \sin x$, $\Delta^3 \sin \overline{x-h}$, $\Delta^5 \sin \overline{x-2h}$, &c. are also in a horizontal line. Hence the numbers in each horizontal line form a geometrical progression, whose ratio is $-4 \sin^2 \frac{h}{2}$. Knowing then $\sin x$ and $\sin \overline{x+h}$, we can calculate all the differences as far as are necessary, and all our sines are then formed by addition and subtraction. If $x = 0$, we have but one series of differences to calculate.

(195.) By a slight alteration in the enunciation of this relation of the differences, we may avoid using any more numbers than are absolutely necessary. Since $\Delta^2 \sin x = -4 \sin^2 \frac{h}{2} . \sin \overline{x+h}$, and $\sin \overline{x+h} = \sin x + \Delta \sin x$, therefore $\Delta^2 \sin x = -4 \sin^2 \frac{h}{2} (\sin x + \Delta \sin x)$; taking the $\overline{n-2}^{th}$ difference of each side $\Delta^n \sin x = -4 \sin^2 \frac{h}{2} (\Delta^{n-2} \sin x + \Delta^{n-1} \sin x)$, a formula which gives any difference in terms of the two differences immediately preceding.

(196.) One important point we must not omit to notice, namely, the number of decimals to which these differences ought to be calculated. For this investigation we shall consider each of them as liable to the same error in the last figure used, (it will never exceed half an unit, if we increase the last figure by 1 when the first rejected is equal to or greater than 5.) Now it is useless to take one difference to so many decimals, that the error from it will be much less than that from any other; we shall then make them as nearly as possible equal. Suppose, now, there are n sines to be calculated by the differences, before our operations are verified by one of the previously calculated sines. The theory of finite differences gives us for the $\overline{n+1}^{th}$ sine, $\sin x + n \Delta \sin x + \frac{n . \overline{n-1}}{2} \Delta^2 . \sin \overline{x-h} + \frac{n . \overline{n-1} . \overline{n+1}}{2 . 3} \Delta^3 . \sin \overline{x-h} + \frac{n . \overline{n-1} . \overline{n+1} . \overline{n-2}}{2 . 3 . 4} \Delta^4 . \sin \overline{x-2h} + \frac{n . \overline{n-1} . \overline{n+1} . \overline{n-2} . \overline{n+2}}{2 . 3 . 4 . 5} \Delta^5 . \sin \overline{x-2h} + \&c.$ The error of each difference will be multiplied, in the n^{th} sine, by the multiplier of that difference. If then $n = 59$, the first difference should be carried to 2 figures more than the sines, the second to 4, the third to 5, the fourth to 6, the fifth to 7, the sixth to 8, the seventh to 9, the eighth to 10, the ninth to 11, the tenth to 12. Or, if we make use of the differences calculated in (195,) as the $\overline{n+1}^{th}$ sine $= \sin x + n \Delta \sin x + \frac{n . \overline{n-1}}{2} \Delta^2 \sin x + \frac{n . \overline{n-1} . \overline{n-2}}{2 . 3} \Delta^3 \sin x + \&c.$, we may in a similar manner find the number of decimal places to which each of these must be calculated. In adding any number to, or subtracting it from, any other number which has not so many decimals, we must not use the superabundant figures, but increase by 1 the first figure used, if the first of the superabundant figures be not less than 5. The sines with which we begin should be taken to 2 or 3 figures more than it is intended to preserve in the tables. In this way we can calculate with great securacy and without any unnecessary labour.

(197.) To interpolate for smaller divisions, as seconds, it is convenient to have a formula for finding the differences for the smaller divisions, by means of the differences for the larger ones. Suppose, now, the smaller divisions to be each $\frac{1}{p}^{th}$ of the large ones. Let Δ', Δ'', &c. be the 1st, 2d, &c. differences for minutes, and δ' δ'', &c. those for seconds. Then, by the common formula, we have

$$\sin x = \sin x$$

$$\sin \overline{x+1''} = \sin x + \frac{1}{p} \Delta' - \frac{1}{p}\left(1-\frac{1}{p}\right)\frac{\Delta''}{1.2} + \frac{1}{p}\left(1-\frac{1}{p}\right)\left(2-\frac{1}{p}\right) . \frac{\Delta'''}{1.2.3} - \&c$$

$$= \sin x + \frac{1}{p} \Delta' - \left(\frac{1}{p} - \frac{1}{p^2}\right) . \frac{\Delta''}{1 . 2} + \left(\frac{2}{p} - \frac{3}{p^2} + \frac{1}{p^3}\right) . \frac{\Delta'''}{1 . 2 . 3} - \left(\frac{6}{p} - \frac{11}{p^2} + \frac{6}{p^3} - \frac{1}{p^4}\right)$$

$$\times \frac{\Delta''''}{1 . 2 . 3 . 4} + \left(\frac{24}{p} - \frac{50}{p^2} + \frac{35}{p^3} - \frac{10}{p^4} + \frac{1}{p^5}\right) . \frac{\Delta'''''}{1 . 2 . 3 . 4 . 5}$$

and the sines of $\overline{x+2''}$, $\overline{x+3''}$, &c. will be found by putting $\frac{2}{p}$, $\frac{3}{p}$, &c. for $\frac{1}{p}$. Upon taking the differences of these successive values, it is clear that the numerator of $\frac{1}{p^m}$ in the n^{th} difference will be $\Delta^n . 0^m$ * multiplied by its factor in $\sin \overline{x+1''}$. Thus we find, (going as far as the 5th differences,)

* By $\Delta^n . 0^m$ is meant the first term of the n^{th} order of differences of the series 0^m, 1^m, 2^m, 3^m, &c.

$$\delta' = \frac{\Delta'}{p} - \frac{p-1}{2p^2}\Delta'' + \frac{\overline{p-1}\,.\,\overline{2p-1}}{6p^3}\Delta''' - \frac{\overline{p-1}\,.\,\overline{2p-1}\,.\,\overline{3p-1}}{24p^4}\Delta'''' + \frac{\overline{p-1}\,.\,\overline{2p-1}\,.\,\overline{3p-1}\,.\,\overline{4p-1}}{120p^5}\Delta'''''.$$

$$\delta'' = \frac{1}{p^2}\Delta'' - \frac{p-1}{p^3}\Delta''' + \frac{\overline{p-1}\,.\,\overline{11p-7}}{12p^4}\Delta'''' - \frac{\overline{p-1}\,.\,\overline{2p-1}\,.\,\overline{5p-3}}{12p^5}\Delta'''''.$$

$$\delta''' = \frac{1}{p^3}\Delta''' - \frac{3\,.\,\overline{p-1}}{2p^4}\Delta'''' + \frac{\overline{p-1}\,.\,\overline{7p-5}}{4p^5}\Delta'''''.$$

$$\delta'''' = \frac{1}{p^4}\Delta'''' - \frac{2\,.\,\overline{p-1}}{p^5}\Delta'''''.$$

$$\delta''''' = \frac{1}{p^5}\Delta'''''$$

These expressions* are quite general; from the relation among tha differences of natural sines mentiooed in (194,) it is not absolutely necessary to calculate more than the first of them; but even there it will be more convenient to use the formulæ.

(198.) The sines up to 60° beiog calculated, those above 60° will be found by simple additioo, from the formula $\sin \overline{60^\circ + A} = \sin \overline{60^\circ - A} + \sin A$. Thus the sines are found for the quadraot; and, consequently, all the cosines are known.

(199.) The tangeots will be found by dividing the sines by the cosines. After 45° they may be found by the formula $\tan \overline{45^\circ + A} = \tan \overline{45^\circ - A} + 2 \tan 2A$.

(200.) The tangents may also be found independently in the following manner. If we expand every fractional term, except the first, of the first series io (157,) and add together the coefficients of similar powers of x, and for x put $\frac{m}{n} \,.\, \frac{\pi}{2}$, we have the following expression, $\tan \frac{m}{n} \,.\, \frac{\pi}{2} =$

$$\frac{2mn}{n^2 - m^2} \times 0{,}6366197723675813$$

$$\begin{array}{ll}
+\frac{m}{r} \times 0{,}2975567820597314 & +\frac{m^3}{n^3} \times 0{,}0186886502773330 \\
+\frac{m^5}{n^5} \times 0{,}001842475203510 & +\frac{m^7}{n^7} \times 0{,}000197580071520 \\
+\frac{m^9}{n^9} \times 0{,}000021697737325 & +\frac{m^{11}}{n^{11}} \times 0{,}000002401136991 \\
+\frac{m^{13}}{n^{13}} \times 0{,}000000266113303 & +\frac{m^{15}}{n^{15}} \times 0{,}000000029586468 \\
+\frac{m^{17}}{n^{17}} \times 0{,}000000003286788 & +\frac{m^{19}}{n^{19}} \times 0{,}000000000365175 \\
+\frac{m^{21}}{n^{21}} \times 0{,}000000000040754 & +\frac{m^{23}}{n^{23}} \times 0{,}000000000004508 \\
+\frac{m^{25}}{n^{25}} \times 0{,}000000000000501 & +\frac{m^{27}}{n^{27}} \times 0{,}000000000000056 \\
+\frac{m^{29}}{n^{29}} \times 0{,}000000000000006 &
\end{array}$$

* The demonstration in the text is the most simple, but the law may be found more generally in this manner. The problem is, from the given differences of the series, u_x, u_{x+p}, u_{x+2p}, &c. to find the differences of u_x, u_{x+1}, u_{x+2}, &c. Let $\phi(t)$ be the *Generating Function* of u_x; the generating functions of Δu_x, $\Delta^2 u_x$, &c. are $\left(\frac{1}{t^p} - 1\right)\phi(t)$, $\left(\frac{1}{t^p} - 1\right)^2 . \phi(t)$ &c.; and those of δu_x, $\delta^2 u_x$, &c. are $\left(\frac{1}{t} - 1\right)\phi(t)$, $\left(\frac{1}{t} - 1\right)^2 . \phi(t)$, &c. For $\delta^n . u_x$, then, we must express $\left(\frac{1}{t} - 1\right)^n$ in powers of $\left(\frac{1}{t^p} - 1\right)$. Let $\frac{1}{t^p} - 1 = z$; $\frac{1}{t} = (1+z)^{\frac{1}{p}}$; $\left(\frac{1}{t} - 1\right)^n = \left(\overline{1+z}^{\frac{1}{p}} - 1\right)^n$; let this $= A z^n + B z^{n+1} + $ &c.; therefore $\left(\frac{1}{t} - 1\right)^n = A . \left(\frac{1}{t^p} - 1\right)^n + B . \left(\frac{1}{t^p} - 1\right)^{n+1} + $ &c., and $\left(\frac{1}{t} - 1\right)^n . \phi(t) = A . \left(\frac{1}{t^p} - 1\right)^n \phi(t) + B . \left(\frac{1}{t^p} - 1\right)^{n+1} . \phi(t) + $ &c.; and taking the quantities of which these are the generating functions, $\delta^n . u_x = A . \Delta^n . u_x + B . \Delta^{n+1} . u_x + $ &c., where A, B, &c. are the coefficients of z^n, z^{n+1}, &c. in the expansion of $\left(\overline{1+z}^{\frac{1}{p}} - 1\right)^n$.

Similarly, from the second series in (157,) $\cot \frac{m}{n} \cdot \frac{\pi}{2} =$

$$\frac{n}{m} \times 0{,}6366197723675813 \quad - \frac{4\,m\,n}{4\,n^2 - m^2} \times 0{,}3183098861837907$$
$$- \frac{m}{n} \times 0{,}205288889414508 \quad - \frac{m^3}{n^3} \times 0{,}0065510747588218$$
$$- \frac{m^5}{n^5} \times 0{,}0003450292555397 \quad - \frac{m^7}{n^7} \times 0{,}0000202791060 52$$
$$- \frac{m^9}{n^9} \times 0{,}0000012366582718 \quad - \frac{m^{11}}{n^{11}} \times 0{,}0000000764958 82$$
$$- \frac{m^{13}}{n^{13}} \times 0{,}0000000047597 38 \quad - \frac{m^{15}}{n^{15}} \times 0{,}0000000002969 05$$
$$- \frac{m^{17}}{n^{17}} \times 0{,}000000000018541 \quad - \frac{m^{19}}{n^{19}} \times 0{,}000000000001158$$
$$- \frac{m^{21}}{n^{21}} \times 0{,}000000000000072 \quad - \frac{m^{23}}{n^{23}} \times 0{,}000000000000005$$

The first fractional term in each expression is not expanded, as the series by that means are made to converge much more rapidly than if it were. It will never be necessary to take $\frac{m}{n}$ greater than $\frac{1}{2}$.

(201.) It is plain, however, that this process is too laborious to be applied to every one of the small divisions, and that it cannot with ease be exteoded farther than to every degree. But the calculation of differences of tangeots admits of none of those simplifieations which assisted us so much io forming tables of sines; we proceed, therefore, to give a oiethod whieh applies to all eases whatever.

(202.) Let u be a funetion of $x = \phi(x)$; suppose x to reeeive the iocrements h, $2h$, &c.; then

$$\phi(x) = u.$$
$$\phi(x+h) = u + \frac{d\,u}{d\,x} \cdot \frac{h}{1} + \frac{d^2\,u}{d\,x^2} \cdot \frac{h^2}{1\,.\,2} + \frac{d^3\,u}{d\,x^3} \cdot \frac{h^3}{1\,.\,2\,.\,3} + \text{\&c.}$$
$$\phi(x+2h) = u + \frac{d\,u}{d\,x} \cdot \frac{2\,.\,h}{1} + \frac{d^2\,u}{d\,x^2} \cdot \frac{2^2\,.\,h^2}{1\,.\,2} + \frac{d^3\,u}{d\,x^3} \cdot \frac{2^3\,.\,h^3}{1\,.\,2\,.\,3} + \text{\&c.}$$

Upoo taking the differenees it is evident, that (observing that $\Delta^n\,.\,0^m$ is 0 wheo n is greater than m) $\Delta^n\,.\,u = \frac{\Delta^n\,.\,0^n}{1\,.\,2\,\ldots\,n} \cdot \frac{d^n\,u}{d\,x^n} h^n + \frac{\Delta^n\,.\,0^{n+1}}{1\,.\,2\,\ldots\,\overline{n+1}} \cdot \frac{d^{n+1}\,u}{d\,x^{n+1}} h^{n+1} + \text{\&c.}$ Now the numbers $\frac{\Delta^n\,.\,0^n}{1\,.\,2\,\ldots\,n} h^n$, $\frac{\Delta^n\,.\,0^{n+1}}{1\,.\,2\,\ldots\,\overline{n+1}}$ h^{n+1}, &c., can be conveniently calculated first (as they will be the same for every difference calculated thus) h being expressed supposing the radius $= 1$; and the formula for $\frac{d^n\,u}{d\,x^n}$ ean also be found, aod it will only be necessary at each calculation of differences to substitute numerical values in the expressioo for $\frac{d^n\,u}{d\,x^n}$. For the tangent, $u = \tan x$, $\frac{d\,u}{d\,x} = \sec^2 x$, $\frac{d^2 u}{d\,x^2} = 2 \sec^2 x \,.\, \tan x$, &c.; and for intervals of a minute each, $h = 0{,}000290888208665721596$. The following table contains the values of $\frac{\Delta^n\,.\,0^m}{1\,.\,2\ldots m}$ from $n = 1$ to $n = 12$, and from $m = 1$ to $m = 12$.

	$\frac{0^1}{1}$	$\frac{0^2}{1.2}$	$\frac{0^3}{1.2.3}$	$\frac{0^4}{1.2.3.4}$	$\frac{0^5}{1\ldots5}$	$\frac{0^6}{1\ldots6}$	$\frac{0^7}{1\ldots7}$	$\frac{0^8}{1\ldots8}$	$\frac{0^9}{1\ldots9}$	$\frac{0^{10}}{1\ldots10}$	$\frac{0^{11}}{1\ldots11}$	$\frac{0^{12}}{1\ldots12}$
Δ^1	1	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{24}$	$\frac{1}{120}$	$\frac{1}{720}$	$\frac{1}{5040}$	$\frac{1}{40320}$	$\frac{1}{362880}$	$\frac{1}{3628800}$	$\frac{1}{39916800}$	$\frac{1}{479001600}$
Δ^2		1	1	$\frac{7}{12}$	$\frac{1}{4}$	$\frac{31}{360}$	$\frac{1}{40}$	$\frac{127}{20160}$	$\frac{17}{12096}$	$\frac{73}{259200}$	$\frac{31}{604800}$	$\frac{2047}{239500800}$
Δ^3			1	$\frac{3}{2}$	$\frac{5}{4}$	$\frac{3}{4}$	$\frac{43}{120}$	$\frac{23}{160}$	$\frac{605}{12096}$	$\frac{311}{20160}$	$\frac{2591}{604800}$	$\frac{437}{403200}$
Δ^4				1	2	$\frac{13}{6}$	$\frac{5}{3}$	$\frac{81}{80}$	$\frac{37}{72}$	$\frac{6821}{30240}$	$\frac{265}{3024}$	$\frac{55591}{1814400}$
Δ^5					1	$\frac{5}{2}$	$\frac{10}{3}$	$\frac{25}{8}$	$\frac{331}{144}$	$\frac{45}{32}$	$\frac{2243}{3021}$	$\frac{1045}{3024}$
Δ^6						1	3	$\frac{19}{4}$	$\frac{21}{4}$	$\frac{1087}{240}$	$\frac{259}{80}$	$\frac{30083}{15120}$
Δ^7							1	$\frac{7}{2}$	$\frac{77}{12}$	$\frac{49}{6}$	$\frac{1939}{240}$	$\frac{4753}{720}$
Δ^8								1	1	$\frac{25}{3}$	12	$\frac{4819}{360}$
Δ^9									1	$\frac{9}{2}$	$\frac{21}{2}$	$\frac{135}{8}$
Δ^{10}										1	5	$\frac{155}{12}$
Δ^{11}											1	$\frac{11}{2}$
Δ^{12}												1

(203.) The same cautions as in (196) must be observed with regard to the number of decimals. And for the calculation for smaller divisions, as seconds, the formulæ of (197) must be used. Thus the table of tangents is completed.

(204.) The secants are calculated from the formula $\tan A + \cot A = 2 \operatorname{cosec} 2A$. This gives the cosecants or secants only for every second division; but the interpolation for every division will be sufficiently easy.

(205.) Thus then our tables of natural sines, tangents, and secants, is completed. The tables of their logarithms might be formed by taking from logarithmic tables the logarithms of these numbers; and many writers have considered this as being upon the whole the easiest way. As they may, however, be found independently, and therefore free from all errors of previous computations, and as the method appears to us the easiest, we shall give it here.

(206.) It has been seen (155) that $\sin x = x\left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right)$. &c., and therefore $\log \sin x = \log x + \log\left(1 - \frac{x^2}{\pi^2}\right) + \log\left(1 - \frac{x^2}{4\pi^2}\right) + \log\left(1 - \frac{x^2}{9\pi^2}\right) + \&c.$ Expanding all the fractions but the first, and putting M for the modulus of common logarithms, $\log \sin x = \log x + \log\left(1 - \frac{x^2}{\pi^2}\right)$

$$- M \left\{ \begin{array}{l} \frac{x^2}{4\pi^2} + \frac{1}{2} \cdot \frac{x^4}{16\pi^4} + \frac{1}{3} \cdot \frac{x^6}{64\pi^6} + \&c. \\ + \frac{x^2}{9\pi^2} + \frac{1}{2} \cdot \frac{x^4}{81\pi^4} + \frac{1}{3} \cdot \frac{x^6}{729\pi^6} + \&c. \\ + \&c. \end{array} \right\}$$

Adding the coefficients of similar powers of x, and putting $\frac{m}{n} \cdot \frac{\pi}{2}$ for x, we find the following series,

$$\log \sin \frac{m}{n} \cdot \frac{\pi}{2} = \log m + \log (2n - m) + \log (2n + m) - 3 \log n + 9{,}594059885702190$$

$$-\frac{m^2}{n^2}\times 0,070022826605902 \qquad -\frac{m^4}{n^4}\times 0,001117266441662$$
$$-\frac{m^6}{n^6}\times 0,000039229146454 \qquad -\frac{m^8}{n^8}\times 0,000001729270798$$
$$-\frac{m^{10}}{n^{10}}\times 0,000000084362986 \qquad -\frac{m^{12}}{n^{12}}\times 0,000000004345716$$
$$-\frac{m^{14}}{n^{14}}\times 0,000000000231931 \qquad -\frac{m^{16}}{n^{16}}\times 0,000000000012659$$
$$-\frac{m^{18}}{n^{18}}\times 0,000000000000703 \qquad -\frac{m^{20}}{n^{20}}\times 0,000000000000040$$
$$-\frac{m^{22}}{n^{22}}\times 0,000000000000002.$$

And similarly $\log\cos\frac{m}{n}\cdot\frac{\pi}{2}=\log\overline{n-m}+\log\overline{n+m}-2\log n$

$$-\frac{m^2}{n^2}\times 0,101494859341893 \qquad -\frac{m^4}{n^4}\times 0,003187294065451$$
$$-\frac{m^6}{n^6}\times 0,000209485800017 \qquad -\frac{m^8}{n^8}\times 0,000016848348598$$
$$-\frac{m^{10}}{n^{10}}\times 0,000001480193987 \qquad -\frac{m^{12}}{n^{12}}\times 0,000000136502272$$
$$-\frac{m^{14}}{n^{14}}\times 0,000000012981715 \qquad -\frac{m^{16}}{n^{16}}\times 0,000000001261471$$
$$-\frac{m^{18}}{n^{18}}\times 0,000000000124567 \qquad -\frac{m^{20}}{n^{20}}\times 0,000000000012456$$
$$-\frac{m^{22}}{n^{22}}\times 0,000000000001258 \qquad -\frac{m^{24}}{n^{24}}\times 0,000000000000128$$
$$-\frac{m^{26}}{n^{26}}\times 0,000000000000013 \qquad -\frac{m^{28}}{n^{28}}\times 0,000000000000001.$$

(207.) If $\frac{m}{n}$ be small, the first terms in the last expression, which together $=\log\overline{1-\frac{m^2}{n^2}}$, may be expanded into the series $-2\,M\times\left(\frac{m^2}{2\,n^2-m^2}+\frac{1}{3}\cdot\left(\frac{m^2}{2\,n^2-m^2}\right)^3+\&c\right)$ where $M=$ modulus $=0{,}434294481903252$.

To make the logarithm positive, 10 must be added. This makes our operations entirely independent of logarithmic tables.

(208.) It is sufficient to find the log sines of the arcs between 45° and 90°, or the log cosines of arcs less than 45°. The remainder may be found thus, $\log\sin A=10+\log\sin 2\,A-\log\cos A-\log 2=\log\sin 2\,A-\log\cos A+9{,}698970004336019$. By properly applying this theorem we may descend successively from log sin 45° to the log sines of all arcs less than 45°. By this method then the log sines and log cosines, and consequently the log tangents (since $\log\tan A=10+\log\sin A-\log\cos A$) may be calculated for every degree.

(209.) If, however, the log sines be calculated independently for larger intervals, as for every 10°, the differences for every degree may be thus found, $\log\sin\overline{x+h}-\log\sin x=\log.\frac{\sin(x+h)}{\sin x}=$

$$2\,M\left\{\frac{\sin\overline{x+h}-\sin x}{\sin\overline{x+h}+\sin x}+\frac{1}{3}\cdot\left(\frac{\sin\overline{x+h}-\sin x}{\sin\overline{x+h}+\sin x}\right)^3+\&c.\right\}$$

a series which converges rapidly. Or when one first difference is thus found, the second differences may be calculated by this series, $\Delta^2\log\sin x=\log\frac{\sin x\sin\overline{x+2h}}{\sin^2\overline{x+h}}$

$$=-2\,M\left\{\frac{\sin^2\overline{x+h}-\sin x.\sin\overline{x+2\,h}}{\sin^2\overline{x+h}+\sin x.\sin\overline{x+2\,h}}+\frac{1}{3}\left(\frac{\sin^2\overline{x+h}-\sin x.\sin\overline{x+2\,h}}{\sin^2\overline{x+h}+\sin x.\sin\overline{x+2\,h}}\right)^3+\&c.\right\};$$

which, since

$$\frac{\sin^2\overline{x+h}-\sin x.\sin\overline{x+2\,h}}{\sin^2\overline{x+h}+\sin x.\sin\overline{x+2\,h}}=\frac{\sin^2 h}{\cos^2 h+\cos 2\,\overline{x+h}},$$

converges much more rapidly.

(210.) Before proceeding farther, it will be proper to verify the numbers already calculated; and here the formula of (159) will be found very useful. For taking the logarithms of both sides of that equation, $\log \sin n\beta = \overline{n-1} \times 0{,}3010299956639812 + \log \sin \beta + \log \sin \overline{\beta + \frac{\pi}{n}} + \log \sin \overline{\beta + \frac{3\pi}{n}} +$ &c. to n terms, where n and β may be taken at pleasure.

(211.) It is then best to fill them up by differences; and the differences may be calculated in the same manner as in (202.) Here $\frac{du}{dx} = \mathrm{M} \cot x$; $\frac{d^2 u}{dx^2} = -\mathrm{M}(1 + \cot^2 x)$; $\frac{d^3 u}{dx^3} = 2\,\mathrm{M} \cot x\,(1 + \cot^2 x)$, &c. The calculation of the differences is rather tedious, but the tables are formed then with great ease, and the certainty that any error will be discovered at the next place of verification makes this method superior to any other.

(212.) For the smaller divisions, the differences will be found from these differences by the formulæ in (197.) Thus our tables of logarithmic sines, cosines, and tangents will be completed.

(213.) It is unnecessary to examine by any formula of verification the accuracy of the numbers for the small divisions of the arc. It is scarcely possible to have a better verification, than the agreement of the last of a series of numbers computed by differences with one which has previously been calculated by an independent process.

(214.) In (165) we have alluded to tables of the logarithms of $\frac{\sin x}{x}$ for a few degrees. These are calculated very easily from the series $\log \frac{\sin x}{x} = -\mathrm{M}\left\{\frac{x^2}{6} + \frac{x^4}{3^2.4.5} + \frac{x^6}{3^4.5.7} + \text{\&c.}\right\}$. When $x = 5°$ the third term has no significant figure in the ten first decimal places. For tables to 10 decimals the first term is sufficient up to 1°, and the two first terms to 5°. For tables to 7 decimals, the first term is sufficient, as the second term produces 1 in the last place when $x = 5°$. This therefore is easily calculated by second differences. If the $\log \frac{\tan x}{x}$ be required, since $\frac{\tan x}{x} = \frac{\sin x}{x} \cdot \frac{1}{\cos x}$, we have $\log \frac{\tan x}{x} = \log \frac{\sin x}{x} +$ ar. comp. $\log \cos x$; or it can be calculated in the same way.

(215.) Since $\sec x = \frac{1}{\cos x}$, its logarithm will be immediately found. And since $\operatorname{versin} x = 1 - \cos x = 2 \sin^2 \frac{x}{2}$, the natural and logarithmic versed sines are found. They are seldom inserted in tables, except in those employed in Nautical Astronomy.

(216.) The principal tables commonly in use are the following: Sherwin's, containing, besides the logarithms of numbers, sines, cosines, tangents, &c., natural and logarithmic, for every minute, to 7 decimals; Hutton's, containing the same, with an interesting and valuable Introduction; Gardiner's, with log. sines, &c. for every 10 seconds to 7 decimals; Taylor's, with log. sines, &c. to 7 decimals for every second; of these, the most common is Hutton's. Many smaller collections of tables are in use. Of the foreign tables, the best are Vega's, containing the logarithms of numbers and log. sines, &c. for every 10″ to 10 decimals; Callet's logarithms of numbers, log. sines, &c. for every 10″ to 7 decimals, with some tables for the decimal division of the circle. This is a very convenient and useful collection. An abridged form of the *Tables du Cadastre*, revised by Delambre, has (we believe) been edited by Borda; and must form a useful collection for the decimal division.

(217.) Trigonometrical tables have generally sines, cosines, tangents, cotangents, &c. up to 45°; the cotangent of an arc being the tangent of its complement, &c. What is gained by this arrangement, except perhaps in the use of subsidiary angles, it is not easy to say; and in taking out the sine, &c. of an arc greater than 45°, or greater than 90°, there is frequently some confusion. We should prefer the more natural arrangement of sines, tangents, &c. up to 90°; these read in the reverse order (as shown by the figures and titles at the bottom of the page,) would give the cosines, cotangents, &c.

ANALYTICAL GEOMETRY.

The Application of Algebra to Geometry forms two distinct branches of Science. The object of the first is to investigate the Theorems, and resolve the determinate Problems, of Elementary Geometry; that of the second, to assign the Figure, and determine the Properties of Curves and of Surfaces. The first of these is of very limited extent, and of comparatively trifling importance; we shall, therefore, confine our attention to the second, which is of great use, *as an instrument of investigation*, in various departments of Pure and Mixed Mathematics. This branch of the subject is usually distinguished by the name of Algebraic, or Analytical, Geometry. It may, with propriety, be divided into two parts, of which the one will embrace the Theory of Curves, and the other, the Theory of Surfaces.

PART I.

ON THE APPLICATION OF ALGEBRA TO THE THEORY OF CURVES.

(1.) *Geometrical magnitude may be represented by the characters of Algebra.*

For let A and B be any two straight lines which are to each other as $a : 1$, a being an abstract number.

Then $A = a\,B$, or if B be taken $= 1$, $A = a$; that is, the straight line A is represented by the algebraic character a. The line B thus assumed equal to unity is called the *linear unit.*

Similarly, if the square and cube described upon B be taken as the respective units of surface and solidity, any abstract number which expresses how often either of them is contained in any proposed surface, or solid, may be conceived to represent the surface or solid itself.

Hence, if a, b, c represent any three straight lines, $a \times b$ will represent a rectangle whose area is $a\,b$ times B^2, and $a \times b \times c$ will represent a rectangular parallelepiped whose solid content is $a\,b\,c$ times B^3.

(2.) *A variable quantity in Algebra may be represented in Geometry by an indefinite straight line.*

Fig 1. Let x be any variable quantity, and X X′ an indefinite straight line, (fig. 1.)

In X X′ assume A as the point from which the lines are to be measured. Then any finite portion A P may be taken to represent a given value of x. Thus, if the point P fall upon A, the distance A P will correspond to $x = 0$; and by increasing A P we may evidently represent all the determinate values of x.

It is immaterial whether the values of x be measured to the right, or to the left of the point A, since the line extends indefinitely in both directions. But if we begin to measure the *positive* values of x to the *right*, then the *negative* values must be measured to the *left*, of A. To illustrate this, let A′ be the point from which the values of a second variable x' are to be measured; take

$$A'P = a,\ A'P' = x',\ \text{and } A\,P \text{ as before} = x.$$

Then $$x' = a + x,$$

or $$x = x' - a.$$

Now if x' be positive, and less than a, the values of x will plainly be negative; but in this case the point P falls to the *left* of A, as at P′; hence the negative values of x ought to be measured to the left of A.

We may therefore lay down this general rule, "When distance is to be estimated from a fixed point, along a straight line given in position, if the positive values of any quantity be measured in either direction from the fixed point, the negative values must be measured in the opposite direction from the same point."

(3.) The application of Algebra to the theory of curves is founded on this principle, *that an indeterminate equation between two variables is capable of being represented by a geometrical locus, and conversely.*

Let $f(x, y)$ be any indeterminate equation between x and y; a any arbitrary value of x, and b the corresponding value of y.

Draw two straight lines A X, A Y of indefinite length, at right angles to each other, and meeting in A, (fig. 2.) Fig. 2.
In A X take $A\,M = a$, and in A Y, $A\,N = b$; through M and N draw M P and N P parallel respectively to A Y, A X, meeting in P; then the point P corresponds to the solution of the proposed equation.

Since the equation admits of an unlimited number of solutions, the points P furnished by each solution will also be infinite in number; and their assemblage will therefore form a certain line, straight or curved, which is called the *locus* of the equation $f(x, y) = 0$.

When the equation admits of only one solution, it represents a point; and when it has no real solution, it indicates an imaginary curve.

(4.) Of the two quantities a and b which represent A M and M P, the former is called the *abscissa*, the latter the *ordinate* of the point P; they are both included under the general appellation of *the coordinates* of that point.

The lines A X, A Y are called the *axes*, and the point A the *origin*, of the coordinates. We have supposed, for simplicity, that the axes are at right angles to each other, or *rectangular*; but they may have any inclination whatever.

When the point P is not given, its coordinates are represented by the letters x and y; of which the former denotes an abscissa measured along A X, the latter an ordinate measured along A Y. Hence, A X is usually called the axis of x, and A Y the axis of y.

If a point be situated on the axis of x, then $y = 0$; if on the axis of y, then $x = 0$; and if it coincide with the origin, then x and y each $= 0$.

By applying the conventional rule with respect to the signs laid down in Art. 2, it is evident, that if the

values of x to the right of A be supposed positive, those to the left of A must be considered negative. In like manner, if the values of y measured along A Y be positive, those in the direction A Y′ must be reckoned negative.

(5.) Let a curve be now supposed to be traced upon a plane, then, reversing the process by which the locus of a given equation is determined, we may deduce from some known property of the curve, the relation subsisting between the coordinates of any one of its points. The equation which expresses this relation, supposing it to be the same for every point, is called the *equation to the curve.*

It is convenient to distinguish curves by the generic appellation of *lines.* They are divided into orders, according to the dimension of the equation by which they are represented.

Thus, *a line of the first order* is the locus of the equation

$$a y + b x + c = 0.$$

A line of the second order is the locus of the equation

$$a y^2 + b x y + c x^2 + d y + e x + f = 0;$$

and so on.

(6.) The position of a point upon a plane may be determined in a manner somewhat different from that which has just been explained; namely, by means of its distance from a given point, and the angle which that distance makes with a line given in position. The given point is called the *pole*, and the variable distance, *the radius vector.*

Thus, referring to fig. 2, A is the pole, and A P the radius vector r. The angle which A P makes with the line A X given in position, is usually denoted by ω.

The quantities r and ω are called *polar coordinates*, and the equation which expresses the relation subsisting between them at any point in a curve, is called *the polar equation to the curve.*

ON THE STRAIGHT LINE.

(7.) *To find the equation to a straight line.*

Let B Z be a straight line of indefinite length, and suppose it referred to the rectangular axes A X, A Y, (fig. 3.) Fig. 3.

Assume any point in it P, draw P M parallel to A Y, meeting A X in M; through B draw B Q parallel to A X, meeting P M in Q.

Let A M $= x$, M P $= y$, A B $= b$.

Now in the right angled triangle B Q P,

$$\frac{\mathrm{PQ}}{\mathrm{QB}} = \frac{\sin \mathrm{PBQ}}{\cos \mathrm{PBQ}} = \tan \mathrm{PBQ};$$

$$\therefore \mathrm{PQ} = \mathrm{QB} \tan \mathrm{PBQ}.$$

But $$y = \mathrm{MP} = \mathrm{MQ} + \mathrm{QP},$$
$$= \mathrm{AB} + \mathrm{QP},$$
and $$\therefore = b + \mathrm{QB} \tan \mathrm{PBQ}.$$

Now as the position of B Z, with respect to A X, is supposed to be given, the angle P B Q or P C X will be known. Hence, denoting tan P B Q by a, we have

$$y = a x + b.$$

The same relation may be shown to subsist between the coordinates of any other point in the line. Hence the equation required is

$$y = a x + b.$$

(8.) *Cor.* 1. When the straight line passes through the origin, $b = 0$; therefore the equation becomes

$$y = a x.$$

(9.) *Cor.* 2. The general equation of the first degree between two variables is

$$\mathrm{A} y + \mathrm{B} x + \mathrm{C} = 0;$$

or dividing by A, and transposing,

$$y = -\frac{\mathrm{B}}{\mathrm{A}} x - \frac{\mathrm{C}}{\mathrm{A}}.$$

Let $$-\frac{\mathrm{B}}{\mathrm{A}} = a, \; -\frac{\mathrm{C}}{\mathrm{A}} = b,$$

then $$y = a x + b,$$

which coincides with the equation deduced in the last article. Whence it appears, conversely, *that the locus of the general equation of the first degree between two variables is a straight line.*

(10.) *To draw the straight line which is the locus of any given equation of the first degree.*

Since two points serve to fix the position of a straight line, it will only be requisite for the solution of the proposed problem to find in each case the points in which the line meets the axes. This will be done by making x and y successively $= 0$ in the general equation.

The equation in its most general form is

$$y = \pm (a x \pm b).$$

1. Let $$y = a x + b.$$
Then it $$x = 0, \; y = b,$$
and if $$y = 0, \; x = -\frac{b}{a}.$$

Hence in A Y (fig. 4) take A B $= b$, and in X A *produced*, A C $= \frac{b}{a}$, join C, B, then C B Z is the line required. Fig. 4.

2. Let $$y = a x - b.$$
Then if $$x = 0, \; y = -b,$$
and if $$y = 0, \; x = \frac{b}{a}.$$

Hence in Y A *produced* take A B′ $= b$, and in A X, A C′ $= \frac{b}{a}$, join B′, C′, and C′ B′ Z′ is the line required.

3. Let $$y = -a x + b.$$

Then, as before, take A B $= b$, A C′ $= \frac{b}{a}$, and the line required is C′ B Z.

4. Let $$y = -a x - b$$

If A B′ be taken $= b$, and A C $= \frac{b}{a}$, the line required will be C B′ Z.

(11.) The general equation to a straight line is $y = a x + b$, which involves two constants, a and b. The equation will therefore occur under various forms, cor-

responding to the conditions which serve to determine these constants. Now a straight line is given in position when it passes through two given points, or when it passes through one given point, and makes a known angle with another straight line. We shall investigate the form of the equation in each of these cases.

(12.) *To find the equation to a line which passes through two given points.*

Let the coordinates of the given points be x', y', and x'', y''; then the general equation is

$$y = a\,x + b \;\ldots\; (1.)$$

and since the given coordinates must satisfy this, we also have

$$y' = a\,x' + b \;\ldots\; (2.)$$

and

$$y'' = a\,x'' + b \;\ldots\; (3.)$$

Subtracting the second from the first, and the third, successively, we have

$$y - y' = a\,(x - x') \;\ldots\; (4.)$$

and

$$y'' - y' = a\,(x'' - x') \;\ldots\; (5.)$$

but from (5.) $a = \dfrac{y'' - y'}{x'' - x'}$; substituting this in (4) we have for the equation sought

$$y - y' = \frac{y'' - y}{x'' - x'}\,(x - x'),$$

which may easily be reduced to the same form as the last.

Cor. Equation (4.) $y - y' = a\,(x - x')$ is the equation to a straight line passing through *one* given point (x', y'); in which the coefficient a is indeterminate, since an indefinite number of lines can be drawn passing through the same point.

(13.) *To find the equation to a straight line which passes through a given point, and makes a given angle with a given straight line.*

Let the equation to the given line be $y = a\,x + b$, then the form of the equation required will be (Art. 12)

$$y - y' = a'\,(x - x') \;\ldots\; (1.)$$

in which a' is to be determined.

Fig. 5 If A M, A N be drawn through the origin parallel to the two lines, (fig. 5.) the angle contained between them is M A X − N A X;

$$\therefore \tan \mathrm{M\,A\,N} = \frac{\tan \mathrm{M\,A\,X} - \tan \mathrm{N\,A\,X}}{1 + \tan \mathrm{M\,A\,X}\,\tan \mathrm{N\,A\,X}}.$$

or assuming tan M A N, which is supposed to be given, $= m$,

$$m = \frac{a - a'}{1 + a\,a'},$$

$$\therefore a - a' = m + m\,a\,a,$$

$$\therefore a - m = (1 + a\,m)\,a',$$

$$\therefore a' = \frac{a - m}{1 + a\,m},$$

hence, by substitution in (1.)

$$y - y' = \frac{a - m}{1 + a\,m}\,(x - x'),$$

which is the equation required.

(14.) *Cor.* 1. If the two lines are perpendicular to each other, m is infinite, therefore $a' = -\dfrac{1}{a}$, and the equation becomes

$$y - y' = -\frac{1}{a}\,(x - x').$$

(15.) *Cor.* 2. If the two lines are parallel, then $m = 0$, therefore $a' = a$, and the equation becomes

$$y - y' = a\,(x - x').$$

Observation. If ρ, ρ' denote the angle of intersection of two lines ρ and ρ', whose equations are

$$y = a\,x + b,$$

and

$$y = a'\,x + b';$$

we then have

$$\tan \rho, \rho' = \frac{a - a'}{1 + a\,a'} \;\ldots\; (1.)$$

In like manner,

$$\sin \rho, \rho' = \frac{a - a'}{\sqrt{(1 + a^2)\,(1 + a'^2)}} \;\ldots\; (2.)$$

and

$$\cos \rho, \rho' = \pm \frac{1 + a\,a'}{\sqrt{(1 + a^2)\,(1 + a'^2)}} \;\ldots\; (3.)$$

in which the positive sign is to be used when the angle is acute, and the negative when it is obtuse.

(16.) *The equation to a straight line may be expressed in terms of the perpendicular let fall upon it from the origin.*

Let p represent this perpendicular, then if the symbols ρ, x and ρ, y be taken to denote the angles which the straight line forms with the axes of x and y, we have

$$p = b \sin \rho, y = b \cos \rho, x,$$

$$\therefore b = \frac{p}{\cos \rho, x},$$

therefore, by substitution in the general equation,

$$y = a\,x + b,$$

we have

$$y = a\,x + \frac{p}{\cos \rho, x},$$

but

$$a = \tan \rho, x = \frac{\sin \rho, x}{\cos \rho, x},$$

$$\therefore y = \frac{x \sin \rho, x + p}{\cos \rho, x},$$

$$\therefore p = y \cos \rho, x - x \sin \rho, x \;\ldots\; (1.)$$

which is the equation sought.

Again, if the angles which p makes with the axes be denoted by p, x and p, y, we have

$$\cos \rho, x = \sin p, x,$$

$$\sin \rho, x = -\cos p, x,$$

$$\therefore p = y \sin p, x + x \cos p, x.$$

(17.) *To find the length of the perpendicular let fall from a given point upon a given straight line.* Fig. 6.

Let P be the given point, (fig. 6.) B Z the given line, and P Q the perpendicular whose value (p) is to be expressed.

Let the coordinates of P be x', y', and the equation to the given line

$$y = a\,x + b.$$

Through P draw P R parallel to B Z, and from A let fall the perpendicular A p upon R, meeting Q B produced in q.

Then, since P R is parallel to B Q, the equation to

Analytical Geometry.

P R is $y = a x + b'$, (15.)

and since (x', y') is a point in it,

$$y' = a x' + b',$$

whence, by the last article,

$$A p = y' \cos \rho, x - x' \sin \rho, x;$$

but

$$A q = b \cos \rho, x;$$

$$\therefore A p - A q, \text{ or } p = (y' - b) \cos \rho, x - x' \sin \rho, x,$$

$$= \cos \rho, x \left\{ y' - b - \frac{\sin \rho, x}{\cos \rho, x} \cdot x' \right\}$$

$$= \cos \rho, x \{ y' - a x' - b \}$$

$$= \therefore \pm \frac{y' - a x' - b}{\sqrt{1 + a^2}}.$$

(18.) *Cor.* 1. If the line pass through the origin, $b = 0$,

$$\therefore p = + \frac{y' - a x'}{\sqrt{1 + a^2}}.$$

(19.) *Cor.* 2. If the given point be the origin, then x' and $y' = 0'$, and

$$p = \mp \frac{b}{\sqrt{1 + a^2}},$$

according as the line is situated below or above, the axis A X.

(20.) *To find the analytical value of the line which joins two given points.*

Fig. 7 Let P and Q be the given points, (fig. 7,) join them. Draw P M, Q N parallel to A Y, and P R parallel to A X, meeting Q N in R.

Let the coordinates of P be x', y', and those of Q x'', y'', and assume $P Q = r$.

Then in the right angled triangle P R Q,

$$P Q^2 = P R^2 + R Q^2,$$

$$= M N^2 + R Q^2,$$

$$= (A N - A M)^2 + (N Q - P M)^2,$$

$$= \therefore (x'' - x')^2 + (y'' - y')^2,$$

$$\therefore r = \pm \sqrt{\{ (x'' - x')^2 + (y'' - y')^2 \}},$$

which is the expression required.

(21.) *Cor.* If either point coincide with the origin, the coordinates of that point will $= 0$, and the above expression will be simplified.

Then let P coincide with A, then x' and $y' = 0$, and

$$r \text{ or } A Q = \sqrt{x''^2 + y''^2}.$$

(22.) *To find the coordinates of the point of intersection of two lines.*

Let the equation to the first be $y = a x + b \ldots (1.)$

that to the second $y = a' x + b' \ldots (2.)$

Two lines which cut each other have evidently the same coordinates at their point of intersection. The coordinates sought will therefore be found by supposing x and y to have the same value in both equations, and then eliminating them.

Hence subtracting (2) from (1) we have

$$(a - a') x + b - b' = 0, \therefore x = - \frac{b - b'}{a - a'},$$

$$\therefore y \text{ which } = a x + b, \quad = - \frac{a' b - b' a}{a - a'},$$

hence the coordinates required are found.

When the lines are parallel, the coordinates are infinitely great, therefore the denominators of the above fractions $= 0$, or $a = a'$, which is the condition of parallelism already established in (15.)

(23.) For the sake of simplicity we have hitherto employed rectangular axes, but it is frequently convenient to suppose them inclined at any angle whatever. We shall therefore present in a tabular form the foregoing results adapted to this hypothesis, leaving the investigation to be supplied by the reader.

1. The equation to any straight line is

$$y = a x + b,$$

in which $a = \dfrac{\sin \rho, x}{\sin \rho, y}$, and $b =$ the ordinate drawn at the origin.

2. The equation to a line passing through *one* given point (x', y') is

$$y - y' = a (x - x').$$

3. The equation to a line passing through *two* given points (x', y') and (x'', y'') is

$$y - y' = \frac{y'' - y'}{x'' - x'} (x - x').$$

4. If ρ, ρ' denote the angle of intersection of two lines whose equations are

$$y = a x + b,$$

$$y = a' x + b'$$

Then $\tan \rho, \rho' = \sin x, y \dfrac{a - a'}{1 + (a + a') \cos x, y + a a'}$.

5. The equation to a straight line drawn through a given point (x', y') at right angles to a given line $y = a x + b$ is

$$y - y' = - \frac{1 + a \cos x, y}{a + \cos x, y} (x - x').$$

When the lines are parallel,

$$y - y' = a (x - x'),$$

as in the case of rectangular coordinates.

6. The equations to a straight line in terms of the perpendicular dropped upon it from the origin, are

$$(1.) \; p = y \sin \rho, y - x \sin \rho, x,$$

$$(2.) \; p = y \cos \rho, y + x \cos \rho, x.$$

7. The value of the perpendicular (p) let fall from a given point (x', y') on the line $y = a x + b$, is

$$p = (y' - a x' - b) \sin \rho, y \ldots (1.)$$

or $$= (y - b) \sin \rho, y - x' \sin \rho, x \ldots (2.)$$

8. The analytical value of the distance (r) between two given points (x', y') and (x'', y'') is

$$r = \sqrt{\{ (x'' - x')^2 + 2 (x'' - x') (y'' - y') \cos x, y + (y'' - y')^2 \}};$$

and when the point (x', y') coincides with the origin

$$r = \sqrt{x''^2 + 2 x'' y'' \cos x, y + y''^2}.$$

(24.) We shall now exemplify the principles laid down in this chapter, by applying them to the following propositions:

1. *Required the equation to a line which bisects the angle contained by two given lines.*

Let A X, A Y be rectangular axes, (fig. 5.)

Draw through the origin two lines A M, A N parallel respectively to the given lines, then M A N is the angle to be bisected. Let A P be the line whose equation is required.

Suppose the equations to A M, A N, A P respectively to be $y = a x$, $y = a' x$, and $y = m x$, in which m is to be determined.

The tangent of the angle P A N $= \frac{m - a'}{1 + m a'}$.

The tangent of the angle P A M $= \frac{a - m}{1 + m a}$,

but these being equal, by hypothesis,

$$\frac{m - a'}{1 + m a'} = \frac{a - m}{1 + m a},$$

or $m + m^2 a - a' - m a a' = a + m a a' - m^2 a'$,

$$\therefore m^2 (a + a') + m (2 - 2 a a') - (a + a') = 0,$$

$$\therefore m^2 - 2 \,.\, \frac{1 - a a'}{1 + a a'} m - 1 = 0.$$

Whence two real values of m may be found. Two lines therefore may be drawn, one of which bisects the angle itself, the other its supplement.

The equations of these lines are

$$y = m x,$$

and $$y = - \frac{1}{m} x,$$

therefore they are at right angles to each other.

2. *Through a given point* P *in a given angle* Y A X *to draw a line* M N, *such that the triangle so cut off may be of given area*, $(a^2,)$ (fig. 8.)

Fig. 8.

Assuming A X, A Y as oblique axes, draw P Q parallel to A Y, and let A Q $= x'$, Q P $= y'$; and A M, which is unknown, $= x_{,}$.

Then the equation to M N is

$$y = \frac{\sin \rho, x}{\sin \rho, y} (x - x_{,}),$$

$$= \therefore \frac{y'}{x' - x_{,}} (x - x_{,}).$$

Let now $x = 0$, $\therefore y$ or A N $= - \frac{y'}{x' - x_{,}} x_{,}$,

$\therefore$ area of the triangle A M N $= \frac{1}{2}$ A M . A N sin A

$$= \tfrac{1}{2} \frac{x_{,}^2 y'}{x' - x_{,}} \sin A = a^2, \text{ by the question,}$$

$$\therefore x_{,}^2 y' \sin A = 2 a^2 x' - 2 a^2 x_{,},$$

$$\therefore x_{,}^2 y' \sin A + 2 a^2 x_{,} = 2 a^2 x',$$

$$\therefore x_{,}^2 + \frac{2 a^2}{y' \sin A} x_{,} = \frac{2 a^2 x'}{y' \sin A},$$

whence a value of $x_{,}$ may be obtained.

If it were required to draw M N such that A M may = A N,

We should then have $- \frac{y' x_{,}}{x' - x_{,}} = x_{,}$,

$$\therefore - y' = x' - x_{,}$$

or Q P = Q M.

3. *If the sides of a plane triangle be bisected, and lines be drawn from the points of bisection at right angles to the sides, they will meet in the same point.*

Let the lines M m, N n, P p, be drawn at right angles to the sides of the triangle A B C from the points of bisection M, N, P, (fig. 9,) these lines will intersect in the same point. Fig. 9

The triangle being referred to rectangular axes A X, A Y, originating at A,

Let the coordinates of A be x', y', and those of B, x'', 0; then the coordinates of N will be $\frac{x'}{2}$, $\frac{y'}{2}$, and those of P $\frac{x' + x''}{2}$, $\frac{y'}{2}$.

In order to prove the proposition we shall find the ordinates of the points in which the lines N n, P p meet M m; and shall then show that these ordinates are identical.

Now N n being drawn through the point N $\left(\frac{x'}{2}, \frac{y'}{2}\right)$ its equation is of the form

$$y - \frac{y'}{2} = a\left(x - \frac{x'}{2}\right);$$

but N n is supposed to be perpendicular to A C, whose inclination to A X $= \tan^{-1} \frac{y'}{x'}$*, $\therefore a = - \frac{x'}{y'}$; $\therefore$

the equation to N n is $y - \frac{y'}{2} = \frac{- x'}{y'}\left(x - \frac{x'}{2}\right) \ldots (1.)$

Similarly, the equation to P p is

$$y - \frac{y'}{2} = \frac{x'' - x'}{y'}\left\{x - \frac{x'' + x'}{2}\right\} \ldots (2.)$$

Let N n, P p be now supposed to meet M m; in which case, x in both equations will become A M or $\frac{x''}{2}$; making this substitution therefore, we have for the ordinates at the point of intersection,

$$y - \frac{y'}{2} = - \frac{x'}{y'} \,.\, \frac{x'' - x'}{2}$$

in the first case, and

$$y - \frac{y'}{2} = - \frac{x'' - x'}{y'} \,.\, x'$$

in the second; but these values are evidently the same, therefore N n, P p, and M m, intersect in the same point.

On precisely similar principles the two following theorems may be proved:

(i.) *The perpendiculars let fall from the angular points of a triangle on the opposite sides intersect one another in the same point.*

(ii.) *The lines drawn from the angles of a triangle to the middle points of the opposite sides intersect in the same point.*

4. *To prove, by reference to the figure of Euclid*, I. 47, *that the lines* A M, N B, *and* C P, *meet in the same point*, (fig. 10.) Fig. 10.

From M and N let fall the perpendiculars M m, N n, on A B produced; let the figure be referred to rectangular axes originating at A, the axis of x being supposed to coincide with A B.

Let the coordinates of C be x', y'; then if A B $= x''$, P B will $= x'' - x'$.

* By the expression $\tan^{-1} \frac{y'}{x'}$ is meant *the angle whose tangent is* $\frac{y'}{x'}$.

Now the triangles A n N and A P C being evidently equal, A n = C P = y', and N n = A P = x'.

Similarly, B $m = y'$ and M $m = x'' - x'$.

Again the equation to A M is $y = \frac{\text{M}\,m}{m\,\text{A}}\,x = \frac{x''-x'}{x''+y'}\,x$. (1)

and that to B N is $y = -\frac{\text{N}\,n}{n\,\text{B}}\,(x - x') =$

$$= -\frac{x'}{x''+y'}\,(x - x') \ldots (2)$$

Now let A M and B N meet C P; in which case, $x = x'$ in each equation.

Then (1) becomes $y = \frac{x''-x'}{x''+y'}\,x'$,

and (2) $y = -\frac{x'-x''}{x''+y'}\,x'$,

which are identical, therefore the three lines meet in the same point.

ON THE TRANSFORMATION OF COORDINATES.

(25.) *The position of a point with respect to a given system of axes being known, to find its position when referred to a new system of axes parallel to the former.*

Fig 11, 12. Let P be the point, A X, A Y the old, A′ X′, A′ Y′ the new, axes, (fig. 11 and 12,) draw P M parallel to A Y meeting A′ X′ in M′, and produce X′ A′, Y′ A′ to meet A Y, A X in the points C, B.

Let the coordinates of P when referred to A X, A Y be x, y, and when referred to A′ X′, A′ Y′, x', y'; also assume A B = a, B A′ = b.

Then M A = M B + B A
= M′A′ + B A,

or $x = x' + a$ \
Similarly, $y = y' + b$ } .... (1.)

Hence if $f\,(x, y) = 0$ be the equation to the poiot P when referred to the old axes, we have only to substitute in it for x and y the values just obtained, aod we shall have the equation to P when referred to the new axes.

The signs of a and b will depend on the position of the new origio A′; this circumstance being attended to, the formulas (1) are quite general.

(26.) *The position of a point with respect to any system of axes being known, to find its position when referred to any other system originating at the same point with the former.*

Let A X, A Y be the primitive,

Fig. 13. A X′, A Y′, the new axes, (fig. 13;)

x, y the coordinates of the given point when referred to the former, x', y' its coordinates wheo referred to the latter.

It may be remarked in general, that whatever be the axes to which a curve is referred, the nature of that curve must remain unchanged; since the object for which axes are employed is merely to determine the relative position of the points of any line. Hence it is evident, that in passing from one system of coordinates to another, the new ones must be *linear functions* of the old ones; for, otherwise, the degree of the equation by which the curve is represented, and therefore the nature of the curve itself, would be altered.

We shall assume, therefore, that the relation between the old and new coordinates may be thus expressed,

$$\left.\begin{aligned} x &= m\,x' + n\,y' \\ \text{and}\quad y &= m'x' + n'y' \end{aligned}\right\} \ldots (1.)$$

m, n and m', n' being independent of either system of coordinates.

In order, therefore, to determine these quantities: Let $y' = 0$, in which case the point will be situated oo A X′, as at P; draw P M parallel to A Y.

$$\text{Theo } x = m\,x', \text{ or } m = \frac{x}{x'} = \frac{\text{A M}}{\text{A P}} = \frac{\sin x', y}{\sin x, y}$$

$$\text{and } m' = \frac{y}{x'} = \frac{\text{P M}}{\text{A P}} = \frac{\sin x', x}{\sin x, y}.$$

In like manoer, by supposing $x = 0$, we obtain

$$n = \frac{\sin y', y}{\sin x, y}, \text{ and } n' = \frac{\sin y', x}{\sin x, y};$$

hence, substituting io (1) for m, n and m', n' these values, we have

$$x = \frac{1}{\sin x, y,}\,\{\,x' \sin x', y + y' \sin y', y\,\}$$

$$y = \frac{1}{\sin x, y,}\,\{\,x' \sin x', x + y' \sin y', x\,\}.$$

If, therefore, these values of x and y be substituted in $f\,(x, y) = 0$, the equation to the point P wheo referred to the new system of axes will be found.

The general problem being thus resolved, we shall consider the following particular cases:

1. Let the primitive axes be rectangular, aod tha new ones oblique.

Then $\sin x, y = 1$

$$\sin x', y = \sin\left(\frac{\pi}{2} - x', x\right) = \cos x', x$$

$$\sin y'\,y = \sin\left(\frac{\pi}{2} + y', x\right) = \cos y', x;$$

therefore the formulas to be used in this case, are

$$x = x' \cos x', x + y' \cos y', x$$
$$y = x' \sin x', x + y' \sin y', x.$$

2. Let both systems be rectangular.

Then these formulas become

$$x = x' \cos x', x - y' \sin x', x$$
$$y = x' \sin x, x + y' \cos x', x.$$

3. Let the primitive oxes be oblique, and the new ones rectaogular.

Then $\sin y', y = \cos x', y$
$\sin y', x = \cos x', x$;

therefore the general formulas become

$$x = \frac{1}{\sin x, y}\,\{\,x' \sin x', y + y' \cos x', y\,\}$$

$$y = \frac{1}{\sin x, y}\,\{\,x' \sin x', x + y' \cos x', x\,\}.$$

If the origin and direction of the axes be both changed at the same time, we have only in the preceding formulas to add the new abscissa to the value of x, and the new ordinate to the value of y.

Analytical Geometry.

Part I.

ON THE CIRCLE.

(27.) *To find the equation to the circle*

Fig. 14. Let D P be a circle, (fig. 14,) P any point in its circumference; and let it be referred to the rectangular axes A X and A Y.

Let A B, B C be the coordinates of the centre C, and A N, N P those of the point P.

Let $AB = x'$, $BC = y'$

$AN = x$, $NP = y$, and $CP = r$;

then $$CP^2 = CM^2 + MP^2$$
$$= (AN - AB)^2 + (PN - BC)^2,$$

therefore, by substitution,

$$r^2 = (x - x')^2 + (y - y')^2,$$

which is the equation required.

(28.) This equation may be simplified as follows:

1. When the axis of x passes through the centre, then $y' = 0$, and the equation is

$$y^2 + (x - x')^2 = r^2.$$

Similarly, when the axis of y passes through the centre,

$$x^2 + (y - y')^2 = r^2.$$

2. When the origin is on the circumference, then $x'^2 + y'^2 = r^2$, and the equation therefore becomes

$$x^2 + y^2 - 2xx' - 2yy' = 0.$$

3. When the origin is on the circumference, and either axis passes through the centre,

$$x^2 + y^2 - 2rx = 0,$$

or $$x^2 + y^2 - 2ry = 0.$$

4. When the origin is at the centre, x' and y' both $= 0$;

$$\therefore x^2 + y^2 = r^2.$$

(29.) The general form of the equation to the circle when referred to rectangular coordinates is

$$x^2 + y^2 + Ax + By + C = 0.$$

Let it now be required to assign the *position* and *magnitude* of the circle to which this belongs.

Comparing it with the general equation

$$(x - x')^2 + (y - y')^2 = r^2,$$

that is, with

$$x^2 + y^2 - 2xx' - 2yy' + x'^2 + y'^2 - r^2 = 0$$

we have

$$\left.\begin{aligned} A &= -2x', \quad \text{or } x' = -\frac{A}{2} \\ B &= -2y', \quad y' = -\frac{B}{2} \end{aligned}\right\};$$

therefore the coordinates of the centre, in other words, the *position* of the circle, is known.

Again, $C = x'^2 + y'^2 - r^2$,

$$\therefore r^2 = x'^2 + y'^2 - C = \tfrac{1}{4}(A^2 + B^2 - 4C),$$

$$\therefore r = \tfrac{1}{2}\sqrt{A^2 + B^2 - 4C},$$

therefore the value of the radius, or the *magnitude* of the circle C is found.

Example.

Find the position and magnitude of the circle whose equation is

$$y^2 + x^2 + 2y - \frac{3}{2}x - \frac{1}{2} = 0.$$

Comparing this with

$$y^2 + x^2 - 2yy' - 2xx' + x'^2 + y'^2 - r^2 = 0$$

We have $$y' = -1$$
$$x' = \frac{3}{4}$$
$$x'^2 + y'^2 - r^2 = -\frac{1}{2},$$
$$\therefore 1 + \frac{9}{16} + \frac{1}{2} = r^2,$$
$$\therefore r = \frac{\sqrt{33}}{4}$$

Assume A X, A Y as axes, (fig. 15,) $AB = \frac{3}{4}$ and $BC = -1$, then from C as centre with rad = the nearest whole number to $\frac{\sqrt{33}}{4}$ describe a circle, and it will be the circle required. Fig. 15

(30.) The general equation of the second degree between two variables is

$$Ay^2 + Bx^2 + Cxy + Dy + Ex + F = 0,$$

which differs from the general equation to the circle. It will afterwards become an important inquiry, to ascertain what class of curves is represented by that equation.

(31.) *To find the polar equation to the circle.*

Let any point S within the circle be assumed as the pole, and draw through it S Z parallel to C X, meeting M P in N, and let the ordinate of S meet C X in Q.

Let $SP = \rho$, angle $PSZ = \omega$, and the coordinates of S, a and b.

Then $$x = a + \rho\cos\omega,$$

and $$y = b + \rho\sin\omega.$$

Therefore by substitution of these values in the equation

$$x^2 + y^2 = r^2,$$

there results

$$(a + \rho\cos\omega)^2 + (b + \rho\sin\omega)^2 = r^2,$$

$$\therefore \rho^2 + 2(a\cos\omega + b\sin\omega)\rho + a^2 + b^2 - r^2 = 0,$$

which is the equation required.

If the point S be without the circle, then the polar equation is

$$\rho^2 - 2(a\cos\omega + b\sin\omega)\rho + a^2 + b^2 - r^2 = 0.$$

(32.) *To find the equation to a tangent applied at a given point (x', y') of a circle.*

Let the equation to the circle be $x^2 + y^2 = r^2$.

Now the equation to any line is

$$p = x\cos\rho, x + y\cos\rho, y.$$

Let the line touch the circle at the point (x', y') then $p = r$, and

$$\cos\rho, x = \frac{x'}{r} \text{ and } \sin\rho, x = \frac{y'}{r},$$

$$\therefore r = \frac{xx'}{r} + \frac{yy'}{r}, \text{ or}$$

$$xx' + yy' = r^2,$$

which is the equation required.

Analytical Geometry.

(33.) If the given point be without the circle, let x_0, y_0 be the unknown coordinates of the point of contact.

Then since the point (x_0, y_0) is on the circumference

$$x_0^2 + y_0^2 = r^2 \ldots . \ (1.)$$

and since the point (x', y') is on the tangent

$$x' x_0 + y' y_0 = r^2 \ \ldots (2.)$$

therefore x_0, y_0 may be found by elimination between these two equations. This process, however, which is tedious, may be superseded by an operation founded on the principle, that *elimination between any two equations corresponds to the intersection of the geometrical loci which they represent.*

The points of contact are therefore determined by the intersection of the loci whose equations are (1) and (2). But the locus of (1) is the given circle, and the locus of (2) is a straight line; and since the points in which it meets the circle are the points of contact, *equation* (2) *must be the equation to the line joining those points.* Its position is thus found:

Fig. 17. Let $x_0 = 0, \therefore y_0 = \dfrac{r^2}{y'} = \text{A C}$, (fig. 17.)

$$y_0 = 0, \therefore x_0 = \frac{r^2}{x'} = \text{A B}.$$

Join B, C meeting the circle in Q, P; these are the points of contact required.

(34.) The points of contact may be found in a different manner, as follows:

Subtracting (2) from (1) we have

$$y_0^2 - y_0 y' + x_0^2 - x_0 x' = 0 \ldots . \ (3.)$$

which (art. 28) is the equation to a circle, the coordinates of whose centre are $\dfrac{x'}{2}, \dfrac{y'}{2}$, and whose radius $= \frac{1}{2}\sqrt{x'^2 + y'^2}$. Hence the locus of (3) is the equation described on C T as a diameter; and its intersection with the given circle determines the points of contact.

This is the construction of Euclid, iii. 17.

(35.) *To find the equation of a common tangent to two circles.*

Let δ be the distance between the centres of the two circles, r and r' their radii, and suppose the axis of x to pass through the centres of both circles.

Then the equations to the circles are,

$$x'^2 + y'^2 = r^2 \ldots . \ (1.)$$

$$(x'' - \delta)^2 + y''^2 = r'^2 \ldots . \ (2.)$$

The equation of the tangent to the first is

$$x x' + y y' = r^2 \ldots . \ (3.)$$

and in order that this line may touch the second circle also, the perpendicular dropped upon it from the centre must $= r'$.

Now the length of this perpendicular $= -\dfrac{a\delta - b}{\sqrt{1 + a^2}}$

but $a = \dfrac{-x'}{y'}$ and $b = \dfrac{r^2}{y'}$,

$$\therefore p = -\frac{\delta x' - r^2}{\sqrt{x'^2 + y'^2}} = \therefore -\frac{\delta x' - r^2}{r},$$

$$\therefore r' = -\frac{\delta x' - r^2}{r} \therefore r r' = -\delta x' + r^2,$$

Part I.

$$\therefore x' = \frac{r}{\delta}(r - r') \quad \therefore y = \frac{\delta r - (r - r')x}{\sqrt{\{\delta^2 - (r - r')^2\}}},$$

hence by substitution in (3) $\dfrac{r x}{\delta}(r - r') + y y' = r^2$,

which is the equation required.

(36.) If the axes to which the circle is referred be inclined at any angle whatever to each other, then

1. The general equation is

$$r^2 = (x - x')^2 + 2(x - x')(y - y') \cos x, y + (y - y')^2,$$

and when the centre is the origin,

$$r^2 = x^2 + 2 x y \cos x, y + y^2.$$

2. The equation to the tangent drawn at a given point (x', y') of the circumference is

$$\{y' + x' \cos x, y\}\, y + \{x' + y' \cos x, y\}\, x = r^2,$$

the origin being at the centre.

ON LINES OF THE SECOND ORDER.

(37.) The general equation of the second degree between two variables is,

$$a y^2 + b x y + c x^2 + d y + e x + f = 0,$$

in which $a, b, c \ldots$ are independent of x and y.

The locus of this equation is called a *line of the second order.* In the following investigations we shall use oblique axes, unless the contrary be specified.

The characteristic property of a line of the second order is, that *a straight line cannot intersect it in more than two points.* To prove this,

Let the curve be supposed to be cut by a straight line whose equation is

$$y = m x + n \ldots . \ (1.)$$

then the points of intersection will be determined by eliminating y between this equation and the general equation

$$a y^2 + b x y + c x^2 + d y + e x + f = 0 \ldots . \ (2.)$$

Hence, substituting in (2) the value of y derived from (1.) we have

$$a (m x + n)^2 + b x (m x + n) + c x^2 + d (m x + n) + e x + f = 0,$$

or developing the terms, and arranging the result according to the powers of x,

$$(a m^2 + b m + c) x^2 + \{(2 a m + b) n + d m + e\} x + a n^2 + d n + f = 0.$$

This equation being of the second degree can have only two roots, which, when real, represent the abscissas of the points of intersection. Whence it follows, that a straight line cannot cut the curve in *more* than two points.

If the roots be imaginary, the straight line does not meet the curve; if they be equal, the two points of section coincide, and the line *touches* the curve.

Definition. A straight line being supposed to cut a line of the second order, the portion of it contained within the curve is called a *chord.*

(38.) *To find the locus of the middle points of any number of parallel chords.*

Let Q P q (fig. 18) be conceived to represent a portion of a line of the second order; and let it be referred to any oblique system of axes A X, A Y. Fig. 18.

Through the origin draw any line A P p, cutting the curve in P, p; then its equation will be of the form

$$y = m x \ldots . (1.)$$

Let Q q be any chord parallel to A P p, bisect it in O, and draw O M parallel to A Y.

Then the object of the problem is to determine the relation between A M and M O, the coordinates of the point O.

Assume $\quad$ A M $= x'$, M O $= y'$.

Let the origin be now transferred to O, in which case we shall have to substitute $x + x'$ and $y + y'$ for x and y in the general equation; we have therefore

$$a (y + y')^2 + b (x + x') (y + y') + c (x + x')^2$$
$$+ d (y + y') + e (x + x') + f = 0 \ldots . (2,)$$

which is the equation of the curve.

Now when the origin is at A the equation of Q q which is drawn parallel to A P p is $y = m x + n'$; but when the origin is removed to O, the equation of Q q will (Art. 6) be

$$y = m x.$$

Hence, the points in which Q q intersects the curve will be found by eliminating y between this and equation (2,) whence we have

$$a (m x + y')^2 + b (x + x') (m x + y') + c (x + x')$$
$$+ d (m x + y') + e (x + x') + f = 0,$$

which becomes on reduction $(a m^2 + b m + c) x^2$

$$+ \{ (2 a m + b) y' + (b m + 2 c) x' + d m + e \} x$$
$$+ a y'^2 + b x' y' + e x'^2 + d y' + e x' + f = 0.$$

But since Q q is bisected in O, it is plain that the roots of this equation are equal, with contrary signs; therefore the coefficient of the second term must $= 0$.

Hence, suppressing the accents which were only employed to distinguish the coordinates of O from those of any point whatever, we have

$$(2 a m + b) y + (b m + 2 c) x + d m + e = 0 \ldots (3.)$$

The relation between x and y being thus expressed by an equation of the first degree, it follows, that the locus of the point O is a *straight line*.

The straight line which bisects any number of parallel chords is called a *diameter*, and each of the points in which it meets the curve is called a *vertex*.

(39.) *Cor.* If the equation to any other chord be

$$y = m' x,$$

then the equation to the corresponding diameter will be

$$(2 a m' + b) y + (b m' + 2 c) x + d m' + e = 0.$$

Fig 19. Draw any two chords $m\,n$, $p\,q$, (fig. 19,) and their corresponding diameters M N, P Q; then if either chord be parallel to the diameter of the other, reciprocally the diameter of the first will be parallel to the chord of the second.

For if $y = m x + n$ be the equation of $m\,n$,

and $\quad y = m' x + n'$ that of $p\,q$,

then $$y = -\frac{b m + 2 e}{2 a m + b} x - \frac{d m + e}{2 a m + b}$$

will be the equation of M N, and

$$y = -\frac{b m' + 2 c}{2 a m' + b} x - \frac{d m' + a}{2 a m' + b}$$

that of P Q.

Let $m\,n$ be now supposed parallel to P Q,

then $$m = -\frac{b m' + 2 c}{2 a m' + b},$$

$$\therefore 2 a m m' + b m = - b m' - 2 c,$$

or $$m' = -\frac{b m + 2 c}{2 a m + b},$$

whence $p\,g$ is parallel to M N, (Art. 15.)

In like manner, if $p\,q$ be supposed parallel to M N, it may be shown that P Q will be parallel to $m\,n$.

Whence it appears that *each diameter bisects the chords drawn parallel to the other.* Diameters thus related to each other are called *conjugate diameters.*

If $y = m x + n$ be any diameter, the equation of the diameter conjugate to it is

$$y = -\frac{b m + 9 c}{2 a m + b} x - \frac{d m + e}{2 a m + b},$$

whence it is evident that an infinite number of pairs of conjugate diameters can be drawn.

We shall now investigate whether any of these systems can be at right angles to each other.

Suppose, for the sake of simplicity, that the axes are rectangular, and let

$$y = m x + n$$
$$y = m' x + n'$$

be any system of conjugate diameters.

Then $$m' = -\frac{b m + 2 c}{2 a m + b},$$

and since the conjugate diameters are by hypothesis at right angles to each other,

$$m' = -\frac{1}{m}, \text{ (Art. 14,)}$$

$$\therefore \frac{b m + 2 c}{2 a m + b} = \frac{1}{m},$$

$$\therefore b m^2 + 2 c m = 2 a m + b,$$

$$\therefore m^2 + 2 \frac{c - a}{b} m = 1,$$

$$\therefore m = -\frac{c - a}{2 b} \pm \sqrt{1 + \left(\frac{c - a}{2}\right)^2},$$

a quantity which is manifestly always real.

Let m and μ be the two roots of this equation; then, since its last term $= -1$,

$$m \times \mu = -1,$$

$$\therefore \mu = -\frac{1}{m} = \therefore m',$$

hence it appears that m and m' are the roots of the same quadratic: *wherefore there can be only one system of rectangular conjugate diameters.*

These are called the *principal diameters.*

(40.) *To find the form which the equation to lines of the second order assumes, when the axes of coordinates are parallel to a system of conjugate diameters.*

Let $y = m x + n$ be the equation of any chord, then

$$y = -\frac{b m + 2 e}{2 a m + b} x - \frac{d m + e}{2 a m + b}$$

will be the equation to its corresponding diameter.

Suppose now that the chord is parallel to the axis of x; then $m = 0$, and the equation of the diameter becomes

$$y = -\frac{2e}{b}x - \frac{e}{b} \dots (1.)$$

Again, let the chord be parallel to the axis of y, then m is ∞, and the equation of the corresponding diameter is

$$y = -\frac{b}{2a}x - \frac{d}{2a} \dots (2.)$$

Hence, when these diameters are conjugate to each other, and the axes are parallel to them, the first will bisect the chords parallel to A X, and ought therefore to involve x alone; and the second ought, for a like reason, to involve y alone; therefore in each case b must equal 0.

Hence, when the axes of coordinates are parallel to a system of conjugate diameters, the coefficient of the second term vanishes, and the general equation assumes the form

$$a y^2 + c x^2 + d y + e x + f = 0.$$

(41.) *To find the coordinates of the centre.*

The centre being the point in which any two diameters cut each other, we have, eliminating y between (1) and (2,) in Art. 40,

$$-\frac{2c}{b}x - \frac{e}{b} = -\frac{b}{2a}x - \frac{d}{2a},$$

or,
$$\frac{b^2 - 4ac}{2ab}x = \frac{2ae - bd}{2ab}$$

$$\therefore x = \frac{2ae - bd}{b^2 - 4ae},$$

and similarly,
$$y = \frac{2cd - be}{b^2 - 4ac}.$$

Cor. If the axes be parallel to a system of conjugate diameters, then $b = 0$, and

$$x = \frac{2ae}{-4ac} = -\frac{e}{2c},$$

$$y = \frac{2cd}{-4ac} = -\frac{d}{2a}.$$

(42.) *Let the origin be now transferred to a point* (α, β), *which is done by substituting* $x + \alpha$ *and* $y + \beta$ *for* x *and* y *in the equation,*

$$a y^2 + c x^2 + d y + e x + f = 0.$$

Then

$$a(y+\beta)^2 + c(x+\alpha)^2 + d(y+\beta) + e(x+\alpha) + f = 0,$$

therefore, developing, and arranging the result,

$$a y^2 + c x^2 + (2a\beta + d)y + (2c\alpha + e)x + a\beta^2 + c\alpha^2 + d\beta + e\alpha + f = 0.$$

Now since α, β are arbitrary quantities, we may fix their value by making the coefficients of y and $x = 0$, we thus have

$$2a\beta + d = 0, \quad 2c\alpha + e = 0,$$

$$\therefore \alpha = -\frac{e}{2c},$$

and
$$\beta = -\frac{d}{2a},$$

which (41, *Cor.*) are the coordinates of the centre.

Hence the general equation is reducible to the form

$$a y^2 + c x^2 + f' = 0 \dots (1.)$$

This reduction is only practicable on the supposition that the equation contains both the terms involving $a y^2$ and $c x^2$; for if either of them, as

$$c x^2 \text{ be } = 0, \text{ then } \alpha = -\frac{e}{0} = \infty,$$

and the term $e x$ cannot be taken away, and the equation therefore assumes the form

$$a y^2 + e x + f = 0.$$

Now, by taking away the term involving y, we have determined only one of the quantities α, β; we may fix the value of the second, α, by supposing the last term to $= 0$. This supposition is always possible, because $c\alpha^2$ vanishing, the last term is only of one dimension in α.

The equation thus reduced will be of the form

$$a y^2 + e x = 0 \dots (2.)$$

Hence, *Lines of the second order are divisible into two classes, according as they have or have not, a centre, the corresponding equations being*

$$a y^2 + c x^2 = \text{F},$$

and
$$a y^2 + e x = 0.$$

(43.) In the first of these equations the coefficients of y^2 and x^2 may have either the same or different signs, the constant quantity F being supposed indeterminate.

I. Let them have the same signs, and

1. Let both be positive.

Then, according as F is negative or positive,

$$\text{A} y^2 + \text{C} x^2 = \text{F} \dots (1.)$$

or
$$\text{A} y^2 + \text{C} x^2 = -\text{F} \dots (2.)$$

but since the sum of two quantities essentially positive cannot equal a negative quantity, the line represented by this equation must be imaginary.

2. Let both be negative, then

$$-\text{A} y^2 - \text{C} x^2 = \text{F},$$

and
$$-\text{A} y^2 - \text{C} x^2 = -\text{F};$$

therefore, changing the signs of the terms in each equation

$$\text{A} y^2 + \text{C} x^2 = -\text{F},$$
$$\text{A} y^2 + \text{C} x^2 = \text{F},$$

which are identical with (1) and (2.)

II. Let them have different signs, and

1. Let A be positive and C negative.

Then
$$\text{A} y^2 - \text{C} x^2 = \text{F} \dots (3.)$$

and
$$\text{A} y^2 - \text{C} x^2 = -\text{F} \dots (4.)$$

2. Let A be negative and C positive.

Then
$$\text{C} x^2 - \text{A} y^2 = \text{F},$$

and
$$\text{C} x^2 - \text{A} y^2 = -\text{F};$$

or, changing the signs in both equations,

$$\text{A} y^2 - \text{C} x^2 = -\text{F},$$
$$\text{A} y^2 - \text{C} x^2 = \text{F},$$

which coincide respectively with (4) and (3.)

Lines of the first class, therefore, may be subdivided into two species.

The first, represented by the equation

$$A y^2 + C x^2 = F,$$

is called the *Ellipse*.

The second, represented by the equation

$$A y^2 - C x^2 = F,$$

or $$C x^2 - A y^2 = F,$$

is called the *Hyperbola*.

It hence appears, that the equation to the hyperbola is deduced from that to the ellipse by changing the sign of x^2 or of y^2.

ON THE ELLIPSE.

(44.) *To find the equation to the ellipse, in terms of a given system of conjugate diameters.*

Fig. 20. Let CP, CD be the given semi-conjugate diameters, (fig. 20,) and the ellipse be referred to these as axes.

Let $CP = a'$, $CD = b'$.

Then the general equation being

$$A y^2 + C x^2 = F.$$

$$\text{Let } y = 0, \therefore x^2 = \frac{F}{C} = CP^2 = \therefore a'^2, \therefore C = \frac{F}{a'^2},$$

$$x = 0, \therefore y^2 = \frac{F}{A} = CD^2 = \therefore b'^2, \therefore A = \frac{F}{b'^2};$$

therefore, substituting these values of A and C in the above equation, and dividing by F, we have

$$\frac{y^2}{b'^2} + \frac{x^2}{a'^2} = 1 \ldots (1.)$$

or $$a'^2 y^2 + b'^2 x^2 = a'^2 b'^2 \ldots (2.)$$

either of which is the equation required.

(45.) *Cor.* 1. If the origin be transferred to P, we must substitute in (2) $a' - x$ for x; the equation therefore becomes

$$a'^2 y^2 + b'^2 x^2 - 2 a' b'^2 x = 0,$$

or $$y^2 = \frac{b'^2}{a'^2} (2 a' x - x^2) \ldots (3.)$$

(46.) *Cor.* 2. Let $2a$, $2b$ represent the principal diameters, then the equation of the ellipse becomes

1. When the centre is the origin,

$$\frac{y^2}{b^2} + \frac{x^2}{a^2} = 1 \ldots (1.)$$

or $$a^2 y^2 + b^2 x^2 = a^2 b^2 \ldots (2.)$$

2. When the extremity of $2a$ is the origin,

$$y^2 = \frac{b^2}{a^2} (2ax - x^2) \ldots (3.)$$

(47.) *To find the equation to the tangent drawn at a given point (x', y') in the ellipse.*

If a straight line be drawn cutting the ellipse, and the two points of section be then supposed to coincide, the secant will become a tangent.

Now the equation to a secant drawn through the given point is

$$y - y' = m (x - x') \ldots (1.)$$

But x', y' being the coordinates of a point in the curve,

$$a^2 y'^2 + b^2 x'^2 = a^2 b^2;$$

and, in general,

$$a^2 y^2 + b^2 x^2 = a^2 b^2,$$

$$\therefore a^2 (y^2 - y'^2) + b^2 (x^2 - x'^2) = 0,$$

$$\therefore a^2 (y + y') (y - y') = b^2 (x - x') (x + x'),$$

or, substituting for $y - y'$ its value in (1,) and dividing each side by $x - x'$,

$$a^2 m (y + y') = b^2 (x + x').$$

If the points of section be now supposed to coincide, $x = x'$ and $y = y'$, and the secant becomes a tangent;

$$\therefore m = -\frac{b^2}{a^2} \cdot \frac{x'}{y'},$$

therefore, by substitution, the equation to the tangent becomes

$$y - y' = -\frac{b^2}{a^2} \cdot \frac{x'}{y'} (y - y') \ldots (2.)$$

or $$a^2 y y' + b^2 x x' = a^2 b^2 \ldots (3.)$$

(48.) *To determine the figure of the ellipse.*

Resuming the equation $a^2 y^2 + b^2 x^2 = a^2 b^2$, we have

$$y = \pm \frac{b}{a} \sqrt{a^2 - x^2}.$$

Now let $y = 0$, $\therefore$ (fig. 21) $x = \pm a = CA$ or CV, Fig. 21.

$x = 0$, $\therefore y = \pm b = CB$ or Cb.

So long as x remains positive, and increases from 0 to a, y is real, and decreases from b to 0.

When $x > a$, the values of y are imaginary, and the curve therefore extends to the right no farther than A.

Let x be negative, then since x^2 is positive, it may in like manner be proved that the curve does not extend beyond V to the left.

Again, $$x = \pm \frac{a}{b} \sqrt{(b^2 - y^2)},$$

and by a process similar to that which has just been followed, it may be shown, that the curve does not extend beyond B or b.

Hence the ellipse has the form assigned to it in the figure, and is wholly contained within the parallels M N, P Q and M P, N Q.

Of the two principal diameters A V, B b, the former is commonly called the *major*, the latter the *minor*, axis.

(49.) *Definition.* The *focus* is a point in the major axis, such that its distance from any point in the ellipse is a rational function of the abscissa.

To determine the focus.

The curve being represented by the equation

$$a^2 y^2 + b^2 x^2 = a^2 b^2.$$

Let the abscissa of the focus be x', its ordinate being necessarily $= 0$.

Then if r denote the distance of the focus from any point (x, y) of the curve, we shall have

$$r^2 = (x - x')^2 + y^2,$$

$$= x^2 - 2 x x' + x'^2 + \frac{b^2}{a^2} (a^2 - x^2),$$

$$= x^2 - 2xx' + x'^2 + b^2 - \frac{b^2}{a^2}x^2,$$

$$= \frac{a^2 - b^2}{a^2}x^2 - 2xx' + x'^2 + b^2.$$

Now in order that r may be rational, the quantity on the right must be a perfect square; therefore we have

$$4\frac{a^2 - b^2}{a^2}x^2(b^2 + x'^2) = 4x^2x'^2,$$

or $\quad (a^2 - b^2)(b^2 + x'^2) - a^2x'^2,$

$$\therefore a^2b^2 + a^2x'^2 - b^4 - b^2x'^2 = a^2x'^2,$$

$$\therefore (a^2 - b^2)b^2 = b^2x'^2,$$

$$\therefore x' = \pm\sqrt{a^2 - b^2}.$$

Whence there are *two foci*, on opposite sides of the centre, and equidistant from it by the quantity $\sqrt{a^2 - b^2}$, which is called the *eccentricity*

Fig. 22. Assume $\sqrt{a^2 - b^2} = a e = \mathrm{CS}$ or CH, then S and H are the foci, (fig. 22.)

Also $\quad \mathrm{SP} = a - e x.$

Similarly, if H be the other focus,

$$\mathrm{HP} = a + e x.$$

Hence $\quad \mathrm{SP} + \mathrm{PH} = 2a;$

or, the distances of any point in the curve from the foci are together equal to the major axis.

Cor. Since $\quad a e = \sqrt{a^2 - b^2},$

$$a^2e^2 = a^2 - b^2,$$

$$\therefore \frac{b^2}{a^2} = 1 - e^2,$$

therefore the equation to the ellipse becomes, by substitution, $\quad y^2 = (1 - e^2)(a^2 - x^2).$

(50.) *To find the value of the ordinate passing through the focus.*

In general $\quad y^2 = \frac{b^2}{a^2}(a^2 - x^2) \ldots\ldots (1.)$

but $\quad x^2 = a^2 - b^2,$

$$\therefore y^2 = \frac{b^2}{a^2}\{a^2 - (a^2 - b^2)\}$$

$$= \frac{b^4}{a^2},$$

$$\therefore y = \pm\frac{b^2}{a},$$

twice this quantity, or $\frac{2b^2}{a}$ is called the *principal parameter*; let it be denoted by $2p$, then the equation to the ellipse in terms of its principal parameter becomes by substitution in equation (1)

$$y^2 = 2px - \frac{p}{a}x^2.$$

(51.) *To find the polar equation to the ellipse.*

The pole may either be the *centre* or *one of the foci.*

1. Let it be the centre.

Assume $\mathrm{CP} = \rho$, angle $\mathrm{PCA} = \omega$, (fig. 22.)

Then $\quad \rho^2 = x^2 + y^2,$

$$= x^2 + (1 - e^2)(a^2 - x^2) \text{ (Art. 49, Cor.)}$$

$$= e^2x^2 + a^2(1 - e^2),$$

but $\quad x = \rho\cos\omega,$

$$\therefore \rho^2 = e^2\rho^2\cos^2\omega + a^2(1 - e^2),$$

$$\therefore \rho = a\sqrt{\frac{1 - e^2}{1 - e^2\cos^2\omega}}.$$

2. Let the pole be the focus S.

Assume $\quad \mathrm{SP} = r$, angle $\mathrm{PSA} = \nu.$

Then $\quad r = a - e x$, (Art. 49.)

but $\quad x = \mathrm{CM} = \mathrm{CS} - \mathrm{SM},$

$$= a e + r\cos\nu,$$

$$\therefore r = a - e(a e + r\cos\nu),$$

$$\therefore r(1 + e\cos\nu) = a(1 - e^2),$$

$$\therefore r = a\frac{1 - e^2}{1 + e\cos\nu} \ldots\ldots (1.)$$

Similarly, if H be the pole, and $\mathrm{HP} = r'$, angle $\mathrm{PHA} = \nu'$, then

$$r' = a\frac{1 - e^2}{1 - e\cos\nu'}.$$

ON THE HYPERBOLA.

(52.) The same notation being retained, the equations to the hyperbola, deduced from the corresponding equations to the ellipse, are

I. When the axes are a given system of conjugate diameters,

$$\left.\begin{aligned} \frac{y^2}{b'^2} - \frac{x^2}{a'^2} &= -1 \\ \frac{x^2}{a'^2} - \frac{y^2}{b'^2} &= -1 \end{aligned}\right\}\ldots\ldots (1.)$$

or $$\left.\begin{aligned} a'^2y^2 - b'^2x^2 &= -a'^2b'^2 \\ b'^2x^2 - a'^2y^2 &= -a'^2b'^2 \end{aligned}\right\}\ldots\ldots (2.)$$

$$y^2 = \frac{b'^2}{a'^2}(x^2 - 2a'x)\ldots\ldots (3.)$$

II. When the axes are the principal diameters,

$$\left.\begin{aligned} \frac{y^2}{b^2} - \frac{x^2}{a^2} &= -1 \\ \frac{x^2}{a^2} - \frac{y^2}{b^2} &= -1 \end{aligned}\right\}\ldots\ldots (1'.)$$

or $$\left.\begin{aligned} a^2y^2 - b^2x^2 &= -a^2b^2 \\ b^2x^2 - a^2y^2 &= -a^2b^2 \end{aligned}\right\}\ldots\ldots (2'.)$$

and $\quad y^2 = \frac{b^2}{a^2}(x^2 - 2ax)\ldots\ldots (3'.)$

III. The equation to the tangent, applied at a given point (x', y') of the hyperbola, is

$$a^2yy' - b^2xx' = -a^2b^2.$$

(53.) *To determine the figure of the hyperbola.*

Taking the first of equations (2')

$$a^2y^2 - b^2x^2 = -a^2b^2,$$

we have $$y = \pm \frac{b}{a} \sqrt{(x^2 - a^2)},$$

Fig. 23. (fig. 23.)

Let $y = 0, \therefore x = \pm a = \text{C A or C V},$

$x = 0, \therefore y = \pm b \sqrt{-1}.$

Hence it is evident that the axis of y can never meet the curve.

Let $x < a$, then the values of y being still imaginary, no part of the curve can lie between C and A.

Let $x > a$; the values of y are now real, and to each assumed value of x there correspond two equal values of y with opposite signs.

As x increases, y also increases; when x is supposed infinite, the values of y are also infinite. Hence, to the right of C the curve extends indefinitely, and consists of two branches A Z, A z symmetrically placed with respect to the axis.

In the same manner it may be shown, by supposing x to be negative, that to the left of C the curve has two infinite branches V Z', V z'.

Again, taking the second equation,

$$b^2 x^2 - a^2 y^2 = - a^2 b^2,$$

we have

$$x = \pm \frac{a}{b} \sqrt{y^2 - b^2}.$$

Fig. 24. Let $x = 0, \therefore y = \pm b = \text{C B, or C} b$, (fig. 24.)

$$y = 0, \therefore x = \pm a \sqrt{-1}.$$

Hence it is evident that the axis of x cannot meet the curve.

Nor can any part of the curve be situated between B and b; for so long as y is less than b, the values of x are imaginary.

The investigation being conducted as in the last case, it will be found that there are two infinite branches B U, B u; b U', bu', on each side of the centre, symmetrically situated with respect to the axis of y.

The two hyperbolas represented by the figures 23 and 24, are said to be *conjugate* to each other.

Since the line B b, in the first case, and A V in the second, never intersect the curve, they cannot, correctly speaking, be called *diameters*. They are so named in order that the analogy between the hyperbola and ellipse may be preserved.

(54.) *To find the coordinates of the points in which any diameter meets the curve.*

Let the equation to any diameter be

$$y = m x,$$

and that to the curve

$$a'^2 y^2 - b'^2 x^2 = - a'^2 b'^2,$$

then, by elimination,

$$a'^2 m^2 x^2 - b'^2 x^2 = - a'^2 b'^2,$$

$$\therefore x^2 = \frac{a'^2 b'^2}{b'^2 - a'^2 m^2},$$

$$\therefore x = \pm \frac{a' b'}{\sqrt{b'^2 - a'^2 m^2}},$$

and $$\therefore y = \pm \frac{m a' b'}{\sqrt{b'^2 - a'^2 m^2}}.$$

So long, therefore, as $b'^2 - a'^2 m^2$ is positive, the diameter meets the curve; if $b'^2 < a'^2 m^2$, or $m > \frac{b'}{a'}$, the diameter does not meet it; if $b'^2 = a'^2 m^2$, or $m = \frac{b'}{a'}$, the diameter intersects the curve at an infinite distance from the centre.

Let P p, D d (fig. 25) be any two conjugate diameters to which the curve is referred as axes; through P draw Q q parallel to and equal to D d; join C, Q: C q; then the lines C Q, C q being produced to Z, z will meet the hyperbola at an infinite distance. Fig. 25.

The lines C Z, C z are called *asymptotes*; and their equation is $$y = \pm \frac{b'}{a'} x.$$

The asymptotes may be considered as separating those diameters which meet the curve from those which never meet it.

(55.) It may be proved, as in the ellipse, that there are two foci S, H situated on the transverse axis at a distance $= \sqrt{a^2 + b^2}$ from the centre. And in like manner it may be shown,

1. That $\text{S P} = e x - a,$

$\text{H P} = e x + a,$

and therefore *that the difference of the focal distances equal the transverse axis.*

2. That the polar equations of the hyperbola are

(1.) When the centre is the pole,

$$\rho = a \sqrt{\frac{e^2 - 1}{e^2 \cos^2 \omega - 1}}.$$

(2.) When the focus S is the pole,

$$r = a \frac{e^2 - 1}{1 + e \cos \nu}.$$

(3.) When the focus H is the pole,

$$r = - a \frac{e^2 - 1}{1 - e \cos \nu}.$$

The equation to the hyperbola in terms of its principal parameter is

$$y^2 = 2 p x + \frac{p}{a} x^2.$$

ON THE PARABOLA.

(56.) The equation to Lines of the second order, when the centre is infinitely distant, is

$$a y^2 + e x = 0,$$

or $y^2 = m x$; if m be taken $= - \frac{e}{a}$.

The curve which is the locus of this equation is called the *parabola.*

(57.) *To find the equation to the tangent drawn at a given point (x', y') of the parabola.*

If a straight line be drawn cutting the parabola, and the two points of section be then supposed to coincide, the secant will become a tangent

Now the equation to a secant drawn through the given point is

$$y - y' = a\,(x - x') \ldots\ldots (1.)$$

But x', y' being the coordinates of a point in the curve, $y'^2 = m\,x'$;

and, in general, $y^2 = m\,x$,

$$\therefore y^2 - y'^2 = m\,(x - x'),$$

$$\therefore (y - y')\,(y + y') = m\,(x - x');$$

or, substituting for $y - y'$ its value in (1,) and dividing each side by $x - x'$,

$$a\,(y + y') = m,$$

$$\therefore a = \frac{m}{y + y'}.$$

If the points of section be now supposed to coincide, $x = x'$ and $y = y'$, and the secant becomes a tangent; therefore, by substitution, the equation to the tangent becomes

$$y - y' = \frac{m}{2\,y'}\,(x - x') \ldots\ldots (2,)$$

or

$$y = \frac{m}{2\,y'}\,(x + x') \ldots\ldots (3.)$$

(58.) *To determine the figure of the parabola.*

Since $y^2 = m\,x$,

$$\therefore y = \pm\sqrt{m\,x},$$

therefore for each assumed value of x, there are two equal values of y with opposite signs, therefore the curve is divided into two equal parts by the axis A X, (fig. 26.)

Fig. 26.

Let $x = 0$, then $y = 0$,

therefore the curve passes through the origin A.

Let x be supposed to increase, then y also increases; let x become infinitely great, then y is infinitely great also.

Let x be negative, then y being imaginary, no part of the curve is situated to the left of A.

Hence the parabola consists of two infinite branches A Z, A z, symmetrically placed with respect to A X.

(59.) The focus being defined as in the ellipse and hyperbola, let it be required to find its position.

Let S be the focus, A S $= x'$, (fig. 26,) and let the coordinates of any point in the curve be x, y; then

$$r^2 = y^2 + (x - x')^2,$$

$$= m\,x + x^2 - 2\,x\,x' + x'^2,$$

$$= x^2 + (m - 2\,x')\,x + x'^2.$$

Now as this is to be a rational quantity, it must be a complete square;

$$\therefore 4\,x^2 x'^2 = x^2\,(m - 2\,x')^2,$$

$$\therefore 4\,x'^2 = (m - 2\,x')^2,$$

$$\therefore 2\,x' = m - 2\,x',$$

$$\therefore x' = \frac{m}{4}$$

Hence there is only *one* focus in the parabola.

Cor. 1. The distance of any point P from S $= x + \frac{m}{4}$.

Cor. 2. Let S L be perpendicular to A X, then since $y^2 = m\,x$, we have

$$\text{S L}^2 = \frac{m^2}{4},$$

$$\therefore \text{S L} = \frac{m}{2},$$

or

$$2\,\text{S L} = m;$$

the quantity 2 S L is called the *latus rectum*, or *principal parameter*.

(60.) *To find the polar equation to the parabola.*

Let $\text{A S P} = \omega$, $\text{S P} = r$,

then

$$r = x + \frac{m}{4},$$

$$= \frac{m}{4} - r\cos\omega,$$

$$\therefore r = \frac{\frac{1}{2}\,m}{1 + \cos\omega},$$

which is the equation required.

(61.) The parabola may be considered as a species of the ellipse, or hyperbola, and its equation deduced from that of either of these curves, by supposing the centre removed to an infinite distance.

Thus the equation of the ellipse and hyperbola, in terms of their principal parameters, is

$$y^2 = 2\,p\,x \mp \frac{p}{a}\,x^2.$$

Now

$$p = \frac{b^2}{a} = \frac{a^2 - a^2 e^2}{a} = \frac{(a + a\,e)\,(a - a\,e)}{a}.$$

But $a - a\,e =$ A S, and $a + a\,e = 2\,a$, when the centre is at an infinite distance,

$$\therefore p = \text{A S}\,.\,\frac{2\,a}{a} = 2\,\text{A S},$$

therefore, by substitution,

$$y^2 = 4\,\text{A S}\,.\,x \mp \frac{2\,\text{A S}}{a}\,x^2,$$

but a being infinitely great, $\frac{2\,\text{A S}}{a}$ is infinitely small, and may therefore be neglected,

$$\therefore y^2 = 4\,\text{A S}\,.\,x.$$

By comparing this with the equation $y^2 = m\,x$, it appears that the constant quantity m is equal to four times the distance of the vertex from the focus.

For the analytical investigation of the properties of Lines of the second order, the reader is referred to the works on *Analytical Geometry* enumerated at the end of this Article, and particularly to Dr. Lardner's *Algebraic Geometry*, vol. i., which contains a variety of Problems resolved with great elegance and simplicity.

PART II.

APPLICATION OF ALGEBRA TO THE THEORY OF SURFACES.

Analytical Geometry. (62.) The application of Algebra to the Theory of Surfaces is founded on this principle, *that an indeterminate equation between three variables may be represented by a geometrical locus, and conversely.*

Fig. 27. Let $f(x, y, z) = 0$ be any indeterminate equation between x, y, and z; let X A Y, X A Z, Y A Z (fig. 27) be three planes, each of which is at right angles to the other two, and A X, A Y, A Z the lines in which they intersect. In A X take A M equal to any arbitrary value of x, draw M N parallel to A Y, and equal to any arbitrary value of y; then if N P be drawn parallel to A Z, and equal to the resulting value of z, each point P so determined will correspond to a solution of the equation $f(x, y, z) = 0$. The assemblage, therefore, of all the points P will form a surface, plane or curved, which is called the locus of the equation

$$f(x, y, z) = 0.$$

The lines A M, M N, N P are called the coordinates of the point P, and A is said to be the *origin*, A X, A Y, A Z the *axes* of the three coordinate planes X A Y, X A Z, Y A Z.

The coordinates are usually denoted by x, y, z respectively; whence A X is called the axis of x, A Y that of y, and A Z that of z; also X A Y is called the plane of $x y$, X A Z the plane of $x z$, and Y A Z the plane of $y z$.

The equation which expresses the relation between the coordinates of any point of a surface is called the *equation to the surface.*

(63.) Complete the rectangular parallelepiped A P, then it is evident that A M = P m = the distance of P from Y A Z, estimated in the direction A X, and also that M N, P N are respectively equal to P's distance from the planes X A Z, X A Y measured in the directions A Y, A Z. Hence it appears, that *the position of a point in space depends on its distances from three rectangular coordinate planes estimated in the direction of the lines in which they intersect.*

(64.) The points N, n, m in which the lines P N, P n, P m meet the planes of $x y$, $x z$, and $y z$ are called the *projections* of the point P upon these planes respectively.

It is manifest, that if any two of these projections be given the third will be known. *Hence the position of a point in space is determined when its projections on any two of the coordinate planes are given.*

In like manner, if the several points of a straight line be projected upon any plane, the line so formed is called the *projection* of the given line, and the plane in which the perpendiculars are situated is called the *projecting plane.*

Surfaces, in the same manner as lines, are divided into orders according to the dimension of the equations by which they are represented.

Thus, a surface of the first order is the locus of the equation

$$a x + o y + c z + d = 0.$$

Part II. A surface of the second order is the locus of the equation

$$a x^2 + b y^2 + c z^2 + 2 a' y z + 2 b' x z + 2 c' x y + 2 a'' x + 2 b'' y + 2 c'' z + d = 0,$$

and so on.

ON THE STRAIGHT LINE IN SPACE.

(65.) If the projections of a straight line upon any two of the coordinate planes be given, the position of the line itself will be determined; because it will evidently be the intersection of the two projecting planes.

We may hence find the equations of a straight line in space.

Let P Q be the given line, $p q$, $p' q'$ its projections on the planes $x z$, $y z$ respectively, (fig. 28.) Also let Fig. 28.

$x = a z + \alpha$ be the equation to $p q$,

and $y = b z + \beta$ be the equation to $p' q'$.

Now, since the first of these is independent of y, it is the equation not only to $p q$ but also to every line in the projecting plane p P Q q. In like manner, the second equation is the equation to every line in the plane p' P Q q'. Therefore the system of equations

$$x = a z + \alpha,$$
$$y = b z + \beta,$$

being common to the two projecting planes, must also be the equation to P Q, which is the line of their intersection.

The quantities a, b denote the tangents of the angles at which $p q$, $p' q'$ are inclined to A Z; and α, β represent the portions of A Z intercepted between A and the points in which the same lines intersect A Z.

(66.) *To find the equations to a straight line passing through a given point.*

Let the coordinates of the given point be x', y', z'.

Then, since they must satisfy the general equations

$$\left.\begin{array}{l} x = a z + \alpha \\ y = b z + \beta \end{array}\right\} \ldots . (1.)$$

we have $x' = a z' + \alpha$, and $y' = b z' + \beta$,

$$\therefore \alpha = x' - a z', \text{ and } \beta = y' - b z'.$$

Substituting these values of α, β in (1)

$$x = a z + x' - a z',$$
$$y = b z + y' - b z',$$

or and

$$\left.\begin{array}{l} x - x' = a (z - z') \\ y - y' = b (z - z') \end{array}\right\} \ldots . (2.)$$

which are the equations required.

(67.) *To find the equations to a straight line passing through two given points.*

Let the coordinates of the second point be x'', y'', z''.

Substituting these for x, y, z in equation (2) in the last article, we have

$$x'' - x' = a\,(z'' - z'),\ \therefore a = \frac{x'' - x'}{z'' - z'},$$

$$y'' - y' = b\,(z'' - z'),\ \therefore b = \frac{y'' - y'}{z'' - z'},$$

therefore replacing a and b by these values in the same equation, we have

$$x - x' = \frac{x'' - x'}{z'' - z'}\,(z - z'),$$

$$y - y' = \frac{y'' - y'}{z'' - z'}\,(z - z'),$$

which are the equations required.

(68.) *To find the coordinates of the point of intersection of two straight lines.*

Let the equation of the lines be

$$x = a\,z + \alpha,\ y = b\,z + \beta,$$

and $\quad x = a'\,z + \alpha',\ y = b'\,z + \beta'.$

When the lines intersect, the coordinates at the point of their intersection will be identical; therefore subtracting the latter equations from the former,

$$(a - a')\,z + \alpha - \alpha' = 0,$$

$$(b - b')\,z + \beta - \beta' = 0;$$

whence, eliminating z,

$$\frac{\alpha - \alpha'}{a - a'} = \frac{\beta - \beta'}{b - b'},$$

which equation expresses the condition under which the two lines intersect.

Now z being $= \dfrac{\alpha' - \alpha}{a - a'}$ we immediately obtain

$$x = \frac{a\,\alpha' - a'\,\alpha}{a - a'} \text{ and } y = \frac{b\,\beta' - b'\,\beta}{a - a'}.$$

Cor. It thence follows, that when the lines are parallel

$$a = a' \text{ and } b = b'.$$

(69.) *To express analytically the distance of a given point (x', y', z') from the origin.*

Fig. 29. Let P be the given point, (fig. 29,) A M, M N, N P its coordinates; join A, N; and let $A\,P = \rho$.

Then the triangles A N P, A M N being evidently right angled in N and M, we have from the first

$$A\,P^2 = A\,N^2 + N\,P^2,$$

$$= \therefore A\,M^2 + M\,N^2 + N\,P^2 \text{ and from}$$

the second, $\quad \therefore \rho^2 = x'^2 + y'^2 + z'^2.$

Cor. In a rectangular parallelepiped, the square of the diagonal is equivalent to the sum of the squares of the three edges.

(70.) *To express analytically the distance between two given points.*

Let x'', y'', z'' be the coordinates of the second point Q, and take $P\,Q = \delta$.

Then P Q is evidently the diagonal of a rectangular parallelepiped, whose three contiguous edges are

$$x' - x'',\ y' - y'',\ \text{and } z' - z'';$$

we have, therefore, by the last article,

$$\delta^2 = (x' - x'')^2 + (y' - y'')^2 + (z' - z'')^2.$$

Cor. If $A\,Q = \rho'$, we have, by expanding the value of δ^2,

$$\delta^2 = x'^2 + y'^2 + z'^2 + x''^2 + y''^2 + z''^2 - 2\,\{x'\,x'' + y'\,y'' + z'\,z''\}$$

$$= \because \rho^2 + \rho'^2 - 2\,(x'\,x'' + y'\,y'' + z'\,z'').$$

(71.) *Given the equations to a straight line, to find its inclination to each of the axes.*

Draw through the origin a straight line parallel to the given line, and let its equations be

$$x = a\,z,\ y = b\,z.$$

In the triangle A P M,

$$A\,M = A\,P \cos P\,A\,X,$$

or $\quad x = \rho \cos \rho, x,$

$$\therefore \cos \rho, x = \frac{x}{\rho} = \frac{x}{\sqrt{x^2 + y^2 + z^2}} = \frac{a\,z}{\sqrt{z^2(1 + a^2 + b^2)}}$$

$$= \therefore \pm \frac{a}{\sqrt{1 + a^2 + b^2}}.$$

In like manner,

$$\cos \rho, y = \pm \frac{b}{\sqrt{(1 + a^2 + b^2)}},$$

and $\quad \cos \rho, z = \pm \dfrac{1}{\sqrt{(1 + a^2 + b^2)}}.$

The line ρ forms with each of the axes two angles which are supplements of each other; hence in the above formulas the positive sign indicates the acute, and the negative sign the obtuse angle.

Cor. 1. Squaring these values, and adding the results, we have

$$\cos^2 \rho, x + \cos^2 \rho, y + \cos^2 \rho, z = 1.$$

Cor. 2. If the angles which ρ makes with the planes $x\,y$, $x\,z$, $y\,z$ be respectively denoted by the symbols $\rho, x\,y$; $\rho, x\,z$, $\rho, y\,z$, we shall have by the last article

$$\sin^2 \rho, y\,z + \sin^2 \rho, x\,z + \sin^2 \rho, x\,y = 1.$$

(72.) *To find the inclination of two lines in terms of their separate inclinations to the axes.*

Through the origin draw two lines respectively parallel to the given lines. In these take any two points P and Q, join P, Q, and let $A\,P = \rho$, $A\,Q = \rho'$; then by Art.

$$P\,Q^2 = \rho^2 + \rho'^2 - 2\,(x'\,x'' + y'\,y'' + z'\,z''),$$

but $\quad P\,Q^2 = \rho^2 + \rho'^2 \quad 2\,\rho\,\rho' \cos \rho, \rho'.$

Woodhouse's *Trig.* ch. ii. Prob. 1; Lardner's *Trig.* Art. 75,

$$\therefore \rho\,\rho' \cos \rho\,\rho' = x'\,x'' + y'\,y'' + z'\,z'' \ldots\ldots (1.)$$

But $\quad x' = \rho \cos \rho, x;\ y' = \rho \cos \rho, y;\ z' = \rho \cos \rho, z.$

Similarly,

$x'' = \rho' \cos \rho, x; \; y'' = \rho' \cos \rho, y; \; z'' = \rho \cos \rho, z;$

therefore, substituting in (1) and dividing by $\rho \rho'$, we have

$\cos \rho, \rho' = \cos \rho, x \cos \rho', x + \cos \rho, y \cos \rho', y + \cos \rho, z \cos \rho', z.$

Cor. When the lines are at right angles to each other,

$\cos \rho, x \cos \rho', x + \cos \rho, y . \cos \rho', y + \cos \rho, z \cos \rho', z = 0.$

(73.) *The equations to two lines being given, to find their mutual inclination.*

Draw two lines through the origin parallel to the given lines, then their equations will be

$$x = a z, y = b z \ldots . (1,)$$
$$x = a' z, y = b' z \ldots . (2.)$$

Now, by last Art.

$\cos \rho, \rho' = \cos \rho, x \cos \rho', x + \cos \rho, y \cos \rho', y + \cos \rho, z \cos \rho', z;$

but

$$\cos \rho, x = \frac{a}{\sqrt{1 + a^2 + b^2}}; \; \cos \rho', x = \frac{a'}{\sqrt{1 + a'^2 + b'^2}}$$

and similarly with respect to $\cos \rho, y$, $\cos \rho, z$, &c.; therefore, by substitution,

$$\cos \rho, \rho' = \frac{1 + a a' + b b'}{\sqrt{(1 + a^2 + b^2)(1 + a'^2 + b'^2)}},$$

which is the expression required.

ON THE PLANE.

(74.) A plane is generated by a straight line which moves parallel to itself, along a straight line given in position.

Of these straight lines the former is called the *generating line*, the latter the *directrix*.

The equation to a plane may be obtained by expressing analytically the mode in which it is generated.

Let the equations to the generating line be

$$x = a z + a, \; y = b z + \beta \ldots . (1,)$$

and the equation to the directrix, which we shall suppose to be in the plane of $x y$,

$$Y = m X + n \ldots . (2.)$$

Now, since the generating line is always parallel to itself, its equations in any position will be

$$x = a z + a', \; y = b z + \beta'.$$

But because it passes, by hypothesis, through a point in the directrix whose coordinates are X, Y, 0, we shall have

$X = a', \; Y = \beta';$

but $a' = x - a z$, and $\beta' = y - b z$,

$$\therefore X = x - a z, \; Y = y - b z.$$

Substituting these values of X, Y in equation (2) we have,

$$y - b z = m (x - a z) + n,$$
$$\therefore y - m x + (m a - b) z - n = 0$$

which is the equation required.

A more symmetrical form may be given to the equation by assuming $\frac{A}{B} = - m$, $\frac{A}{C} = b - m a$, and $\frac{A}{D} = n$.

Then we have

$$A x + B y + C z + D = 0,$$

for the general equation to a plane.

(75.) *Cor.* 1. When the plane passes through the origin, $D = 0$, and the equation becomes

$$A x + B y + C z = 0.$$

(76.) *Cor.* 2. If the plane meet any one of the axes, for example A Z, then x and $y = 0$, therefore $z = - \frac{D}{C}$.

If the plane be perpendicular to A Z, then each of its points is equidistant from the plane $x y$, and therefore z is constant.

If the plane be parallel to A Z, then $\frac{D}{C}$ being infinitely great, $C = 0$.

(77.) *Cor.* 3. If the plane meet any one of the coordinate planes, $x y$, for example, then $z = 0$, and the equation to their intersection is

$$A x + B y + D = 0.$$

(78.) The intersection of a plane with any one of the coordinate planes is called the *trace* of the given plane.

If the plane be perpendicular to $x y$, then since it must be parallel to A Z, $C = 0$, therefore the equation to the *trace* is $A x + B y + D = 0$.

As the same reasoning is applicable to the remaining two coordinate planes, we conclude that *when a plane is perpendicular to any one of the coordinate planes, its equation is that of its trace upon the same plane.*

(79.) If the plane be parallel to that of $x y$, then the coordinates of its intersection with A X and A Y, namely, $\frac{-D}{A}$ and $\frac{-D}{B}$ will be infinitely great, therefore A and B each $= 0$, hence, the equation becomes $C z + D = 0$.

(80.) *To find the equation to a plane in terms of the perpendicular* (p) *dropped upon it from the origin, and the angles which that perpendicular forms with the axes.*

Let the plane meet the axes in the points B, C, D, and take $A B = a$, $A C = b$, $A D = c$; then, by the last article,

$$a = - \frac{D}{A}, b = - \frac{D}{B}, c = - \frac{D}{C}.$$

But the general equation is $A x + B y + C z = - D$,

$$\text{or } - \frac{A}{D} x - \frac{B}{D} y - \frac{C}{D} z = 1,$$

therefore, by substitution, $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \ldots . (1,)$

but it is evident that $\rho = a \cos \rho, x$, or $a = \dfrac{\rho}{\cos \rho, x}$; in like manner, $b = \dfrac{\rho}{\cos \rho, y}$, $c = \dfrac{\rho}{\cos \rho, z}$;

therefore, replacing a, b, c, in (1) by these values, we have $\quad x \cos \rho, x + y \cos \rho, y + z \cos \rho, z = \rho,$

which is the equation required.

(81.) *Cor.* $\quad \text{Cos}^2 \rho, x + \cos^2 \rho, y + \cos^2 \rho, z$

$$= \frac{\rho^2}{a^2} + \frac{\rho^2}{b^2} + \frac{\rho^2}{c^2},$$

$$= \frac{\rho^2}{D^2}(A^2 + B^2 + C^2) = 1,$$

$$\therefore \rho = \pm \frac{D}{\sqrt{A^2 + B^2 + C^2}};$$

$$\therefore \cos \rho, x = -\rho \cdot \frac{A}{D} = \pm \frac{A}{\sqrt{A^2 + B^2 + C^2}}.$$

In like manner,

$$\cos \rho, y = \pm \frac{B}{\sqrt{A^2 + B^2 + C^2}},$$

$$\cos \rho, z = \pm \frac{C}{\sqrt{A^2 + B^2 + C^2}}.$$

(82.) *To find the equations to a perpendicular let fall from a given point* (x', y', z') *upon a given plane* $Ax + By + Cz + D = 0$.

By Art. 63 the equations sought will be of the form

$$\left.\begin{aligned} x - x' &= a(z - z') \\ y - y' &= b(z - z') \end{aligned}\right\} \dots (1,)$$

in which a and b are to be determined.

Suppose the perpendicular and the plane to be projected upon any one of the coordinate planes, then these projections will evidently be at right angles to each other; because the projecting plane of the perpendicular being at right angles to the given plane, their intersections with any of the coordinate planes, in other words, the projections in question, will also be at right angles to each other.

The given plane, then, being projected on the planes of $x\,z$ and $y\,z$, the equations to its traces are

$$\left.\begin{aligned} Ax + Cz + D = 0, \text{ or } x &= -\frac{C}{A}z - \frac{D}{A} \\ By + Cz + D = 0, \text{ or } y &= -\frac{C}{B}z - \frac{D}{B} \end{aligned}\right\} \dots (2.)$$

But since the projections of the perpendicular are at right angles to these traces, we have (Art. 14)

$$a = \frac{A}{C} \text{ and } b = \frac{B}{C};$$

therefore the equations required are

$$x - x' = \frac{A}{C}(z - z'),$$

$$y - y' = \frac{B}{C}(z - z').$$

(83.) *To find the length* (p) *of the perpendicular dropped from a given point on a given plane.*

Conceive a plane drawn through the given point parallel to the given plane, and let fall upon it from the origin A a perpendicular A Q meeting the given plane in P; then P Q will $= p$.

Now $AQ = x' \cos \rho, x + y' \cos \rho, y + z' \cos \rho, z,$

$$= \pm \frac{Ax' + By' + Cz'}{\sqrt{A^2 + B^2 + C^2}};$$

$$\therefore p = AQ - AP = AQ - \rho,$$

$$= \pm \frac{Ax' + By' + Cz' - D}{\sqrt{A^2 + B^2 + C^2}}.$$

(84.) *To find the inclination of a given straight line to a given plane.*

Let the equations to the line be

$$x = az + \alpha, \quad y = bz + \beta,$$

and the equation to the plane $Ax + By + Cz + D = 0$. Now the inclination of a line to a plane is the angle contained by the line and its projection upon the plane, and is therefore equal to the complement of the angle formed by the line and a perpendicular let fall from any point of it upon the plane.

Let the equations to the perpendicular be

$$x = a'z + \alpha', \quad y = b'z + \beta',$$

then $\quad a' = \dfrac{A}{C}$, and $b' = \dfrac{B}{C}$, by (Art. 82.)

Let ρ be the given line, ρ' the perpendicular dropped from any point of it on the given plane, and let the symbol ρ, Π denote the angle at which the line is inclined to the plane.

Then in general $\cos \rho, \rho' = \dfrac{1 + aa' + bb'}{\sqrt{(1 + a^2 + b^2)(1 + a'^2 + b'^2)}}$, (Art. 73.)

Therefore, substituting for a', b' their values obtained above, we have

$$\sin \rho, \Pi = \frac{Aa + Bb + C}{\sqrt{(1 + a^2 + b^2)(A^2 + B^2 + C^2)}}.$$

(85.) *Cor.* When the line is parallel to the plane, then $\quad Aa + Bb + C = 0.$

(86.) *To find the inclination of two planes, in terms of their separate inclination to the axes.*

Draw through the origin two planes parallel respectively to the given planes; then if two lines ρ, ρ' be drawn from the origin at right angles to the planes, their inclination ρ, ρ' will equal the angle Π, Π', and their equations will be

$$x \cos \rho, x + y \cos \rho, y + z \cos \rho, z = 0,$$

$$x \cos \rho', x + y \cos \rho', y + z \cos \rho', z = 0.$$

Now in general,

$$\cos \rho, \rho' = \cos \rho, x \cos \rho', x + \cos \rho, y \cos \rho', y + \cos \rho, z \cos \rho', z \dots (1,)$$

but $\cos \rho, x = \cos \Pi, yz$; $\cos \rho, y = \cos \Pi, xz$; $\cos \rho, z = \cos \Pi, xy$, and so on; therefore, substituting these values in (1,) we have

$$\cos \Pi, \Pi' = \cos \Pi, yz \cdot \cos \Pi', yz + \cos \Pi, xz \cos \Pi', xz + \cos \Pi, xy \cos \Pi', xy,$$

which is the inclination required.

(87.) *To find the inclination of two planes whose equations are*

$$A x + B y + C z + D = 0,$$
$$A' x + B' y + C' z + D' = 0.$$

The inclination required will equal that of two planes drawn through the origin parallel to the given planes.

In general,

$$\cos \rho, \rho' = \cos \rho, x \cos \rho', x + \cos \rho, y \cos \rho', y + \cos \rho, z \cos \rho', z \ldots . \ (1;)$$

$$\text{but} \cos \rho, x = \frac{A}{\sqrt{A^2 + B^2 + C^2}}, \cos \rho, y = \frac{B}{\sqrt{A^2 + B^2 + C^2}},$$

and so on; therefore, by substitution in (1,) we have

$$\cos \Pi, \Pi' = \frac{A A' + B B' + C C'}{\sqrt{(A^2 + B^2 + C^2)(A'^2 + B'^2 + C'^2)}}.$$

(88.) *Cor.* When the planes are perpendicular to each other,

$$A A' + B B' + C C' = 0.$$

ON THE TRANSFORMATION OF COORDINATES IN SPACE.

(89.) *The position of a point with respect to a given system of planes being given, to find its position when referred to a new system of planes parallel to the former.*

Let a, b, c be the coordinates of the new origin: and x, y, z; $x' y', z'$ those of any point P when referred to the old and new system respectively.

Then it is evident, that in order to obtain the equation of P in relation to the new system, we have only to substitute for x, y, z in the given equation

$$f(x, y, z) = 0,$$

the quantities $x' + a$, $y' + b$, $z' + c$.

The *position* of the new origin relatively to that of the old one will be indicated by the signs of a, b, and c.

(90.) *The position of a point with respect to any system of planes being known, to find its position when referred to any other system whatever, originating at the same point with the former.*

Since the new coordinates must evidently be *linear functions* of the old ones, let us assume

$$x = m x' + n y' + p z',$$
$$y = m' x' + n' y' + p' z',$$
$$z = m'' x' + n'' y' + p'' z';$$

the quantities $m, m', m'' \ldots$ being independent of $x, y \ldots$

In order to determine their value, Let y' and z' each equal 0, in which case the point is situated on A X'.

Then
$$m = \frac{x}{x'} = \frac{\sin x', y z}{\sin x, y z},$$
$$m' = \frac{y}{x'} = \frac{\sin x', x z}{\sin y, x z};$$
$$m'' = \frac{z}{x'} = \frac{\sin x', x y}{\sin z, x y}.$$

In like manner, supposing the point to be successively on the axes A Y', A Z', we have

$$n = \frac{\sin y', y z}{\sin x, y z}; \qquad p = \frac{\sin z', y z}{\sin x, y z}.$$

$$n' = \frac{\sin y', x z}{\sin y, x z}; \qquad p' = \frac{\sin z', x z}{\sin y, x z}.$$

$$n'' = \frac{\sin y', x y}{\sin z, x y}; \qquad p'' = \frac{\sin z', x y}{\sin z, x y}.$$

Hence we have, by substitution,

$$x = \frac{1}{\sin x, y z} \{ x' \sin x', y z + y' \sin y', y z + z' \sin z', y z \};$$

$$y = \frac{1}{\sin y, x z} \{ x' \sin x', x z + y' \sin y', x z + z' \sin z', x z \};$$

$$z = \frac{1}{\sin z, x y} \{ x' \sin x', x y + y' \sin y', x y + z' \sin z', x y \}.$$

(91.) Such are the general formulas to be used in passing from one oblique system to another. We shall now deduce from them the following particular cases:

1. Let the old axes be rectangular, and the new ones oblique.

Then the denominators become each $= 1$; also in the first line,

$$\sin x', y z = \cos x', x; \ \sin y', y z = \cos y', x;$$
$$\sin z', y z = \cos z', x;$$

the remaining two lines being in like manner modified, we have

$$\left.\begin{aligned} x &= x' \cos x', x + y' \cos y', x + z' \cos z', x \\ y &= x' \cos x', y + y' \cos y', y + z' \cos z', y \\ z &= x' \cos x', z + y' \cos y', z + z' \cos z', z \end{aligned}\right\} \ldots (1;)$$

but because the primitive axes are rectangular, the following equations also hold true,

$$\left.\begin{aligned} \cos^2 x', x + \cos^2 x', y + \cos^2 x', z = 1 \\ \cos^2 y', x + \cos^2 y', y + \cos^2 y', z = 1 \\ \cos^2 z', x + \cos^2 z', y + \cos^2 z', z = 1 \end{aligned}\right\} \ldots . (2.)$$

It appears, therefore, that of the nine angles involved in the formulas (1) *six* alone are independent, since three of them are evidently determined by equations (2.)

2. Let both systems be rectangular.

Then, since each two of the coordinates x', y', z' are at right angles to each other, we have, by (Art. 72,)

$$\cos x', x \cos y', x + \cos x', y \cos y', y + \cos x', z \cos y', z = 0$$
$$\cos x', x \cos z', x + \cos x', y \cos z', y + \cos x', z \cos z', z = 0$$
$$\cos y', x \cos z', x + \cos y', y \cos z', y + \cos y', z \cos z', z = 0$$
$$\ldots . (3,)$$

by means of which equations the six angles that enter into the formulas (1) are now reduced to three. Whence it follows, that in order to pass from one rectangular system to another, three independent angles alone are required.

ON THE SPHERE.

(92.) *To find the equation to a spherical surface.*

Let a sphere whose radius is r be referred to any system of oblique axes; suppose x', y', z' to be the coordinates of the centre, and x, y, z those of any point on the surface.

Now, since all the points on the surface are equidistant from the centre, we have

$$(x-x')^2+(y-y')^2+(z-z')^2+2(x-x')(y-y')\cos x,y+2(x-x')(z-z')\cos x,z+2(y-y')(z-z')\cos y,z=r^2.$$

(93.) *Cor.* Let the origin be at the centre; then x', y', z' being $=0$, the equation becomes

$$x^2+y^2+z^2+2xy\cos x,y+2xz\cos x,z+2yz\cos y,z=r^2.$$

The general equation to the sphere may, as in the case of the circle, be simplified by changing the origin and direction of the axes.

Let the axes be now supposed rectangular, then the general equation becomes

$$(x-x')^2+(y-y')^2+(z-z')^2=r^2\ldots.(1.)$$

and when the origin is at the centre

$$x^2+y^2+z^2=r^2\ldots.(2.)$$

Let the origin be on the surface of the sphere, then since

$$x'^2+y'^2+z'^2=r^2,$$

equation (1) becomes

$$x^2+y^2+z^2-2xx'-2yy'-2zz'=0\ldots.(3.)$$

Let the origin be on one of the coordinate planes.

If it be upon the plane of xy, then $z'=0$, and the equation becomes

$$(x-x')^2+(y-y')^2+z^2=r^2\ldots.(4.)$$

Let the origin be upon one of the axes.

If it be upon the axis of x, then y' and $z'=0$, and the equation becomes

$$(x-x')^2+y^2+z^2=0\ldots.(5.)$$

(94.) The general form to the equation to a sphere when referred to rectangular coordinates is,

$$x^2+y^2+z^2+Ax+By+Cz+D=0\ldots.(1.)$$

Let it now be required to assign the position and magnitude of the sphere which it represents.

Comparing equation (1) with the general equation

$$(x-x')^2+(y-y')^2+(z-z')^2=r^2,$$

that is, with

$$x^2+y^2+z^2-2xx'-2yy'-2zz'+x'^2+y'^2+z'^2-r^2=0\ldots.(2.)$$

we have

$$A=-2x',\text{ or }x'=-\frac{A}{2},$$

$$B=-2y',\text{ or }y'=-\frac{B}{2},$$

$$C=-2z',\text{ or }z'=-\frac{C}{2};$$

also

$$D=x'^2+y'^2+z'^2-r^2,$$
$$=\tfrac{1}{4}(A^2+B^2+C^2)-r^2,$$
$$\therefore r=\pm\tfrac{1}{2}\sqrt{(A^2+B^2+C^2-4D)}.$$

Hence it follows, that equation (1) belongs to a sphere whose radius is $=\frac{1}{2}\sqrt{\{A^2+B^2+C^2-4D\}}$, and the coordinates of whose centre are $-\frac{A}{2}, -\frac{B}{2}, -\frac{C}{2}$.

(95.) *To find the equation to a tangent plane, drawn through a given point (x', y', z') of the sphere.*

The origin being at the centre, the equation to the sphere will be

$$x^2+y^2+z^2+2xy\cos x,y+2xz\cos x,z+2yz\cos y,z=r^2\ldots.(1.)$$

Now if a secant be drawn through the given point, its equations will be

$$\left.\begin{aligned}x-x'&=m(z-z')\\ \text{and}\quad y-y'&=n(z-z')\end{aligned}\right\}\ldots.(2.)$$

But since the given coordinates x', y', z' must satisfy equation (1) we have

$$x'^2+y'^2+z'^2+2x'y'\cos x,y+2x'z'\cos x,z+2y'z'\cos y,z=r^2\ldots.(3.)$$

whence subtracting this from (1)

$$x^2-x'^2+y^2-y'^2+z^2-z'^2+2(xy-x'y')\cos x,y+2(xz-x'z')\cos x,z+2(yz-y'z')\cos y,z=0\ldots.(4.)$$

But
$$x^2-x'^2=(x+x')(x-x')=(x+x')m(z-z'),$$
$$y^2-y'^2==(y+y')n(z-z'),$$
$$z^2-z'^2==(z+z')\ (z-z'),$$

also,

$$\begin{aligned}xy-x'y'&=x(y-y')+y'(x-x')\\&=x.n(z-z')+y'm.(z-z'),\\&=(z-z')(my'+nx),\\ xz-x'z'&=(z-z')(x+mz'),\\ yz-y'z'&=(z-z')(y+nz').\end{aligned}$$

therefore substituting in (4) and dividing each term of the result by $z-z'$, we have

$$\left.\begin{aligned}m(x+x')+n(y+y')+(z+z')+2(my'+nx)\cos x,y\\+2(x+mz')\cos x,z\\+2(y+nz')\cos y,z\end{aligned}\right\}\ldots.=0.$$

Suppose now that $x=x'$, $y=y'$, and $z=z'$, then the points of section coincide, and the secant becomes a tangent; we have therefore in this case, after dividing each term by 2,

$$\left.\begin{aligned}mx'+ny'+z'+(my'+nx')\cos x,y\\+(x'+mz')\cos x,z\\+(y'+nz')\cos y,z\end{aligned}\right\}=0,$$

or collecting the terms involving m and n,

$$(x'+y'\cos x,y+z'\cos x,z)\,m$$
$$+(y'+x'\cos x,y+z'\cos y,z)\,n$$
$$+z'+x'\cos x,z+y'\cos y,z=0;$$

eliminating m and n by means of equation (2)

$$\{x'+y'\cos x,y+z'\cos x,z\}\frac{x-x'}{z-z'}$$
$$+(y'+x'\cos x,y+z'\cos y,z)\frac{y-y'}{z-z'}$$
$$+z'+x'\cos x,z+y'\cos y,z=0,$$

which on being reduced by means of equation (3) becomes

$$\{x'+y'\cos x,y+z'\cos x,z\}\,x$$
$$+\{y'+x'\cos y,x+z'\cos y,z\}\,y$$
$$+\{z'+x'\cos z,x+y'\cos z,y\}\,z=r^2,$$

which is the equation required.

(96.) *Cor.* When the axes are rectangular, this equation becomes

$$x x' + y y' + z z' = r^2.$$

As this is the equation commonly used, we shall investigate it by a method analogous to that employed in the case of the circle.

The equation to the sphere is

$$x^2 + y^2 + z^2 = r^2 \ldots . (1.)$$

and the equation to any plane is

$$x \cos \rho, x + y \cos \rho, y + z \cos \rho, z = \rho \ldots . (2.)$$

Now when this plane touches the sphere, we have $\rho = r$, also $\cos \rho, x = \frac{x'}{r}$, $\cos \rho, y = \frac{y'}{r}$, $\cos \rho, z = \frac{z'}{r}$, x', y' and z' being the coordinates of the point of contact; hence, by substitution,

$$\frac{1}{r} \{ x x' + y y' + z z' \} = r,$$

or $$x x' + y y' + z z' = r^2,$$

as before.

In like manner, if the equation to the sphere be

$$(x - a)^2 + (y - \beta)^2 + (z - \gamma)^2 = r^2,$$

the equation to a tangent plane applied at a point x', y', z' will be

$$(x - a)(x' - a) + (y - \beta)(y' - \beta) + (z - \gamma)(z' - \gamma) = r^2.$$

(97.) *To find the equation of a plane that shall be a common tangent to two given spheres.*

Let the axes be rectangular, and let us suppose for simplicity that the plane of $x y$ passes through the centres of the two spheres, and that the axis of x coincides with the line joining their centres.

Hence if r, r' be the radii and of the spheres, and δ the distance between their centres, the equations to the spheres will be

$$x'^2 + y'^2 + z'^2 = r^2 \ldots . (1.)$$

$$(x'' - \delta)^2 + y''^2 + z''^2 = r'^2 \ldots . (2.)$$

The equation of the tangent plane to the first sphere will be

$$x x' + y y' + z z' = r^2 \ldots . (3.)$$

And in order that this plane may also touch the second sphere, the perpendicular let fall upon it from the centre of the latter must $= r'$.

Now the coordinates of the second sphere being

$$x = \delta, y = 0, z = 0,$$

and r' being $= p$, we have

$$r' = \pm \frac{\delta x' - r^2}{\sqrt{x'^2 + y'^2 + z'^2}} = \pm \frac{\delta x' - r^2}{r},$$

but as the spheres are situated between the tangent plane and the plane of $x y$, the lower sign must be taken,

$$\therefore r' = - \frac{\delta x' - r^2}{r},$$

$$\therefore x' = \frac{r}{\delta} (r - r') \ldots . (4.)$$

therefore, substituting this value of x' in (3) and transposing, we have

$$z z' = r^2 - y y' - \frac{r}{\delta} (r - r') x,$$

but $$z' = \sqrt{r^2 - x'^2 - y'^2},$$

$$\therefore z = \frac{\delta (r^2 - y y') - r (r - r') x}{\sqrt{(r^2 - x'^2 - y'^2)}},$$

or substituting for x'^2 its value in (4.)

$$z = \frac{\delta (r^2 - y y') - r (r - r') x}{\sqrt{\{ \delta^2 (r^2 - y'^2) - r^2 (r - r')^2 \}}},$$

which is the equation required.

If z be made $= 0$, we obtain

$$y = \frac{\delta r - (r - r') x}{\sqrt{\delta^2 - (r - r')^2}},$$

which was proved in Art. 35 to be the equation of the common tangent to two circles.

ON THE CYLINDER.

(98.) *To find the equation to a cylindrical surface.*

A cylindrical surface is generated by a straight line which moves parallel to itself, and with its extremity describes the perimeter of a given curve.

The straight line is called the *generating line*, and the given curve, the *directrix*, or *base*.

Let the equations to the generating line, when in any position, be

$$\left.\begin{array}{l} x = a z + \alpha \\ y = b z + \beta \end{array}\right\} \ldots . (1.)$$

in which α, β are variable, and a, b constant, since the line is supposed to move parallel to itself.

Let the equation to the directrix, which we shall assume in the plane of $x y$, be

$$f (\mathrm{X}, \mathrm{Y}) = 0 \ldots . (2.)$$

Then, since the generating line always moves through a point of the directrix ($x = \mathrm{X}$, $y = \mathrm{Y}$, $z = 0$), we have

$\mathrm{X} = \alpha$, and therefore from (1) $\mathrm{X} = x - a z$,
$\mathrm{Y} = \beta$, and therefore $\quad \mathrm{Y} = y - b z$,

whence, by substitution in (2),

$$f \{ x - a z, y - b z \} = 0,$$

which is the equation required.

The surface generated is said to be that of a *right*, or of an *oblique* cylinder, according as the generating line is perpendicular, or inclined, to the plane of the directrix.

Example. The equation to an oblique cylinder whose base is a circle $\mathrm{X}^2 + \mathrm{Y}^2 = 2 r \mathrm{X}$,

is $$(x - a z)^2 + (y - b z)^2 = 2 r (x - a z),$$

the origin being at the extremity of a diameter.

ON THE CONE.

(99.) *To find the equation to a conical surface.*

A conical surface is generated in the same manner as a cylindrical surface, except that the generating line instead of moving parallel to itself always passes through a given point. The given point is called the *vertex* of the cone.

Let the coordinates of the vertex be x', y', z'; then the equations to the generating line will be

$$\left.\begin{array}{l} x - x' = a (z - z') \\ y - y' = b (z - z') \end{array}\right\} \ldots . (1.)$$

Let the equation to the directrix, which we shall suppose, as before, to lie in the plane of $x y$, be

$$f (\mathrm{X}, \mathrm{Y}) = 0 \ldots . (2.)$$

Then, since the generating line must always pass through a point of the directrix ($x = X$, $y = Y$, $z = 0$) we shall have

$$X - x' = -a z', \text{ or } X = x' - a z',$$
$$Y - y' = -b z', \text{ or } Y = y' - b z',$$

therefore by substitution in (2)

$$f\{x' - a z', y' - b z'\} = 0,$$

or $$f(a, b) = 0;$$

but $$a = \frac{x - x'}{z - z'}, \; b = \frac{y - y'}{z - z'},$$

therefore $$f\left\{\frac{x - x'}{z - z'}, \frac{y - y'}{z - z'}\right\} = 0,$$

which is the equation required.

Since the generating line may be extended indefinitely upwards, the conical surface will be composed of two similar portions, one above, and the other below the vertex; each portion is called a *sheet*, this term being understood to bear the same relation to *surface*, that *branch* does to *curve*.

The surface generated is said to be that of a right or of an oblique cone, according as the generating line is perpendicular, or inclined, to the plane of the directrix.

Example 1. The equation to a *right* cone whose base is a circle

$$(X - x')^2 + (Y - y')^2 = r^2,$$

is $$(x - x')^2 + (y - y')^2 = \frac{r^2}{z'^2}(z - z')^2.$$

Example 2. The equation to an *oblique* cone whose base is a circle

$$(X - \alpha)^2 + (Y - \beta)^2 = r^2,$$

is

$$\{z x' - x z' - \alpha (z - z')^2\} + \{z y' - y z' - \beta (z - z')^2\}^2 = r^2 (z - z')^2.$$

ON SURFACES OF REVOLUTION.

(100.) *To find the equation to a surface of revolution.*

A surface of revolution is generated by a curve which revolves about a fixed line or *axis*, in such a manner that each point of the curve may describe a circle whose centre is on that line, and whose plane is perpendicular to it.

Hence if the surface be cut by a plane perpendicular to the axis, the intersection will be a circle. The surface may, therefore, be considered as formed by a circle of variable magnitude, which moves parallel to itself and meets the generating curve.

Let the equations to the generating curve be

$$\left.\begin{array}{l} f(x, y, z) = 0 \\ f'(x', y', z') = 0 \end{array}\right\} \ldots . \text{(1.)}$$

Then if x', y', z' be the coordinates of any point in the axis of revolution, the equations to the axis will be

$$\left.\begin{array}{l} x - x' = a (z - z') \\ y - y' = b (z - z') \end{array}\right\} \ldots . \text{(2.)}$$

Hence the equation to a plane perpendicular to the axis is (Art. 82)

$$a x + b y + z = e \ldots . \text{(3,)}$$

and that to a sphere whose centre is (x', y', z')

is $$(x - x')^2 + (y - y')^2 + (z - z')^2 = r^2.$$

Now we may conceive the circle which results from the intersection of this sphere by the plane (3) to be one of those which compose the surface. Therefore e and r^2, or their equals, must be constant or variable together, for the same points. In other words, one of them must be a function of the other. Hence

$$(x - x')^2 + (y - y')^2 + (z - z')^2 = F(a x + b y + z)$$

will be the equation required.

(101.) *Cor.* Let the axis of revolution coincide with the axis of z, then the equation to the variable circle will be

$$z = e, \; x^2 + y^2 = r^2.$$

Whence the equation to the surface of revolution becomes

$$x^2 + y^2 = F(z).$$

In like manner, the equation to the surface will be

$$x^2 + z^2 = F(y),$$

or $$y^2 + z^2 = F(x),$$

according as the axis of revolution coincides with the axis of y or of x.

(102.) Let the generating curve be

Example 1. A parabola, $x^2 = 2 p z$.

Then the equation to a *paraboloid of revolution* is

$$x^2 + y^2 = 2 p z.$$

Example 2. An ellipse, $a^2 z^2 + b^2 x^2 = a^2 b^2$.

Then according as the revolution is performed about the *major* or the *minor* axis, the equation to the *ellipsoid of revolution*, or of the *spheroid*, will be

$$b^2 z^2 + a^2 (y^2 + x^2) = a^2 b^2 \ldots . \text{(1.)}$$

or, $$a^2 z^2 + b^2 (x^2 + y^2) = a^2 b^2 \ldots . \text{(2.)}$$

The spheroid is of two kinds, the *prolate* and the *oblate*; the former is represented by equation (1,) and the latter by equation (2.)

Example 3. In like manner, the equation to the *hyperboloid of revolution* is

$$b^2 (x^2 + y^2) - a^2 z^2 = a^2 b^2.$$

ON SURFACES OF THE SECOND ORDER IN GENERAL.

(103.) The general equation of the second degree between three variables is,

$$a x^2 + b y^2 + c z^2 + 2 a' y z + 2 b' x z + 2 c' x y + 2 a'' x + 2 b'' y + 2 c'' z + d = 0.$$

The surfaces which are the loci of this equation are called surfaces of the second order.

In the following investigations the coordinate planes are supposed to have any inclination whatever, except in those cases which are expressly mentioned.

The characteristic property of surfaces of the second order is, that *they cannot be intersected by a straight line in more than two points.*

For let the surface be cut by the straight line

$$x = m z + \alpha,$$
$$y = n z + \beta.$$

Then at the points of intersection the coordinates of the line and surface are identical; therefore by substituting the values of x and y in the general equation

$$a x^2 + b y^2 + c z^2 + 2 a' y z + 2 b' x z + 2 c' x y + 2 a'' x + 2 b'' y + 2 c'' z + d = 0.$$

we shall obtain a quadratic equation, which can only have two roots; hence the surface cannot be intersected by a straight line in more than two points.

Def. The portion of the line intercepted between the two points of section is called a *chord*.

(104.) *To find the locus of the middle points of any number of parallel chords.*

Let $x = m z,\ y = n z \ldots\ (g)$

be the equations to any straight line drawn through the origin, and cutting the surface in the points P, p; take O (x', y', z') the middle point of any chord Q q parallel to P p; then the object of the proposition is to find the relation between x', y', and z'.

Let the origin be transferred to the point O, then the equation to the surface becomes

$$a(x+x')^2 + b(y+y')^2 + c(z+z')^2 + 2a'(y+y')(z+z') + 2b'(x+x')(z+z') + 2c'(x+x')(y+y') + 2a''(x+x') + 2b''(y+y') + 2c''(z+z') + d = 0,$$

and the equations to Q q will then become

$$x = m z,\ y = n z.$$

Now the points in which Q q intersects the surface will be found by supposing the variables x, y, z identical in the two equations; we thus have

$$a(mz+x')^2 + b(nz+y')^2 + c(z+z')^2 + 2a'(nz+y')(z+z') + 2b'(mz+x')(z+z') + 2c'(mz+x')(nz+y') + 2a''(mz+x') + 2b''(nz+y') + 2c''(z+z') + d = 0.$$

But since the chord Q q is bisected in O, the two values of z in this quadratic equation will be equal, with contrary signs; therefore the coefficient of the second term will vanish; whence collecting the terms involving z we have

$$2amx' + 2bny' + 2a'nz' + 2a'y' + 2b'mz' + 2b'x' + 2c'nx' + 2c'ny' + 2a''m + 2b''n + 2c'' = 0,$$

therefore, dividing by 2, suppressing the accents of the variables, and arranging the result with reference to x, y, and z, we have

$$(am + c'n + b')x + (c'm + bn + a')y + (b'm + a'n + c)z + a''m + b''n + c'' = 0 \ldots (1)$$

the equation to *a plane*, which is therefore the locus required.

That plane which bisects a system of parallel chords is called a *diametral plane*.

(105.) In like manner, if there be two other chords

$$x = m'z + \alpha',\ y = n'z + \beta' \ldots (h)$$

$$x = m''z + \alpha'',\ y = n''z + \beta'' \ldots (k)$$

the corresponding diametral planes will be

$$(am' + c'n' + b')x + (c'm' + bn' + a')y + (b'm' + a'n' + c)z + a''m' + b''n' + c'' = 0 \ldots (2.)$$

$$(am'' + c'n'' + b')x + (c'm'' + bn'' + a')y + (b'm'' + a'n'' + c)z + a''m'' + b''n'' + c'' = 0 \ldots (3.)$$

The direction of the plane (1) depends on the direction of the chord (g) which was drawn at pleasure. We shall now fix the relative position of the other two chords (h) and $(k,)$ by supposing, first, that each of them is parallel to the plane (1,) and next that either of them (k) is parallel to the diametral plane (2) of the other.

Then, since (h) and (k) are each of them parallel to (1) we have

$$m'(am + c'n + b') + n'(c'm + bn + a') + b'm + a'n + c = 0,$$

$$m''(am + c'n + b') + n''(c'm + bn + a') + b'm + a'n + c = 0;$$

or,

$$b'(m + m') + a'(n + n') + c'(mn' + nm') + amm' + bnn' + c = 0 \ldots (4.)$$

$$b'(m + m'') + a'(n + n'') + c'(mn'' + nm'') + amm'' + bnn'' + c = 0 \ldots (5;)$$

and since k also is parallel to (2,)

$$b'(m' + m'') + a'(n' + n'') + c'(m'n'' + n'm'') + am'm'' + bn'n'' + c = 0 \ldots (6.)$$

But since (k) is parallel to (1) and (2) it must be parallel to the line of their intersection, therefore the diametral plane (3) of the chord (k) bisects all chords which are parallel to the intersection of the other two diametral planes.

And since (4,) (5,) and (6) are equations of symmetrical forms, the diametral planes of (g) and (h) will bisect the chords which are parallel to the respective intersections of the planes (2) and (3,) and (1) and (3.)

It appears, therefore, that each of these three diametral planes bisects the chords which are parallel to the intersections of the other two.

Diametral planes thus related, are said to be *conjugate* to one another, and the intersections of each two of them are called *conjugate diameters*.

The point in which any three diametral planes intersect one another is called the *centre*.

(106.) *To find whether any system of conjugate diametral planes can be rectangular.*

In this problem we shall suppose, for the sake of brevity, that the coordinate planes are rectangular.

When three diametral planes are conjugate to one another, their equations are

$$\left.\begin{array}{l} b'(m + m') + a'(n + n') + c'(mn' + nm') \\ \quad + amm' + bnn' + c = 0 \\ b'(m + m'') + a'(n + n'') + c'(mn'' + nm'') \\ \quad + amm'' + bnn'' + c = 0 \\ b'(m' + m'') + a'(n' + n'') + c'(m'n'' + n'm'') \\ \quad + am'm'' + bn'n'' + c = 0 \end{array}\right\} \ldots (1)$$

Now, if these be supposed rectangular, the following equations must hold true, Art. 88,

$$\left.\begin{array}{l} 1 + mm' + nn' = 0 \\ 1 + m'm'' + n'n'' = 0 \\ 1 + mm'' + nn'' = 0 \end{array}\right\} \ldots (2.)$$

The object therefore is to derive from equations (1) and (2) the values of m and n, of m' and n', and of m'' and n''.

Multiplying the first of equations (1) by n'', and the last by n', and taking the difference of the products, we have

$$(c + b'm + a'n)(n'' - n') + (a + c'm + bn)(m'n'' - n'm'') = 0.$$

The same nperation being performed no the first and last of equatioos (2,) there results

$$n'' - n' + m\,(m' n'' - n' m'') = 0;$$

therefore eliminating $n'' - n'$ from the last two equations, we have

$$m = \frac{b' + a\,m + c'\,n}{c + b'\,m + a'\,n} \,\ldots.(3.)$$

In like manner, if the first and last of equations (1) and (2) be successively multiplied by m'', m', aod the difference of the respective products taken, we shall have

$$n = \frac{a' + c'\,m + b\,n}{c + b'\,m + a'\,n} \,\ldots.(4.)$$

From (3) is derived

$$n = \frac{m\,(a + c) + b'\,(1 - m^2)}{a'\,m - c} \ldots.(5.)$$

and from (4)

$$a'\,n^2 + (b'\,m + c - b)\,n - c'\,m - a' = 0.$$

If the value of n in (5) be substituted in this equation, the result is a cubic equation which must contain at least one real value of m, tu which there corresponds a real value of n deducible from equation (3.)

It may be proved, in like manner, that there must exist at least one real value of m' and n', and of m'' and n''.

Now the cubic equations involving m, m' and m'' will be found identical, as may at once be inlcrred from the symmetrical form of equatioos (1) and (2;) therefore m, m', m'' are the three roots of the same cubic equation. Hence it follows, that there cao be only one system of conjugate diametral planes that are rectangular.

The intersections of each two of these planes are called *the principal diameters*, and the points in which they cut the surface are called *the vertices*.

(107.) *To find the form of the equation to surfaces of the second order, when the coordinate planes are parallel to a system of conjugate diametral planes.*

The equation to any diametral plane is

$$(a\,m + c'\,n + b')\,x + (c'\,n + b\,n + a)\,y + (b'\,m + a'\,n + c)\,z + a''\,m + b''\,n + c'' = 0.$$

If m and n, successively, be now supposed first to be infinitely great, and next to be equal to 0, the resulting equations will be the equations to the diametral planes which bisect the chords parallel to the axes of x, of y, and of z.

Heoce when the coordinate planes are parallel to a system of coojugate diametral planes, of the three equations

$$a\,x + c'\,y + b'\,z + a'' = 0,$$
$$c'\,x + b\,y + a'\,z + b'' = 0,$$
$$b'x + a'\,y + c\,z + c'' = 0;$$

the first ought only to involve x, the second y, and the third z;

$$\therefore\ a' = 0,\ b' = 0,\ c' = 0;$$

wherefore the general equation becomes

$$a\,x^2 + b\,y^2 + c\,z^2 + 2\,a''\,x + 2\,b''\,y + 2\,c''\,z + d = 0,$$

which is the equation required.

Let the origin be now transferred to a point (α, β, γ) which is effected by substituting in the last equation

$$x + \alpha,\ y + \beta,\ z + \gamma,$$

for

$$x,\ y,\ z; \quad \text{hence}$$

$$a\,(x + \alpha)^2 + b\,(y + \beta)^2 + c\,(z + \gamma)^2 + 2\,a''\,(x + \alpha) + 2\,b''\,(y + \beta) + 2\,c''\,(z + \gamma) + d = 0;$$

or, developing, and arranging the result,

$$a\,x^2 + b\,y^2 + c\,z^2 + 2\,(a\,\alpha + a'')\,x + 2\,(b\,\beta + b'')\,y + 2\,(c\,\gamma + c'')\,z + a\,\alpha^2 + b\,\beta^2 + c\,\gamma^2 + 2\,a''\,\alpha + 2\,b\,\beta'' + 2\,c''\,\gamma + d = 0;$$

hence if the last term be represented by f,

$$a\,x^2 + b\,y^2 + c\,z^2 + 2\,(a\,\alpha + a'')\,x + 2\,(b\,\beta + b'')\,y + 2\,(c\,\gamma + c'')\,z + f = 0.$$

Now α, β, γ being arbitrary quantities, we may fix their valoe by supposing that the coefficients of x, of y, of $z = 0$; we thus have

$$a\,\alpha + a'' = 0,\ b\,\beta + b'' = 0,\ c\,\gamma + c'' = 0,$$

or

$$\alpha = -\frac{a''}{a};\ \beta = -\frac{b''}{b};\ \gamma = -\frac{c''}{c},$$

which are evidently the coordinates of the *centre*, Art. 105

Hence the general equation is reducible to the form

$$a\,x^2 + b\,y^2 + c\,z^2 + f = 0 \ldots.(1.)$$

This reduction has been effected on the supposition that the general equation contains the terms $a\,x^2$, $b\,y^2$, $c\,z^2$; if any one of these, $a\,x^2$ for instance, be wantiog, then since $a = 0$, we have

$$\alpha = -\frac{a''}{0} = \infty;$$

therefore, since the term $2\,a''\,(x + \alpha)$ caonot be taken away, the equation assumes the form

$$b\,y^2 + c\,z^2 + 2\,a''\,x + f = 0.$$

Now, by taking away the terms involving y and z, we have determined only two of the three quantities α, β, and γ; we may fix the valoe of the third, α, by supposing the last term to equal 0; a supposition which is always possible, since $a\,\alpha^2$ vanishing, that term is only of one dimension in α. The equatioo thus reduced will be of the form

$$b\,y^2 + c\,z^2 + 2\,a''\,x = 0 \ldots.(2.)$$

Hence surfaces of the second order are divisible into two classes, characterised by equations of the form

$$A\,z^2 + B\,y^2 + C\,x^2 + D = 0,$$

and

$$A\,z^2 + B\,y^2 + E\,x \qquad = 0.$$

ON SURFACES OF THE SECOND ORDER WHICH HAVE A CENTRE.

(108.) In order to ascertain the differeot species of surfaces represeoted by the equation

$$A\,z^2 + B\,y^2 + C\,x^2 + D = 0,$$

we shall make successively x, y, and z equal to some coostant quantity, which amounts to the same thing as cutting the surface by planes respectively parallel to the coordioate planes. Now the nature of the intersection will depend, as was shown in Part I., on the signs of the coefficients A, B, C. Hence, by assigning to these quantities all the varieties of sign which they admit of, the above equation will assume the following different forms:

(1.) $A\,z^2 + B\,y^2 + C\,x^2 + D = 0,$

when A, B, C are all positive.

(2.) $A\,z^2 + B\,y^2 - C\,x^2 + D = 0,$

Analytical Geometry.

when two of the coefficients are positive, and one negative.

$$(3.)\ A x^2 - B y^2 - C z^2 + D = 0,$$

when one of the coefficients is positive, and the other two negative.

$$(4.)\ -A x^2 - B y^2 - C z^2 + D = 0,$$

when the coefficients are all negative.

We shall now discuss these equations in succession.

(109.) I. When A, B, C are all positive, the equation is

$$A x^2 + B y^2 + C z^2 + D = 0,$$

in which the last term D may be *negative*, *positive*, or *zero*.

First. Let D be negative.

Then the equation is

$$A x^2 + B y^2 + C z^2 = D.$$

Let the surface be cut by planes parallel respectively to the coordinate planes; then if $x = \alpha$, the equation becomes

$$A z^2 + B y^2 = D - C \alpha^2 \ldots (1,)$$

which is the equation to the section made by a plane parallel to the plane of $y\,z$. In like manner, the equations

$$A x^2 + C z^2 = D - B \beta^2 \ldots (2,)$$
$$B y^2 + C z^2 = D - A \gamma^2 \ldots (3,)$$

belong to the sections made by planes parallel respectively to the planes of $x\,z$ and $x\,y$.

These sections therefore are ellipses, which become imaginary when

$$\alpha > \pm\sqrt{\frac{D}{C}},\ \beta > \pm\sqrt{\frac{D}{B}},\ \gamma > \pm\sqrt{\frac{D}{A}},$$

and are reduced to a point when

$$\alpha = \pm\sqrt{\frac{D}{C}},\ \beta = \pm\sqrt{\frac{D}{B}},\ \gamma = \pm\sqrt{\frac{D}{A}}.$$

This surface is limited in all directions, and is called, from the nature of its sections, *the ellipsoid*.

(110.) To find the *traces*, or *principal sections* of the ellipsoid.

These are determined by making x, y, and z successively equal 0 in the general equation, whence we have

$$A z^2 + B y^2 = D,$$
$$A x^2 + C z^2 = D,$$
$$B y^2 + C z^2 = D.$$

It appears, therefore, that the principal sections are *ellipses*.

(111.) *To find the points in which the surface intersects the three axes.*

Referring to the last article,

In the first equation, let

$$y = 0,\ \therefore z = \pm\sqrt{\frac{D}{A}} = OC,\ (\text{fig. } 30;)$$

in the second,

$$x = 0,\ \therefore y = \pm\sqrt{\frac{D}{B}} = OB;$$

in the third,

$$z = 0,\ \therefore x = \pm\sqrt{\frac{D}{A}} = OA.$$

The lines $A\,a$, $B\,b$, $C\,c$ are called the *principal diameters*, or *axes* of the ellipsoid.

Part II.

(112.) *To express the equation to the ellipsoid in terms of its principal diameters.*

Let $\quad OA = a,\ OB = b,\ OC = c.$

Then $\quad A = \dfrac{D}{c^2},\ B = \dfrac{D}{b^2},\ C = \dfrac{D}{a^2};$

hence we have, by substitution,

$$\frac{z^2}{c^2} + \frac{y^2}{b^2} + \frac{x^2}{a^2} = 1 \ldots (1,)$$

or

$$a^2 b^2 z^2 + a^2 c^2 y^2 + b^2 c^2 x^2 = a^2 b^2 c^2 \ldots (2.)$$

If any two of the coefficients be equal, for example, those of x^2 and y^2, then the equation becomes

$$b^2 z^2 + c^2 (x^2 + y^2) = a^2 c^2,$$

which is the equation to an ellipsoid of revolution about the axis of c. In like manner, if $a = c$, or $b = c$, the resulting equation will be that to an ellipsoid of revolution about the axis of b, or of a.

If $a = b = c$, the equation will be

$$z^2 + y^2 + x^2 = a^2,$$

which is the equation to a spherical surface.

(113.) Secondly. We have hitherto supposed D to be negative, let it now be considered positive, then

$$A x^2 + B y^2 + C z^2 = -D,$$

which is impossible; therefore the surface is in this case imaginary.

(114.) Thirdly. Let $D = 0$,

$$\therefore A x^2 + B y^2 + C z^2 = 0,$$

which is the equation to a point.

Hence, the first species of surfaces that have a centre is an ellipsoid, which in particular cases becomes an *ellipsoid of revolution*, *a sphere*, *a point*, *and an imaginary surface*.

(115.) II. When A and B are positive and C negative, the equation becomes

$$A x^2 + B y^2 - C z^2 + D = 0.$$

First. Let D be negative.

Then $\quad A x^2 + B y^2 - C z^2 = D.$

Let

$$x = \alpha,\ \therefore A z^2 + B y^2 = D + C \alpha^2 \ldots (1,)$$
$$y = \beta,\ \therefore A x^2 - C z^2 = D - B \beta^2 \ldots (2,)$$
$$z = \gamma,\ \therefore B y^2 - C x^2 = D - A \gamma^2 \ldots (3.)$$

It appears, therefore, that the sections of the surface made by planes parallel to $y\,z$ are in all cases ellipses; the two remaining sections are hyperbolas. Hence the surface is continuous, and is called the *hyperboloid of one sheet.**

The principal sections of the surface are

1. An ellipse, whose equation is

$$A x^2 + B y^2 = D.$$

2. An hyperbola, whose equation is

$$A x^2 - C z^2 = D.$$

3. An hyperbola, whose equation is

$$B y^2 - C z^2 = D.$$

(116.) *To express the equation to the hyperboloid of one sheet, in terms of its principal diameters.*

If $\quad a = \sqrt{\dfrac{D}{A}},\ b = \sqrt{\dfrac{D}{B}},\ c = \sqrt{\dfrac{D}{C}}$

* For the explanation of the term *sheet*, see Art. 99.

then the equation becomes, by substitution,

$$a^2 b^2 z^2 + a^2 c^2 y^2 - b^2 c^2 x^2 = a^2 b^2 c^2.$$

If in this equation $b = c$, we shall have

$$a^2 (z^2 + y^2) - b^2 x^2 = a^2 b^2,$$

which is the equation to a hyperboloid of revolution about the axis of a.

(117.) Secondly. Let D be positive.

The equation then becomes

$$A z^2 + B y^2 - C x^2 = - D.$$

Let

$$x = a, \therefore A z^2 + B y^2 = C a^2 - D \ldots. (1.)$$
$$y = \beta, \therefore A z^2 - C x^2 = - B \beta^2 - D \ldots. (2.)$$
$$z = \gamma, \therefore B y^2 - C x^2 = - A \gamma^2 - D \ldots. (3.)$$

It appears from equations (2) and (3) that the sections of the surface by planes parallel to the planes of $x z$ and $x y$ are hyperbolas. The section parallel to the plane of $y z$ is an ellipse so long as $a^2 > \frac{D}{C}$, or $a > \pm\sqrt{\frac{D}{C}}$.

Hence, if two planes be drawn parallel to the plane of $y z$ at the distance $+\sqrt{\frac{D}{C}}$, and $-\sqrt{\frac{D}{C}}$, the surface will have no point situated between those planes, but will extend indefinitely above and below them. This surface is composed, therefore, of two distinct parts, and is for that reason called the *hyperboloid of two sheets*.

The principal sections are

1. An imaginary line, $A z^2 + B y^2 = - D$,
2. An hyperbola, $A z^2 - C x^2 = - D$,
3. An hyperbola, $B y^2 - C x^2 = - D$

These hyperbolas have evidently a common transverse axis which coincides with the axis of x.

(118.) Proceeding as in the former case, we obtain for the equation of the hyperboloid of two sheets referred to its principal diameters,

$$a^2 b^2 z^2 + a^2 c^2 y^2 - b^2 c^2 x^2 = - a^2 b^2 c^2.$$

Of the three diameters $2 a$, $2 b$, $2 c$, the first alone meets the surface.

If in this equation $b = c$, we shall have

$$a^2 (z^2 + y^2) - b^2 x^2 = - a^2 b^2,$$

which is the equation of a hyperboloid of revolution about the axis of x.

(119.) Thirdly. If $D = 0$.

Then the equations of the two species of hyperboloid become

$$A z^2 + B y^2 - C x^2 = 0,$$

or

$$\frac{y^2}{z^2} = \frac{C}{B} \frac{x^2}{z^2} - \frac{A}{B},$$

$$\therefore \frac{y}{z} = f\left(\frac{x}{z}\right),$$

whence the surface becomes that of a cone.

The equation to the two species of hyperboloid becomes, therefore, in particular cases, the equation to an hyperbola of revolution, and to a conical surface.

ON SURFACES WHICH HAVE NOT A CENTRE.

(120.) The general equation is

$$A z^2 + B y^2 + E x = 0,$$

in which A and B may have the same, or different signs.

1. Let A and B be both positive; and since E may be either negative or positive: first, let it be negative.

Then the equation is

$$A z^2 + B y^2 = E x.$$

Let

$$x = a, \therefore A z^2 + B y^2 = E a \ldots. (1.)$$
$$y = \beta, \therefore A z^2 = E x - B \beta^2 \ldots. (2.)$$
$$z = \gamma, \therefore B y^2 = E x - A \gamma^2 \ldots. (3.)$$

The section made by the first plane is evidently an ellipse, which is real so long as a remains positive. When $a = 0$, the ellipse is reduced to a point, and it becomes imaginary when a is negative. The surface, therefore, extends indefinitely to the right of the origin, in the direction of the axis of x, and is limited towards the left by the plane of $y z$, which it touches. The remaining sections are evidently parabolas.

Let $y = 0$, then $z^2 = \frac{E}{A} x$,

$z = 0$, $y^2 = \frac{E}{B} x$.

Hence the principal sections by the planes of $x z$ and $x y$ are parabolas.

This surface is called the *elliptic paraboloid*.

Next, let E be positive, then the equation is

$$A z^2 + B y^2 = - E x,$$

which becomes, if x be supposed negative,

$$A z^2 + B y^2 = E x;$$

hence the surface is the same as before, only it now extends indefinitely to the *left* of the origin.

When $A = B$, the equation becomes

$$z^2 + y^2 = \frac{E}{A} x,$$

which belongs to a *paraboloid of revolution* about the axis of x.

(121.) 2. Let A and B have different signs, the equation will then assume the form

$$A z^2 - B y^2 = E x.$$

Let

$$x = a, \text{ then } A z^2 - B y^2 = E a \ldots. (1.)$$
$$y = \beta, \quad A z^2 = B \beta^2 + E x \ldots. (2.)$$
$$z = \gamma, \quad B y^2 = A \gamma^2 - E x \ldots. (3.)$$

The first of these equations is that to an hyperbola, whose transverse axis is parallel to the axis of x or of y, according as a is positive or negative. The remaining two equations are those to parabolas whose axes are parallel to the axis of x.

The principal sections have for their equations,

(1.) $A z^2 - B y^2 = 0$,

(2.) $A z^2 = E x$,

(3.) $B y^2 = - E x$.

The first equation is that to two straight lines which intersect at the origin, and the remaining two are the

equations to parabolas which have a common origin, and whose axes are A X, A X′.

This surface is called the *hyperbolic paraboloid.*

(122.) *No section of the hyperbolic paraboloid made by a plane can be an ellipse.*

For if x be eliminated between the equation to any plane, and the equation

$$A z^2 - B y^2 = E x,$$

no term of the resulting equation can involve $y z$. Hence, since the terms containing z^2 and y^2 are of different signs, the equation cannot represent an ellipse.

It thence follows, that this species of paraboloid can never become a surface of revolution.

(123.) We shall conclude this discussion of surfaces of the second order with proving, that

The equation to the paraboloid may be deduced from the equation to surfaces of the first class.

The equation to the ellipsoid, or hyperboloid of one sheet, is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \pm \frac{z^2}{c^2} = 1$$

Let the origin be transferred to the extremity of the diameter $2\,a$, which is done by substituting $a - x$ for x; then the above equation becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \pm \frac{z^2}{c^2} = \frac{2\,x}{a},$$

therefore, multiplying each term by a, we have

$$\frac{x^2}{a} + \frac{a}{b^2} y^2 \pm \frac{a}{c^2} z^2 = 2\,x \ldots . (1.)$$

Now the principal sections made by the planes of $x\,y$, $x\,z$ are in the ellipsoid, ellipses; and, in the hyperboloid of one sheet, an ellipse and hyperbola. Let m and m' denote the distances from the foci of these sections to the vertex of the diameter $2\,a$.

Then if the centre be supposed infinitely distant, $\frac{x^2}{a} = 0$, also $\frac{b^2}{a} = 2\,m$, and $\frac{c^2}{a} = 2\,m'$.

Therefore, by substitution, equation (1) becomes

$$m' y^2 \pm m\, z^2 = 4\, m\, m'\, x,$$

which is the equation to the elliptic or hyperbolic paraboloid, according as the upper or lower sign is used.

For farther information on the subject of Curves and of Surfaces, the reader is referred to the following works:

Annales de Mathématiques; Biot's *Essai de Géométrie Analytique*, sixth edition; Bourdon's *Application de l'Algèbre à la Géométrie*; Boucharlat's *Théorie des Courbes et des Surfaces du Second Ordre*; *Correspondance sur l'Ecole Polytechnique*; Cramer's *Introduction à l'Analyse des Lignes Courbes Algébriques*; Euler's *Introductio in Analysin Infinitorum*, tom. ii.; Garnier's *Géométrie Analytique*; Hamilton's *Principles of Analytical Geometry*; *Journal de l'Ecole Polytechnique*; Lacroix's *Traité Elémentaire de Trigonométrie Rectiligne et Sphérique, et d'Application de l'Algèbre à la Géométrie*; Lardner's *Algebraic Geometry*; Maclaurin's *Algebra*; Monge's *Application de l'Algèbre à la Géométrie*; Peacock's *Examples on the Differential and Integral Calculus*; Poullet-Delisle's *Application de l'Algèbre à la Géométrie*; Reynaud's *Traité d'Application d'Algèbre à la Géométrie.*

LINES OF THE SECOND ORDER,

OR

CONIC SECTIONS.

Conic Sections.

THE principal properties of Lines of the Second Order, or, as they are more generally termed, of Conic Sections, may be derived with great facility from their equations, which have been already obtained in the Article entitled Analytical Geometry. But with a view of rendering the following Treatise independent of any previous investigation, we propose to deduce those equations from a general definition first assumed by Boscovich,* and subsequently adopted by other writers of celebrity, as the basis of geometrical systems of Conic Sections.

(1.) *Definition.* A Conic Section is the locus of a point, whose distances from a fixed point and a straight line given in position, are to each other in a constant ratio.

Fig. 1. Thus, let S be a fixed point, K *k* a straight line given in position, P any point; join P, S, and let fall the perpendicular P Q upon K *k*; then, if P be always taken to P Q in the same constant ratio, the locus of P will be a conic section.

The fixed point S is called the *focus*, and the straight line K *k*, given in position, the *directrix*.

(2.) The particular species of the conic section will depend upon the constant ratio of P S : P Q, which may be either a ratio of equality, or of lesser or greater inequality.

1. Let $PS = PQ$.

Then the locus of P is called the *Parabola*.

2. Let $PS < PQ$.

Then the locus of P is called the *Ellipse*.

3. Let $PS > PQ$.

Then the locus of P is called the *Hyperbola*.

ON THE PARABOLA.

CHAPTER I.

ON THE PARABOLA REFERRED TO ITS AXIS.

(3.) *To find the equation to the parabola.*

The parabola is the locus of a point whose distance from the focus is always equal to its perpendicular distance from the directrix.

Fig 2. Let S be the focus, K *k* the directrix, P any point in the parabola; through S draw the indefinite line E S X perpendicular to the directrix; from P let fall the perpendiculars P M and P Q on E X, K *k* respectively, and join P, S. Parabola.

Then, if E S be bisected in A, the point A is, agreeably to the definition, a point in the parabola; from A draw A Y at right angles to A X, and assume A X and A Y as the rectangular axes, to which the parabola is to be referred.

Let $AM = x$, $MP = y$, and $AS = m$.

Then $SP^2 = PM^2 + MS^2 = y^2 + (x - m)^2 \ldots\ldots (1.)$

but $SP^2 = PQ^2 = EM^2 = (EA + AM)^2$

$= (m + x)^2 \ldots\ldots (2.)$

Whence, equating these two values of SP^2,

$$y^2 + (x - m)^2 = (x + m)^2,$$

$$\therefore y^2 = 4\,m\,x,$$

which is the equation required.

(4.) *To determine the figure of the parabola from its equation.*

The same axes being employed, the equation to the parabola is

$$y^2 = 4\,m\,x,$$

or

$$y = \pm 2\sqrt{m\,x}.$$

Let $x = 0$, $\therefore y = 0$; Fig. 3.

therefore the curve passes through the origin A.

Let x be supposed to have any *positive* value.

Then, for each assumed value of x, there are *two* equal values of y, with contrary signs; as x increases, the values of y increase; and when x is taken indefinitely great, the values of y will also become indefinitely great.

Let x be now supposed to have any *negative* value.

Then the values of y being in this case imaginary, it is plain that no part of the curve can lie to the *left* of A.

The parabola consists, therefore, of two infinite branches A Z, A *z* situated to the *right* of the point A, and symmetrically placed with respect to the straight line A X.

The point A is called the *vertex*, and the line A X the *axis*, of the parabola.

Cor. 1. The parabola can have but one focus, and one directrix.

Cor. 2. To find the value of S L, the ordinate passing through the focus.

At the point S, $x = AS = m$,

$$\therefore y^2 = 4\,m^2,$$

$$\therefore y \text{ or } SL = \pm 2\,m.$$

* *Element. Univers. Mathes.* tom. iii. p. 1.

The double ordinate $L\,l$ passing through the focus, is called the *principal parameter*, or *latus rectum*, of the parabola.

Cor. 3. Hence, if P be any point in the parabola,

$$PM^2 = L\,l \,.\, AM;$$

that is, *the square of the ordinate is equal to the latus rectum multiplied into the corresponding abscissa.*

(5.) *To find the intersection of a straight line with a parabola.*

Let the equation to the proposed straight line be

$$y = \alpha x + \beta \ldots (1.)$$

Then the coordinates of the point or points of intersection with the parabola will be determined by combining this equation with the equation

$$y^2 = 4\,m\,x \ldots (2.)$$

Substituting, then, in (2) the value of x derived from (1,) we have

$$y^2 \quad 4\,m \,.\, \frac{y-\beta}{\alpha},$$

or

$$y^2 - \frac{4\,m}{\alpha}\,y + \frac{4\,m\,\beta}{\alpha} = 0.$$

This quadratic gives two values of y, which, substituted in (1), furnish two corresponding values of x; therefore the coordinates required may be determined.

When the two roots of the quadratic are equal, the points of section coincide, and the straight line $P\,p$ will then *touch* the parabola; and when the two roots are *imaginary*, the straight line $P\,p$ falls entirely without the parabola.

Hence it appears, that a straight line cannot cut a parabola in more than two points.

That part of the straight line contained within the parabola, is called a *chord*; when it passes through the focus, it is then called the *focal chord.*

(6.) *To find the equation to a straight line that touches a parabola in a given point.*

Let x', y' be the coordinates of the given point, and x'', y'' those of any other point in the parabola near the first.

Then the equation to the straight line drawn through these two points, and cutting the parabola, is

$$y - y' = \frac{y''-y'}{x''-x'}\,(x - x') \ldots (1.)$$

But these points being in the parabola we have

$$y'^2 = 4\,m\,x',$$

and

$$y''^2 = 4\,m\,x'',$$

$$\therefore y''^2 - y'^2 = 4\,m\,(x'' - x'),$$

$$\therefore \frac{y''-y'}{x''-x'} = \frac{4\,m}{y''+y'},$$

therefore equation (1) becomes, by substitution,

$$y - y' = \frac{4\,m}{y''+y'}\,(x - x').$$

Let the points (x', y') and (x'', y'') be now supposed to coincide, then $x'' = x'$, $y'' = y'$, and the *secant* will become a *tangent.*

Hence the equation to the tangent is

$$y - y' = \frac{2\,m}{y'}\,(x - x'),$$

in which x', y' are the coordinates of the point of contact, and x, y *the variable coordinates of any point whatever, in the tangent.*

Cor. The equation to the tangent may be presented under a more commodious form; for multiplying each side by y' we have

$$y\,y' - y'^2 = 2\,m\,x - 2\,m\,x',$$

but

$$y'^2 = 4\,m\,x',$$

$$\therefore y\,y' = 2\,m\,x + 4\,m\,x' - 2\,m\,x',$$

$$= 2\,m\,(x + x'),$$

which is the equation most commonly used.

(7.) *To find the intersection of the tangent with the axis.*

In the equation $y\,y' = 2\,m\,(x + x')$.

Let $y = 0$, as at T, then $x + x' = 0$, Fig 4.

or $x = -x'$,

that is, $AT = AM$;

the negative sign merely implying that AT must be measured in the contrary direction to AM.

Cor. 1. Hence $MT = 2\,MA$.

Def. The line MT intercepted between the foot of the ordinate, and the point where the tangent meets the axis, is called the *subtangent.*

It appears, therefore, that the subtangent is equal to twice the abscissa.

Cor. 2. Hence is derived a simple method of drawing a tangent to a parabola at a given point.

Let P be the given point, and AM, MP its coordinates; in MA produced take $AT = AM$, join TP, then TP touches the parabola in P.

Def. The straight line which is drawn from the point of contact at right angles to the tangent, is called the *normal.*

(8.) *To find the equation to the normal.*

Let TP touch the parabola in P, and from this point draw $P\,g$ at right angles to PT.

Then, since $P\,g$ is at right angles to PT, whose equation is

$$y - y' = \frac{2\,m}{y'}\,(x - x'),$$

the equation to $P\,g$ will be

$$y - y' = -\frac{y'}{2\,m}\,(x - x').$$

(9.) *To find the intersection of the normal with the axis.*

When the normal cuts the axis, as at G, then $y = 0$,

$$\therefore -y' = -\frac{y'}{2\,m}\,(x - x',)$$

$$\therefore x - x' = 2\,m;$$

that is, $AG - AM$, or $MG = 2\,m$.

Def. The line MG, intercepted between the foot of the ordinate and the point where the normal cuts the axis, is called the *subnormal.*

Hence it appears, that *the subnormal is equal to half the latus rectum.*

We have considered the normal $P\,g$ as an indefinite line, but it is customary to give that name to the straight line PG intercepted between the point of contact and the point in which $P\,g$ cuts the axis.

(10.) *To draw a tangent to a parabola from a given point (x'', y'') without it.*

Let x', y' be the *unknown* coordinates of the point of contact.

The equation to the tangent being in general

$$y = \frac{2m}{y'}(x + x'),$$

and the point (x'', y'') being, by hypothesis, a point in the tangent, we have

$$y'' = \frac{2m}{y'}(x'' + x') \dots (1.)$$

Also, the point of contact (x', y') being in the parabola,
$$y'^2 = 4mx' \dots (2;)$$

hence, by means of these two equations, the coordinates x', y' of the point of contact may be determined.

Since the equation which results from the elimination of x' between (1) and (2) is of the second degree, it follows, that there are two points of contact, or that two tangents may be drawn to a parabola from a given point without it.

In general, the values of x and y, obtained by elimination between any two equations, are the coordinates of the point or points in which the loci of such equations intersect. We may at once, therefore, in the question under consideration, determine the position of the points of contact by constructing the loci of (1) and (2) in which x' and y' are the variable quantities.

Now, the locus of (2) is the given parabola, and that of (1) is evidently a straight line, whose position may be assigned by making x' and y' successively $= 0$. ANALYTICAL GEOMETRY, Art. 10.

If $\quad x' = 0$, then $y' = 2m\dfrac{x''}{y''}$,

$y' = 0$, then $x' = -x''$.

Fig. 5. Hence, take A T in the opposite direction to A M $= x''$, and in A Y take $\text{A B} = 2m\dfrac{x''}{y''}$; join T, B, and let T B cut the parabola in the points P, p; these will be the points of contact required.

Cor. 1. Since the straight line which has just been constructed determines, by its intersection with the parabola, the points of contact, it follows that the equation

$$y''y' = 2m(x' + x''),$$

in which x' and y' are variable, is the equation to the indefinite straight line joining the points of contact.

Cor. 2. Since A T is independent of y'', it will remain the same for all points whose abscissas are $= x''$, that is, for all points in the indefinite line Q q drawn through Q parallel to A Y. Hence the following theorem:

If from the several points of a line, perpendicular to the axis, pairs of tangents be drawn to a parabola, the chords joining the points of contact in each case will all pass through the same point.

Cor. 3. If the given line be the directrix, then $x'' = -m$; therefore $\text{A T} = m = \text{A S}$, and *all the chords will in this case pass through the focus.* Hence the equation to the focal chord of contact is

$$y''y' = 2m(x' - m).$$

CHAPTER II.

ON THE PARABOLA REFERRED TO THE FOCUS.

(11.) *To find the polar equation to the parabola, the focus being the pole.*

Let P be any point in the parabola, P Q and P M perpendiculars on the directrix and axis respectively, and join P, S. Fig. 6.

Let $\quad \text{P S} = r$, angle $\text{A S P} = \omega$.

Then $\quad \text{S P} = \text{P Q} = \text{E M} = \text{E S} + \text{S M},$

$$\therefore r = 2m + r\cos \text{PSM},$$

$$= 2m - r\cos\omega,$$

$$\therefore r = \frac{2m}{1 + \cos\omega} \dots (1.)$$

or
$$= \frac{m}{\cos^2\dfrac{\omega}{2}} \dots (2.)$$

Cor. 1. If P S be produced to meet the parabola in p, and S p be denoted by r', then since $\text{A S}p = \pi - \omega$,

$$r' = \frac{2m}{1 - \cos\omega},$$

or
$$= \frac{m}{\sin^2\dfrac{\omega}{2}}$$

Cor. 2. Hence

$$\frac{1}{r} + \frac{1}{r'} = \frac{1 + \cos\omega}{2m} + \frac{1 - \cos\omega}{2m} = \frac{2}{2m},$$

$$\text{or } \frac{1}{\text{S P}} + \frac{1}{\text{S}p} = \frac{2}{\text{S L}}.$$

That is, the *principal semi-parameter is an harmonic mean between the segments of any chord drawn through the focus.*

Cor. 3. Since $\dfrac{1}{r} + \dfrac{1}{r'} = \dfrac{r + r'}{rr'}$ and also $= \dfrac{2}{2m}$,

$$\therefore rr' = m(r + r'),$$

that is, $\quad \text{S P} \cdot \text{S}p = m \cdot \text{P}p.$

(12.) *If from the point of contact two straight lines be drawn, one parallel to the axis, and the other to the focus, they will make equal angles with the tangent.*

Let T P be a tangent, from P draw P X′ parallel to A X, and join P S; the angles $t\text{P X}' = \text{S P T}$. Fig. 7.

For since $\text{A T} = \text{A M}$,

$$\text{S T} = \text{S A} + \text{A T} = m + x = \therefore \text{S P}, \ (11,)$$

therefore angle S P T = angle S T P,

$$= t\text{P X}',$$

because P X′ is parallel to A X.

(13.) *The tangent at any point, and the perpendicular let fall upon it from the focus, intersect* A Y *in the same point.*

Let T P be a tangent at P, from S let fall the perpendicular S Q upon it, to prove that Q is a point in A Y. Fig. 8.

The equation to P T is

$$y = \frac{2m}{y'}(x + x') \dots (1.)$$

and the equation to S Q drawn from S ($x = m$, $y = o$) perpendicular to T P is

$$y = -\frac{y'}{2m}(x - m) \ldots. (2.)$$

Now when P T, S Q meet A Y, x must $= o$ in both equations, we have therefore from (1)

$$y = 2m\frac{x'}{y'} = 2m \cdot \frac{y'^2}{4my'} = \frac{y'}{2},$$

and from (2) $y = \frac{y'}{2}$;

and as these values of y are identical, P T and S Q meet A Y in the same point.

Cor. Hence S T . S A = S Q², or since S T = S P,

$$SP \cdot SA = SQ^2.$$

(14.) *If two lines be drawn from the focus, one to the point of contact, and the other to the point in which the tangent meets the directrix, they will be at right angles to each other.*

Fig. 9. Let the tangent at P meet the directrix in Q, then drawing S P, S Q, it is required to prove that S P is perpendicular to S Q.

The equation to the tangent being

$$y = \frac{2m}{y'}(x + x').$$

When it meets the directrix, $x = -m$,

$$\therefore y \text{ or } EQ = \frac{2m}{y'}(x' - m).$$

Now the equation to S Q is

$$y = -\tan QSE\,(x - m),$$

$$= -\frac{QE}{2m}(x - m),$$

$$= \therefore -\frac{x' - m}{y'}(x - m) \ldots. (1.)$$

Also, the equation to S P is

$$y = \tan PSX\,(x - m),$$

$$= \frac{PM}{MS}(x - m),$$

$$= \therefore \frac{y'}{x' - m}(x - m) \ldots. (2;)$$

therefore comparing the coefficients in (1) and (2) it follows (Analytical Geometry, (Art. 14,)) that S P is perpendicular to S Q.

The proposition may be at once proved by taking the equation to the focal chord of contact. For that equation being

Fig. 9.

$$y' = \frac{2m}{y''}(x' - m), \text{ Art. 10, } Cor. 3,$$

and the equation to S Q being

$$y = -\frac{QE}{ES}(x - m),$$

$$= -\frac{y''}{2m}(x - m), \text{ since } QE = y'',$$

it follows, that S Q is perpendicular to S P.

CHAPTER III.

ON THE PARABOLA REFERRED TO ANY DIAMETER.

(15.) *To find the locus of the middle points of any number of parallel chords.*

Let P p be any chord, O its middle point; from the points P, O, p let fall the perpendiculars P M, O N, $p\,m$ on the axis A X; then if the equation to P p be Fig. 10.

$$y = a x + \beta,$$

the equation containing the values of y at the points P, p will be

$$y^2 - \frac{4m}{a}y + \frac{4m\beta}{a} = 0. \quad \text{Art. 5.}$$

Now, since in any quadratic equation the coefficient of the second term with its proper sign is equal to the sum of the roots with their signs changed,

$$\frac{4m}{a} = PM + pm.$$

But O being the middle point of P p

$$ON = \frac{PM + pm}{2}$$

$$= \therefore \frac{2m}{a}.$$

Now m is constant, and a remains the same for all chords parallel to P p, therefore this value of O N is invariable; in other words, the equation to the middle points of any number of parallel chords is

$$y = \text{constant},$$

therefore (Analytical Geometry, Art. 4) the locus required is a straight line parallel to the axis A X.

Def. 1. The straight line which has just been shown to bisect any number of parallel chords is called a *diameter.*

Def. 2. Each half of the chord, so bisected, is called *an ordinate* to the diameter bisecting it.

Cor. 1. The diameters of the parabola are parallel to the axis, and intersect the curve only in one point.

The truth of the first part of the corollary is evident from the proposition; that of the second may be thus proved.

The equation to any diameter is

$$y = c, \; c \text{ being a constant quantity};$$

therefore the intersection of the diameter with the parabola will be determined by combining this equation with the equation $\quad y^2 = 4mx;$

we therefore have $\quad c^2 = 4mx,$

$$\therefore x = \frac{c^2}{4m}.$$

Hence there is only *one* point of intersection.

Cor. 2. If the equation to any chord be

$$y = a x + \beta \ldots. (1.)$$

the equation to a diameter passing through any point (x', y'), and bisecting that chord, will be

$$y' = \frac{2m}{a} \ldots. (2.)$$

Conversely, since $\quad a = \frac{2m}{y'},$

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the ordinate to a diameter passing through the point (x', y') will have for its equation

$$y = \frac{2m}{y'} x + \beta \ldots (3.)$$

Cor. 3. Comparing equation (3) with the equation to a tangent at the point (x', y'), (6) it appears *that the tangent applied at the vertex of any diameter is parallel to the ordinates of that diameter.*

(16.) *To find the equation to the parabola, when it is referred to any diameter and the tangent at its vertex, as axes.*

Fig. 11 Let PX' be any diameter, and PY' the tangent at its vertex; draw any chord Qq parallel to PY' meeting PX' in V; and let $PV = x$, $VQ = y$

Then since the chord Qq, being parallel to the tangent, is bisected in V, $VQ = Vq$, that is, for any assumed value of x there are two equal values of y with opposite signs. As the same thing holds true for all other chords drawn parallel to PY', the equation required must necessarily be of the form

$$y^2 = Mx \ldots (1.)$$

in which M is some constant quantity.

In order to determine M, let $Q'Sq'$ be the position of the chord when it passes through the focus, join PS, and produce VP to meet the directrix in O.

Then PS and PV making equal angles with PY', and therefore with $Q'q'$, $PV = PS = PO$, therefore OV is bisected in P and $x = PS'$.

Also, $Qq =$ sum of the distances of Q and q from the directrix $= 2\,OV = 4\,OP = 4\,PS$,

$$\therefore y = 2\,PS.$$

Substituting these values of x and y in (1) we have

$$4\,PS^2 = M \,.\, PS,$$

$$\therefore M = 4\,PS;$$

therefore the equation required is

$$y^2 = 4\,PS \,.\, x.$$

Cor. 1. Hence if m denote the distance of the origin from the focus, the equation to the parabola is always

$$y^2 = 4\,m\,x.$$

Cor. 2. It appears that the ordinate passing through the focus is equal to four times the distance of the vertex from the focus,—this quantity is called the *parameter* of the diameter.

Cor. 3. Hence, at any point of the parabola, the square of the ordinate is equal to the parameter multiplied into the corresponding abscissa.

(17.) The equation to the tangent, when the parabola is referred to any diameter, is of the same form as before, namely,

$$y = \frac{2m}{y'}(x + x'),$$

the coefficient $\frac{2m}{y'}$ denoting in this case the ratio of the sines of the angles which the tangent makes with the axis of x and y.

Cor. 1. When the tangent meets the axis of x, then $y = 0$, $\therefore x = -x'$;

that is, *the subtangent is bisected by the curve, whether the coordinates are rectangular or oblique.*

Cor. 2. Hence also, whatever be the inclination of the axes, the equation of an ordinate to a diameter passing through any point (x', y') is Parabola.

$$y = \frac{2m}{y'}x + \beta.$$

See Art. 15, *Cor.* 2.

(18.) *If from the several points of a line given in position, pairs of tangents be drawn to a parabola, the lines joining the corresponding points of contact will all pass through the same point.*

Let MN be the given line, and AX the axis of the parabola; through any point in AX draw a chord mn parallel to MN, let PX' be the diameter which bisects this chord, and at the vertex P apply a tangent PY', which will be parallel to MN.

Then the equation to the parabola when referred to the oblique axes PX' and PY' will be

$$y^2 = 4\,m\,x \ldots (1.)$$

and if from any point (x'', y'') in MN a pair of tangents be drawn to the parabola, it may be shown precisely as in Art. 10, which is a particular case of the question under consideration, that the equation to the line joining the points of contact is

$$y'' y' = 2m(x' + x'') \ldots (2.)$$

in which x' and y' are the variable coordinates of the point of contact.

Let the chord (2) cut the axis of x, then $y' = 0$, and therefore $x' = -x''$;

hence the point of intersection will be the same for all points whose abscissa $= x''$, that is, for all points in the given line MN.

(19.) *If from the point of intersection of two tangents a diameter be drawn, it will bisect the line joining the points of contact.*

From the equation to an ordinate to the diameter passing through (x'', y'') is (15, *Cor.* 2,)

$$y = \frac{2m}{y''}x + \beta \ldots (1.)$$

And the equation to the line joining the points of contact is

$$y' = \frac{2m}{y''}(x' + x'') \ldots (2.)$$

therefore the latter being parallel to the former is also an ordinate, and consequently bisected.

(20.) *If through any point within or without a parabola two straight lines, given in position, be drawn to meet the curve, the rectangle contained by the segments of the one will be to that contained by the segments of the other in a constant ratio.*

Let O be any point within or without a parabola, and let any two straight lines drawn through that point meet the curve in the points R, r and Q, q; to prove that the ratio of

$$OR \,.\, Or : OQ \,.\, Oq \text{ is given.}$$

Through O draw the diameter PX', then the equation to the parabola when referred to that diameter, and the tangent at its vertex, is $y^2 = 4\,m\,x \ldots (1.)$

Let $OR = r$, $PO = \delta$, $\frac{\sin r, x}{\sin x, y} = p$, $\frac{\sin r, y}{\sin x, y} = q$. Fig. 12.

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Then
$$y = p\,r,$$
$$x = \delta - q\,r,$$
therefore by substitution in (1)
$$p^2 r^2 = 4\,m\,\delta - 4\,m\,q\,r,$$
$$\therefore r^2 + \frac{4\,m\,q}{p^2}\,r - \frac{4\,m\,\delta}{p^2} = o,$$
in which the two values of r are O R and O r; therefore by the theory of equation

$\text{O R} . \text{O} r = \frac{-4\,m\,\delta}{p^2}$. In like manner, if $\text{O Q} = r'$, and
$$\frac{\sin r', x}{\sin x, y} = p', \quad \frac{\sin r', y}{\sin x, y} = q', \quad \text{O Q} . \text{O} q = \frac{-4\,m\,\delta}{p'^2};$$
$$\therefore \text{O R} . \text{O} r : \text{O Q} . \text{O} q :: p'^2 : p^2;$$
but the directions of r, r' being by hypothesis given, the quantities p^2 and p'^2 are known, therefore these rectangles are to each other in a given ratio.

CHAPTER IV.

MISCELLANEOUS PROPOSITIONS.

(21.) *A parabola being traced upon a plane, to find the position of its axis.*

Fig. 13. Draw any two parallel chords P p, Q q, and bisect them by the line M N; that line will be a diameter. Art. 15.

In this diameter take any point, and through it draw R r perpendicular to M N, meeting the parabola in R, r; then if R r be bisected in O, and A O X be drawn parallel to M N, it will be the axis required, as is evident.

(22.) *Let* P p *be any chord cutting the axis in* O, *and let* A M, A m *be the respective abscissas of* P *and* p, *to prove that*
$$\text{A M} . \text{A}\,m = \text{A O}^2.$$
Let the equation to P p be
$$y = a\,x + b \ldots . (1.)$$
then the abscissas A M, A m will be found by eliminating y between this equation and
$$y^2 = 4\,m\,x \ldots . (2;)$$
we therefore have
$$(a\,x + b)^2 = 4\,m\,x.$$
But at the point O where P p cuts the axis, $y = o$,
$$\therefore x = -\frac{b}{a} = \text{A O},$$
$$\therefore \text{A O}^2 = \frac{b^2}{a^2} = \therefore \text{A M} . \text{A}\,m,$$
as was to be proved.

(23.) *In the axis* A X *of a given parabola to find a point* O *such that if any chord whatever* P O p *be drawn through it, the angle* P A p *may be a right angle.*

Parabola.

Since the proposed property is, by hypothesis, true of all chords whatever drawn through P, we shall take that which is at right angles to the axis.

Let P O p, therefore, be perpendicular to A X, and join A P, A p.

Then at the point P, if x, y be its coordinates,
$$y^2 = 4\,m\,x \ldots . (1.)$$
But since A X evidently bisects P A p, the angle P A O, and therefore also A P O, is half a right angle,
$$\therefore \text{A O} = \text{O P},$$
or
$$x = y;$$
therefore substituting x for y in equation (1)
$$x^2 = 4\,m\,x,$$
$$\therefore x = 0, \text{ and } = 4\,m.$$
The first value of x corresponds to the origin, the second to the point O; whence it follows, that a point whose distance from the vertex is equal to the *latus rectum*, has the property above mentioned.

(24.) *If pairs of tangents to a parabola be always supposed to intersect at right lines, to find the locus of their intersection.*

Let
$$y = a\,x + \beta \ldots . (1)$$
be any line cutting a parabola
$$y^2 = 4\,m\,x \ldots . (2,)$$
then the equation which contains the values of (y) at the points of intersection is (5)
$$y^2 - \frac{4\,m}{a}\,y + \frac{4\,m\,\beta}{a} = o \ldots . (3;)$$
but when these roots are equal, the intersecting line becomes a tangent; hence equation (3) is in this case a perfect square, the criterion of which is, *that four times the product of the extreme terms is equal to the square of the mean*; we have therefor
$$16\,\frac{m\,\beta}{a} = 16\,\frac{m^2}{a^2},$$
$$\therefore \frac{m}{a} = \beta = y - a\,x,$$
$$\therefore m = a\,y - a^2\,x,$$
$$\therefore a^2 - \frac{y}{x}\,a + \frac{m}{x} = o,$$
in which equation the values of a are the trigonometrical tangents of the angles which the two tangents to the parabola make with the axis; therefore the product of these values $= \frac{m}{x}$, and also $= -1$, since by hypothesis the tangents are at right angles to each other,
$$\therefore \frac{m}{x} = -1,$$
or
$$x = -m,$$
hence the locus of their intersection is the directrix.

ON THE ELLIPSE.

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CHAPTER I.

ON THE ELLIPSE REFERRED TO ITS AXIS.

The ellipse is the locus of a point whose distance from the focus is always less, in a given ratio, than its distance from the directrix.

(25.) *To find the equation to the ellipse.*

Fig. 2. Let S be the focus, K k the directrix, P any point in the ellipse; through S draw the indefinite line E S X perpendicular to the directrix; from P let fall the perpendiculars P M, P Q on E X, K k respectively, and join P, S.

Let the given ratio of P S : P Q be as $e : 1$, e being less than 1; then if S E be divided in A, so that S A : A E :: e : 1, A is a point in the ellipse.

From A draw A Y at right angles to A X, and assume A X, A Y as the rectangular axes to which the ellipse is to be referred.

Let $AM = x$, $MP = y$, and $AS = m$.

Then $SP^2 = PM^2 + MS^2 = y^2 + (x-m)^2 \ldots (1.)$

but $SP^2 = e^2 . PQ^2 \quad = e^2 (AE + AM)^2$

$$= e^2\left(\frac{m}{e} + x^2\right) \ldots (2;)$$

therefore equating (1) and (2)

$$y^2 + (x-m)^2 = m^2 + 2\,m\,e\,x + e^2 x^2,$$

$$\therefore y^2 = 2\,m\,(1+e)\,x - (1-e^2)\,x^2,$$

$$= (1-e^2)\left(\frac{2\,m}{1-e}\,x - x^2\right).$$

or if $\frac{m}{1-e}$ be assumed $= a$,

$$y^2 = (1-e^2)\,(2\,a\,x - x^2),$$

which is the equation required.

Cor. 1. In A X take $A\,a = 2\,a$, and bisect A a in C, then at this point, $x = a$,

$$\therefore y^2 = (1-e^2)\,a^2,$$

$$\therefore y = \pm\, a\,\sqrt{1-e^2},$$

which is always real, since $e < 1$.

Hence if B C b be drawn through C at right angles to A a, and C B, C b be each taken $= a\,\sqrt{1-e^2}$, B, b will be points in the ellipse.

Cor. 2. Let B b be denoted by 2 b,

then $$b = \pm\, a\,\sqrt{1-e^2},$$

$$\therefore \sqrt{1-e^2} = \pm\,\frac{b}{a},$$

therefore by substitution the above equation becomes

$$y = \pm\,\frac{b}{a}\,\sqrt{2\,a\,x - x^2} \ldots (1.)$$

Def. The straight lines A a and B b, represented by 2 a and 2 b, are called respectively the *major* and *minor axes;* the points A, a, B, b in which they meet the ellipse are called the *vertices;* and the point C in which they intersect each other, the *centre.* Ellipse.

(26.) *To find the equation to the ellipse, when the coordinates are measured from the centre.*

Let P be any point in the ellipse, let fall the perpendicular P M on A a, and assume $CM = x'$. Fig. 14.

Then the equation to the ellipse when the coordinates originate at A is

$$y^2 = \frac{b^2}{a^2}\,(2\,a\,x - x^2) \ldots (1.)$$

but $$x = AM = AC + CM,$$

$$= a + x';$$

therefore substituting this value for x, we have

$$y^2 = \frac{b^2}{a^2}\,\left\{2\,a\,(a+x') - (a+x')^2\right\},$$

$$= \frac{b^2}{a^2}\,(a^2 - x'^2) \ldots (2.)$$

which is the equation required.

Cor. Suppressing the accent, which was only used to distinguish the new from the old abscissa, we have by multiplication and transposition

$$a^2\,y^2 + b^2\,x^2 = a^2\,b^2 \ldots (3.)$$

If each term be divided by $a^2\,b^2$, we have

$$\frac{y^2}{b^2} + \frac{x^2}{a^2} = 1 \ldots (4.)$$

Of the three last forms of the equation to the ellipse, the equation marked (3) is the most frequently used.

When $a = b$, these equations represent the circle, which is therefore a species of the ellipse.

(27.) Equations (1) and (2) when translated into geometrical language, express a property of the ellipse. For if P be any point, we have

$$2\,a\,x - x^2 = (2\,a - x)\,x = AM . Ma, \text{ and}$$

$$a^2 - x'^2 = (a + x')\,(a - x') = AM . Ma,$$

$$\therefore MP^2 = \frac{BC^2}{AC^2} . AM . Ma,$$

or $$AM . Ma : MP^2 :: AC^2 : BC^2;$$

that is, *the rectangle contained by the segments of the major axis is to the square of the ordinate, as the square of the semiaxis major is to the square of the semiaxis minor.*

(28.) *To determine the figure of the ellipse, from its equation.*

Resuming the equation $a^2\,y^2 + b^2\,x^2 = a^2\,b^2$, we have either

$$y = \pm \frac{b}{a} \sqrt{a^2 - x^2} \dots . (1.)$$

or

$$x = \pm \frac{a}{b} \sqrt{b^2 - y^2} \dots . (2.)$$

1. In equation (1.)

Let $x = 0$,

then $y = \pm b = \text{C B or C} b$.

Let $y = 0$,

then $x = \pm a = \text{C A or C} a$.

Let $x < \pm a$,

then for each value of x there are *two* equal values of y with contrary signs.

Let $x = \pm a$,

then $y = \pm 0$, that is, the ellipse cuts the axis of x at the points A and a.

Let $x > \pm a$,

then the quantity under the vinculum being negative, the values of y are *imaginary*, and no point of the ellipse can lie beyond A to the *right*, or a to the *left*.

It appears, therefore, that the ellipse is a continuous curve, returning into itself, and divided by the major axis A a into two equal parts.

In the same way it might be shown by discussing equation (2.) that the ellipse has the form just assigned to it, and that it is divided by the minor axis B b into two equal parts.

(29.) *Cor. To find the value of the ordinate passing through the focus.*

When the ordinate passes through the focus

$$x = m = a(1 - e),$$

$$\therefore y^2 = \frac{b^2}{a^2} \{ 2a^2(1 - e) - a^2(1 - e)^2 \},$$

$$= b^2(1 - e)\{ 2 - (1 - e) \},$$

$$= b^2(1 - e^2),$$

$$= \frac{b^4}{a^2};$$

$$\therefore y = \pm \frac{b^2}{a}, \text{ therefore the } \textit{latus rectum} = \frac{2b^2}{a}.$$

The double ordinate passing through the focus is called the *principal parameter*, or *latus rectum*,

therefore the *latus rectum* $\dfrac{2b^2}{a}$.

Def. The line S C $= a e$, is called *the eccentricity* of the ellipse.

(30.) *To find the intersection of a straight line with the ellipse.*

Let the equation to the proposed line be

$$y = \alpha x + \beta \dots . (1.)$$

Then the coordinates of the point or points of intersection with the ellipse will be obtained by combining this equation with that to the ellipse

$$a^2 y^2 + b^2 x^2 = a^2 b^2 \dots . (2.)$$

Substituting, then, in (2) the value of x derived from (1) we have

$$a^2 y^2 + b^2 \left(\frac{y - \beta}{\alpha}\right)^2 = a^2 b^2,$$

$$\therefore (a^2 \alpha^2 + b^2) y^2 - 2 b^2 \beta y + b^2 \beta^2 = a^2 b^2 \alpha^2,$$

$$\therefore y^2 - \frac{2 b^2 \beta}{a^2 \alpha^2 + b^2} y + \frac{b^2(\beta^2 - a^2 \alpha^2)}{a^2 \alpha^2 + b^2} = 0.$$

From this quadratic are obtained two values of y, which substituted in (1) furnish two corresponding values of x; therefore the coordinates required may be determined.

When the two roots of the quadratic are equal, the points of section coincide, and the straight line *touches* the ellipse; and when they are imaginary, the straight line falls entirely without the ellipse.

Hence it appears, that a straight line cannot cut an ellipse in more than two points.

Def. The portion of the straight line contained within the ellipse is called a *chord*; when the chord passes through the focus it is called the *focal chord*.

(31.) *To find the equation to a straight line which touches the ellipse in a given point.*

Let x', y' be the coordinates of the given point, and x'', y'' those of any other point in the ellipse near the first.

Then the equation to the line drawn through these points is (Analytical Geometry, Art. 12)

$$y - y' = \frac{y'' - y'}{x'' - x'}(x - x') \dots . (1.)$$

But these two points being in the ellipse, we have

$$a^2 y'^2 + b^2 x'^2 = a^2 b^2, \text{ and}$$

$$a^2 y''^2 + b^2 x''^2 = a^2 b^2,$$

$$\therefore a^2(y''^2 - y'^2) + b^2(x''^2 - x'^2) = 0,$$

$$\therefore \frac{y''^2 - y'^2}{x''^2 - x'^2} = -\frac{b^2}{a^2},$$

$$\therefore \frac{(y'' + y')(y'' - y')}{(x'' + x')(x'' - x')} = -\frac{b^2}{a^2},$$

$$\therefore \frac{y'' - y'}{x'' - x'} = -\frac{b^2}{a^2} \cdot \frac{x'' + x'}{y'' + y'},$$

therefore equation (1) becomes by substitution

$$y - y' = -\frac{b^2}{a^2} \cdot \frac{x'' + x'}{y'' + y'}(x - x').$$

Let the point (x'', y'') be now supposed to coincide with (x', y'), then $x'' = x'$, $y'' = y'$, and the secant will become a tangent at the point (x', y'); hence the equation to the tangent is

$$y - y' = \frac{-b^2}{a^2} \cdot \frac{x'}{y'}(x - x'),$$

in which x, y are the variable coordinates of any point whatever in the tangent.

Cor. This equation may be presented under a more commodious form, for multiplying each side by $a^2 y'$, we have

$$a^2 y y' - a^2 y'^2 = -b^2 x x' + b^2 x'^2,$$

therefore transposing

$$a^2 y y' + b^2 x x' = a^2 y'^2 + b^2 x'^2,$$

$$= \therefore a^2 b^2,$$

which is the equation most frequently employed.

Cor. Let $a = b$, then the ellipse becomes a circle, and the equation to the tangent at a point (x', y') in the circumference is

$$y y' + x x' = a^2.$$

(32.) *To find the intersection of the tangent with the axes of x and y.*

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Fig. 15. The equation to the tangent being

$$a^2yy' + b^2xx' = a^2b^2,$$

let it cut (1) the axis of x, as at T;

then $y = 0, \therefore b^2xx' = a^2b^2, \therefore x = \frac{a^2}{x'}$,

or $$CT = \frac{CA^2}{CM}$$

(2.) Let the tangent cut the axis of y, as at t;

then $x = 0, \therefore a^2yy' = a^2b^2, \therefore y = \frac{b^2}{y'}$.

or $$Ct = \frac{CB^2}{Cm}.$$

Whence it follows, that *each of the semi-axes is a mean proportional between the abscissa of any point, and the part of the axis intercepted between its intersection with the tangent and the centre.*

Cor. 1. Since $CT = \frac{a^2}{x'}$,

$$\therefore MT = CT - CM, = \frac{a^2}{x'} - x', = \frac{a^2 - x'^2}{x'}.$$

Def. The line MT intercepted between the foot of the ordinate, and the point where the tangent meets the axis, is called the *subtangent*.

Cor. 2. The value of the subtangent being independent of the ordinate y', it will remain the same for all ellipses described upon the same major axis Aa; now the circle is a species of ellipse, (26, *Cor.* 1;) *hence if on the major axis a circle be described, and the ordinate* MP *be produced upwards to meet the circumference in* Q, *the tangents applied at* P *and* Q *will intersect the axis* AX *in the same point* T.

This may be directly proved; for the equation to a line touching the circle at Q, is

$$yy' + xx' = a^2.$$

Let this line cut the axis of x, then $y = 0$,

$$\therefore x = \frac{a^2}{x'} = CT,$$

as in the ellipse.

Def. The straight line which is drawn from the point of contact at right angles to the tangent is called the *normal*.

(33.) *To find the equation to the normal.*

Fig. 15 Let TP touch the ellipse in P, and from this point draw PG at right angles to PT.

Then, because PG is drawn through the point (x', y') at right angles to PT, whose equation is

$$y - y' = -\frac{b^2}{a^2}\,\frac{x'}{y'}\,(x - x'),$$

the equation will be

$$y - y' = \frac{a^2}{b^2}\,.\,\frac{y'}{x'}\,(x - x'),$$

in which x, y are the variable coordinates of any point whatever in the line PG, considered as indefinite. Ellipse.

(34.) *To find the intersection of the normal with the axes of x and y.* Fig. 15.

The equation to the normal being

$$y - y' = \frac{a^2}{b^2}\,.\,\frac{y'}{x'}\,(x - x'),$$

let it first cut the axis of x, as at G

then $y = 0$, and $-y' = \frac{a^2}{b^2}\,.\,\frac{y'}{x'}(x - x')$,

$$\therefore x - x' = -\frac{b^2}{a^2}\,x',$$

$$\therefore x' - x = \frac{b^2}{a^2}\,x',$$

or $$CM - CG,$$

that is, $$MG = \frac{b^2}{a^2}\,x'.$$

Next, conceive the normal to cut the axis of y, as at g; then $x = 0$,

$$\therefore y - y' = -\frac{a^2}{b^2}\,.\,\frac{y'}{x'}\,x',$$

$$= -\frac{a^2}{b^2}\,y',$$

$$\therefore y = -\frac{a^2 - b^2}{a^2}\,y'$$

The negative sign implying that the point g lies *below* the axis AX.

Def. The line MG intercepted between the foot of the ordinate, and the point where the normal cuts the axis of x, is called the *subnormal*.

(35.) *To draw a tangent to an ellipse from a given point (x'', y'') without it.*

Let x', y' be the unknown coordinates of the point of contact.

Then the equation to the tangent being, in general,

$$a^2yy' + b^2xx' = a^2b^2,$$

and the point (x'', y'') being by hypothesis a point in the tangent, we have

$$a^2y''y' + b^2x''x' = a^2b^2 \ldots . (1;)$$

also the point of contact (x', y') being in the ellipse,

$$a^2y'^2 + b^2x'^2 = a^2b^2 \ldots . (2;)$$

hence, by means of these two equations, the coordinates x', y' of the point of contact may be determined.

Since the equation which results from the elimination of x' between (1) and (2) is of the second degree, it follows that there are two points of contact; in other words, *that two tangents may be drawn to an ellipse from a given point without it.*

Instead of going through the operation of eliminating we may, as in the case of the parabola, (Art. 10,) find the position of the points of contact by constructing the loci of (1) and (2,) in which x', y' are the variable coordinates.

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Now the locus of (2) is the given ellipse, and the locus of (1,) which is an equation of the first degree, is a straight line, whose position is determined by making x' and y' successively $= 0$.

Fig. 16.

If, then, in the equation $a^2 y' y'' + b^2 x' x'' = a^2 b^2$,

$$x' = 0, \text{ then } y' = \frac{b^2}{y''},$$

$$y' = 0, \text{ then } x' = \frac{a^2}{x''}.$$

Hence, take $C R = \frac{a^2}{x''}$ and $C r = \frac{b^2}{y''}$, join R, r, and let R r meet the ellipse in P and p, these will be the points of contact required.

Cor. 1. Since the straight line R r, which has just been drawn, determines by its intersection with the ellipse the points of contact, it follows that the equation $a^2 y'' y' + b^2 x'' x = a^2 b^2$,

in which x' and y' are the variables, is the equation to the indefinite line joining the points of contact.

Cor. 2. Because C R is independent of y'' it will remain the same for all points whose abscissas are $= x''$, that is, for all the points in the indefinite line Q q drawn through Q parallel to C Y. Hence, the following theorem:

If from the several points of a straight line perpendicular to the axis C X, *pairs of tangents be drawn to the ellipse, the chords joining the points of contact in each case will all pass through the same point.*

CHAPTER II.

ON THE ELLIPSE REFERRED TO THE FOCUS.

(36.) *To find the distance of any point in the ellipse from the focus.*

Fig. 17.

Let S, H be the foci, P any point in the ellipse, to find the distance of P from S.

Let fall the perpendicular P M on C A.

Then $S P^2 = S M^2 + M P^2$,

$$= (C S - C M)^2 + M P^2,$$
$$= (a e - x)^2 + y^2,$$
$$= a^2 e^2 - 2 a e x + x^2 + (1 - e^2)(a^2 - x^2),$$
$$= a^2 - 2 a e x + e^2 x^2,$$
$$\therefore S P = a - e x.$$

In like manner it may be proved, that

$$H P = a + e x.$$

Cor. Hence, by addition,

$$S P + H P = 2 a = A a.$$

In other words, *the sum of the focal distances is equal to the major axis.*

From this property the equation to the ellipse may be deduced, as in the following article:

(37.) *To find the locus of a point whose distances from two fixed points are together always equal to a given quantity* 2 *a.*

Ellipse.

Let S, H be the two fixed points, P the point whose locus is required.

Join S, H, bisect S H in C, let fall the perpendicular P M on S H, which produce indefinitely towards X; from C draw C Y at right angles to C X, and assume C X and C Y as the axes of coordinates.

Fig. 18.

Let $C M = x$, $M P = y$, and $S C = c$.

$$\left.\begin{array}{ll}\text{Then} & S P^2 = y^2 + (c - x)^2 \\ \text{and} & H P^2 = y^2 + (c + x)^2\end{array}\right\} \ldots (1,)$$

$$\therefore H P^2 - S P^2 = (c + x)^2 - (c - x)^2,$$

or $(H P + S P)(H P - S P) = 4 c x$,

$$\therefore H P - S P = \frac{4 c x}{2 a},$$
$$= \frac{2 c x}{a};$$

but $H P + S P = 2 a$,

$$\therefore H P = a + \frac{c x}{a},$$

and $S P = a - \frac{c x}{a}$.

Squaring these values, and adding the result,

$$S P^2 + H P^2 = 2\left(a^2 + \frac{c^2 x^2}{a^2}\right),$$

and also $= 2 (y^2 + c^2 + x^2)$ from (1,)

$$\therefore y^2 + c^2 + x^2 = a^2 + \frac{c^2 x^2}{a^2},$$
$$\therefore y^2 = a^2 - c^2 + \frac{c^2 x^2}{a^2} - x^2,$$
$$= a^2 - c^2 - \frac{a^2 - c^2}{a^2} x^2,$$
$$= \frac{a^2 - c^2}{a^2}(a^2 - x^2),$$

which is the equation to an ellipse, whose major axis $= 2 a$, and minor axis $= 2 \sqrt{a^2 - c^2}$.

If $x = 0$, then $y^2 = a^2 - c^2 = b^2$ if $b =$ the ordinate drawn from C.

(38.) *To find the polar equation to the ellipse, the focus being the pole.*

(1.) Let S be the pole.

Fig. 17.

Let $S P = r$, angle $P S X = \omega$;

then $r = a - e x$, (Art. 36,)

but $x = C S - S M$,

$$= a e - r \cos(\pi - \omega),$$
$$= a e + r \cos \omega,$$
$$\therefore r = a - a e^2 - e r \cos \omega,$$
$$\therefore (1 + e \cos \omega) r = a (1 - e^2),$$
$$\therefore r = \frac{a(1 - e^2)}{1 + e \cos \omega},$$

which is the equation required.

(2.) Let H be the pole.

Let $H P = r'$, and $P H X = \omega$,

then $r' = a + e x$,

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but $$x = \mathrm{C\,M} = \mathrm{H\,M} - \mathrm{H\,C},$$
$$= r' \cos \omega' - a\,e,$$
$$\therefore r' = a + e\,r' \cos \omega' - a\,e^2,$$
$$\therefore r'(1 - e \cos \omega') = a(1 - e^2),$$
$$\therefore r' = \frac{a(1-e^2)}{1 + e \cos \omega'}$$
which is the equation required.

Cor. 1. If P S be produced to meet the ellipse in p, then since the angle A S $p = \pi - \omega$, we have
$$\mathrm{S}p = \frac{a(1-e^2)}{1 - e \cos \omega}.$$

Cor. 2. Hence
$$\frac{1}{\mathrm{S\,P}} + \frac{1}{\mathrm{S}p} = \frac{1 + e \cos \omega}{a(1-e^2)} + \frac{1 + e \cos \omega}{a(1-e^2)},$$
$$= \frac{2}{a(1-e^2)} = \frac{2}{\mathrm{S\,L}};$$
that is, the principal semi-parameter is an harmonic mean between the segments of any focal chord.

Cor. 3. Since $\dfrac{1}{\mathrm{S\,P}} + \dfrac{1}{\mathrm{S}p} = \dfrac{\mathrm{S\,P} + \mathrm{S}p}{\mathrm{S\,P}\,.\,\mathrm{S}p}$,

and also $$= \frac{2}{a(1-e^2)},$$
$$\therefore \mathrm{S\,P}\,.\,\mathrm{S}p = \frac{a}{2}(1-e^2)(\mathrm{S\,P} + \mathrm{S}p).$$

(39.) *To find the polar equation to the ellipse, the centre being the pole.*

Fig. 17. Let C P $= \rho$ and the angle P C A $= v$.

Then $$\rho^2 = x^2 + y^2,$$
$$= x^2 + (1-e^2)(a^2 - x^2),$$
$$= e^2 x^2 + a^2(1 - e^2),$$
$$= e^2 \rho^2 \cos^2 v + a^2(1 - e^2),$$
$$\therefore \rho^2(1 - e^2 \cos^2 v) = a^2(1 - e^2),$$
$$\therefore \rho = a\sqrt{\frac{1-e^2}{1 - e^2\cos^2 v}},$$
which is the equation required.

(40.) *To prove that the focal distances of any point make equal angles with the tangent at that point.*

Fig. 19. Let T P t be a tangent at the point P (x', y), draw the normal P G, and join S, P and H, P

Then $$\mathrm{C\,G} = \frac{a^2 - b^2}{a^2}\,x' = e^2 x', (34.)$$
$$\therefore \frac{\mathrm{S\,G}}{\mathrm{H\,G}} = \frac{\mathrm{S\,C} - \mathrm{C\,G}}{\mathrm{S\,C} + \mathrm{C\,G}} = \frac{a\,e - e^2 x'}{a\,e + e^2 x'} = \frac{a - e\,x'}{a + e\,x'},$$
$$= \frac{\mathrm{S\,P}}{\mathrm{H\,P}}.$$
$\therefore$ angle S P G = angle H P G. (Euc. vi. 3.)

But G P T = G P t, $\therefore$ S P T = H P t,

as was to be proved.

(41.) *To find the locus of the points in which a perpendicular from the focus upon the tangent at any point intersects the tangent.*

Ellipse.

Fig. 20. Let P T be a tangent at any point P (x', y'), and S Y a perpendicular let fall from S on P T, meeting it in Y, to find the locus of Y.

From C let fall the perpendicular C Q on T P produced, and draw S q parallel to P T meeting C Q in q.

Then $$\mathrm{C\,Y}^2 = \mathrm{C\,Q}^2 + \mathrm{Q\,Y}^2 = \mathrm{C\,Q}^2 + \mathrm{S}q^2,$$
$$= \mathrm{C\,T}^2 \sin^2 \mathrm{T} + \mathrm{C\,S}^2 \cos^2 \mathrm{T};$$
but $\mathrm{C\,T} = \dfrac{a^2}{x'}$ (32) and $\mathrm{C\,S} = a\,e$,
$$\therefore \mathrm{C\,Y}^2 = \frac{a^4}{x'^2} \sin^2 \mathrm{T} + a^2 e^2 \cos^2 \mathrm{T},$$
$$= \frac{a^4}{x'^2}(1 - \cos^2 \mathrm{T}) + a^2 e^2 \cos^2 \mathrm{T},$$
$$= \frac{a^4}{x'^2} - \frac{a^2}{x'^2}(a^2 - e^2 x'^2)\cos^2 \mathrm{T} \ldots (1.)$$
Now $$\tan \mathrm{T} = -\frac{b^2}{a^2}\,\frac{x'}{y'}\ (31) = -\frac{b}{a}\,.\,\frac{x'}{\sqrt{a^2 - x'^2}},$$
$$\therefore 1 + \tan^2 \mathrm{T} = 1 + \frac{b^2 x'^2}{a^2(a^2 - x'^2)},$$
$$= \frac{a^4 - (a^2 - b^2)x'^2}{a^2(a^2 - x'^2)},$$
$$= \frac{a^4 - a^2 e^2 x'^2}{a^2(a^2 - x'^2)}$$
$$= \frac{a^2 - e^2 x'^2}{a^2 - x'^2},$$
$$\therefore \cos^2 \mathrm{T} = \frac{a^2 - x'^2}{a^2 - e^2 x'^2};$$
therefore by substitution in (1)
$$\mathrm{C\,Y}^2 = \frac{a^4}{x'^2} - \frac{a^2}{x'^2}(a^2 - e^2 x'^2)\,.\,\frac{a^2 - x'^2}{a^2 - e^2 x'^2},$$
$$= \frac{a^4}{x'^2} - \frac{a^2}{x'^2}(a^2 - x'^2),$$
$$= \frac{a^2}{x'^2}\{a^2 - a^2 + x'^2\} = a^2,$$
$$\therefore \mathrm{C\,Y} = \pm\, a,$$
therefore the locus of Y is a circle whose radius is a, and which is therefore described on the major axis A a as a diameter.

(42.) *The rectangle contained by the perpendiculars let fall from the foci upon any tangent, is equal to the square of the semi-axis minor.*

For if the perpendiculars S Y and H Z be let fall from S and H on the tangent P T, then Fig. 20.
$$\mathrm{S\,Y} = \mathrm{S\,T} \sin \mathrm{T},$$
but
$$\mathrm{S\,T} = \mathrm{C\,T} - \mathrm{C\,S} = \frac{a^2}{x'} - a\,e = \frac{a}{x'}(a - e\,x'),$$
$$\therefore \mathrm{S\,Y} = \frac{a}{x'}(a - e\,x') \sin \mathrm{T}.$$
Similarly, $$\mathrm{H\,Z} = \frac{a}{x'}(a + e\,x') \sin \mathrm{T},$$
$$\therefore \mathrm{S\,Y}\,.\,\mathrm{H\,Z} = \frac{a^2}{x'^2}(a^2 - e^2 x'^2) \sin^2 \mathrm{T} \ldots (1.)$$

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Now
$$\cos^2 T = -\frac{a^2 - x'^2}{a^2 - e^2 x'^2},$$
$$\therefore \sin^2 T = 1 - \cos^2 T,$$
$$= 1 - \frac{a^2 - x'^2}{a^2 - e^2 x'^2},$$
$$= \frac{x'^2 - e^2 x'^2}{a^2 - e^2 x'^2};$$

therefore by substitution in (1)

$$S Y . H Z = \frac{a^2}{x'^2}(a^2 - e^2 x'^2) . \frac{x'^2(1 - e^2)}{a^2 - e^2 x'^2},$$
$$= a^2(1 - e^2) = b^2.$$

CHAPTER III.

ON THE ELLIPSE REFERRED TO ANY SYSTEM OF CONJUGATE DIAMETERS.

SECTION I.

On Conjugate Diameters in General.

(43.) *To find the locus of the middle points of any number of parallel chords.*

Fig. 21. Let Pp be any chord, O its middle point, and X, Y its coordinates.

From the points O, P, p let fall the perpendiculars O N, P M, $p\,m$ on the axis A X, then if the equation to Pp be

$$y = a x + \beta,$$

the equation containing the values of y at the points P, p will be

$$y^2 - \frac{2 b^2 \beta}{a^2 a^2 + b^2} y + \frac{b^2(\beta^2 - a^2 a^2)}{a^2 a^2 + b^2} = 0, \text{ (Art. 30.)}$$

Now, since in any quadratic equation the coefficient of the second term, with its proper sign, is equal to the sum of the roots with their signs changed,

$$\frac{2 b^2 \beta}{a^2 a^2 + b^2} = PM + p\,m;$$

but O being the middle point of Pp,

$$ON = \frac{PM + p\,m}{2},$$
$$\therefore Y = \frac{b^2 \beta}{a^2 a^2 + b^2} \ldots . (1.)$$

Now
$$X = \frac{1}{a}(Y - \beta),$$
$$= \therefore - \frac{a^2 a \beta}{a^2 a^2 + b^2} \ldots . (2).$$

To obtain the relation between X and Y we must eliminate β between (1) and (2,)

$$\therefore \frac{a^2 a^2 + b^2}{b^2} Y = -\frac{a^2 a^2 + b^2}{a^2 a} X,$$
$$\therefore Y = -\frac{b^2}{a^2 a} X$$

Now, a remains the same for chords parallel to Pp, therefore the equation just found expresses the relation between the coordinates of their middle points, and being of the first degree, the locus required is a *straight line.*

Ellipse.

Def. The straight line which has been proved to be the locus of the middle points of any number of parallel chords is called a *diameter*, and the points in which it intersects the curve are called the *vertices.*

(44.) *Cor.* The equation $Y = -\frac{b^2}{a^2 a} X$ is the equation to a line passing through the origin, which is in this case the centre; hence *every diameter must pass through the centre.*

(45.) *A diameter being drawn through a given point, to find the equation to any one of its ordinates.*

If x', y' be the coordinates of the given point, the equation to the diameter drawn through it will be

$$y = \frac{y'}{x'} x \ldots . (1.)$$

Let
$$y = a x + \beta \ldots . (2,)$$

be the required equation to any ordinate,

then
$$\frac{y'}{x'} = -\frac{b^2}{a^2 a} \text{ (44,)}$$
$$\therefore a = -\frac{b^2}{a^2} . \frac{x'}{y'},$$

therefore any ordinate to a diameter passing through (x', y') has for its equation

$$y = -\frac{b^2}{a^2} . \frac{x'}{y'} + \beta.$$

Cor. Comparing this equation with the equation to the tangent, it appears that the tangent applied at the vertex of any diameter is parallel to the ordinates of that diameter.

(46.) *Any two diameters being given, if the ordinates of one be parallel to the other, the ordinates of the latter will be parallel to the former.*

Let
$$y = a x \ldots . (1,)$$
$$y = a' x \ldots . (2,)$$ Fig. 22

be any two diameters C P, C D, then by the last article the equations of any ordinates M N, Q R to the first and second, respectively, will be

$$y = -\frac{b^2}{a^2 a} x + \beta \ldots . (1',)$$
$$y = -\frac{b^2}{a^2 a'} x + \beta' \ldots . (2'.)$$

Let the ordinate M N be now supposed parallel to the diameter C D,

Then
$$-\frac{b^2}{a^2 a} = a',$$

or
$$a = -\frac{b^2}{a^2 a'},$$

therefore the equation to Q R becomes by substitution in (2')
$$y = a x + \beta',$$

that is, Q R the second ordinate is parallel to the first diameter C P, which was to be proved.

Whence each of these diameters is parallel to the ordinates of the other.

Diameters, thus related, are said to be *conjugate to each other*.

Cor. 1. Hence, when the two diameters

$$y = \alpha x,$$
$$y = \alpha' x,$$

are conjugate to each other,

$$\alpha \alpha' = - \frac{b^2}{a^2}.$$

Cor. 2. Therefore if

$$y = \alpha x$$

be any diameter,

$$y = - \frac{b^2}{a^2 \alpha} x$$

will be the diameter conjugate to it. The number of pairs of conjugate diameters is therefore unlimited.

If $\alpha = 0$, or the first diameter be A a, then

$$y = - \frac{b^2}{a^2 0} x = \infty . x,$$

therefore the diameter conjugate to A a, being at right angles to it, is B b; or, *the axes of the ellipse are conjugate diameters.*

Cor. 3. If (x', y') be any point in the ellipse, the diameter passing through it is

$$y = \frac{y'}{x'} x,$$

$$\therefore y = - \frac{b^2}{a^2} \cdot \frac{x'}{y'}$$

is the corresponding conjugate diameter.

But the equation to a tangent drawn through (x', y') is

$$y - y' = - \frac{b^2}{a^2} \cdot \frac{x'}{y'} (x - x'). \quad (31.)$$

whence it follows, *that the tangent applied at the vertex of any diameter is parallel to the corresponding conjugate diameter.*

(47.) It has just been shown in *Cor.* 2, that the axes of the ellipse are conjugate diameters, we shall now prove *that the axes are the only pair of conjugate diameters which can be at right angles to each other.*

Fig. 23. For, if possible, let C P, C D be a pair of rectangular conjugate diameters different from the axes, and let

$$\text{angle P C A} = \theta, \quad \text{angle D C A} = \theta'.$$

Now $$\theta' = \text{D C A} = \text{D C P} + \text{P C A},$$

$$= \frac{\pi}{2} + \theta \text{ by hypothesis;}$$

but $$- \frac{b^2}{a^2} = \alpha \alpha' \ (46, \textit{Cor.}\ 1) = \therefore \tan \theta \tan \theta',$$

$$= \frac{\sin \theta}{\cos \theta} \cdot \frac{\sin \theta'}{\cos \theta'},$$

$$\therefore a^2 \sin \theta \sin \theta' + b^2 \cos \theta \cos \theta' = 0 \ldots . (1.)$$

but

$$\sin \theta' = \sin \left(\frac{\pi}{2} + \theta\right) = \sin \left(\frac{\pi}{2} - \theta\right) = \cos \theta,$$

$$\cos \theta' = \cos \left(\frac{\pi}{2} + \theta\right) = - \cos \left(\frac{\pi}{2} - \theta\right) = -\sin \theta,$$

therefore by substitution in (1)

$$(a^2 - b^2) \sin \theta \cos \theta = 0,$$

or $$\tfrac{1}{2} (a^2 - b^2) \sin 2 \theta = 0.$$

Now, since a is $>$ b, this equation can be satisfied only by supposing

$$\sin 2 \theta = 0,$$
$$\therefore 2 \theta = 0, \text{ or } = \pi,$$
$$\therefore \quad \theta = 0, \text{ or } = \frac{\pi}{2},$$

and $$\therefore \quad \theta' = \frac{\pi}{2}, \text{ or } = 0;$$

C P and C D must coincide with C A and C B respectively.

(48.) *To find the equation to the ellipse when it is referred to any two conjugate diameters as axes.*

Let C be the centre, C P, C D a given system of conjugate diameters, of which the former is supposed to be the axis of x, the latter the axis of y.

Take any point Q in the ellipse, and draw Q q parallel to C Y meeting C X in V.

Let C V $= x$, V Q $= y$; also C P $= a'$, C D $= b'$. Fig. 24.

Then, since the chord Q q is bisected by C P in V, V Q $=$ V q; and since every other chord parallel to C Y is bisected by C X, it follows that for each assumed value of x there are two equal values of y, with contrary signs. In like manner it may be shown, that for each assumed value of y, there are two equal values of x with contrary signs; therefore the equation required will be of the form

$$\text{M}\, y^2 + \text{N}\, x^2 = \text{P}.$$

It now remains to determine the values of M, N, and P. When the axis C X cuts the ellipse,

$$y = 0, \text{ and } x = \text{C P} = a',$$
$$\therefore \text{N}\, x^2 = \text{P} = \text{N}\, a'^2,$$
$$\therefore \text{N} = \frac{\text{P}}{a'^2}.$$

When the axis C Y cuts the ellipse,

$$x = 0, \text{ and } y = \text{C D} = b',$$
$$\therefore \text{M}\, y^2 = \text{P} = \text{M}\, b'^2,$$
$$\therefore \text{M} = \frac{\text{P}}{b'^2}.$$

Substituting these values of M, N, P in the above equation, and dividing each term of the result by P, we have

$$\frac{y^2}{b'^2} + \frac{x^2}{a'^2} = 1 \ldots . (1.)$$

or $$a'^2 y^2 + b'^2 x^2 = a'^2 b'^2 \ldots . (2.)$$

either of which is the equation required.

Cor. 1. Hence $y = \pm \dfrac{b'}{a'} \sqrt{a'^2 - x^2}$.

Cor. 2. To find the form of the equation when the coordinates originate at P, the vertex of the diameter C P.

Let P M $= x'$, then $x =$ C P $-$ P M $= a' - x'$

Substituting this value of x in *Cor.* 1, we have

$$y = \pm \frac{b'}{a'} \sqrt{2 a' x' - x'^2},$$

or suppressing the accent of x

$$y = \pm \frac{b'}{a'} \sqrt{2a'x - x^2},$$

which is the equation required.

Cor. 3. The equations (1,) (2,) and (3) are of the same form as the equations in terms of the axes, and express, when translated into geometrical language, a property of which that in Art. 29 is only a particular case.

For $a'^2 - x^2 = (a' + x)(a' - x) = PV.VG$,

and $2a'x - x^2 = (2a' - x)x = PV.VG$,

$$\therefore VQ^2 = \frac{PC^2}{CD^2} PV.VG$$

or $PV.VG : QV^2 :: PC^2 : CD^2$;

that is, *the rectangle contained by the segments of any diameter is to the square of the ordinate as the square of the semi-diameter is to the square of its semi-conjugate.*

(49.) It appears from the preceding proposition, that the equation to the ellipse, whether the axes be rectangular or oblique, is always of the same form; whence it follows:

(1.) That if the equation to the major axis A a be

$$y = \alpha x,$$

then

$$y = -\frac{b'^2}{a'^2 \alpha} x$$

will be the equation to the minor axis B b. And

(2.) That the equation to the tangent will be

$$a'^2 y y' + b'^2 x x' = a'^2 b'^2.$$

(50.) *To find the intersection of the tangent with any two conjugate diameters considered as axes.*

Fig. 25. Let a tangent applied at any point Q meet C P in T and C D in t, and draw the ordinates Q V, Q v.

The equation to the tangent being

$$a'^2 y y' + b'^2 x x' = a'^2 b'^2,$$

let the tangent meet C X as at T, then $y = 0$,

$$\therefore x = \frac{a'^2}{x'}, \text{ or } CT = \frac{CP^2}{CV}.$$

Let the tangent meet C Y as at t, then $x = 0$,

$$\therefore y = \frac{b'^2}{y'}, \text{ or } Ct = \frac{CD^2}{Cv},$$

whence the points of intersection required are found.

(51.) *If from the several points of a line given in position, pairs of tangents be drawn to an ellipse, the lines which join the corresponding points of contact will all pass through the same point.*

Let C be the centre of the ellipse, M N the given line.

Draw any chord $m\,n$ parallel to M N, and bisect it by the diameter C X; from C draw C Y parallel to $m\,n$ or M N, then C X, C Y are conjugate diameters; and if the ellipse be referred to these as axes, its equation will be*

$$a'^2 y^2 + b'^2 x^2 = a'^2 b'^2 \ldots. (1.)$$

From any point (x'', y'') in M N let a pair of tangents be drawn to the ellipse, then it may be shown as in Art. 35, which is only a particular case of this proposition, that the equation to the line joining the points of contact is

$$a'^2 y'' y' + b'^2 x' x'' = a'^2 b'^2 \ldots (2,)$$

in which x', y' are the variable coordinates of the point of contact.

Let the line (2) cut the axis of x, then $y' = 0$,

and $\therefore x' = \frac{a'^2}{x''}$;

hence the point of intersection will be the same for all points whose abscissa $= x''$, that is, for all points in the line M N, as was to be proved.

Cor. The point of intersection is situated on the diameter conjugate to that which is parallel to the given line.

(52.) *If from the point of intersection of two tangents a diameter be drawn, it will bisect the line joining the points of contact.*

For the equation to an ordinate to the diameter passing through (x'', y'') is (45)

$$y = -\frac{b'^2}{a'^2} \cdot \frac{x''}{y''} x + \beta \ldots (1,)$$

and the equation to the line which joins the points of contact is

$$y' = -\frac{b'^2}{a'^2} \cdot \frac{x''}{y''} x - \frac{b'^2}{y''} \ldots (2;)$$

hence the latter, being parallel to the former, is also an ordinate, and is therefore bisected.

(53.) *If through any point within or without an ellipse, two straight lines, given in position, be drawn to meet the curve, the rectangle contained by the segments of the one will bear a constant ratio to the rectangle contained by the segments of the other.*

Let O be any point *within* the ellipse, through which draw the two lines P p and Q q, whose position is supposed known, meeting the ellipse in the points P, p and Q, q; to prove that Fig. 26.

O P . O p : O Q . O q in a constant ratio.

Through O draw the diameter C X, and let C Y be the diameter conjugate to it; then if the ellipse be referred to these diameters as axes, its equation will be

$$a'^2 y^2 + b'^2 x^2 = a'^2 b'^2 \ldots. (1.)$$

Through P draw P M parallel to C Y, and let

$$OP = r, \quad CO = \delta;$$

then

$$\frac{PM}{PO} = \frac{\sin POM}{\sin PMO} = \frac{\sin r, x}{\sin x, y} = p \text{ (suppose)}$$

$$\therefore y = p\,r; \text{ in like manner if } \frac{\sin r, y}{\sin x, y} = q,$$

$$x = CO + ON,$$

$$= \delta + q\,r;$$

therefore substituting these values of x and y in (1,)

$$a'^2 p^2 r^2 + b'^2 \{ \delta^2 + 2\delta q r + q^2 r^2 \} = a'^2 b'^2,$$

$$\therefore (a'^2 p^2 + b'^2 q^2) r^2 + 2\delta q b'^2 r + b'^2 (\delta^2 - a'^2) = 0,$$

$$\therefore r^2 + \frac{2\delta q b'^2}{a'^2 p^2 + b'^2 q^2} r + \frac{b'^2 (\delta^2 - a'^2)}{a'^2 p^2 + b'^2 q^2} = 0,$$

in which the values of r are O P, O p,

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$$\therefore O P \,.\, O p = \frac{-b'^2(\delta^2 - a'^2)}{a'^2 p^2 + b'^2 q^2}.$$

In like manner,

$$O Q \,.\, O q = \frac{-b'^2(\delta^2 - a'^2)}{a'^2 p'^2 + b'^2 q'^2},$$

therefore

$O P \,.\, O p : O Q \,.\, Q q :: a'^2 p'^2 + b'^2 q'^2 : a'^2 p^2 + b'^2 q^2$,

which is a constant ratio, as was to be proved.

SECTION II.

On the Properties of Conjugate Diameters.

(54.) *A diameter being drawn through a given point (x', y') to find the coordinates of the point in which the diameter conjugate to it meets the ellipse.*

Fig. 27. Let C P, C D be any two semi-conjugate diameters, then the equation to C P being

$$y = \frac{y'}{x'}\, x \ldots . (1.)$$

the equation to C D will (46, *Cor.* 2) be

$$y = -\frac{b^2}{a^2} \cdot \frac{x'}{y'}\, x \ldots . (2.)$$

therefore the coordinates of the point D in which C D cuts the ellipse will be determined by combining (2) with the equation

$$a^2 y^2 + b^2 x^2 = a^2 b^2 \ldots . (3.)$$

Hence, substituting in (3) the value of y in (2) we have

$$\left\{ \frac{b^4}{a^2} \cdot \frac{x'^2}{y'^2} + b^2 \right\} x^2 = a^2 b^2,$$

or, dividing by b^2,

$$\left(\frac{b^2}{a^2} \cdot \frac{x'^2}{y'^2} + 1 \right) x^2 = a^2,$$

$$\therefore (b^2 x'^2 + a^2 y'^2)\, x^2 = a^4 y'^2,$$

$$\therefore a^2 b^2 x^2 = a^4 y'^2,$$

$$\therefore x^2 = \frac{a^2}{b^2}\, y'^2,$$

or
$$x = \pm \frac{a}{b}\, y',$$

therefore also
$$y = -\frac{b^2}{a^2} \cdot \frac{x'}{y'}\, x,$$

$$= \mp \frac{b}{a}\, x'.$$

The signs of x and y being different, as they ought to be.

(55.) *The sum of the squares of any two semi-conjugate diameters is equal to the sum of the squares of the semi-axes.*

Let C P, C D be any two semi-conjugate diameters; then
$$C P^2 = C M^2 + M P^2 = x'^2 + y'^2,$$

$$C D^2 = C m^2 + m D^2 = \frac{a^2}{b^2}\, y'^2 + \frac{b^2}{a^2}\, x'^2,$$

$$\therefore C P^2 + C D^2 = \left(x'^2 + \frac{a^2}{b^2}\, y'^2\right) + \left(y'^2 + \frac{b^2}{a^2}\, x'^2\right),$$

Ellipse.

$$= \frac{b^2 x'^2 + a^2 y'^2}{b^2} + \frac{a^2 y'^2 + b^2 x'^2}{a^2},$$

$$= \frac{a^2 b^2}{b^2} + \frac{a^2 b^2}{a^2},$$

$$= a^2 + b^2.$$

(56.) *If at the vertices of any two conjugate diameters tangents be applied so as to form a parallelogram, the area of all such parallelograms is constant.*

Let P p, D d be any two conjugate diameters, and let the tangents applied at P and p, D and d be produced to meet, then it is plain (45, *Cor.*) that they will form a parallelogram. Fig. 28.

From P and T let fall the perpendiculars P F, T Q, on D C produced.

Then the area of the whole parallelogram is equal to four times the area of the parallelogram P D

$$= 4\, P C \,.\, C D \sin P C D,$$

$$= 4\, C D \,.\, P F \ldots . (1;)$$

but

$$P F = T Q = C T \sin T C Q = \frac{a^2}{x} \cdot \frac{m\, D}{D\, C}, (32.)$$

$$\therefore P F.\, C D = \frac{a^2}{x} \,.\, m\, D,$$

$$= \frac{a^2}{x} \cdot \frac{b}{a}\, x, (54.)$$

$$= a\, b \ldots . (2.)$$

therefore by substitution in (1.)

The area of the whole parallelogram $= 4\, a\, b$, and is therefore constant.

Cor. 1. By equation (2) $P F \,.\, C D = a\, b$;

but $C D = b'$, and $P F = P C \sin P C D = a' \sin \gamma$ if $\gamma = P C D$,

$$\therefore a\, b = a' b' \sin \gamma.$$

Cor. 2. Hence the value of P F may be found.

For
$$P F = \frac{a\, b}{C D},$$

but
$$C D^2 = a^2 + b^2 - a'^2, \; (55.)$$

$$\therefore P F = \frac{a\, b}{\sqrt{a^2 + b^2 - a'^2}}.$$

(57.) *To find the magnitude and position of two equal conjugate diameters.*

In general, $a^2 + b^2 = a'^2 + b'^2$

Let $a' = b'$,

$$\therefore 2\, a'^2 = a^2 + b \, ,$$

$$\therefore a' = \pm \sqrt{\frac{a^2 + b^2}{2}} \ldots . (1.)$$

therefore the *magnitude* of the equal conjugate diameters is found.

Again, their *position* may be determined.

For $a\, b = a' b' \sin \gamma$,

$= \therefore a'^2 \sin \gamma$, when $a' = b'$,

$$\therefore \sin \gamma = \frac{a\, b}{a'^2}.$$

$$= \frac{2\,a\,b}{a^2 + b^2} \quad \ldots (2.)$$

which is their mutual inclination.

Also, their inclination to the major axis may be found, because, being equal, they are symmetrically placed with respect to the major axis, and are therefore equally inclined to it; but in general

$$\tan P C X \,.\, \tan D C X = -\frac{b^2}{a^2},$$

Fig. 23. or

$$\tan P C X \,.\, \tan D C a = \frac{b^2}{a^2} = \tan^2 P C X,$$

$$\therefore \tan P C X = \pm \frac{b}{a} \ldots . (3.)$$

whence it follows, that the equal conjugate diameters are parallel to the lines B A, B a.

(58.) *Of all systems of conjugate diameters, those that are equal contain the greatest angle.*

For, in general, $\sin \gamma = \dfrac{a\,b}{a'\,b'}$,

therefore the angle P C d is a *minimum*, or P C D *a maximum* when the product $a'\,b'$ is a *maximum;* that is, when $a' = b'$, as was to be proved.

Cor. Hence it may be proved, that of all systems of *conjugate diameters the sum of those that are* rectangular *is the* least, *and of those that are* equal, *the* greatest.

For $\quad a' + b' = \sqrt{(a^2 + b'^2 + 2\,a'\,b')},$

$$= \sqrt{\left(a^2 + b^2 + \frac{2\,a\,b}{\sin \gamma}\right)}.$$

Therefore (1) $a' + b'$ is a *maximum*, when $\sin \gamma$ is a *minimum*, that is, when $a' = b'$.

(2) $a' + b'$ is a *minimum* when $\sin \gamma$ is a *maximum*, that is, when $\gamma = \dfrac{\pi}{2}$, or the conjugate diameters are rectangular.

(59.) *The rectangle contained by the distances of any point from the two foci is equal to the square of the corresponding semi-conjugate diameter.*

Fig. 29. Let P be any point, C D the semi-diameter conjugate to C P, join P, S and P, H; to prove that

$$S P \,.\, H P = C D^2.$$

For

$$\begin{aligned} C D^2 &= a^2 + b^2 - C P^2, \\ &= a^2 + b^2 - (x^2 + y^2), \\ &= a^2 + b^2 - x^2 - (1 - e^2)(a^2 - x^2), \\ &= a^2 + b^2 - x^2 - a^2 + x^2 + e^2 a^2 - e^2 x^2, \\ &= b^2 + a^2 e^2 - a^2 x^2, \\ &= a^2 - e^2 x^2 \ldots . (1.) \end{aligned}$$

But

$$\begin{aligned} a^2 - e^2 x^2 &= (a - e\,x)(a + e\,x) \\ &= S P \,.\, H P, \ (36,) \\ \therefore S P \,.\, H P &= C D^2. \end{aligned}$$

(60.) *Let* C P, C D *be any two semi-conjugate diameters, and let a tangent at* P *meet the axes of the ellipse in* T *and* t, *to prove that* P T . P t = C D^2.

If C P, C D be assumed as the axes of coordinates, then the equations to C A, C B are respectively Fig. 30.

$$y = a\,x,$$

$$y = -\frac{b'^2}{a'^2\,a}\,x. \quad (49.)$$

Let $x = a'$ or C P, then y or P T $= a\,a'$ in the first,

and $\quad y$ or P $t = -\dfrac{b'^2}{a'\,a}$ in the second,

$$\therefore P T \,.\, P t = -b'^2 = C D^2.$$

The product P T . P t is negative because P T, P t being situated on opposite sides of the axis A X have different signs.

SECTION III.

On Supplemental Chords.

Def. If from the vertices of any diameter two straight lines be drawn to any point in the ellipse, they are called *Supplemental Chords.*

(61.) *Any two supplemental chords being drawn, and the equation to either of them being given, to find the equation to the other.*

The ellipse being referred to any two conjugate diameters, its equation will be Fig. 31.

$$a'^2 y^2 + b'^2 x^2 = a'^2 b'^2 \ldots . (1.)$$

Through any point P (x', y') draw the diameter P p, and let P Q, $p\,q$ be any two supplemental chords; then the equation to P Q being

$$y - y' = a\,(x - x') \ldots . (2)$$

it is required to find the equation to p Q.

The coordinates of P being x', y', those of p will be $-x'$, $-y'$, therefore the equation to p Q will be *of the form*

$$y + y' = a'\,(x + x') \ldots . (3.)$$

in which a' is to be found.

Since the lines whose equations are (2) and (3) intersect at Q, the coordinates of Q will be identical; therefore considering x and y as the same in these equations we have by multiplying them together

$$y^2 - y'^2 = a\,a'\,(x^2 - x'^2),$$

$$\therefore a\,a' = \frac{y^2 - y'^2}{x^2 - x'^2} \ldots . (4;)$$

but x and y being the coordinates of Q, a point in the ellipse, they will satisfy equation (1,)

$$\therefore a'^2 y^2 + b'^2 x^2 = a'^2 b'^2.$$

Subtracting (1) from this, we have

$$a'^2 (y^2 - y'^2) + b'^2 (x^2 - x'^2) = 0,$$

$$\therefore \frac{y^2 - y'^2}{x^2 - x'^2} = -\frac{b'^2}{a'^2};$$

therefore by substitution in (4)

$$a\,a = -\frac{b'^2}{a'^2\,a}, \therefore a' = -\frac{b'^2}{a'^2\,a},$$

and the equation to p Q becomes by substitution

$$y - y = -\frac{b'^2}{a'^2\,a}\,(x - x').$$

Cor. Let Pp coincide with the major axis Aa, then the equation to aQ drawn through the point $(-a, 0)$ will be

$$y = \alpha(x + a),$$

therefore the equation to AQ drawn through the point $A(+a, 0)$ will be

$$y = -\frac{b'^2}{a'^2\alpha}(x - a).$$

(62.) *If two diameters be drawn parallel to any supplemental chords, they will be conjugate to each other.*

The equations to any two supplemental chords being

$$y - y' = \alpha(x - x') \dots (1,)$$

and $$y + y' = -\frac{b'^2}{a'^2\alpha}(x + x') \dots (2,)$$

let any diameter be drawn parallel to (1,) then its equation will be

$$y = \alpha x;$$

therefore the equation to its conjugate being

$$y = -\frac{b'^2}{a'^2\alpha} \cdot x,$$

it follows that the latter is parallel to (2), as was to be proved.

Cor. 1. Hence may be drawn a diameter which shall be conjugate to a given diameter.

Let Pp be the given diameter, and

1. Let the major axis of the ellipse be given.

From a draw aR parallel to Pp, and join R, A; then if Dd be drawn through C parallel to RA, it will be conjugate to Pp.

2. If the major axis be not given.

Fig. 33. Draw any diameter whatever, Rr; through r draw rQ parallel to Pp, join Q, R; then if Dd be drawn through C parallel to RQ, it will be conjugate to Pp.

These conclusions are evident.

Cor. 2. Hence also is derived a very simple method of applying a tangent at a given point of the ellipse.

Let P be the given point, and

1. Let the major axis be given.

Draw PC, and the chord aQ parallel to it, join QA; then if PT be drawn parallel to QA, it will touch the ellipse at P.

2. Let the major axis be unknown.

Draw any diameter whatever RCr, join P, C; draw rQ parallel to PC, join Q, R; then if PT be drawn parallel to QR it will be a tangent at P.

(63.) *To find the angle contained by the supplemental chords, drawn from the extremities of the major axis.*

Fig. 34. Let the point $Q(x', y')$ be the intersection of the two chords AQ, aQ, and suppose the ellipse referred to its axes.

Then if the equations to Qa, QA be

$$y = \alpha(x + a),$$
$$y = \alpha'(x - a),$$

$$\tan AQa = \frac{\alpha' - \alpha}{1 + \alpha\alpha'} \text{ (Anal. Geom., Art. 13)}$$

$$= \frac{\alpha' - \alpha}{1 - \frac{b^2}{a^2}} \dots (1,) \text{ since } \alpha' = -\frac{b^2}{a^2\alpha}.$$

Now $$\alpha' = \tan QAX = -\tan QAa = -\frac{y'}{a - x'},$$

and $$\alpha = \tan QaX = \frac{y'}{a + x'},$$

$$\therefore \alpha' - \alpha = -y' \cdot \left(\frac{1}{a - x'} + \frac{1}{a + x'}\right),$$

$$= -\frac{y'}{a^2 - x'^2} \cdot 2a,$$

$$= -\frac{2ab^2}{a^2 y'};$$

therefore by substitution in (1)

$$\tan AQa = -\frac{2ab^2}{y'(a^2 - b^2)},$$

therefore, since the sign of this quantity is negative, the angle is always obtuse.

Cor. 1. The angle AQa will be the greatest possible when y' is so, that is, when $y' = b$, or the point Q coincides with B, the vertex of the minor axis. At this point the supplemental chords are equal, and their inclination to the major axis is

$$= \tan^{-1}\frac{b}{a}.$$

Cor. 2. Hence, the conjugate diameters which are parallel to these chords are also equal, and contain the greatest possible angle. See Art. 57.

(64.) *To draw two conjugate diameters making a given angle with each other.*

The ellipse being referred to its axes, let $2a'$, $2b'$ denote the conjugate diameters required, and γ the angle at which they are inclined to each other.

Then, since $$a'^2 + b'^2 = a^2 + b^2 \dots (1,)$$

and $$a'b' = \frac{ab}{\sin\gamma} \dots\dots (2,)$$

we have, by adding twice the second equation to the first,

$$a'^2 + b'^2 + 2a'b' = a^2 + b^2 + \frac{2ab}{\sin\gamma},$$

therefore extracting the square root,

$$a' + b' = \pm\sqrt{a^2 + b^2 + \frac{2ab}{\sin\gamma}}.$$

In like manner,

$$a' - b' = \pm\sqrt{a^2 + b^2 - \frac{2ab}{\sin\gamma}},$$

therefore by addition and subtraction successively,

$$a' = \pm\tfrac{1}{2}\sqrt{a^2 + b^2 + \frac{2ab}{\sin\gamma}} \pm \tfrac{1}{2}\sqrt{a^2 + b^2 - \frac{2ab}{\sin\gamma}},$$

$$b' = \pm\tfrac{1}{2}\sqrt{a^2 + b^2 + \frac{2ab}{\sin\gamma}} \mp \tfrac{1}{2}\sqrt{a^2 + b^2 - \frac{2ab}{\sin\gamma}};$$

therefore the *magnitude* of the required diameters is determined.

Again, since $PCA = DCA - DCP$,

$$\tan PCA = \frac{\tan DCA - \tan DCP}{1 + \tan DCA \tan DCP},$$

or retaining the notation already used

$$\alpha = \frac{\alpha' + \tan\gamma}{1 - \alpha' \tan\gamma};$$

but $$\alpha\alpha' = -\frac{b^2}{a^2}, \therefore \alpha' = -\frac{b^2}{a^2\alpha},$$

therefore by substitution

$$\alpha = \frac{-\frac{b^2}{a^2\alpha} + \tan\gamma}{1 + \frac{b^2}{a^2\alpha}\tan\gamma},$$

$$\therefore \alpha^2 - \frac{b^2}{a^2}\tan\gamma \,.\, \alpha = -\frac{b^2}{a^2} - \alpha\tan\gamma,$$

or $$\alpha^2 - \left(1 - \frac{b^2}{a^2}\right)\tan\gamma \,.\, \alpha = -\frac{b^2}{a^2},$$

$$\therefore \alpha = \frac{a^2 - b^2}{2a^2}\tan\gamma \pm \frac{1}{2a^2}\sqrt{(a^2 - b^2)^2\tan^2\gamma - 4a^2b^2},$$

therefore the *position*, also, of the diameters is determined.

The problem would be impossible, if

$$\tan^2\gamma < \frac{4a^2b^2}{(a^2 - b^2)^2},$$

or $$\tan\gamma < \frac{2ab}{a^2 - b^2}.$$

But γ being an acute angle it will be a *minimum* when the diameters are equal, and in that case

$$\tan\gamma = \frac{2ab}{a^2 - b^2},$$

therefore $\tan\gamma$ can never be *less* than $\frac{2ab}{a^2 - b^2}$, and therefore the problem is always possible.

The same problem admits of the following geometrical solution.

Fig. 35. Draw any diameter whatever, Rr, and upon it describe a segment of a circle containing an angle equal to the given angle, and cutting the ellipse in Q; join QR, Qr, and parallel to these draw the diameters Pp, Dd; these will be the diameters required.

For being parallel to the supplemental chords QR, Qr, they are conjugate to each other, and the angle $PCD = RQr$, and therefore equal to the given angle.

The problem admits of a second solution: for the circle will cut the ellipse again in some point Q'; draw therefore the supplemental chords $Q'R$, $Q'r$; then if $P'p$ and $D'd$ be drawn through the centre parallel to $Q'R$, $Q'r$, they will be the diameters required.

Fig. 36. For they are evidently conjugate to each other, and $P'CD' = \pi - RQ'r$, and is therefore equal to the given angle.

CHAPTER IV.

MISCELLANEOUS PROPOSITIONS.

(65.) *An ellipse being traced upon a plane, to find its centre and axes.*

1. *To find its centre.*

Draw any two parallel chords MN, PQ, and bisect them in the points m, p respectively, join mp and produce it to meet the ellipse in R, r; then, because mp passes through the centre it is a diameter, and therefore C, the middle point of Rr, is the centre required. Fig. 37.

2. *To find the axes.*

Assume any point P in the ellipse, and

From the point C, just found, as centre, with distance CP, describe a circle cutting the ellipse in p, draw Pp and bisect it at right angles by a straight line ACa meeting the ellipse in A and a; then ACa is the *major* axis: and the *minor* axis is obtained by drawing BCb at right angles to Aa. Fig. 38.

(66.) *To find the locus of the extremity of a straight line which moves on two lines at right angles to each other, so that the parts intercepted by these lines may always be of the same given length.*

Let AX, AY be the given lines, QRP any position of the line, the locus of whose extremity is sought. Fig. 39.

Assuming AX, AY as the axes of coordinates, let fall the perpendicular PM on AX, and produce it to meet in N a parallel to AX drawn through the point Q.

Let $AM = x$, $MP = y$, $QP = a$, $PR = b$;

then $QP^2 = QN^2 + NP^2 \ldots (1;)$

but $QN = AM = x$,

and $NP = \frac{QP}{RP} . MP = \frac{a}{b}\,y$,

therefore by substitution

$$a^2 = x^2 + \frac{a^2}{b^2}y^2,$$

or $$a^2y^2 + b^2x^2 = a^2b^2,$$

which is the equation to an ellipse.

Therefore the locus of P is an ellipse of which A is the centre, and $2a$ and $2b$ the axes.

Cor. 1. Hence may be derived an easy practical method of describing an ellipse by means of an instrument called the *Elliptic Compasses.*

Let QP be assumed equal the semi-major, and NP equal the semi-minor, axis; and let the line QNP be turned round so that the points Q, N may always remain upon the axes of coordinates; then the point P will describe an ellipse, as is evident from the foregoing investigation. Fig. 40.

Cor. 2. By a method precisely similar to the above, it may be proved, that if the axes are inclined to each other at an angle θ, the equation to the locus of P will be

$$a^2y^2 + b^2x^2 + 2ab\cos\theta \,.\, xy - a^2b^2 = 0.$$

(67.) *In the major axis Aa of an ellipse to find a point O, such that if any chord whatever POp be*

Fig. 41.

drawn through it, the angle P A p *may be a right angle.*

Let the equation to A P be $y = \alpha x$, then that to A p will be $y = -\frac{1}{\alpha} x$; therefore the coordinates of P (x', y') and p (x'', y'') will be determined by eliminating (y) between the above equations, and the equation to the ellipse $y^2 = \frac{b^2}{a^2}(2ax - x^2)$; we thus have

$$x' = \frac{2 b^2 a}{a^2 \alpha^2 + b^2}, \quad y' = \frac{2 b^2 a \alpha}{a^2 \alpha^2 + b^2}$$

$$x'' = \frac{2 b^2 a \alpha^2}{a^2 + b^2 \alpha^2}, \quad y'' = -\frac{2 b^2 \text{o} \alpha}{a^2 + b^2 \alpha^2};$$

therefore, denoting $2 b^2 \text{o}$ by c, and the denominator in the first and second lines respectively by m and n, we have for the equation to P p

$$y = -\alpha \frac{m+n}{\alpha^2 m - n}\left(x - \frac{c}{m}\right).$$

Let P p now cut the axis as at O, then $y = 0$, and

$$x - \frac{c}{m} = \frac{c}{m} \frac{\alpha^2 m - n}{m + n},$$

$$= c \frac{\alpha^2 + 1}{m + n};$$

therefore, substituting for m and n, and reducing,

$$x \text{ or AO} = \frac{c}{a^2 + b^2} = \therefore \frac{2 b^2 a}{a^2 + b^2}.$$

(68.) *Pairs of tangents to an ellipse being always supposed to intersect at right angles, to find the locus of the points of intersection.*

If the straight line

$$y = \alpha x + \beta \ldots\ldots (1)$$

be drawn to cut the ellipse

$$a^2 y^2 + b^2 x^2 = a^2 b^2 \ldots\ldots (2,)$$

the ordinates of the two points of section will be obtained from the equation

$$(a^2 \alpha^2 + b^2) y^2 - 2 b^2 \beta y + b^2 (\beta^2 - a^2 \alpha^2). \text{ Art. 30}$$

Let the secant be now supposed to become a tangent, then the two roots of this equation are equal, and the equation being therefore a perfect square,

$$4 (a^2 \alpha^2 + b^2) b^2 (\beta^2 - a^2 \alpha^2) = 4 b^4 \beta^2,$$

or

$$(\text{o}^2 \alpha^2 + b^2)(\beta^2 - a^2 \alpha^2) = b^2 \beta^2,$$

$$\therefore a^2 \alpha^2 \beta^2 - \text{o}^4 \alpha^4 + b^2 a^2 \alpha^2 = b^2,$$

$$\therefore a^2 \alpha^2 \beta^2 = a^4 \alpha^4 + b^2 a^2 \alpha^2,$$

$$\therefore a^2 \alpha^2 + b^2 = \beta^2 = (y - \alpha x)^2 \text{ from (1,)}$$

$$= y^2 - 2 x y \alpha + \alpha^2 x^2,$$

$$\therefore (a^2 - x^2) \alpha^2 + 2 x y . \alpha + b^2 - y^2 = 0,$$

or

$$\alpha^2 + \frac{2 x y}{a^2 - x^2} + \frac{b^2 - y^2}{a^2 - x^2} = 0.$$

Suppose α', α'' to be the roots of this equation, then they denote the trigonometrical tangents of the angle which the tangents to the ellipse form with the axis of x, and by the theory of equations

$$\alpha \alpha' = \frac{b^2 - y^2}{a^2 - x^2},$$

but, by hypothesis, the tangents intersect at right angles,

$$\therefore \alpha \alpha' = -1;$$

hence

$$\frac{b^2 - y^2}{a^2 - x^2} = -1,$$

$$\therefore b^2 - y^2 = -a^2 + x^2,$$

$$\therefore y^2 + x^2 = \text{o}^2 + b^2,$$

which is the equation to a circle.

Hence the locus required is a circle whose radius

$$= \sqrt{a^2 + b^2}.$$

Cor. In the same manner we may find the locus of the intersection of pairs of tangents which are always parallel to conjugate diameters.

For in this case $\alpha \alpha' = -\frac{b^2}{a^2}$,

$$\therefore \frac{b^2 - y^2}{a^2 - x^2} = -\frac{b^2}{a^2},$$

$$\therefore \text{o}^2 b^2 - a^2 y^2 = -b^2 a^2 + b^2 x^2,$$

$$\therefore a^2 y^2 + b^2 x^2 = 2 a^2 b^2.$$

which is the equation to an ellipse.

Hence the locus required is an ellipse whose centre is the same as that of the original.

To find its axes.

Let $x = 0$, $\therefore a^2 y^2 = 2 a^2 b^2$,

$$\therefore y = b \sqrt{2} = \text{the semi-minor axis};$$

and, in like manner,

$$x = a \sqrt{2} = \text{the semi-major axis}.$$

ON THE HYPERBOLA.

CHAPTER I.

ON THE HYPERBOLA REFERRED TO ITS AXIS.

The hyperbola is the locus of a point, whose distance from the focus is always greater, in a given ratio, than its distance from the directrix.

(69.) *To find the equation to the hyperbola.*

Fig. 42 Let S be the focus, K k the directrix, P any point in the hyperbola, through S draw the indefinite line E S X perpendicular to K k, and from P let fall the perpendiculars P M, P Q on A X, K k respectively, and join P, S.

Let the given ratio of P S : P Q be as $e : 1$, e being > 1; then if S E be divided in A, so that S A : A E :: $e : 1$, A will be a point in the hyperbola.

From A draw A Y at right angles to A X, and assume A X and A Y as the axes of coordinates.

Let $\quad AM = x,\ MP = y,\ AS = m$;

then $SP^2 = PM^2 + MS^2 = y^2 + (x-m)^2 \ldots\ (1,)$

but $\quad SP^2 = e^2 . PQ^2 \quad = e^2(AE + AM)^2$

$$= e^2\left(\frac{m}{e} + x\right)^2 \ldots (2;)$$

therefore equating (1) and (2,)

$$y^2 + (x-m)^2 = m^2 + 2\,m\,e\,x + e^2 x^2,$$

$$\therefore y^2 = 2\,m\,(1+e)\,x + (e^2 - 1)\,x^2,$$

$$= (e^2 - 1)\left(\frac{2\,m}{e-1}\,x + x^2\right),$$

or if $\dfrac{m}{e-1}$ be assumed $= a$,

$$y^2 = (e^2 - 1)\,(2\,a\,x + x^2),$$

which is the equation required.

Cor. 1. In X A take $A\,a = \dfrac{2\,m}{e-1}$, bisect A a in C then at this point $x = -a$,

$$\therefore y^2 = (e^2 - 1)\,x - a^2,$$

$$\therefore y = \pm\, a\,\sqrt{-1}\,.\,\sqrt{e^2-1},$$

which is always imaginary, since $e > 1$.

Hence, if B C b be drawn through C at right angles to A a, and C B, C b each taken $= a\,\sqrt{e^2 - 1}$, the points B and b are *not* points in the hyperbola.

Cor. 2. Let B b be denoted by $2\,b$, then

$$b = \pm\, a\,\sqrt{e^2 - 1},$$

$$\therefore \sqrt{e^2-1} = \pm\,\frac{b}{a};$$

therefore, by substitution, the above equation becomes

$$y = \pm\,\frac{b}{a}\,\sqrt{2\,a\,x + x^2}.$$

Def. The straight lines A a, B b represented by $2\,a$ and $2\,b$ are called, respectively, the *transverse* and the *conjugate*, axis; the points A, a in which the former meets the hyperbola, are called the *vertices*; and the point C, in which the axes intersect each other, the *centre*.

(70.) *To find the equation to the hyperbola when the coordinates are measured from the centre.*

Let P be any point in the hyperbola, let fall the perpendicular P M on A a, and assume $CM = x'$.

Then the equation to the hyperbola, when the coordinates originate at A, is

$$y^2 = \frac{b^2}{a^2}(2\,a\,x + x^2) \ldots (1,)$$

but $\quad x = AM = CM - CA,$
$\quad\quad = x' - a.$

Substituting this value for x, we have

$$y^2 = \frac{b^2}{a^2}\left\{2\,a\,(x' - a) + (x' - a)^2\right\},$$

$$= \frac{b^2}{a^2}\,(x'^2 - a^2) \ldots (2,)$$

which is the equation required.

Cor. 1. Suppressing the accent, which was used only to distinguish the new from the old abscissa, we have by multiplying and transposing,

$$a^2 y^2 - b^2 x^2 = -\,a^2 b^2 \ldots (3.)$$

If each term be divided by $a^2 b^2$, we have

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = -1 \ldots (4.)$$

Of the three last forms of the equation to the hyperbola, that marked (3) is the most frequently used.

Cor. 2. These equations when translated into geometrical language express a property of the hyperbola.

For if P be any point, we have

$$2\,a\,x + x^2 = x\,(2\,a + x) = AM\,.\,M\,a,$$

and $\quad x'^2 - a^2 = (x' - a)\,(x' + a) = AM\,.\,M\,a,$

$$\therefore MP^2 = \frac{BC^2}{CA^2}\,AM\,.\,M\,a,$$

or $\quad AM\,.\,M\,a : MP^2 :: AC^2 : BC^2,$

that is, *the rectangle contained by the segments of the transverse axis is to the square of the ordinate, as the square of the semi-transverse axis is to the square of the semi-conjugate.*

Cor. 3. Let $a = b$, then equations (1) and (2) become

$$y^2 = 2\,a\,x + x^2,$$
$$y^2 = x^2 - a^2.$$

Conic Sections.

The hyperbola represented by these equations is called *equilateral*, or *rectangular*, and is to the common hyperbola what the circle is to the ellipse.

By comparing the equation to the hyperbola

$$a^2 y^2 - b^2 x^2 = -a^2 b^2,$$

with the equation to the ellipse

$$a^2 y^2 + b^2 x^2 = a^2 b^2,$$

it is manifest, that to pass from the one curve to the other we have only to change $+ b^2$ into $- b^2$, or b into $b\sqrt{-1}$.

(71.) *To determine the figure of the hyperbola, from its equation.*

Resuming the equation

$$a^2 y^2 - b^2 x^2 = -a^2 b^2,$$

we have either

$$y = \pm \frac{b}{a} \sqrt{x^2 - a^2} \ldots. (1.)$$

or

$$x = \pm \frac{a}{b} \sqrt{y^2 + b^2} \ldots (2.)$$

I. In equation (1,)

let $x = 0$,

then $y = \pm b\sqrt{-1} = \text{C B or C } b$.

Let $y = 0$,

then $x = \pm a = \text{C A or C } a$.

Let $x < \pm a$,

then the values of y are imaginary; therefore no point of the hyperbola is situated between A and a.

Let $x = \pm a$,

then $y = \pm 0$;

that is, the hyperbola cuts the axis A X at the points A, a.

Let $x > \pm a$,

then for each value of x there are *two* equal values of y with contrary signs.

As x increases, the values of y increase; and when x becomes indefinitely great, the values of y become so likewise.

The hyperbola, therefore, consists of two equal and opposite branches extending indefinitely to the *right* of A and to the *left* of a, and symmetrically placed with respect to the axis X A X′.

II. The discussion of equation (2) would lead to the same result.

Observation. In the equation $a^2 y^2 - b^2 x^2 = -a^2 b^2$, let x be changed into y, and y into x; in other words, let the abscissas be now reckoned along C Y and the ordinates along C X; we then have

$$a^2 x^2 - b^2 y^2 = -a^2 b^2,$$

which represents the same hyperbola as before, but differently placed.

Let $x = 0$, $\therefore y = \pm a$,

$y = 0$, $\therefore x = \pm b\sqrt{-1}$,

therefore the transverse axis is now B b, and the conjugate axis A a.

This hyperbola is called, relatively to the former, the *conjugate hyperbola*.

Hyperbola.

Cor. To find the value of the ordinate passing through the focus.

When the ordinate passes through the focus,

$$x = m = a(e - 1),$$

therefore by substitution in (1), Art. 70,

$$y^2 = \frac{b^2}{a^2}\{2a^2(e-1) + a^2(e-1)^2\},$$
$$= b^2(e-1)\{2 + e - 1\},$$
$$= b^2(e^2 - 1),$$
$$= \frac{b^4}{a^2} \text{ (Art. 69, } Cor.\ 2,)$$
$$\therefore y = \pm \frac{b^2}{a}.$$

The double ordinate passing through the focus is called the *principal parameter*, or *latus rectum*,

$$\text{therefore the } latus\ rectum = \frac{2b^2}{a}.$$

Def. The line S C, represented by $a e$, is called the *eccentricity* of the hyperbola.

(72.) *To find the intersection of a straight line with the hyperbola.*

Let the equation to the proposed line be

$$y = \alpha x + \beta \ldots. (1.)$$

Then the coordinates of the point or points of intersection with the hyperbola will be obtained by combining this equation with that to the hyperbola

$$a^2 y^2 - b^2 x^2 = -a^2 b^2 \ldots. (2.)$$

Substituting, then, in (2) the value of x derived from (1) we have

$$a^2 y^2 - b^2\left(\frac{y - \beta}{\alpha}\right)^2 = -a^2 b^2,$$
$$\therefore (a^2\alpha^2 - b^2)y^2 + 2b^2\beta y - b^2\beta^2 = -a^2 b^2 \alpha^2,$$
$$\therefore y^2 + \frac{2b^2\beta}{a^2\alpha^2 - b^2}y - \frac{b^2(\beta^2 - a^2\alpha^2)}{a^2\alpha^2 - b^2} = 0;$$

from this quadratic are obtained two values of y, which substituted in (1,) furnish two corresponding values of x, therefore the coordinates required may be determined.

When the two roots of the quadratic are equal, the points of intersection coincide, and the straight line then *touches* the hyperbola; when they are imaginary, the straight line falls entirely without the hyperbola.

Hence it appears, that a straight line cannot cut an hyperbola in more than two points.

Def. The portion of the straight line contained within the hyperbola is called a *chord;* when the chord passes through the focus it is called the *focal chord.*

(73.) *To find the equation to a straight line which touches an hyperbola in a given point.*

Let x', y' be the coordinates of the given point, and x'', y'' those of any other point in the hyperbola near the first.

Then the equation to the line drawn through these points is

$$y - y' = \frac{y'' - y'}{x'' - x'}(x - x') \ldots. (1)$$

But these two points being in the hyperbola, we have

$$a^2 y'^2 - b^2 x'^2 = -a^2 b^2,$$
$$a^2 y''^2 - b^2 x''^2 = -a^2 b^2;$$

therefore by subtraction

$$a^2 (y''^2 - y'^2) = b^2 (x''^2 - x'^2),$$
$$\therefore \frac{(y'' + y')(y'' - y')}{(x'' + x')(x'' - x')} = \frac{b^2}{a^2},$$
$$\therefore \frac{y'' - y'}{x'' - x'} = \frac{b^2}{a^2} \cdot \frac{x'' + x'}{y'' + y'}$$

and equation (1) becomes by substitution

$$y - y' = \frac{b^2}{a^2} \cdot \frac{x'' + x'}{y'' + y'} (x - x').$$

Let the point (x'', y'') be now supposed to coincide with (x', y'); then $x'' = x'$, $y'' = y'$, and the secant becomes a tangent at the point (x', y'); hence the equation to the tangent is

$$y - y' = \frac{b^2}{a^2} \cdot \frac{x'}{y'} (x - x'),$$

in which x' and y' are the coordinates of the point of contact, and x, y the variable coordinates of any point whatever in the tangent.

Cor. This equation may be simplified, for multiplying each side by $a^2 y'$,

$$a^2 y y' - a^2 y'^2 = b^2 x x' - b^2 x'^2,$$
$$\therefore a^2 y y' - b^2 x x' = a^2 y'^2 - b^2 x'^2,$$
$$= -a^2 b^2,$$

which is the equation most commonly used.

When $a = b$, the hyperbola becomes equilateral, and the equation to the tangent is

$$y y' - x x' = -a^2.$$

(74.) *To find the intersection of the tangent with the axes of x and y.*

Fig. 13. The equation to the tangent being

$$a^2 y y' - b^2 x x' = -a^2 b^2,$$

let it cut

1. The axis of x, as at T.

Then $\quad y = 0, \therefore x = \frac{a^2}{x'}$,

or $\quad CT = \frac{CA^2}{CM}$.

2. The axis of y, as at t.

Then $\quad x = 0, \therefore y = \frac{b^2}{y'}$,

or $\quad Ct = \frac{CB^2}{Cm}$.

Whence it follows, *that each semi-axis is a mean proportional between the abscissa of any point, and the part of it intercepted between its intersection with the tangent, and the centre.*

Cor. Since $CT = \frac{a^2}{x'}$

$$\therefore MT = CM - CT,$$
$$= x' - \frac{a^2}{x'},$$
$$= \frac{x'^2 - a^2}{x'}.$$

Def. The line MT intercepted between the foot of the ordinate, and the point where the tangent meets the axis, is called the *subtangent.*

Def. The straight line drawn from the point of contact at right angles to the tangent, is called the *normal.*

(75.) *To find the equation to the normal.*

Let PT touch the hyperbola in P, from which point draw the line Pg at right angles to PT.

Then, because Pg is drawn through the point (x', y') at right angles to the line,

$$y - y' = \frac{b^2}{a^2} \cdot \frac{x'}{y'} (x - x')$$

its equation will be

$$y - y' = -\frac{a^2}{b^2} \frac{y'}{x'} (x - x'),$$

in which x', y' are the coordinates of the point of contact, and x, y those of any point whatever in the indefinite line Pp.

The term *normal* is usually confined to the line PG. See Art. 9.

(76.) *To find the intersection of the normal with the axes of x and y.*

The equation to the normal being Fig. 43.

$$y - y' = -\frac{a^2}{b^2} \frac{y'}{x'} (x - x'),$$

let it intersect

1. The axis of x as at G.

Then $y = 0$, and $-y' = -\frac{a^2}{b^2} \cdot \frac{y'}{x'} (x - x')$,

$$\therefore x - x' = \frac{b^2}{a^2} x' = MG.$$

2. The axis of y, as at g.

Then $\quad x = 0, \therefore y - y' = \frac{a^2}{b^2} \frac{y'}{x'} x',$

$$= \frac{a^2}{b^2} y',$$
$$\therefore y = \frac{a^2 + b^2}{b^2} y'.$$

Def. The line MG intercepted between the foot of the ordinate, and the point where the normal cuts the axis of x, is called the *subnormal.*

(77.) *To draw a tangent to an hyperbola from a given point without it.*

The equation to the tangent being in general

$$a^2 y y' - b^2 x x' = -a^2 b^2,$$

and the point (x'', y'') being by hypothesis a point in the tangent, we have

$$a^2 y'' y' - b^2 x'' x' = -a^2 b^2 \ldots . (1;)$$

also, the point of contact (x', y') being in the hyperbola

$$a^2 y'^2 - b^2 x'^2 = -a^2 b^2 \ldots . (2,)$$

hence, by means of these two equations, the coordinates x', y' of the point of contact may be determined.

Since the equation resulting from the elimination of x' between (1) and (2) is of the second degree, it follows, that there are two points of contact; in other words, *that two tangents may be drawn to an hyperbola from a given point without it.*

But the position of the points of contact may be directly found by constructing, as in Arts. 10 and 35, the loci of equations of (1) and (2,) in which x' and y' are variable.

Now the locus of (2) is the given hyperbola, and the locus of (1) is a straight line whose position is determined by making x' and y' successively $= 0$.

If, therefore, in the equation

$$a^2 y'' y' - b^2 x'' x' = -a^2 b^2,$$

$$x' = 0, \text{ then } y' = -\frac{b^2}{y''},$$

$$y' = 0, \text{ then } x' = -\frac{a^2}{x''}.$$

Hence if C R be taken $= \frac{b^2}{y''}$, and C $r = \frac{a^2}{x''}$, the line joining R, r will cut the hyperbola in the points of contact required.

Cor. 1. The equation to the chord joining the points of contact is

$$a^2 y'' y' - b^2 x'' x' = -a^2 b^2.$$

Cor. 2. Since C R is independent of y'', it follows *that if from the several points of a line perpendicular to* C X *pairs of tangents be drawn to the hyperbola, the chords joining the points of contact, in each case, will all pass through the same given point.*

CHAPTER II.

ON THE HYPERBOLA REFERRED TO THE FOCUS.

(78.) *To find the distance of any point in the hyperbola from either focus.*

Fig. 44. Let S, H be the foci, P any point (x, y) in the hyperbola, to find the value of S P, or H P.

1. Of S P.

In general, the distance between two points (x, y) and (x', y') is

$$= \sqrt{(x - x')^2 + (y - y')^2};$$

but the coordinates of S, since it is a point on the axis of x, are $x' = a e$, $y' = 0$,

$$\begin{aligned} \therefore \text{S P}^2 &= (x - a e)^2 + y^2, \\ &= (x - a e)^2 + (e^2 - 1)(x^2 - a^2), \\ &= x^2 - 2 a e x + a^2 e^2 + e^2 x^2 - e^2 a^2 - x^2 + a^2, \\ &= a^2 - 2 a e x + e^2 x^2, \end{aligned}$$

$$\therefore \text{S P} = e x - a.$$

2. In like manner,

$$\text{H P} = e x + a.$$

Cor. Hence, subtracting S P from H P,

$$\text{H P} - \text{S P} = 2 a.$$

In other words, *the difference of the focal distances is equal to the transverse axis.*

The distance of any point from the focus is called the *focal distance.*

(79.) From this property the equation to the hyperbola may be deduced, as in the case of the ellipse.

Let S, H be the two fixed points, P the point whose locus is required.

Join S, H, S, P, and H, P; bisect S H in C; let fall the perpendicular P M on S H, which produce indefinitely towards X; from C draw C Y at right angles to C X, and assume C X and C Y as the axes of the coordinates.

Let C M $= x$, M P $= y$, and S C $= c$.

$$\left.\begin{aligned} \text{Then S P}^2 &= \text{S M}^2 + \text{M P}^2 = y^2 + (c - x)^2 \\ \text{H P}^2 &= \text{H M}^2 + \text{M P}^2 = y^2 + (c + x)^2 \end{aligned}\right\} \ldots (1,)$$

$$\therefore \text{H P}^2 - \text{S P}^2 = (c + x)^2 - (c - x)^2,$$

or $\quad (\text{H P} + \text{S P})(\text{H P} - \text{S P}) = 4 c x;$

but $\text{H P} - \text{S P} = 2 a$,

$$\therefore \text{H P} + \text{S P} = \frac{4 c x}{2 a} = \frac{2 c x}{a}, \text{ and}$$

$$\text{H P} - \text{S P} = 2 a,$$

$$\therefore \text{H P} = \frac{c x}{a} + a,$$

and
$$\text{S P} = \frac{c x}{a} - a,$$

squaring these values, and adding the results,

$$\text{H P}^2 + \text{S P}^2 = 2\left(\frac{c^2 x^2}{a^2} + a^2\right),$$

and also $\quad = 2(y^2 + c^2 + x^2)$

by adding equations (1,)

$$\therefore y^2 + c^2 + x^2 = \frac{c^2 x^2}{a^2} + a^2,$$

$$\begin{aligned} \therefore y^2 &= \frac{c^2 x^2}{a^2} + a^2 - c^2 - x^2, \\ &= \frac{x^2}{a^2}(c^2 - a^2) + (a^2 - c^2), \\ &= \frac{c^2 - a^2}{a^2}(x^2 - a^2), \end{aligned}$$

which is the equation to an hyperbole whose transverse axis $= 2 a$, and conjugate axis $= 2\sqrt{c^2 - a^2}$.

If $x = 0$, then $y^2 = -(c^2 - a^2) = -b^2$, if b be the imaginary ordinate drawn from C.

(80.) *To find the polar equation to the hyperbola, the focus being the pole.*

1. Let S be the pole. Fig. 44.

Let S P $= r$, angle A S P $= \omega$.

Then $\quad r = e x - a,$

but
$$\begin{aligned} x &= \text{C S} + \text{S M}, \\ &= a e + r \cos(\pi - \omega), \\ &= a e - r \cos \omega, \end{aligned}$$

$$r = a e^2 - e r \cos \omega - a$$

Conic Sections.

$$\therefore r = a \frac{e^2 - 1}{1 + e \cos \omega},$$

which is the equation required.

2. Let H be the pole.

Let $\quad HP = r'$, angle $PHA = \omega'$;

then $\quad r' = ex + a,$

but $\quad x = CM = HM - HC$

$$= r' \cos \omega' - ae,$$

$$\therefore r' = e r' \cos \omega' - a e^2 + a,$$

$$\therefore r' = \qquad - a \frac{e^2 - 1}{1 - e \cos \omega'},$$

which is the equation required.

Cor. 1. Produce PS to meet the hyperbola in p, then because $ASp = \pi - \omega$,

$$Sp = a \frac{e^2 - 1}{1 - e \cos \omega}.$$

Cor. 2. Hence $\dfrac{1}{SP} + \dfrac{1}{Sp} = \dfrac{1 + e \cos \omega}{a(e^2 - 1)} + \dfrac{1 - e \cos \omega}{a(e^2 - 1)}$

$$= \frac{2}{a(e^2 - 1)},$$

$$= \frac{2}{SL},$$

therefore the principal semi-parameter is an harmonic mean between the segments of any chord drawn through the focus.

Cor. 3. Since $\dfrac{1}{SP} + \dfrac{1}{Sp} = \dfrac{SP + Sp}{SP \cdot Sp},$

and also $\qquad = \dfrac{2}{a(e^2 - 1)},$

$$\therefore SP \cdot Sp = \tfrac{1}{2} a (e^2 - 1)(SP + Sp).$$

(81.) *The focal distances of any point make equal angles with the tangent at that point.*

Fig. 45. Let TPt be a tangent at any point $P(x', y')$ draw the normal PG, and join S, P and H, P.

$$\text{Then } CG = \frac{a^2 + b^2}{a^2} x' = e^2 x', \quad (76.)$$

$$\therefore \frac{SG}{HG} = \frac{CG - CS}{CG + CS} = \frac{e^2 x' - ae}{e^2 x' + ae}$$

$$= \frac{ex' - a}{ex' + a} = \frac{SP}{HP}, \ (78.)$$

therefore PG bisects the angle SPh. Enc. vi. Prop. A.

Now the right angle $GPT = GPt$,

and angle $\qquad GPS = GPt$,

therefore the remaining angle $SPT = hPt = HPT$, that is, SP and HP make equal angles with the tangent at P, as was to be proved.

(82.) *To find the locus of the points in which a perpendicular from the focus upon the tangent at any point intersects the tangent.*

Hyperbola.

Let PT be a tangent at any point (x', y'), and SY a perpendicular let fall from S on PT, to find the locus of Y.

Fig. 46. From C let fall the perpendicular CQ on PT, and draw Sq parallel to PT meeting CQ in q.

Then $\quad CY^2 = CQ^2 + QY^2,$

$$= CQ^2 + Sq^2,$$

$$= CT^2 \sin^2 T + CS^2 \cos^2 T;$$

but $CT = \dfrac{a^2}{x'}$ (74) and $CS = ae$,

$$\therefore CY^2 = \frac{a^4}{x'^2} \sin^2 T + a^2 e^2 \cos^2 T,$$

$$= \frac{a^4}{x'^2}(1 - \cos^2 T) + a^2 e^2 \cos^2 T,$$

$$= \frac{a^4}{x'^2} + \frac{a^2}{x'^2}(e^2 x'^2 - a^2) \cos^2 T \ldots . (1.)$$

But $\qquad \tan T = \dfrac{-b^2}{a^2} \cdot \dfrac{x'}{y'},$

therefore, as in Art. 41, $\cos^2 T = \dfrac{x'^2 - a^2}{e^2 x'^2 - a^2}$;

and substituting in (1)

$$CY^2 = \frac{a^4}{x'^2} + \frac{a^2}{x'^2}(e^2 x'^2 - a^2) \cdot \frac{x'^2 - a^2}{e^2 x'^2 - a^2},$$

$$= \frac{a^4}{x'^2} + \frac{a^2}{x'^2}(x'^2 - a^2),$$

$$= \frac{a^4}{x'^2} + a^2 - \frac{a^4}{x'^2},$$

$$= a^2,$$

$$\therefore CY = \pm a,$$

therefore the locus of Y is a circle described on the transverse axis.

(83.) *The rectangle contained by the perpendiculars let fall from the foci upon the tangent at any point, is equal to the square of the semi-conjugate axis.*

For $SY = ST \sin T$,

but $\quad ST = CS - CT = ae - \dfrac{a^2}{x'} = \dfrac{a}{x'}(ex' - a),$

$\therefore SY = \dfrac{a}{x'}(ex' - a) \sin T.$ In like manner,

$$HZ = \frac{a}{x'}(ex' + a) \sin T,$$

$$\therefore SY \cdot HZ = \frac{a^2}{x'^2}(e^2 x'^2 - a^2) \sin^2 T;$$

but, as in the ellipse, $\sin^2 T = \dfrac{e^2 x'^2 - x'^2}{e^2 x'^2 - a^2}$

$$\therefore SY \cdot HZ = \frac{a^2}{x'^2}(e^2 x'^2 - a^2) \cdot x'^2 \cdot \frac{(e^2 - 1)}{e^2 x'^2 - a^2}$$

$$= a^2(e^2 - 1) = b^2.$$

Conic Sections.

CHAPTER III.

ON THE HYPERBOLA REFERRED TO ANY SYSTEM OF CONJUGATE DIAMETERS.

SECTION I.

ON CONJUGATE DIAMETERS IN GENERAL.

(84.) *To find the locus of the middle points of any two parallel chords.*

Fig. 47 Let Pp be any chord, O its middle point; from the points O, P, p let fall the perpendiculars O N, P M, $p\,m$ on the axis A X.

Let $\quad AN = X,\ NO = Y;$

then if the equation to Pp be

$$y = \alpha x + \beta \ldots . (1.)$$

the equation containing the values of y at the points P, p will be

$$y^2 + \frac{2b^2\beta}{a^2\alpha^2 - b^2}\,y - b^2\,\frac{\beta^2 - a^2\alpha^2}{a^2\alpha^2 - b^2} = 0.$$

Now since in any quadratic equation the coefficient of the second term with its proper sign is equal to the sum of the roots with their signs changed,

$$\frac{2b^2\beta}{a^2\alpha^2 - b^2} = -(PM + pm).$$

But O being the middle point of Pp,

$$ON = \frac{PM + pm}{2},$$

$$\therefore\ Y = \frac{-b^2\beta}{a^2\alpha^2 - b^2} \ldots . (2.)$$

Now X and Y satisfy equation (1,) since they are the coordinates of a point in Pp, therefore

$$X = \frac{1}{\alpha}(Y - \beta),$$

$$= \therefore \frac{-a^2\alpha\beta}{a^2\alpha^2 - b^2} \ldots . (3.)$$

To obtain the relation between X and Y we must eliminate β between (2) and (3,)

$$\therefore \frac{a^2\alpha^2 - b^2}{b^2}\,Y = \frac{a^2\alpha^2 - b^2}{a^2\alpha}\,X,$$

$$\therefore\ Y = \frac{b^2}{a^2\alpha}\,X.$$

Now α remains the same for all chords parallel to Pp, therefore the equation just found expresses the relation between the coordinates of their middle points, and being of the first degree, the locus required is a *straight line.*

Def. The straight line which has been proved to be the locus of the middle points of any number of parallel chords is called a *diameter*, and the points in which it intersects the curve are called the *vertices.*

The letters X and Y are introduced to distinguish the two sets of coordinates, and the equation to the diameter bisecting any chord,

$$y = \alpha x + \beta,$$

may always be written Hyperbola.

$$y = \frac{b^2}{a^2\alpha}\,x.$$

From the form of this, it is plain that every diameter passes through the centre.

(85.) *To find the intersection of any diameter with the hyperbola.*

The equation to any diameter being Fig. 48.

$$y = \alpha x,$$

and that to the hyperbola

$$a^2y^2 - b^2x^2 = -a^2b^2,$$

the coordinates of the points of intersection will be obtained by combining these two equations; we thus have

$$(a^2\alpha^2 - b^2)\,x^2 = -a^2b^2,$$

$$\therefore\ x = \pm\frac{ab}{\sqrt{b^2 - a^2\alpha^2}} = CM \text{ or } Cm,$$

and

$$\therefore\ y = \pm\frac{ab\alpha}{\sqrt{b^2 - a^2\alpha^2}} = PM \text{ or } pm,$$

the coordinates required.

Cor. I. *Since* $AM = Am$, *and* $PM = pm$, *it follows that every diameter is bisected at the centre.*

Cor. 2. In order that the diameter may meet the hyperbola,

b^2 must be $> a^2\alpha^2$,

or $\quad \pm b$ must be $> a\alpha$,

therefore $\quad \alpha$ must be $< \pm \frac{b}{a}$.

From the vertex A draw A E and A e perpendicular to A C, and each equal b; join C E, C e, and produce them indefinitely towards Z and z.

Then since $\quad \tan ZCX = \dfrac{EA}{AC} = \dfrac{b}{a}$, Fig. 49.

and $\quad \tan zCX = \dfrac{eA}{AC} = -\dfrac{b}{a}$,

it follows that the diameters C Z, C z will never meet the curve at any finite distance.

The lines C Z, C z are from this circumstance called *asymptotes.*

(86.) *A diameter being drawn through a given point to find the equation to any one of its ordinates.*

If x', y' be the coordinates of the given point, the equation to the diameter drawn through it will be

$$y = \frac{y'}{x'}\,x \ldots . (1.)$$

Let $\quad y = \alpha x + \beta \ldots . (2.)$

be the required equation to any ordinate, then

$$\frac{y'}{x'} = \frac{b^2}{a^2\alpha},$$

$$\therefore\ \alpha = \frac{b^2}{a^2}\,\frac{x'}{y'},$$

therefore the equation to any ordinate to a diameter passing through (x', y') is

$$y = \frac{b^2}{a^2}\,\frac{x'}{y'}\,x + \beta.$$

Conic Sections

Cor. Comparing this equation with the equation to the tangent, it appears *that the tangent applied at the vertex of any diameter is parallel to the ordinates of that diameter.*

(87.) *Two diameters being drawn such that the ordinates of one may be parallel to the other, to prove that the ordinates of the latter will be parallel to the former.*

Fig. 50. Let C P, C D be two diameters, and M N, Q R chords bisected by each respectively; then if M N be supposed parallel to C D, we are to prove that Q R will be parallel to C P.

If the equations to C P, C D be

$$y = \alpha x \dots (1,)$$

$$y = \alpha' x \dots (2,)$$

then the equations to M N, Q R, respectively, will, by Art. 84, be

$$y = \frac{b^2}{a^2 \alpha} x + \beta \dots (1',)$$

$$y = \frac{b^2}{a^2 \alpha'} x + \beta' \dots (2'.)$$

But if M N be parallel to C D, then

$$\alpha' = \frac{b^2}{a^2 \alpha},$$

$$\therefore \alpha = \frac{b^2}{a^2 \alpha'},$$

therefore by substitution in (2') the equation to Q R becomes

$$y = \alpha x + \beta',$$

that is, Q R is parallel to C P, as was to be proved.

Whence each of the diameters C P, C D *is parallel to the ordinates of the other.*

Diameters thus related to each other are called *conjugate diameters.*

Cor. 1. Therefore when the two diameters

$$y = \alpha x,$$

$$y = \alpha' x,$$

are conjugate to each other,

$$\alpha \alpha' = \frac{b^2}{a^2}.$$

Cor. 2. Hence if $y = \alpha x$ be any diameter,

$$y = \frac{b^2}{a^2 \alpha}$$

will be the diameter conjugate to it.

Cor. 3. Since (α) may have any value between 0 and ∞, the number of pairs of conjugate diameters is infinite.

If $\alpha = 0$, or the first diameter be the transverse axis A a, then

$$\alpha' = \frac{b^2}{a^2 . 0} = \infty,$$

therefore the diameter conjugate to A a being at right angles to it, is the conjugate axis B b; whence the axes are conjugate diameters; and it may be shown, precisely as in Art. 47, that they are the only conjugate diameters which are at right angles to each other.

Hyperbola.

Cor. 4. If (x', y') be any point in the hyperbola, the diameter passing through it is

$$y = \frac{y'}{x'} x,$$

$$\therefore y = \frac{b^2}{a^2} \frac{x'}{y'} x$$

is the corresponding conjugate diameter.

But the equation to a tangent applied at the point (x', y') is

$$y - y' = \frac{b^2}{a^2} \frac{x'}{y'} (x - x'),$$

whence it follows, that *the tangent at the vertex of any diameter is parallel to the corresponding conjugate diameter.*

(88.) *Of any two conjugate diameters, only one can meet the curve.*

For let

$$y = \alpha x,$$

$$y = \alpha' x$$

be any two conjugate diameters.

It was shown that no diameter can meet the curve unless

$$\alpha < \frac{b}{a}.$$

Suppose, then, in the given system, that the first diameter meets the curve, then

$$\alpha < \frac{b}{a},$$

but

$$\alpha \alpha' = \frac{b^2}{a^2},$$

$$\therefore \alpha' > \frac{b}{a},$$

and consequently the second diameter *cannot meet the hyperbola.*

(89.) *To find the equation to the hyperbola when it is referred to any two conjugate diameters as axes.*

Let C be the centre, C P, C D a given system of conjugate diameters, of which the former is supposed to be the axis of x, the latter the axis of y. Fig. 51

Take any point Q in the hyperbola, and draw Q q parallel to C Y, meeting C X in V.

Let C V $= x$, V Q $= y$, C P $= a'$; and since C Y does not meet the hyperbola, let C D $= b' \sqrt{-1}$. Because the chord Q q is bisected by C P in V, V Q $=$ V q, and since every other chord parallel to C Y is bisected by C X, it follows that for each assumed value of x there are two equal values of y with contrary signs. In like manner it may be shown, that for each assumed value of y there are two equal values of x with contrary signs; also, when $x = 0$ the values of y ought to be imaginary, and when $y = 0$ the values of x are real; therefore the equation required must be of the form

$$M y^2 - N x^2 = - P.$$

We are now to determine the values of M, N, and P.

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When the axis C X meets the hyperbola, $y = 0$, and $x = CP = a'$,

$$\therefore N x^2 = P = N a'^2,$$

$$\therefore N = \frac{P}{a'^2}.$$

When $x = 0$, the axis C Y does not meet the curve,

but $$y \text{ or } CD = b' \sqrt{-1},$$

$$M y^2 = -P = -N b'^2,$$

$$\therefore M = \frac{P}{b'^2}.$$

Substituting these valnes of M and N in the above equatiou, and dividing each term of the result hy P, we have

$$\frac{y^2}{b'^2} - \frac{x^2}{a'^2} = -1 \ldots (1.)$$

or $$a'^2 y^2 - b'^2 x^2 = -a'^2 b'^2 \ldots (2.)$$

either of which is the equation required.

Cor. 1. Hence $$y = \pm \frac{b'}{a'} \sqrt{x^2 - a'^2}.$$

Cor. 2. *To find the form of the equation, when the coordinates originate at* P, *the vertex of the diameter* C P.

Let $PM = x'$, then $x = CP + PM = a' + x'$.

Substitoting this value of x in *Cor.* 1, we have

$$y = \pm \frac{b'}{a'} \sqrt{(x' + a')^2 - a'^2},$$

or suppressing the accent of x,

$$y = \pm \frac{b'}{a'} \sqrt{x^2 + 2a'x},$$

which is the equation required.

Cor. 3. The equations (1.) (2.) and (3.) are of the same form as the equation in terms of the axes, (Art. 70,) and express a property of the hyperbola.

For $x^2 - a'^2 = (x + a')(x - a') = PV . VG$,

and $2a'x + x^2 = (2a' + x)x = PV . VG$,

$$\therefore VQ^2 = \frac{PC^2}{CD^2} PV . VG.$$

or $$PV . VG : QV^2 :: PC^2 : CD^2$$

that is, *the rectangle contained by the segments of any diameter is to the square of the ordinate as the square of the semidiameter is to the square of its semiconjugate.*

(90.) It appears from the preceding article, that the equation to the hyperbola retains the same form, whether the axes of coordinates be rectangular or oblique. Whence it follows, when the axes are oblique,

(1.) That if the equation to the transverse axis $A a$ be $$y = \alpha x,$$
the equation to the conjugate axis $B b$ will be

$$y = -\frac{b'^2}{a'^2 \alpha} x.$$

(2.) That the equation to the tangent at any point (x', y')

$$a'^2 y y' - b'^2 x x' = -a'^2 b'^2.$$

Hyperbola.

(91.) *To find the intersection of the tangent with any two conjugate diameters, considered as axes.*

Let a tangent applied at any point Q (x', y') meet C P in T, and C D in t, and draw the ordinates Q V, Q v. Then the equation to the tangent being Fig. 52.

$$a'^2 y y' - b'^2 x x' = -a'^2 b'^2,$$

Let the tangent meet C X as at T, then $y = 0$,

and $$x = \frac{a'^2}{x'}, \text{ or } CT = \frac{CP^2}{CV}.$$

Let the tangent meet C Y as at t, then $x = 0$,

$$\therefore y = \frac{-b'^2}{y'}, \text{ or } Ct = \frac{CD^2}{CV}.$$

Whence the points of intersection are known. See (74,) which is only a particular case of this Article.

(92.) *If from the several points of a straight line given in position, pairs of tangents be drawn to an hyperbola, the lines which join the corresponding points of contact will all pass through the same point.*

Let C be the centre of the hyperbola, M N the given line, draw any chord $m\,n$ parallel to M N, and bisect it hy the diameter C X; from C draw C Y parallel to $m\,n$, or M N, then C X, C Y are conjugate diameters, and if the hyperbola be referred to these as axes, its equation will be

$$a'^2 y^2 - b'^2 x^2 = -a'^2 b'^2 \ldots (1.)$$

From any point (x'', y'') in M N let a pair of tangents be drawn to the hyperbola, then it may be shown, that the equation to the line joining the points of contact is

$$a'^2 y'' y' - b'^2 x'' x' = -a'^2 b'^2 \ldots (2.$$

in which x', y' are the variable coordinates of the point of contact.

Let the straight line (2) cut the axis of x, then $y = 0$,

and $$\therefore x' = \frac{a'^2}{x''},$$

hence the point of intersection will be the same for all points whose abscissas equal x'', that is, for all points in the line M N, as was to be proved.

Cor. The point of intersection is situated on the diameter conjugate to that which is parallel to the given line.

(93.) *If from the point of intersection of two tangents a diameter be drawn, it will bisect the line joining the points of contact.*

For the equation to an ordinate to the diameter passing through (x'', y'') is (86)

$$y = \frac{b'^2}{a'^2} \frac{x''}{y''} x + \beta \ldots (1.)$$

and the equation to the line joining the points of contact is

$$y' = \frac{b'^2}{a'^2} \frac{x''}{y''} x + \frac{b'^2}{y''} \ldots (2.)$$

therefore the latter being parallel to the former is also an ordinate and consequently is bisected.

Conic Sections.

Hyperbola.

(94.) *If through any point within or without an hyperbola, two straight lines, given in position, be drawn to meet the curve, the rectangle contained by the segments of the one will bear a constant ratio to the rectangle contained by the segments of the other.*

Fig. 53. Let O be any point within the hyperbola, through which draw the two lines P p, Q q, whose position is supposed known, to meet the hyperbola in the points P, p and Q, q, to prove that

$$\mathrm{O\,P} \,.\, \mathrm{O}\,p : \mathrm{O\,Q} \,.\, \mathrm{O}\,q \text{ in a constant ratio.}$$

Through O draw the diameter C X, and let C Y be the diameter conjugate to it; then, if the hyperbola be referred to these diameters as axes, its equation will be

$$a'^2 y^2 - b'^2 x^2 = - a'^2 b'^2 \dots (1.)$$

Through P draw P M parallel to C Y, and let O P $= r$, C O $= \delta$; then $\frac{\mathrm{P\,M}}{\mathrm{P\,O}} = \frac{\sin \mathrm{P\,O\,M}}{\sin \mathrm{P\,M\,O}} = \frac{\sin r, x}{\sin x, y}$, $\therefore y = \frac{\sin r, x}{\sin x, y} r$.

Similarly, $x = \delta + \frac{\sin r, y}{\sin x, y} r$,

or, denoting the coefficient of r by p in the first case, and q in the second, and substituting these values of x and y in (1,)

$$a'^2 p^2 r^2 - b'^2 \{ \delta^2 + 2 \delta q r + q^2 r^2 \} = - a'^2 b'^2,$$

$$\therefore r^2 - \frac{2 \delta q b'^2}{a'^2 p^2 - b'^2 q^2} r - \frac{b'^2 (\delta^2 - a'^2)}{a'^2 p^2 - b'^2 q^2} = 0,$$

in which the values of r are O P, O p,

$$\therefore \mathrm{O\,P} \,.\, \mathrm{O}\,p = \frac{b'^2 (\delta^2 - a'^2)}{a'^2 p^2 - b'^2 q^2}.$$

In like manner, if O Q $= r'$, and p' and q' denote $\frac{\sin r', x}{\sin x, y}$ and $\frac{\sin r', y}{\sin x, y}$; $\mathrm{O\,Q} \,.\, \mathrm{O}\,q = \frac{b'^2 (\delta^2 - a'^2)}{a'^2 p'^2 - b'^2 q'^2}$; therefore

$$\mathrm{O\,P} \,.\, \mathrm{O}\,p : \mathrm{O\,Q} \,.\, \mathrm{O}\,q :: a'^2 p'^2 - b'^2 q'^2 : a'^2 p^2 - b'^2 q^2,$$

which is a constant ratio, as was to be proved.

SECTION II.

ON THE PROPERTIES OF CONJUOATE DIAMETERS.

(95.) *A diameter being drawn through a given point* (x', y') *to find the imaginary coordinates of the extremity of the diameter conjugate to it.*

Let C P, C D be any two conjugate diameters, of which the former is drawn through the given point P (x', y'); then the latter C D will not meet the hyperbola.

If $y = \frac{y'}{x'} x \dots (1)$ be the equation to C P, then $y = \frac{b^2}{a^2} \frac{x'}{y'} x \dots (2)$ will be the equation to C D; therefore the imaginary coordinates of the point D, will be found by combining (2) with the equation

$$a^2 y^2 - b^2 x^2 = - a^2 b^2 \dots (3.)$$

Hence, substituting in (3) the value of y in (2,) and dividing the result by b^2, we have

$$\left(- \frac{b^2}{a^2} \frac{x'^2}{y'^2} + 1 \right) x^2 = a^2,$$

$$\therefore (a^2 y'^2 - b^2 x'^2) x^2 = a^4 y'^2,$$

or

$$- a^2 b^2 x^2 = a^4 y'^2,$$

$$\therefore x = \pm \frac{a}{b \sqrt{-1}} \cdot y',$$

$$\therefore y = \frac{b^2}{a^2} \frac{x'}{y'} x,$$

$$= \pm \frac{b \sqrt{-1}}{a} x'.$$

(96.) *The difference of the squares of any two semiconjugate diameters is equal to the difference of the squares of the semiaxes.*

Let C P, C D be any two semiconjugate diameters, then denoting them by a' and $b' \sqrt{-1}$ respectively,

$$a'^2 = x'^2 + y'^2,$$

$$- b'^2 = - \frac{a^2}{b^2} y'^2 - \frac{b^2}{a^2} x'^2,$$

$$\therefore a'^2 - b'^2 = x'^2 - \frac{a^2}{b^2} y'^2 + y'^2 - \frac{b^2}{a^2} x'^2,$$

$$= \frac{b^2 x'^2 - a^2 y'^2}{b^2} + \frac{a^2 y'^2 - b^2 x'^2}{a^2}$$

$$= + \frac{a^2 b^2}{b^2} - \frac{a^2 b^2}{a^2},$$

$$= a^2 - b^2.$$

(97.) *If at the extremities of any two conjugate diameters, tangents be applied so as to form a parallelogram, the area of all such parallelograms is constant.*

Let P p, D d be any two conjugate diameters, and let the tangents at P and p, D and d, be produced to meet, then it is plain, (Art. 87. *Cor.* 3) that they will form a parallelogram. Fig. 54.

From P and T let fall the perpendiculars P F, T Q on D C. Then the area of the whole parallelogram is equal to four times the area of the parallelogram P D

$$= 4 \,\mathrm{P\,C} \,.\, \mathrm{C\,D} \sin \mathrm{P\,C\,D},$$

$$= 4 \,\mathrm{C\,D} \,.\, \mathrm{P\,F} \dots (1.)$$

But $\mathrm{P\,F} = \mathrm{T\,Q} = \mathrm{C\,T} \sin \mathrm{T\,C\,Q} = \frac{a^2}{x'} \frac{m\,\mathrm{D}}{\mathrm{D\,C}}$ (95,)

$$\therefore \mathrm{P\,F} \,.\, \mathrm{C\,D} = \frac{a^2}{x^2} \,.\, m\,\mathrm{D},$$

$$= \frac{a^2}{x} \frac{b \sqrt{-1}}{a} x,$$

$$= a\, b \sqrt{-1} \dots (2;)$$

therefore by substitution in (1) the area of the whole parallelogram $= 4\, a\, b \sqrt{-1}$, and is therefore constant. The imaginary quantity involved in this expression indicates that the parallelogram does not, as in the case of the ellipse, *circumscribe the curve.*

Cor. 1. From equation (2) $\mathrm{P\,F} \,.\, \mathrm{C\,D} = a\, b \sqrt{-1}$, but $\mathrm{C\,D} = b' \sqrt{-1}$, and $\mathrm{P\,F} = \mathrm{P\,C} \sin \mathrm{P\,C\,D}$,

$$= a' \sin \gamma, \text{ if } \mathrm{P\,C\,D} = \gamma,$$

$$\therefore a\, b = a'\, b' \sin \gamma.$$

Conic Sections.

Cor. 2. Hence the value of P F may be found; for

$$PF = \frac{ab}{CD} = \frac{ab}{\sqrt{a'^2 - (a^2 - b^2)}}.$$

Cor. 3. Since $a^2 - b^2 = a'^2 - b'^2$, the conjugate diameters cannot be equal to each other in the hyperbola.

(98.) *The rectangle contained by the focal distances of any point is equal to the square of the semidiameter conjugate to that which passes through the proposed point.*

Let P be any point, C D the semidiameter conjugate to C P, join P, S and P, H; to prove that

$$SP \cdot PH = CD^2.$$

For $\quad CP^2 - CD^2 = a^2 - b^2,$

$$\therefore CD^2 = CP^2 - a^2 + b^2,$$
$$= x^2 + y^2 - a^2 + b^2,$$
$$= x^2 + (e^2 - 1)(x^2 - a^2) - a^2 + b^2,$$
$$= e^2 x^2 - e^2 a^2 + b^2,$$
$$= e^2 x^2 - a^2,$$
$$= (e x - a)(e x + a),$$
$$= \therefore SP \cdot HP.$$

(99.) *Let C P, C D be any two semiconjugate diameters, and let a tangent at P meet the axes of the hyperbola in T, t; to prove that* $PT \cdot Pt = CD^2$.

If C P, C D be assumed as the axes of coordinates, then the equations to C A, C B are respectively

$$y = a x,$$
$$y = \frac{b^2}{a^2 a} x.$$

Let $x = a'$ or C P, then y or $PT = a a'$ from (1,)

and $\quad y$ or $Pt = \frac{b'^2}{a'^2 a} \ldots (2,)$

$$\therefore PT \cdot Pt = b'^2 = CD^2.$$

SECTION III.

ON SUPPLEMENTAL CHORDS

Def. If from the vertices of any diameter two straight lines be drawn to any point in the hyperbola, they are called *supplemental chords.*

(100.) *Any two supplemental chords being drawn, and the equation to either of them being given, to find the equation to the other.*

Fig. 55. The hyperbola being referred to any two conjugate diameters, its equation will be

$$a'^2 y^2 - b'^2 x^2 = - a'^2 b'^2 \ldots (1.)$$

Through any point P (x', y') draw the diameter P p, and let P Q, p Q be any two supplemental chords, then if the equation to P Q be

$$y - y' = a (x - x') \ldots (2,)$$

it is required to find the equation to p Q.

The coordinates of P being x', y' those of p will be $-x', -y'$, therefore the equation to p Q will be of the form

$$y + y' = a' (x + x') \ldots (3)$$

in which a' is to be found.

Since the lines whose equations are (2) and (3) intersect at Q, the coordinates of that point will be identical in both; therefore considering x and y as the same in these equations, we have by multiplying them together

Hyperbola.

$$y^2 - y'^2 = a a' (x^2 - x'^2),$$
$$\therefore a a' = \frac{y^2 - y'^2}{x^2 - x'^2} \ldots (4,)$$

but because x', y are the coordinates of P, a point in the hyperbola, they will satisfy equation (1,)

$$\therefore a'^2 y'^2 - b'^2 x'^2 = - a'^2 b'^2.$$

Subtracting this from (1) we have

$$a'^2 (y^2 - y'^2) - b'^2 (x^2 - x'^2) = 0,$$
$$\therefore \frac{y^2 - y'^2}{x^2 - x'^2} = \frac{b'^2}{a'^2},$$

therefore by substitution in (4)

$$a a' = \frac{b'^2}{a'^2}, \text{ and } a' = \frac{b'^2}{a'^2 a},$$

and the equation to p Q becomes by substitution in (3)

$$y + y' = \frac{b'^2}{a'^2 a} (x + x').$$

Cor. 1. Let P p coincide with the transverse axis A a, then the equation to a Q drawn through the point a $(-a, 0)$ will be

$$y = a (x + a),$$

therefore the equation to A Q drawn through the point A $(a, 0)$ will be

$$y = \frac{b'^2}{a'^2}(x - a).$$

Cor. 2. If the hyperbola be referred to its axes, we have only to substitute a and b for a' and b' in the above equation.

(101.) *If two diameters be drawn parallel to any two supplemental chords, they will be conjugate to each other.*

The equations to any two supplemental chords being

$$y - y' = a (x - x') \ldots (1,)$$

and $\quad y + y' = \frac{b'^2}{a'^2 a} (x + x') \ldots (2,)$

let a diameter be drawn parallel to the chord whose equation is (1,) then its equation will be

$$y = a x,$$

therefore the equation to its conjugate being

$$y = \frac{b'^2}{a'^2 a} x,$$

it follows that the latter is parallel to (2,) as was to be proved.

Cor. 1. Hence may be drawn a diameter which shall be conjugate to a given diameter.

Let P p be the given diameter, and Fig. 56.

First, Let the transverse axis be given.

From a draw a R parallel to P p, and join R A; then if D d be drawn through C parallel to R A, it will be conjugate to P p.

Secondly, If the transverse axis be not given

Conic Sections. Fig. 56.

Draw any diameter whatever $R\,r$, through r draw $r\,Q$ parallel to $P\,p$, join Q, R; then if $D\,d$ be drawn through C parallel to $R\,Q$, it will be conjugate to $P\,p$.

These conclusions are evident.

Cor. 2. Hence also is derived a very simple method of applying a tangent at a given point of the hyperbola.

Let P be the given point, and

First, Let the transverse axis be given.

Draw $P\,C$ and the chord $a\,Q$ parallel to it, join Q, A; then if $P\,T$ be drawn parallel to $Q\,A$, it will touch the hyperbola at P.

Secondly, If the transverse axis be not given.

Draw any diameter $R\,C\,r$, meeting the hyperbola in R, r, join P, C, draw $r\,Q$ parallel to $P\,C$, join Q, R; then if $P\,T$ be drawn parallel to $Q\,R$, it will be a tangent at P.

(102.) *To find the angle contained by the principal supplemental chords.**

Fig. 57. Let the point $Q\ (x', y')$ be the intersection of the chords $A\,Q$, $a\,Q$, and suppose the hyperbola referred to its axes; then if the equations to $Q\,a$, $Q\,A$ be

$$y = \alpha\,(x + a),$$

$$y = \alpha'\,(x + a),$$

$$\tan A\,Q\,a \text{ will} = \frac{\alpha' - \alpha}{1 + \alpha\,\alpha'};$$

or since
$$\alpha\,\alpha' = \frac{b^2}{a^2},$$

$$\tan A\,Q\,a = \frac{\alpha' - \alpha}{1 + \frac{b^2}{a^2}} \ldots . \ (1.)$$

Now
$$\alpha' = \tan Q\,A\,X = \frac{y'}{x' - a}$$

and
$$\alpha = \tan Q\,a\,X = \frac{y'}{x' + a},$$

$$\therefore \alpha' - \alpha = y' \cdot \left(\frac{1}{x' - a} - \frac{1}{x' + a}\right),$$

$$= y' \cdot \frac{2\,a}{x'^2 - a^2}.$$

therefore by substitution in (1)

$$\tan A\,Q\,a = \frac{2\,a\,b^2}{y'\,(a^2 + b^2)};$$

and since the sign of this quantity is positive, the angle is always acute.

Cor. When $y' = 0$, $\tan A\,Q\,a = \infty$, therefore the angle is a right angle.

When $y' = \infty$, $\tan A\,Q\,a = 0$, therefore the angle is $= 0$; hence the acute angle contained by any two supplemental chords in the hyperbola may pass through all states of magnitude from 0 to $\frac{\pi}{2}$.

(103.) *To draw two conjugate diameters making a given angle.*

Fig. 58. The analytical solution of the problem is similar to that for the ellipse, except that the reducing equation will be a quadratic of the fourth degree. We shall therefore proceed to give the geometrical construction.

* The chords drawn from the vertices of the transverse axis to any point in the hyperbola, are called the *principal supplemental chords.*

Hyperbola.

Draw any diameter $R\,r$, meeting the hyperbola in R, r, and upon it describe a segment of a circle containing an angle equal to the given angle and cutting the hyperbola in Q, join $Q\,R$, $Q\,r$, and parallel to these draw the diameters $P\,p$, $D\,d$; these will be the diameters required.

For being parallel to the supplemental chords $Q\,R$, $Q\,r$, they are conjugate to each other, and the angle $P\,C\,D = R\,Q\,r$, and therefore equal the given angle.

The problem admits also, as in the ellipse, of a second solution. See Art. 64.

In the case of the ellipse, the given angle formed by two conjugate diameters must be confined within certain limits, but in the hyperbola no such restriction is necessary.

From the principles already laid down, the reader will have no difficulty in adapting the miscellaneous propositions on the ellipse, ch. iv. p. 753, to the case of the hyperbola.

CHAPTER IV.

ON THE ASYMPTOTES OF THE HYPERBOLA.

It was shown in Art. 85, *Cor.* 2, that certain diameters of the hyperbola meet the curve only at an infinite distance, and are for that reason termed *Asymptotes.* Since the asymptotes, therefore, pass through the centre, and are inclined to the transverse axis at an angle whose $\tan = \pm\frac{b}{a}$, their equation will be

$$y = \pm\frac{b}{a}\,x.$$

(104.) *Let it now be required to find the position of the asymptotes when the hyperbola is referred to any two conjugate diameters.*

For this purpose it is only necessary to find the intersection of any diameter

$$y = \alpha\,x \ldots . \ (1.)$$

with the hyperbola

$$a'^2 y^2 - b'^2 x^2 = -a'^2 b'^2 \ldots . \ (2.)$$

Eliminating y between (1) and (2)

$$(a'^2 \alpha^2 - b'^2)\,x^2 = a'^2 b'^2,$$

$$\therefore x = \pm\frac{a'\,b'}{\sqrt{b'^2 - a'^2 \alpha^2}},$$

Fig. 59.

and
$$\therefore y = \pm\frac{a'\,b'\,\alpha}{\sqrt{b'^2 - a'^2 \alpha^2}}.$$

Now so long as $b'^2 > a'^2 \alpha^2$, or $\alpha < \pm\frac{b'}{a'}$, the diameter meets the curve; but when $\alpha = \pm\frac{b'}{a'}$, the diameter becomes an asymptote.

Hence, if through P the line $E\,e$ be drawn equal and parallel to $D\,d$, and $C\,E$, $C\,e$ be joined; the lines $C\,E\,X'$, $C\,e\,Y'$ will be asymptotes.

The equation to $C\,X'$ is $y = \frac{b'}{a'}\,x$,

and that to $C\,Y'$ is $y = \frac{-b}{a'}\,x$.

Conic Sections.

Cor. 1. Since E e touches the hyperbola at P, it follows that the part of the tangent intercepted by the asymptotes is bisected at the point.

Cor. 2. If P N, P n be drawn parallel to C X′, C Y′, then since e P equal P E, e N will equal N C, and C n = n E.

(105.) The equation to the asymptotes may be deduced from that to the curve; for we have

$$y = \pm \frac{b'}{a'} \sqrt{x^2 - a'^2},$$

in which y is the ordinate to the hyperbola, and x the corresponding abscissa. Now in tracing the figure of the hyperbola from it sequation, it was shown that for each value of x, however great, there are two equal values of y with contrary signs. If x therefore be assumed infinitely great, the ordinate to the curve ought to coincide with that to the asymptote.

Hence in the above equation, expanding the value of y, we have

$$y = \pm \frac{b'}{a'} x \left(1 - \frac{a'^2}{x^2}\right)^{\frac{1}{2}},$$

$$= \pm \frac{b'}{a'} x \left(1 - \tfrac{1}{2} \frac{a'^2}{x^2} - \text{\&c.}\right),$$

$$= \pm \frac{b'}{a'} x \mp \frac{a' b'}{2} \cdot \frac{1}{x} \dots .$$

Let $x = \infty$, therefore all the terms containing x in the denominator vanish, and we have

$$y = \pm \frac{b'}{a'} x,$$

which is the equation required,

(106.) *The asymptote may be considered as a tangent to the hyperbola at a point infinitely distant.*

For the equation to a tangent at any point (x', y') is

$$a'^2 y y' - b'^2 x x' = - a'^2 b'^2,$$

or $$y = \frac{b'^2}{a'^2} \frac{x'}{y'} x - \frac{b'^2}{y'} \dots . (1.)$$

Now $$y' = \pm \frac{b'}{a'} \sqrt{x'^2 - a'^2}.$$

Suppose x' to be infinitely great, then a'^2 vanishes when compared with x'^2,

$$\therefore y' = \pm \frac{b'}{a'} x',$$

therefore by substitution in (1) the equation to the tangent, when the point (x', y') is infinitely distant, becomes

$$y = \pm \frac{b'}{a'} x \mp \frac{a' b'}{x'};$$

or since $\frac{a' b'}{x'} = 0$,

$$y = \pm \frac{b'}{a'} x,$$

which is the equation to the asymptotes: whence the truth of the proposition.

Hyperbola.

(107.) *If any chord of the hyperbola be produced to meet the asymptotes, the parts of it intercepted between the curve and the asymptotes will be equal.*

Let the chord Q q be produced to meet the asymptotes in R, r, to prove that Fig. 59.

$$\text{Q R} = q\, r.$$

Bisect Q q by the diameter C X, and draw C D conjugate to it; then the equation to the hyperbola being

$$y = \pm \frac{b'}{a'} \sqrt{x^2 - a'^2} \dots . (1.)$$

that to the asymptotes will be

$$y = \pm \frac{b'}{a'} x \dots . (2.)$$

Now to the same abscissa C M, we have

$$\text{M Q} = \text{M}\, q$$

from the first of these equations, and

$$\text{M R} = \text{M}\, r$$

from the second;
therefore by subtraction

$$\text{R Q} = r\, q,$$

as was to be proved.

Cor. Hence $$\text{P R} . \text{P}\, r = (\text{M R} - \text{M P})(\text{M R} + \text{M P}) = \text{M R}^2 - \text{M P}^2,$$

but $$\text{M R}^2 = \frac{b'^2}{a'^2} x^2,$$

and $$\text{M P}^2 = \frac{b'^2}{a'^2} (x^2 - a'^2),$$

$$\therefore \text{M R}^2 - \text{M P}^2 = \frac{b'^2}{a'^2} \{ x^2 - x^2 + a'^2 \},$$

$$= b'^2 = \text{C D}^2,$$

$$\therefore \text{P R} . \text{P}\, r = \text{C D}^2.$$

(108.) *To find the equation to the hyperbola when referred to its asymptotes.*

Let P be any point whatever in the hyperbola, join C P, and draw the conjugate diameter D d; through P draw R P r equal and parallel to D d, and join C R, C r, which produce indefinitely towards Y and X, then C Y, C X are asymptotes. Assuming these as the axes of coordinates, it is required to find the equation to the hyperbola. Fig. 59.

From P draw P M, P m parallel to C Y, C X, respectively, and let C M = x, M P = y, angle R C P = θ.

Then $$\text{C}\, r = 2\, \text{C M} = 2\, x,$$

and $$\text{C R} = 2\, \text{C}\, m = 2\, y,$$

$$\therefore \text{C}r . \text{C R} = 4\, x y \dots . (1;)$$

but C r . C R sin 2θ = twice the triangle R C r = twice the parallelogram D P

$$= 2\, a' b' \sin \gamma = \therefore 2\, a\, b,$$

$$\therefore \text{C R} . \text{C}\, r = \frac{2\, a\, b}{\sin 2\theta};$$

hence $$x y = \frac{2\, a\, b}{4 \sin 2\theta} = \frac{a\, b}{4 \sin\theta \cos\theta} \dots . (1.)$$

Now $\tan \theta = \frac{b}{a}$,

$$\therefore \frac{b^2}{a^2} = \tan^2\theta = \frac{\sin^2\theta}{\cos^2\theta} = \frac{\sin^2\theta}{1-\sin^2\theta}$$

$$\therefore b^2 - b^2\sin^2\theta = a^2\sin^2\theta$$

$$\therefore \sin\theta = \frac{a}{\sqrt{a^2+b^2}}.$$

Similarly $\cos\theta = \dfrac{a}{\sqrt{a^2+b^2}}$,

therefore substituting in (1)

$$x\,y = \frac{a\,b}{4\,a\,b}\,(a^2+b^2),$$

$$= \therefore \frac{a^2+b^2}{4},$$

which is the equation required.

(109.) *Having given the equation to the hyperbola in terms of its axes, to find the equation when it is referred to the asymptotes.*

Fig. 60. Let C X′ be the transverse axis, C X, C Y the asymptotes, P any point in the hyperbola; let fall the perpendicular P on C X′, and draw P N parallel to C Y.

Let C M $= x$, M P $= y$; C N $= x'$, N P $= y'$, and X C Y $= 2\,\theta$, it is required from the equation between x and y,

$$a^2y^2 - b^2x^2 = -a^2b^2 \ldots. (1,$$

o deduce that between x' and y'.

From N and P, draw N m and P n parallel to P M, and A M, respectively.

Then $y = $ P M $=$ N m $-$ N n,
$=$ N C sin N C m $-$ N P sin N P n,
$=$ (N C $-$ N P) sin N C m,
$= (x' - y')\sin\theta$.

In like manner,

$$x = (x' + y')\sin\theta,$$

$$\therefore a^2y^2 = (x'-y')^2a^2\sin^2\theta = (x'-y')^2\,\frac{a^2b^2}{a^2+b^2},$$

$$b^2x^2 = (x'+y')^2b^2\sin^2\theta = (x'+y')^2\,\frac{a^2b^2}{a^2+b^2},$$

$$\therefore a^2y^2 - b^2x^2 = \frac{a^2b^2}{a^2+b^2}\left\{(x'-y')^2 - (x'+y')^2\right\}$$

$$= -\frac{a^2b^2}{a^2+b^2}\,4\,x'y';$$

but $a^2y^2 - b^2x^2 = -a^2b^2$,

$$\therefore 4\,x'y' = a^2 + b^2,$$

or $x'y' = \dfrac{a^2+b^2}{4}$,

which is the equation required.

(108.) *The asymptotes being assumed as axes, to find the equation to the tangent at a given point* (x', y').

Any other point (x'', y'') being taken in the hyperbola near the first, the equation to a line drawn through (x', y') and (x'', y'') is

$$y - y' = \frac{y''-y'}{x''-x'}\,(x - x'), \ldots (1;)$$

but because these points are in the hyperbola, we have

$$x'y' = m^2,$$

and $x''y'' = m^2$,

$$\therefore x''y'' - x'y' = 0,$$

$$\therefore y'' = \frac{x'y'}{x''},$$

$$\therefore y'' - y' = \frac{x'y'}{x''} - y',$$

$$= -\frac{y'}{x''}\,(x'' - x'),$$

$$\therefore \frac{y''-y'}{x''-x'} = -\frac{y'}{x''},$$

therefore by substitution in (1) the equation to the secant becomes

$$y - y' = -\frac{y'}{x''}\,(x - x').$$

Let the point (x'', y'') be now supposed to coincide with (x', y'), then $x'' = x'$, $y'' = y'$, and the secant becomes a tangent, therefore the equation to the tangent is

$$y - y' = -\frac{y'}{x'}\,(x - x').$$

Cor. Multiplying each side by x',

$$y\,x' - x'y' = -x\,y' + y'x'$$

$$\therefore y\,x' + x\,y' = 2\,x'y',$$

$$= 2\,m^2.$$

(111.) *To find the intersection of the tangent with the asymptotes.*

The equation to the tangent being

$$y\,x' + x\,y' = 2\,m^2.$$

Let the tangent cut the axis of x, as at T,

then $y = 0$ and $x = \dfrac{2\,m^2}{y'}$;

and when it cuts the axis of y, as at t,

then $x = 0$ and $y = \dfrac{2\,m^2}{x'}$.

Cor. Hence C T . C $t = \dfrac{4\,m^4}{x'y'} = \dfrac{4\,m^4}{m^2} = 4\,m^2$;

$\therefore \frac{1}{2}$ C T . C t $\times$ sin T C $t = 2\,m^2$ sin T C t,

that is, area of the triangle T C t $= 2\,m^2$ sin T C t. In other words, if the tangent at any point be produced to meet the asymptotes, the area of the triangle so cut off will be constant.

(112.) *Having given one point in the hyperbola, and the position of the asymptotes, to find the direction and magnitude of the transverse and conjugate diameters.*

Let C be the centre, C X′, C Y′ the asymptotes, and P the given point in the hyperbola. Fig. 60.

1. To find the direction of the axes.

Bisect the angle X′ C Y′ by the line C X, and the angle X′ C y', the supplement of the former, by the line C Y; then C X, C Y will evidently be the direction of the transverse and conjugate axes, respectively.

2. To find their magnitude.

If the coordinates of the given point P be (x', y') we

have
$$x' y' = \frac{a^2 + b^2}{y},$$
$$\therefore a^2 + b^2 = 4 x' y' \ldots\ldots (1,)$$
but
$$\pm \frac{b}{a} = \tan \tfrac{1}{2} X' C Y' = \tan \theta,$$
$$\therefore \frac{b^2}{a^2} = \frac{\sin^2 \theta}{\cos^2 \theta}$$
or $b^2 \cos^2 \theta = a^2 \sin^2 \theta$, therefore substituting for b^2 its value in (1)
$$(4 x' y' - a^2) \cos^2 \theta = a^2 \sin^2 \theta;$$
$$\therefore 4 x' y' \cos^2 \theta = a^2 (\sin^2 \theta + \cos^2 \theta),$$
$$= a^2,$$
$$\therefore a = \pm 2 \cos \theta \sqrt{x' y'},$$
and
$$b = \pm 2 \sin \theta \sqrt{x' y'},$$
therefore their magnitude is found.

ON THE SECTIONS OF THE CONE.

Conic Sections. Fig. 61. Cone.

Def. Let C be a fixt point above the plane of a given circle B E D, and B C Z an indefinite straight lino which always passes through C, whilst its extremity A moves nver the circumference B E D; then B C Z will describe by its revolutino a snlid figure called a *cone.*

The point C is called the *vertex*, the circle B E D the *base*, and the lino C O, which joins the vertex with the centre of the base, the *axis* of the cone.

The cone is denominated a *right*, or an *oblique* cone, according as the axis is at right angles or inclined to the plane of the base.

The surface of a cone is composed of twn similar portions, one above, and the other below the vertex; each of these portions is called a *sheet.**

It is evident from the manner in which a cone is generated, that every section made by a plane parallel to the base is a circle.

(113.) *To find the nature of the curve which results from the intersection of a right cone by a plane.*

Fig. 62. Let A P p be the curve formed by the intersectino of a right cone by a plane; through the axis C O draw a plane B C D perpendicular to the given plane, then their intersectinn will be the straight line A a. In A a take any point M, through which draw a plane parallel to the base, then its intersectinns with the cone and the given plane will be, respectively, the circle N P Q and the straight line M P, which being perpendicular to A a and N Q, will be a commoo ordinate to both curves.

Assume A a as the axis of x, and A Y, at right angles to A a, as the axis of y, and let A M $= x$, M P $= y$; also take A C $= \delta$, angle B C D $= \alpha$, and angle C A $a = \theta$.

Then $$\frac{A a}{A C} = \frac{\sin A C a}{\sin A a C} = \frac{\sin \alpha}{\sin (\alpha + \theta)},$$

$$\therefore A a = \frac{\sin \alpha}{\sin (\alpha + \theta)} \delta,$$

$$\therefore M a = A a - x = \frac{\delta \sin \alpha}{\sin (\alpha + \theta)} - x \ldots . \text{ (1.)}$$

Now, by the property of the circle,

$$M P^2 \text{ or } y^2 = N M . M Q;$$

but $$N M = M A \frac{\sin N A M}{\sin A N M} = x \frac{\sin C A a}{\cos N C M}$$

$$= \therefore \frac{x \sin \theta}{\cos \frac{\alpha}{2}},$$

and $$M Q = M a \frac{\sin A a C}{\sin M Q a} = M a \frac{\sin (\alpha + \theta)}{\sin N Q a}$$

$$= \therefore \left\{ \frac{\delta \sin \alpha}{\sin (\alpha + \theta)} - x \right\} \frac{\sin (\alpha + \theta)}{\cos \frac{1}{2} \alpha};$$

therefore by substitution

$$y^2 = \frac{x \sin \theta}{\cos \frac{1}{2} \alpha} . \frac{\sin (\alpha + \theta)}{\cos \frac{1}{2} \alpha} \left\{ \frac{\delta \sin \alpha}{\sin (\alpha + \theta)} - x \right\},$$

$$= \frac{\sin \theta}{\cos^2 \frac{1}{2} \alpha} \left\{ \delta \sin \alpha . x - \sin (\alpha + \theta) x^2 \right\},$$

which is the eqnation required.

1. Let the plane be parallel to C D, then $\alpha + \theta = \pi$, therefore sin $(\alpha + \theta) = \sin \pi = 0$; also $\sin \theta = \sin (\pi - \alpha) = \sin \alpha$, therefore the equation becomes

$$y^2 = \frac{\delta \sin^2 \alpha}{\cos^2 \frac{1}{2} \alpha} . x = \frac{4 \delta \sin^2 \frac{1}{2} \alpha \cos^2 \frac{1}{2} \alpha}{\cos^2 \frac{1}{2} \alpha} x = 4 \delta \sin^2 \frac{\alpha}{2} x,$$

which is the equation to a parabola whose *latus rectum* $= 4 \delta \sin^2 \frac{\alpha}{2}$.

If the plane pass through the vertex of the cone, then $\delta = 0$, and the equation to the section becomes $y^2 = 0$; which is the eqoation to the line C D.

2. Let the plane meet C B and C D, then $\alpha + \theta < \pi$, and therefore sin $(\alpha + \theta)$ is positive; therefore the equation to the section is

$$y^2 = \frac{\sin \theta}{\cos^2 \frac{1}{2} \alpha} \left\{ \delta \sin \alpha . x - \sin (\alpha + \theta) x^2 \right\},$$

which is the equation to the *ellipse.*

Comparing this with the equatioo

$$y^2 = \frac{b^2}{a^2} \left\{ 2 a x - x^2 \right\}, \text{ or with}$$

$$y^2 = \frac{2 b^2}{a} x - \frac{b^2}{a^2} x^2,$$

we have $$\frac{2 b^2}{a} \text{ or } \textit{latus rectum} = \frac{\delta \sin \alpha \sin \theta}{\cos^2 \frac{1}{2} \alpha}$$

$$= 2 \delta \tan \frac{\alpha}{2} \sin \theta$$

Let $$a = \frac{\delta \sin \alpha}{2 \sin (\alpha + \theta)},$$

$$\therefore b^2 = \frac{a}{2} . 2 \delta \tan \frac{\alpha}{2} \sin \theta,$$

$$= \frac{\delta \sin \alpha}{2 \sin (\alpha + \theta)} . \delta \tan \frac{\alpha}{2} \sin \theta,$$

$$= \delta^2 \sin^2 \frac{\alpha}{2} . \frac{\sin \theta}{\sin (\alpha + \theta)}.$$

$$\therefore b = \delta \sin \frac{\alpha}{2} \sqrt{\frac{\sin \theta}{\sin (\alpha + \theta)}}.$$

If the plane pass through the vertex, then $\delta = 0$, and the equation becomes

$$y^2 = - \frac{\sin (\alpha + \theta) \sin \theta}{\cos^2 \frac{1}{2} \alpha} x,$$

* *Sheet* is to a *surface*, what *branch* is to a *curve.*

which is the equation to the point C, since the equation can be satisfied only by $x = 0$, $y = 0$.

3. Let the plane meet both sheets of the surface.

Then $a + \theta > \pi$, and because $\sin(a + \theta)$ is negative, therefore the equation to the section is

$$y^2 = \frac{\sin\theta}{\cos^2\frac{1}{2}a}\{\delta\sin a \, . \, x + \sin(a+\theta)\,x^2\},$$

which is the equation to the hyperbola.

The *latus rectum* and axes of the hyperbola may be determined in the same manner as in the ellipse.

If the plane pass through the vertex, then $\delta = 0$, and the equation becomes

$$y^2 = \frac{\sin\theta}{\cos^2\frac{1}{2}a}\sin(a+\theta)\,x^2,$$

$$\therefore\ y = \pm\frac{\sqrt{\sin\theta\sin(a+\theta)}}{\cos\frac{1}{2}a}\,x,$$

which are the equations to C B, C D; hence the section becomes in this case the two generating lines of the cone.

It appears, therefore, that if a right cone be cut by a plane, the section will be

1. A *parabola*, when the plane is parallel to the generating line.
2. An *ellipse*, when the plane meets only one sheet of the cone.
3. An *hyperbola*, when the plane meets both sheets of the cone.

(114.) *To find the nature of the curve which results from the intersection of an oblique cone by a plane.*

Let A P p be the curve formed by the section of an oblique cone by a plane.

Fig. 63. The construction is the same as before, excepting that the line M P is no longer perpendicular both to A a and N Q, but only to the latter; we shall assume therefore, as oblique axes, A a and A Y parallel to M P.

Hence, as before, $\mathrm{A}a = \dfrac{\delta\sin a}{\sin(a+\theta)}$.

$$\mathrm{M}a = \frac{\delta\sin a}{\sin(a+\theta)} - x \ldots\ (1;)$$

also $y^2 = \mathrm{N\,M}\,.\,\mathrm{M\,Q}$;

but $\mathrm{N\,M} = x\dfrac{\sin\theta}{\sin\mathrm{B}}$,

and $\mathrm{M\,Q} = \mathrm{M}a\dfrac{\sin(x+\theta)}{\sin(a+\mathrm{B})}$,

$$= \frac{\sin(a+\theta)}{\sin(a+\mathrm{B})}\left\{\frac{\delta\sin a}{\sin(a+\theta)} - x\right\},$$

$$\therefore\ y^2 = \frac{\sin\theta}{\sin\mathrm{B}\sin(a+\mathrm{B})}\{\delta\sin a\,.\,x - \sin(a+\theta)x^2\},$$

which, according as the given plane is parallel to C D, or meets one or both sheets of the cone, is the equation to a parabola, ellipse, or hyperbola, referred to oblique axes.

Cor. To find in what cases the section is a circle.

Having put the equation under the form

$$y^2 = \frac{\sin\theta\sin(a+\theta)}{\sin\mathrm{B}\sin(a+\mathrm{B})}\left\{\frac{\delta\sin a}{\sin(a+\theta)}x - x^2\right\},$$

it is evident that the section will be a circle when the coefficient

$$\frac{\sin\theta\sin(a+\theta)}{\sin\mathrm{B}\sin(a+\mathrm{B})} = 1,$$

or $\quad \sin\theta\sin(a+\theta) = \sin\mathrm{B}\sin(a+\mathrm{B})$,

or $\quad \cos a - \cos(a+2\theta) = \cos a - \cos(a+2\mathrm{B})$,

$$\therefore\ \cos(a+2\theta)\ \text{must} = \cos(a+2\mathrm{B}),$$

$$\therefore\ a + 2\theta = a + 2\mathrm{B}\ \ldots\ldots\ (1),$$

or $\quad = 2\pi - (a+2\mathrm{B}) \ldots\ldots\ (2.)$

First, if $\quad a + 2\theta = a + 2\mathrm{B}$,

$$\theta = \mathrm{B},$$

that is, when the plane is parallel to the base the section is a circle.

Secondly, if $\quad a + 2\theta = 2\pi - (a+2\mathrm{B})$,

$$2a + 2\theta + 2\mathrm{B} = 2\pi,$$

or $\quad a + \theta + \mathrm{B} = \pi = \therefore\ a + \mathrm{D} + \mathrm{B}$,

$$\therefore\ \theta = \mathrm{D};$$

hence, when the angle C A X is = C D B, the section is also a circle. This is called the *subcontrary section of the cone.*

DIFFERENTIAL AND INTEGRAL CALCULUS.

PART I.

DIFFERENTIAL CALCULUS.

Differential Calculus. (1.) In the investigations of the relations which exist between several quantities, those which are supposed to retain the same value are said to be *constant*, and those to which several values may be assigned are said to be *variable*. The first are usually represented by the first letters of the alphabet, and the others by the last. The words *constant* and *variable* are also frequently used substantively, to express constant and variable quantities. Part I.

(2.) When variable quantities are so connected that the value of one of them is determined by the values ascribed to the others, that variable quantity is said to be a *function* of the others. Thus, for instance, the sum of the terms of a geometrical progression is a function of the first term, of the ratio, and of the number of the terms. In a like manner, when an equation subsists between several variable quantities, any one of them is a function of all the others.

To express in a general manner a function of one or more variables, one of the letters F, f, ϕ, ψ, &c. is usually prefixed to the letters by which the variables are represented, enclosing them at the same time between parentheses, and separating them, when there are several, by commas. Thus, $F(x)$, $\phi(x, y, z)$, signify, the first a function of the variable x, and the second a function of the three variables x, y, z. Another notation, also employed, consists in placing simply the variable on the right side of, and a little below, the letter U, or any other. Thus, U_x, $Z_{x,y}$ denote, the first a function of x, and the other a function of x and y.

(3.) A function of one or more variables is said to be *explicit*, when the operations, to be performed on the variables, to obtain the value of the function, are immediately expressed by means of algebraical signs, or by means of notations previously defined. But when the relation between a function and the variables is only expressed by means of an equation, it is said to be an *implicit* function, as long as the equation is not resolved.

(4.) Functions receive different denominations, according to the nature of the operations which produce them. Those which are formed by means of the operations of Algebra, *viz.* addition, subtraction, multiplication, division, involution, and evolution, are called *algebraical functions:* those which contain variable exponents are called *exponential functions;* they receive the name of *logarithmic functions*, when they contain variable logarithms; and they are designated by the name of *circular* or *trigonometrical functions*, when some of the operations of Trigonometry are required to form them. All those which cannot be reduced to some of the preceding are called *transcendental functions.*

(5.) Algebraical functions are again divided into *rational* and *irrational functions;* the first being those which contain only integral powers of the variables, and the last containing fractional powers of the variables, or radical quantities, under which the variables enter. An *integral function* is a polynom which contains only integral powers of the variable; and the quotient of two such functions is a *fractional function.*

(6.) Different values of a function often correspond to a set of values of the variables. In the function $A(x-a)^{\frac{1}{2}} + B$, for instance, two values correspond to every value of x, except to the value $x = a$; in the function log. x, an infinite number of values correspond to every value of x, one of which is real, and all the others imaginary, when x is a positive quantity; and all of which are imaginary, when x is negative. The arc being considered as a function of its sine, is another instance in which, to every value of the variable, corresponds an infinite number of values of the function, but in this case all the values are real.

In establishing any conclusion with respect to any particular function, it is always necessary to examine whether it is true for every value of the function, and if not to state for which of the values it obtains.

(7.) When a function of one variable $f(x)$ takes a single and finite value for every value of x equal to or greater than a, but less than, or, at most, equal to b; and that, at the same time, for every value of x between these limits, the difference $f(x+h) - f(x)$ may be made less than any assignable quantity, by taking h sufficiently small, $f(x)$ is said to be a *continuous function*, between the limits a and b.

A function is also said to be continuous for values differing but little from a particular value a, when it is continuous between two limits, nearly equal, the one greater and the other less than a.

(8.) A quantity A is said to be *the limit* of a function of one variable x, when the values of that function corresponding to a series of increasing or decreasing values of the variable, continually approach to A, and that a value of x may always be assigned such as to make the difference between the limit and the function less than any given quantity.

The function $A + Bx$, for instance, has obviously for its limit A, for decreasing values of x; and $A + \frac{B}{x}$ has the same limit, for increasing values of the variable.

It is not always easy to find the limits of a given function of x, but various simple remarks frequently facilitate their determination. If for every value assigned to x, for instance, the value of $f(x)$ is always included between the corresponding values of two other functions of the same variable, which have for their common limit A, it is evident that A will equally be the limit of $f(x)$.

It follows also from the above definition, that if A and B represent the limits of two functions of x, then $A + B$, $A - B$, $A B$, $\frac{A}{B}$, will respectively be the limits of the sum, the difference, the product, or the quotient of the two functions.

(9.) All functions can undergo, without changing their values, an infinite number of transformations; from the comparison of some of which their properties arise. When they are transformed in a finite or an infinite series of terms connected together by a certain law, they are said to be developed, and the series is called the developement of the function. Among the various developements of a function, that which proceeds according to the powers of the variable has been most considered, and appears to be of a greater importance than any other.

The binomial theorem, demonstrated in Algebra, furnishes examples of finite and infinite developements of functions according to the powers of a variable.

In order to render the nature and object of the Differential and Integral Calculus better understood, we shall begin by demonstrating the following theorem, relative to the transformation or developement of a function of the sum of two variables into a series of terms containing the successive powers of one of them.

(10.) *Let* u *represent any function of* x, *and* u' *what that function becomes when in it* x *is changed into* x + h. *Then, provided* x *remains an indeterminate quantity,* u' *may always be developed in a series of the following forms:*

$$u + P h + Q h^2 + R h^3 + S h^4 + \&c.$$

where P, Q, R, S, &c. *do not contain* h.

Let us first suppose $\quad u' = N + P h^{\alpha} + Q h^{\beta} + R h^{\gamma} + S h^{\delta} + \&c. \ldots\ldots (a,)$

N, P, Q, R, S, &c. being unknown functions of x, and $\alpha, \beta, \gamma, \delta$, &c. indeterminate exponents, arranged in ascending order.

It is first obvious that all these exponents must be positive; for if any of them were negative, the supposition $h = 0$ would render u infinite, while by that hypothesis it becomes equal to u. The supposition of $h = 0$, in both sides of equation $(a,)$ proves now that $N = u$, since it makes the left side equal to u, and the other equal to N. The equation (1) will therefore have the form

$$u' = u + P h^{\alpha} + Q h^{\beta} + R h^{\gamma} + S h^{\delta} + \&c. \ldots (b.)$$

Let us now change h into $h + k$, and let u'' represent what u' becomes by that substitution, we shall have

$$u'' = u + P (h + k)^{\alpha} + Q (h + k)^{\beta} + R (h + k)^{\gamma} + S (h + k)^{\delta} + \&c. \ldots (c.)$$

But if in equation (2) we change x into $x + k$, u' will also become equal to u''; for the result of this substitution will be again the same function of $x + h + k$, as u is of x. The quantities u, P, Q, &c. which are functions of x, will become functions of $x + k$; and we may represent their developements according to the powers of this last quantity respectively by

$$\begin{array}{l} u + P k^{\alpha} + \&c. \\ P + P' k^{\alpha'} + \&c. \\ Q + Q' k^{\alpha''} + \&c. \\ R + R' k^{\alpha'''} + \&c. \\ \quad \&c. \end{array}$$

The first differing only in the developement of u' in equation (b) by the change of k for h. Thus by the substitution of $x + k$ for h in equation (b) we shall have

$$\left.\begin{array}{l} u'' = u + P h^{\alpha} + Q h^{\beta} + R h^{\gamma} + S h^{\delta} + \&c. \\ \qquad + P k^{\alpha} + P' k^{\alpha'} h^{\alpha} + Q' k^{\alpha''} h^{\beta} + R' k^{\alpha'''} h^{\gamma} + S' k^{\alpha''''} h^{\delta} + \&c. \\ \qquad + \&c. + \&c. \quad + \&c. \quad + \&c. \qquad\qquad + \&c. \end{array}\right\} \ldots (d.)$$

The two values we have obtained for u'', must be equal; let us first compare them in the supposition of $k = h$, where they become respectively

$$\begin{array}{l} u'' = u + P\, 2^{\alpha} h^{\alpha} + Q\, 2^{\beta} h^{\beta} + \&c. \\ u'' = u + P h^{\alpha} + Q h^{\beta} + R h^{\gamma} + S h^{\delta} + \&c. \\ \qquad + P h^{\alpha} + P' h^{\alpha + \alpha'} + Q' h^{\beta + \alpha''} + \&c. \\ \qquad + \&c. \ + \&c. \quad + \&c. \end{array}$$

And these cannot be equal unless the terms which multiply the same power of h be separately equal to each other, since the equality must subsist, h remaining an indeterminate quantity. We shall have, consequently,

$$P . 2^{\alpha} = 2 P, \text{ hence } 2^{\alpha} = 2, \; 2^{\alpha - 1} = 1, \text{ and } \alpha = 1.$$

The equation (b) will thus become,

$$u' = u + P h + Q h^{\beta} + R h^{\gamma} + S h^{\delta} + \&c$$

and therefore *the second term of the developement of any function u′ of the sum of two variables, according to the powers of one of them, contains the first power of that variable.*

(11.) The immediate consequence of this proposition is, thet the exponents α', α'', α''', &c. are each equal to nnity.

This understood, the equations (c) and (d) will become respectively,

$$\begin{aligned} u'' = u &+ \mathrm{P}\,h + \mathrm{Q}\,h^{\beta} + \mathrm{R}\,h^{\gamma} + \mathrm{S}\,h^{\delta} + \&c. \\ &+ \mathrm{P}\,k + \beta\,\mathrm{Q}\,h^{\beta-1}k + \gamma\,\mathrm{R}\,h^{\gamma-1}k + \delta\,\mathrm{S}\,h^{\delta-1}k + \&c. \\ &+ \&c. \\ u'' = u &+ \mathrm{P}\,h + \mathrm{Q}\,h^{\beta} + \mathrm{R}\,h^{\gamma} + \mathrm{S}\,h^{\delta} + \&c. \\ &+ \mathrm{P}\,k + \mathrm{P}'h\,k + \mathrm{Q}'\,h^{\beta}k + \mathrm{R}'\,h^{\gamma}k + \mathrm{S}'h^{\delta}\,k + \&c. \\ &+ \&c. \end{aligned}$$

The first lines of these two values of u'' are the eame; the second lines are composed of all the terms which contaio the first power of k, they must eonsequently be equal. Dividing each of these lioes by k, and suppressing P, which le common to both, we shall have the following eqnation,

$$\beta\,\mathrm{Q}\,h^{\beta-1} + \gamma\,\mathrm{R}\,h^{\gamma-1} + \delta\,\mathrm{S}\,h^{\delta-1} + \&c. = \mathrm{P}'\,h + \mathrm{Q}'\,h^{\beta} + \mathrm{R}'\,h^{\gamma} + \mathrm{S}'\,h^{\delta} + \&c.$$

In both eides of which the exponents of h being in ascendiog order, the terms of the same rank must be equel to one another, and therefore we ehall have

$$\beta\,\mathrm{Q}\,h^{\beta-1} = \mathrm{P}'\,h, \quad \gamma\,\mathrm{R}\,h^{\gamma-1} = \mathrm{Q}'\,h^{\beta}, \quad \delta\,\mathrm{S}\,h^{\delta-1} = \mathrm{S}'\,h^{\delta}, \ \&c.$$

From which wa get $\quad \beta = 2, \ \gamma = 3, \ \delta = 4, \ \&c. \quad \mathrm{Q} = \dfrac{\mathrm{P}'}{2}, \ \mathrm{R} = \dfrac{\mathrm{Q}'}{3}, \ \mathrm{S} = \dfrac{\mathrm{S}'}{4}, \ \&c.$

Substituting the values of the exponeots, the equation (2) becomes

$$u' = u + \mathrm{P}\,h + \mathrm{Q}\,h^2 + \mathrm{R}\,h^3 + \mathrm{S}\,h^4 + \&c. \ldots\ldots (7.)$$

which proves the theorem stated (10,) aod shows, moreover, from the above values of Q, R, S, that Q ls equal to half the coefficient of the second term of the developement aecordiog to the powers of h, of what the fuoction represented by P becomes when x is changed into $x + h$; that R is equal to the third of the coefficient of the eecond term of the developement according to the power of h, of what the fuoctioo represeoted by Q becomes when x is changed into $x + h$, &c.

We are indebted for this very important theorem to Dr. Brook Taylor. We shall soon see with what eleganee it may be analytically expressed by means of somn notations we shull now proceed to explaio.

(12.) *The difference* u′ − n *between any function of one variable* x *represented by* u, *and the value* o′ *assumed by that function when in it* x *is changed into* x + h, *is called the* DIFFERENCE *of the function* u, *and is represented by* Δ u. So that, according to what precedes,

$$\Delta\,u = \mathrm{P}\,h + \mathrm{Q}\,h^2 + \mathrm{R}\,h^3 + \mathrm{S}\,h^4 + \&c.$$

The first term P h *of this difference is the* DIFFERENTIAL *of the function* u, *and is designated by* d u. Thue

$$d\,u = \mathrm{P}\,h.$$

According to these notations we shall have $\Delta\,x = h$ and $d\,x = h$, since h is at the same time the whole differenco betweeo the functioo x and $x + h$, and the first term of that difference.

The coefficient of h *in the differential of a function, or the coefficient of* h *in the first term of the developement of the difference, is called the differential coefficient of that function.* It is therefore equal to $\dfrac{d\,u}{h}$ or $\dfrac{d\,u}{d\,x}$, since h and $d\,x$ represent the eame quantity. This understood, equation (7) may already be writteo in the following manner,

$$u' = u + \frac{d\,u}{d\,x}\,h + \frac{d\,\mathrm{P}}{d\,x}\,\frac{h^2}{1.2} + \frac{d\,\mathrm{Q}}{d\,x}\,\frac{h^3}{1.2.3} + \frac{d\,\mathrm{R}}{d\,x}\,\frac{h^4}{1.2.3.4} + \&c. \ldots\ldots (8.)$$

(13.) The differential eoefficient $\dfrac{d\,u}{d\,x}$ of a function of x is geuerally another funetion of x, which has also a differential and a differential coefficient. By means of the agreed notations they will respectively be represented by $d\,.\,\dfrac{d\,u}{d\,x}$ and $\dfrac{d\,.\,\dfrac{d\,u}{d\,x}}{d\,x}$. It has been agreed upon to write the first $\dfrac{d^2\,u}{d\,x}$, and coosequeotly the second $\dfrac{d^2\,u}{d\,x^2}$. It should be remembered, that in these expressions the figure 2 placed a little above d is not an exponent, but only indicates the differential of a differeotial; and that $d\,x^2$ does oot eignify the differeotial of x^2, but the square of $d\,x$. The function $\dfrac{d^2\,u}{d\,x^2}$, or the differential coefficient of the differential coefficient of the functioo u, is called the second differeotial coefficient of that function. It has also a differeotial, and differential coefficient,

which will be expressed by $d \,.\, \frac{d^2 u}{d x^2}$ and $\frac{d \,.\, \frac{d^2 u}{d x^2}}{d x}$, or in using the preceding notation by $\frac{d^3 u}{d x^3}$, and $\frac{d^3 u}{d x^3}$.

This last quantity is the third differentiol coefficient.

It is now easy to understand what is meant by the fourth, &c. or generally by the n^{th} differential coefficient of the function u, and that they may be represented by

$$\frac{d^3 u}{d x^3},\ \frac{d^4 u}{d x^4} \ldots\ldots\ldots \frac{d^{n-1} u}{d x^{n-1}},\ \frac{d^n u}{d x^n}.$$

(14.) We may now make use of these notations, to express more simply the developement of the difference of the function u. It is plain from the relatioos we have found between the successive coefficients P, Q, R, &c. of that developement, that

$$\mathrm{P} = \frac{d u}{d x},\ \ \mathrm{Q} = \frac{d^2 u}{d x^2},\ \ \mathrm{R} = \frac{d^3 u}{d x^3},\ \ \mathrm{S} = \frac{d^4 u}{d x^4}, \text{ \&c.}$$

and consequently thot

$$u' = u + \frac{d u}{d x}\,\frac{h}{1} + \frac{d^2 u}{d x^2}\,\frac{h^2}{1\,.\,2} + \frac{d^3 u}{d x^3}\,\frac{h^3}{1\,.\,2\,.\,3} + \frac{d^4 u}{d x^4}\,\frac{h^4}{1\,.\,2\,.\,3\,.\,4} + \text{\&c.} \ldots\ldots \ (9.)$$

This may still receive another form, by substituting $d x$ for h, and writing u in the left side of the equation. It becomes then

$$u' - u = \Delta u = \frac{d u}{1} + \frac{d^2 u}{1\,.\,2} + \frac{d^3 u}{1\,.\,2\,.\,3} + \text{\&c.}$$

which expresses the difference of a function by means of its successive differentials.

(15.) The solutions of a great many importaot aod interesting questions have been fouod to depend upon the differential coefficicnts of functions. This has given rise to a separate branch of Analysis, the object of which is, first, to show how the differential coefficients of functions may be obtained; and, secoodly, how, from the knowledge of the differential coefficients, or from knowo relations between the functions and their differential coefficieuts, the values of these functioos may be determined. The methods bitherto discovered to resolve the different cases of this double problem, constitute *the Differential and Integral Calculus.*

To obtain the value of the differential coefficieots, various considerations have been used; sometimes that of the rate of increase of functions for increasing values of the variables; sometimes that of limits, &c. Each of these views may be employed exclusively, to establish the priociples of the differential calculus; and hence have arisen the divers methods which have been proposed, each possessing some advantage io particular cases, but all arriving at the same end, though by differeot means.

(16.) From what has already been stated, we may deduce a general method to find the differential coefficient of any explicit function. It will be sufficient to substitute $x + h$ for x in the function, then to develope according to the powers of h, and the coefficient of the first power of that letter will be the quantity required. If, therefore, we kocw how to find the developement of every such function, the problem of the differentiation of explicit functions would present no difficulties. When this cannot be done easily, the value of the differeotial coefficient may be determined by means of the following proposition.

(17.) We have found

$$u' - u = \frac{d u}{d x}\,h + \frac{d^2 u}{d x^2}\,\frac{h^2}{1\,.\,2} + \frac{d^3 u}{d x^3}\,\frac{h^3}{1\,.\,2\,.\,3} + \text{\&c.}$$

Henca

$$\frac{u' - u}{h} = \frac{d u}{d x} + \frac{d^2 u}{d x^2}\,\frac{h}{1\,.\,2} + \frac{d^3 u}{d x^3}\,\frac{h^2}{1\,.\,2\,.\,3} + \text{\&c.}$$

The right side of this equation has obviously for limit $\frac{d u}{d x}$ (8), therefore $\frac{d u}{d x}$ is also the limit of the left side.

Thus, *the differential coefficient* $\frac{d u}{d x}$ *of any function* u *is equal to the limit of the ratio between* $u' - u$ *or the difference of the functions, and* h *or the difference of the variable.*

We shall oow proceed to the investigation of the value of the differeotial coefficient of the various explicit functions of one variable.

(18.) *The differential coefficient of* $u + \mathrm{A}$, A *being any constant quantity, and* u *any function of* x *is the same as the differential coefficient of* u; *and the differential coefficient of* $\mathrm{A}\,u$ *is equal to the differential coefficient of* u *multiplied by* A.

If in u we change x into $x + h$, we shall have

$$u' = u + \frac{d u}{d x}\,h + \frac{d^2 u}{d x^2}\,\frac{h}{1\,.\,2} + \text{\&c.}$$

 Consequently the developements of what the functions $u+A$ become, and $A u$, when in them $x+h$ is substituted for x will be

$$u+A+\frac{d u}{d x}h+\frac{d^2 u}{d x^2}\frac{h^2}{1.2}+\&c.$$

$$A u+A\frac{d u}{d x}h+\frac{A d^2 u}{d x^2}\frac{h^2}{1.2}+\&c.$$

And since the coefficient of the first power of h in the first is $\frac{d u}{d x}$, and in the second $A\frac{d u}{d x}$, we shall have

$$\frac{d(u+A)}{d x}=\frac{d u}{d x}, \text{ and } \frac{d A u}{d x}=\frac{A d u}{d x}.$$

(19.) *When two functions of the same variable are equal, their differential coefficients are also equal.*

Let u and v be two equal functions of x. If in each we change x into $x+h$, and represent by u' and v' the results of this substitution, we shall have $u'=v'$. But, by Taylor's Theorem,

$$u'=u+\frac{d u}{d x}.h+\frac{d^2 u}{d x^2}.\frac{h^2}{1.2}+\&c., \text{ and } v'=v+\frac{d v}{d x}.h+\frac{d^2 v}{d x^2}.\frac{h^2}{1.2}+\&c.$$

Hence $$u+\frac{d u}{d x}.h+\frac{d^2 u}{d x^2}.\frac{h^2}{1.2}+\&c.=v+\frac{d v}{d x}.h+\frac{d^2 v}{d x^2}.\frac{h^2}{1.2}+\&c.$$

but $u=v$, and as h remains indeterminate, the coefficients of the terms which contain the same power of that quantity in both sides of the equation must consequently be equal. Therefore

$$\frac{d u}{d x}=\frac{d v}{d x}, \quad \frac{d^2 u}{d x^2}=\frac{d^2 v}{d x^2}, \&c.$$

(20.) The reciprocal of this proposition is not true, that is to say, that from the equality between the differential coefficients of the same rank, of two functions of the same variable, we cannot infer the equality of the functions. If, for instance, $\frac{d^3 u}{d x^3}=\frac{d^3 v}{d x^3}$, it will result, it is true, from the preceding proposition that $\frac{d^4 u}{d x^4}=\frac{d^4 v}{d x^4}$, and, consequently, that in the developements of u' and v' all the terms, beginning with the fourth, are equal to each other; but we cannot say any thing about the equality of the preceding terms, and, consequently, about that of u and v. We shall be able, hereafter, to give the form of their difference.

(21.) *The differential coefficients of a function, composed of the sum or difference of several functions of the same variable, is equal to the sum or difference of the differential coefficients of these functions.*

Let $u=y_1+y_2-z_1-z_2$, u, y_1, y_2, z_1, z_2, being functions of x. If for x we substitute $x+h$, and designate by u', y_1', &c. what these different functions become, we shall have

$$u'=y_1'+y_2'-z_1'-z_2',$$

and by Taylor's theorem

$$u'=u+\frac{d u}{d x}.h+\frac{d^2 u}{d x^2}.\frac{h^2}{1.2}+\&c.$$

$$y_1'=y_1+\frac{d y_1}{d x}.h+\frac{d^2 y_1}{d x^2}\frac{h^2}{1.2}+\&c.$$

&c. &c. &c.

Therefore

$$\begin{aligned}u+\frac{d u}{d x}.h+\frac{d^2 u}{d x^2}.\frac{h^2}{1.2}+\&c.=\;&y_1+\frac{d y_1}{d x}.h+\frac{d^2 y_1}{d x^2}.\frac{h^2}{1.2}+\&c.\\&+y_2+\frac{d y_2}{d x}.h+\frac{d^2 y_2}{d x^2}.\frac{h^2}{1.2}+\&c.\\&-z_1-\frac{d z_1}{d x}.h-\frac{d^2 z_1}{d x^2}.\frac{h^2}{1.2}-\&c.\\&-z_2-\frac{d z_2}{d x}.h-\frac{d^2 z_2}{d x^2}.\frac{h^2}{1.2}-\&c.\end{aligned}$$

And, consequently,

$$\frac{d u}{d x}=\frac{d y_1}{d x}+\frac{d y_2}{d x}-\frac{d z_1}{d x}-\frac{d z_2}{d x},$$

and also

$$\frac{d^2 u}{d x^2} = \frac{d^2 y_1}{d x^2} + \frac{d^2 y_2}{d x^2} - \frac{d^2 z_1}{d x^2} - \frac{d^2 z_2}{d x^2},$$

&c. &c. &c.

(22.) *The differential coefficient of the product of two functions of the same variable, is equal to the sum of the products of each of them by the differential coefficient of the other.*

Let $u = y_1 y_2$, we shall have $u' = y_1' y_2'$, but

$$u' = u + \frac{d u}{d x} . h + \frac{d^2 u}{d x^2} \cdot \frac{h^2}{1 . 2} + \text{\&c.}$$

$$y_1' = y_1 + \frac{d y_1}{d x} . h + \frac{d^2 y_1}{d x^2} \cdot \frac{h^2}{1 . 2} + \text{\&c.}$$

$$y_2' = y_2 + \frac{d y_2}{d x} . h + \frac{d^2 y_2}{d x^2} \cdot \frac{h^2}{1 . 2} + \text{\&c.}$$

Multiplying the two last equations, it is easy to see that in the product of the two right sides, the coefficient of the first power of h will be

$$y_1 \frac{d y_2}{d x} + y_2 \frac{d y_1}{d x};$$

and since this product is equal to the developement of u', h remaining an indeterminate quantity, we shall have

$$\frac{d u}{d x} = y_1 \frac{d y_2}{d x} + y_2 \frac{d y_1}{d x}.$$

If we suppose u to be the product of three functions y_1, y_2, y_3, the preceding proposition will give

$$\frac{d u}{d x} = y_1 \frac{d y_2 y_3}{d x} + y_2 y_3 \frac{d y_1}{d x},$$

but

$$\frac{d y_2 y_3}{d x} = y_2 \frac{d y_3}{d x} + y_3 \frac{d y_2}{d x}.$$

Hence by substitution

$$\frac{d u}{d x} = y_1 y_2 \frac{d y_3}{d x} + y_1 y_3 \frac{d y_2}{d x} + y_2 y_3 \frac{d y_1}{d x},$$

and, generally, if $u = y_1 y_2 y_3 \ldots . y_n$

$$\frac{d u}{d x} = y_2 y_3 \ldots . y_n \frac{d y_1}{d x} + y_1 y_3 \ldots . y_n \frac{d y_2}{d x} + \ldots . y_1 y_2 \ldots . y_{n-1} \frac{d y_n}{d x}.$$

This equation and the preceding ones may receive another form, by dividing both sides by u. The last becomes then

$$\frac{1}{u} \cdot \frac{d u}{d x} = \frac{1}{y_1} \frac{d y_1}{d x} + \frac{1}{y_2} \frac{d y_2}{d x} + \ldots\ldots + \frac{1}{y_n} \frac{d y_n}{d x}.$$

(23.) *The differential coefficient of a fraction whose numerator and denominator are functions of the same variable, is equal to the denominator multiplied by the differential coefficient of the numerator, less the product of the numerator by that differential coefficient of the denominator, the whole divided by the square of the denominator.*

Let $u = \frac{y_1}{z_1}$, where y_1 and z_1 are functions of x. Multiplying both sides by z_1, we shall have $u z_1 = y_1$, consequently $\frac{d u z_1}{d x} = \frac{d y_1}{d x}$, but $\frac{d u z_1}{d x} = u \frac{d z_1}{d x} + z_1 \frac{d u}{d x}$, therefore $\frac{u d z_1}{d x} + z_1 \frac{d u}{d x} = \frac{d y_1}{d x}$, and hence

$$\frac{d u}{d x} = \frac{d \frac{y_1}{z_1}}{d x} = \frac{\frac{d y_1}{d x} - u \frac{d z_1}{d x}}{z_1} = \frac{z_1 \frac{d y_1}{d x} - y_1 \frac{d z_1}{d x}}{z_1^2}.$$

This last value may be written under another form by multiplying and dividing it by $\frac{y_1}{z_1}$. It then becomes

$$\frac{d . \frac{y_1}{z_1}}{d x} = \frac{y_1}{z_1} \left\{ \frac{1}{y_1} \cdot \frac{d y_1}{d x} - \frac{1}{z_1} \cdot \frac{d z_1}{d x} \right\}.$$

If the numerator is constant, equal to a for instance, then $\dfrac{d\frac{a}{z_1}}{dx} = -\dfrac{a\frac{dz_1}{dx}}{z_1^2}$.

(24.) *If u is a function of y, and y a function of x, then the differential coefficient of u considered as a function of x, is equal to the differential coefficient of u considered as a function of y, multiplied by the differential coefficient of y considered as a function of x.*

Let $u = \mathrm{F}(y)$, and $y = f(x)$. To prove the truth of this proposition, we must show that when x is changed into $x + h$, the developement of the corresponding value of u according to the powers of h, has for the coefficient of the first power of that quantity $\frac{du}{dy} \cdot \frac{dy}{dx}$. In that supposition let y' be what y becomes,

$$y' = f(x+h) = y + \frac{dy}{dx} \cdot h + \frac{d^2y}{dx^2} \cdot \frac{h^2}{1.2} + \&c.$$

Let $\frac{dy}{dx} \cdot h + \frac{d^2y}{dx^2} \cdot \frac{h^2}{1.2} + \&c.$, the increase of y corresponding to the substitution of $x + h$ for x, be represented by k. Then if we change in the function u, y into $y + k$, we shall have the value of that function corresponding to $x + h$. Let u' be this value

$$u' = \mathrm{F}(y+k) = u + \frac{du}{dy} \cdot k + \frac{d^2u}{dy^2}\frac{k^2}{1.2} + \&c.$$

It is easy to see, now, that if we substitute in this developement for k its value $\frac{dy}{dx} \cdot h + \frac{d^2y}{dx^2}\frac{h^2}{1.2} + \&c.$, the only term which will contain the first power of h will be $\frac{du}{dy} \cdot \frac{dy}{dx}$. Therefore

$$\frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx}.$$

When u is a function of y, reciprocally y may be considered as a function of u, and an immediate consequence of the proposition just demonstrated is, that *the product of the differential coefficient of u considered as a function of y, by the differential coefficient of y considered as a function of u, is equal to unity.*

(25.) *The differential coefficient of the function $a x^m + b$ is equal to $m a x^{m-1}$, for every value of m, positive or negative, integral or fractional.*

This results evidently from the binomial theorem demonstrated in Algebra. For if we change x into $(x + h)$, in the function we shall have by that theorem

$$a(x+h)^m + b = a x^m + b + m a x^{m-1} h + m(m-1) x^{m-2}\frac{h^2}{1.2} + \&c.,$$

a developement in which the coefficient of the first power of h is equal to $m a x^{m-1}$. Therefore if we suppose $u = a x^m + b$, we shall have

$$\frac{du}{dx} = m a x^{m-1}, \frac{d^2u}{dx^2} = \frac{m(m-1)}{1.2} a x^{m-2}, \ldots.$$

$$\frac{d^nu}{dx^n} = \frac{m(m-1)(m-2)\ldots\ldots(m-n+1)}{1 \cdot 2 \cdot 3 \ldots\ldots\ldots\ldots n} a x^{m-n}.$$

If m be an integer, it is obvious, from these formulæ, that the m^{th} differential coefficient will be equal to a, and, consequently, all those of a higher order equal to nothing.

It will be easy, by means of the preceding rule, to find the differential coefficients of every algebraical function of one variable. We shall apply it to a few examples.

Example 1. Let $u = \sqrt[n]{(a + bx + cx^2 + \&c.)^m}$. Assume $a + bx + cx^2 + \&c. = z$, then $u = \sqrt[n]{z^m} = z^{\frac{m}{n}}$, $\frac{dz}{dx} = b + 2cx + \&c.$, and $\frac{du}{dz} = \frac{m}{n} z^{\frac{m}{n}-1}$. But by (24) $\frac{du}{dx} = \frac{du}{dz} \cdot \frac{dz}{dx}$, therefore $\frac{du}{dx} = \frac{m}{n} z^{\frac{m}{n}-1}(b + 2cx + \&c.)$; or in substituting for z its value

$$\frac{du}{dx} = \frac{m}{n}(b + 2cx + \&c.)\sqrt[n]{(a + bx + cx^2 + \&c.)^{m-n}}.$$

If $m = 1$, $n = 2$,

$$\frac{du}{dx} = \frac{(b + 2cx + \&c.)}{2\sqrt{(a + bx + cx^2 + \&c.)}}.$$

Example 2. Let $u=(a+bx+cx^2)^m\,(a'+b'x+c'x^2)^n$.

We shall have, by (22,)

$$\frac{du}{dx}=(a+bx+cx^2)^m\cdot\frac{d.(a'+b'x+c'x^2)^n}{dx}+(a'+b'x+c'x^2)^n\cdot\frac{d.(a+bx+cx^2)^m}{dx}.$$

But,

$$\frac{d(a'+b'x+c'x^2)^n}{dx}=n(b'+2c'x)(a'+b'x+c'x^2)^{n-1} \text{ and } \frac{d.(a+bx+cx^2)^m}{dx}=m(b+2cx)(a+bx+cx^2)^{m-1}$$

substituting

$$\frac{du}{dx}=(a+bx+cx^2)^{m-1}(a'+b'x+c'x^2)^{n-1}\left\{(b'+2c'x)(a+bx+cx^2)+(b+2cx)(a'+b'x+c'x^2)\right\}.$$

Example 3. Let
$$u=\frac{(a+bx+cx^2)^m}{(a'+b'x+c'x^2)^n}$$
we shall find, in applying the rule given (22,)

$$\frac{du}{dx}=\frac{(a+bx+cx^2)^{m-1}\left\{(b'+2c'x)(a+bx+cx^2)+(b+2cx)(a'+b'x+c'x^2)\right\}}{(a'+b'x+c'x^2)^{n+1}}.$$

Example 4. Let $u=\sqrt[4]{\left\{a-\frac{b}{\sqrt{x}}+\sqrt[3]{(c^2-x^2)}\right\}^3}$. Assume $a-\frac{b}{\sqrt{x}}+\sqrt[3]{(c^2-x^2)}=y$; then

$$u=\sqrt[4]{y^3}=y^{\frac{3}{4}}, \text{ and therefore } \frac{du}{dy}=\frac{3}{4}\,y^{-\frac{1}{4}},$$

but
$$\frac{dy}{dx}=\frac{d\left(a-\frac{b}{\sqrt{x}}+\sqrt[3]{(c^2-x^2)}\right)}{dx}=\frac{-d\frac{b}{\sqrt{x}}}{dx}+\frac{d\sqrt[3]{(c^2-x^2)}}{dx},$$

and
$$\frac{d\frac{b}{\sqrt{x}}}{dx}=\frac{d\,b\,x^{-\frac{1}{2}}}{dx}=-\frac{1}{2}\,b\,x^{-\frac{3}{2}}=\frac{-b}{2x\sqrt{}}$$

$$\frac{d\sqrt[3]{(c^2-x^2)}}{dx}=\frac{d\,(c^2-x^2)^{\frac{1}{3}}}{dx}=\frac{1}{3}(c^2-x^2)^{-\frac{2}{3}}\;-2x=\frac{-2x}{3\sqrt[3]{(c^2-x^2)^2}}.$$

And since $\frac{du}{dx}=\frac{du}{dy}\cdot\frac{dy}{dx}$, we shall have, by substitution,

$$\frac{du}{dx}=\frac{\frac{3b}{2x\sqrt{x}}-\frac{2x}{3\sqrt[3]{(c^2-x^2)^2}}}{4\sqrt[4]{\left\{a-\frac{b}{\sqrt{x}}+\sqrt[3]{(c^2-x^2)}\right\}}}.$$

We shall give two more examplee, in which we propose to find the second, third, and differential coefficients es well as the first.

Example 5. Let
$$u=a+bx+cx^2+dx^3+\ldots\ldots tx^m, \text{ then}$$
$$\frac{du}{dx}=b+2cx+3dx^2+\ldots\ldots\ldots mx^{m-1},$$
$$\frac{d^2u}{dx^2}=2c+2.3dx+\ldots\ldots\ldots m(m-1)x^{m-2},$$
$$\frac{d^3u}{dx^3}=2.3d+\ldots\ldots\ldots m(m-1)(m-2)x^{m-3},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\frac{d^mu}{dx^m}=m(m-1)(m-2)\ldots\ldots\ldots 1.$$

Example 6. Let $u=(a+bx+cx^2)^r$, and let it be required to find the n^{th} differentiel coefficient of u.

Instead of calculating successively the first, second, third, and differential coefficients, it is obvious from Taylor's theorem, that we shall obtain the n^{th} differential coefficient at once; if, after having substituted $x+h$ for x in the function u, we can find the coefficient of h^n in the developement. For it will be eufficient to multiply it by $1.2.3\ldots.n$ to have the value of the n^{th} differential coefficient. The result of the eubstitution of $(x+h)$ for x in u, gives
$$u'=(a+bx+bh+cx^2+2cxh+ch^2)^r.$$

Assume
$$a+bx+cx^2=p, \text{ and } b+2cx=q,$$

then
$$u' = (p + q\,h + c\,h^2)^r,$$
and by the binomial theorem
$$u' = (p + q\,h)^r + \frac{r}{1}\,(p + q\,h)^{r-1}\,c\,h^2 + \frac{r\,(r-1)}{1\,.\,2}\,(p + q\,h)^{r-2}\,c^2\,h^4 + \frac{r\,(r-1)\,(r-2)}{1\,.\,2\,.\,3}(p+q\,h)^{r-3}\,c^3\,h^6 + \&c.$$

If we develope now the powers of $(p + q\,h)$ which are indicated in this series, and collect afterwards under the same coefficient, the terms which contain the same power of h, we shall have the developement of u' according to the powers of that letter. But since we only want the coefficient of h^n, it will be sufficient to calculate the coefficient of h^n in the developement of $(p + q\,h)^r$; that of h^{n-2} in the developement of $(p + q\,h)^{r-1}$, since that binomial in the above series is multiplied by h^2; that of h^{n-4} in the developement of $(p + q\,h)^{r-2}$, &c.

These coefficients are respectively,
$$\frac{r\,.\,(r-1)\,(r-2)\ldots\ldots(r-n+1)}{1\,.\,2\,.\,3\ldots\ldots\ldots n}\,p^{r-n}\,q^n,$$
$$\frac{(r-1)\,(r-2)\ldots\ldots\ (r-n+2)}{1\,.\,2\,.\,3\ldots\ldots\ldots n-2}\,p^{r-n+1}\,q^{n-2},$$
$$\frac{(r-2)\,(r-3)\ldots\ldots\ldots(r-n+3)}{1\,.\,2\,.\,3\ldots\ldots\ldots n-4}\,p^{r-n+2}\,q^{n-4},$$
&c.

Hence the value of the coefficient of h^n in the developement of u', will be the sum of these quantities, which being multiplied by $1\,.\,2\,.\,3\ldots.\,n$ will give
$$\frac{d^n u}{d\,x^n} = r\,(r-1)\,(r-2)\,.\,.\,(r-n+1)\,p^{r-n}\,q^n\left\{1 + \frac{n\,(n-1)}{r-n+1}\;\frac{c\,p}{q^2} + \frac{n\,(n-1)\,(n-2)\,(n-3)}{1\,.\,2\,(r-n+1)\,(r-n+2)}\;\frac{c^2\,p^2}{q^4}\right.$$
$$\left. + \frac{n\,(n-1)\,(n-2)\,(n-3)\,(n-4)\,(n-5)}{1\,.\,2\,.\,3\,(r-n+1)\,(r-n+2)\,(r-n+3)}\;\frac{c^3\,p^3}{q^6} + \&c.\right\}$$

The value of $\frac{d^n u}{d\,x^n}$ may be put under a simpler form, which it will not be useless to give here, as an example of analytical transformation.

We have, first, $\quad u' = (p + q\,h + c\,h^2)^r = p^r\left(1 + \frac{2\,q}{2\,p}\,h + \frac{4\,p\,c}{4\,p^2}\,h^2\right)^r.$

But $\quad p = a + b\,x + c\,x^2$, hence $4\,p\,c = 4\,a\,c + 4\,b\,c\,x + 4\,c^2\,x^2$,

$q = b + 2\,c\,x$, hence $q^2 = b^2 + 4\,b\,c\,x + 4\,c^2\,x^2$,

therefore $\quad 4\,p\,c - q^2 = 4\,a\,c - b^2$. Assume $4\,a\,c - b^2 = e^2$, then $4\,p\,c = q^2 + e^2$,

substituting in the value of u', we shall have
$$u' = p^r\left(1 + \frac{2\,q}{2\,p}\,h + \frac{q^2 + e^2}{4\,p^2}\,.\,h^2\right)^r = p^r\left\{\left(1 + \frac{q}{2\,p}\,h\right)^2 + \frac{e^2}{4\,p^2}\,h^2\right\}^r.$$

Hence
$$u' = p^r\left\{\left(1 + \frac{q}{2\,p}\,h\right)^{2r} + \frac{r}{1}\left(1 + \frac{q}{2\,p}\,h\right)^{2r-2}\frac{e^2\,h^2}{4\,p^2} + \frac{r\,(r-1)}{1\,.\,2}\left(1 + \frac{q}{2\,p}\,h\right)^{2r-4}\frac{e^4\,h^4}{4^2\,p^4} + \frac{r\,(r-1)\,(r-2)}{1\,.\,2\,.\,3}\right.$$
$$\left.\left(1 + \frac{q}{2\,p}\,h\right)^{2r-6}\frac{e^6\,h^6}{4^3\,p^6} + \&c.\right\}.$$

Developing each of the binomials, collecting the terms which multiply h^n, and multiplying their aggregate by $1\,.\,2\,.\,3\ldots.\,n$, we shall have
$$\frac{d^n u}{d\,x^n} = 1\,.\,2\,.\,3\,.\,.\,p^r\left\{\frac{2\,r\,(2\,r-1)\,.\,.\,(2\,r-n+1)}{1\,.\,2\ldots\ldots\ldots n}\;\frac{q^n}{2^n\,p^n} + \frac{r\,(2\,r-2)\,(2\,r-3)\,.\,.\,(2\,r-n+1)}{1\,.\,2\ldots\ldots\ldots n-2}\;\frac{q^{n-2}}{2^{n-2}\,p^{n-2}}\;\frac{e^2}{4\,p^2}\right.$$
$$\left. + \frac{r\,.\,(r-1)}{1\,.\,2}\;\frac{(2\,r-4)\,(2\,r-5)\ldots.(2\,r-n+1)}{1\,.\,2\,.\,3\ldots\ldots\ldots n-4}\;\frac{q^{n-4}}{2^{n-4}\,p^{n-4}}\;\frac{e^4}{4^2\,p^4} + \&c.\right\}$$

This value may be written in the following manner,
$$\frac{d^n u}{d\,x^n} = 2\,r\,(2\,r-1)\ldots.\,(2\,r-n+1)\;\frac{q^n}{2^n}\,p^{r-n}$$
$$\left\{1 + \frac{r}{1}\;\frac{n\,(n-1)}{2\,r\,(2\,r-1)}\;\frac{e}{q^2} + \frac{r\,(r-1)}{1\,.\,2}\;\frac{n\,(n-1)\,(n-2)\,(n-3)}{2\,r\,(2\,r-1)\,(2\,r-2)\,(2\,r-3)}\;\frac{e^2}{q^4} + \&c.\right\}.$$

When n is an even number this series has $\frac{n}{2} + 1$ terms, and $\frac{n+1}{2}$ when n is odd. This formulæ and the other found before, were first given by Lagrange. They led to important and curious results, when various values are assumed for u and n. (See a collection of examples on the application of the Differential and Integral Calculus, p. 12.)

We shall now proceed to investigate the rules to find the values of the differential coefficients of the exponential, logarithmic and trigonometrical functions.

(26.) *The differential coefficient of the function* a^x *is equal to* $a^x \,\mathrm{l}\, a$, $\mathrm{l}\, a$ *being the hyperbolic logarithm of the base.*

Let x be changed into $x+h$, the difference of the function will be $a^{x+h} - a^x = a^x(a^h - 1)$, and it is the coefficient of the first power of h in the development of that difference that we are to determine. Assume $a = 1 + b$, then $a^h = (1+b)^h$, and therefore the difference of a^x takes the form $a^x\{(1+b)^h - 1\}$. Expanding $(1+b)^h$ by the binomial theorem, we shall have

$$a^x\{(1+b)^h - 1\} = a^x\left(\frac{h}{1}.b + \frac{h(h-1)}{1.2}b^2 + \frac{h(h-1)(h-2)}{1.2.3}b^3 + \&c.\right).$$

If we arrange now the terms between the parenthesis according to the powers of h, we shall have for the coefficient of the first power of that quantity the following series,

$$\frac{b}{1} - \frac{b^2}{2} + \frac{b^3}{3} - \frac{b^4}{4} + \&c$$

Let us represent it by k, then we shall have

$$\frac{d\,a^x}{d\,x} = k\,a^x.$$

Hence
$$\frac{d^2\,a^x}{d\,x^2} = k^2 a^x, \quad \frac{d^3\,a^x}{d\,x^3} = k^3 a^x, \ \&c.$$

Therefore by Taylor's theorem

$$a^{x+h} = a^x + k\,a^x.\frac{h}{1} + k^2 a^x.\frac{h^2}{1.2} + k^3 a^x.\frac{h^3}{1.2.3} + \&c.$$

Dividing both sides by a^x,

$$a^h = 1 + \frac{k\,h}{1} + \frac{k^2.h^2}{1.2} + \frac{k^3.h^3}{1.2.3} + \&c.$$

This equation being true for every value of h. Assume $h = \frac{1}{k}$, it will become

$$a^{\frac{1}{k}} = 1 + 1 + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} + \&c.$$

The ratio of two successive terms of this series decreases rapidly. Therefore we can approximate indefinitely to its value. The ten first terms equal 2.7182818. Let the whole be represented by e, then

$$e = 1 + 1 + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} + \&c.$$

and
$$a^{\frac{1}{k}} = e.$$

The number e is of frequent use in analysis; and it will not be useless to prove, before we proceed, that it is incommensurable. First, e cannot be a whole number; for, evidently,

$$\frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} + \&c.\ldots < \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \&c.$$

but the last series is equal to one. Hence

$$\frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} + \&c. < 1.$$

Therefore e is not an integer, and its value lies between 2 and 3. Secondly, no fractional number can be equal to e; for, if possible, let $\frac{m}{n} = e$, n being an integer less than m, but greater than one. Then

$$\frac{m}{n} = 1 + 1 + \frac{1}{1.2} + \frac{1}{1.2.3} + \ldots\ldots + \frac{1}{1.2.3\ldots.n} + \frac{1}{1.2.3\ldots.n(n+1)} + \&c.$$

Multiplying both sides by $1.2.3\ldots.n$, it becomes

$$1.2.3..n-1.m = 1.2.3..n + 1.2.3..n + 4.5.6..n + .. + n + 1 + \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \&c.$$

The left side being an integer, the other side should also be one. This cannot take place unless $\frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \&c.$ be a whole number. But it is impossible, for the series is obviously less than $\frac{1}{n+1} + \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} + \&c.$, which is equal to $\frac{1}{n}$. Therefore e cannot be equal to a commensurable number.

We resume now the investigation of the value of k. We have found $a^{\frac{1}{k}} = e$. Taking the logarithms of both sides, we find $\mathrm{L}\,a = k\,\mathrm{L}\,e$, or $k = \frac{\mathrm{L}\,a}{\mathrm{L}\,e}$, and since a and e are known, k is known. If a be the base of the system of logarithms, $\mathrm{L}\,a = 1$, and therefore $k = \frac{1}{\mathrm{L}\,e}$. If e be the base then $\mathrm{L}\,e = 1$, and $k = \mathrm{L}\,a$.

The logarithms corresponding to the base e, are called *Naperian* or *hyperbolic* logarithms. They are of great use; and it will be found convenient to denote them in a particular manner. We shall therefore prefix the letter l to a quantity to express its hyperbolic logarithm, and the letter L to represent the logarithm related to any other base. Thus the value of k will be represented by $l\,a$; therefore

$$\frac{d\,a^x}{d\,x} = a^x\,l\,a.$$

By substituting x for h, and for k its value, in the series we have obtained for a^h we shall have

$$a^x = 1 + \frac{l\,a}{1}\,.\,x + \frac{(l\,a)^2}{1\,.\,2}\,.\,x^2 + \frac{(l\,a)^3}{1\,.\,2\,.\,3}\,.\,x^3 + \&c$$

If $a = e$, this series becomes

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{1\,.\,2} + \frac{x^3}{1\,.\,2\,.\,3} + \&c.$$

(27.) *The differential coefficient of* $\mathrm{L}\,x$ *is equal to* $\frac{m}{x}$, *m being the modulus corresponding to the base of the system of logarithms; that is to say, equal to one divided by the hyperbolic logarithm of that base.*

Let $u = \mathrm{L}\,x$, and a be the base. Then $x = a^u$, and therefore, by (26), $\frac{d\,x}{d\,u} = a^u\,.\,l\,a = x\,l\,a$. But, by (24), $\frac{d\,u}{d\,x}\,.\,\frac{d\,x}{d\,u} = 1$, consequently $\frac{d\,.\,\mathrm{L}x}{d\,x} = \frac{d\,u}{d\,x} = \frac{1}{x\,l\,a} = \frac{m}{x}$, m being equal to $\frac{1}{l\,a}$.

Hence, $$\frac{d^2\,\mathrm{L}\,x}{d\,x^2} = \frac{-\,m}{x^2},\quad \frac{d^3\,.\,\mathrm{L}\,x}{d\,x^3} = \frac{2\,m}{x^3},\quad \frac{d^4\,\mathrm{L}\,x}{d\,x^4} = \frac{-\,2\,.\,3\,m}{x^4},\ \&c.$$

If the logarithms were hyperbolics, $m = \frac{1}{l\,a}$ would be equal to one, and therefore

$$\frac{d\,l\,x}{d\,x} = \frac{1}{x},\quad \frac{d^2\,l\,x}{d\,x^2} = \frac{-\,1}{x^2},\quad \frac{d^3\,l\,x}{d\,x^3} = \frac{2}{x^3},\quad \frac{d^4\,l\,x}{d\,x^4} = \frac{-\,2\,.\,3}{x^4}.$$

Having thus found the values of the successive differential coefficients of the functions $\mathrm{L}\,x$ and $l\,x$, we may apply Taylor's theorem to the developements of $\mathrm{L}\,(x+h)$ and $l\,(x+h)$. We shall find

$$\mathrm{L}\,(x+h) = \mathrm{L}\,x + m\left(\frac{h}{x} - \frac{1}{2}\,\frac{h^2}{x^2} + \frac{1}{3}\,\frac{h^3}{x^3} - \frac{1}{4}\,\frac{h^4}{x^4} + \&c.\right).$$

$$l\,(x+h) = l\,x + \frac{h}{x} - \frac{1}{2}\,\frac{h^2}{x^2} + \frac{1}{3}\,\frac{h^3}{x^3} - \frac{1}{4}\,\frac{h^4}{x^4} + \&c.$$

Or, assuming $\frac{h}{x} = z$,

$$\mathrm{L}\,(x+h) - \mathrm{L}\,x = \mathrm{L}\left(\frac{x+h}{x}\right) = \mathrm{L}\,(1+z) = m\left(\frac{z}{1} - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \&c.\right).$$

$$l\,(x+h) - l\,x = l\left(\frac{x+h}{x}\right) = l\,(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \&c.$$

The two preceding rules, combined with those previously explained, will enable us to find the differential coefficients of any function in which logarithms or exponentials, depending on the variable, enter.

Example 1. Let it be proposed to find the differential coefficient of $u = l\,(x + \sqrt{(1+x^2)})$.

Assume $x + \sqrt{(1+x^2)} = z$, then $u = l\,z$, $\frac{d\,u}{d\,z} = \frac{1}{z} = \frac{1}{x+\sqrt{(1+x^2)}}$, and $\frac{d\,z}{d\,x} = 1 + \frac{x}{\sqrt{(1+x^2)}} = \frac{\sqrt{(1+x^2)} + x}{\sqrt{(1+x^2)}}$, hence

$$\frac{d\,u}{d\,x} = \frac{d\,u}{d\,z}\,.\,\frac{d\,z}{d\,x} = \frac{1}{\sqrt{(1+x^2)}}.$$

Example 2. $u = (l\,x)^n$. Let $l\,x = z$, then $u = z^n$, $\frac{d\,u}{d\,z} = n\,z^{n-1} = n\,(l\,x)^{n-1}$, $\frac{d\,z}{d\,x} = \frac{1}{x}$, and therefore

$$\frac{d\,u}{d\,x} = \frac{n\,(l\,x)^{n-1}}{x}.$$

Example 3. $u = l\,(l\,x)$. Let $l\,x = z$; then $u = l\,z$, $\frac{d\,u}{d\,z} = \frac{1}{z} = \frac{1}{l\,x}$, $\frac{d\,z}{d\,x} = \frac{1}{x}$, and consequently $\frac{d\,u}{d\,x} = \frac{1}{x\,l\,x}$.

Example 4. $u = a^{b^x}$. Let $b^x = z$, then $u = a^z$, $\frac{d\,u}{d\,z} = a^z\,l\,a = a^{b^x}\,l\,a$, $\frac{d\,z}{d\,x} = b^x\,l\,b$, hence $\frac{d\,u}{d\,x} = a^{b^x}\,b^x\,l\,a\,l\,b$.

Example 5. Let $u = z^y$, z and y being any functions of x. Taking the hyperbolic logarithms, we have $l\,u = y\,l\,z$, and therefore

$$\frac{1}{u} \cdot \frac{d\,u}{d\,x} = \frac{y}{z}\,\frac{d\,z}{d\,x} + l\,z\,\frac{d\,y}{d\,x}, \text{ or}$$

$$\frac{d\,u}{d\,x} = u\left\{\frac{y}{z}\,\frac{d\,z}{d\,x} + l\,z\,\frac{d\,y}{d\,z}\right\}$$

If $y = x$ and $z = x$, this formula becomes

$$\frac{d\,x^x}{d\,x} = x^x\,(1 + l\,x).$$

(28.) *The differential coefficient of the sine of an arc, considered as a function of the arc itself, is equal to the cosine of the same arc; and the differential coefficient of the cosine of an arc, is equal to minus the sine of the same arc.*

To prove this proposition we shall make use of the property of the differential coefficient demonstrated (17), viz. that it is the limit of the ratio between the difference of the function and the difference of the variable. It is necessary to show previously, that *the limit of the ratio of the sine of an arc to the arc itself, the arc being supposed to decrease indefinitely, is equal to unity.*

First, it is obvious, that the arc is always greater than its sine; for it is greater than the chord, and the chord is an oblique with respect to the sine. Secondly, the arc is always less than its tangent; for the product of the arc by half the radius is the surface of a sector contained in the triangle measured by the product of the tangent by half the radius. Hence, if we designate by x any arc, the ratio $\frac{\sin x}{x}$ will always be included between the two $\frac{\sin x}{\sin x} = 1$, and $\frac{\sin x}{\tan x} = \cos x$, since they have all the same numerator, and since the denominator of the first is greater than that of the second, and less than that of the third. But the second is equal to unity, and the third $= \cos x$, has clearly for limit *one*. Therefore $\frac{\sin x}{x}$, included between the two has also the same limit.

This understood, let $u = \sin x$, the difference of the function is $\sin\,(x + h) - \sin x$; and we must determine the limit of the ratio $\frac{\sin\,(x + h) - \sin x}{h}$. We shall observe to that effect, that

$$\sin\,(x + h) - \sin x = 2 \sin \tfrac{1}{2}\,h \cos\,(x + \tfrac{1}{2}\,h).$$

Hence the ratio becomes $\frac{\sin \frac{1}{2}\,h}{\frac{1}{2}\,h} \cos\,(x + \frac{1}{2}\,h)$. But we have just proved that the limit of $\frac{\sin \frac{1}{2}\,h}{\frac{1}{2}\,h}$ was equal to unity. The limit of $\cos\,(x + \frac{1}{2}\,h)$ is clearly $\cos x$; consequently the limit of $\frac{\sin\,(x + h) - \sin x}{h}$ is equal to $\cos x$.

Therefore
$$\frac{d \sin x}{d\,x} = \cos x.$$

To find the differential coefficient of $\cos x$, we observe that $\cos x = \sin\left(\frac{\pi}{2} - x\right)$, and assuming $\frac{\pi}{2} - x = z$,

$$\sin\left(\frac{\pi}{x} - x\right) = \sin z. \quad \frac{d\,.\sin z}{d\,z} = \cos z = \cos\left(\frac{\pi}{2} - x\right), \text{ and } \frac{d\,z}{d\,x} = -1.$$

Therefore
$$\frac{d\,.\sin\left(\frac{\pi}{2} - x\right)}{d\,x} = \frac{d \cos x}{d\,x} = -\cos\left(\frac{\pi}{2} - x\right) = -\sin x.$$

The second, third, and differential coefficients of $\cos x$ and $\sin x$, may now be easily calculated. We shall find

$$\frac{d^2 \sin x}{d\,x^2} = \frac{d \cos x}{d\,x} = -\sin x, \frac{d^3 \sin x}{d\,x^3} = -\frac{d \sin x}{d\,x} = -\cos x, \frac{d^4 \sin x}{d\,x^4} = -\frac{d \cos x}{d\,x} = \sin x, \text{ \&c.};$$

and
$$\frac{d^2 \cos x}{d\,x^2} = -\frac{d \sin x}{d\,x} = -\cos x, \frac{d^3 \cos x}{d\,x^3} = -\frac{d \cos x}{d\,x} = \sin x, \frac{d^4 \cos x}{d\,x^4} = \frac{d \sin x}{d\,x} = \cos x, \text{ \&c.}$$

These values, combined with Taylor's theorem, give

$$\sin\,(x + h) = \sin x + \cos x\,.\,\frac{h}{1} - \sin x\,.\,\frac{h^2}{1\,.\,2} - \cos x\,.\,\frac{h^3}{1\,.\,2\,.\,3} + \sin x\,.\,\frac{h^4}{1\,.\,2\,.\,3\,.\,4} - \text{\&c.}$$

$$\cos\,(x + h) = \cos x - \sin x\,.\,\frac{h}{1} - \cos x\,.\,\frac{h^2}{1\,.\,2} + \sin x\,.\,\frac{h^3}{1\,.\,2\,.\,3} + \cos x\,.\,\frac{h^4}{1\,.\,2\,.\,3\,.\,4} - \text{\&c.}$$

If we change h into $-h$, in the first, it becomes

$$\sin(x-h) = \sin x - \cos x \, . \, \frac{h}{1} - \sin x \, . \, \frac{h^2}{1.2} + \cos x \, . \, \frac{h^3}{1.2.3} + \sin x \, . \, \frac{h^4}{1.2.3.4} - \&c.$$

Subtracting this last equation from the first, and dividing by $2\cos x$, we obtain

$$\sin h = \frac{h}{1} - \frac{h^3}{1.2.3} + \frac{h^5}{1.2.3.4.5} - \frac{h^7}{1.2.3.4.6.7} + \&c.$$

By the addition of the same equations, and in dividing by $2\sin x$,

$$\cos h = 1 - \frac{h^2}{1.2} + \frac{h^4}{1.2.3.4} - \frac{h^6}{1.2.3.4.5.6} + \&c.$$

(29.) The differential coefficients of the other trigonometrical lines, considered as functions of the arc, may now easily be found.

1st. Let $u = \tan x$. Since $\tan x = \frac{\sin x}{\cos x}$, we shall have, by (23),

$$\frac{du}{dx} = \frac{d \, . \, \frac{\sin x}{\cos x}}{(\cos x)^2} = \frac{\cos x \frac{d \sin x}{dx} - \sin x \frac{d \cos x}{dx}}{(\cos x)^2} = \frac{(\cos x)^2 + (\sin x)^2}{(\cos x)^2} = \frac{1}{(\cos x)^2}.$$

2d. Let $u = \cot x$. Since $\cot x = \frac{\cos x}{\sin x}$, we shall find, in a similar manner,

$$\frac{du}{dx} = \frac{1}{(\sin x)^2}.$$

3d. Let $u = \sec x$. Since $\sec x = \frac{1}{\cos x}$, by (23) we shall get

$$\frac{du}{dx} = \frac{\sin x}{(\cos x)^2} = \tan x \sec x.$$

4th. Let $u = \operatorname{cosec} x$. Since $\operatorname{cosec} x = \frac{1}{\sin x}$, we find

$$\frac{du}{dx} = \frac{-\cos x}{(\sin x)^2} = -\cot x \operatorname{cosec} x.$$

(30.) *The differential coefficient of an arc considered as a function of its sine, is equal to one divided by the square root of the difference between one and the square of the arc.*

We shall represent the arc whose sine is equal to x by $\sin^{-1} x$. This manner of denoting such a function results from a notation lately introduced in the higher branches of analysis, to express the repetition of the operation indicated by the nature of a function upon the function itself. It has been proposed to represent such functions as $l\,l\,x$, $\sin \sin x$, $\tan \tan \tan \tan x$, by $l^2 x$, $\sin^2 x$, $\tan^4 x$, the index denoting the number of times the operation must be repeated. In general, $f^2(x)$, $f^3(x)$, ... $f^n(x)$ will be equal respectively to $f(f(x))$, $f(f(f(x)))$, &c. An immediate consequence of this notation is, that $f^m(f^n(x)) = f^{m+n}(x)$. To find the meaning of such expressions as $f^0(x)$, $f^{-1}(x)$, it will be sufficient in the last formula, first, to make $n = 0$ and $m = 1$, and, secondly, $m = 1$ and $n = -1$. The first supposition gives $f(f^0(x)) = f(x)$, and, consequently, $f^0(x) = x$. The second supposition gives $f(f^{-1}(x)) = f^0(x) = x$. Let $x = f(y)$, and let the value of y, derived from this equation, be $F(x)$, then F will be the inverse function of f; but in substituting for y its value $F(x)$ we have $x = f(F(x))$, which equation compared to $f(f^{-1}(x)) = x$ gives $f^{-1}(x) = F(x)$, therefore f^{-1} denotes the inverse function of f, and, consequently, $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, will stand respectively for arc whose sine $= x$, arc whose cosine $= x$, arc whose tangent $= x$. The symmetry of this notation, and above all the new views it opens of the nature of analytical operations, seem to authorize its universal adoption.

Let therefore $u = \sin^{-1} x$, then $x = \sin u$ and $\frac{dx}{du} = \cos u$; therefore, by (24),

$$\frac{du}{dx} = \frac{1}{\cos u} = \frac{1}{\sqrt{(1-x^2)}}.$$

In a similar manner, if we suppose $u = \cos^{-1} x$, we shall have $x = \cos u$, consequently $\frac{dx}{du} = -\sin u$, and

$$\frac{du}{dx} = \frac{-1}{\sin u} = \frac{-1}{\sqrt{(1-x^2)}}.$$

(31.) *The differential coefficient of an arc, considered as a function of its tangent, is equal to a fraction whose numerator is one, and whose denominator is one plus the square of the tangent.*

would be expressed by $\frac{d\left(\frac{du}{dx}\right)}{dy}$; the third partial differential coefficient of $\frac{d^2u}{dx^2}$, with respect to y, by $\frac{d^3\left(\frac{d^2u}{dx^2}\right)}{dy^3}$;

and the n^{th} partial differential coefficient of $\frac{d^m u}{dx^m}$ with respect to y, by $\frac{d^n\left(\frac{d^m u}{dx^m}\right)}{dy^n}$ To simplify these results, it has been agreed to represent them respectively by the following symbols;

$$\frac{d^2u}{dy\,dx},\quad \frac{d^5u}{dy^3\,dx^2},\quad \frac{d^{n+m}u}{dy^n\,dx^m},$$

which, by means of the numerical index in the numerator, and the exponents of dy, dx, in the denominator, can leave no doubt with respect to their real meaning. Thus, for instance, to form the value of $\frac{d^3u}{dy^2\,dx}$, we ought to take the first partial differential coefficient of u, with respect to x, and then the second partial differential coefficient of this result with respect to y. To find the value of $\frac{d^3u}{dx\,dy^2}$ the order of the operations should be inverted.

In general, to form the value of $\frac{d^{n+m}u}{dy^n\,dx^m}$ we should first find the m^{th} partial differential coefficient of u, with respect to x, and afterwards the n^{th} partial differential coefficient of this result with respect to y. And to form the value of $\frac{d^{m+n}u}{dx^m\,dy^n}$ the order of the operations should be the reverse.

No farther explanation will be necessary, to understand the meaning of expressions such as

$$\frac{d^{m+n+p}u}{dx^m\,dy^n\,dz^p}$$

where u is supposed to contain the three variables x, y, z, and to extend the same notation to any number of variables.

The determination of the values of these various partial differential coefficients, can present no difficulty, each operation being performed in the supposition that all the variables but one are constant, and being consequently assimilated to the case of functions of one variable.

(34.) Let $f(x, y, z, \&c.)$ be a function of any number of variables, if we change x into $x+h$, y into $y+k$, &c., h, k, &c. being indeterminate quantities, it becomes $f(x+h, y+k, \&c.)$ and $f(x+h, y+k, \&c.) - f(x, y)$ is called the difference of the function. In making use of the preceding notations, and of Taylor's theorem, this difference may be developed, under a symmetrical form, in a series of terms containing the successive powers of h, k, &c.

We shall first consider the case of a function of two variables, and then it will be easy to extend the results we shall obtain to a function of any number of variables.

Let $u = f(x, y)$, and substitute $x+h$ for x, we shall have, by Taylor's theorem,

$$f(x+h, y) = u + \frac{du}{dx}\cdot h + \frac{d^2u}{dx^2}\cdot\frac{h^2}{1.2} + \frac{d^3u}{dx^3}\cdot\frac{h^3}{1.2.3}.$$

Change now y into $y+k$ in both sides. Each of the coefficients of the powers of h in the right side of the equation will become a function of $y+k$, and may, consequently, be developed according to the powers of k.

Thus u, $\frac{du}{dx}$, &c. will give rise to the following series respectively:

$$u + \frac{du}{dy}\cdot k + \frac{d^2u}{dy^2}\cdot\frac{k^2}{1.2} + \frac{d^3u}{dy^3}\cdot\frac{k^3}{1.2.3} + \&c.$$

$$\frac{du}{dx} + \frac{d^2u}{dy\,dx}\cdot k + \frac{d^3u}{dy^2\,dx}\cdot\frac{k^2}{1.2} + \frac{d^4u}{dy^3\,dx} + \&c.$$

$$\frac{d^2u}{dx^2} + \frac{d^3u}{dy\,dx^2}\cdot k + \frac{d^4u}{dy^2\,dx^2}\cdot\frac{k^2}{1.2} + \&c.$$

$$\frac{d^3u}{dx^3} + \frac{d^4u}{dy\,dx^3}\cdot k + \frac{d^5u}{dy^2\,dx^3}\cdot\frac{k^2}{1.2} + \&c.$$

$$+ \&c.$$

$\frac{du}{dy}k$, and this will be obtained by determining the first partial differential coefficient of du with respect to x, and that with respect to y, multiplying the first by h, and the second by k, and adding the two results.

We shall have successively

$$\frac{d(du)}{dx}=\frac{d^2u}{dx^2}h+\frac{d^2u}{dx\,dy}k,\quad \frac{d(du)}{dy}=\frac{d^2u}{dy\,dx}h+\frac{d^2u}{dy^2}k,$$

and $$d^2u=\frac{d(du)}{dx}k+\frac{d(du)}{dy}k=\frac{d^2u}{dx^2}h^2+2\frac{d^2u}{dx\,dy}hk+\frac{d^2u}{dy^2}k^2;$$

or, substituting dx for h, and dy for k, we find

$$d^2u=\frac{d^2u}{dx^2}dx^2+2\frac{d^2u}{dx\,dy}dx\,dy+\frac{d^2u}{dy^2}dy^2.$$

The value of d^3u will be obtained in a similar manner: first,

$$d^3u=\frac{d(d^2u)}{dx}h+\frac{d(d^2u)}{dy}k,$$

but $$\frac{d(d^2u)}{dx}=\frac{d^3u}{dx^3}h^2+\frac{2d^3u}{dx^2dy}hk+\frac{d^3u}{dx\,dy^2}k$$

and $$\frac{d(d^2u)}{dy}=\frac{d^3u}{dy\,dx^2}h^2+\frac{2d^3u}{dx\,dy^2}hk+\frac{d^3u}{dy^3}k^2,$$

therefore $$d^3u=\frac{d^3u}{dx^3}h^3+\frac{3d^3u}{dx^2dy}h^2k+\frac{3d^3u}{dx\,dy^2}hk^2+\frac{d^3u}{dy^3}k^3.$$

or $$d^3u=\frac{d^3u}{dx^3}dx^3+\frac{3d^3u}{dx^2dy}dx^2dy+\frac{3d^3u}{dx\,dy^2}dx\,dy^2+\frac{d^3u}{dy^3}dy^3.$$

The analogy of the numerical coefficients and exponents of k and k, in the expression of the successive differentials of u, to the coefficients and exponents of the same letters in the developements of the first, second, and third power of the binomial $h+k$, is obvious. We may prove that the same analogy subsists for any order. For let us suppose

$$d^nu=\frac{d^nu}{dx^n}h^n+\frac{A\,d^nu}{dx^{n-1}dy}h^{n-1}k+\frac{B\,d^nu}{dx^{n-2}dy^2}h^{n-2}k^2+\frac{C\,d^nu}{dx^{n-3}dy^3}h^{n-3}k^3+\&c.$$

We shall have $$d^{n+1}u=\frac{d(d^nu)}{dx}h+\frac{d(d^nu)}{dy}k.$$

But $$\frac{d(d^nu)}{dx}=\frac{d^{n+1}u}{dx^{n+1}}h^n+\frac{A\,d^{n+1}u}{dx^ndy}h^{n-1}k+\frac{B\,d^{n+1}u}{dx^{n-1}dy^2}h^{n-2}k^2+\frac{C\,d^{n+1}u}{dx^{n-2}dy^3}h^{n-3}k^3+\&c.$$

and $$\frac{d(d^nu)}{dy}=\frac{d^{n+1}u}{dy\,dx^n}h^n+\frac{A\,d^{n+1}u}{dx^{n-1}dy^2}h^{n-1}k+\frac{B\,d^{n+1}u}{dx^{n-2}dy^3}h^{n-2}k^2+\frac{C\,d^{n+1}u}{dx^{n-3}dy^4}h^{n-3}k^3+\&c.$$

Multiplying the first series by h, the second by k, and adding them, we find

$$d^{n+1}u=\frac{d^{n+1}u}{dx^{n+1}}h^{n+1}+(A+1)\frac{d^{n+1}u}{dx^ndy}h^nk+(B+A)\frac{d^{n+1}u}{dx^{n-1}dy^2}h^{n-1}k^2+(C+B)\frac{d^{n+1}u}{dx^{n-2}dy^3}h^{n-2}k^3+\&c.$$

From the manner in which this last series has been obtained, it is evident that the numerical coefficients and exponents are precisely the same as if we had multiplied the value of d^nu by $h+k$. But we have already proved that for the first, second, and third differentials, these coefficients and exponents are the same as in the developements of the three first powers of $k+h$, therefore for the n^{th} differential they will be equal to those of the developement of $(h+k)^n$. Thus

$$d^nu=\frac{d^nu}{dx^n}h^n+\frac{n\,d^nu}{dx^{n-1}dy}h^{n-1}k+\frac{n(n-1)}{1.2}\frac{d^nu}{dx^{n-2}dy^2}h^{n-2}k^2+\frac{n(n-1)(n-2)}{1.2.3}\frac{d^nu}{dx^{n-3}dy^3}h^{n-3}k^3+\&c.$$

or $$d^nu=\frac{d^nu}{dx^n}dx^n+\frac{n\,d^nu}{dx^{n-1}dy}dx^{n-1}dy+\frac{n(n-1)}{1.2}\frac{d^nu}{dx^{n-2}dy^2}dx^{n-2}dy^2+\&c.$$

(37.) The foregoing expressions o. the successive differentials of a function of two variables, will enable us to give a very symmetrical form to the developement of the difference of that function. For that purpose let us bring to the same denominator, in the developement of $f(x+h, y+k)$, the terms in which the sum of the exponents of these two letters is the same. We shall have

$$f(x+h,\ y+k) = u + \frac{1}{1}\left(\frac{d\,u}{d\,x}h + \frac{d\,u}{d\,y}k\right)$$
$$+ \frac{1}{1\,.\,2}\left(\frac{d^2 u}{d\,x^2}h^2 + \frac{2\,d^2 u}{d\,x\,d\,y}h\,k + \frac{d^2 u}{d\,y^2}k^2\right)$$
$$+ \frac{1}{1\,.\,2\,.\,3}\left(\frac{d^3 u}{d\,x^3}h^3 + \frac{3\,d^3 u}{d\,x^2\,d\,y}h^2 k + \frac{3\,d^3 u}{d\,x\,d\,y^2}h\,k^2 + \frac{d^3 u}{d\,y^3}k^3\right)$$
$$+ \text{\&c.}$$

To form the n^{th} horizontal line of this series, we must collect all the terms of the developement of $f(x+h, y+k)$, in which the sum of the exponents of h and k is equal to n; or, which is the same thing, those which contain the partial differential coefficients of the n^{th} order. These will clearly be the first term of the developement of $\frac{d^n u}{d\,x^n}\ \frac{h^n}{1\,.\,2\,\ldots\,n}$, when in it y is changed into $y + k$; the second term of the developement of $\frac{d^{n-1} u}{d\,x^{n-1}}\ \frac{h^{n-1}}{1\,.\,2\,..\,n-1}$, in the same supposition; the third of the developement of $\frac{d^{n-2} u}{d\,x^{n-2}}\ \frac{h^{n-2}}{1\,.\,2\,..\,n-2}$ 0, &c.... down to the $(n+0)^{th}$ of the developement of $f(n, y+k)$. Therefore if $\frac{1}{1\,.\,2\,.\,3\,\ldots\,n}$ is considered as a common factor to all these terms, the n^{th} horizontal line will be

$$\frac{1}{1\,.\,2\,\ldots\,n}\left(\frac{d^n u}{d\,x^n}h^n + \frac{n\,d^n u}{d\,x^{n-1}\,d\,y}h^{n-1}k + \frac{n\,(n-1)}{1\,.\,2}\ \frac{d^n u}{d\,x^{n-2}\,d\,y^2} + \ldots \frac{d^n u}{d\,y^n}k^n\right)$$

If we compare the developement of $f(x+h,\ y+k)$ under this form, to the values we have given for the successive differentials of u, we shall immediately observe that the last are equal to the quantities enclosed between parentheses in the first. Hence

$$\Delta\,u = d\,u + \frac{d^2 u}{1\,.\,2} + \frac{d^3 u}{1\,.\,2\,.\,3} + \frac{d^4 u}{1\,.\,2\,.\,3\,.\,4} + \text{\&c.}$$

which is the same formula we have obtained in (14), only applied to functions of two variables.

(38.) All that has been said with respect to functions of two variables may easily be extended to functions of any number of variables.

Let u be a function of n variables x, y, z, &c. There will be n first partial differential coefficients represented by

$$\frac{d\,u}{d\,x},\quad \frac{d\,u}{d\,y},\quad \frac{d\,u}{d\,z},\ \text{\&c.}$$

and the partial differential coefficient of the m^{th} order will be expressed generally by

$$\frac{d^{p+q+r+\text{\&c.}}\,u}{d\,x^p\,d\,y^q\,d\,z^r\ \text{\&c.}},$$

where $p + q + r + \text{\&c.} = m$.

If x, y, z, &c. are changed into $x + h$, $y + k$, $z + l$, &c.; and if u' represent the value assumed by u in that supposition, $u' - u$ will be the difference of u, and we shall be able to develope it in a series containing the successive powers of h, k, l, &c. by substituting first $x + h$ for x, in u, then developing by Taylor's theorem, and changing in the developement successively y into $y + k$, z into $z + l$, &c.; and after each substitution developing each term by means of the same theorem.

It is obvious that the terms which will multiply the first powers of h, k, l, &c. will be

$$\frac{d\,u}{d\,x}h + \frac{d\,u}{d\,y}k + \frac{d\,u}{d\,z}l + \text{\&c.}$$

They are respectively the *partial differentials* with respect to x, y, z, &c., and their sum constitutes the *total differential*, or simply *the differential* of u. Thus

$$d\,u = \frac{d\,u}{d\,x}h + \frac{d\,u}{d\,y}k + \frac{d\,u}{d\,z}l + \text{\&c.}$$

or

$$d\,u = \frac{d\,u}{d\,x}d\,x + \frac{d\,u}{d\,y}d\,y + \frac{d\,u}{d\,z}d\,z + \text{\&c.}$$

The developement of u' must remain the same, whatever be the order of the substitutions of $x + h$ to x, $y + k$ to k, $z + l$ to z, &c. Hence we shall infer, that if we take p times the partial differential coefficient of u with respect to x, q times with respect to y, r times with respect to z, &c., the result will be the same whatever be the order of these successive operations.

The formation of the successive differentials of u will present no difficulty. It will be sufficient to operate upon $d\,u$, $d^2 u$, $d^3 u$, &c., precisely in the same manner as we have operated upon u to form $d\,u$. Thus, we shall have

$$d^2 u = \frac{d\,(d\,u)}{d\,x}h + \frac{d\,(d\,u)}{d\,y}k + \frac{d\,(d\,u)}{d\,z}l + \text{\&c.}$$

and if we observe that we shall have to multiply the partial differential coefficients of

$$du = \frac{du}{dx}\,h + \frac{du}{dy}\,k + \frac{du}{dz}\,l + \&c.$$

successively by h, k, l, &c. it will appear evident, with a little attention, that the numerical coefficients and exponents of h, k, l, &c. in the value of $d^2 u$, will be the same as in the product of $(h + k + l + \&c.)$, by $(h + k + l + \&c.)$, or in $(h + k + l + \&c.)^2$. Hence, in using a similar reasoning, we shall conclude that in the value of $d^3 u$, these coefficients and exponents will be the same as in $(h + k + l + \&c.)^3$, and generally in the expression of $d^n u$ the same as in the developement of $(h + k + l + \&c.)^n$.

The comparison of the values of the successive differentials of u, with the developement of u', after having written in a line the terms in which the sum of the exponents of h, k, l, &c. is the same, will lead, as before, to the formulæ

$$\Delta\,u = \frac{du}{1} + \frac{d^2 u}{1\,.\,2} + \frac{d^3 u}{1\,.\,2\,.\,3} + \&c.$$

which therefore in general, whatever be the number of variables of the function u.

(39.) We have considered hitherto all the variables x, y, z, &c. which enter the function u, as independent of each other. Let us suppose now that some of them are functions of some of the others; and, first, let y, z, &c. be all functions of x. Then $u = f(x, y, z, \&c.)$ will be a function composed of functions of x; and when x is changed into $x + h$, &c. y, z, &c. will become

$$y + \frac{dy}{dx}\,h + \frac{d^2 y}{dx^2}\,\frac{h^2}{1\,.\,2} + \frac{d^3 y}{dx^3}\,\frac{h^3}{1\,.\,2\,.\,3} + \&c.$$

$$z + \frac{dz}{dx}\,h + \frac{d^2 z}{dx^2}\,\frac{h^2}{1\,.\,2} + \frac{d^3 z}{dx^3}\,\frac{h^3}{1\,.\,2\,.\,3} + \&c.$$

But by (38) we know that generally if we change in u, x into $x + h$, y into $y + k$, z into $z + l$, &c. and develope, the terms of the series containing the first powers of h, k, l, &c. are

$$\frac{du}{dx}\,h + \frac{du}{dy}\,k + \frac{du}{dz}\,l + \&c.$$

In the present case, the substitution of x into $x + h$ in u will make these terms assume the following form,

$$\frac{du}{dx}\,h + \frac{du}{dy}\left(\frac{dy}{dx}\,h + \frac{d^2 y}{dx^2}\,\frac{h^2}{1\,.\,2} + \&c.\right) + \frac{du}{dz}\left(\frac{dz}{dx} + h\,\frac{d^2 z}{dx^2}\,\frac{h^2}{1\,.\,2} + \&c.\right) + \&c.$$

Hence the coefficient of the first power of h, or the differential coefficient of u, is

$$\frac{1}{dx}\,.\,du = \frac{du}{dx} + \frac{du}{dy}\,.\,\frac{dy}{dx} + \frac{du}{dx}\,\frac{dz}{dx} + \&c.$$

But $\frac{du}{dy}\,.\,\frac{dy}{dx}$, $\frac{du}{dz}\,.\,\frac{dz}{dx}$, &c. are by (24) the partial differential coefficients of u with respect to y, z, &c., these variables being considered as functions of x, therefore *the differential coefficient of any function of* x, y, z, *&c. in which* y, z, *&c. are the representatives of functions of* x, *is equal to the sum of the partial differential coefficients of that function with respect to* x, y, z, *&c. separately.*

This rule applied to the first differential coefficient, in which $\frac{dy}{dx}$, $\frac{dz}{dx}$, &c. are to be considered as new variables, functions of x, will give the second, and then third, fourth, &c. differential coefficients.

The manner in which the partial differential coefficients of u may be obtained, in any other supposition, relative to the dependency of the variables x, y, z, &c. is now sufficiently indicated by the preceding investigation.

(40.) A few examples will be sufficient to show the application of the rules, to find the values of the differentials and differential coefficients of a function of several variables

Example 1. Let $u = (x^m + y^n + z^p)^r$, then

$$\frac{du}{dx} = r\,(x^m + y^n + z^p)^{r-1}\,m\,x^{m-1},\quad \frac{du}{dy} = r\,(x^m + y^n + z^p)^{r-1}\,n\,y^{n-1},\quad \frac{du}{dz} = r\,(x^m + y^n + z^p)\,p\,z^{p-1},$$

and $$du = r\,(x^m + y^n + z^p)^{r-1}\,(m\,x^{m-1}\,dx + n\,y^{n-1}\,dy + p\,z^{p-1}\,dz).$$

Example 2. Let $u = (a + b\,x^m)^p\,(c + d\,y^n)^q$, then

$$\frac{du}{dx} = (c + d\,y^n)^q\,.\,p\,(a + b\,x^m)^{p-1}\,b\,m\,x^{m-1},\quad \frac{du}{dy} = (a + b\,x^m)^p\,q\,(c + d\,y^n)^{q-1}\,d\,n\,y^{n-1},$$

and $$du = (a + b\,x^m)^{p-1}\,(c + d\,y^n)^{q-1}\,\{b\,m\,p\,(c + d\,y^n)\,x^{m-1}\,dx + d\,n\,q\,(a + b\,x^m)\,y^{n-1}\,dy\}.$$

Example 3. Let $u = x^y$. Then

$$\frac{d\,u}{d\,x} = y\,x^{y-1},\ \frac{d\,u}{d\,y} = x^y\,l\,x,\ d\,u = x^y\left(\frac{y}{x}\,d\,x + l\,x\,d\,y\right).$$

$$\frac{d^2\,u}{d\,x^2} = y\,(y-1)\,x^{y-2},\ \frac{d^2\,u}{d\,x\,d\,y} = x^{y-1} + y\,x^{y-1}\,l\,x,\ \frac{d^2\,u}{d\,y^2} = x^y\,(l\,x)^2,$$

$$d^2\,u = y\,(y-1)\,x^{y-2}\,d\,x^2 + 2\,x^{y-1}\,(1 + y\,l\,x)\,d\,x\,d\,y + x^y\,(l\,x)^2\,d\,y^2,$$

$$= x^{y-2}\left\{y\,(y-1)\,d\,x^2 + 2\,x\,(1 + y\,l\,x)\,d\,x\,d\,y + x^2\,(l\,x)^2\,d\,y^2\right\}.$$

Example 4. Let $u = x \sin y + y \sin x$, then

$$\frac{d\,u}{d\,x} = \sin y + y \cos x,\ \frac{d\,u}{d\,y} = x \cos y + \sin x,$$

$$\frac{d^2\,u}{d\,x^2} = -\,y \sin x,\ \frac{d^2\,u}{d\,y^2} = -\,x \sin y,\ \frac{d^2\,u}{d\,x\,d\,y} = \cos y + \cos x.$$

$$d^2\,u = 2\,(\cos y + \cos x)\,d\,x\,d\,y - y \sin x\,d\,x^2 - x \sin y\,d\,y^2.$$

Example 5. Let $u = (x + l\,x + e^x + \sin x)^m$.

Assume $\quad l\,x = y,\ e^x = z,\ \sin x = v$, then $u = (x + y + z + v)^m$, and

by (39)

$$\frac{1}{d\,x}\,d\,u = \frac{d\,u}{d\,x} + \frac{d\,u}{d\,y}\,\frac{d\,y}{d\,x} + \frac{d\,u}{d\,z}\,\frac{d\,z}{d\,x} + \frac{d\,u}{d\,v}\,\frac{d\,v}{d\,x} =$$

$$m\,(x + y + z + v)^{m-1}\left\{1 + \frac{1}{x} + e^x + \cos x\right\}.$$

The two last examples we propose to give, will afford a verification of a theorem of considerable importance, relative to homogeneous functions of several variables, and which for that reason we shall first demonstrate.

(41.) *If* n *be the sum of the exponents in each term of an homogeneous function* u *of the variables* x, y, z, &c., then

$$n\,u = \frac{d\,u}{d\,x}\,x + \frac{d\,u}{d\,y}\,y + \frac{d\,u}{d\,z}\,z + \&c.$$

Let us change the variables x, y, z, &c. into $x + g^x$, $y + g^y$, &c., or $x\,(1 + g)$, $y\,(1 + g)$, &c. The function u will become $(1 + g)^n\,u$. Hence

$$\begin{aligned}(1+g)^n\,u = u &+ \frac{d\,u}{d\,x}\,g\,x + \frac{d^2\,u}{d\,x^2}\,\frac{g^2\,x^2}{1\,.\,2} + \&c.\\ &+ \frac{d\,u}{d\,y}\,g\,y + \frac{d^2\,u}{d\,y\,d\,x}\,g^2\,x\,y,\\ &+ \frac{d\,u}{d\,z}\,g\,z + \frac{d^2\,u}{d\,y^2}\,\frac{g^2\,y^2}{1\,.\,2},\\ &+ \&c. \qquad + \&c.\end{aligned}$$

But $\quad (1+g)^n\,u = u\left(1 + n\,g + \frac{n\,(n-1)}{1\,.\,2}\,g^2 + \frac{n\,(n-1)\,(n-2)}{1\,.\,2\,.\,3}\,g^3 + \&c.\right)$

The terms which multiply the same powers of g in these two developements of $(1 + g)^n\,u$, must be equal;

therefore

$$n\,u = \frac{d\,u}{d\,x}\,x + \frac{d\,u}{d\,y}\,y + \frac{d\,u}{d\,z}\,z + \&c.$$

and also

$$n\,(n-1)\,u = \frac{d^2\,u}{d\,x^2}\,x^2 + \frac{2\,d^2\,u}{d\,x\,d\,y}\,x\,y + \frac{d^2\,u}{d\,y^2}\,y^2 + \&c.$$

The relations between a function and its partial differential coefficients, is sometimes called the *theorem of homogeneous functions*, it was discovered by Fontaine; the preceding demonstration was given by Lagrange.

Example 6. Let $u = \dfrac{x\,y\,z}{x + y + z}$; where $n = 2$. We shall find

$$\frac{d\,u}{d\,x} = \frac{(x+y+z)\,y\,z - x\,y\,z}{(x+y+z)^2},\ \frac{d\,u}{d\,y} = \frac{(x+y+z)\,x\,z - x\,y\,z}{(x+y+z)^2},\ \frac{d\,u}{d\,z} = \frac{(x+y+z)\,x\,y - x\,y\,z}{(x+y+z)^2}.$$

Hence

$$\frac{d\,u}{d\,x}\,x + \frac{d\,u}{d\,y}\,y + \frac{d\,u}{d\,z}\,z = \frac{2\,x\,y\,z}{x + y + z} = 2\,u.$$

Example 7. Let $u = (x + y)\,\sqrt{(x - y)}$; where $n = \frac{3}{2}$. We shall have

$$\frac{d\,u}{d\,x} = \sqrt{(x-y)} + \frac{(x+y)}{2\,\sqrt{(x-y)}},\ \frac{d\,u}{d\,y} = \sqrt{(x-y)} - \frac{(x+y)}{2\,\sqrt{(x-y)}}.$$

Hence $\frac{du}{dx}x + \frac{du}{dy}y = (x+y)\sqrt{(x-y)} + \frac{(x+y)(x-y)}{2\sqrt{(x-y)}} = \frac{3}{2}(x+y)\sqrt{(x-y)} = \frac{3}{2}u.$

We shall find also,

$$\frac{d^2u}{dx^2} = \frac{1}{\sqrt{(x-y)}} - \frac{x+y}{4(x-y)\sqrt{(x-y)}}, \frac{d^2u}{dxdy} = \frac{x+y}{4(x-y)\sqrt{(x-y)}}, \frac{d^2u}{dy^2} = \frac{-1}{\sqrt{(x-y)}} - \frac{x+y}{4(x-y)\sqrt{(x-y)}}.$$

Hence

$$\frac{d^2u}{dx^2}x^2 + 2\frac{d^2u}{dydx}xy + \frac{d^2u}{dy^2}y^2 = \frac{x^2-y^2}{\sqrt{(x-y)}} - \frac{(x+y)(x-y)^2}{4(x-y)\sqrt{(x-y)}} = \frac{3}{4}(x+y)\sqrt{(x-y)} = \frac{3}{2}\left(\frac{3}{2}-1\right)u.$$

(42.) When two variables x and y are connected by an equation, such as $f(x, y) = 0$, either of them may be considered as a function of the other; y, for instance, as a function of x; and if the equation cannot be resolved with respect to y, then, as we have before stated, y is said to be an implicit function of x.

We shall now examine how, in that supposition, we may determine the values of the successive differential coefficients of the function of x represented by y, or rather how we may discover the relations which subsist between x, y, and the differential coefficients $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, &c.

Let $f(x, y) = u = 0$ be the proposed equation, and let $\phi(x)$ be the function of x which y represents; that is to say, let us suppose that $\phi(x)$ is the value we would obtain for y, if we were able to resolve the equation $f(x, y) = 0$. If we substitute $\phi(x)$ for y in $f(x, y)$, we shall have therefore $f(x, \phi(x))$ equal nothing independently of any particular value of x. Consequently, if we change x into $x + h$, we shall also have $f(x+h, \phi(x+h))$ equal to nothing. Now, since $\phi(x)$ is the value of y, by Taylor's theorem

$$\phi(x+h) = y + \frac{dy}{dx}.h + \frac{d^2y}{dx^2}\frac{h^2}{1.2} + \frac{d^3y}{dx^3}\frac{h^3}{1.2.3} + \text{\&c. or} = y + k;$$

in assuming

$$\frac{dy}{dx}h + \frac{d^2y}{dx^2}\frac{h^2}{1.2} + \frac{d^3y}{dx^3}\frac{h^3}{1.2.3} + \text{\&c.} = k.$$

So that $f(x+h, y+k) = f\{x+h, \phi(x+h)\}$ is equal to nothing, whatever be the values of x and h, when for y and k the above values are substituted. But by (34),

$$\begin{aligned} f(x+h, y+k) = u &+ \frac{du}{dx}\frac{h}{1} + \frac{d^2u}{dx^2}\frac{h^2}{1.2} + \frac{d^3u}{dx^3}\frac{h^3}{1.2.3} + \text{\&c.} \\ &+ \frac{du}{dy}\frac{k}{1} + \frac{d^2u}{dydx}hk + \frac{d^3u}{dydx^2}\frac{h^2}{1.2}k, \\ &+ \frac{d^2u}{dy}\frac{k^2}{1.2} + \frac{d^3u}{dy^2dx}k\frac{k^2}{1.2}, \\ &+ \frac{d^3u}{dy^3}\frac{k^3}{1.2.3}, \\ &+ \text{\&c.} \end{aligned}$$

Or, in substituting for k its value,

$$\begin{aligned} f(x+h, y+k) = u &+ \left(\frac{du}{dx} + \frac{du}{dy}\cdot\frac{dy}{dx}\right)h, \\ &+ \left(\frac{d^2u}{dx^2} + \frac{2d^2u}{dydx}\cdot\frac{dy}{dx} + \frac{d^2u}{dy^2}\frac{dy^2}{dx^2} + \frac{du}{dy}\frac{d^2y}{dx^2}\right)\frac{h^2}{1.2}, \\ &+ \text{\&c.} \end{aligned}$$

Therefore this series must be equal to nothing, whatever be the value of h; hence the coefficients of the different powers of h must separately equal nothing. Consequently the following equations will obtain

$$u = 0,$$

$$\frac{du}{dx} + \frac{du}{dy}\cdot\frac{dy}{dx} = 0,$$

$$\frac{d^2u}{dx^2} + 2\frac{d^2u}{dydx}\frac{dy}{dx} + \frac{d^2u}{dy^2}\frac{dy^2}{dx^2} + \frac{du}{dy}\frac{d^2y}{dx^2} = 0.$$

The first is only the proposed equation $f(x, y) = u = 0$. The second is the expression of the relation which exists between x, y, and the first differential coefficient $\frac{dy}{dx}$; and from it we may determine the value of that differential coefficient in function of x and y. The third expresses the relation between x, y, $\frac{dy}{dx}$, and the second

differential coefficient $\frac{d^2 y}{d x^2}$, and would give the value of the last quantity in function of the three others, or simply in function of x and y, if we had previously determined the value of $\frac{d y}{d x}$, by means of the second equation. The following equations would, in a similar manner, give the values of $\frac{d^3 y}{d x^3}$, $\frac{d^4 y}{d x^4}$, &c.

A very simple rule may be given to form the equation $\frac{d u}{d x} + \frac{d u}{d y} \frac{d y}{d x} = 0$; for, by multiplying both sides by $d x$, it becomes $\frac{d u}{d x} d x + \frac{d u}{d y} d y = 0$, and then the left side is obviously the total differential of u.

It follows from this remark, that *to find the value of the first differential coefficient of an implicit function represented by* y, *and connected with the variable* x *by an equation* $u = f(x, y) = 0$, *we must find the total differential of* u, *as if the two variables* x *and* y *were independent of each other, then write this differential equal to zero, and deduce from the equation so formed, the value of* $\frac{d y}{d x}$.

(43.) The equations, by means of which the values of the second and successive differential coefficients of the implicit function y may be determined, can easily be derived from the preceding.

We may observe, that the left sides of these equations are the coefficients of h, $\frac{h^2}{1 \cdot 2}$, $\frac{h^3}{1 \cdot 2 \cdot 3}$, &c., in the developement of $f(x + h, y + k), = f(x + h, \phi(x + h))$, that is, in the developement of a function of $x + h$. Hence it will result, from Taylor's theorem, that the second, which is the coefficient of $\frac{h^2}{1 \cdot 2}$, must be the differential coefficient of the first; that the third, which is the coefficient of $\frac{h^3}{1 \cdot 2 \cdot 3}$, must be the differential coefficient of the second; and, in the same manner, each succeeding one the differential coefficient of that which precedes it. But in taking these successive differential coefficients, it must not be forgotten that x is the only independent variable, and that y, $\frac{d y}{d x}$, $\frac{d^2 y}{d x^2}$, &c., are only the representatives of implicit functions of x. Therefore we should operate as in the case of functions of functions (39), that is to say, after having taken the partial differential coefficient of the quantity under consideration, with respect to x, we ought to take the partial differential coefficients of the same quantity, with respect to each of the other variables y, $\frac{d y}{x}$, $\frac{d^2 y}{d x^2}$, &c., multiplying each of them respectively by the differential coefficient of each of these variables with respect to x, that is, by $\frac{d y}{d x}$, $\frac{d^2 y}{d x^2}$, $\frac{d^3 y}{d x^3}$, &c., and then add all these partial differential coefficients together.

Let us form, according to this rule, the equation upon which depends the determination of the value of $\frac{d^2 y}{d x^2}$. We know that the equation from which the value of the preceding differential coefficient may be derived, is

$$\frac{d u}{d x} + \frac{d u}{d y} \frac{d y}{d x} = 0 \ldots\ldots\ldots\ldots\ldots (a).$$

Hence, if we suppose the left side of this equation equal to u', $\frac{d y}{d x} = p$, and consequently $\frac{d^2 y}{d x^2} = \frac{d p}{d x}$, we shall have to determine this last quantity from the equation

$$\frac{d u'}{d x} + \frac{d u'}{d y} \cdot \frac{d y}{d x} + \frac{d u'}{d p} \cdot \frac{d p}{d x} = 0;$$

but
$$\frac{d u'}{d x} = \frac{d^2 u}{d x^2} + \frac{d^2 u}{d y \, d x} \cdot \frac{d y}{d x}, \quad \frac{d u'}{d y} = \frac{d^2 u}{d y \, d x} + \frac{d^2 u}{d y^2} \frac{d y}{d x}, \quad \frac{d u'}{d p} = \frac{d u}{d y}.$$

Substituting these values, and $\frac{d^2 y}{d x^2}$ instead of $\frac{d p}{d x}$, we find

$$\frac{d^2 u}{d x^2} + 2 \frac{d^2 u}{d y \, d x} \cdot \frac{d y}{d x} + \frac{d^2 u}{d y^2} \frac{d y^2}{d x^2} + \frac{d u}{d y} \frac{d^2 y}{d x^2} = 0 \ldots\ldots\ldots\ldots\ldots (b)$$

an equation identical with that obtained in (41).

In following the same process, we shall find that the equation expressing the relation between $\frac{d^3 y}{d x^3}$, the preceding differential coefficients, $\frac{d^2 y}{d x^2}$, $\frac{d y}{d x}$, the function y, and the variable x is

$$\frac{d^3 u}{d x^3} + \frac{3\, d^3 u}{d y\, d x^2} \frac{d y}{d x} + \frac{3\, d^3 u}{d y^2\, d x} \frac{d y^2}{d x^2} + \frac{d^3 u}{d y^3} \frac{d y^3}{d x^3} + \frac{3\, d^2 u}{d y\, d x} \frac{d^2 y}{d x^2} + \frac{3\, d^2 u}{d y^2} \cdot \frac{d y}{d x} \cdot \frac{d^2 y}{d x^2} + \frac{d u}{d y} \frac{d^3 y}{d x^3} = 0 \,.\,. (c)$$

The formation of the equations relative to the differential coefficients of higher orders can present no difficulty.

(44.) All these equations, and the equation $u = 0$, would be verified; that is, that in each, the left side would become identical with the right side, if we were to substitute for y the function of x, it represents, and for the differential coefficients of y, the differential coefficients of that function. This is expressed by saying, that these various equations *subsist or obtain together*. Hence, by combining them in any way whatever, other equations will be formed, which will subsist or obtain with them.

(45.) An equation which contains one or several differential coefficients is called a *differential equation;* and *a primitive equation* is that which does not contain any.

A differential equation of the *first order* is that which contains no other differential coefficient than the first, and generally it is said to be of the n^{th} *order*, when the n^{th} differential coefficient is the highest it contains.

The degree of a differential equation is the highest power of the differential coefficient, which marks its order, it contains. Thus a differential equation of the n^{th} order in which the highest power of the n^{th} differential coefficient would be the m^{th}, would be of the m^{th} degree.

(46.) From the remark we have made in (44), we already perceive that several differential equations of the same order may correspond to the same primitive equation. Thus, it is obvious that from each combination of a differential equation of the m^{th} order, with the differential equations of the preceding orders, will result another differential equation of the m^{th} order. But among the various differential equations of the same order which may be so obtained, some require a peculiar attention, because they express more general relations between x, y, and the differential coefficients of y, than the others.

We must first observe, that by differentiating a primitive equation between x and y, that is, by applying the rule given (42) to form the equation which gives the value of $\frac{d y}{d x}$, it may happen that one of the constants contained in the equation should disappear. It would obviously be the case, for instance, with respect to the constant a, if the primitive equation had the form $f(x, y) = a$; and if a were not contained in $f(x, y)$. But in all cases, by combining the primitive equation with the differential equation of the first order, so as to eliminate one of the constants, it will always be easy to obtain a differential equation of the first order, containing one constant less than the primitive equation.

Such a differential equation does not only correspond to the proposed primitive equation, but to all those which differ from it by the value of the constant. Hence it expresses a relation between x, y, and $\frac{d y}{d x}$ more general, than a differential equation of the first order containing that constant.

If the constant eliminated enter the primitive equation in a degree higher than the first, the result to which we shall arrive will contain the differential coefficient of the first order in a degree higher than the first.

(47.) These considerations may easily be extended to differential equations of higher orders. We shall be able, for instance, to eliminate two constants between the primitive equation, the differential equation of the first order, and that upon which depends the value of the differential coefficient of the second order; and the result will be a differential equation of the second order containing two constants less than the primitive equation. Generally, we see that we may obtain a differential equation of the m^{th} order, containing m constants less than the primitive equation.

(48.) Instead of eliminating constants between the primitive equation, and its differential equations, they might be combined so as to make other quantities disappear in the result. The variables x or y, for instance, or any function of them entering the primitive and differential equations might be eliminated.

We shall now apply the foregoing rules and observations relative to implicit functions of x, or to equations between the two variables x and y, to a few examples.

Example 1. Let it be proposed to determine the values of the first and second differential coefficients of the implicit function of x which y represents in the equation

$$a y^3 + b x^3 = c x y + d.$$

We shall have, by (41), to determine the first differential coefficient

$$3 a y^2 \frac{d y}{d x} + 3 b x^2 = c x \frac{d y}{d x} + c y \ldots\ldots\ldots\ldots (a),$$

hence

$$\frac{d y}{d x} = - \frac{3 b x^2 - c y}{3 a y^2 - c x}.$$

To find the second differential coefficient, we shall take the differential coefficient of both sides of (a), considering y and $\frac{d y}{d x}$ as implicit functions of x. We find

$$3 a y^2 \frac{d^2 y}{d x^2} + 6 a y \frac{d y^2}{d x^2} + 6 b x = c \frac{d y}{d x} + c x \frac{d^2 y}{d x^2} + c \frac{d y}{d x},$$

and, supposing $\frac{dy}{dx} = p$,

$$\frac{d^2 y}{dx^2} = \frac{-(6bx + 6ayp^2 + 2cp)}{3ay^2 - cx};$$

or substituting for p its value

$$\frac{d^2 y}{dx^2} = \frac{-\{6bx(3ay^2 - cx)^2 + 6ay(3bx^2 - cy)^2 - 2c(3bx^2 - cy)(3ay^2 - cx)\}}{(3ay^2 - cx)^3}.$$

Example 2. Let the proposed equation be $y^2 - 2mxy + x^2 - a^2 = 0$.

We shall have to determine $\frac{dy}{dx}$

$$2y\frac{dy}{dx} - 2mx\frac{dy}{dx} - 2my + 2x = 0;$$

and, consequently,

$$\frac{dy}{dx} = \frac{my - x}{y - mx}.$$

In this case the primitive equation, containing no higher power of y than the second, may be resolved with respect to that variable. It gives

$$y = mx \pm \sqrt{(a^2 - x^2 + m^2 x^2)}.$$

Substituting these values for y, in the expression of $\frac{dy}{dx}$, we shall find

$$\frac{dy}{dx} = m \pm \frac{-x + m^2 x}{\sqrt{(a^2 - x^2 + m^2 x^2)}}.$$

It is easy to verify that the two values we have thus obtained for the differential coefficient of y are identical with those we might derive from the value of y.

There is still another manner in which we might arrive at the value of $\frac{dy}{dx}$, expressed in terms of x alone. We might eliminate y between the primitive equation, and the differential equation of the first order, by taking the value of y in the last, where it enters only in the first degree, and substituting it in the other, we shall have, by this process, the following equation,

$$\frac{dy^2}{dx^2} - 2m\frac{dy}{dx} + \frac{x^2 - m^2 x^2 - a^2 m^2}{x^2 - a^2 - m^2 x^2} = 0,$$

which, being resolved, will lead to the same values of $\frac{dy}{dx}$ as before.

Example 3. Let the primitive equation be

$$x \sin y + y \cos x = a,$$

then

$$\sin y + x \cos y \frac{dy}{dx} + \cos x \frac{dy}{dx} - y \sin x = 0,$$

and

$$\frac{dy}{dx} = \frac{y \sin x - \sin y}{\cos x + x \cos y}$$

Example 4. Let the primitive equation be

$$y^2 = ax + b,$$

then

$$2y\frac{dy}{dx} = a;$$

and by eliminating a we find for differential equation of the first order independent of a

$$y^2 - 2xy\frac{dy}{dx} - b = 0.$$

Differentiating this equation, b will disappear, and we shall obtain a differential equation of the second order independent of the two constants a and b,

$$\frac{dy^2}{dx^2} + y\frac{d^2 y}{dx^2} = 0.$$

It is easy to verify that this equation is satisfied by the value of y given by the primitive equation. For this value is $y = (ax + b)^{\frac{1}{2}}$, hence $\frac{dy}{dx} = a(ax + b)^{\frac{1}{2}}$ and $\frac{d^2 y}{dx^2} = -\frac{1}{2}a^2(ax + b)^{-\frac{3}{2}}$, which being substituted in the differential equation of the second order, makes one side identically equal to the other.

Example 5. Let the primitive equation be

$$y^2 - 2\,a\,y + x^2 = a^2,$$

where the constant a enters in the second power. We find

$$(y - a)\frac{d\,y}{d\,x} + x = 0,$$

taking the value of a in that equation, and substituting it in the primitive equation, we shall have

$$(x^2 - 2\,y^2)\frac{d\,y^2}{d\,x^2} - 4\,x\,y\frac{d\,y}{d\,x} - x^2 = 0,$$

for the differential equation of the first order, independent of the constant a.

Example 6. Let the primitive equation be

$$y^3 + y = (a^2 + x^2)^{\frac{m}{n}},$$

we shall have for the differential equation of the first order

$$(3\,y^2 + 1)\frac{d\,y}{d\,x} = \frac{m}{n}(a^2 + x^2)^{\frac{m}{n}-1}\,.\,2\,x = \frac{m\,(a^2 + x^2)^{\frac{m}{n}}}{n\,(a^2 + x^2)}\,.\,2\,x.$$

We may now substitute in this equation, instead of $(a^2 + x^2)^{\frac{m}{n}}$, its value taken in the primitive equation, and then we shall obtain a differential equation independent of that irrational function of x,

$$(3\,y^2 + 1)\frac{d\,y}{d\,x} = \frac{m\,(y^3 + y)}{n\,(a^2 + x^2)}\,.\,2\,x.$$

Example 7. We shall take for the last example an equation containing logarithmic, exponential, and trigonometrical functions; and we shall propose to eliminate them by means of the differential equations.

Let the primitive equation be

$$y + l\,y + e^{-x} + \sin x = c,$$

we shall find

$$\frac{d\,y}{d\,x} + \frac{1}{y}\frac{d\,y}{d\,x} - e^{-x} + \cos x = 0,$$

and by differentiating again

$$\frac{d^2\,y}{d\,x^2}\left(1 + \frac{1}{y}\right) - \frac{1}{y^2}\frac{d\,y^2}{d\,x^2} + e^{-x} + \sin x = 0,$$

subtracting this from the primitive equation, the functions e^{-x} and $\sin x$ will be destroyed. We shall have

$$y + l\,y - \frac{d^2\,y}{d\,x^2}\left(1 + \frac{1}{y}\right) - \frac{1}{y^2}\frac{d\,y^2}{d\,x^2} = 0;$$

and, it is obvious, that by a new differentiation $l\,y$ will disappear.

(49.) When m variables are connected together by $m - 1$ equations, any one of them may be considered as the independent variable, and all the others as implicit functions of it. Hence it may be required to find the values of the differential coefficients of these implicit functions.

Let $u = 0$, $v = 0$, $w = 0$, &c. be the proposed equations between the variables t, x, y, z, &c., in which t is supposed to be the independent variable, and x, y, z, &c. implicit functions of t. Then u, v, w, &c. may be considered as functions of t, and therefore their first differential coefficients, with respect to that variable, will be respectively (39.)

$$\frac{d\,u}{d\,t} + \frac{d\,u}{d\,x}\,\frac{d\,x}{d\,t} + \frac{d\,u}{d\,y}\,\frac{d\,y}{d\,t} + \frac{d\,u}{d\,z}\,\frac{d\,z}{d\,t} + \text{\&c.}$$

$$\frac{d\,v}{d\,t} + \frac{d\,v}{d\,x}\,\frac{d\,x}{d\,t} + \frac{d\,v}{d\,y}\,\frac{d\,y}{d\,t} + \frac{d\,v}{d\,z}\,\frac{d\,z}{d\,t} + \text{\&c.}$$

$$\frac{d\,w}{d\,t} + \frac{d\,w}{d\,x}\,\frac{d\,x}{d\,t} + \frac{d\,w}{d\,y}\,\frac{d\,y}{d\,t} + \frac{d\,w}{d\,z}\,\frac{d\,z}{d\,t} + \text{\&c.}$$

But the functions of functions of t, denoted by u, v, w, are equal to zero; since by the hypothesis, if we substitute for x, y, z in them, the functions of t they represent, the equations $u = 0$, $v = 0$, $w = 0$ must be verified. Therefore the differential coefficients of these functions must also be equal to zero. Hence

$$\frac{d\,u}{d\,t} + \frac{d\,u}{d\,x}\,\frac{d\,x}{d\,t} + \frac{d\,u}{d\,y}\,\frac{d\,y}{d\,t} + \frac{d\,u}{d\,z}\,\frac{d\,z}{d\,t} + \text{\&c.} = 0,$$

$$\frac{d\,v}{d\,t} + \frac{d\,v}{d\,x}\,\frac{d\,x}{d\,t} + \frac{d\,v}{d\,y}\,\frac{d\,y}{d\,t} + \frac{d\,v}{d\,z}\,\frac{d\,z}{d\,t} + \text{\&c.} = 0,$$

$$\frac{d\,w}{d\,t} + \frac{d\,w}{d\,x}\,\frac{d\,x}{d\,t} + \frac{d\,w}{d\,y}\,\frac{d\,y}{d\,t} + \frac{d\,w}{d\,z}\,\frac{d\,z}{d\,t} + \text{\&c.} = 0,$$

&c.

Equations by means of which the values of $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$, &c. may be determined.

The formation of the equations upon which depends the determination of the differential coefficients $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, $\frac{d^2z}{dt^2}$, &c. will present no difficulty. It is clear that the differential coefficients of the second order of the functions u, v, w, &c., considered as functions of functions of t, will be obtained by taking the differential coefficients of their first differential coefficients, in which, it must be remembered, $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$, &c. are new variables representing functions of t. These differential coefficients of the second order must, as well as those of the first, be equal to zero. Hence will result a sufficient number of equations to calculate the values of $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, $\frac{d^2z}{dt^2}$, &c., which quantities they will involve in the first degree. That which should be done to determine the values of $\frac{d^3x}{dt^3}$, $\frac{d^3y}{dt^3}$, $\frac{d^3z}{dt^3}$, &c., and the differential coefficients of still higher orders, are sufficiently indicated by what precedes, and require no further explanation.

(50.) The observations which have been made before in the case of a single equation between two variables, with respect to the combinations of the primitive equation, and its differential equations, apply clearly here. Between the $m-1$ primitive equations, and the $m-1$ differential equations of the first order $2m-3$, constant or variable quantities may be eliminated; and generally between the $m-1$ primitive equations and $nm-n$ differential equations of the n first orders, $(n+1)m-n-2$ quantities may be eliminated.

Let us take, for example, the two equations

$$y^3 + 3atx = bc^2,$$
$$x^3 + 3cty = a^2b.$$

We shall have to determine $\frac{dx}{dt}$, and $\frac{dy}{dt}$, the two following

$$y^2\frac{dy}{dt} + at\frac{dx}{dt} + ax = 0,$$
$$x^2\frac{dx}{dt} + ct\frac{dy}{dt} + cy = 0.$$

And by taking the differential coefficients of the left sides of these equations, we shall have

$$y^2\frac{d^2y}{dt^2} + 2y\frac{dy^2}{dt^2} + at\frac{d^2x}{dt^2} + 2a\frac{dx}{dt} = 0,$$
$$x^2\frac{d^2x}{dt^2} + 2x\frac{dx^2}{dt^2} + ct\frac{d^2y}{dt^2} + 2c\frac{dy}{dt} = 0.$$

Equations by means of which we shall be able to find the values of $\frac{d^2y}{dt^2}$, $\frac{d^2x}{dt^2}$.

(51.) We shall now proceed to examine implicit functions of two or more variables.

Let $u=0$ be an equation containing three variables x, y, z. Either of them may be considered as a function of the other two; z, for instance, as a function of x and y. Let us propose to find the values of the partial differential coefficients of z.

This will be very easy; for if z were expressed by an explicit function of x and y, to determine $\frac{dz}{dx}$ we should consider y as a constant in that function, and then differentiate it as a function of x alone. Hence, in the present case, we shall first suppose that z and x are the only variables in u, and we shall have by (42)

$$\frac{du}{dx} + \frac{du}{dz}\frac{dz}{dx} = 0 \ldots (a).$$

Equation which will give the value of $\frac{dz}{dx}$.

Secondly. We shall consider x as a constant, and we shall have to determine the value of $\frac{dz}{dy}$ the equation

$$\frac{du}{dy} + \frac{du}{dz}\frac{dz}{dy} = 0 \ldots (b).$$

The research of the values of $\frac{d^2z}{dx^2}$, $\frac{d^2z}{dy\,dx}$, $\frac{d^2z}{dy^2}$ will present no difficulty. To find the first we shall

take, as in (43), the differential coefficient of the left side of (a), y being still supposed to be a constant, but $\frac{dz}{dx}$ being considered as a variable, we shall have to determine $\frac{d^2z}{dx^2}$ the equation,

$$\frac{d^2u}{dx^2}+2\frac{d^2u}{dz\,dx}\frac{dz}{dx}+\frac{d^2u}{dz^2}\frac{dz^2}{dx^2}+\frac{du}{dz}\frac{d^2z}{dx^2}=0.$$

Operating upon (b) in a similar manner, x being then the constant, and y, z, $\frac{dz}{dy}$ the variables, we shall find for the equation which gives the value of $\frac{d^2z}{dy^2}$,

$$\frac{d^2u}{dy^2}+2\frac{d^2u}{dz\,dy}\frac{dz}{dy}+\frac{d^2u}{dz^2}\frac{dz^2}{dy^2}+\frac{du}{dy}\frac{d^2z}{dy^2}=0.$$

To obtain the equation upon which depends the value of $\frac{d^2z}{dx\,dy}$, we may take either the differential coefficient of the left side of (a) with respect to y, or the differential coefficient of the left side of (b) with respect to x. The two results will be found to be

$$\frac{d^2u}{dx\,dy}+\frac{d^2u}{dz\,dy}\frac{dz}{dx}+\frac{d^2u}{dz\,dx}\frac{dz}{dy}+\frac{d^2u}{dz^2}\frac{dz}{dx}\frac{dz}{dy}+\frac{du}{dz}\frac{d^2z}{dx\,dy}=0.$$

No farther explanation is required to understand how the partial differential coefficients of a superior order may be determined.

(52.) If instead of one equation between three variables x, y, z, we had m equations between $m+n$ variables, it is obvious that any m of them could be considered as implicit functions of the n remaining.

Let x, y, z, &c. represent the n independent variables, and x', y', z', t', &c. the m variables which are considered as functions of them. Each of the m equations may be differentiated in the supposition of x being the only independent variable, and lead to m equations involving the differential coefficients $\frac{dx'}{dx}$, $\frac{dy'}{dx}$, $\frac{dz'}{dx}$, $\frac{dt'}{dx}$, and sufficient to determine their values. The same operation may be repeated on the given equations, y being then considered as the independent variable, and lead to m new equations, by means of which the values of the m differential coefficients $\frac{dx'}{dy}$, $\frac{dy'}{dy}$, $\frac{dz'}{dy'}$, &c. may be obtained. In following the same process we shall find the mn partial differential coefficients of the first order. It will be sufficient to differentiate the mn preceding equations in a similar manner; and in considering the partial differential coefficients already involved in them, as new variable functions of x, y, z, to arrive at new equations which will give the partial differential coefficients of superior order.

These various equations would be verified, as well as the proposed equation, if for x', y', z', &c. the functions of x, y, z, which they represent, were substituted. Hence they may be combined in any way, and lead to new equations which will also be satisfied by the same values of x', y', z', &c. Consequently, constant or variable quantities may be eliminated between them.

The denominations of partial differential equations of the first, second order, &c., and the degree of a partial differential equation of a given order, can be easily understood from what has been said (45), and do not require any further explanation.

The elimination between partial differential equations, presents important results, which we shall now examine.

Let $u=0$ be an equation between three variables x, y, z, and let t denote a certain function of x and y, a function of which $f(t)=s$ is involved in u. So that if $t=\phi(x, y)$, u may be represented by $F(s, x, y, z)$, or by $F(f(\phi(x, y)), x, y, z)$. Hence, if we apply to the equation $u=0$ the rules for differentiating functions of functions, we shall have

$$\frac{du}{dx}+\frac{du}{dz}\frac{dz}{dx}+\frac{du}{ds}\cdot\frac{ds}{dt}\cdot\frac{dt}{dx}=0,$$

and

$$\frac{du}{dy}+\frac{du}{dz}\frac{dz}{dy}+\frac{du}{ds}\cdot\frac{ds}{dt}\cdot\frac{dt}{dy}=0.$$

These two equations contain s and $\frac{ds}{dt}$, therefore by combining them with the proposed equation $u=0$, the two quantities s and $\frac{ds}{dt}$ may be eliminated. The result will be a partial differential equation of the first order, not containing $s=f(t)$, and which therefore will be verified by the primitive equation $u=0$, whatever be the form of the function of t designated by f. Thus it appears, that by means of the two partial differential

equations derived from $u = 0$, it is always possible to eliminate a function of a certain function of x and y involved in u, and to obtain a relation between x, y, z, $\frac{dz}{dx}$, and $\frac{dz}{dy}$, true for every form which may be assigned to that function.

A similar reasoning would prove, that, generally, m arbitrary functions may be eliminated, with the assistance of the $m\,n$ partial differential equations of the first order derived from m equations between $m + n$ variables.

(53.) We cannot however infer, by analogy from what precedes, that a partial differential equation of the second order may, in all cases, be obtained, containing two arbitrary functions less than the primitive equation.

Let us suppose that the equation $u = 0$ between the three variables x, y, z, involves two functions s and s', the first of t and the other of t'. The two partial differential equations of the first order will contain the quantities $\frac{ds}{dt}$ and $\frac{ds'}{dt'}$. Differentiating again, we shall obtain three partial differential equations of the second order, in which will be found, in general, besides the two coefficients $\frac{d^2 s}{dt^2}$, $\frac{d^2 s'}{dt'^2}$, the quantities s, s', $\frac{ds}{dt}$ and $\frac{ds'}{dt'}$. Thus, to make s and s' disappear, we should eliminate these six quantities, between the primitive and the five differential equations, but this will be generally impossible. We should have recourse therefore to the partial differential equations of the fourth order. These will be four in number, and will only contain the two new arbitrary functions $\frac{d^3 s}{dt^3}$, and $\frac{d^3 s'}{dt'^3}$. We shall have, then, ten equations between eight arbitrary quantities; and consequently we shall be able to arrive at two partial differential equations of the fourth order, entirely independent of the arbitrary functions s and s'.

The same considerations will easily show what order of partial differential equations it is necessary to use, to eliminate any given number of arbitrary functions, and how many differential equations of that order may be obtained independent of those functions. In the case of m arbitrary functions to be eliminated from a given equation between three variables, it will be easy to see that the partial differential equations of the $(2m - 1)^{th}$ order must be used, and that m differential equations of that order may be obtained independent of those functions.

The order of differentiation indicated by the preceding observations, is the highest which can be required, to perform the elimination, but it may happen that such relations should exist between the terms of the proposed equations, that the arbitrary functions might be made to disappear without having recourse to it.

Let us take for first example the equation

$$z = (x + y)^n \phi (x^2 - y^2),$$

and let us represent the differential coefficient of $\phi (x^2 - y^2)$ taken with respect to the function between the parenthesis, considered as a variable, be denoted by $\phi' (x^2 - y^2)$. We shall have for the two partial differential equations of the first order,

$$\frac{dz}{dx} = n (x + y)^{n-1} \phi (x^2 - y^2) + 2x (x + y)^n \phi' (x^2 - y^2),$$

$$\frac{dz}{dy} = n (x + y)^{n-1} \phi (x^2 - y^2) - 2y (x + y)^n \phi' (x^2 - y^2).$$

Eliminating $\phi' (x^2 - y^2)$ between these two equations, we find

$$y \frac{dz}{dx} + x \frac{dz}{dy} = n (x + y)^n \phi (x^2 - y^2).$$

Substituting now for $\phi (x^2 \quad y^2)$ its value taken in the primitive equation, we shall have

$$y \frac{dz}{dx} + x \frac{dz}{dy} = n z,$$

for the partial differential equation of the first order, independent of the function ϕ, and expressing therefore a relation between x, y, z, $\frac{dz}{dx}$, $\frac{dz}{dy}$, verified by the equation $z = (x + y)^n \phi (x^2 - y^2)$, and by all those which differ only from it by the form of the function ϕ.

Example 2. Let the equation be

$$z = \frac{y^2}{2} + \phi\left(\frac{1}{x} + \log y\right).$$

Then $$\frac{dz}{dx} = -\phi'\left(\frac{1}{x} + \log y\right)\frac{1}{x^2}, \quad \frac{dz}{dy} = y + \phi'\left(\frac{1}{x} + \log y\right)\frac{1}{y}.$$

Eliminating $\phi'\left(\frac{1}{x} + \log y\right)$ between these two equations, we get

$$x^2 \frac{dz}{dx} + y \frac{dz}{dy} - y^2 = 0.$$

Example 3. Let the equation be $z = \phi(y + x) + x y \psi(y - x)$, containing the two functions $\phi(y + x)$ and $\psi(y - x)$, which are to be eliminated by means of the differential equations. To simplify, we shall write ϕ, ψ, ϕ', and ψ', &c., iostead of $\phi(y+x)$, $\psi(y-x)$, $\phi'(y+x)$, $\psi'(y-x)$, &c.

We shall find
$$\frac{dz}{dx} = \phi' + y\psi = xy\psi', \quad \frac{dz}{dy} = \phi' + x\psi + xy\psi'.$$

Between these two equatioos, and the primitive equation, we cannot eliminate the four quantities ϕ, ψ, ϕ', and ψ', and therefore we proceed to the partial differential coefficients of the second order. We shall have
$$\frac{d^2 z}{dx^2} = \phi'' - 2y\psi' + xy\psi'', \quad \frac{d^2 z}{dy^2} = \phi'' + 2x\psi' + xy\psi'',$$
$$\frac{d^2 z}{dx\,dy} = \phi'' + \psi + (y - x)\psi' - xy\psi''.$$

These three new equations contain two oew indeterminate functions ϕ'' and ψ'', so that we have six equations, and six quantities to eliminate, which is impossible. We shall therefore determine the partial differential coefficients of the third order. We get
$$\frac{d^3 z}{dx^3} = \phi''' + 3y\psi'' - xy\psi''', \quad \frac{d^3 z}{dx^2\,dy} = \phi''' - 2\psi' - (2y - x)\psi'' + xy\psi'''$$
$$\frac{d^3 z}{dy^2\,dx} = \phi''' + 2\psi' + (y - 2x)\psi'' - xy\psi''', \quad \frac{d^3 z}{dy^3} = \phi''' + 3x\psi'' + xy\psi'''.$$

We have now ten equations and only eight arbitrary functions ϕ, ϕ', ϕ'', ϕ''', ψ, ψ', ψ'', ψ''', therefore the elimination is possible, and will lead to two partial differential equations of the third order. The values of the differential coefficients of the third order give, by adding the two last and subtracting the two first,
$$\frac{d^3 z}{dy^3} + \frac{d^3 z}{dy^2\,dx} - \frac{d^3 z}{dx^2\,dy} - \frac{d^3 z}{dx^3} = 4\psi', \text{ but } \frac{d^2 z}{dy^2} - \frac{d^2 z}{dx^2} = 2(x + y)\psi',$$
hence
$$2\left(\frac{d^2 z}{dy^2} - \frac{d^2 z}{dx^2}\right) - (x + y)\left(\frac{d^3 z}{dy^3} + \frac{d^3 z}{dy^2\,dx} - \frac{d^3 z}{dx^2\,dy} - \frac{d^3 z}{dx^3}\right) = 0.$$

The other equation, independent of the arbitrary functions, would be obtained in making use of the differential coefficients of the first order, but it is much more complicated thao that just obtained, and is useless to our present purpose.

Example 4. We shall take for the last example an equatioo containing two arbitrary functions, which will disappear io making use only of the partial differential coefficients of the second order.

Let
$$z = x\phi\left(\frac{y}{x}\right) + \psi\left(\frac{y}{x}\right),$$
we shall have in writing again ϕ and ψ instead of $\phi\left(\frac{y}{x}\right)$ and $\psi\left(\frac{y}{x}\right)$,
$$\frac{dz}{dx} = \phi - \frac{y}{x}\phi' - \frac{y}{x^2}\psi', \quad \frac{dz}{dy} = \phi' + \frac{1}{x}\psi';$$
ϕ' and ψ' may be eliminated at the same time from these two equations, but multiplying the second by $\frac{y}{x}$, and then adding. We find thus
$$\frac{dz}{dx} + \frac{y}{x}\frac{dz}{dy} = \phi.$$

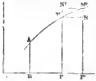
Taking the partial differential coefficients of this equation, first with respect to x, and then with respect to y, we shall have
$$\frac{d^2 z}{x} + \frac{y}{x}\frac{d^2 z}{dy\,dx} - \frac{y}{x^2}\frac{dz}{dy} = -\frac{y}{x^2}\phi',$$
$$\frac{d^2 z}{dx\,dy} + \frac{y}{x}\frac{d^2 z}{dy^2} - \frac{1}{x}\frac{dz}{dy} = \frac{1}{x}\phi',$$
multiplyiog the last eqoation by $\frac{y}{x}$, and adding it with the first, we shall eliminate ϕ', and find
$$x^2\frac{d^2 z}{dx^2} + 2xy\frac{d^2 z}{dx\,dy} + y^2\frac{d^2 z}{dy^2} = 0.$$

A partial differential equation of the second order, which is satisfied by the equation $z = x\phi\left(\frac{y}{x}\right) + \psi\left(\frac{y}{x}\right)$ whatever be the forms of the functions ϕ and ψ.

(54.) We have investigated the rules to determine the values of the differential eoefficients of every giveo explicit or implicit function of one or more variables. To complete the subject, it remains only to show how, io some cases, the valoe of the differential coefficients may be determined, although the relatioo of the function to the variables is not known either explicitly, or by unresolved equations which connect them together.

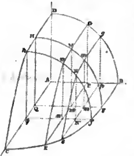
This can be done in various cases by means of some circumstances which cannot be expressed analytically, and which, without any knowledge of the nature of the function, allow us to determine the limit of the ratio between its difference and the difference of the variable. We have had already an example of this method, in the manner which we have used to find the differential coefficient of sin x.

Let us propose, for another example, to investigate the value of the differential coefficient of the area A B M P, included between the axis of the abscissæ, two ordinates A B, M P, and a curve A M.



This area is clearly a function of the abscissa $OP = x$, since the point A being supposed a given point, the value of A B M P will be determined for every value assigned to x. Let the ordinate be called y, and let $y = f(x)$ be the equation of the curve. If we suppose $P P' = h$, and if we change x into $x + h$, the unknown function of x that is represented by A B M P will become A B M′ P′, and the difference of the function will be M M′ P P′. Let us draw the two lines M N, M′ N′ parallel to O P′, it is obvious, that by taking h sufficiently small, the area P M P′ M′ may always be considered as greater than the rectangular parallelogram P M N P′, and less than P P′ N′ M′. Therefore the ratio of the difference of the unknown function to the difference of the variable, that is, $\frac{P P' M M'}{P P'}$, is greater than $\frac{P P' M N}{P P'}$ and less than $\frac{P P' M' N'}{P P'}$. But $\frac{P P' M N}{P P'} = M P = y$, and $\frac{P P' M' N'}{P P'}$ $= M' P'$, which is the value of the ordinate corresponding to the abscissa $x + h$, and consequently equal to $y + \frac{d y}{d x} h + \frac{d^2 y}{d x^2} \frac{h^2}{1.2} + \&c.$, the limit of which, with respect to decreasing values of h, is y. Hence the ratio of the difference of the function A B M P, to the difference of the variable, is included between two quantities, one of which is y, and the other has for its limit y; consequently the limit of that ratio, or the differential coefficient of the area, is also equal to y.

We may apply the same method to a function of two variables. Let D A B, D A C, C A B be three coordinate planes, cut by a curve surface D C B H G M E F, whose equation is $z = f(x, y)$. If by any point M of that surface, whose coordinates M M′, M′ P, A P, are respectively z, y, x, two planes are drawn, F M H Q and E M G P, parallel to the coordinate planes D A B, D A C, they will form a solid D H M G A P M Q, whose volume is clearly a function of x and y. Let u be that unknown function, and let it be required to find the value of $\frac{d^2 u}{d x d y}$. Let $P p = h$ and $Q q = k$, and by the points p and q let planes be drawn parallel to D A C and D A B, and meeting in N N′. If in the function u we change x into $x + h$, it becomes D H m g A Q m' p $= u + \frac{d u}{d x} h + \frac{d^2 u}{d x^2} \frac{h^2}{1.2} + \&c.$, and the partial difference of u with respect to x is M m M′ m' G g P p $= \frac{d u}{d x} h + \frac{d^2 u}{d x^2} \frac{h^2}{1.2} + \&c.$ Let us change in this difference y into $y + k$, it will become N n N′ n' G g P p, and will have for its own difference M N m n M′ N′ m' n', the first term of the



developement of which will clearly be $\frac{d^2 u}{d x d y} h k$. But we may easily see that the first term of the expression of M N m n M′ N′ m' n' is also $z h k$. For $M M' = z$,

$$m m' = z + \frac{d z}{d x} h + \frac{d^2 z}{d x^2} \frac{h^2}{1.2}, \quad n n' = z + \frac{d z}{d y} k + \frac{d^2 z}{d x^2} \frac{k^2}{1.2} + \&c.,$$

$$N N' = z + \frac{d z}{d x} h + \frac{d z}{d y} k + \frac{d^2 z}{d x^2} \frac{h^2}{1.2} + \&c.$$

Consequently, if by the points M, N, m, n, we draw four planes parallel to the plane A B C, we shall form four rectangular parallelopipedons having the same base $M' N' m' n' = h k$, and the first terms of the expressions of the volumes of which will all be $z h k$. Now, it is obvious, that the parallelopipedons are always some greater and some less than the solid N N m n M′ N′ m' n', therefore the first term $\frac{d^2 u}{d x d y} k$ of the expression of the volume of that last solid, must be equal to the first term $z h k$, common to the expressions of the volumes of the four parallelopipedons. Hence $\frac{d^2 u}{d x d y} = z f(x, y)$.

(55.) It does not unfrequently happen that it becomes necessary to substitute for the differential coefficients of one or several functions, with respect to one or more variables, involved in a formula, the differential coefficients of the same function, with respect to other variables connected with the first by given relations.

Let us first suppose that the formula contains only the differential coefficients of y with respect to the variable x, and that it is required to substitute for them the differential coefficients of y with respect to another variable t, x being a function of t. The values of $\frac{d y}{d x}, \frac{d^2 y}{d x^2}$, &c., in terms of $\frac{d y}{d t}, \frac{d^2 y}{d t^2}$, &c., $\frac{d x}{d t}, \frac{d^2 x}{d t^2}$, may readily be formed, by means of what precedes.

We shall have first, by (24),

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}, \text{ and hence } \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}, \text{ from which}$$

we get

$$\frac{d^2y}{dx^2} = \frac{\frac{d^2y}{dt^2}\frac{dx}{dt} - \frac{d^2x}{dt^2} \cdot \frac{dy}{dt}}{\frac{dx^3}{dt^3}};$$

and, by taking again the differential coefficients of both sides of this equation with respect to t, we find

$$\frac{d^3y}{dx^3} = \frac{\frac{d^3y}{dt^3}\frac{dx^2}{dt^2} - 3\frac{d^2y}{dt^2}\frac{d^2x}{dt^2}\frac{dx}{dt} + 3\frac{dy}{dt}\frac{d^2x^2}{dt^4} - \frac{d^3x}{dt^3}\frac{dy}{dt}\frac{dx}{dt}}{\frac{dx^5}{dt^5}}$$

In a similar manner, the values of the differential coefficients of higher orders may be found.

If, in these formulæ, we suppose $t = y$, then

$$\frac{dy}{dt} = 1, \ \frac{d^2y}{dt^2} = 0 \quad \frac{d^3y}{dt^3} = 0;$$

and they become respectively,

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}, \ \frac{d^2y}{dx^2} = \frac{-\frac{d^2x}{dy^2}}{\frac{dx^3}{dy^3}}, \text{ and}$$

$$\frac{d^3y}{dx^3} = \frac{3\frac{d^2x^2}{dy^4} - \frac{d^3x}{dy^3}\frac{dx}{dy}}{\frac{dx^5}{dy^5}}$$

With these values we shall be able to transform any analytical expression involving the differential coefficients of y with respect to x, into another, in which they will be replaced by the differential coefficients of x with respect to y.

(56.) No greater difficulty will be found to change the differential coefficients of any number of functions y_1, y_2, y_3, &c., with respect to any number of variables x_1, x_2, x_3, &c., into the differential coefficients of the same functions with respect to an equal number of variables z_1, z_2, z_3, &c., connected with the first by as many equations as there are variables. The first partial differential coefficients will be as before,

$$\frac{dy_1}{dx_1} = \frac{\frac{dy_1}{dz_1}}{\frac{dx_1}{dz_1}}, \quad \frac{dy_1}{dx_2} = \frac{\frac{dy_1}{dz_2}}{\frac{dx_2}{dz_2}}, \ \&c.$$

$$\frac{dy_2}{dx_1} = \frac{\frac{dy_2}{dz_1}}{\frac{dx_1}{dz_1}}, \quad \frac{dy_2}{dx_2} = \frac{\frac{dy_2}{dz_2}}{\frac{dx_2}{dz_2}}, \ \&c.$$

And, taking the differential coefficients of each of those with respect to each of the new variables, the values of the partial differential coefficients of higher orders will be obtained.

We have now explained all the general rules of the Differential Calculus, and sufficiently illustrated the meaning of the notations which are used in it. We shall therefore proceed to the Integral Calculus, intending to show afterwards the application of both to analytical and geometrical investigations.

PART II

INTEGRAL CALCULUS.

Integral Calculus. (57.) We have before stated, that the Integral Calculus was the inverse of the Differential Calculus, and had for its object to determine the value of a function, the differential coefficient of which is known, or, more generally, to discover the relations which exist between the variables and the functions, from given equations between them and their differential coefficients. Part II.

(58.) We shall first consider the simplest case; which is, to find the value of a function of one variable, when the first differential coefficient is given explicitly in terms of that variable.

Let X be the given differential coefficient, and let y designate the unknown function, then $\frac{d\,y}{d\,x} = \mathrm{X}$, or $d\,y = \mathrm{X}\,d\,x$. The required function is generally represented by $\int \mathrm{X}\,d\,x$, the characteristic $\int$ denoting an operation precisely the inverse of that indicated by d in the differential calculus. Hence, if the two characteristics $\int$ and d were prefixed to the same function u, they would neutralize each other, and we would have $\int d\,u = u$. It follows also from (30) that $d^{-1}\,\mathrm{X}\,d\,x$ would signify the same thing as $\int \mathrm{X}\,d\,x$; and consequently that we might dispense with the use of a new sign. But as it is universally employed, we shall retain it here. In the sequel we shall have occasion to mention the origin of this notation, and also of the name *integral*, applied to $\int \mathrm{X}\,d\,x$, or to the function whose differential coefficient is X. The operation, by means of which the integral of a given differential is determined, is called *integration*. To *integrate* a differential, is to find the value of its integral.

These definitions and notations understood, we shall deduce without any difficulty from the observations and rules stated in the differential calculus, the following results.

(59.) If y represent a function of x, and if $d\,y = \mathrm{X}\,d\,x$, then, from (18), $\int \mathrm{X}\,d\,x = y + \mathrm{A}$, where A is an arbitrary constant. Hence we may always add to the integral of a given differential a constant quantity, whose value remains in general indeterminate. If however the value of the integral corresponding to a particular value of x, happen to be known, then the constant may be determined. Let us suppose, for instance, that we know that the integral becomes equal to B, when x is assumed equal to b. Then, if we designate by C the value of y corresponding to the same supposition, we must have $\mathrm{C} + \mathrm{A} = \mathrm{B}$, and consequently $\mathrm{A} = \mathrm{B} - \mathrm{C}$.

(60.) We shall also have, by (18), M being a constant,

$$\int \mathrm{M}\,\mathrm{X}\,d\,x = \mathrm{M} \int \mathrm{X}\,d\,x = \mathrm{M}\,y + \mathrm{A}.$$

Hence, when a constant factor multiplies a given differential function, it may be written out of the sign of integration.

(61.) Let y_1, y_2, y_3, &c. be functions of x, and $d\,y_1 = \mathrm{X}_1\,d\,x$, $d\,y_2 = \mathrm{X}_2\,d\,x$, $d\,y_3 = \mathrm{X}_3\,d\,x$, &c; then, by (21),

$$\int (\mathrm{X}_1\,d\,x + \mathrm{X}_2\,d\,x - \mathrm{X}_3\,d\,x) = \int \mathrm{X}_1\,d\,x + \int \mathrm{X}_2\,d\,x - \int \mathrm{X}_3\,d\,x = y_1 + y_2 - y_3 + \mathrm{A}.$$

Hence the integral of the sum, or difference, of the several differential functions of the same variable is equal to the sum or difference of the integrals of these differentials.

(62.) The rule given (22) to find the differential coefficient of the product of two functions of the same variable, will give

$$\int y_1\,\mathrm{X}_2\,d\,x = y_1\,y_2 - \int y_2\,\mathrm{X}_1\,d\,x, \text{ or } \int y_1\,d\,y_2 = y_1\,y_2 - \int y_2\,d\,y_1.$$

This result shows, that when the differential function may be decomposed into two factors y_1 and $\mathrm{X}_2\,d\,x$, and that the integral of one of these may be obtained, the integration of the proposed formula will depend upon that of another function equal to the product of the integral already found by the differential of the factor not yet integrated. This method is called *integration by parts*. We shall frequently have occasion to make use of it.

(63.) Each of the rules given in the differential calculus, to obtain the differential coefficients of the functions of one variable, we have examined, being inverted, will clearly lead to a corresponding rule of integration. In consequence, to avoid repetitions, we shall write down the values we have determined for the differential coefficients of the various species of functions, and opposite to each, the integral formula which is deduced from it. Thus we shall form the following *tableau*.

We have found $\frac{d\,a\,x^m}{d\,x} = m\,x^{m-1}$ ...... hence ...... $\int a\,x^m = \frac{a\,x^{m+1}}{m+1} + c$ .... (a).

$\frac{d\,a^x}{d\,x} = a^x\,l\,a$ .... $\int a^x\,d\,x = \frac{a^x}{l\,a} + c$ .... (b).

$\frac{d\,e^x}{d\,x} = e^x$ .... $\int e^x\,d\,x = e^x + c$ .... (c).

We have found

$$\frac{d\,\mathrm{L}\,x}{d\,x}=\frac{m}{x}\ \ldots\ldots \text{ hence } \ldots\ldots \int\frac{m\,d\,x}{x}=\mathrm{L}\,x+c \ldots\ldots (d).$$

$$\frac{d\,l\,x}{d\,x}=\frac{1}{x}\ \ldots\ldots \int\frac{d\,x}{x}=l\,x+c \ldots\ldots (e).$$

$$\frac{d\sin x}{d\,x}=\cos x \ldots\ldots \int\cos x\,d\,x=\sin x+c \ldots\ldots (f).$$

$$\frac{d\cos x}{d\,x}=-\sin x \ldots\ldots \int\sin x\,d\,x=-\cos x+c \ldots\ldots (g).$$

$$\frac{d\tan x}{d\,x}=\frac{1}{(\cos x)^2} \ldots\ldots \int\frac{d\,x}{(\cos x)^2}=\tan x+c \ldots\ldots (h).$$

$$\frac{d\cot x}{d\,x}=\frac{1}{(\sin x)^2} \ldots\ldots \int\frac{d\,x}{(\sin x)^2}=\cot x+c \ldots\ldots (i).$$

$$\frac{d\sec x}{d\,x}=\tan x\sec x \ldots\ldots \int\tan x\sec x\,d\,x=\sec x+c \ldots\ldots (k).$$

$$\frac{d\,\mathrm{cosec}\,x}{d\,x}=-\cot x\,\mathrm{cosec}\,x \ldots\ldots \int\cot x\,\mathrm{cosec}\,x\,d\,x=-\mathrm{cosec}\,x+c \ldots (l).$$

$$\frac{d\sin^{-1}x}{d\,x}=\frac{1}{\sqrt{(1-x^2)}} \ldots\ldots \int\frac{d\,x}{\sqrt{(1-x^2)}}=\sin^{-1}x+c \ldots\ldots (m).$$

$$\frac{d\cos^{-1}x}{d\,x}=\frac{-1}{\sqrt{(1-x^2)}} \ldots\ldots \int\frac{-d\,x}{\sqrt{(1-x^2)}}=\cos^{-1}x+c \ldots\ldots (n).$$

$$\frac{d\tan^{-1}x}{d\,x}=\frac{1}{1+x^2} \ldots\ldots \int\frac{d\,x}{1+x^2}=\tan^{-1}x+c \ldots\ldots (o).$$

$$\frac{d\cot^{-1}x}{d\,x}=\frac{-1}{1+x^2} \ldots\ldots \int\frac{-d\,x}{1+x^2}=\cot^{-1}x+c \ldots\ldots (p).$$

$$\frac{d\sec^{-1}x}{d\,x}=\frac{1}{x\sqrt{(x^2-1)}} \ldots\ldots \int\frac{d\,x}{x\sqrt{(x^2-1)}}=\sec^{-1}x+c \ldots\ldots (q)$$

$$\frac{d\,\mathrm{cosec}^{-1}x}{d\,x}=\frac{-1}{x\sqrt{(x^2-1)}} \ldots\ldots \int\frac{-d\,x}{x\sqrt{(x^2-1)}}=\mathrm{cosec}^{-1}x+c \ldots\ldots (r).$$

(64.) Each of these formulæ is the analytical expression of a rule of integration. The first, which is one of the most important, shows, that the integral of the m^{th} power of a variable multiplied by the differential of that variable, is obtained by increasing the exponent by one, then dividing by the new exponent and by the differential of the variable, and adding an arbitrary constant to the result. This rule does not apply when $m=-1$; that is, to the integration of $\frac{d\,x}{x}$, but the formula (e) gives the integral in that case.

(65.) Whenever we are able by some transformation to change the formula $\mathrm{X}\,d\,x$ into one of the preceding, the value of $\int\mathrm{X}\,d\,x$ will become known. Hence our object now must be to examine successively the various forms X may have, and to endeavour to reduce each of them to one of those we already know how to integrate.

(66.) When X is a rational and integral algebraical function of x, its most general form is

$$\mathrm{A}\,x^a+\mathrm{B}\,x^b+\mathrm{C}\,x^c+\ldots\ldots+\mathrm{T},$$

a, b, c, &c. being positive integers. By comparing, in that supposition, $\mathrm{X}\,d\,x$ with the formula (a) of the preceding paragraph, we shall have, obviously,

$$\int(\mathrm{A}\,x^a+\mathrm{B}\,x^b+\mathrm{C}\,x^c+\ldots\ldots+\mathrm{T})\,d\,x=\frac{\mathrm{A}\,x^{a+1}}{a+1}+\frac{\mathrm{B}\,x^{b+1}}{b+1}+\frac{\mathrm{C}\,x^{c+1}}{c+1}+\ldots\ldots+\mathrm{T}\,x+\mathrm{V},$$

V being the arbitrary constant.

It is not necessary that a, b, c, &c. should be positive integers; they might be negative or fractional, and the integral would still be obtained in the same manner, except in the case in which one of the terms of X should be of the form $\frac{s}{x}$, and then the corresponding term of the integral would be $s\,l\,x$ by (e), (63).

The same mode of integration succeeds when $\mathrm{X}=(\mathrm{A}+\mathrm{B}\,x)^a$, or equal the sum of terms similar to that. First, if a be a positive integer, the binomial $(\mathrm{A}+\mathrm{B}\,x)^a$ may be developed into a finite series of terms of the form $\mathrm{M}\,x^m$, and then $\mathrm{X}\,d\,x$ may be integrated as above. But we may obtain the integral in a simpler manner, which has, besides, the advantage of being applicable whatever be the value of a. Assume $(\mathrm{A}+\mathrm{B}\,x)=y$, then $d\,x=\frac{d\,y}{\mathrm{B}}$, and $(\mathrm{A}+\mathrm{B}\,x)^a=y^a$; therefore $\mathrm{X}\,d\,x=\frac{y^a\,d\,y}{\mathrm{B}}$, and $\int\mathrm{X}\,d\,x=\frac{y^{a+1}}{\mathrm{B}\,(a+1)}+c$, substituting now

for y its value, we shall find $\int X\,d\,x = \int (A + B\,x)^a\,d\,x = \frac{(A + B\,x)^{a+1}}{B\,(a + 1)} + c.$
In the case of $a = -1$, the same transformation gives

$$\int \frac{d\,x}{(A + B\,x)} = \frac{l\,(A + B\,x)}{B} + c = l\,(A + B\,x)^{\frac{1}{B}} + c.$$

This transformation will succeed, again, when $X = (A + B\,x^b)^a\,x^{b-1}\,d\,x$, whatever be the exponents a and b. We shall assume $A + B\,x^b = y$, then $(A + B\,x^b)^a = y^a$, $x^{b-1}\,d\,x = \frac{d\,y}{B}$; therefore $X\,d\,x = \frac{y^a\,d\,y}{B}$, and $\int X\,d\,x = \frac{y^{a+1}}{B\,(a + 1)} + c$; or, substituting for y its value,

$$\int X\,d\,x = \int (A + B\,x^b)^a\,x^{b-1}\,d\,x = \frac{(A + B\,x^b)^{a+1}}{B\,(a + 1)} + c;$$

and when $a = -1$,

$$\int \frac{x^{b-1}\,d\,x}{(A + B\,x^b)} = \frac{l\,(A + B\,x^b)}{B} + c = l\,(A + B\,x^b)^{\frac{1}{B}} + c.$$

(67.) Let us next consider the case in which the function X is a rational, but fractional function. Its most general form will then be

$$\frac{A\,x^{n-1} + B\,x^{n-2} + C\,x^{n-3} + \dots\dots\dots T}{x^n + A'\,x^{n-1} + B'\,x^{n-2} + \dots\dots\dots\dots T'}.$$

The denominator of this expression may always be put under the following form:

$$(x-a)\,(x-b)\,\&c.\dots\dots \times (x-a')^p\,(x-b')^q\,\&c.\dots\dots \times (x^2 - 2\,\alpha\,x + \alpha^2 + \beta^2)\,\&c.\dots\dots \times (x^2 - 2\,\alpha'\,x + \alpha'^2 + \beta'^2)^r\,\&c.$$

If we suppose, that by resolving the equation

$$x^n + A'\,x^{n-1} + B'\,x^{n-2} + \&c.\dots\dots + T' = 0,$$

we have found it had the unequal roots a, b, &c., p roots equal to a', q roots equal to b', &c., a pair of imaginary roots equal to $\alpha \pm \beta\,\sqrt{-1}$, &c., and r pairs of imaginary roots equal to $\alpha' \pm \beta'\,\sqrt{-1}$.

The denominator of the fraction being so decomposed into factors, we may transform the proposed function into the sum of the following simple fractions:

$$\begin{aligned}
&\frac{N}{x-a} + \frac{N_1}{x-b} + \&c. \\
&+ \frac{P}{(x-a')^p} + \frac{P_1}{(x-a')^{p-1}} + \dots\dots\dots\dots + \frac{P_{p-1}}{(x-a')}, \\
&+ \frac{Q}{(x-b')^q} + \frac{Q_1}{(x-b')^{q-1}} + \dots\dots\dots\dots + \frac{Q_{q-1}}{(x-b')}, \\
&+ \&c. \\
&+ \frac{K\,x + L}{x^2 - 2\,\alpha\,x + \alpha^2 + \beta^2} \\
&+ \&c. \\
&+ \frac{R\,x + S}{(x^2 - 2\,\alpha'\,x + \alpha'^2 + \beta'^2)^r} + \frac{R_1\,x + S_1}{(x^2 - 2\,\alpha'\,x + \alpha'^2 + \beta'^2)^{r-1}} + \dots\dots + \frac{R_{r-1}\,x + S_{r-1}}{x^2 - 2\,\alpha'\,x + \alpha'^2 + \beta'^2}, \\
&+ \&c.
\end{aligned}$$

N, N', ... R_{r-1}, S_{r-1}, being constant quantities. To determine their values we should bring all these fractions to the same denominator, which will obviously be the denominator of the proposed fraction. Then the sum of their numerators must be equal to the numerator of X; and as this equality must subsist independently of any particular value assigned to x, the coefficients of the same powers of that variable in both quantities must be equal. This will furnish precisely the same number of equations as there are unknown quantities. For it is easy to see, that n being the degree of the denominator of X, $n - 1$ will be the degree of the numerator of the sum of the fractions $\frac{N}{x-a}$, $\frac{N_1}{x-b}$, &c., and n the number of the unknown quantities. It is clear, moreover, that these last quantities will enter the equations only in the first degree, and, consequently, that their values will be real, and that they may always be assigned. Therefore the transformation of the proposed fraction, indicated above, may always take place, and the difficulty of its integration is reduced to that of the four following formulæ,

$$\frac{N\,d\,x}{x-a},\quad \frac{P\,d\,x}{(x-a')^p},\quad \frac{(K\,x + L)\,d\,x}{x^2 + 2\,\alpha\,x + \alpha^2 + \beta^2},\quad \frac{(R\,x + S)\,d\,x}{(x^2 - 2\,\alpha'\,x + \alpha'^2 + \beta'^2)^r},$$

which include all the forms of the fractions in which it is decomposed.

(68.) The determination of the values of the numerators N, N_1, &c. of the partial fractions, by the method we have explained, will, in general, be very laborious. It may be simplified in making use of the differential calculus.

Let us propose to find the numerators of the fractions

$$\frac{P}{(x-a')^p},\ \frac{P_1}{(x-a')^{p-1}}\ \ldots\ldots\ldots\ldots\ \frac{P_{p-1}}{x-a'},$$

correspondleg to the factor $(x - a')^p$. All that we shall say will apply to the numerators of the simple factors $x - a$, $x - b$, &c. in supposing $p = 1$. To simplify, let us represent the numerator of X by U, and its denominator by V; then we shall have

$$\frac{U}{V} = \frac{P}{(x-a')^p} + \frac{P_1}{(x-a')^{p-1}} + \ldots\ldots\ \frac{P_{p-1}}{x-a'} + \frac{U_p}{Q},$$

$\frac{U_p}{Q}$ beleg the sum of all the other partial fractions. Multiplying both sides of this equation by Q, and observing that $V = Q\,(x - a')^p$, we find

$$U_p = \frac{Q\left\{\frac{U}{Q} - P - P_1\,(x-a') - P_2\,(x-a')^2 \ldots\ldots\ldots - P_{p-1}\,(x-a')^{p-1}\right\}}{(x-a')^p}.$$

The part of the numerator of this expression contained within the parenthesis, since Q is not divisible by $(x - a')$, must be divisible by $(x - a')^p$. It may therefore be represented by $y\,(x - a')^p$, y being an integral function of x. Hence all the differential coefficients of that quantity, till that of the $(p - 1)^{th}$ order inclusively, must become $= 0$, when $x = a'$. We shall heve therefore in denoting by $\frac{u}{q}$, $\frac{d\,\frac{u}{q}}{d\,x}$, $\frac{d^2\,\frac{u}{q}}{d\,x^2}$, &c., the values assumed by $\frac{U}{Q}$, and its differential coefficients when a' is substituted for x

$$P = \frac{u}{q},\ P_1 = \frac{d\,\frac{u}{q}}{d\,x},\ P_2 = \frac{1}{1\,.\,2}\ \frac{d^2\,\frac{u}{q}}{d\,x^2},\ P_3 = \frac{1}{1\,.\,2\,.\,3}\ \frac{d^3\,\frac{u}{q}}{d\,x^3},\ \text{\&c.}$$

The values of Q, $\frac{d\,Q}{d\,x}$, $\frac{d^2\,Q}{d\,x^2}$, and for $x = a'$, may even be derived from the values of the differential coefficients of V of the p^{th} and following orders, corresponding to the same hypothesis, in nsing the relation $V = Q\,(x - a')^p$. So that the determination of P, P_1, &c. may he made to depend upon the differential coefficients of the numerator U, and denominator V of the proposed fraction.

(69.) To find the values of the numerators of the fractions corresponding to the imaginary roots, we shall proceed nearly in the same manner. In that case, we have

$$\frac{U}{V} = \frac{R\,x + S}{(x^2 - 2\,a'\,x + a'^2 + \beta''^2)^r} + \frac{R_1\,x + S_1}{(x^2 - 2\,a'\,x + a'^2 + \beta''^2)^{r-1}} + \ldots\ldots\ \frac{R_{r-1}\,x + S_{r-1}}{x^2 - 2\,a'\,x + a'^2 + \beta'^2} + \frac{U\,r}{Q}.$$

Reducing to the same denominator, and observing that $V = Q\,(x^2 - 2\,a'\,x + a'^2 + \beta''^2)^r$, we get

$$U_r = \frac{Q\left\{\frac{U}{Q} - (R\,x + S) - (R_1\,x + S_1)\ (x^2 - 2\,a'\,x + a'^2 + \beta'^2) - \text{\&c.}\ldots\ldots\right\}}{(x^2 - 2\,a'\,x + a'^2 + \beta''^2)^r}.$$

The part of the numerator of this expression between the parenthesis, must be divisible by $(x^2 - 2\,a'\,x + a''^2 + \beta'^2)^r$; if therefore we suppose it equal to W, we must have

$$W,\ \frac{d\,W}{d\,x},\ \frac{d^2\,W}{d\,x^2},\ldots\ldots\ldots\ldots\ \frac{d^{r-1}\,W}{d\,x^{r-1}},$$

equal zero when x is one of the values which makes $x^2 - 2\,a'\,x + a'^2 + \beta^2 = 0$.

By substituting for them W, and its differential coefficients, each of these quantities will assume two forms, such as $G + H\,\sqrt{-1}$, and $G - H\,\sqrt{-1}$, which cannot be both equal to zero, unless $G = 0$ and $H = 0$. Hence we shall have just as many equations as there are quantities to determine.

(70.) Let us now examine the four formulæ, to the integration of which may be reduced that of any rational fractional function, as we heve seen (67).

To integrate the first $\frac{N\,d\,x}{x - a}$, it is sufficient to observe that the numerator is equal to a constant N multiplied by the differential of the denominator, therefore by (e) (63),

$$\int\frac{N\,d\,x}{x-a} = N\,l\,(x-a) + c = l\,(x-a)^N + c.$$

(71.) The second formula, $\frac{P\,d\,x}{(x - a')^p}$ is also integrated immediately. Assuming $x - a' = z$, then $d\,x = d\,z$.

Substituting, we find $\frac{P\,d\,z}{z^p} = P\,z^{-p}\,d\,z$; hence, by (a) (63), $\int \frac{P\,d\,z}{z^p} = \frac{-P}{(p-1)\,z^{p-1}} + c$; and, putting again for z its value,

$$\int \frac{P\,d\,x}{(x-a')^p} = \frac{-P}{(p-1)\,(x-a')^{p-1}} + c.$$

(72.) To find the integral of the third, $\frac{(K\,x + L)\,d\,x}{x^2 - 2\,a\,x + a^2 + \beta^2}$. We assume $x - a = z$, then $d\,x = d\,z$, and substituting, the formula becomes $\frac{(K\,z + K\,a + L)\,d\,z}{z^2 + \beta^2}$. This may be resolved into the two following,

$$\frac{K\,z\,d\,z}{z^2 + \beta^2} \text{ and } \frac{(K\,a + L)\,d\,z}{z^2 + \beta^2},$$

which may be written

$$\frac{K}{2} \cdot \frac{2\,z\,d\,z}{z^2 + \beta^2} \text{ and } \frac{K\,a + L}{\beta} \frac{d\left(\frac{z}{\beta}\right)}{1 + \frac{z^2}{\beta^2}}.$$

The first is equal to a constant multiplied by a fraction, the numerator of which is the differential of the denominator. Hence we shall have by (c) (63),

$$\int \frac{K}{2} \frac{2\,z\,d\,z}{z^2 + \beta^2} = \frac{K}{2}\,l\,(z^2 + \beta^2) + c.$$

The second is equal to a constant multiplied by a formula; which, being compared to (0) (63), gives

$$\int \frac{K\,a + L}{\beta} \cdot \frac{d\left(\frac{z}{\beta}\right)}{1 + \frac{z^2}{\beta^2}} = \frac{K\,a + L}{\beta} \tan^{-1}\frac{z}{\beta} + c.$$

Hence, by substituting for z its value, and adding the two results, we get

$$\int \frac{(K\,x + L)\,d\,x}{x^2 - 2\,a\,x + a^2 + \beta^2} = \frac{K}{2}\,l\,(x^2 - 2\,a\,x + a^2 + \beta^2) + \frac{K\,a + L}{\beta}\tan^{-1}\frac{x - a}{\beta} + c.$$

(73.) To integrate the fourth formula $\frac{(R\,x + S)\,d\,x}{(x^2 - 2\,a'\,x + a'^2 + \beta'^2)^r}$, we shall use the same transformation as for the preceding; it will then assume the form

$$\frac{(R\,z + R\,\beta' + S)\,d\,z}{(z^2 + \beta'^2)^r},$$

which may also be resolved into the two

$$\frac{R\,z\,d\,z}{(z^2 + \beta'^2)^r}, \quad \frac{(R\,\beta' + S)\,d\,z}{(z^2 + \beta'^2)^r}.$$

The first being written in the following manner, $\frac{R}{2}\,\frac{2\,z\,d\,z}{(z^2 + \beta'^2)^r}$, it is obvious that the numerator is the differential of the quantity, within the parenthesis, in the denominator.

Hence $$\int \frac{R}{2}\,\frac{2\,z\,d\,z}{(z^2 + \beta'^2)^r} = \frac{-R}{2\,(r-1)\,(z^2 + \beta'^2)^{r-1}}.$$

Since $(R\,\beta' + S)$ is a constant quantity, to find the integral of the second part, it will be sufficient to determine that of $\frac{d\,z}{(z^2 + \beta'^2)^r}$.

Assume $$\int \frac{d\,z}{(z^2 + \beta'^2)^r} = \frac{G\,z}{(z^2 + \beta'^2)^{r-1}} + \int \frac{H\,d\,z}{(z^2 + \beta'^2)^{r-1}}.$$

G and H being two indeterminate quantities. To find their values, take the differentials of both sides of this equation, then bringing all the terms to the same denominator, and dividing by $d\,z$, we shall find

$$1 = G\,(z^2 + \beta'^2) - 2\,(r-1)\,G\,z^2 + H\,(z^2 + \beta'^2).$$

The comparison of the terms containing the same powers of z, will give the two equations

$$1 = G\,\beta'^2 + H\,\beta'^2, \quad (3 - 2\,r)\,G + H = 0.$$

From which $$G = \frac{1}{(2\,r-2)\,\beta'^2} \text{ and } H = \frac{2\,r-3}{(2\,r-2)\,\beta'^2}.$$

Hence $$\int \frac{dz}{(z^2+\beta'^2)^r} = \frac{z}{(2r-2)\beta'^2(z^2+\beta'^2)^{r-1}} + \frac{2r-3}{2r-2}\int\frac{dz}{(z^2+\beta'^2)^{r-1}}.$$

With this formula we shall be able to obtain the value of $\int\frac{dz}{(z^2+\beta'^2)^r}$, if we can determine $\int\frac{dz}{(z^2+\beta'^2)^{r-1}}$, this might be made to depend, in a similar manner, on the integration of $\frac{dz}{(z^2+\beta'^2)^{r-2}}$, and the same process being pursued until the exponent of $(z^2+\beta'^2)$ shall be reduced to unity, we shall have, finally, to find the integral of $\frac{dz}{z^2+\beta'^2}$, which by (72) is equal to $\frac{1}{\beta'}\tan^{-1}\frac{z}{\beta'}$.

The value of $\int\frac{dz}{(z^2+\beta'^2)^r}$ being thus calculated, we shall add to it $\frac{R}{2(r-1)(z^2+\beta'^2)^{r-1}}$, and substituting then for z its value $x-a$, we shall have the integral of

$$\frac{(Rx+S)\,dx}{(x^2-2a'x+a'^2+\beta'^2)^r}.$$

The last transformation we have used, and by which the value of an integral is made to depend on another, must be noticed as being frequently employed, and often with success.

It follows from the preceding investigations, that every differential whose coefficient is a rational function of x, may always be integrated, and that the integral will be composed of rational functions, logarithms, and arcs of circle.

(74.) To illustrate the rules we have already given, we shall apply them to a few examples.

Example 1. Let $$X = \frac{1}{x^8+x^7-x^4-x^3}.$$

The factors of the denominator of this fraction are easily found. We have clearly

$$x^8+x^7-x^4-x^3 = x^3(x+1)(x^4-1) = x^3(x+1)^2(x-1)(x^2+1).$$

We shall therefore assume

$$\frac{1}{x^8+x^7-x^4-x^3} = \frac{N}{x-1} + \frac{P}{(x+1)^2} + \frac{P_1}{x+1} + \frac{R}{x^3} + \frac{R_1}{x^2} + \frac{R_2}{x} + \frac{kx+L}{x^2+1}.$$

To determine the values of the numerators N, P, &c. we shall also use the formulæ given (68.)

Let us first consider the numerator N, corresponding to the factor $x-1$.

In this case $U=1$, $Q=x^3(x+1)^2(x^2+1)$, therefore $N=\frac{u}{q}=\frac{1}{8}$.

To find P and P_1, we have $U=1$, $Q=x^3(x-1)(x^2+1)$, and $x=-1$. Hence

$$P=\frac{u}{q}=\frac{1}{4}, \qquad P_1=\frac{d\frac{u}{q}}{dx}=\frac{9}{8}.$$

To obtain the values of R, R_1, R_2, we must suppose $U=1$, $Q=(x-1)(x+1)^2(x^2+1)$, and $x=c$, and we get

$$R=\frac{u}{q}=-1, \qquad R_1=\frac{d\frac{u}{q}}{dx}=1, \qquad R_2=\frac{q}{1.2}\,\frac{d^2\frac{u}{q}}{dx^2}=-1.$$

Finally, for the numerator $kx+L$, we assume the proposed fraction equal to

$$\frac{Rx+L}{x^2+1}+\frac{U_1}{x^3(x+1)^2(x-1)}.$$

Hence the value of $$U_1=\frac{1-(Rx+L)x^3(x+1)^2(x-1)}{x^2+1},$$

the numerator of this expression must be divisible by x^2+1, and consequently should become nothing when $x^2+1=0$, or when $x=\pm\sqrt{-1}$. Hence the two equations $2R+2L=1$, and $R=L$, from which we get $R=\frac{1}{4}$, $L=\frac{1}{4}$.

Thus the differential $X\,dx$ is resolved into the following

$$\frac{1}{8}\frac{dx}{x-1}+\frac{1}{4}\frac{dx}{(x+1)^2}+\frac{9}{8}\frac{dx}{x+1}-\frac{dx}{x^3}+\frac{dx}{x^2}-\frac{dx}{x}+\frac{1}{4}\frac{(x+1)\,dx}{x^2+1}$$

The integrals of which are obtained without difficulty by (70), (71), (72), (73); and are respectively

$$\frac{1}{8}\,l\,(x-1),\quad \frac{-1}{4}\,\frac{1}{(x+1)},\quad \frac{9}{8}\,l\,(x+1),\quad \frac{1}{2x^2},\quad \frac{-1}{x},\quad -l\,x,\quad -\frac{1}{8}\,l\,(x^2+1),\quad \frac{-1}{4}\tan^{-1}x.$$

After reduction we shall find

$$\int\frac{dx}{x^8+x^7-x^4-x^3} = \frac{2-2x-5x^2}{4x^2(1+x)} + \frac{1}{8}\,l\left(\frac{x^2-1}{x^2+1}\right) + l\left(\frac{x+1}{x}\right) - \frac{1}{4}\tan^{-1}x + c.$$

Example 2. Let
$$X = \frac{1}{a+bx+cx^2} = \frac{1}{c\left(\frac{a}{c}+\frac{b}{c}x+x^2\right)}.$$

When b^2-4ac is positive, $\frac{a}{c}+\frac{b}{c}x+x^2$ may be decomposed into the two real factors

$$x+\frac{b+\sqrt{(b^2-4ac)}}{2c},\qquad x+\frac{b-\sqrt{(b^2-4ac)}}{2c},$$

and then we shall easily find

$$\frac{1}{a+bx+cx^2} = \frac{2c}{\sqrt{(b^2-4ac)}}\left\{\frac{-1}{2cx+b+\sqrt{(b^2-4ac)}}+\frac{1}{2cx+b-\sqrt{(b^2-4ac)}}\right\};$$

and hence
$$\int\frac{dx}{a+bx+cx^2} = \frac{1}{\sqrt{(b^2-4ac)}}\;l\;\frac{2cx+b-\sqrt{(b^2-4ac)}}{2cx+b+\sqrt{(b^2-4ac)}}.$$

But if b^2-4ac is negative, the factors of $\frac{a}{c}+\frac{b}{c}+x^2$ are imaginary; and instead of resolving the fraction into two others, it is preferable to assume $x+\frac{b}{2c}=z$, then $dx=dz$, and $\frac{a}{c}+\frac{bx}{c}+x^2=z^2+\frac{4ac-b^2}{4c^2}$ and we shall have

$$\frac{dx}{a+bx+cx^2} = \frac{dz}{c\left(z^2+\frac{4ac-b^2}{4c^2}\right)},$$

but
$$\int\frac{dz}{c\left(z^2+\frac{4ac-b^2}{4c^2}\right)} = \frac{2}{\sqrt{(4ac-b^2)}}\tan^{-1}\frac{2cz}{\sqrt{(4ac-b^2)}},$$

and by putting again for z its value

$$\int\frac{dx}{a+bx+cx^2} = \frac{2}{\sqrt{(4ac-b^2)}}\tan^{-1}\frac{2cx+b}{\sqrt{(4ac-b^2)}}.$$

Example 3. Let
$$X = \frac{x^m}{x^n+1}.$$

All the factors of the second degree of the denominator are included in the general form

$$x^2 - 2x\cos\frac{(2i+1)\pi}{n}+1,$$

in which i is an integer. Let us propose to find the numerator of the partial fraction corresponding to that denominator. If we represent it by $kx+L$, we shall have to determine k and L,

$$\frac{x^m}{x^n+1} = \frac{kx+L}{x^2-2x\cos\frac{(2i+1)\pi}{n}+1}+\frac{U_1}{Q},$$

and hence
$$U_1 = \frac{x^m-(kx+L)\,Q}{x^2-2x\cos\frac{(2i+1)\pi}{n}+1}.$$

The numerator of the value of U_1 must vanish when x is equal to one of the roots of the equation $x^2-2x\cos\frac{(2i+1)\pi}{n}+1=0$, that is to say, one of the two quantities $\cos\frac{(2i+1)\pi}{n}\pm\sqrt{-1}\,\sin\frac{(2i+1)\pi}{n}$.

The result of the substitution of the first of these two values in x^m will be

$$\cos\frac{m(2i+1)\pi}{n}+\sqrt{-1}\,\sin\frac{m(2i+1)\pi}{n}$$

but $x^n + 1 = Q\left(x^2 - 2x\cos\frac{(2i+1)\pi}{n} + 1\right)$, hence, taking the differential coefficients,

$$n x^{n-1} = Q\left(2x - 2\cos\frac{(2i+1)\pi}{n}\right) + \frac{dQ}{dx}\left(x^2 - 2x\cos\frac{(2i+1)\pi}{n} + 1\right).$$

Therefore the same value of x being put in Q will give

$$\frac{n\left\{\cos\frac{(n-1)(2i+1)\pi}{n} + \sqrt{-1}\sin\frac{(n-1)(2i+1)\pi}{n}\right\}}{2\sqrt{-1}\sin\frac{(2i+1)\pi}{n}},$$

and the numerator of the value U_1 will become by this substitution,

$$\cos\frac{m(2i+1)\pi}{n} + \sqrt{-1}\sin\frac{m(2i+1)\pi}{n}$$

$$-\frac{nk\left\{\cos\frac{n(2i+1)\pi}{n} + \sqrt{-1}\sin\frac{n(2i+1)\pi}{n}\right\}}{2\sqrt{-1}\sin\frac{(2i+1)\pi}{n}} - \frac{n\mathrm{L}\left\{\cos\frac{(n-1)(2i+1)\pi}{n} + \sqrt{-1}\sin\frac{(n-1)(2i+1)\pi}{n}\right\}}{2\sqrt{-1}\sin\frac{(2i+1)\pi}{n}}$$

This quantity must be equal to zero, as well as the result we would have obtained if we had put for x the other value $\cos\frac{(2i+1)\pi}{n} - \sqrt{-1}\sin\frac{(2i+1)\pi}{n}$ We shall in consequence obtain the two equations

$$\cos\frac{m(2i+1)\pi}{n} - \frac{\frac{n}{2}k\sin\frac{n(2i+1)\pi}{n}}{\sin\frac{(2i+1)\pi}{n}} - \frac{\frac{n\mathrm{L}}{2}\sin\frac{(n-1)(2i+1)\pi}{n}}{\sin\frac{(2i+1)\pi}{n}} = 0,$$

$$\sin\frac{m(2i+1)\pi}{n} + \frac{\frac{n}{2}k\cos\frac{n(2i+1)\pi}{n}}{\sin\frac{(2i+1)\pi}{n}} + \frac{\frac{n\mathrm{L}}{2}\cos\frac{(n-1)(2i+1)\pi}{n}}{\sin\frac{(2i+1)\pi}{n}} = 0.$$

From which we derive

$$k = \frac{2}{n}\cos\frac{(n-m-1)(2i+1)\pi}{n}, \quad \text{and } \mathrm{L} = -\frac{2}{n}\cos\frac{(n-m)(2i+1)\pi}{n}.$$

The partial fraction corresponding to the factor $x^2 - 2x\cos\frac{(2i+1)\pi}{n} + 1$, is therefore

$$\frac{\frac{2}{n}\left(x\cos\frac{(n-m-1)(2i+1)\pi}{n} - \cos\frac{(n-m)(2i+1)\pi}{n}\right)}{x^2 - 2x\cos\frac{(2i+1)\pi}{n} + 1}.$$

By comparing it with the fraction integrated (72), we shall have

$$\int \frac{2}{n}\,\frac{\left(x\cos\frac{(n-m-1)(2i+1)\pi}{n} - \cos\frac{(n-m)(2i+1)\pi}{n}\right)dx}{x^2 - 2x\cos\frac{(2i+1)\pi}{n} + 1} =$$

$$\frac{2}{n}\left\{\cos\frac{(n-m-1)(2i+1)\pi}{n}\, l\sqrt{\left(x^2 - 2x\cos\frac{(2i+1)\pi}{n} + 1\right)}\right.$$

$$\left. + \sin\frac{(n-m-1)(2i+1)\pi}{n}\tan^{-1}\frac{x - \cos\frac{(2i+1)\pi}{n}}{\sin\frac{(2i+1)\pi}{n}}\right\} + c,$$

adding to this formula the constant $\sin\frac{(n-m-1)(2i+1)\pi}{n}\tan^{-1}\frac{\cos\frac{(2i+1)\pi}{n}}{\sin\frac{(2i+1)\pi}{n}}$ it becomes

$$\frac{2}{n}\left\{\cos\frac{(n-m-1)(2i+1)\pi}{n}\, l\sqrt{\left(x^2-2x\cos\frac{(2i+1)\pi}{n}+1\right)}\right.$$

$$\left.+\sin\frac{(n-m-1)(2i+1)\pi}{n}\tan^{-1}\frac{x\sin\frac{(2i+1)\pi}{n}}{1-x\cos\frac{(2i+1)\pi}{n}}\right\}+c.$$

Let us first suppose n to be an even number, then by taking in this formula $i=0$, $i=1$, &c.... $i=\frac{n}{2}$, it will give the integrals of the $\frac{n}{2}$ partial fractions into which $\frac{x^m\,dx}{x^n+1}$, may be resolved; therefore, by adding these values we shall have

$$\int\frac{x^m\,dx}{x^n+1}=-\frac{2}{n}\cos\frac{(m+1)\pi}{n}\, l\sqrt{\left(x^2-2x\cos\frac{\pi}{n}-1\right)},$$

$$+\frac{2}{n}\sin\frac{(m+1)\pi}{n}\tan^{-1}\frac{x\sin\frac{\pi}{n}}{1-x\cos\frac{\pi}{n}}$$

$$-\frac{2}{n}\cos\frac{3(m+1)\pi}{n}\, l\sqrt{\left(x^2-2x\cos\frac{3\pi}{n}+1\right)},$$

$$+\frac{2}{n}\sin\frac{3(m+1)\pi}{n}\tan^{-1}\frac{x\sin\frac{3\pi}{n}}{1-x\cos\frac{3\pi}{n}},$$

$$-\frac{2}{n}\cos\frac{5(m+1)\pi}{n}\, l\sqrt{\left(x^2-2x\cos\frac{5\pi}{n}+1\right)},$$

$$+\frac{2}{n}\sin\frac{5(m+1)\pi}{n}\tan^{-1}\frac{x\sin\frac{5\pi}{n}}{1-x\cos\frac{5\pi}{n}},$$

&c.

This series being pursued to the terms corresponding to the value $i=\frac{n}{2}$.

When n is an odd number, the series must only be calculated for all the values of i, from $i=0$ to $i=\frac{n-1}{2}$; and it is necessary to add to it the integral corresponding to the real factor $x+1$ of the denominator, which it is easy to see is $\frac{(-1)^m\, l\,(x+1)}{n}$.

Similar steps would lead to the integral of $\frac{x^m\,dx}{x^n-1}$, and we would find

$$\int\frac{x^m\,dx}{x^n-1}=\frac{2}{n}\cos\frac{2(m+1)\pi}{n}\, l\sqrt{\left(x^2-2x\cos\frac{2\pi}{n}+1\right)}$$

$$-\frac{2}{n}\sin\frac{2(m+1)\pi}{n}\tan^{-1}\frac{x\sin\frac{2\pi}{n}}{1-x\cos\frac{2\pi}{n}}$$

$$+\frac{2}{n}\cos\frac{4(m+1)\pi}{n}\, l\sqrt{\left(x^2-2x\cos\frac{4\pi}{n}+1\right)},$$

$$-\frac{2}{n}\sin\frac{4(m+1)\pi}{n}\tan^{-1}\frac{x\sin\frac{4\pi}{n}}{1-x\cos\frac{4\pi}{n}},$$

$$\int \frac{x^m\,d\,x}{x^n - 1} = + \frac{2}{n} \cos \frac{6\,(m+1)\,\pi}{n}\; l \sqrt{\left(x^2 - 2\,x \cos \frac{6\,\pi}{n} + 1\right)},$$

$$- \frac{2}{n} \sin \frac{6\,(m+1)\,\pi}{n} \operatorname{tang}^{-1} \frac{x \sin \frac{6\,\pi}{n}}{1 - x \cos \frac{6\,\pi}{n}},$$

$$+ \text{\&c.}$$

This series being continued, if n is an even number, to $i = \frac{n-2}{2}$, and then adding to it the two terms $\frac{(-1)^{m+1}}{n}\, l\,(x+1)$, $\frac{1}{n}\, l\,(x-1)$, which are the integral of the partial fractions corresponding to the two real factors of the denominator of $\frac{x^m}{x^n - 1}$. When n is an odd number the series is carried no farther than the term corresponding to $i = \frac{n-1}{2}$, and $\frac{1}{n}\, l\,(x-1)$, which is the integral of the real factor of the denominator added to it.

Analogous formulæ might be obtained, in a similar way, for the integral of $\frac{x^m\,d\,x}{x^{2n} - 2\,a^n\,x^n \cos\phi + a^{2n}}$, since we know the general form of the real factors of the second degree of the denominator.

Having proved that the integral of $X\,d\,x$ may always be obtained whenever X is a rational function of x, any differential must be considered as integrated, when, by some transformation, we shall have been able to reduce it to a rational form.

No general rule can be given for these transformations, they depend on the form of X. We must therefore examine successively the few classes of functions, for which some means have been discovered to make them rational.

First, let
$$X = \frac{A\,x^{\frac{m}{n}} + B\,x^{\frac{p}{q}} + \text{\&c.}}{A'\,x^{\frac{m'}{n'}} + B'\,x^{\frac{p'}{q'}} + \text{\&c.}}.$$

If we reduce the fractional exponents of x to a common denominator N, it is obvious that in assuming $x = y^N$, X will become rational. But, then $d\,x = N\,y^{N-1}\,d\,y$, therefore $X\,d\,x$ will also be reduced to a rational form.

Secondly, let X equal a rational function of x, and of terms such as $(a + b\,x)^{\frac{m}{n}}$. If we suppose all the fractional exponents of $a + b\,x$ to be reduced to a common denominator N, and if we assume $a + b\,x = y^N$, we shall have $x = \frac{y^N - a}{b}$, and $d\,x = \frac{N\,y^{n-1}\,d\,y}{b}$; therefore, by substitution, $X\,d\,n$ will become a rational differential.

If instead of $(a + b\,x)$, in the preceding function, we had $\frac{a + b\,x}{a' + b'\,x}$, the same transformation would succeed. For if we suppose $\frac{a + b\,x}{a' + b'\,x} = y^n$, we find $x = \frac{a'\,y^n - x}{b - b'\,y^n}$ and $d\,x = n\,y^{n-1}\,\frac{\{a'\,(b' - b'\,y^n) + b'\,(a'\,y^n - a)\}}{(b - b'\,y^n)^2}\,d\,y$, which values being substituted in $X\,d\,x$ will reduce it to a rational form.

(75.) We shall next suppose X to be a rational function of x, and $\sqrt{(a + b\,x + c\,x^2)}$. In order to make it rational, we must distinguish two cases, that in which the roots of the equation $a + b\,x + c\,x^2 = 0$ are real, and that in which they are imaginary. In the first supposition $a + b\,x + c\,x^2$ may be decomposed into two real factors, which we shall represent by $p - q\,x$, and $p' - q'\,x$; then let us assume

$$a + b\,x + c\,x^2 = (p - q\,x)^2\,z^2.$$

We shall find in putting instead of $a + b\,x + c\,x^2$, the product $(p - q\,x)\,(p' - q'\,x)$,

$$x = \frac{p\,z^2 - q'}{q\,z^2 - q'}, \qquad d\,x = \frac{2\,(p'\,q - p\,q')\,z\,d\,z}{(q\,z^2 - q')^2}, \qquad \text{and } \sqrt{(a + b\,x + c\,x^2)} = \frac{(p'\,q - p\,q')\,z}{q\,z^2 - q'}.$$

Hence $X\,d\,x$ will be transformed into a rational differential function of z.

In the second case we shall suppose

$$a + b\,x + c\,x^2 = (x\,\sqrt{c} + z)^2,$$

which gives

$$x = \frac{z^2 - a}{b - 2z\sqrt{c}}, \quad dx = \frac{2(bz - z^2\sqrt{c} - a\sqrt{c})\,dz}{(b - 2z\sqrt{c})^2}, \quad \text{and } \sqrt{(a + bx + cx^2)} = \frac{bz - z^2\sqrt{c} - a\sqrt{c}}{b - 2z\sqrt{c}},$$

which values will reduce $X\,dx$ to a rational form. This transformation would also apply, although the factors of $a + bx + cx^2$ should not be imaginary, provided C be positive.

The same transformations will succeed to make the formula

$$x^{ir-1}\,dx\,(a + bx^r + cx^{2r})^{\frac{n}{2}},$$

in which i is any integer, positive or negative; for if we suppose $x^r = y$, we shall have $x^{ir-1}\,dx = \frac{y^{i-1}\,dy}{r}$,

it will become
$$\frac{1}{r}\,y^{i-1}\,dy\,(a + by + cy^2)^{\frac{n}{2}},$$

which may be made rational by the means used above.

(76.) The next class of irrational functions we shall examine is represented by the formula

$$x^{m-1}\,dx\,(a + bx^n)^{\frac{p}{q}}\,dx.$$

We shall observe, that without making it less general, we may always suppose m and n to be integers. For if they were fractional, they might be brought to a common denominator N; and by assuming $x = z^N$, the formula would be transformed into a similar one, in which the exponents corresponding to $m - 1$ and n would be integers. We may also consider n as positive; for in the contrary case, it would be sufficient to make $\frac{1}{x} = z$, and then the exponent of the variable between the parenthesis would become positive. The formula $x^{m-1}\,dx\,(ax^r + bx^n)^{\frac{p}{q}}$ is consequently included in the preceding, since it may be written in the following manner,

$$x^{m+\frac{pr}{q}-1}\,dx\left(a + bx^{n-r}\right)^{\frac{p}{q}}.$$

This understood, let $a + bx^n = y^q$, we shall have

$$(a + bx^n)^p = y^p, \qquad x^n = \frac{y^q - a}{b}, \qquad x^m = \left(\frac{y^q - a}{b}\right)^{\frac{m}{n}}, \quad \text{and } x^{m-1}\,dx = \frac{q}{nb}\,y^{q-1}\,(y^q - a)^{\frac{m}{n}-1}\,dy.$$

So that the formula $x^{m-1}\,dx\,(a + bx^n)^{\frac{p}{q}}$ will become

$$\frac{q}{nb}\,y^{p+q-1}\,d\left(\frac{y^q - a}{b}\right)^{\frac{m}{n}-1},$$

which is rational when $\frac{m}{n}$ is equal to an integer.

Secondly, let us assume in the same formula $a + bx^n = x^n z^q$. Then we shall find

$$x^n = \frac{a}{z^q - b}, \quad a + bx^n = \frac{az^q}{z^q - b}, \quad (a + bx^n)^{\frac{p}{q}} = \frac{a^{\frac{p}{q}}\,z^p}{(z^q - b)^{\frac{p}{q}}}, \quad x^m = \frac{a^{\frac{m}{n}}}{(z^q - b)^{\frac{m}{n}}},$$

$$x^{m-1}\,dx = -\frac{q}{n}\,a^{\frac{m}{n}}\,z^{q-1}\,(z^q - b)^{-\frac{m}{n}-1}.$$

These values being substituted, the proposed formula becomes

$$\frac{q}{n}\,a^{\frac{m}{n}+\frac{p}{q}}\,z^{p+q-1}\,dz\,(z^q - b)^{-\frac{m}{n}-\frac{p}{q}+1}$$

which is rational when $\frac{m}{n} + \frac{p}{q}$ is an integer

There are, therefore, two cases in which we shall be able to transform the binomial differential $x^{m-1}\,dx\,(a + bx^n)^{\frac{p}{q}}$ into a rational formula. They are the only two which have hitherto been assigned.

(77.) The formulæ

$$f\{x^m,\ (a + bx^n)^{\frac{p}{q}},\ (a + bx^n)^{\frac{r}{s}},\ (a + bx^n)^{\frac{t}{u}},\ \&\text{c.}\}\,x^{n-1}\,dx,$$

$$f\left\{x^m,\ \left(\frac{a + bx^n}{a' + b'x^n}\right)^{\frac{p}{q}},\ \left(\frac{a + bx^n}{a' + b'x^n}\right)^{\frac{r}{s}},\ \left(\frac{a + bx^n}{a' + b'x^n}\right)^{\frac{t}{u}}\right\}x^{n-1}\,dx,$$

in which f denotes a rational function of the quantities contained within the parenthesis, may also be made

rational by a simple transformation. For the first, it is sufficient to assume $a + b\,x^n = y^{qsu\,\&c.}$. From which we get

$$x^n = \frac{y^{qsu\,\&c.} - a}{b}, \qquad x^m = \left(\frac{y^{qsu\,\&c.} - a}{b}\right)^{\frac{m}{n}}, \qquad \text{and } x^{n-1}\,d\,x = \frac{q^{su\,\&c.}}{n\,b}\,y^{qsu\,\&c.-1}\,d\,y.$$

For the second we shall make $\dfrac{a + b\,x^n}{a' + b'\,x^n} = z^{qsu\,\&c.}$, and we shall find

$$x^n = \frac{a - a'\,z^{qsu\,\&c.}}{b'\,z^{qsu\,\&c.} - b}, \qquad x^m = \left(\frac{a - a\,z^{qsu\,\&c.}}{b'\,z^{qsu\,\&c.} - b}\right)^{\frac{m}{n}}, \qquad \text{and } x^{n-1}\,d\,x = \frac{-\,q\,s\,u\,\&c.\;z^{qsu\,\&c.-1}\,(a\,b' - a\,b)}{(b'\,z^{qsu\,\&c.} - b)^2},$$

which values, being substituted in the proposed formulæ, will make them rational.

(78.) There is no other irrational form of X, besides those already considered, for which general rules may be given to transform X $d\,x$ into a rational differential. However, for various particular cases not included in these forms, some transformations have been, or may be, found to succeed. Much practice in analysis enables us to foresee, without going through every intermediate step, the result of a substitution, and consequently indicates that which is most likely to lead to the sought result. In the examples which will be given of the integration of irrational functions, will be found two or three instances of such cases.

When X $d\,x$ cannot be made irrational, its integral may not unfrequently be made to depend upon that of a simpler formulæ. This, in many cases, is effected in using the integration by parts (62), which gives

$$\int y_1\,d\,y_2 = y_1\,y_2 - \int y_2\,d\,y_1 \ldots . (1).$$

(79.) We shall apply this method to the binomial differential, which, in order to simplify the calculation, we shall write

$$x^{m-1}\,d\,x\,(a + b\,x^n)^p,$$

p denoting then a fractional number.

Let $y_1 = (a + b\,x^n)^p$, and $d\,y_2 = x^{m-1}\,d\,x$, then $d\,y_1 = p\,n\,b\,x^{n-1}\,(a + b\,x^n)^{p-1}\,d\,x$, and $y_2 = \dfrac{x^m}{m}$. These values being substituted in the equation (1) will give

$$\int x^{m-1}\,d\,x\,(a + b\,x^n)^p = \frac{x^m\,(a + b\,x^n)^p}{m} - \frac{p\,n\,b}{m}\int x^{m+n-1}\,d\,x\,(a + b\,x^n)^{p-1} \ldots . (a).$$

A formula by means of which the integration of the proposed differential is made to depend on another, in which the exponent of the binomial is less by one, and the exponent of x out of the parenthesis increased by n.

Let us suppose now

$$y_1 = x^{m-n}, \qquad \text{and } d\,y_2 = x^{n-1}\,(a + b\,x^n)^p\,d\,x,$$

then

$$d\,y_1 = (m - n)\,x^{m-n-1}\,d\,x, \qquad \text{and } y_2 = \frac{(a + b\,x^n)^{p+1}}{n\,b\,(p + 1)},$$

We shall find

$$\int x^{m-1}\,d\,x\,(a + b\,x^n)^p = \frac{x^{m-n}\,(a + b\,x^n)^{p+1} - (m - n)\int x^{m-n-1}\,d\,x\,(a + b\,x^n)^{p+1}}{n\,b\,(p + 1)} \ldots . (b).$$

A formula presenting a reduction of the exponent of x without the parenthesis.

In order to obtain farther reductions, we shall observe that

$$(a + b\,x^n)^p = (a + b\,x^n)\,(a + b\,x^n)^{p-1} = a\,(a + b\,x^n)^{p-1} + b\,x^n(a + b\,x^n)^{p-1}.$$

Hence $\quad \int x^{m-1}\,d\,x\,(a + b\,x^n)^p = a\int x^{m-1}\,d\,x\,(a + b\,x^n)^{p-1} + b\int x^{m+n-1}\,(a + b\,x^n)^{p-1} \ldots . (c).$

This value being substituted in equation (a), gives, after reduction,

$$\int x^{m+n-1}\,d\,x\,(a + b\,x^n)^{p-1} = \frac{\dfrac{x^m}{m}\,(a + b\,x^n)^p - a\int x^{m-1}\,d\,x\,(a + b\,x^n)^{p-1}}{b + \dfrac{p\,n\,b}{m}} \ldots\ldots (d)$$

We may, in this, change p into $p + 1$, and $p - 1$ into p, m into $m - n$, and $m + n$ into m, and we find

$$\int x^{m-1}\,d\,x\,(a + b\,x^n)^p = \frac{x^{m-n}\,(a + b\,x^n)^{p+1} - a\,(m - n)\int x^{m-n-1}\,d\,x\,(a + b\,x^n)^p}{b\,(m + p\,n)} \ldots . (e).$$

Hence the integral of $x^{m-1}\,d\,x\,(a + b\,x^n)^p$ depends on that of $x^{m-n-1}\,d\,x\,(a + b\,x^n)^p$; and if we change in (e), successively m into $m - n$, $m - 2\,n$, &c. we shall have

$$\int x^{m-n-1}\,d\,x\,(a+b\,x^n)^p = \frac{x^{m-2n}\,(a+b\,x^n)^{p+1} - a\,(m-2\,n)\int x^{m-2n-1}\,d\,x\,(a+b\,x^n)^p}{b\,(m-n+n\,p)},$$

$$\int x^{m-2n-1}\,d\,x\,(a+b\,x^n)^p = \frac{x^{m-3n}\,(a+b\,x^n)^{p+1} - a\,(m-3\,n)\int x^{m-3n-1}\,d\,x\,(a+b\,x^n)^p}{b\,(m-2\,n+n\,p)},$$

....

and generally

$$\int x^{m-(i-1)n-1}\,d\,x\,(a+b\,x^n)^p = \frac{x^{m-in}\,(a+b\,x^n)^{p+1} - a\,(m-i\,n)\int x^{m-in-1}\,d\,x\,(a+b\,x^n)^p}{b\,(m-(i-1)\,n+n\,p)},$$

i being an integer.

If we substitute the first of these values in (e), then the second in the result of this first substitution, and so on, we shall find that $\int x^{m-1}\,d\,x\,(a+b\,x^n)^p$ depends on the value of $\int x^{m-in-1}\,d\,x\,(a+b\,x^n)^p$. The coefficient of this last integral is $m-i\,n$, therefore it will disappear in the result, when $\frac{m}{n}$ is an integer, and $i=\frac{m}{n}$. So that in that case, in which we have already proved that the integration of the binomial differential mey be effected, the above process will give us the general expression of the integral.

Let us now substitute in (c) the value of $\int x^{m+n-1}\,d\,x\,(a+b\,x^n)^{p-1}$ obtained in (d). We shall find

$$\int x^{m-1}\,d\,x\,(a+b\,x^n)^p = \frac{x^m\,(a+b\,x^n)^p + p\,n\,a\int x^{m-1}\,d\,x\,(a+b\,x^n)^{p-1}}{m+p\,n}\ldots\ldots(f).$$

Changing p into $p-1$, in this equation, we shall obtain

$$\int x^{m-1}\,d\,x\,(a+b\,x^n)^{p-1} \text{ by means of } \int x^{m-1}d\,x\,(a+b\,x^n)^{p-2},$$

and by each step we shall decrease the exponent of $(a+b\,x^n)$ by one, until it becomes less than one, if p be fractional, or equal zero, if p be an integer.

The formulæ (e) and (f) will thus enable us to reduce in every case the integration of

$$x^{m-1}\,d\,x\,(a+b\,x^n)^p \text{ to that of } x^{m-in-1}\,d\,x\,(a+b\,x^n)^{p-r},$$

$i\,n$ being the highest multiple of n contained in m, and r the greatest integer contained in p.

If the exponents m and p were negative, these formulæ would increase the exponents of the factors of the binomial differential instead of diminishing them; but it will be sufficient to invert (e) and (f) to obtain the formulæ answering to this case.

We derive from equation (e)

$$\int x^{m-n-1}\,d\,x\,(a+b\,x^n)^p = \frac{x^{m-n}\,(a+b\,x^n)^{p+1} - b\,(m+n\,p)\int x^{m-1}\,d\,x\,(a+b\,x^n)^p}{a\,(m-n)},$$

and if we change m into $-m+n$, we shall find

$$\int x^{-m-1}\,d\,x\,(a+b\,x^n)^p = \frac{x^{-m}\,(a+b\,x^n)^{p+1} - b\,(n-m+n\,p)\int x^{-m+n-1}\,d\,x\,(a+b\,x^n)^p}{-a\,m}\ldots(g).$$

Writing successively in this formula $-m+n$, $-m+2\,n\ldots\ldots-m+i\,n$, instead of m we shall find that $\int x^{-m-1}\,d\,x\,(a+b\,x^n)^p$ depends on $\int x^{-m+in-1}\,d\,x\,(a+b\,x^n)^p$.

When p is negative, we shall take the value of $\int x^{m-1}\,d\,x\,(a+b\,x^n)^{p-1}$ in (f), and we shall get

$$\int x^{m-1}\,d\,x\,(a+b\,x^n)^{p-1} = \frac{x^m\,(a+b\,x^n)^p - (m+n\,p)\int x^{m-1}\,d\,x\,(a+b\,x^n)^p}{a\,n\,p}.$$

Changing p into $-p+1$, we find

$$\int x^{m-1}\,d\,x\,(a+b\,x^n)^{-p} = \frac{x^m\,(a+b\,x^n)^{-p+1} - (m+n-n\,p)\int x^{m-1}\,d\,x\,(a+b\,x^n)^{-p+1}}{a\,n\,(p-1)}\ldots(h).$$

This formula, combined with the preceding (g), will reduce the integration of

$$x^{-m-1}\,d\,x\,(a+b\,x^n)^{-p} \text{ to that of } x^{-m+in-1}\,d\,x\,(a+b\,x^n)^{-p+r},$$

$i\,n$ being the highest multiple of n contained in m, and r the greatest integer contained in p.

(80.) There are some cases in which these various formulæ cannot be used, because their denominators become equal zero, but then the binomiel differentiels may be easily integrated.

When $m=0$, the denominator of (a) equals zero. In that case the binomial differential is reduced to $\int\frac{d\,x\,(a+b\,x^n)^p}{x}$ which becomes rational by assuming $a+b\,x^n$ equal z raised to a power equal to the denominator of p.

The denominator of (b) mey vanish by three different suppositions, when $n=0$, $b=0$, or $p=-1$. In the two first the binomial differenee becomes $x^{m-1}\,d\,x$ multiplied by a constant, and is therefore immediately integrated. In the third it is reduced to $\frac{x^{m-1}\,d\,x}{a+b\,x^n}$, that is to a rational fraction.

The supposition of $b = 0$, or $p = -\frac{m}{n}$ makes the denominator of (d) equal zero. We have already seen what becomes the differential in the first; in the second it is reduced to $x^{m-1}\, d\, x\, (a + b\, x)^{-\frac{m}{n}}$; and by assuming $a + b\, x^n = x^n\, z^n$, it is transformed into a rational function.

The hypotheses which make the denominators of the other formulæ (e), (f), (g), (h) vanish, are the same as those we have examined.

(81.) Similar reductions to those which have been effected in (79) upon the binomial differentials, may be also applied to some other functions. The integration of the general formula

$$x^{m-1}\, d\, x\, (a + b\, x^n + c\, x^{2n} + e\, x^{3n} + \&c.)^p,$$

may be made to depend on those

$$x^{m-n-1}\, d\, x\, X^p, \qquad x^{m-2n-1}\, d\, x\, X^p, \qquad x^{m-pn-1} d\, x\, X^p, \&c.$$

where X is equal to $a + b\, x^n + c\, x^{2n} + e\, x^{3n} + \&c.$

The steps of the calculation are entirely analogous to those used in (79.)

(82.) The expression $x^{m-1}\, d\, x\, (a + b\, x^n + c\, x^{2n})^p$ may sometimes be reduced to binomial differentials. Let us assume $x^n = y - \frac{b}{2\, c}$, it becomes $\frac{1}{n}\left(y - \frac{b}{2\, c}\right)^{\frac{m}{n}-1}\left(\frac{4\, a\, c - b^2}{4\, c} + c\, y^2\right)^p$, and, consequently, will be reduced to a limited number of binomial differentials, if $\frac{m}{n}$ be an integer. This will also be the case, if $\frac{-m}{n} - 2\, p$ be a positive integer. For the proposed formula may be written $x^{m+2np-1}\, d\, x\, (a\, x^{-2n} + b\, x^{-n} + c)^p$, and if we assume $x^{-n} = y - \frac{b}{2\, a}$, it will be changed into

$$\frac{-1}{n}\left(y - \frac{b}{2\, a}\right)^{\frac{-m}{n}-2p-1} d\, y\left(a\, y^2 + \frac{4\, a\, c - b^2}{4\, a}\right)^p.$$

(83.) The integration of X $d\, x$, where X is a rational function of x and $\sqrt{a + b\, x + c\, x^2 + d\, x^3 + e\, x^4}$, may be proved to depend on that of the three following formulæ,

$$\frac{d\, x}{\mathrm{R}}, \quad \frac{x^2\, d\, x}{\mathrm{R}}, \text{ and } \frac{d\, x}{(x^2 + a)\, \mathrm{R}};$$

R being equal to $\sqrt{\alpha + \beta\, x^2 + \gamma\, x^4}$.

We cannot give here the details of this investigation, which has been the object of the labours of Euler, Lagrange, and Legendre. We only mention it, to take the opportunity to observe, that with respect to the integration of functions of one variable, in the present state of analytical science, it is principally the reduction of differentials to a few really distinct formulæ, which must be aimed at. The integrals of these must be considered as new transcendants differing from logarithmic and trigonometrical functions; but which may be equally important in analytical researches.

We shall now give a few examples of the integration of irrational functions.

Example 1. Let

$$\mathrm{X} = \frac{(1 + x^{\frac{1}{2}} - x^{\frac{2}{3}})\, d\, x}{1 + x^{\frac{1}{3}}}$$

The common denominator of the fractional exponents of x is 6. Hence we assume $x = y^6$, then $d\, x = 6\, y^5\, d\, y$, and X becomes $6\, y^5\, d\, y\, \frac{1 + y^3 - y^4}{1 + y^2} = -6\, d\, y\left(y^7 - y^6 - y^5 + y^4 - y^3 + 1 - \frac{1}{1 + y^2}\right)$. Integrating each term, and putting for y its value, we shall find

$$\int \frac{1 + x^{\frac{1}{2}} - x^{\frac{2}{3}}}{1 + x^{\frac{1}{3}}} = \frac{-3}{4}\, x^{\frac{4}{3}} + \frac{6}{7}\, x^{\frac{7}{6}} + x - \frac{6}{5}\, x^{\frac{5}{6}} + 2\, x^{\frac{1}{2}} - 6\, x^{\frac{1}{6}} + \tan^{-1} x^{\frac{1}{6}} + \mathrm{C}.$$

Example 2. Let $\mathrm{X} = \frac{1}{\sqrt{(a + b\, x + c\, x^2)}}$, c being supposed a positive quantity.

Assume as in (75) $a + b\, x + c\, x^2 = (x\, \sqrt{c} + z)^2$, then we shall find

$$\mathrm{X}\, d\, x = \frac{2\, d\, z}{b - 2\, z\, \sqrt{c}}, \text{ but } \int \frac{2\, d\, z}{b - 2\, z\, \sqrt{c}} = \frac{-1}{\sqrt{c}}\, l\, (b - 2\, z\, \sqrt{c})$$

substituting for z its value, we get

$$\int \mathrm{X}\, d\, x = \int \frac{d\, x}{\sqrt{(a + b\, x + c\, x^2)}} = \frac{-1}{\sqrt{c}}\, l\, (b + 2\, c\, x - 2\, \sqrt{c}\, \sqrt{(a + b\, x + c\, x^2)}) + \mathrm{C}.$$

This value may be put under another form, in multiplying and dividing the quantity under the sign l by $b + 2\, c\, x + 2\, \sqrt{c}\, \sqrt{(a + b\, x + c\, x^2)}$. We shall have

$$\int \frac{dx}{\sqrt{(a+bx+cx^2)}} = \frac{-1}{\sqrt{C}}\, l\left(\frac{b^2-4ac}{b+2cx+2\sqrt{c}\sqrt{(a+bx+cx^2)}}\right) + C, \text{ or equal to}$$

$$\frac{1}{\sqrt{c}}\, l(b+2cx+2\sqrt{c}\,.\sqrt{(a+bx+cx^2)}) + C,$$

including in the arbitrary constant C the quantity $\frac{-1}{\sqrt{c}}\, l\,(b^2-4ac)$.

When $b=0$, and $a=c=1$, this formula becomes, in adding $l\,2$ to the arbitrary constant,

$$\int \frac{dx}{\sqrt{1+x^2}} = l\,(x+\sqrt{1+x^2}) + C.$$

Example 3. Let $X = \frac{1}{\sqrt{(a+bx-cx^2)}}$, and suppose a and c to be both positive, so that the roots of the equation $a+bx-cx^2=0$ are real and of contrary signs. The two factors of $a+bx-cx^2$ may then be represented by $p-qx$, and $p'+q'x$, where p, q, p', q', are all positive.

Assuming as in (75), $a+bx-cx^2=(p-qx)^2 z^2$. We shall find

$$X\,dx = \frac{2\,dz}{qz^2+q'}, \quad \text{and since} \int \frac{2\,dz}{q\,z^2+q'} = \frac{2}{\sqrt{qq'}} \tan^{-1} z \sqrt{\frac{q}{q'}},$$

then

$$\int X\,dx = \int \frac{dx}{\sqrt{(a+bx-cx^2)}} = \frac{2}{\sqrt{qq'}} \tan^{-1} \frac{\sqrt{(p'+q'x)}\sqrt{q}}{\sqrt{(p-qx)}\sqrt{q'}} + C.$$

Since $\tan 2s = \frac{2\tan s}{1-\tan^2 s}$, this value may be presented under the following form

$$\int \frac{dx}{\sqrt{(a+bx-cx^2)}} = \frac{1}{\sqrt{qq'}} \tan^{-1} \frac{2\sqrt{qq'}\sqrt{(p'+q'x)(p-qx)}}{pq'-p'q-2qq'x} + C;$$

and in observing that $qq'=c$, $pq'-p'q=b$, and $\sqrt{(p'+q'x)(p-qx)} = \sqrt{(a+bx-cx^2)}$ it becomes

$$\int \frac{dx}{\sqrt{(a+bx-cx^2)}} = \frac{1}{\sqrt{c}} \tan^{-1} \frac{2\sqrt{c}\sqrt{(a+bx-cx^2)}}{b-2cx} + C,$$

or, again,

$$\int \frac{dx}{\sqrt{(a+bx-cx^2)}} = \frac{1}{\sqrt{c}} \cos^{-1} \frac{b-2cx}{\sqrt{(b^2+4ac)}} + C.$$

When $b=0$, and $a=c=1$, then $p=q=p'=q'=1$, and the formula becomes

$$\int \frac{dx}{\sqrt{(1-x^2)}} = 2\tan^{-1} \frac{\sqrt{(1+x)}}{\sqrt{(1-x)}} + C = \sin^{-1} x + C.$$

This result agrees with the value of $\int \frac{dx}{\sqrt{(1-x^2)}}$, given (*m*) (63). We shall observe, that the formula given in *Example* 2, may be applied to $\int \frac{dx}{\sqrt{(1-x^2)}}$, by supposing in it $b=0, a=1, c=-1$. It gives then, in including $\frac{l\,2\sqrt{-1}}{\sqrt{-1}}$ in the constant,

$$\int \frac{dx}{\sqrt{(1-x^2)}} = \frac{1}{\sqrt{-1}}\, l\,(x\sqrt{-1}+\sqrt{(1-x^2)}) + C.$$

Hence

$$\sin^{-1} x = \frac{1}{\sqrt{-1}}\, l\,(x\sqrt{-1}+\sqrt{(1-x^2)}) + C,$$

and if we suppose $x=\sin y$

$$y = \frac{1}{\sqrt{-1}}\, l\,(\cos y + \sqrt{-}\sin y) + C.$$

It is easy to see that this constant equal zero; for if $y=0$, we have $\sin y=0$, $\cos y=1$, and $l\,1=0$. Therefore we shall have simply

$$y = \frac{1}{\sqrt{-1}}\, l\,(\cos y + \sqrt{-1}\sin y),$$

and by changing the sign of y

$$-y = \frac{1}{\sqrt{-1}}\, l\,(\cos y - \sqrt{-1}\sin y).$$

If we suppose in the first of these two equations $y=\frac{\pi}{2}$, we get

$$\frac{\pi}{2} = \frac{l\sqrt{-1}}{\sqrt{-1}}, \text{ or } -\frac{\pi}{2} = \sqrt{-1}\, l \sqrt{-1} = l\,(\sqrt{-1})^{\sqrt{-1}}, \text{ or } \pi = \frac{l\,(-1)}{\sqrt{-1}} = -\,l\,(-1)^{\sqrt{-1}}$$

formulæ, which may also be expressed in the following manner:

$$e^{-\frac{\pi}{2}} = (\sqrt{-1})^{\sqrt{-1}}, \text{ or } e^{-\pi} = (-1)^{\sqrt{-1}}, \text{ or } l\,(-1) = \pi\sqrt{-1}.$$

The numerical values of e and π are known, it is easy to find that $e^{-\frac{\pi}{2}} = .207879$, hence

$$(\sqrt{-1})^{\sqrt{-1}} = .207879.$$

These various singular results must be considered as the symbolic expressions of relations between infinite series.

Example 4. We shall propose for the next example to find the value of

$$\int \frac{x^m\, d x}{\sqrt{(1-x^2)}}.$$

If we compare it with the binomial differentials, we find that here we have m instead of $m-1$, $n=2$, and $\frac{p}{q} = -\frac{1}{2}$. Hence if m be an even number $\frac{m}{n}$ will be an integer, and if m be odd, $\frac{m}{n} + \frac{p}{q}$ will be equal to a whole number. Therefore, provided m be an integer positive or negative, it will always be possible to transfer the above formula into a rational one, and, consequently, to obtain its integral under a finite form. For that purpose we shall make use of the formula of reduction given (79).

By substituting $m-1$ for m, and assuming $n=2$, $p=-\frac{1}{2}$, $a=1$, and $b=-1$, the formulæ (e) (76) will give

$$\int \frac{x^m\, d x}{\sqrt{(1-x^2)}} = -\frac{x^{m-1}\sqrt{(1-x^2)}}{m} + \frac{(m-1)}{m}\int \frac{x^{m-2}\, d x}{\sqrt{(1-x^2)}} \ldots\ldots (a).$$

Making successively $m=1$, $m=3$, we shall obtain in

$$\int \frac{x\, d x}{\sqrt{(1-x^2)}} = -\sqrt{(1-x^2)} + \text{C},$$

$$\int \frac{x^3\, d x}{\sqrt{(1-x^2)}} = -\frac{1}{3}x^2\sqrt{(1-x^2)} + \frac{2}{3}\int \frac{x\, d x}{\sqrt{(1-x^2)}},$$

$$\int \frac{x^5\, d x}{\sqrt{(1-x^2)}} = -\frac{1}{5}x^4\sqrt{(1-x^2)} + \frac{4}{5}\int \frac{x^3\, d x}{\sqrt{(1-x^2)}},$$

$$\int \frac{x^7\, d x}{\sqrt{(1-x^2)}} = -\frac{1}{7}x^6\sqrt{(1-x^2)} + \frac{6}{7}\int \frac{x^5\, d x}{\sqrt{(1-x^2)}},$$

&c. &c. &c.

Hence, by substituting in each the value of the preceding integral, we find

$$\int \frac{x\, d x}{\sqrt{(1-x^2)}} = -\sqrt{(1-x^2)} + \text{C},$$

$$\int \frac{x^3\, d x}{\sqrt{(1-x^2)}} = -\left(\frac{1}{3}x^2 + \frac{1\,.\,2}{1\,.\,3}\right)\sqrt{(1-x^2)} + \text{C},$$

$$\int \frac{x^5\, d x}{\sqrt{(1-x^2)}} = -\left(\frac{1}{5}x^4 + \frac{1\,.\,4}{3\,.\,5}x^2 + \frac{1\,.\,2\,.\,4}{1\,.\,3\,.\,5}\right)\sqrt{(1-x^2)} + \text{C},$$

$$\int \frac{x^7\, d x}{\sqrt{(1-x^2)}} = -\left(\frac{1}{7}x^6 + \frac{1\,.\,6}{5\,.\,7}x^4 + \frac{1\,.\,4\,.\,6}{3\,.\,5\,.\,7}x^2 + \frac{1\,.\,2\,.\,4\,.\,6}{1\,.\,3\,.\,5\,.\,7}\right)\sqrt{(1-x^2)} + \text{C},$$

&c. &c. &c.

The law of these values is very obvious, and we may easily form the general formulæ.

$$\int \frac{x^{2r+1}\, d x}{\sqrt{(1-x^2)}} = -\sqrt{(1-x^2)}\left\{\frac{1}{2r+1}x^{2r} + \frac{1\,.\,2r}{(2r-1)(2r+1)}x^{2r-2} + \frac{1\,.\,2r\,.\,2r-2}{(2r-3)(2r-1)(2r+1)}x^{2r-4} + \right.$$

$$\left. \ldots\ldots\ldots\ldots + \frac{1\,.\,2\,.\,4 \ldots\ldots 2r}{1\,.\,3\,.\,5 \ldots (2r+1)}\right\} + \text{C}.$$

Let us assume now successively $m = 0$, $m = 2$, $m = 4$, &c. We shall have

$$\int \frac{dx}{\sqrt{(1-x^2)}} = \sin^{-1} x + c \text{ by } (m)\ (63),$$

$$\int \frac{x^2\,dx}{\sqrt{(1-x^2)}} = -\frac{x\sqrt{(1-x^2)}}{2} + \frac{1}{2}\int \frac{dx}{\sqrt{(1-x^2)}},$$

$$\int \frac{x^4\,dx}{\sqrt{(1-x^2)}} = -\frac{x^3\sqrt{(1-x^2)}}{4} + \frac{3}{4}\int \frac{x^2\,dx}{\sqrt{(1-x^2)}},$$

$$\int \frac{x^6\,dx}{\sqrt{(1-x^2)}} = -\frac{x^5\sqrt{(1-x^2)}}{6} + \frac{5}{6}\int \frac{x^4\,dx}{\sqrt{(1-x^2)}},$$

and by substitution,

$$\int \frac{dx}{\sqrt{(1-x^2)}} = \sin^{-1} x + \mathrm{C},$$

$$\int \frac{x^2\,dx}{\sqrt{(1-x^2)}} = -\frac{1}{2}x\sqrt{(1-x^2)} + \frac{1}{2}\sin^{-1} x + \mathrm{C},$$

$$\int \frac{x^4\,dx}{\sqrt{(1-x^2)}} = -\left(\frac{1}{4}x^3 + \frac{1.3}{2.4}x\right)\sqrt{(1-x^2)} + \frac{1.3}{2.4}\sin^{-1} x + \mathrm{C},$$

$$\int \frac{x^6\,dx}{\sqrt{(1-x^2)}} = -\left(\frac{1}{6}x^5 + \frac{1.5}{4.6}x^3 + \frac{1.3.5}{2.4.6}x\right)\sqrt{(1-x^2)} + \frac{1.3.5}{2.4.6}\sin^{-1} x + \mathrm{C},$$

&c. &c. &c.

and, generally,

$$\int \frac{x^{2r}\,dx}{\sqrt{(1-x^2)}} = -\left(\frac{1}{2r}x^{2r-1} + \frac{1.(2r-1)}{(2r-2)2r}x^{2r-3} + \frac{1.(2r-3)(2r-1)}{(2r-4)(2r-2)2r}x^{2r-5} + \ldots\ldots\right.$$

$$\left.\ldots\ldots + \frac{1.3.5\ldots.(2r-1)}{2.4.6\ldots.2r}\right)\sqrt{(1-x^2)} + \frac{1.3.5\ldots.(2r-1)}{2.4.6\ldots.2r}\sin^{-1} x + \mathrm{C}.$$

When m is negative, the differential assumes the form $\frac{dx}{x^m\sqrt{(1-x^2)}}$, and by substituting m for $m-1$, and making $a = 1$, $b = -1$, $n = 2$, and $p = -\frac{1}{2}$, in the formula (g) (76), we shall find

$$\int \frac{dx}{x^m\sqrt{(1-x^2)}} = -\frac{\sqrt{(1-x^2)}}{(m-1)x^{m-1}} + \frac{m-2}{m-1}\int \frac{dx}{x^{m-2}\sqrt{(1-x^2)}} \ldots\ldots\ldots\ldots (b).$$

If we suppose $m = 1$, the left side of the equation becomes infinite. To obtain the integral in this case, let $1 - x^2 = z^2$, then $x = \sqrt{(1-z^2)}$ and $dx = \frac{-z\,dz}{\sqrt{(1-z^2)}}$. Therefore we shall have

$$\int \frac{dx}{x\sqrt{(1-x^2)}} = \int \frac{-dz}{1-z^2} = -\frac{1}{2}\,l\,\frac{1+z}{1-z} = -\frac{1}{2}\,l\,\frac{1+\sqrt{(1-x^2)}}{1-\sqrt{(1-x^2)}} + \mathrm{C},$$

which, by multiplying both terms of the fraction under the sign l, by the numerator, becomes, after reduction,

$$-\,l\,\frac{1+\sqrt{(1-x^2)}}{x} + \mathrm{C}.$$

We shall therefore obtain from equation (b), by substituting successively for m the terms of the series of odd numbers, the following results:

$$\int \frac{dx}{x\sqrt{(1-x^2)}} = -\,l\,\frac{1+\sqrt{(1-x^2)}}{x} + \mathrm{C},$$

$$\int \frac{dx}{x^3\sqrt{(1-x^2)}} = -\frac{\sqrt{(1-x^2)}}{2x^2} + \frac{1}{2}\int \frac{dx}{x\sqrt{(1-x^2)}},$$

$$\int \frac{dx}{x^5\sqrt{(1-x^2)}} = -\frac{\sqrt{(1-x^2)}}{4x^4} + \frac{3}{4}\qquad \frac{dx}{x^3\sqrt{(1-x^2)}},$$

$$\int \frac{dx}{x^7\sqrt{(1-x^2)}} = -\frac{\sqrt{(1-x^2)}}{6x^6} + \frac{5}{6}\int \frac{dx}{x^5\sqrt{(1-x^2)}},$$

&c.

and consequently by substitution,

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$$\int \frac{dx}{x\sqrt{(1-x^2)}} = -\, l\, \frac{1+\sqrt{(1-x^2)}}{x} + c,$$

$$\int \frac{dx}{x^3\sqrt{(1-x^2)}} = -\frac{\sqrt{(1-x^2)}}{2x^2} - \frac{1}{2}\, l\, \frac{1+\sqrt{(1-x^2)}}{x} + c,$$

$$\int \frac{dx}{x^5\sqrt{(1-x^2)}} = -\sqrt{(1-x^2)}\left(\frac{1}{4x^4} + \frac{1.3}{2.4x^2}\right) - \frac{1.3}{2.4}\, l\, \frac{1+\sqrt{(1-x^2)}}{x} + c,$$

$$\int \frac{dx}{x^7\sqrt{(1-x^2)}} = -\sqrt{(1-x^2)}\left(\frac{1}{6x^6} + \frac{1.5}{4.6x^4} + \frac{1.3.5}{2.4.6x^2}\right) - \frac{1.3.5}{2.4.6}\, l\, \frac{1+\sqrt{(1-x^2)}}{x} + c,$$

and generally

$$\int \frac{dx}{x^{2r+1}\sqrt{(1-x^2)}} = -\sqrt{(1-x^2)}\left(\frac{1}{2r.x^{2r}} + \frac{1.(2r-1)}{(2r-2)\,2r\,x^{2r-2}} + \frac{1.(2r-3)(2r-1)}{(2r-4)(2r-2)\,2r\,x^{2r-4}} + \ldots\ldots \right.$$
$$\left. \ldots\ldots \frac{1.3.5\ldots.(2r-1)}{2.4.6\ldots\ldots 2r\,x^2}\right) - \frac{1.3.5\ldots.(2r-1)}{2.4.6\ldots\ldots.. 2r}\, l\, \frac{1+\sqrt{(1-x^2)}}{x} + c.$$

We shall obtain in the same manner, by supposing in equation (b) m successively equal to 0, 2, 4, 6, &c.

$$\int \frac{dx}{\sqrt{(1-x^2)}} = \sin^{-1} x + c,$$

$$\int \frac{dx}{x^2\sqrt{(1-x^2)}} = -\frac{\sqrt{(1-x^2)}}{x} + c,$$

$$\int \frac{dx}{x^4\sqrt{(1-x^2)}} = -\frac{\sqrt{(1-x^2)}}{3x^3} + \frac{2}{3}\int \frac{dx}{x^2\sqrt{(1-x^2)}}$$

$$\int \frac{dx}{x^6\sqrt{(1-x^2)}} = -\frac{\sqrt{(1-x^2)}}{5x^5} + \frac{4}{5}\int \frac{dx}{x^4\sqrt{(1-x^2)}},$$

&c.

Hence

$$\int \frac{dx}{x^2\sqrt{(1-x^2)}} = -\frac{\sqrt{(1-x^2)}}{x} + c,$$

$$\int \frac{dx}{x^4\sqrt{(1-x^2)}} = -\sqrt{(1-x^2)}\left(\frac{1}{3x^3} + \frac{2}{1.3x}\right) + c,$$

$$\int \frac{dx}{x^6\sqrt{(1-x^2)}} = -\sqrt{(1-x^2)}\left(\frac{1}{5x^5} + \frac{4}{3.5x^3} + \frac{2.4}{1.3.5x}\right) + c,$$

and generally

$$\int \frac{dx}{x^{2r}\sqrt{(1-x^2)}} = -\sqrt{(1-x^2)}\left(\frac{1}{(2r-1)x^{2r-1}} + \frac{1.(2r-2)}{(2r-3)(2r-1)x^{2r-3}} + \frac{1.(2r-4)(2r-2)}{(2r-5)(2r-3)(2r-1)x^{2r-5}} + \right.$$
$$\left. \ldots\ldots.. \frac{1.2.4\ldots.(2r-2)}{1.3.5\ldots.(2r-1)x}\right) + c.$$

If in the formula $\int \frac{x^{2r}\,dx}{\sqrt{(1-x^2)}}$, we suppose $x^2 = \frac{y}{2c}$, we shall have $dx = \frac{1}{2\sqrt{2c}}\frac{dy}{\sqrt{y}}$, and $\int \frac{x^{2r}\,dx}{\sqrt{(1-x^2)}}$ $= \frac{1}{2(2c)^r}\int \frac{y^r\,dy}{\sqrt{(2cy-y^2)}}$, and in making the same substitution in the value we have found above, we shall obtain that of $\int \frac{y^r\,dy}{\sqrt{(2cy-y^2)}}$, at which we might arrive in a direct manner by analogous reductions.

Example 5. Let the proposed differential be $\frac{dx}{x(a+bx)^{\frac{2}{3}}}$. To make it rational, we assume $a + bx = ay^3$. We find $dx = \frac{3ay^2\,dy}{b}$, $x = \frac{a(y^3-1)}{b}$, $(a+bx)^{\frac{2}{3}} = a^{\frac{2}{3}}y^2$, these values being substituted, we get for the transformed differential $\frac{3\,dy}{a^{\frac{2}{3}}(y^3-1)}$. Then $\frac{3}{y^3-1} = \frac{1}{y-1} - \frac{y+2}{y^2+y+1}$, but $\int \frac{dy}{y-1} = l(y-1)+c$,

and

$$\int \frac{-(y+2)\,dy}{y^2+y+1} = -\frac{1}{2}\, l(y^2+y+1) - \sqrt{3}\tan^{-1}\frac{2y+1}{\sqrt{3}} + c. \text{ Therefore}$$

$$\int \frac{3\,dy}{a^{\frac{2}{3}}(y^3-1)} = \frac{1}{a^{\frac{2}{3}}}\left\{l(y-1) - \frac{1}{2}\, l(y^2+y+1) - \sqrt{3}\tan^{-1}\frac{2y+1}{\sqrt{3}}\right\} + c.$$

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Observing that $l(y-1) - \frac{1}{2} l(y^2+y+1) = l\frac{y-1}{\sqrt{(y^2+y+1)}} = l\frac{(y-1)^{\frac{3}{2}}}{(y^3-1)^{\frac{1}{2}}} = \frac{3}{2}\, l\frac{y-1}{(y^3-1)^{\frac{1}{3}}}$,

It becomes
$$\int \frac{3\,dy}{a^{\frac{1}{3}}(y^3-1)} = \frac{1}{a^{\frac{1}{3}}}\left\{\frac{3}{2}\, l\frac{y-1}{(y^3-1)^{\frac{1}{3}}} - \sqrt{3}\tan^{-1}\frac{2y+1}{\sqrt{3}}\right\} + c.$$

Substituting now for y its value, we find
$$\int \frac{dx}{x(a+bx)^{\frac{1}{3}}} = \frac{1}{a^{\frac{1}{3}}}\left\{\frac{3}{2}\, l\frac{(a+bx)^{\frac{1}{3}} - a^{\frac{1}{3}}}{x^{\frac{1}{3}}} - \sqrt{3}\tan^{-1}\frac{2(a+bx)^{\frac{1}{3}} + a^{\frac{1}{3}}}{a^{\frac{1}{3}}\sqrt{3}}\right\} + c.$$

We shall now give examples of some of the irrational formulæ which may be integrated, although the preceding rules cannot be applied to make them rational.

Example 6. Let $X = \frac{1}{(1-x^n)\sqrt[2n]{(2x^n-1)}}$, and let $\frac{\sqrt[2n]{(2x^n-1)}}{x} = y$, we shall have $y^{2n} = \frac{2x^n-1}{x^{2n}}$,

and $1 - y^{2n} = \frac{x^{2n}-2x^n+1}{x^{2n}} = \frac{(x^n-1)^2}{x^{2n}}$. Hence $y^{2n-1}\,dy = \frac{dx(1-x^n)}{x^{2n+1}}$.

Dividing both sides of this last equation, respectively by the sides of the preceding, we find
$$\frac{y^{2n-1}\,dy}{1-y^{2n}} = \frac{dx}{x(1-x^n)},$$
and dividing again the left side of this by y, and the right side by the assumed value of y, we get
$$\frac{dx}{(1-x^n)\sqrt[2n]{(2x^n-1)}} = \frac{y^{2n-2}\,dy}{1-y^{2n}}.$$
The integration of the proposed formula is thus reduced to that of $\frac{y^{2n-2}\,dy}{1-y^{2n}}$, which is rational.

Example 7. Let $X = \frac{x^{n-1}}{(1-x^n)\sqrt[2n]{(2x^n-1)}}$, we shall make $\sqrt[2n]{(2x^n-1)} = y$, from which we deduce $y^{2n} = 2x^n - 1$, or $x^n = \frac{1}{2}(y^{2n}-1)$.

Hence $x^{n-1}\,dx = y^{2n-1}\,dy$, and therefore
$$\frac{2y^{2n-1}\,dy}{1-y^{2n}} = \frac{x^{n-1}\,dx}{1-x^n},$$
and dividing by $y = \sqrt[2n]{(x^{2n}-1)}$, we find
$$\frac{x^{n-1}\,dx}{(1-x^n)\sqrt[2n]{x^{2n}-1}} = \frac{2y^{2n-2}\,dy}{1-y^{2n}},$$
which last expression is rational.

(84.) The number of cases in which $X\,dx$ may be integrated when X involves logarithmic functions of the variable is very limited.

We shall first consider the function $X\,dx\,(lx)^n$, in which X is supposed to represent a rational function of x. $\int X\,dx$ may therefore be determined, and we shall represent it by y. We shall have, in integrating the proposed formula by parts,
$$\int X\,dx\,(lx)^n = y\,(lx)^n - n\int \frac{dx}{x}\, y\,(lx)^{n-1}.$$

If $\frac{y}{x}$ be again a rational function, we may, in the same manner, make the integration of $\int \frac{dx}{x}\, y\,(lx)^{n-1}$, to depend on that of another in which the exponent of lx will be still less by one. By continuing the same process, if n be an integer, and if at each operation we may integrate the function which multiplies the power of lx, we shall be able to effect completely the integration.

Let us suppose $X = x^m$, we shall obtain
$$\int x^m\,dx\,(lx)^n = \frac{x^{m+1}}{m+1}(lx)^n - \frac{n}{m+1}\int \frac{dx}{x}(lx)^{n-1}x^{m+1} = \frac{x^{m+1}}{m+1}(lx)^n - \frac{n}{m+1}\int x^m\,dx\,(lx)^{n-1}.$$

In the same manner,
$$\int x^m\,dx\,(lx)^{n-1} = \frac{x^{m+1}}{m+1}(lx)^{n-1} - \frac{n-1}{m+1}\int x^m\,dx\,(lx)^{n-2},$$
$$\int x^m\,dx\,(lx)^{n-2} = \frac{x^{m+1}}{m+1}(lx)^{n-2} - \frac{n-2}{m+1}\int x^m\,dx\,(lx)^{n-3},$$
&c.

And substituting each value, in the preceding equation, we shall get the general formula

$$\int x^m\,dx\,(lx)^n = \frac{x^{m+1}}{m+1}\Big\{(lx)^n - \frac{n}{m+1}(lx)^{n-1} + \frac{n(n-1)}{(m+1)^2}(lx)^{n-2} - \frac{n(n-1)(n-2)}{(m+1)^3}(lx)^{n-3} + \&c.\Big\} + c, \; \ldots\ldots(a).$$

This series is limited when n is an integer. When $m = -1$ the denominators become equal to zero, and consequently the series cannot be used. But in that case the proposed formula is $\frac{dx\,(lx)^n}{x}$, and by supposing $lx = z$, we have $\frac{dx}{x} = dz$; consequently it is changed into $z^n\,dz$, the integral of which is $\frac{z^{n+1}}{n+1}$. Substituting again for z its value, we find

$$\int \frac{dx}{x}(lx)^n = \frac{(lx)^{n+1}}{n+1} + c$$

It is obvious that the transformation we have used here, would be equally successful for any differential function $\frac{X\,dx}{x}$, in which X would only involve lx; the supposition $x = lx = z$, would make it algebraical. When n is either fractional or negative, the series a is unlimited. If $n = -\frac{1}{2}$ for instance, it becomes

$$\int\frac{x^m\,dx}{\sqrt{lx}} = x^{m+1}\Big\{\frac{1}{(m+1)(lx)^{\frac{1}{2}}} + \frac{1}{2(m+1)^2(lx)^{\frac{3}{2}}} + \frac{1.3}{4(m+1)^3(lx)^{\frac{5}{2}}} + \frac{1.3.5}{8(m+1)^4(lx)^{\frac{7}{2}}} + \&c.\ldots\Big\} + c$$

We may obtain formulæ of reduction, corresponding to the case in which the exponent of lx is negative, analogous to those obtained above, and by means of which the integration is made to depend on that of differentials, in which that exponent is less.

The expression $X\,dx\,(lx)^n$ may then be written $X\,x\frac{dx}{x}(lx)^{-n}$, and by integration by parts,

$$\int X\,x\frac{dx}{x}(lx)^{-n} = -\frac{X\,x}{(n-1)(lx)^{n-1}} + \frac{1}{(n-1)}\int\frac{1}{(lx)^{n-1}}d(X\,x).$$

And by applying to this last integral the same decomposition, we shall reduce it to the integration of a formula in which the exponent of (lx) will be $-n+2$; the same process being continued will lead by successive substitutions to the following expression,

$$\int\frac{X\,dx}{(lx)^n} = -\frac{X\,x}{(n-1)(lx)^{n-1}} - \frac{X_1\,x}{(n-1)(n-2)(lx)^{n-2}} - \frac{X_2\,x}{(n-1)(n-2)(n-3)(lx)^{n-3}} - \&c.$$

in which $d(X\,x) = X_1\,dx$, $d(X_1\,x) = X_2\,dx$, &c. .... The last term of the series will be

$$+\frac{1}{(n-1)(n-2)\ldots.1}\int\frac{X_{n-1}\,dx}{lx}, \text{ if } n \text{ be an integer, and } + \frac{1}{(n-1)(n-2)\ldots.1}\int\frac{X_{m}\,dx}{(lx)^{n-m}},$$

if n be a fractional number, and m the greatest integer it contains.

Let $X = x^m$, the above formula will give

$$\int\frac{x^m\,dx}{(lx)^n} = -\frac{x^{m+1}}{(n-1)(lx)^{n-1}} - \frac{(m+1)\,x^{m+1}}{(n-1)(n-2)(lx)^{n-2}} - \frac{(m+1)^2\,x^{m+1}}{(n-1)(n-2)(n-3)(lx)^{n-3}} + \frac{(m+1)^{n-1}}{(n-1)(n-2)\ldots.1}\int\frac{x^m\,dx}{lx},$$

n being supposed to be an integer.

This last integral may be reduced to a simpler form by assuming $x^{m+1} = y$, for then $x^m\,dx = \frac{dy}{m+1}$, $lx = \frac{ly}{m+1}$, and hence, $\int\frac{x^m\,dx}{lx} = \int\frac{dy}{ly}$. No further reduction can be effected upon the expression $\int\frac{dy}{ly}$, which thus appears to be a new transcendant.

The preceding method of integration would not apply to the differential $\frac{dx}{x\,lx}$, but then the integral is obviously $l\,(lx) + c$.

(85.) When X involves exponential functions of x, the differential expression $X\,dx$ may also be completely integrated in a few cases.

We shall first observe, that if X be an algebraical function of a^x, $X\,dx$ may be reduced to an algebraical differential expression of one variable. For by assuming $a^x = u$, we get $\frac{a^x\,dx}{la} = du$, $dx = \frac{la.\,du}{u}$, and by

substituting for a^x and $d\,x$ their values in $X\,d\,x$, it will become an algebraical function of u, which may be integrated by methods previously explained.

When X contains at the same time the variable x, and a^x, the expression $X\,d\,x$ may easily be transformed into another involving only the variable, and the logarithm of that variable, by supposing $a^x = u$. Then the rules given in (84) may be applied to the new differential. In most cases, however, it will be simpler to integrate without making use of this transformation.

The expression $X\,e^x\,d\,x$ will be integrated immediately, whenever we shall be able to decompose X into two parts, one of which shall be the differential coefficient of the other. Let Y designate one of the parts, and, consequently, $\frac{d\,Y}{d\,x}$ the other, it is obvious that we shall have

$$\int X\,e^x\,d\,x = \int\left(Y + \frac{d\,Y}{d\,x}\right)e^x\,d\,x = Y\,e^x + c.$$

No general rule can be given now for this decomposition, the discovery of the transformations and artifices calculated to facilitate it depends entirely upon the habit of analytical investigations. We shall find in the sequel that such a decomposition may be effected by means of the integration of an equation; but this means brings back the difficulty precisely to the same point.

When X is an integral and rational function of x, the expression $X\,a^x\,d\,x$ may be completely integrated. We shall have first, by integrating by parts,

$$\int X\,a^x\,d\,x = \frac{1}{l\,a}\,X\,a^x - \frac{1}{l\,a}\int a^x\,d\,X.$$

Let $d\,X = X_1\,d\,x$, $d\,X_1 = X_2\,d\,x$, $d\,X_2 = X_3\,d\,x$, &c. We shall have successively,

$$\int X_1\,a^x\,d\,x = \frac{1}{l\,a}\,X_1\,a^x - \frac{1}{l\,a}\int a^x\,d\,X_1, \quad \int X_2\,a^x\,d\,x = \frac{1}{l\,a}\,X_2\,a^x - \frac{1}{l\,a}\int a^x\,d\,X_2, \text{ \&c.}$$

and by substitution,

$$\int X\,a^x\,d\,x = \frac{1}{l\,a}\,X\,a^x - \frac{1}{(l\,a)^2}\,X_1\,a^x + \frac{1}{(l\,a)^3}\,X_2\,a^x \ldots\ldots - \text{\&c.} + c.$$

A series which will obviously be limited, since X being by supposition an integral and rational function of x, one of the successive differential coefficients X_1, X_2, &c. will necessarily be equal to nothing.

Let $X = x^m$, m being an integer, we shall have

$$\int x^m\,a^x\,d\,x = a^x\left\{\frac{x^m}{l\,a} - \frac{m\,x^{m-1}}{(l\,a)^2} + \frac{m\,(m-1)\,x^{m-2}}{(l\,a)^3} - \ldots\ldots \mp \frac{m\,(m-1)\ldots\ldots 1}{(l\,a)^{m+1}}\right\} + c,$$

the sign of the last term being $-$ when m is an odd number, and $+$ when it is even.

Another transformation may sometimes be used to obtain the integral of $X\,a^x\,d\,x$. Let $\int X\,d\,x = X_1$, $\int X_1\,d\,x = X_2$, $\int X_2\,d\,x = X_3$, &c.; and let us begin the integration by parts of the differential $X\,a^x\,d\,x$, by the factor $X\,d\,x$, instead of the factor $a^x d\,x$, we shall have successively,

$$\int X\,a^x\,d\,x = X_1\,a^x - l\,a\int X_1\,a^x\,d\,x, \quad \int X_1\,a^x\,d\,x = X_2\,a^x - l\,a\int X_2\,a^x\,d\,x, \text{ \&c.}$$

and, consequently,

$$\int X\,a^x\,d\,x = X_1\,a^x - l\,a\,X_2\,a^x + (l\,a)^2\,X_3\,a^x \ldots\ldots\ldots \pm (l\,a)^n \int X_n\,a^x\,d\,x,$$

the sign of the last term being $+$ when n is an even number, and $-$ when it is odd.

Let in the last equation $X = x^{-m}$, and it will become

$$\int \frac{a^x\,d\,x}{x^m} = -\frac{a^x}{(m-1)\,x^{m-1}} - \frac{a^x\,l\,a}{(m-1)\,(m-2)\,x^{m-2}} - \frac{a^x\,(l\,a)^2}{(m-1)\,(m-2)\,(m-3)\,x^{m-3}} \ldots\ldots\ldots$$
$$+ \frac{(l\,a)^{m-1}}{(m-1)\,(m-2)\ldots\ldots 1}\int \frac{a^x\,d\,x}{x}.$$

The last term of this series cannot be reduced any further, but it may easily be shown that it does not differ from the new transcendant $\int \frac{d\,x}{l\,x}$ to which we have been led in (84), for if we suppose $a^x = y$, we shall have $a^x\,d\,x = \frac{d\,y}{l\,a}$, $x = \frac{l\,y}{l\,a}$, and, consequently, $\int \frac{a^x\,d\,x}{x} = \int \frac{d\,y}{l\,y}$.

We shall apply the rules for integrating logarithmic and exponential functions to two examples.

Example 1. Let $X\,d\,x = \frac{d\,x}{x^{\frac{1}{2}}}\,l\left(\frac{1}{1-x}\right)$. This differential expression is included in the general formula $Y\,d\,x\,l\,Z$, in which Y and Z are algebraical functions of x, and which, by integration by parts, is reduced to $l\,Z\int Y\,d\,x \int\left(\frac{d\,Z}{z}\int Y\,d\,x\right)$. When the quantity between the parenthesis happens to be an algebraical function,

the whole integration may be performed by the rules given for this kind of functions. In the example chosen this will take place. We shall have

$$\int \frac{dx}{x^{\frac{3}{2}}}\, l\left(\frac{1}{1-x}\right) = -\frac{2}{x^{\frac{1}{2}}}\, l\left(\frac{1}{1-x}\right) + \int \frac{2\,dx}{x^{\frac{1}{2}}(1-x)}, \text{ but}$$

$$\int \frac{2\,dx}{x^{\frac{1}{2}}(1-x)} = \int \frac{4\,d(x^{\frac{1}{2}})}{1-x} = 2\,l\left(\frac{1+x^{\frac{1}{2}}}{1-x^{\frac{1}{2}}}\right) + c, \text{ and therefore,}$$

$$\int \frac{dx}{x^{\frac{3}{2}}}\, l\left(\frac{1}{1-x}\right) = -\frac{2}{x^{\frac{1}{2}}}\, l\left(\frac{1}{1-x}\right) + 2\,l\left(\frac{1+x^{\frac{1}{2}}}{1-x^{\frac{1}{2}}}\right) + c.$$

Example 2. Let $X\,dx = \frac{e^x x\,dx}{(1+x)^2}$. We shall try to decompose $\frac{x}{(1+x)^2}$ in two parts, one of which shall be the differential coefficient of the other. With a little attention we see that

$$\frac{x}{(1+x)^2} = \frac{1+x}{(1+x)^2} - \frac{1}{(1+x)^2} = \frac{1}{(1+x)} - \frac{1}{(1+x)^2},$$

and that under this last form the proposed decomposition has been effected. Consequently $\int \frac{e^x x\,dx}{(1+x)^2} = \frac{e^x}{1+x} + c.$

(86.) We proceed now to the integration of differential expressions containing circular functions of the variable.

The formulæ f, g, h, &c. (63) will enable us to integrate any differential of the following form,

$$dx\,\{A + B\sin x + C\sin 2x + \&c. \ldots\ldots + A_1 + B_1\cos x + C_1\cos 2x + \&c.\}$$

And therefore, since any rational and integral function of the sine and cosine may be transformed into series similar to that between the parenthesis, we shall be able to integrate the differential $X\,dx$ whenever X will be such a function. In many cases, however, it will not be necessary to make use of these developements. The formula $(\sin x)^m (\cos x)^n\,dx$, for instance, may easily be integrated in several cases by the method of integration by parts. We have first

$$\int dx\,(\sin x)^m (\cos x)^n = \int dx\,\sin x\,(\sin x)^{m-1} (\cos x)^n, \text{ but}$$

$$\int dx\,\sin x\,(\cos x)^n = -\frac{(\cos x)^{n+1}}{n+1}, \text{ since } d\cos x = -\sin x\,dx, \text{ therefore}$$

$$\int dx\,(\sin x)^m (\cos x)^n = -\frac{(\cos x)^{n+1} (\sin x)^{m-1}}{n+1} + \frac{(m-1)}{n+1}\int dx\,(\cos x)^{n+2} (\sin x)^{m-2},$$

and because $(\cos x)^{n+2} = (\cos x)^2 (\cos x)^n = \{1 - (\sin x)^2\}(\cos x)^n = (\cos x)^n - (\sin x)^2 (\cos x)^n$, we shall have by substituting, and then taking the value of $\int dx\,(\sin x)^m (\cos x)^n$, a quantity which will be, in both sides of the equation,

$$\int dx\,(\sin x)^m (\cos x)^n = -\frac{(\sin x)^{m-1} (\cos x)^{n+1}}{m+n} + \frac{m-1}{m+n}\int dx\,(\sin x)^{m-2} (\cos x)^n \ldots\ldots\ldots (a).$$

Operating upon $\cos x$ as we have upon $\sin x$, we shall arrive by similar steps to the following expression,

$$\int dx\,(\sin x)^m (\cos x)^n = \frac{(\sin x)^{m+1} (\cos x)^{n-1}}{m+n} + \frac{n-1}{m+n}\int dx\,(\cos x)^{n-2} (\sin x)^m \ldots\ldots\ldots (b).$$

By means of the formula (*a*) the integration of $dx\,(\sin x)^m (\cos x)^n$ will be reduced to that of $dx\,\sin x\,(\cos x)^n$ if m be a positive odd number, and to that of $dx\,(\cos x)^n$ if it be a positive even number. In the first case, the expression will be completely integrated, whatever be the value of n, since $\int dx\,\sin x\,(\cos x)^n = -\frac{(\cos x)^{n+1}}{n+1} + c$. Similar remarks apply to the formula (*b*). If both m and n are positive integers, the complete integral of $dx\,(\sin x)^m (\cos x)^n$ may be obtained by the use of the formulæ (*a*) and (*b*), for they will reduce the integration to that of one of the following differentials dx, $dx\sin x$, $dx\cos x$, $dx\sin x\cos x$, the integrals of which are respectively x, $\cos x$, $-\sin x$, $\frac{(\sin x)^2}{2}$. If $m + n = 0$, these formulæ will be of no use, even in the supposition that m should be an odd number, because the coefficient $\frac{n-1}{m+n}$ becomes then infinite.

If in the formulæ (*a*) and (*b*) we take the values of $\int dx\,(\sin x)^{m-2} (\cos x)^n$, and $\int dx\,(\sin x)^m (\cos x)^{n-2}$, and then substitute in the first m for $m-2$, and in the second n for $n-2$, we shall find

$$\int dx\,(\sin x)^m (\cos x)^n = \frac{(\sin x)^{m+1}(\cos x)^{n+1}}{m+1} + \frac{m+n+2}{m+1}\int dx\,(\sin x)^{m+2}(\cos x)^n \ldots\ldots (c).$$

$$\int dx\,(\sin x)^m (\cos x)^n = -\frac{(\sin x)^{m+1}(\cos x)^{n+1}}{n+1} + \frac{m+n+2}{n+1}\int dx\,(\sin x)^m(\cos x)^{n+2} \ldots\ldots (d).$$

The formula (*c*) will reduce the integration of $dx\,(\sin x)^m\ (\cos x)^n$ to that of $dx\,(\sin x)^{-1}(\cos x)^n$, if m be a negative odd number, and to that of $dx\,(\cos x)^n$, if it be even. The first of these may easily be transformed into an algebraical and rational formula, for if we assume $\cos x = y$, it becomes $\frac{-y^n\,dy}{1-y^2}$, and therefore can always be integrated whatever be the value of n. Similar remarks apply to the formula (*d*). If m and n be both negative integers, the formulæ (*c*) and (*d*) will reduce the integration of $dx\,(\sin x)^m\ (\cos x)^n$ to one of the four following differentials, dx, $\frac{dx}{\sin x}$, $\frac{dx}{\cos x}$, $\frac{dx}{\sin x \cos x}$. Of these we have only to consider the three last, and they may all be easily reduced to the same form. By Trigonometry we have $\sin x = 2\sin\frac{1}{2}x\cos\frac{1}{2}x$, and

$\cos x = \sin\left(\frac{\pi}{2}-x\right) = \sin\left(\frac{\pi}{2}+x\right) = 2\sin\frac{1}{2}\left(\frac{\pi}{2}+x\right)\cos\frac{1}{2}\left(\frac{\pi}{2}+x\right)$, therefore $\frac{dx}{\sin x} = \frac{\frac{dx}{2}}{\sin\frac{1}{2}x\cos\frac{1}{2}x}$, and

$\frac{dx}{\cos x} = \frac{\frac{dx}{2}}{\sin\frac{1}{2}\left(\frac{\pi}{2}+x\right)\cos\frac{1}{2}\left(\frac{\pi}{2}+x\right)}$. To integrate $\frac{dx}{\sin x \cos x}$, we divide both numerator and denominator

by $(\cos x)^2$, it becomes then $\frac{\frac{dx}{(\cos x)^2}}{\frac{\sin x}{\cos x}} = \frac{\frac{dx}{(\cos x)^2}}{\tan x}$. Under this form, it is obvious that the numerator is the differential of the denominator, hence $\int\frac{dx}{\sin x\cos x} = l\tan x + c$, and consequently

$$\int\frac{dx}{\sin x} = \int\frac{\frac{dx}{2}}{\sin\frac{1}{2}x\cos\frac{1}{2}x} = l\tan\tfrac{1}{2}x + c, \text{ and}$$

$$\int\frac{dx}{\cos x} = \int\frac{\frac{dx}{2}}{\sin\frac{1}{2}\left(\frac{\pi}{2}+x\right)\cos\frac{1}{2}\left(\frac{\pi}{2}+x\right)} = l\,.\tan\tfrac{1}{2}\left(\frac{\pi}{2}+x\right) + c.$$

We have already stated that the formulæ (*a*) and (*b*) were of no use to integrate $dx\,(\sin x)^m\ (\cos x)^n$, when $m+n=0$, or $n=-m$. In that case, the formulæ (*c*) and (*d*) may be employed with success if m be an integer. But in that same supposition the differential may always be completely integrated, whatever be the value of m. It assumes then the form $\frac{dx\,(\sin x)^m}{(\cos x)^m} = dx\,(\tan x)^m$, or $\frac{dx\,(\cos x)^m}{(\sin x)^m} = \frac{dx}{(\tan x)^m}$. These two cannot be considered as distinct from one another, since m is supposed to be any quantity whatever. Therefore, we shall only consider the first. If we make $\tan x = y$, we shall have $dx = \frac{dy}{1+y^2}$, and substituting, the proposed differential will become $\frac{y^m\,dy}{1+y^2}$, a formula which is rational, if m be an integer, or which may always, without difficulty, be transformed into a rational one, if m be fractional. Therefore when $m+n=0$ the differential $dx\,(\sin x)^m\ (\cos x)^n$ can always be integrated. We may even generalize this result; for since by means of the formulæ (*a*), (*b*), (*c*), (*d*), the exponents m and n may be increased or decreased by any multiple of two, the differential $dx\,(\sin x)^m\ (\cos x)^n$ can also be integrated if $m+n$ equal, plus, or minus any multiple of two, or in other words if $\frac{m+n}{2}$ be an integer. If we recapitulate now the various cases in which we have proved that the differential expression $dx\,(\sin x)^m\ (\cos x)^n$ can be integrated, we shall find that they are all included in the two following conditions: First, when one of the two exponents m and n is a positive or negative odd number, or, which is the same thing, when $\frac{m+1}{2}$, or $\frac{n+1}{2}$, is an integer. Secondly, When $\frac{m+n}{2}$ is an integer.

We would have arrived precisely to the same result if we had first transformed the differential $dx(\sin x)^m(\cos x)^n$ into an algebraical expression. For that it would have been sufficient to make either $\sin x = y$, or $\cos x = y$. In the first supposition we have $\cos x = \sqrt{1-y^2}$, and $dx = \frac{dy}{\sqrt{1-y^2}}$. Substituting these values the differential becomes $y^m\,dy\,(1-y^2)^{\frac{n-1}{2}}$. Under this form it is easy to compare it with binomial differentials, and applying to it what has been proved (76), we find that it may be made rational when $\frac{m+1}{2}$, or $\frac{m+n}{2}$, are integers.

(87.) By substitutions similar to the last it will always be possible to transform a differential function of trigonometrical lines into an algebraical one. They may also be transformed into exponential functions by substituting for the trigonometrical lines their values in terms of the exponential of the arc. It requires much practice in analysis to determine in each particular case upon the means which are most likely to lead to the required result in the simplest manner. To complete what can be said here upon the integration of circular functions, we shall give a few examples, chosen with the intention to show the various artifices which have been used hitherto, and which will include nearly all the cases which have been integrated, besides the general differential expression we have already examined in (86).

Example 1. Let $X\,dx = dx\sin(a+bx)\sin(a'+b'x)$. The difficulty of integrating here, arises from the circumstance that the sines of two different angles are multiplied; but we have generally, $\sin y \sin z = \frac{\cos(y-z)-\cos(y+z)}{2}$. Making use of this reduction the proposed differential will become

$$\frac{dx\cos\{(a-a')+(b-b')x\}}{2} - \frac{dx\cos\{(a+a')+(b+b')x\}}{2},$$

and hence we get

$$\int dx\sin(a+bx)\sin(a'+b'x) = \frac{\sin\{(a-a')+(b-b')x\}}{2(b-b')} - \frac{\sin\{(a+a')+(b+b')x\}}{2(b+b')} + c$$

Example 2. Let $X\,dx = x^n\,dx\sin x$. Integrating by parts, we shall have successively,

$$\int x^n\,dx\sin x = -x^n\cos x + n\int x^{n-1}\,dx\cos x,$$
$$\int x^{n-1}\,dx\cos x = x^{n-1}\sin x - n - 1\int x^{n-2}\,dx\sin x,$$
&c. &c. &c.

and by substitution,

$$\int x^n\,dx\sin x = -x^n\cos x + n\,x^{n-1}\sin x + n(n-1)\,x^{n-2}\cos x - \text{\&c.}\ldots\ldots + c.$$

A series which will be limited when n is a positive integer.

Example 3. Let $X\,dx = \frac{dx\sin x}{x^n}$. We shall again integrate by parts, but we shall begin with the factor $\frac{dx}{x^n}$. We shall find

$$\int\frac{dx\sin x}{x^n} = -\frac{\sin x}{(n-1)\,x^{n-1}} + \frac{1}{n-1}\int\frac{dx\cos x}{x^{n-1}},$$
$$\int\frac{dx\cos x}{x^{n-1}} = -\frac{\cos x}{(n-2)\,x^{n-2}} - \frac{1}{n-2}\int\frac{dx\sin x}{x^{n-2}},$$
&c. &c. &c.

and by substitution,

$$\int\frac{dx\sin x}{x^n} = -\frac{\sin x}{(n-1)\,x^{n-1}} - \frac{\cos x}{(n-1)(n-2)\,x^{n-2}} + \frac{\sin x}{(n-1)(n-2)(n-3)\,x^{n-3}} - \text{\&c.}$$

If n be an integer, this series will have its n^{th} term infinite, and, consequently, can be of no use to represent the integral. The integration by parts shows, however, that the integral of $\frac{dx\sin x}{x^n}$, may be made to depend upon that of $\frac{dx\sin x}{x}$, n being an integer. If we substitute in this last expression for $\sin x$ its value $\frac{e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}}{2\sqrt{-1}}$, it will appear that the transcendant $\int\frac{dx\sin x}{x}$ does not essentially differ from $\int\frac{e^x\,dx}{x}$, or $\int\frac{dx}{lx}$.

Example 4. Let $X\,dx = e^{ax}\,dx\,(\sin x)^n$. The integration by parts will succeed here, if m be an integer. We shall find

$$\int e^{ax}\,dx\,(\sin x)^m = \frac{1}{a}\,e^{ax}(\sin x)^m - \frac{m}{a}\int e^{ax}\,dx\cos x\,(\sin x)^{m-1},$$

$$\int e^{ax}\,dx\cos x\,(\sin x)^{m-1} = \frac{1}{a}\,e^{ax}\cos x\,(\sin x)^{m-1} - \frac{1}{a}\int e^{ax}\,dx\,\{(m-1)(\cos x)^2(\sin x)^{m-2} - (\sin x)^m\},$$

writing in this last integral for $(\cos x)^2$ its value $1-(\sin x)^2$, and substituting the value of $\int e^{ax}\,dx\cos x\,(\sin x)^{m-1}$, which will arise, in the equation above, $\int e^{ax}\,dx\,(\sin x)^m$ will then be in both sides, taking its value we shall find

$$\int e^{ax}\,dx\,(\sin x)^m = \frac{e^{ax}(\sin x)^{m-1}(a\sin x - n\cos x)}{a^2+n^2} + \frac{m(m-1)}{a^2+n^2}\int e^{ax}\,dx\,(\sin x)^{m-2}.$$

The integral contained in the right side of this equation disappears when $m=1$, and when $m=0$, consequently the integral of $e^{ax}\,dx\,(\sin x)^m$ is known in those two cases; and since the above equation shows that this integral may always be reduced to one of these two cases when m is an integer, we may conclude that the proposed differential can always be integrated in that supposition.

Example 5. Let $X\,dx = e^{ax}\,dx\,(\sin x)^m\,(\cos x)^n$. Here it is necessary to recollect that when m and n are integers, a series of terms, such as $\sin b\,x$, or $\cos c\,x$, may be substituted for such an expression as $(\sin x)^m\,(\cos x)^n$. The required integration will therefore be reduced to integrate differential expressions of the form $e^{ax}\,dx\sin b\,x$, or $e^{ax}\,dx\cos c\,x$, which will be effected in the manner indicated in the last example.

Example 6. Let $X\,dx = \dfrac{dx}{a+b\cos x}$. This example is very remarkable by the reductions it presents, and the various manners to express the integral. An algebraical form may be given to the differential by assuming $\cos x = y$, but to avoid radicals it will be simpler to make $\cos x = \dfrac{1-y^2}{1+y^2}$. Then we shall have $dx = \dfrac{2\,dy}{1+y^2}$, and consequently, $\dfrac{dx}{a+b\cos x} = \dfrac{2\,dy}{a+b+(a-b)\,y^2}$. Comparing this last differential with that integrated, *Example* 2. (74), we find, immediately, the two following expressions for the integral,

$$\int\frac{2\,dy}{a+b+(a-b)\,y^2} = \frac{1}{\sqrt{(b^2-a^2)}}\;l\,\frac{(a-b)\,y - \sqrt{(b^2-a^2)}}{(a-b)\,y+\sqrt{(b^2-a^2)}} + c, \text{ and}$$

$$\int\frac{2\,dy}{a+b+(a-b)\,y^2} = \frac{2}{\sqrt{(a^2-b^2)}}\tan^{-1}\frac{(a-b)\,y}{\sqrt{(a^2-b^2)}} + c.$$

But since $\cos x = \dfrac{1-y^2}{1+y^2}$, we have $y = \dfrac{\sqrt{(1-\cos x)}}{\sqrt{(1+\cos x)}} = \tan\dfrac{x}{2}$.

Substituting these values, we find

$$\int\frac{dx}{a+b\cos x} = \frac{1}{\sqrt{(b^2-a^2)}}\;l\,\frac{\sqrt{(b+a)}\,(1+\cos x)+\sqrt{(b-a)}\,(1-\cos x)}{\sqrt{(b+a)}\,(1+\cos x)-\sqrt{(b-a)}\,(1-\cos x)} + c$$

Multiplying both terms of the fraction under the sign l, by the numerator, we shall have

$$\int\frac{dx}{a+b\cos x} = \frac{1}{(\sqrt{b^2-a^2})}\;l\,\frac{b+a\cos x+\sin x\sqrt{(b^2-a^2)}}{a+b\cos x} + c.$$

We shall have also by the substitution of $\tan\dfrac{x}{2}$ to y in the second of the two first values obtained for the integral,

$$\int\frac{dx}{a+b\cos x} = \frac{2}{\sqrt{(a^2-b^2)}}\tan^{-1}\frac{\sqrt{(a-b)}}{\sqrt{(a+b)}}\tan\tfrac{1}{2}x + c.$$

If for $\tan\frac{1}{2}x$ its value be substitoted, the value of the integral becomes

$$\frac{2}{\sqrt{(a^2-b^2)}}\tan^{-1}\frac{\sqrt{(a-b)\,(1-\cos x)}}{\sqrt{(a+b)\,(1+\cos x)}} + c.$$

But twice the arc whose tangent $= k$, equal the arc whose tangent $= \dfrac{2k}{1-k^2}$, therefore

$$\int\frac{dx}{a+b\cos x} = \frac{1}{\sqrt{(a^2-b^2)}}\tan^{-1}\frac{\sin x\sqrt{(a^2-b^2)}}{b+a\cos x} + c = \frac{1}{\sqrt{(a^2-b^2)}}\cos^{-1}\frac{b+a\cos x}{a+b\cos x} + c.$$

These various values of the integral become $\frac{0}{0}$ when $a=b$.

In that case, we can integrate without any difficulty. We find

$$\int\frac{dx}{a+a\cos x} = \frac{1}{a}\int\frac{dx}{1+\cos x} = \frac{1}{a}\int\frac{dx}{2\left(\cos\frac{x}{2}\right)^2} = \frac{1}{a}\tan\frac{x}{2} + c.$$

Example 7. Let $X\,dx = \dfrac{dx\,(a' + b'\cos x)}{(a + b\cos x)^m}$. We shall make use here of an artifice we have already had occasion to employ. We shall assume

$$\int \frac{dx\,(a' + b'\cos x)}{(a + b\cos x)^m} = \frac{A\sin x}{(a + b\cos x)^{m-1}} + \int \frac{dx\,(B + C\cos x)}{(a + b\cos x)^{m-1}}.$$

A, B, C being three unknown constant quantities which are to be determined so as to satisfy this equation.

If we differentiate both sides of this equation, and then divide by dx, we shall find

$$a' + b'\cos x = A\cos x\,(a + b\cos x) + (m-1)\,A\,b\,(\sin x)^2 + (B + C\cos x)\,(a + b\cos x).$$

Developing, and putting $(\sin x)^2$ instead of $1 - (\cos x)^2$, we shall have

$$\left.\begin{array}{rrr} (m-1)\,Ab & +\,Aa \\ + \quad Ba & +\,Bb \\ & +\,Ca \\ - \quad a' & -\,b' \end{array}\right|\begin{array}{rr} \cos x & + \quad Ab \\ & -\,(m-1)\,Ab \\ & + \quad Cb \\ & \end{array}\left|\begin{array}{l} (\cos x)^2 = 0. \\ \\ \\ \\ \end{array}\right.$$

Making equal to nothing the coefficients of similar terms, we shall get three equations, in which the coefficients A, B, C enter only in the first degree, and from which we shall obtain the following values,

$$A = \frac{ab' - ba'}{(n-1)\,(a^2 - b^2)}, \qquad B = \frac{aa' - bb'}{a^2 - b^2}, \qquad C = \frac{(n-2)\,(ab' - ba')}{(n-1)\,(a^2 - b^2)},$$

and consequently

$$\int \frac{dx\,(a' + b'\cos x)}{(a + b\cos x)^n} = \frac{(ab' - ba')\sin x}{(n-1)\,(a^2 - b^2)\,(a + b\cos x)^{n-1}} + \frac{1}{(n-1)\,(a^2 - b^2)}$$
$$\int \frac{dx\,(n-1)\,(aa' - bb') + (n-2)\,(ab' - ba')\cos x}{(a + b\cos x)^{n-1}}.$$

By means of this formula, if n be an integer, the required integration will be reduced to that of a differential of the form $\dfrac{dx\,(p + q\cos x)}{a + b\cos x}$; and this presents no difficulty, for we easily get

$$\int \frac{dx\,(p + q\cos x)}{a + b\cos x} = \int dx \left\{ \frac{q}{b} + \frac{bp - aq}{b\,(a + b\cos x)} \right\} = \frac{q}{b}\,x + \frac{bq - aq}{b} \int \frac{dx}{a + b\cos x}.$$

(88.) Differential expressions containing the circular functions $\sin^{-1} x$, $\cos^{-1} x$, &c. can also be integrated in a few cases. The means by which the integrals are obtained are nearly the same as those which have been used with functions of sines and cosines, &c. We shall, therefore, show simply upon some general examples, including most of the formulæ for which the integration may be completed, which are the substitutions and transformations most likely to succeed.

Example 1. Let the differential be $X\,dx\sin^{-1} x$, and let $\int X\,dx = X_1$; then, integrating by parts, we shall find $\int X\,dx\sin^{-1} x = X_1 \sin^{-1} x - \int \dfrac{X_1\,dx}{\sqrt{(1 - x^2)}}$. If, therefore, X_1 be an algebraical function, the integration of $X\,dx\sin^{-1} x$ is reduced to that of an algebraical function. Let $X = x^m$, for instance, we shall have $\int x^m\,dx\sin^{-1} x = \dfrac{x^{m+1}}{m+1}\sin^{-1} x - \dfrac{1}{m+1}\int \dfrac{x^{m+1}\,dx}{\sqrt{(1 - x^2)}}$, and when m is an integer this last integral is obtained, as in *Example* 4. (83).

In a similar manner we shall have

$$\int x^m\,dx\tan^{-1} x = \frac{x^{m+1}}{m+1}\tan^{-1} x - \frac{1}{m+1}\int \frac{x^{m+1}\,dx}{1 + x^2}.$$

Example 2. Let the differential be $\dfrac{x^3\,dx}{\sqrt{(1 - x^2)}}\sin^{-1} x$. We have found before $\int \dfrac{x^3\,dx}{\sqrt{(1 - x^2)}} = -\left(\dfrac{1}{3}x^2 + \dfrac{1.2}{1.3}\right)\sqrt{(1 - x^2)}$; hence, integrating by parts, we shall have

$$\int \frac{x^3\,dx}{\sqrt{(1 - x^2)}}\sin^{-1} x = -\left(\frac{1}{3}x^2 + \frac{1.2}{1.3}\right)\sqrt{(1 - x^2)}\,.\,\sin^{-1} x + \int \left(\frac{1}{3}x^2 + \frac{1.2}{1.3}\right) dx,$$

and reducing

$$\int \frac{x^3\,dx}{\sqrt{(1 - x^2)}}\sin^{-1} x = -\left(\frac{1}{3}x^2 + \frac{1.2}{1.3}\right)\sqrt{(1 - x^2)}\sin^{-1} x + \frac{x^3}{9} + \frac{2x}{3} + c.$$

Example 3. Let the differential be $\frac{x^4 d x}{\sqrt{(1-x^2)}} \sin^{-1} x$.

We have found $$\int \frac{x^4 d x}{\sqrt{(1-x^2)}} = -\left(\frac{1}{4} x^3 + \frac{1.3}{2.4} x\right) \sqrt{(1-x^2)} + \frac{1.3}{2.4} \sin^{-1} x,$$

hence we shall find by integration by parts

$$\int \frac{x^4 d x}{\sqrt{(1-x^2)}} \sin^{-1} x = -\left\{\left(\frac{1}{4} x^3 + \frac{1.3}{2.4} x\right) \sqrt{(1-x^2)} - \frac{1.3}{2.4} \sin^{-1} x\right\} \sin^{-1} x +$$

$$\int \left\{\left(\frac{1}{4} x^3 + \frac{1.3}{2.4} x\right) d x - \frac{1.3}{2.4} \frac{d x}{\sqrt{(1-x^2)}} \sin^{-1} x\right\},$$

and consequently by reduction

$$\int \frac{x^4 d x}{\sqrt{(1-x^2)}} \sin^{-1} x = -\left\{\left(\frac{1}{4} x^3 + \frac{1.3}{2.4} x\right) \sqrt{(1-x^2)} \quad \frac{3}{16} \sin^{-1} x\right\} \sin^{-1} x + \frac{1}{16} x^4 + \frac{3}{16} x^2 + c.$$

Example 4. We shall take for the last example the differential $d x (\sin^{-1} x)^m$. Integrating by parts, we shall have successively

$$\int d x (\sin^{-1} x)^m = x (\sin^{-1} x)^m - m \int \frac{x d x}{\sqrt{(1-x^2)}} (\sin^{-1} x)^{m-1},$$

$$\int \frac{x d x}{\sqrt{(1-x^2)}} (\sin^{-1} x)^{m-1} = -\sqrt{(1-x^2)} (\sin^{-1} x)^{m-1} + m - 1 \int d x (\sin^{-1} x)^{m-2},$$

&c. &c.

and by substitution

$$\int d x (\sin^{-1} x)^m = x (\sin^{-1} x)^m + m \sqrt{(1-x^2)} (\sin^{-1} x)^{m-1} - m (m-1) x (\sin^{-1} x)^{m-2} - m (m-1)(m-2) \sqrt{(1-x^2)} (\sin^{-1} x)^{m-3} + \&c.$$

a series which will be limited when m is a positive integer.

(89.) By means of series it is always possible to represent the value of the integral of a differential expression; and these, especially when none of the preceding rules can be applied, may sometimes be used with advantage.

From the theorem of Taylor, it obviously results that if we designate by y the integral of $X d x$, and by y_1 the value of y when in it x is changed into $x_1 + h$, we shall have

$$y_1 = y + X h + \frac{d X}{d x} \frac{h^2}{1.2} + \frac{d^2 X}{d x^2} \frac{h^3}{1.2.3} + \&c. \ldots (a).$$

If in this series we change h into $-x$, y_1 will become an arbitrary constant c equal to the value of the integral corresponding to $x = 0$, and by writing y in the left side of the equation we shall have

$$y = \int X d x = c + X x - \frac{d X}{d x} \frac{x^2}{1.2} + \frac{d^2 X}{d x^2} \frac{x^3}{1.2.3} - \&c. \ldots (b).$$

This series has been given for the first time by Jean Bernouilli, and it is known under the name of the series of Bernouilli. It may be obtained by applying to the differential $X d x$, the process of integration by parts. We shall have successively

$$\int X d x = X x - \int \frac{d X}{d x} . x d x, \qquad \int \frac{d X}{d x} . x d x = \frac{d X}{d x} \frac{x^2}{1.2} - \int \frac{d^2 X}{d x^2} . \frac{x^2 d x}{1.2},$$

$$\int \frac{d^2 X}{d x^2} . \frac{x^2 d x}{1.2} = \frac{d^2 X}{d x^2} . \frac{x^3}{1.2.3} - \int \frac{d^3 X}{d x^3} \frac{x^3 d x}{1.2.3}, \&c.$$

and by substitution

$$\int X d x = X x - \frac{d X}{d x} \frac{x^2}{1.2} + \frac{d^2 X}{d x^2} \frac{x^3}{1.2.3} \ldots\ldots\ldots \pm \int \frac{d^n X}{d x^n} \frac{x^n d x}{1.2 \ldots . n}.$$

The arbitrary constant being included in the last integral.

Another developement of the integral may be derived from the series of Taylor. If in the above equation (a) we suppose $x = 0$, and afterwards change x into h, we shall have in representing by Z_0, Z_1, Z_2, &c. the values of y, X, $\frac{d X}{d x}$, $\frac{d^2 X}{d x^2}$, &c. corresponding to $x = 0$,

$$y = \int X d x = Z_0 + Z_1 \frac{x}{1} + Z_2 \frac{x^2}{1.2} + Z_3 \frac{x^3}{1.2.3} + \&c. \ldots$$

This series has the disadvantage of being only applicable when none of the quantities $\frac{d X}{d x}, \frac{d^2 X}{d x^2}$, &c. becomes infinite on the supposition of $x = 0$.

 (90.) It will always be easy to find a developement of the value of the integral of any differential $X\,d\,x$, whenever we are able to transform the function X into a series of terms, each of which can be integrated. This will be better elucidated by some examples. We shall begin with one or two differentials which we have already integrated, in order to be able to compare together the results of various methods.

Example 1. Let $\frac{d\,x}{a+x}$ be the proposed differential. Here the function $\frac{1}{a+x}$ is easily developed according to the powers of x. We have

$$\frac{1}{a+x} = \frac{1}{a} - \frac{x}{a^2} + \frac{x^2}{a^3} - \frac{x^3}{a^4} + \&c.$$

and consequently

$$\int \frac{d\,x}{a+x} = \int d\,x \left\{ \frac{1}{a} - \frac{x}{a^2} + \frac{x^2}{a^3} - \frac{x^3}{a^4} + \&c. \right\} = \frac{x}{a} - \frac{x^2}{2\,a^2} + \frac{x^3}{3\,a^3} - \frac{x^4}{4\,a^4} + \&c. + c.$$

Comparing this last series with one of those obtained in (27) we see that it is equal to $l\,(x+a) - l\,a$. Hence $\int \frac{d\,x}{a+x} = l\,(x+a) - l\,a + c$, or simply $l\,(x+a) + c$, including $-\,l\,a$ in the constant. A result which is identical with the known value of $\int \frac{d\,x}{a+x}$.

Example 2. Let $\frac{d\,x}{\sqrt{(1-x^2)}}$ be the proposed differential. We can develope $\frac{1}{\sqrt{(1-x^2)}} = (1-x^2)^{-\frac{1}{2}}$ by the binomial theorem. We get

$$\frac{1}{\sqrt{(1-x^2)}} = 1 + \frac{1}{2}\,x^2 + \frac{1.3}{2.4}\,x^4 + \frac{1.3.5}{2.4.6}\,x^6 + \frac{1.3.5.7}{2.4.6.8}\,x^8 + \&c.$$

and
$$\int \frac{d\,x}{\sqrt{(1-x^2)}} = \frac{x}{1} + \frac{1}{2}\,\frac{x^3}{3} + \frac{1.3}{2.4}\,\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\,\frac{x^7}{7} + \&c. \quad + c.$$

But we know that $\int \frac{d\,x}{\sqrt{(1-x^2)}} = \sin^{-1} x + c$, consequently

$$\sin^{-1} x = \frac{x}{1} + \frac{1}{2}\,\frac{x^3}{3} + \frac{1.3}{2.4}\,\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\,\frac{x^7}{7} + \&c. \quad + c.$$

We arrive thus, in a very simple manner, to the developement of $\sin^{-1} x$, and by similar means we might obtain those of $\cos^{-1} x$, $\tan^{-1} x$, &c., and generally of all those functions, the differential coefficient of which may be developed according to the powers of the variable.

Example 3. We have found that $\int \frac{d\,x}{\sqrt{(1+x^2)}} = l\,(x + \sqrt{(1+x^2)}) + c$. We may easily get the developement of this logarithm, according to the powers of x. For we have

$$\frac{1}{\sqrt{(1+x^2)}} = (1+x^2)^{-\frac{1}{2}} = 1 - \frac{1}{2}\,x^2 + \frac{1.3}{2.4}\,x^4 - \frac{1.3.5}{2.4.6}\,x^6 + \&c.$$

Therefore
$$\int \frac{d\,x}{\sqrt{(1+x^2)}} = l\,(x + \sqrt{(1+x^2)}) = x - \frac{1}{2}\,\frac{x^3}{3} + \frac{1.3}{2.4}\,\frac{x^5}{5} - \frac{1.3.5}{2.4.6}\,\frac{x^7}{7} + \&c. + c.$$

It is especially when the integral cannot be obtained under a finite form, that it may be useful to find its developement. We shall take for the following examples differentials which cannot be integrated by the rules previously given.

Example 4. In this example, the integral of each term of the developement of the differential will be composed of several terms. The proposed differential is $\frac{d\,x\sqrt{(1-e^2\,x^2)}}{\sqrt{(1-x^2)}}$.

We have first
$$\sqrt{(1-e^2\,x^2)} = 1 - \frac{1}{2}\,e^2\,x^2 + \frac{1.1}{2.4}\,e^4\,x^4 - \frac{1.1.3}{2.4.6}\,e^6\,x^6 + \&c.$$

and
$$\int \frac{d\,x\,\sqrt{(1-e^2\,x^2)}}{\sqrt{(1-x^2)}} = \int \frac{d\,x}{\sqrt{(1-x^2)}} \left\{ 1 - \frac{1}{2}\,e^2\,x^2 + \frac{1.1}{2.4}\,e^4\,x^4 - \frac{1.1.3}{2.4.6}\,e^6\,x^6 + \&c. \right\}$$

each term of which may be integrated by means of the formulæ given in *Example* 4. (83.) Substituting these integrals we shall find

$$\int\frac{dx\,\sqrt{(1-e^2x^2)}}{\sqrt{(1-x^2)}}=\sin^{-1}x,$$
$$+\frac{1}{2}e^2\left\{\frac{1}{2}x\sqrt{(1-x^2)}-\frac{1}{2}\sin^{-1}x\right\},$$
$$+\frac{1.1}{2.4}e^4\left\{\left(\frac{1}{4}x^3+\frac{1.3}{2.4}x\right)\sqrt{(1-x^2)}-\frac{1.3}{2.4}\sin^{-1}x\right\},$$
$$+\frac{1.1.3}{2.4.6}e^6\left\{\left(\frac{1}{6}x^5+\frac{1.5}{4.6}x^3+\frac{1.3.5}{2.4.6}x\right)\sqrt{(1-x^2)}-\frac{1.3.5}{2.4.6}\sin^{-1}x\right\},$$
$$+\&c. \qquad +c.$$

Example 5. We shall take for this example the new transcendant $\int\frac{a^x\,dx}{x}$. We have found (26) that

$$a^x=1+\frac{la}{1}x+\frac{(la)^2}{1.2}x^2+\frac{(la)^3}{1.2.3}x^3+\&c.$$

Consequently $$\int\frac{a^x\,dx}{x}=\int dx\left\{\frac{1}{x}+\frac{la}{1}+\frac{(la)^2}{1.2}x+\frac{(la)^3}{1.2.3}x^2+\&c.\right\},$$
$$=lx+\frac{la}{1}x+\frac{(la)^2}{1.2}\frac{x^2}{2}+\frac{(la)^3}{1.2.3}\frac{x^3}{3}+\frac{(la)^4}{1.2.3.4}\frac{x^4}{4}+\&c.\quad+c.$$

If we substitute e for a, this formula becomes

$$\int\frac{e^x\,dx}{x}=lx+x+\frac{1}{1.2}\frac{x^2}{2}+\frac{1}{1.2.3}\frac{x^3}{3}+\frac{1}{1.2.3.4}\frac{x^4}{4}+\&c.\quad+c.$$

and making $x=ly$, in which case $\int\frac{e^x\,dx}{x}=\int\frac{dy}{ly}$, we find

$$\int\frac{dy}{ly}=lly+ly+\frac{1}{1.2}\frac{(ly)^2}{2}+\frac{1}{1.2.3}\frac{(ly)^3}{3}+\frac{1}{1.2.3.4}\frac{(ly)^4}{4}+\&c.\quad+c.$$

Example 6. Several series may be obtained for $\int\frac{a^x\,dx}{1-x}$. By developing a^x according to the powers of x, the value of $\int\frac{a^x\,dx}{1-x}$ would be expressed in a series of integrals of the form $\int\frac{x^m\,dx}{1-x}$. By successive integration by parts and substitutions, the following developement may easily be found

$$\int\frac{a^x\,dx}{1-x}=a^x\left\{\frac{1}{(1-x)\,la}-\frac{1}{(1-x)^2\,(la)^2}+\frac{1.2}{(1-x)^3\,(la)^3}-\frac{1.2.3}{(1-x)^4\,(la)^4}+\&c.\right\}+c.$$

To find a series arranged according to the powers of x, we shall observe that

$$\frac{1}{1-x}=1+x+x^2+x^3+\&c.\quad\text{and } a^x=1+\frac{x\,la}{1}+\frac{x^2\,(la)^2}{1.2}+\frac{x^3\,(la)^3}{1.2.3}+\&c.$$

these being multiplied will give a product of the form

$$A+Bx+Cx^2+Dx^3+\&c.$$

in which

$$A=1,\quad B=1+\frac{la}{1},\quad C=1+\frac{la}{1}+\frac{(la)^2}{1.2},\quad D=1+la+\frac{(la)^2}{1.2}+\frac{(la)^3}{1.2.3}+\&c.$$

Hence

$$\int\frac{a^x\,dx}{1-x}=x+\left(1+\frac{la}{1}\right)\frac{x^2}{2}+\left(1+\frac{la}{1}+\frac{(la)^2}{1.2}\right)\frac{x^3}{3}+\left(1+\frac{la}{1}+\frac{(la)^2}{1.2}+\frac{(la)^3}{1.2.3}\right)\frac{x^4}{4}+\&c.+c.$$

Example 7. We shall take for the last example the differential $x^{nx}\,dx$. Applying to x^{nx} the known developement of a^x, we find

$$x^{nx}=1+\frac{nx\,lx}{1}+\frac{n^2x^2\,(lx)^2}{1.2}+\frac{n^3x^3\,(lx)^3}{1.2.3}+\&c.$$

We have integrated (84) differentials of the form $x^m\,dx\,(lx)^n$. If we make use here of these formulæ, we shall have

$$\int x^{nx}\,dx=x+\frac{1}{1}\frac{1}{2}nx^2\left(lx-\frac{1}{2}\right),$$
$$+\frac{1}{1.2}\frac{1}{3}n^2x^3\left((lx)^2-\frac{2}{3}(lx)+\frac{2.1}{3^2}\right),$$
$$+\frac{1}{1.2.3}\frac{1}{4}n^3x^4\left((lx)^3-\frac{3}{4}(lx)^2+\frac{3.2}{4^2}(lx)-\frac{3.2.1}{4^3}\right),$$
$$+\&c.\quad+c.$$

Or if the terms ere erranged according to the powers of $(l\,x)$.

$$\int x^{nx}\,dx = x\left(1 - \frac{n\,x}{2^2} + \frac{n^2\,x^2}{3^3} - \frac{n^3\,x^3}{4^4} + \&c.\right)$$
$$+ \frac{n\,x^2\,l\,x}{1}\left(\frac{1}{2} - \frac{n\,x}{3^2} + \frac{n^2\,x^2}{4^3} - \frac{n^3\,x^3}{5^4} + \&c.\right)$$
$$+ \frac{n^2\,x^3\,(l\,x)^2}{1\,.\,2}\left(\frac{1}{3} - \frac{n\,x}{4^2} + \frac{n^2\,x^2}{5^3} - \frac{n^3\,x^3}{6^4} + \&c.\right)$$
$$+ \&c. \qquad + c.$$

(91.) When a differential $X\,d\,x$ is decomposed into an infinite series of terms, which we shall represent generally by

$$(A + A_1\,Y + A_2\,Y^2 + A_3\,Y^3 + \&c.)\,Z\,d\,x.$$

In which Y and Z are two functions of x. The integration of each term separately may be avoided when the differential $\frac{Z\,d\,x}{1 - a\,Y}$ can be integrated.

Let U be this integral, and let its developement according to the powers of a be

$$a^0\,V + a^1\,V_1 + a^2\,V_2 + a^3\,V_3 + \&c.$$

The differential $\frac{Z\,d\,x}{1 - a\,Y}$ developed according to the powers of the same letter gives

$$Z\,d\,x\,(1 + a\,Y + a^2\,Y^2 + a^3\,Y^3 + \&c.)$$

Therefore we must have

$$a^0\,V + a^1\,V_1 + a^2\,V_2 + \&c. = \int Z\,d\,x + a\int Z\,Y\,d\,x + a^2\int Z\,Y^2\,d\,x + \&c.,$$

and, consequently,

$$V = \int Z\,d\,x, \quad V_1 = \int Z\,Y\,d\,x, \quad V_2 = \int Z\,Y^2\,d\,x, \quad \&c.$$

Hence

$$\int X\,d\,x = \int Z\,d\,x\,(A + A_1\,Y + A_2\,Y^2 + A_3\,Y^3 + \&c.) = A\,V + A_1\,V_1 + A_2\,V_2 + \&c. + c.$$

Thus after heving integrated the differential $\frac{Z\,d\,x}{1 - a\,Y}$, end developed the integral according to the powers of a, it will be sufficient to substitute, in this series, for the successive powers of a, the coefficients A, A_1, A_2, &c. of the differential $(A + A_1\,Y + A_2\,Y^2 + A_3\,Y^3 + \&c.)\,Z\,d\,x$ to obtain its integral.

Let us apply this to the differential $\frac{e^x\,d\,x}{1 - x}$, a particular case of the differential $\frac{a^x\,d\,x}{1 - x}$ which we heve already integrated in *Example* 6. (90). We have

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{1\,.\,2} + \frac{x^3}{1\,.\,2\,.\,3} + \&c.$$

and

$$\int \frac{e^x\,d\,x}{1 - x} = \int\left(1 + \frac{x}{1} + \frac{x^2}{1\,.\,2} + \frac{x^3}{1\,.\,2\,.\,3} + \frac{x^4}{1\,.\,2\,.\,3\,.\,4} + \&c.\right)\frac{d\,x}{1 - x}.$$

But instead of integrating each differential term of this series, we shall, according to the preceding remark, observing that here $Y = x$, and $Z = \frac{1}{1 - x}$, integrate first $\frac{d\,x}{(1 - x)\,(1 - a\,x)}$. We easily get

$$\int \frac{d\,x}{(1 - x)\,(1 - a\,x)} = \frac{1}{(1 - a)}\,\{\,l\,(1 - a\,x) - l\,(1 - x)\,\} + c.$$

To develope this according to the powers of a, we have

$$\frac{1}{1 - a} = 1 + a + a^2 + a^3 + \&c., \text{ and } l\,(1 - a\,x) = -\frac{a\,x}{1} - \frac{a^2\,x^2}{2} - \frac{a^3\,x^3}{3} - \&c.$$

Consequently the developement of $\int \frac{d\,x}{(1 - x)\,(1 - a\,x)}$, according to the powers of a, will be

$$- l\,(1 - x)$$
$$+ a\left\{- l\,(1 - x) - \frac{x}{1}\right\}$$
$$+ a^2\left\{- l\,(1 - x) - \frac{x}{1} - \frac{x^2}{2}\right\}$$
$$+ a^3\left\{- l\,(1 - x) - \frac{x}{1} - \frac{x^2}{2} - \frac{x^3}{3}\right\}$$
$$+ \&c. \ldots\ldots\ldots + c.$$

And it is now sufficient to substitute, in this series, to the successive powers of a, the coefficients of the powers of x in the developement

$$1+\frac{x}{1}+\frac{x^2}{1.2}+\frac{x^3}{1.2.3}+\frac{x^4}{1.2.3.4}+\&c.$$

and we shall find

$$\begin{aligned}\int\frac{e^x\,d\,x}{1-x} &= -l(1-x)\\ &+\frac{1}{1}\left\{-l(1-x)-\frac{x}{1}\right\}\\ &+\frac{1}{1.2}\left\{-l(1-x)-\frac{x}{1}-\frac{x^2}{2}\right\}\\ &+\frac{1}{1.2.3}\left\{-l(1-x)-\frac{x}{1}-\frac{x^2}{2}-\frac{x^3}{3}\right\}\\ &+\&c. \qquad\qquad +c.\end{aligned}$$

Or by putting for $l(1-x)$ its developement, arranging then the terms according to the powers of x,

$$\int\frac{e^x\,d\,x}{1-x}=\frac{x}{1}+\left(1+\frac{1}{1}\right)\frac{x^2}{2}+\left(1+\frac{1}{1}+\frac{1}{1.2}\right)\frac{x^3}{3}+\left(1+\frac{1}{1}+\frac{1}{1.2}+\frac{1}{1.2.3}\right)\frac{x^4}{4}+\&c.+c.$$

A result agreeing with that obtained in *Example* 6. (90.)

(92.) In the applications of the Integral Calculus, it is not, in most cases, under the general and undeterminate form that we have obtained them, that the integrals of differential expressions are required. What is wanted generally, is the difference of the values assumed by the integral, such as we have found it, when for the variable two particular values are successively substituted. In taking this difference the arbitrary constant disappears, and a result is obtained in which nothing remains undeterminate. This result is called a *definite integral*, and the two quantities substituted for the variable, are the *limits* of the integral. *Indefinite integrals*, on the contrary, are like those we have hitherto considered, in which the variable and the constant remain undeterminate. Thus we have found that the general or indefinite integral of $x^m\,d\,x$ was $\frac{x^{m+1}}{m+1}+c$; the definite integral of the same differential between the limits a and b will be the difference of the values $\frac{a^{m+1}}{m+1}+c$, $\frac{b^{m+1}}{m+1}+c$, of the general integral corresponding to $x=a$ and $x=b$, and is therefore equal to $\frac{a^{m+1}-b^{m+1}}{m+1}$.

To designate a definite integral the sign $\int$ is still used, and the two limits are placed by the side of it, the one corresponding to the value of the integral which is substracted below, and the other above. Thus we have

$$\int_b^a x^m\,d\,x=\frac{a^{m+1}-b^{m+1}}{m+1}.$$

In the same manner we shall have

$$\int_b^a\frac{d\,x}{x}=l\left(\frac{a}{b}\right)\int_m^{m+1}e^x\,d\,x=e^m(e-1)\int_0^1\frac{d\,x}{\sqrt{(1-x^2)}}=\frac{\pi}{4}.$$

(93.) When the indefinite integral is known, the determination of the value of the definite integral presents no difficulty, since, to find it, it is sufficient to take the difference between the values of the indefinite integral corresponding to the two limits. But, in many cases, the value of the definite integral may be obtained, although that of the indefinite integral cannot. These determinations form one of the most important parts of the Integral Calculus, and will be treated separately with all that relates to definite integrals. In this place we shall limit ourselves to a few remarks which will be necessary to understand the analytical and geometrical applications of the Differential and Integral Calculus.

(94.) Let $X\,d\,x$ be a differential of which it is required to find the definite integral between the limits a and b. Let the indefinite integral be represented by $f(x)$, and let $a-b=h$. We shall have by Taylor's theorem

$$f(x+h)=f(x)+X\frac{h}{1}+\frac{d\,X}{d\,x}\frac{h^2}{1.2}+\frac{d^2X}{d\,x^2}\frac{h^3}{1.2.3}+\&c.$$

If we make in both sides of this equation $x=b$, and if we designate by Y', Y'', Y''', &c. the values assumed by X, $\frac{d\,X}{d\,x}$, $\frac{d^2X}{d\,x^2}$, &c. in that supposition, we shall get

$$f(b+h)=f(a)=f(b)+Y'h+Y''\frac{h^2}{1.2}+Y'''\frac{h^3}{1.2.3}+\&c.$$

and consequently

$$f(a)-f(b)=\int_b^a X\,d\,x=Y'h+Y''\frac{h^2}{1.2}+Y'''\frac{h^3}{1.2.3}+\&c.$$

and if none of the quantities X, $\frac{dX}{dx}$, $\frac{d^2X}{dx^2}$, has become infinite by making in them $x = b$, we have a series representing the value of the definite integral, which, if converging, may be used to find its approximate value.

(95.) It is therefore necessary to examine in what cases the series of Taylor is converging, or more generally to determine the limits of the series beginning with any term. We shall first demonstrate the following proposition.

Every function U of x which vanishes for $x = 0$, and the first differential coefficient of which, designated by U′, neither becomes infinite, nor changes its sign, for any value of the variable between the two limits $x = 0$, and $x = b$, is of the same sign as the differential coefficient, if b be positive, and of a contrary sign if b be negative.

Let b be divided into any number of equal parts represented by i, and let

$$0,\ U_1,\ U_2,\ U_3,\ \&c. \qquad U_0',\ U_1',\ U_2',\ U_3',\ \&c.$$

be the values of U and U′ corresponding to

$$x = 0,\ = i,\ = 2i,\ = 3i,\ \&c.$$

By Taylor's theorem, we have for the developement of what U becomes when $x + i$ is substituted for x,

$$U + U'i + \frac{dU'}{dx}\cdot\frac{i^2}{1.2} + \frac{d^2U'}{dx^2}\,\frac{i^3}{1.2.3} + \&c.$$

and since the first differential coefficient does not become infinite for any of the above values of x, we shall have in substituting them successively in this series, and representing by $V_0 i^a$, $V_1 i^a$, $V_2 i^a$, $V_3 i^a$, the corresponding values of the part which follows the two first terms, in the following equations

$$\begin{aligned} U_1 &= U_0'\,i + i^a V_0,\\ U_2 - U_1 &= U_1'\,i + i^a V_1,\\ U_3 - U_2 &= U_2'\,i + i^a V_2,\\ &+ \dots\dots\dots\dots\dots\\ + U_n - U_{n-1} &= U'_{n-1} + i^a V_{n-1}. \end{aligned}$$

We must first observe that the exponent a is necessarily greater than one, and secondly, that since when $x = 0$, U vanishes, and consequently, that when $i = 0$, U_1, U_2, U_3, &c. become also equal to nothing, none of the quantities V_0, V_1, V_2, &c. can become infinite in the supposition of $i = 0$. Hence by taking a value for i sufficiently small, the second terms of the right sides of each of these equations may be made less than the first terms in any proportion whatever; the signs of the quantities $U_0'\,i + i^a V_0$, $U_1'\,i + i^a V_1$, &c. will therefore be the same as those of $U_0'\,i$, $U_1'\,i$, $U_2'\,i$, &c. But we have supposed that U_0', U_1', U_2', &c. had all the same sign, hence this is also the case with $U_0'\,i$, $U_1'\,i$, $U_2'\,i$, &c. and consequently with U_1, $U_2 - U_1$, $U_3 - U_2$, &c. Therefore, finally, the quantities U_1, U_2, ... U_n will have the same sign as the differential coefficient U′, if i or b be positive, and a contrary sign if b be negative.

(96.) We shall suppose now that in the series of Taylor a particular value has been substituted for x, but we shall continue to represent the developement by

$$u' = u + \frac{du}{dx}h + \frac{d^2u}{dx^2}\frac{h^2}{1.2} + \frac{d^3u}{dx^3}\frac{h^3}{1.2.3} + \&c.$$

The value of the series will then vary only with the value of h.

We have proved before that generally

$$\frac{d^n u'}{dx^n} = \frac{d^n u'}{dh^n},$$

these differential coefficients are functions of h, and vary accordingly with the value of that variable. Let m be the least, and M the greatest value of $\frac{d^n u'}{dx^n} = \frac{d^n u'}{dh^n}$, corresponding to the values of h between the limits $h = 0$, and $h =$ any constant quantity, so that

$$M - \frac{d^n u'}{dh^n},\ \text{and}\ \frac{d^n u'}{dh^n} - m,$$

are functions of h which will remain positive for any value of h between these limits. These quantities are respectively the first differential coefficients of

$$Mh - \frac{d^{n-1}u'}{dh^{n-1}},\ \text{and}\ \frac{d^{n-1}u'}{dh^{n-1}} - mh,$$

and consequently, of

$$Mh - \left(\frac{d^{n-1}u'}{dh^{n-1}} - \frac{d^{n-1}u}{dx^{n-1}}\right),\ \text{and}\ \frac{d^{n-1}u'}{dh^{n-1}} - \frac{d^{n-1}u}{dx^{n-1}} - mh,$$

since $\frac{d^{n-1}u}{dx^{n-1}}$ does not contain h. But these new expressions vanish for $h = 0$, for then $u' = u$, and we have

besides $\frac{d^{n-1} u'}{d h^{n-1}} = \frac{d^{n-1} u'}{d x^{n-1}}$. Therefore by the theorem demonstrated (95), these expressions are of the same sign as their first differential coefficients respectively, that is to say, both positive, for all the values of h between the assigned limits. Again, they may be considered as the differential coefficients of

$$\mathrm{M}\,\frac{h^2}{1\,.\,2} - \left(\frac{d^{n-2} u'}{d h^{n-2}} - \frac{d^{n-2} u}{d x^{n-2}} - \frac{d^{n-1} u}{d x^{n-1}} h\right), \text{ and } \frac{d^{n-2} u'}{d h^{n-2}} - \frac{d^{n-2} u}{d x^{n-2}} - \frac{d^{n-1} u}{d x^{n-1}} h - m\,\frac{h^2}{1\,.\,2}$$

After observing that these new expressions become nothing when $h = 0$, we shall conclude as before that they remain positive for all the values of h between the assigned limits.

Proceeding with the same reasoning, we shall be able to prove that the two following quantities are both positive between the same limits,

$$\frac{\mathrm{M}\,h^n}{1\,.\,2\,.\,3 \ldots . n} - \left(u' - u - \frac{d u}{d x} h - \frac{d^2 u}{d x^2}\frac{h^2}{1\,.\,2} \ldots . - \frac{d^{n-1} u}{d x^{n-1}}\frac{h^{n-1}}{1\,.\,2 \ldots . n-1}\right), \text{ and}$$

$$u' - u - \frac{d u}{d x} h - \frac{d^2 u}{d x^2}\frac{h^2}{1\,.\,2} - \ldots\ldots\ldots \frac{d^{n-1} u}{d x^{n-1}}\frac{h^{n-1}}{1\,.\,2 \ldots . n-1} - \frac{m\,h^n}{1\,.\,2 \ldots . n}.$$

Let us now substitute for u' its value, given by the series of Taylor, and we shall find

$$\frac{\mathrm{M}\,h^n}{1\,.\,2\,.\,3 \ldots . n} > \frac{d^n u}{d x^n}\frac{h^n}{1\,.\,2\,.\,3 \ldots . n} - \frac{d^{n+1} u}{d x^{n+1}}\frac{h^{n+1}}{1\,.\,2\,.\,3 \ldots . n+1} - \&\text{c. and}$$

$$\frac{m\,h^n}{1\,.\,2\,.\,3 \ldots . u} < \frac{d^n u}{d x^n}\frac{h^n}{1\,.\,2\,.\,3 \ldots . n} + \frac{d^{n+1} u}{d x^{n+1}}\frac{h^{n+1}}{1\,.\,2\,.\,3 \ldots . n+1} + \&\text{c.}$$

Therefore $\frac{\mathrm{M}\,h^n}{1\,.\,2\,.\,3 \ldots . n}$, and $\frac{m\,h^n}{1\,.\,2\,.\,3 \ldots . u}$, are the limits between which are included the whole of that part of the series of Taylor beginning with the $(n+1)^{st}$ term; or, in other words, remembering what M and m are intended to represent, we conclude that when the series of Taylor is limited to the n first terms, the part neglected has for limits the greatest and least values of $\frac{d^n u'}{d h^n} = \frac{d^n u'}{d x^n}$, multiplied by $\frac{h^n}{1\,.\,2\,.\,3 \ldots . u}$.

(97.) It is easy to infer from the preceding investigation, that, in the series of Taylor, a value may always be assigned to h which will make any term greater than the sum of all those which follow it. For

$$\frac{d^n u}{d x^n}\frac{h^n}{1\,.\,2 \ldots . n} + \frac{d^{n+1} u}{d x^{n+1}}\frac{h^{n+1}}{1\,.\,2 \ldots . n+1} + \&\text{c.}$$

being included between the greatest and the least value of $\frac{d^n u'}{d x^n}$, multiplied by $\frac{h^n}{1\,.\,2 \ldots . n}$, it is obvious that there must exist an intermediate value of this quantity, which being multiplied by $\frac{h^n}{1\,.\,2 \ldots . n}$, will be precisely equal to $\frac{d^n u}{d x^n}\frac{h^n}{1\,.\,2 \ldots . n} + \frac{d^{n+1} u}{d x^{n+1}}\frac{h^{n+1}}{1\,.\,2 \ldots . u+1} + \&\text{c.}$

Let U_n be this value, then we shall have exactly

$$u' = u + \frac{d u}{d x} h + \frac{d^2 u}{d x^2}\frac{h^2}{1\,.\,2}\ldots\ldots\ldots + \frac{d^{n-1} u}{d x^{n-1}}\frac{h^{n-1}}{1\,.\,2 \ldots . n-1} + \frac{\mathrm{U}_n\,h^n}{1\,.\,2 \ldots . n}.$$

To find the value of h which will make $\frac{d^{n-1} u}{d x^{n-1}}\frac{h^{n-1}}{1\,.\,2 \ldots . u-1}$ greater than the remainder of the series, it is therefore sufficient to find that which will make that term greater than $\frac{\mathrm{U}_n\,h^n}{1\,.\,2 \ldots . n}$, and we shall clearly satisfy this condition by taking

$$h < \frac{n}{\mathrm{U}_n}\frac{d^{n-1} u}{d x^{n-1}}.$$

Thus to obtain the required value of h it will not be necessary to know the value of U_n, but simply any quantity greater than it, for instance the greatest value of $\frac{d^n u'}{d x^n}$.

When we are at liberty to take any value for the quantity h, and that none of the differential coefficients become infinite for the particular value of x, the series of Taylor may always be rendered a converging series, since each term may be made greater than all those which follow it.

In the series we have found (94) for $\int_b^a \mathrm{X}\,d\,x$, h has a determinate value equal to $a - b$, and therefore what has just been stated cannot be applied; but we shall always be able to determine the limits of the error made by taking only a limited number of terms of the series.

(98.) Before proceeding to investigate other series for the value of $\int_b^a \mathrm{X}\,d\,x$, we must briefly state that with respect to the developement of a function of two or more variables, limits of the series may also be determined.

Let $u = f.(x, y)$, $u' = f(x + h, y + k)$. For h and k put $h\,t$ and $k\,t$, then u' may be considered as a function of t, and if we substitute instead of it $t + a$, the two following developements, which must be equal to each other, are obtained:

$$u' = u + \frac{d\,u}{d\,t}\frac{a}{1} + \frac{d^2 u}{d\,t^2}\frac{a^2}{1\,.\,2} + \frac{d^3 u}{d\,t^3}\frac{a^3}{1\,.\,2\,.\,3} + \&c.$$

$$u' = u + \frac{a}{1}\left(\frac{d\,u}{d\,x}h + \frac{d\,u}{d\,y}\,k\right) + \frac{a^2}{1\,.\,2}\left(\frac{d^2 u}{d\,x^2}h^2 + 2\,\frac{d^2 u}{d\,x\,d\,y}h\,k + \frac{d^2 u}{d\,y^2}k^2\right) + \&c.$$

If we take only n terms of the first, the sum of the remaining terms will have for limits the greatest and smallest value of $\frac{d^n u'}{d\,t^n}$ multiplied by $\frac{a^n}{1\,.\,2\,.\,.\,.\,.\,n}$. In the same supposition, the limits of the remaining terms of the second developement will therefore be the greatest and the least value of

$$\frac{d^n u}{d\,x^n}h^n + n\,\frac{d^{n-1} u}{d\,x^{n-1}}h^{n-1}k + \ldots.\ \frac{d^n u}{d\,y^n}k^n,$$

multiplied by $\frac{a^n}{1\,.\,2\,.\,.\,.\,n}$. Making in this last result $a = 1$, we find the limits of the developement of $f(x + h, y + k)$.

(99.) Let us return now to the definite integral $\int_b^a X\,d\,x$, if we suppose $a - b = n\,i$, n being an integer, by taking it sufficiently large, i may be made less than any assigned quantity. Let Y, Y_1, Y_2, &c. be the values of the indefinite integral, corresponding to $x = b, = b + i, = b + 2\,i$, &c., and Y', Y_1', Y_2', &c., Y'', Y_1'', Y_2'', &c. Y''', Y_1''', Y_2''', &c. the values of X, $\frac{d\,X}{d\,x}$, $\frac{d^2 X}{d\,x^2}$, &c. corresponding to the same values. Then we shall get, in the same manner as in (94), the following equations:

$$Y_1 = Y + Y'\ i + Y''\,\frac{i^2}{1\,.\,2} + Y'''\,\frac{i^3}{1\,.\,2\,.\,3} + \&c.$$

$$Y_2 = Y_1 + Y_1'\,i + Y_1''\,\frac{i^2}{1\,.\,2} + Y_1'''\,\frac{i^3}{1\,.\,2\,.\,3} + \&c.$$

$$Y_3 = Y_2 + Y_2'\,i + Y_2''\,\frac{i^2}{1\,.\,2} + Y_2'''\,\frac{i^3}{1\,.\,2\,.\,3} + \&c.$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$Y_n = Y_{n-1} + Y_{n-1}'\,i + Y_{n-1}''\,\frac{i^2}{1\,.\,2} + Y_{n-1}'''\,\frac{i^3}{1\,.\,2\,.\,3} + \&c.$$

If we add all these equations, suppress the terms which would be common to the two sides of the sum, and place Y on the left side, we find

$$\begin{aligned} Y_n - Y = \int_b^{b+ni} X\,d\,x = {} & i\,(Y' + Y_1' + Y_2' + \ldots.\ + Y_{n-1}') \\ & + \frac{i^2}{1.2}\,(Y'' + Y_1'' + Y_2'' + \ldots.\ + Y_{n-1}'')\ldots\ldots\ldots(e) \\ & + \frac{i^3}{1.2.3}\,(Y''' + Y_1''' + Y_2''' + \ldots.\ + Y_{n-1}''') \\ & + \&c. \end{aligned}$$

Instead of substituting successively for x the values, $b, b + i, b + 2\,i$, &c., we might have followed an inverted order, beginning with $b + n\,i$, $b + (n - 1)\,i$, &c.... down to b. We find in this manner

$$\begin{aligned} Y_n - Y = \int_b^{b+ni} X\,d\,x = {} & i\,(Y_1' + Y_2' + \ldots\ldots\ldots\ldots + Y_n') \\ & - \frac{i^2}{1.2}\,(Y_1'' + Y_2'' + Y_3'' + \ldots\ldots\ldots\ Y_n'')\ldots\ldots\ldots(f) \\ & + \frac{i^3}{1.2.3}\,(Y_1''' + Y_2''' + Y_3''' + \ldots\ldots\ldots\ Y_n''') \\ & - \&c. \end{aligned}$$

The two series (e) and (f) are formed by the addition of a limited number of series, in each of which any term may be made greater than the sum of all the following ones, by taking i sufficiently small. (97.) Hence it is easy to infer that these two series will have the same property, and consequently, that they may be considered as converging series.

(100.) Another important consequence, relative to the equations (e) and (f), may be derived from the theorem demonstrated (97.) It results from this proposition, not only that a value may always be assigned to i which will make any term greater than the sum of all those which follow, but also, that by taking for i values less and less than this, the remainder of the series may be rendered less than any assigned quantity, however small. Hence, in the equations (e) and (f) the quantities in the left side are, at the same time, the sums of

the series; and with respect to the decreasing values of i, the limits of any limited number of their terms, beginning with the first. Therefore the definite integral $\int_b^{b+ni} X\,d\,x$, with respect to decreasing values of i, is the limit of the first term of either of these series. But the quantities $i\,y'$, $i\,y''$, $i\,y'''$, &c., of which the first terms are composed, are the values of $X\,d\,x$ corresponding to $x = b$, $= b + i$, $= b + 2\,i$, &c.... and $d\,x = i$, consequently the definite integral of a differential expression $X\,d\,x$, taken between two limits $a = b + n\,i$ and b, may be considered, with respect to increasing values of n, or decreasing values of i, as the limit of the sum of the values assumed by that differential when i is substituted for $d\,x$, and x successively replaced by the terms of either of the two following arithmetical progressions:

$$b,\quad b+i,\quad b+2\,i\,.....\,b+(n-i)\,i.$$

$$b+i,\quad b+2\,i,\quad b+3\,i\,.....\,b+n\,i.$$

(101.) When it is known or assumed that for a particular value $x = c$ the integral of a differential expression vanishes, this value is said to be the *origin* of the integral. In that case the integral may be considered as the definite integral between the limits x and c; it may be represented by $\int_c^x X\,d\,x$, and all that has been said hitherto, with respect to definite integrals, applies to it.

(102.) When instead of the first differential coefficient, it is that of a higher order that is known, the determination of the primitive function will require several integrations, and as many arbitrary constants will be introduced.

Let X be the given differential coefficient, which we shall suppose to be of the n^{th} order, and let y represent the primitive function, then

$$\frac{d^n y}{d\,x^n} = X.$$

Multiplying both sides by $d\,x$, and integrating, we get

$$\int \frac{d^n y}{d\,x^{n-1}} = \int \frac{d^{n-1} y}{d\,x^{n-1}} = \int X\,d\,x + c.$$

If we multiply again by $d\,x$, and integrate, and we find

$$\int \frac{d^{n-1} y}{d\,x^{n-2}} = \frac{d^{n-2} y}{d\,x^{n-2}} = \int \{\,d\,x \int X\,d\,x\,\} + c\,x + c_1.$$

The same operation being repeated n times will give the value of y with n arbitrary constants.

A symmetrical form may be given to the value of y, by means of the integration by parts. We first observe that $d^n y = X\,d\,x^n$, and consequently

$$d^{n-1} y = \int X\,d\,x^n,\quad d^{n-2} y = \int\!\!\int X\,d\,x^n \text{ or } \int^2 X\,d\,x^n,\quad d^{n-3} y = \int\!\!\int^2 X\,d\,x^n = \int^3 X\,d\,x^n,$$

and

$$y = \int^n X\,d\,x^n.$$

We shall now examine the transformations which may be made upon

$$\int^2 X\,d\,x^2,\ \int^3 X\,d\,x^3,\ \text{\&c.}$$

We find, according to the rules of integration by parts, and recollecting that $d\,x$ is constant,

$$\int^2 X\,d\,x^2 = \int\!\!\int X\,d\,x^2 = \int d\,x \int X\,d\,x = x \int X\,d\,x - \int x\,X\,d\,x;$$

by means of this value we shall have

$$\int^3 X\,d\,x^3 = \int d\,x \int d\,x \int X\,d\,x = \int x\,d\,x \int X\,d\,x - \int d\,x \int x\,X\,d\,x.$$

But

$$\int x\,d\,x \int X\,d\,x = \frac{1}{2}\,x^2 \int X\,d\,x - \frac{1}{2} \int x^2\,X\,d\,x,$$

and

$$\int d\,x \int x\,X\,d\,x = x \int x\,X\,d\,x - \int x^2\,X\,d\,x.$$

These values being substituted, we shall have after reduction

$$\int^3 X\,d\,x^3 = \frac{1}{2}\,(x^2 \int X\,d\,x - 2\,x \int x\,X\,d\,x + \int x^2\,X\,d\,x).$$

By the same means, we shall find the value of $\int^4 X\,d\,x^4$, &c. Thus we have

$$\int X\,d\,x = \int X\,d\,x,$$

$$\int^2 X\,d\,x^2 = \frac{1}{1}\,(x \int X\,d\,x - \int x\,X\,d\,x),$$

$$\int^3 X\,d\,x^3 = \frac{1}{1.2}\,(x^2 \int X\,d\,x - 2\,x \int x\,X\,d\,x + \int x^2\,X\,d\,x),$$

$$\int^4 X\,d\,x^4 = \frac{1}{1.2.3}\,(x^3 \int X\,d\,x - 3\,x^2 \int x\,X\,d\,x + 3\,x \int x^2\,X\,d\,x + \int x^3\,X\,d\,x),$$

&c. &c.

The law of these values is obvious, and we may, without difficulty, form, by analogy, the developement of $\int^n X\,d\,x^n$; and we shall find

$$\int^n X\,dx^n = \frac{1}{1.2.3.\ldots.n-1}\left(x^{n-1}\int X\,dx - (n-1)\,x^{n-2}\int x\,X\,dx + \frac{(n-1)(n-2)}{1.2}\,x^{n-3}\int x^2\,X\,dx - \text{\&c.} \ldots \pm \int x^{n-1}\,X\,dx\right),$$

the last term being $+$ when $n-1$ is an even number, and $-$ when it is odd.

To prove that this value is exact, it will be sufficient to show that the law, according to which it has been formed, is true, it being admitted that $\int^n X\,dx^n$ will subsist for $\int^{n+1} X\,dx^{n+1}$. For since it has been verified for $\int^2 X\,dx^2$, $\int^3 X\,dx^3$, it will, of course, be true for all the following orders.

We have $\int^{n+1} X\,dx^{n+1} = \int dx \int^n X\,dx^n$, substituting for $\int^n X\,dx^n$ the above developement, integrating by parts each term, and uniting under the same coefficient similar integrals we find

$$\int^{n+1} X\,dx^{n+1} = \frac{1}{1.2.\ldots.n}\left(x^n\int X\,dx - \frac{n}{1}\,x^{n-1}\int x\,X\,dx + \frac{n(n-1)}{1.2}\,x^{n-2}\int x^2\,X\,dx \ldots\ldots \pm n\,x\int x^{n-1}\,X\,dx\right) - \frac{1}{1.2.\ldots.n}\left(1 - \frac{n}{1} + \frac{n(n-1)}{1.2}\ \ldots\ldots\ldots \pm n\right)\int x^n\,X\,dx.$$

The coefficient of $\int x^n X\,dx$ is equal to $(1-1)^n \pm 1 = \pm 1$, and therefore the assumed law is verified for $\int^{n+1} X\,dx^{n+1}$.

There is another developement of $\int^n X\,dx^n$, no term of which requires to be integrated. By applying to $\int X\,x^n\,dx$ the integration by parts, we shall easily find

$$\int X\,x^n\,dx = X\,\frac{x^{n+1}}{n+1} - \frac{dX}{dx}\,\frac{x^{n+2}}{(n+1)(n+2)} + \frac{d^2X}{dx^2}\,\frac{x^{n+3}}{(n+1)(n+2)(n+3)} - \text{\&c.} + c.$$

If in this series we suppose successively $n = 0, = 1, = 2, \ldots = n-1$, and substitute the values of $\int X\,dx$, $\int x\,X\,dx$, $\int x^2\,X\,dx$, &c. which will arise, in the value found above for $\int^n X\,dx^n$, we shall have, after reduction,

$$\int^n X\,dx^n = \frac{X\,x^n}{1.2.\ldots.n} - \frac{dX}{dx}\,\frac{\frac{n}{1}\,x^{n+1}}{1.2.\ldots.n+1} + \frac{d^2X}{dx^2}\,\frac{\frac{n(n+1)}{2}\,x^{n+2}}{1.2.\ldots\ldots n+2} - \frac{d^3X}{dx^3}\,\frac{\frac{n(n+1)(n+2)}{1.2.3}\,x^{n+3}}{1.2.3.\ldots.n+3} + \text{\&c.}$$

To complete this developement, it is necessary to add to it the terms containing the arbitrary constants, which are clearly

$$\frac{C\,x^{n-1}}{1.2.\ldots.n-1} + \frac{C_1\,x^{n-2}}{1.2.\ldots.n-2} + \frac{C_2\,x^{n-3}}{1.2.\ldots.n-3} + \text{\&c.}$$

or simply $C\,x^{n-1} + C_1\,x^{n-2} + C_2\,x^{n-3} + \text{\&c.}$ including the denominators in the constants C, C_1, C_2, &c.

(103.) We proceed now to the integration of differentials containing more than one variable.

With respect to functions of more than one variable, two cases may occur. In the first it may be required to find the value of the primitive function, when one of the partial differential coefficients is given; and in the second to determine the primitive function when the complete differential is known.

The first case presents no greater difficulty than the integration of the differential coefficient of a function of one variable. If it be the partial differential coefficient with respect to x that is given, then all the other variables must be considered as constant, and the integration is to be performed by means of the preceding rules; but instead of adding an arbitrary constant, it will be an arbitrary function of all the other variables that will be added.

(104.) Let us next examine the second case. We have seen (35) that the differential of a function of several variables, of three, for instance, is of the form

$$X\,dx + Y\,dy + Z\,dz,$$

X, Y, Z being respectively the partial differential coefficients of the function with respect to x, to y, and to z. If in the given differential it happen that X contains neither y nor z, Y neither x nor z, Z neither x nor y, then the integration will present no difficulty, for we shall have obviously

$$\int (X\,dx + Y\,dy + Z\,dz) = \int X\,dx + \int Y\,dy + \int Z\,dz + c.$$

(105.) When the variables are mixed in the quantities X, Y, Z, &c. this method of integration cannot be applied. To begin with the simplest case, let $M\,dx + N\,dy$ be the differential of a function of two variables, in which M and N are each functions of the two variables x and y. If u represent the primitive function, then

$$\frac{du}{dx} = M, \quad \text{and}\ \frac{du}{dy} = N;$$

and because it has been proved (35) that $\frac{d^2 u}{dx\,dy} = \frac{d^2 u}{dy\,dx}$, the two quantities M and N must be such that

$$\frac{d\,M}{dy} = \frac{d\,N}{dx}.$$

Unless this condition be satisfied, $M\,dx + N\,dy$ cannot be the result of the differentiation of e function of two variables. When the equation $\frac{d\,M}{dy} = \frac{d\,N}{dx}$ obtains, we shall find the integral in the following menner. Since $\frac{du}{dx} = M$, we have $u = \int M\,dx + Y$, the integration being performed with respect to the variable x alone, and Y being an arbitrary function of y. To determine the value of Y, we observe that $\frac{du}{dy}$ must be equal to N, hence we must have

$$\frac{d\int M\,dx}{dy} + \frac{d\,Y}{dy} = N;$$

or, if we represent $\int M\,dx$ by v, $\frac{dv}{dy} + \frac{d\,Y}{dy} = N$, and consequently $Y = \int\left(N - \frac{dv}{dy}\right) dy + c$.
Therefore we shall have

$$u = \int M\,dx + \int\left(N - \frac{dv}{dy}\right) dy + c.$$

We may derive from this result the condition of integrability, already determined. It is obvious, that M being the partial differential coefficient of u with respect to x, $N - \frac{dv}{dy}$ must be independant of x, therefore its differential coefficient with respect to that variable must be equal to nothing; that is to say, we must have

$\frac{d\,N}{dx} - \frac{d^2 v}{dy\,dz} = 0$, or $\frac{d\,N}{dx} = \frac{d^2 v}{dy\,dx} = \frac{d\frac{dv}{dx}}{dy}$, but v being equal to $\int M\,dx$, $\frac{dv}{dx} = M$, therefore $\frac{d\,N}{dx} = \frac{d\,M}{dy}$, which is the condition previously found.

(106.) Differentials of functions of more than two variables may be integrated by generalizing the rules already given. It will be sufficient to consider the case of a function of three variables; and then it will be easy to extend the same method to any other number.

Let $M\,dx + N\,dy + P\,dz$ be the proposed differentiel, M, N, P being functions of x, y, and z, and let u represent the primitive function. Then

$$\frac{du}{dx} = M, \quad \frac{du}{dy} = N, \quad \frac{du}{dz} = P,$$

and consequently unless we have

$$\frac{d\,M}{dy} = \frac{d\,N}{dx}, \quad \frac{d\,M}{dz} = \frac{d\,P}{dx}, \quad \frac{d\,N}{dz} = \frac{dz}{dy},$$

$M\,dx + N\,dy + P\,dz$ cannot be the differential of a function of three variables. But if these conditions are fulfilled, then the integral may readily be obtained. In that hypothesis each of the three quantities $M\,dx + N\,dy$, $M\,dx + P\,dz$, $N\,dy + P\,dz$, represents a complete differential of u, corresponding respectively to the supposition of z, y, x, being considered as constant. Any one of them may therefore be integrated by the preceding rule. Let v be the integral of $M\,dx + N\,dy$, for instance, we shall have

$$\int (M\,dx + N\,dy + P\,dz) = v + Z,$$

Z being a function of z alone, which must be determined by the condition that the partial differential coefficient of $v + Z$, with respect to z shall be equal to P, that is $P = \frac{dv}{dz} + \frac{dz}{dz}$. From this last equation we find

$$\frac{dz}{dz} = P - \frac{dv}{dz} \text{ and } Z = \int\left(P - \frac{dv}{dz}\right) dz + c.$$

Hence it is necessary, in order to be able to integrate the proposed differential, that $P - \frac{dv}{dz}$ should contain neither x nor y, which condition we shall express by making the differential coefficient of $P - \frac{dv}{dz}$ with respect to either of these variables equal to nothing; thus we must have

$$\frac{d\,P}{dx} - \frac{d^2 v}{dx\,dz} = 0, \text{ and } \frac{d\,P}{dy} - \frac{d^2 v}{dy\,dz} = 0.$$

But
$$\frac{d^2 v}{dx\,dz} = \frac{d\,M}{d\,z}, \quad \text{and} \quad \frac{d^2 v}{dy\,dz} = \frac{d\,N}{d\,z},$$
therefore
$$\frac{d\,P}{d\,x} = \frac{d\,M}{d\,z}, \quad \text{and} \quad \frac{d\,P}{d\,y} = \frac{d\,N}{d\,z}.$$

These two equations, together with the supposition we have already made that $M\,dx + N\,dy$ was a complete differential; or, which is the same thing, that $\frac{d\,M}{dy} = \frac{d\,N}{d\,x}$, are precisely the expression of the conditions which should be fulfilled in order that $M\,dx + N\,dy + P\,dz$ might be the differential of a function of three variables; therefore, when they are satisfied, the integral may always be found.

After having proved that the conditions of integrability are fulfilled, the value of the primitive function may be obtained by integrating with respect to x alone the term $M\,dx$; with respect to y the terms of $N\,dy$, which do not contain x; and, finally, with respect to z the terms of z, which contain neither x nor y.

It is obvious, now, that the number of conditions of integrability relative to differentials of n variables is $\frac{n(n-1)}{2}$, and that to obtain the integral of such a differential, it will be sufficient to determine, first, the integral of the differential, considering one of the variables as a constant, and to add to it an arbitrary function of that variable, which will be determined by the method already used.

(107.) The differentials of an order higher than the first, may be considered, with respect to functions of several variables, as well as with respect to functions of one variable, as the first differentials of the differentials of the order immediately preceding. Hence when the differential of any order n is given, and that it is proposed to pass from it to the differential of the order $n - 1$, the conditions of integrability we have found above, and the methods of integration which have been explained, may still be used. However, they require some modifications, which will be sufficiently elucidated by the following remarks upon a differential of the second order of a function of two variables.

Let $Q\,dx^2 + R\,dx\,dy + S\,dy^2$ be the proposed differential. We must observe, first, that the term $R\,dx\,dy$ is the aggregate of two terms, the one resulting from a differentiation with respect to x, and the other from a differentiation with respect to y. In order to put the proposed differential under the form $M\,dx + N\,dy$, we shall assume $R = R' + R''$, and we shall be able then to write it in the following manner,
$$(Q\,dx + R'\,dy)\,dx + (R''\,dx + S\,dy)\,dy.$$
The condition expressing that this is the differential of a differential of the first order will be
$$\frac{d\,.\,(Q\,dx + R'\,dy)}{d\,y} = \frac{d\,(R''\,dx + S\,dy)}{d\,x},$$
or developing
$$\frac{d\,Q}{d\,y}\,dx + \frac{d\,R'}{d\,y}\,dy = \frac{d\,R''}{d\,x}\,dx + \frac{d\,S}{d\,x}\,dy,$$
and because x and y are variables independent of each other, and consequently dx and dy are in the same case, this equation will give
$$\frac{d\,Q}{d\,y} = \frac{d\,R''}{d\,x}, \quad \frac{d\,R'}{d\,y} = \frac{d\,S}{d\,x}.$$
But $R'' = R - R'$, and consequently $\frac{d\,R''}{d\,x} = \frac{d\,R}{d\,x} - \frac{d\,R'}{d\,x}$. Substituting this value, we find
$$\frac{d\,Q}{d\,y} = \frac{d\,R}{d\,x} - \frac{d\,R'}{d\,x}, \quad \frac{d\,R'}{d\,y} = \frac{d\,S}{d\,x}.$$
If we differentiate the first equation with respect to x, and the second with respect to y, $\frac{d^2 R'}{d\,x\,d\,y}$ will be in both equations, and by eliminating it, we shall have
$$\frac{d^2 Q}{d\,y^2} + \frac{d^2 S}{d\,x^2} = \frac{d^2 R}{d\,x\,d\,y}.$$
Such is the condition to be fulfilled, in order that $Q\,dx^2 + R\,dx\,dy + S\,dy^2$ should be the differential of a differential of the first order. Similar means would lead to the conditions relative to higher orders.

When the above condition is satisfied, the first integral of $Q\,dx^2 + R\,dx\,dy + S\,dy^2$ is readily obtained. We know that it must be of the form $U\,dx + V\,dy$, therefore the term $Q\,dx^2$ must be the differential of $U\,dx$ taken with respect to x, and consequently $U = \int Q\,dx$. In the same manner V must be equal to $\int S\,dy$. Thus
$$\int (Q\,dx^2 + R\,dx\,dy + S\,dy^2) = dx \int Q\,dx + dy \int S\,dy.$$
We have now to verify that this integral is exact, when $\frac{d^2 Q}{d\,y^2} + \frac{d^2 S}{d\,x^2} = \frac{d^2 R}{d\,x\,d\,y}$. For that we must prove that its complete differential is equal to $Q\,dx^2 + R\,dx\,dy + S\,dy^2$. By differentiating, we obtain $Q\,dx^2 +$

$S\,d\,y^2 + d\,x\,d\,y\left(\frac{d\,.\int Q\,d\,x}{d\,y} + \frac{d\,.\int S\,d\,y}{d\,x}\right)$. It will be sufficient to show, therefore, that

$$R = \frac{d\,.\int Q\,d\,x}{d\,y} + \frac{d\,.\int S\,d\,y}{d\,x}.$$

Differentiating this equation first with respect to x and then with respect to y, we shall find

$$\frac{d^2\,R}{d\,x\,d\,y} = \frac{d^3\,Q}{d\,y^3} + \frac{d^3\,S}{d\,x^3},$$

which is precisely the condition of integrability.

It may be also observed, that since the first integral of $Q\,d\,x^2 + R\,d\,x\,d\,y + S\,d\,y^2$ is $d\,x\int Q\,d\,x + d\,y\int S\,d\,y$, the conditions to be fulfilled, in order that $Q\,d\,x^2 + R\,d\,x\,d\,y + S\,d\,y^2$ should be the second differential of a function of x and y, are

$$\frac{d^2\,Q}{d\,y^2} + \frac{d^2\,S}{d\,x^2} = \frac{d^2\,R}{d\,x\,d\,y}\text{, and }\frac{d\int Q\,d\,x}{d\,y} = \frac{d\int S\,d\,y}{d\,x}.$$

Or differentiating the last equation twice, first with respect to x, and afterwards with respect to y, the two conditions will become

$$\frac{d^2\,Q}{d\,y^2} = \frac{1}{2}\,\frac{d^2\,R}{d\,x\,d\,y},\quad \frac{d^2\,Q}{d\,y^2} = \frac{d^2\,S}{d\,x^2}.$$

These conditions are verified, for instance, for the differential $y^2\,d\,x^2 + 4\,x\,y\,d\,x\,d\,y + x^2\,d\,y^2$, and we find by means of the preceding rules, that the integral is $x^2\,y^2$ without the constants.

(108.) We have proved (41) that if n be the sum of the exponents in each term of a homogeneous function u of the variables x, y, z, &c. then

$$n\,u = \frac{d\,u}{d\,x}\,x + \frac{d\,u}{d\,y}\,y + \frac{d\,u}{d\,z} + \text{\&c.}$$

This theorem may sometimes facilitate the integration of the complete differential of a function of several variables. It follows from the rules given for the differentiation of algebraical functions, that the differentials of homogeneous functions are themselves homogeneous. Hence if a given differential $M\,d\,x + N\,d\,y +$ &c. be homogeneous, we may infer that the integral is in the same case. If, therefore, $M\,d\,x + N\,d\,y +$ &c. fulfil the conditions of integrability, if u represent the integral, and m the sum of the exponents in each of its terms, we shall have

$$m\,u = M\,x + N\,y + \text{\&c. since } M = \frac{d\,u}{d\,x},\quad N = \frac{d\,u}{d\,y}, \text{\&c.}$$

This value of $m\,u$ proves also that $m = n + 1$, n being the degree of the functions M, N, &c., consequently

$$u = \int M\,d\,x + N\,d\,y + \text{\&c.} = \frac{M\,x + N\,y + \text{\&c.}}{n+1}.$$

This method of integration cannot be used when $n = -1$, since then the denominator of the value of u becomes nothing.

The relations which have been found (41) between a homogeneous function of several variables and the partial differential coefficients of orders higher than the first, might also be used, in some cases, to find the integrals of differentials of higher orders.

We shall now apply the rules which have been given for the integration of differentials of functions of several variables to a few examples.

Example 1. Let the differential be $(x^2 + x\,y + y^2)\,d\,x + (x^2 - x\,y + y^2)\,d\,y$.
Here $M = x^2 + x\,y + y^2$, $N = x^2 - x\,y + y^2$, therefore

$$\frac{d\,M}{d\,y} = x + 2\,y,\quad \frac{d\,N}{d\,x} = 2\,x - y;$$

these two quantities are not equal, therefore $(x^2 + x\,y + y^2)\,d\,x + (x^2 - x\,y + y^2)\,d\,y$ is not the differential of a function of two variables.

Example 2. $(a\,x + b\,y + c)\,d\,x + (b\,x + e\,y + f)\,d\,y$.

In this case $\frac{d\,M}{d\,y} = \frac{d\,N}{d\,x} = b.$

We shall have the primitive function by integrating first $(a\,x + b\,y + c)\,d\,x$, considering y as a constant, and adding to it the integrals of the terms of $(b\,x + e\,y + f)\,d\,y$ which do not contain y. We shall find

$$\int (a\,x + b\,y + c)\,d\,x + (b\,x + e\,y + f)\,d\,y = \frac{a\,x^2}{2} + b\,x\,y + c\,x + \frac{e\,y^2}{2} + f\,y + c.$$

Example 3. $$d\,u = \frac{d\,x}{y} + \frac{d\,y}{d\,x} - \frac{y\,d\,x}{x^2} - \frac{x\,d\,y}{y^2}.$$

we have
$$M = \frac{1}{y} - \frac{y}{x^2}, \qquad N = \frac{1}{x} - \frac{x}{y^2},$$
$$\frac{d\,M}{d\,y} = \frac{d\,N}{d\,x} = -\frac{1}{x^2} - \frac{1}{y^2}, \qquad \int M\,d\,x = \int\left(\frac{1}{y} - \frac{y}{x^2}\right)d\,x = \frac{x}{y} + \frac{y}{x} + Y,$$
in which Y is an arbitrary function of x.
$$N = \frac{1}{x} - \frac{x}{y^2} = \frac{d\,.\left(\frac{x}{y} + \frac{y}{x} + Y\right)}{d\,y} = -\frac{x}{y^2} + \frac{1}{x} + \frac{d\,Y}{d\,y},$$
hence
$$\frac{d\,Y}{d\,y} = 0, \quad Y = c, \quad \text{and} \int\left(\frac{d\,x}{y} + \frac{d\,y}{x} - \frac{y\,d\,x}{x^2} - \frac{x\,d\,y}{y^2}\right) = \frac{x}{y} + \frac{y}{x} + c.$$

Example 4. $$d\,u = (3\,x^2 + 2\,a\,x\,y)\,d\,x + (a\,x^2 + 3\,y^2)\,d\,y.$$

In this example the functions M and N are homogeneous and of the same degree $n = 2$, and moreover, the condition of integrability is satisfied, for
$$\frac{d\,M}{d\,y} = \frac{d\,N}{d\,x} = 2\,a\,x, \text{ therefore } \int(3\,x^2 + 2\,a\,x\,y)\,d\,x + (a\,x^2 + 3\,y^2)\,d\,y = \frac{(3\,x^2 + 2\,a\,x\,y)\,x + (a\,x^2 + 3\,y^2)\,y}{3}$$
$$= x^3 + a\,x^2\,y + y^3 + c.$$

Example 5. $$d\,u = \frac{y\,z\,(y+z)}{(x+y+z)^2}\,d\,x + \frac{x\,z\,(x+z)}{(x+y+z)^2}\,d\,y + \frac{x\,y\,(x+y)}{(x+y+z)^2}\,d\,z.$$

The conditions of integrability are satisfied, and M, N, are homogeneous functions the degree of which is one, therefore we shall have
$$u = \tfrac{1}{2}\left\{\frac{y\,z\,(y+z)\,x + x\,z\,(x+z)\,y + x\,y\,(x+y)\,z}{(x+y+z)^2}\right\} = \frac{x\,y\,z}{x+y+z} + c.$$

(109.) When a function of several variables x, y, &c. is differentiated in the supposition that x, y, &c. are functions of other variables, there arise differential expressions containing x, y, &c. $d\,x$, $d\,y$, &c. $d^2 x$, $d^2 y$, &c. $d^n x$, $d^n y$, &c. Reciprocally, when such a differential expression occurs, it may be required to determine the function from the differentiation of which it is supposed to have resulted.

We shall consider the case of two variables only. Let U be any function of x, y, $d\,x$, $d\,y$, $d^2 x$, $d^2 y$ .... $d^n x$, $d^n y$, and let U_1 be its integral, that is a function of the same kind, but containing neither $d^n x$ nor $d^n y$; if we make
$$d\,x = x_1, \quad d^2 x = x_2, \quad d^3 x = x_3 \ldots\ldots\ldots$$
$$d\,y = y_1, \quad d^2 y = y_2, \quad d^3 y = y_3 \ldots\ldots\ldots$$
U_1 will become a function of the variables x, y, x_1, y_1, ... x_{n-1}, y_{n-1}, and its complete differential will be
$$d\,U_1 = \left\{\begin{aligned} &\frac{d\,U_1}{d\,x}\,d\,x + \frac{d\,U_1}{d\,x_1}\,d\,x_1 + \frac{d\,U_1}{d\,x_2}\,d\,x_2 + \ldots\ldots\ldots + \frac{d\,U_1}{d\,x_{n-1}}\,d\,x_{n-1} \\ &+ \frac{d\,U_1}{d\,y}\,d\,y + \frac{d\,U_1}{d\,y_1}\,d\,y_1 + \frac{d\,U_1}{d\,y_2}\,d\,y_2 + \ldots\ldots\ldots + \frac{d\,U_1}{d\,y_{n-1}}\,d\,y_{n-1}\,; \end{aligned}\right.$$
but $d\,U_1 = U$, substituting also for $d\,x$, $d\,x_1$, &c. their values x_1, x_2, &c. we shall have
$$U = \left\{\begin{aligned} &\frac{d\,U_1}{d\,x}\,x_1 + \frac{d\,U_1}{d\,x_1}\,x_2 + \frac{d\,U_1}{d\,x_2}\,x_3 + \ldots\ldots\ldots + \frac{d\,U_1}{d\,x_{n-1}}\,x_n \\ &+ \frac{d\,U_1}{d\,y}\,y_1 + \frac{d\,U_1}{d\,y_1}\,y_2 + \frac{d\,U_1}{d\,y_2}\,y_3 + \ldots\ldots\ldots + \frac{d\,U_1}{d\,y_{n-1}}\,y_n \end{aligned}\right.$$

If we differentiate U successively with respect to each of the variables x, y, ... x_n, y_n, we shall have first with respect to x
$$\frac{d\,U}{d\,x} = \left\{\begin{aligned} &\frac{d^2 U_1}{d\,x^2}\,x_1 + \frac{d^2 U_1}{d\,x\,d\,x_1}\,x_2 + \frac{d^2 U_1}{d\,x\,d\,x_2}\,x_3 + \ldots\ldots\ldots + \frac{d^2 U_1}{d\,x\,d\,x_{n-1}}\,x_n \\ &+ \frac{d^2 U_1}{d\,x\,d\,y}\,y_1 + \frac{d^2 U_1}{d\,x\,d\,y_1}\,y_2 + \frac{d^2 U_1}{d\,x\,d\,y_2}\,y_3 + \ldots\ldots\ldots + \frac{d^2 U_1}{d\,x\,d\,y_{n-1}}\,y_n\,; \end{aligned}\right.$$

and by inverting the order of differentiation in each term, we may easily see that the right side of the equation

is the complete differential of $\frac{dU_1}{dx}$, therefore $\frac{dU}{dx} = d\frac{dU_1}{dx}$. Let us differentiate now with respect to x_1, we shall find

$$\frac{d^n U}{d x_1} = \begin{cases} \frac{dU_1}{dx} + \frac{d^n U_1}{dx_1\,dx}x_1 + \frac{d^n U_1}{dx_1\,dx_1}x_2 + \frac{d^n U_1}{dx_1\,dx_2}x_3 + \ldots\ldots\ldots + \frac{d^n U_1}{dx_1\,dx_{n-1}}x_n \\ + \frac{d^n U_1}{dx_1\,dy}y_1 + \frac{d^n U_1}{dx_1\,dy_1}y_2 + \frac{d^n U_1}{dx_1\,dy_2}y_3 + \ldots\ldots\ldots + \frac{d^n U_1}{dx_1\,dy_{n-1}}y_n; \end{cases}$$

inverting the order of the differentiations in every term in which U_1 is differentiated twice, we shall have $\frac{dU}{dx_1} = \frac{dU_1}{dx} + d\frac{dU_1}{dx_1}$, and we shall find in the same manner

$$\frac{dU}{dx_2} = \frac{dU_1}{dx_1} + d\frac{dU_1}{dx_2}$$

$$\frac{dU}{dx_3} = \frac{dU_1}{dx_2} + d\frac{dU_1}{dx_3}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{dU}{dx_{n-1}} = \frac{dU_1}{dx_{n-2}} + d\frac{dU_1}{dx_{n-1}}.$$

But when we come to $x_n = d^n x$, which is not in U_1, since that function is only of the $(n-1)^{th}$ order, we shall have simply $\frac{dU}{dx_n} = \frac{dU_1}{dx_{n-1}}$.

By differentiating with respect to y, y_1, &c. we should obtain similar results. We may without difficulty eliminate U_1 from these equations; for that it will be sufficient to subtract from the equation $\frac{dU}{dx} = d\frac{dU_1}{dx}$, the differential of $\frac{dU}{dx_1} = \frac{dU_1}{dx} + d\frac{dU_1}{dx_1}$, then add the differential of the second order of the following equation, subtract the differential of the third order of the next, &c. &c. We shall find

$$\frac{dU}{dx} - d\frac{dU}{dx_1} + d^2\frac{dU}{dx_2} - d^3\frac{dU}{dx_3} + \text{\&c.} = 0,$$

and in the same manner

$$\frac{dU}{dy} - d\frac{dU}{dy_1} + d^2\frac{dU}{dy_2} - d^3\frac{dU}{dy_3} + \text{\&c.} = 0.$$

We should have had as many similar equations as there were variables, if instead of two we had supposed any other number. These equations will be verified whenever U is the differential of a function U_1 of an order less by one than U. If, therefore, we wish to ascertain whether a differential function, of the n^{th} order, be the differential of another function of the $(n-1)^{th}$ order, we shall assume $dx = x_1$, $d^2x = x_2, \ldots dy = y_1$, $d^2y = y_2$, &c. $\ldots dz = z_1$, $d^2z = z_2$, &c., and then, the function being represented by U, we shall form the quantities

$$\frac{dU}{dx}, \frac{dU}{dx_1} \ldots\ldots \frac{dU}{dy}, \frac{dU}{dy_1} \ldots\ldots \frac{dU}{dz}, \frac{dU}{dz_1}, \text{\&c. \&c.},$$

and we shall substitute them in the equations we have found; if they are not satisfied, we may safely conclude that the function U is not the differential of a function of the $(n-1)^{th}$ order.

Let us take for example the function $x\,d^2y - y\,d^2x$, it will be changed into $xy_2 - yx_2 = U$, and we shall have

$$\frac{dU}{dx} = y_2, \quad \frac{dU}{dx_1} = 0, \quad \frac{dU}{dx_2} = -y,$$

$$\frac{dU}{dy} = -x_2, \quad \frac{dU}{dy_1} = 0, \quad \frac{dU}{dy_2} = x,$$

which give the following equations,

$$y_2 - d^2y = 0, \quad -x_2 d^2 x = 0;$$

these being satisfied, we may infer that $x\,d^2y - y\,d^2x$ is the differential of a function of the first order; and it is, in fact, the differential of $x\,dy - y\,dx$.

When U is of an order superior to the first, then it may be required to determine whether U_1 be the differential of a function U_2 of an order less by one, or in other words to determine whether U be the second differential

of a function U_2 of an order less by two, &c. The equations which express these conditions may easily be formed by means of what precedes. If U_1 be the result of the differentiation of U_2, then the equation

$$\frac{d\,U_1}{d\,x} - d\,\frac{d\,U_1}{d\,x_1} + d^2\frac{d\,U_1}{d\,x_2} - d^3\frac{d\,U_1}{d\,x_3} + \text{\&c.} = 0,$$

limited to the differential x_{n-1}, and others similar with respect to the other variables, must be satisfied. The values of the differential coefficients may be readily found in terms of the differential coefficients of U, by means of the relations found before; if we substitute them in the equation above, we shall find

$$\frac{d\,U}{d\,x_1} - 2\,d\,\frac{d\,U}{d\,x_2} + 3\,d^2\frac{d\,U}{d\,x_3} - 4\,d^3\frac{d\,U}{d\,x_4} + \text{\&c.}$$

and we shall have similar equations for each of the other variables; all these joined to the equations which express that U is the differential of a function U_1 must be satisfied, in order that U should be a second differential of a function U_2. Similar considerations will prove that U, in order to be a third differential of a function U_3, must satisfy, besides the preceding, the equation

$$\frac{d\,U}{d\,x_2} - 3\,d\,\frac{d\,U}{d\,x_3} + 6\,d^2\,\frac{d\,U}{d\,x_4} - \text{\&c.} = 0,$$

and those alike applicable to the other variables.

We have supposed, in order to be more general, none of the first differentials $d\,x$, $d\,y$, &c. to be constant; if it were not the case, then the equations relative to the variables, the differentials of which are supposed to be constant, should be suppressed. If we suppose $d\,x$, for instance, to be constant, it is obvious that all the differential coefficients taken with respect to x_1 would vanish.

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OF THE

PRINCIPAL MATTERS

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TREATISE OF DIFFERENTIAL AND INTEGRAL CALCULUS

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